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STATIONARY SOLUTIONS OF OUTFLOW PROBLEM FOR FULL COMPRESSIBLE NAVIER-STOKES-POISSON SYSTEM: EXISTENCE, STABILITY AND CONVERGENCE RATE*

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Abstract. In this paper, we study the asymptotic behavior of solution to the initial boundary value problem for the non-isentropic Navier-Stokes-Poisson system in a half line $(0,\infty)$. We consider an outflow problem where the gas blows out the region through the boundary for general gases including ideal polytropic gas. First, we give necessary condition for the existence of stationary solution by use of the center manifold theory. Second, using energy method we show the asymptotic stability of the solutions under assumptions that the boundary value and the initial perturbation is small. Third, we prove that the algebraic and exponential decay of the solution toward supersonic stationary solution is obtained, when the initial perturbation belongs to Sobolev space with algebraic and exponential weight respectively.

Keywords. Navier-Stokes-Poisson equations; outflow problem; stationary solution; stability; convergence rate.

AMS subject classifications. 76N06; 34K21; 41A25; 39A30; 76N10.

1. Introduction and main results

The compressible Navier-Stokes-Poisson (called NSP in the sequel for simplicity) system may be used to simulate the transport of charged particles (e.g., electrons and ions) under the influence of the electro-static potential force governed by the self-consistent Poisson equation. In this paper, we consider the following compressible NSP equations in Eulerian coordinate:

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p)_x=\rho E+\mu u_{xx}, \\ [\rho(e+\frac{u^2}{2})]_t+[\rho u(e+\frac{u^2}{2})+pu]_x=\rho uE+\kappa\theta_{xx}+\mu(uu_x)_x, \\ \partial_x E=\rho-\rho_+, \end{cases} \tag{1.1}$$

where the unknown functions are the density $\rho(x,t)>0$, the velocity $u(x,t)$, the temperature $\theta(x,t)>0$, and the electron field $E(x,t)$. Also, $p=p(\rho,\theta)$ is the pressure and $e=e(\rho,\theta)$ is the internal energy, while $\mu>0$, $\kappa>0$ and $\rho_+>0$ denote the viscosity, the heat-conductivity and the doping profile, respectively.

We consider the system (1.1) supplemented with the initial data and far field condition

$$\begin{cases} (\rho,u,\theta)\,|_{t=0}=(\rho_0,u_0,\theta_0)(x), \quad x>0, \\ \lim_{x\to+\infty}(\rho,u,\theta,E)(x,t)=(\rho_+,u_+,\theta_+,0), \end{cases} \tag{1.2}$$

and the outflow boundary condition

$$u\,|_{x=0}=u_-<0, \quad \theta\,|_{x=0}=\theta_-, \tag{1.3}$$

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where $u_\pm, \theta_\pm > 0$ are prescribed constants. We are interested in the large-time behavior of solutions for the initial-boundary value problem (1.1)-(1.3) in the case of $u_- < 0$, that is, outflow problem.

Throughout this paper, we assume that

$$p_\rho(\rho,\theta) > 0, \quad e_\theta(\rho,\theta) > 0. \tag{1.4}$$

Setting $v = \rho^{-1}$, it is well-known that given any two variables of the thermodynamical variables (v, p, e, θ and s), the remaining three are smooth functions of the others, where s is entropy of gas. The second law of thermodynamics $\theta ds = de + pdv$ asserts that, if we choose (v,θ) or (v,s) as the independent variables and write $(p,e,s) = (p(v,\theta), e(v,\theta), s(v,\theta))$ or $(p,e,\theta) = (\widetilde{p}(v,s), \widetilde{e}(v,s), \widetilde{\theta}(v,s))$ respectively. Then, by (1.5) in [5] and (1.4), we can deduce

$$\widetilde{p}_v(v,s) = p_v(v,\theta) - \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)} < 0, \tag{1.5}$$

and

$$\begin{cases} \widetilde{e}_{ss}(v,s) = \frac{\theta}{e_\theta(v,\theta)} > 0, \quad \widetilde{e}_{vs}(v,s) = -\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}, \\ \widetilde{e}_{vv}(v,s) = -p_v(v,\theta) + \frac{\theta(p_\theta(v,\theta))^2}{e_\theta(v,\theta)} > 0, \end{cases} \tag{1.6}$$

which means that $\widetilde{e}(v,s)$ is convex with respect to (v,s).

Notation: Throughout this paper, $O(1), c$ or C denote a generic positive constant independent of t, x and $c_i(\cdot,\cdot)$ or $C_i(\cdot,\cdot)(i \in Z_+)$ stands for generic constant that depends only on the quantities listed in parentheses. $H^k := H^k(0,\infty)$ denote the Sobolev space with norm $\|\cdot\|_k$ and $\|\cdot\|_0 = \|\cdot\|$ will be used to denote the usual L^2-norm.

Now, we state the results of this paper. The stationary solution $(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E})(x)$ of the problem (1.1)-(1.3) must satisfy the system (1.7):

$$\begin{aligned} &(\hat{\rho}\hat{u})_x = 0, \qquad x > 0, \\ &(\hat{\rho}\hat{u}^2 + \hat{p})_x = \hat{\rho}\hat{E} + \mu \hat{u}_{xx}, \\ &[\hat{\rho}\hat{u}(\hat{e} + \frac{\hat{u}^2}{2}) + \hat{p}\hat{u}]_x = \hat{\rho}\hat{u}\hat{E} + \kappa\hat{\theta}_{xx} + \mu(\hat{u}\hat{u}_x)_x, \\ &\hat{E}_x = \hat{\rho} - \rho_+ \\ &(\hat{u},\hat{\theta})(0) = (u_-,\theta_-), \quad \lim_{x\to\infty} (\hat{\rho},\hat{u},\hat{\theta},\hat{E})(x) = (\rho_+, u_+, \theta_+, 0), \end{aligned} \tag{1.7}$$

where $\hat{p} = p(\hat{\rho},\hat{\theta})$, $\hat{e} = e(\hat{\rho},\hat{\theta})$.

Theorem 1.1 (Existence of stationary solution). *Let $\rho_+ > 0, \theta_\pm > 0, u_- < 0$. The necessary condition for the existence of a solution to the system (1.7) is*

$$\hat{\rho}(x)\hat{u}(x) = \rho_+ u_+ = \hat{\rho}(0) u_-, \, \forall x > 0. \tag{1.8}$$

If $u_+ \geq 0$, then there is no stationary solution for the system (1.7).

If $u_+ < 0$ and (1.4) holds, then there exists a positive constant δ_0 and a local manifold $\mathcal{M} \subset \mathcal{M}_{\delta_0} := \{(u,\theta) \in (-\infty,0)\times(0,\infty) \,|\, 0 < |(u-u_+, \theta-\theta_+)| \leq \delta_0\}$ such that if $(u_-,\theta_-) \in \mathcal{M}$, then the system (1.7) has a unique solution $(\hat{\rho},\hat{u},\hat{\theta},\hat{E})(x)$ satisfying

$$|\partial_x^k(\hat{\rho}-\rho_+, \hat{u}-u_+, \hat{\theta}-\theta_+, \hat{E})| \leq C\delta \exp(-\hat{c}x), \quad k = 0,1,2,\cdots, \tag{1.9}$$

where $\delta = |(u_- - u_+, \theta_- - \theta_+)|$ *and* $C, \hat{c}$ *are positive constants independent of* x, δ.

REMARK 1.1. If $\delta \neq 0$, then $\hat{E}$ cannot vanish. Compared with the previous results (see [17]), $\hat{E} \neq 0$ is a fundamental difference, and the stationary solution of the system (1.7) converges to the spatial asymptotic state with an exponential rate even for the case $M_+ = 1$, where M_+ denotes the Mach number at far field $x = \infty$ defined by

$$M_+ = \frac{|u_+|}{\sqrt{p_\rho(\rho_+, s_+)}}.$$

Next, we state the result for the stability toward the stationary solution for the problem (1.1)-(1.4).

THEOREM 1.2 (Asymptotic stability of solution). *Let* $\rho_\pm > 0, u_\pm < 0, \theta_\pm > 0$. *Suppose that there exists the solution* $(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E})(x)$ *to the system* (1.7) *satisfying* (1.9). *In addition, suppose that the initial data* (ρ_0, u_0, θ_0) *satisfies*

$$(\rho_0 - \hat{\rho}, u_0 - \hat{u}, \theta_0 - \hat{\theta}) \in H^1(0,\infty),\ u_0(0) = u_-,\ \theta_0(0) = \theta_-. \tag{1.10}$$

Then, there is a positive constant ε_0 *such that if*

$$\|(\rho_0 - \hat{\rho}, u_0 - \hat{u}, \theta_0 - \hat{\theta})\|_1 + \|E(\cdot,0) - \hat{E}\| + \delta \le \varepsilon_0, \tag{1.11}$$

where $\delta = |(u_- - u_+, \theta_- - \theta_+)|$, *then the problem* (1.1)-(1.4) *has a unique global solution* $(\rho, u, \theta, E)(x,t)$ *satisfying*

$$\begin{aligned}&(\rho - \hat{\rho}, u - \hat{u}, \theta - \hat{\theta}) \in C([0,\infty); H^1(0,\infty)), \quad E - \hat{E} \in C([0,\infty); L^2(0,\infty)),\\ &\rho_x, E_x \in L^2(0,\infty; L^2(0,\infty)),\ u_x, \theta_x \in L^2(0,\infty; H^1(0,\infty)).\end{aligned}$$

Moreover, the solution $(\rho, u, \theta, E)(x,t)$ *tends time-asymptotically to the stationary solution* $(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E})(x)$ *in the sense that*

$$\lim_{t\to\infty} \sup_{x\in(0,\infty)} |(\rho, u, \theta, E)(x,t) - (\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E})(x)| = 0.$$

THEOREM 1.3 (Convergence rate of solutions). *Let* $\rho_\pm > 0, u_\pm < 0, \theta_\pm > 0$. *Suppose that there exists the solution* $(\hat{\rho}, \hat{u}, \hat{\theta})(x)$ *to the* (1.7) *satisfying* (1.9) *for the case of* $M_+ > 1$. *If* (1.10) *and* (1.11) *hold, we have*

(1) exponential decay of solutions *if* $(\rho - \hat{\rho}, u - \hat{u}, \theta - \hat{\theta}, E - \hat{E})(\cdot, 0) \in L^2_{\varsigma,\exp}(0,\infty)$, *there is a constant* $\beta > 0$ *depending on the* ς *such that the solution* $(\rho, u, \theta)(x,t)$ *to the problem* (1.1)-(1.4) *satisfies*

$$\sup_{x\in(0,\infty)} |(\rho, u, \theta, E)(x,t) - (\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E})(x)| \le Ce^{-\beta t}.$$

(2) algebraic decay of solutions *if* $(\rho - \hat{\rho}, u - \hat{u}, \theta - \hat{\theta}, E - \hat{E})(\cdot, 0) \in L^2_\varsigma(0,\infty)$, *then the solution* $(\rho, u, \theta)(x,t)$ *to the problem* (1.1)-(1.4) *satisfies*

$$\sup_{x\in(0,\infty)} |(\rho, u, \theta, E)(x,t) - (\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E})(x)| \le C(1+t)^{-\frac{\varsigma}{2}},$$

where $\varsigma > 0$ *and*

$$\begin{aligned}L^2_{\varsigma,\exp}(0,\infty) &:= \{f \in L^2_{loc}(0,\infty); \int_0^\infty e^{\varsigma x} f^2(x)dx < \infty\},\\ L^2_\varsigma(0,\infty) &:= \{f \in L^2_{loc}(0,\infty); \int_0^\infty (1+x)^\varsigma f^2(x)dx < \infty\}.\end{aligned}$$

REMARK 1.2. In the case of isentropic gas, similar results as in Theorem 1.2 and Theorem 1.3 were recently obtained by [31] and [16].

Related results: From a physical point of view, the motion of the ion-dust plasma ([10, 19]), the self-gravitational viscous gaseous stars [4] and the charged particles in semiconductor devices [22] can be governed by the NSP system. In recent years, there have been a great number of mathematical studies about the compressible NSP system. In what follows, we only mention some of them related to our interest. Recently, the global existence and convergence rates for the three-dimensional NSP system around a non-vacuum constant state were studied through carrying out the spectrum analysis by [13,14,20,21,34] etc. The pointwise estimate of the solution for the multi-dimensional NSP system was discussed in [30]. In addition, the global strong solution to the one-dimensional non-isentropic NSP system with large data for density-dependent viscosity was established by [27] and nonexistence was discussed in [3]. The above works show that the momentum of the NSP system decays at a slower rate than that of the compressible Navier-Stokes system in the absence of the electric field. This fully demonstrates that the electric field could affect the large-time behavior of the solution.

For the Cauchy problem of the NSP system, the large-time behavior around nonlinear wave patterns began to be studied (see [2, 8, 9, 26]). For the multi-dimensional isentropic NSP system, the stability of stationary states was studied by Tan-Wang-Wang [26] and Cai-Tan [2] in the case with non-flat doping profile and with an external force, respectively, under the assumption in that the gas states at far fields $\pm\infty$ are equal. For the one dimensional two-fluid NSP system with different gas states at far fields, the stability of rarefaction waves was studied by Duan-Liu [8] and Duan-Liu-Yin-Zhu [2] in isothermal and non-isentropic cases, respectively, with the nontrivial electric potential.

Another interesting and challenging problem is to study the stability of the NSP system on half space with different gas states at boundary and far field. In general, it is well known that the large-time behavior of solutions to the NSP system in the half space is much more complicated than that for the corresponding Cauchy problem due to boundary effect.

Recently, the mathematical studies to clarify the stability of nonlinear wave patterns for outflow and inflow problem on the compressible NSP system was begun (see [6, 7, 11, 12, 16, 31–33]). Duan-Yang [7] first proved the stability of rarefaction wave and boundary layer for outflow problem on the two-fluid isentropic NSP system. Later, for outflow problem on the two-fluid isentropic NSP system, Zhou-Li [33] showed convergence rate toward stationary solutions and Yin-Zhang-Zhu [32] proved the stability of the superposition of stationary solutions and rarefaction wave. Also, for outflow problem on the two-fluid non-isentropic NSP system, Cui-Gao-Yin-Zhang [6] proved the stability and convergence rate toward stationary solutions. Recently, for inflow problem on the two-fluid non-isentropic NSP system, Hong-Shi-Wang [12] proved the stability of stationary solutions. One important point used in [6, 7, 12, 32, 33] was that the large-time behavior of the electric field was trivial and hence the two-fluids indeed had the same asymptotic profiles which were constructed from Navier-Stokes equations without any force under the assumptions that all physical parameters in the model must be unity, which was obviously impractical since ions and electrons generally had different masses. On the other hand, for outflow problem on the unipolar isentropic NSP system with doping profile, Jiang-Lai-Yin-Zhu [16] and Wang-Zhang-Zhang [31] studied the existence, stability and convergence rate of stationary solutions. Recently, Hong [11] obtained the stability result for inflow problem on the unipolar non-isentropic NSP system with doping profile. However, to the best of our knowledge, there is little research about the stability of nonlinear wave patterns for the outflow problem on the unipolar non-isentropic NSP system with doping profile which is of interest in our paper.

Here, we briefly review some main difficulties. For the outflow problem of the single quasineutral Navier-Stokes system $(1.1)_1$-$(1.1)_3$ with $E=0$, there have been many mathematical studies about the existence, stability and convergence rate of the stationary solutions, please refer to [5, 15, 17, 18, 23–25, 28, 29] and the references therein. Compared to the Navier-Stokes model in [17], our problem is more general and more complex for the electric field is taken into account. For instance, in order to obtain the existence of stationary solutions, we have to introduce the new variable to deduce the stationary equations to a 4×4 system of autonomous ordinary differential equations, and examine dynamics around an equilibrium by applying the manifold theory (see Section 2). Next, to deduce our results desired for the stability of the stationary solutions by the elementary energy method as in [17], it is sufficient to deduce certain uniform (with respect to the time t) a priori estimates on the perturbations $(\varphi,\psi,\zeta,\chi)$ around stationary solutions. In the first step of a priori estimates, comparing with [17], the main difficulty is to control the energy form (3.4) so that we get the uniform estimate for L_2-norm of the perturbations, which is not trivial for the general gas including ideal polytropic gas (see Section 3). Finally, the main point of the proof for the convergence rate of the solutions is how to get the lower estimate for the term $-w_xG^1$ in weighted energy form (4.3). To do this, we derive a lower estimate (4.9) for the leading order term $-G^1_1$ for Taylor expansion of the $-G^1$ and rely on the fact that the other terms are smaller than the leading order term (see Section 4).

The remainder of this paper consists of the following: In Section 2, we discuss the existence of the stationary solutions and present the proof of Theorem 1.1 with the aid of the stable manifold theory. Section 3 is devoted to showing the asymptotic stability result (Theorem 1.2) of the stationary solutions. In Section 4, for the supersonic case, the convergence rate mentioned in Theorem 1.3 is obtained by a time- and space-weighted energy method.

2. The existence of stationary solutions

2.1. Reformulation of stationary problem.

Integrating the first, second and third equations in (1.7) over $[x,\infty)$ yields

$$\begin{aligned}
&\hat\rho\hat u=\rho_+u_+,\ x>0,\\
&\mu\hat u_x=\rho_+u_+(\hat u-u_+)+(\hat p-p_+)+\int_x^\infty\hat\rho\hat E dy,\\
&\kappa\hat\theta_x=\rho_+u_+\left((\hat e-e_+)+\frac{1}{2}(\hat u^2-u_+^2)-\hat E_1\right)+(\hat p\hat u-p_+u_+)-\mu\hat u\hat u_x,
\end{aligned}\tag{2.1}$$

where $p_+=p(v_+,\theta_+)$, $e_+=e(v_+,\theta_+)$, $\hat E_1(x)=-\int_x^\infty\hat E(y)dy$.

Integrating the first equation in (1.7) over $[0,x)$ yields

$$\hat\rho\hat u=\hat\rho(0)u_-,\ x>0.\tag{2.2}$$

By (2.1) and (2.2), there holds (1.8). So, if $u_+\geq 0$, then there is no solution to the system (1.7). Using $\hat\rho=\hat E_x+\rho_+$, we have

$$\int_x^\infty\hat\rho\hat E dy=-\frac{1}{2}\hat E^2-\rho_+\hat E_1.\tag{2.3}$$

We set $\hat{u}=\frac{u_+}{v_+}\hat{v}\,(\hat{v}=\hat{\rho}^{-1},v_+=\rho_+^{-1})$, $\hat{u}_1=\hat{u}_x$. Then, we have from (2.1) and (2.3)

$$\begin{cases} \hat{v}_x=\frac{u_+}{\mu v_+}(\hat{v}-v_+)+\frac{v_+}{\mu u_+}(\hat{p}-p_+)-\frac{v_+}{2\mu u_+}\hat{E}^2-\frac{1}{\mu u_+}\hat{E}_1, \\ \hat{\theta}_x=\dfrac{u_+}{\kappa v_+}(\hat{e}-e_+)-\dfrac{u_+^3}{2\kappa v_+^3}(\hat{v}-v_+)^2+\dfrac{u_+}{\kappa v_+}p_+(\hat{v}-v_+) \\ \qquad +\dfrac{u_+}{2\kappa v_+}\hat{v}\hat{E}^2+\dfrac{u_+}{\kappa v_+^2}\hat{E}_1(\hat{v}-v_+), \\ \hat{E}_x=\frac{1}{\hat{v}}-\frac{1}{v_+}, \qquad (\hat{E}_1)_x=\hat{E}. \end{cases} \tag{2.4}$$

Also, we have

$$(\hat{v},\hat{\theta})(0)=(v_-,\theta_-)\text{ with }v_-=\frac{u_-}{u_+}v_+,\quad (\hat{v},\hat{\theta},\hat{E},\hat{E}_1)(\infty)=(v_+,\theta_+,0,0). \tag{2.5}$$

To discuss the solvability of the system (2.4), (2.5) near the infinity asymptotic state $(v_+,\theta_+,0,0)$, we need to introduce the stationary perturbation variables given by

$$(\widetilde{v},\widetilde{\theta},\widetilde{E},\widetilde{E}_1):=(\hat{v},\hat{\theta},\hat{E},\hat{E}_1)-(v_+,\theta_+,0,0).$$

Then, the system (2.4), (2.5) is transformed into the vector equations for $(\widetilde{v},\widetilde{\theta},\widetilde{E},\widetilde{E}_1)$

$$\frac{d}{dx}\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{E}\\ \widetilde{E}_1\end{pmatrix}=J_+\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{E}\\ \widetilde{E}_1\end{pmatrix}+\begin{pmatrix}g_1(\widetilde{v},\widetilde{\theta},\widetilde{E})\\ g_2(\widetilde{v},\widetilde{\theta},\widetilde{E},\widetilde{E}_1)\\ g_3(\widetilde{v})\\ 0\end{pmatrix},\quad x>0$$
$$(\widetilde{v},\widetilde{\theta})(0)=(v_--v_+,\theta_--\theta_+),\quad (\widetilde{v},\widetilde{\theta},\widetilde{E},\widetilde{E}_1)(\infty)=(0,0,0,0),$$

where J_+ is the Jacobian matrix at an equilibrium point $(0,0,0,0)$ defined by

$$J_+=\begin{pmatrix} \frac{v_+}{\mu u_+}\left(\frac{u_+^2}{v_+^2}+p_v^+\right) & \frac{v_+}{\mu u_+}p_\theta^+ & 0 & -\frac{1}{\mu u_+}\\ \frac{u_+}{\kappa v_+}(e_v^++p_+) & \frac{u_+}{\kappa v_+}e_\theta^+ & 0 & 0\\ -\frac{1}{v_+^2} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\equiv\begin{pmatrix} a_{11} & a_{12} & 0 & a_{14}\\ a_{21} & a_{22} & 0 & 0\\ a_{31} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{2.6}$$

and $g_i(i=1,\cdots,4)$ are nonlinear terms such that

$$g_1(\widetilde{v},\widetilde{\theta},\widetilde{E})=\frac{v_+}{\mu u_+}(\hat{p}-p_+-p_v^+\widetilde{v}-p_\theta^+\widetilde{\theta})-\frac{v_+}{2\mu u_+}\widetilde{E}^2=O(\widetilde{v}^2+\widetilde{\theta}^2+\widetilde{E}^2),$$
$$g_2(\widetilde{v},\widetilde{\theta},\widetilde{E},\widetilde{E}_1)=\frac{u_+}{\kappa v_+}(\hat{e}-e_+-e_v^+\widetilde{v}-e_\theta^+\widetilde{\theta})-\frac{u_+^3}{2\kappa v_+^3}\widetilde{v}^2+\frac{u_+}{2\kappa v_+}(\widetilde{v}+v_+)\widetilde{E}^2$$
$$+\frac{u_+}{\kappa v_+^2}\widetilde{E}_1\widetilde{v}=O(\widetilde{v}^2+\widetilde{\theta}^2+\widetilde{E}^2+\widetilde{E}_1\widetilde{v}),$$
$$g_3(\widetilde{v})=\frac{1}{\widetilde{v}+v_+}-\frac{1}{v_+}+\frac{\widetilde{v}}{v_+^2}=O(\widetilde{v}^2),$$

where $p_v^+=p_v(v_+,\theta_+)$, $e_v^+=e_v(v_+,\theta_+)$ and so on.

2.2. Proof of Theorem 1.1. By (2.6), we have

$$J_+ - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & 0 & a_{14} \\ a_{21} & a_{22}-\lambda & 0 & 0 \\ a_{31} & 0 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}$$

and the characteristic determinant of J_+ is

$$\begin{aligned} |J_+ - \lambda I| &= (-\lambda)\begin{vmatrix} a_{11}-\lambda & a_{12} & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & 0 & -\lambda \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22}-\lambda & 0 \\ a_{31} & 0 & -\lambda \\ 0 & 0 & 1 \end{vmatrix} \\ &= \lambda^2 \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} + a_{14}a_{31}(a_{22}-\lambda). \end{aligned}$$

Assume that $u_+ < 0$ and (1.4). Then, the eigenvalues $\lambda_i (i = 1, \cdots, 4)$ of J_+ must be satisfied

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0, \tag{2.7}$$

where

$$\begin{aligned} b_1 &= -(a_{11}+a_{22}) = -\frac{v_+}{\mu u_+}\left(\frac{u_+^2}{v_+^2} + p_v^+\right) - \frac{u_+}{\kappa v_+} e_\theta^+, \\ b_2 &= a_{11}a_{22} - a_{12}a_{21} = \frac{1}{\mu\kappa}\left(\frac{u_+^2}{v_+^2} + p_v^+\right) e_\theta^+ - \frac{1}{\mu\kappa}\left(e_v^+ + p_+\right) p_\theta^+, \\ b_3 &= -a_{14}a_{31} = \frac{-1}{\mu v_+^2 u_+} > 0, \\ b_4 &= a_{14}a_{31}a_{22} = \frac{1}{\kappa\mu v_+^3} e_\theta^+ > 0. \end{aligned}$$

From Vieta's formula, its roots have the following properties:

$$\begin{aligned} &\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -b_1, \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = b_2, \\ &\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = -b_3 < 0, \\ &\lambda_1\lambda_2\lambda_3\lambda_4 = b_4 > 0. \end{aligned}$$

By using

$$\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4 > 0, \\ &\lambda_1\lambda_2(\lambda_3 + \lambda_4) + (\lambda_1 + \lambda_2)\lambda_3\lambda_4 < 0, \end{aligned}$$

we obtain that (2.7) does not have any zero real root and the following possible cases :

$$\begin{aligned} &(1) \quad \lambda_1 < 0,\ \lambda_2 < 0,\ \lambda_3 > 0,\ \lambda_4 > 0 \\ &(2) \quad \lambda_1 = a + bi,\ \lambda_2 = a - bi (a < 0),\ \lambda_3 > 0,\ \lambda_4 > 0, \\ &(3) \quad \lambda_1 = a + bi,\ \lambda_2 = a - bi,\ \lambda_3 < 0,\ \lambda_4 < 0, \\ &(4) \quad \lambda_1 = a + bi, \lambda_2 = a - bi,\ \lambda_3 = c + di, \lambda_4 = c - di (a < 0 \,\text{or}\, c < 0), \end{aligned} \tag{2.8}$$

where $b \neq 0, d \neq 0$ and i denotes the imaginary number.

Without the loss of generality, we can assume $Re\lambda_1 < 0$ and $Re\lambda_2 < 0$ due to (2.8). Therefore, applying the center manifold theory (see [1]), we can prove Theorem 1.1 by the same lines as in [11]. We will omit it for brevity. Thus Theorem 1.1 is proved.

3. Asymptotic stability of stationary solutions

We can rewrite (1.1) and (1.7) as

$$\begin{cases} \rho_t+(\rho u)_x=0, \qquad x>0,\ t>0,\\ \rho(u_t+uu_x)+p_x=\rho E+\mu u_{xx},\\ \rho(e_t+ue_x)+pu_x=\kappa\theta_{xx}+\mu u_x^2,\\ \rho\theta(s_t+us_x)=\kappa\theta_{xx}+\mu u_x^2,\ s=s(\rho,\theta),\\ E_x=\rho-\rho_+ \end{cases} \tag{3.1}$$

and

$$\begin{cases} (\hat\rho\hat u)_x=0, \qquad x>0,\ t>0,\\ \hat\rho\hat u\hat u_x+\hat p_x=\hat\rho\hat E+\mu\hat u_{xx},\ \hat p=p(\hat\rho,\hat\theta),\\ \hat\rho\hat u\hat e_x+\hat p\hat u_x=\kappa\hat\theta_{xx}+\mu\hat u_x^2,\ \hat e=e(\hat\rho,\hat\theta),\\ \hat\rho\hat\theta\hat u\hat s_x=\kappa\hat\theta_{xx}+\mu\hat u_x^2,\ \hat s=s(\hat\rho,\hat\theta),\\ \hat E_x=\hat\rho-\rho_+. \end{cases} \tag{3.2}$$

Let us set the perturbation $(\varphi,\psi,\zeta,\chi)$ as

$$(\varphi,\psi,\zeta,\chi)(x,t)=(\rho,u,\theta,E)(x,t)-(\hat\rho,\hat u,\hat\theta,\hat E)(x)$$

and the solution space $X(I)$ as

$$\begin{aligned} X(I)=\{(\varphi,\psi,\zeta,\chi)\,|(\varphi,\psi,\zeta)\in C(I;H^1),\ \chi\in C(I;L^2),\\ (\varphi_x,\chi_x)\in L^2(I;L^2),\ (\psi_x,\zeta_x)\in L^2(I;H^1)\} \end{aligned}$$

for any interval $I\subset[0,\infty)$.

The local existence of the solution to the outflow problem (1.1)-(1.4) can be established by the standard iteration argument and hence will be skipped in the paper. To prove Theorem 1.2, a crucial step is to show the following a priori estimate.

PROPOSITION 3.1 (Priori estimate of the solution). *Besides the assumptions of the Theorem 1.2, suppose that (ρ,u,θ,E) is the solution to the problem (1.1)-(1.4) satisfying $(\phi,\psi,\zeta,\chi)\in X([0,T])$. Then, there is a positive constant ε_1 such that if*

$$\sup_{0\le t\le T}(\|(\varphi,\psi,\zeta)(t)\|_1+\|\chi(t)\|)\le\varepsilon_1, \quad and \quad \delta=|(u_--u_+,\theta_--\theta_+)|\le\varepsilon_1,$$

for any $t\in[0,T]$, it holds that

$$\begin{aligned} &\|(\varphi,\psi,\zeta)(t)\|_1^2+\|\chi(t)\|^2+\int_0^t\left(\|(\varphi_x,\chi_x)(\tau)\|^2+\|(\psi_x,\zeta_x)(\tau)\|_1^2\right)d\tau\\ &\quad+\int_0^t|(\varphi,\varphi_x,\chi)(0,\tau)|^2d\tau\le C\left(\|(\varphi,\psi,\zeta)(0)\|_1^2+\|\chi(0)\|^2\right), \end{aligned} \tag{3.3}$$

where C is a positive constant independent of t,T,ε_1.

We will prove Proposition 3.1 by the following four steps.

Step 1: Energy estimate.

For notational simplicity, we introduce $A\lesssim B$ if $A\le C_0B$ holds uniformly on the constant C_0 independent of t,x,T,ε_1.

Setting

$$\mathcal{E}=(e-\hat{e})-\hat{\theta}(s-\hat{s})+\frac{\psi^2}{2}+\hat{p}(v-\hat{v}), \tag{3.4}$$

we have (see (3.10) and (3.18) in [5])

$$(\rho\mathcal{E})_t+(\rho u\mathcal{E})_x+\mu\frac{\hat{\theta}}{\theta}\psi_x^2+\kappa\frac{\hat{\theta}}{\theta^2}\zeta_x^2=\Delta_{1x}+\Delta_2+\Delta_3+\Delta_4+\rho\psi\chi, \tag{3.5}$$

where

$$\begin{aligned}
\Delta_1&=\mu\psi\psi_x+\kappa\frac{\zeta\zeta_x}{\theta}-(p-\hat{p})\psi,\\
\Delta_2&=\kappa\frac{\hat{\theta}_x\zeta\zeta_x}{\theta^2}-(\kappa\hat{\theta}_{xx}+\mu\hat{u}_x^2)\frac{\zeta^2}{\theta\hat{\theta}}+2\mu\frac{\zeta\psi_x}{\theta}\hat{u}_x+\mu\hat{\rho}\psi\hat{u}_{xx}(v-\hat{v})-\rho\psi^2\hat{u}_x,\\
\Delta_3&=-\hat{u}_x\left(p-\hat{p}-\widetilde{p}_\rho(\hat{\rho},\hat{s})(\rho-\hat{\rho})-\widetilde{p}_s(\hat{\rho},\hat{s})(s-\hat{s})\right),\\
&\quad+\hat{\rho}\hat{u}\hat{s}_x\Big(\theta-\hat{\theta}-\widetilde{\theta}_\rho(\hat{\rho},\hat{s})(\rho-\hat{\rho})-\widetilde{\theta}_s(\hat{\rho},\hat{s})(s-\hat{s})\Big)+(\hat{\rho}\hat{u}-\rho u)\hat{\theta}_x(s-\hat{s}).
\end{aligned}$$

Due to (1.6) and the assumptions of Proposition 3.1, it is easy to check that

$$(\varphi^2+\psi^2+\zeta^2)\lesssim\mathcal{E}(x,t)\lesssim(\varphi^2+\psi^2+\zeta^2). \tag{3.6}$$

By using (3.6), $u|_{x=0}=u_-<0$ and $(\psi,\zeta)|_{x=0}=0$, we have

$$\Delta_1|_{x=0}=0,\quad -(\rho u\mathcal{E})|_{x=0}\gtrsim\varphi^2(0,t). \tag{3.7}$$

Noticing that $\chi=E-\hat{E}=-\int_x^\infty(\rho-\tilde{\rho})dy$ and

$$\chi_x=\varphi,\ \chi_t=\hat{\rho}\hat{u}-\rho u=-\rho\psi-\varphi\hat{u}, \tag{3.8}$$

and by using integration by parts, we have

$$\begin{aligned}
\int_0^\infty\rho\psi\chi dx&=-\int_0^\infty\chi\chi_t dx-\int_0^\infty\chi\chi_x\hat{u}dx\\
&=-\frac{1}{2}\frac{d}{dt}\int_0^\infty\chi^2dx+\frac{u_-}{2}\chi^2(0,t)+\frac{1}{2}\int_0^\infty\hat{u}_x\chi^2dx.
\end{aligned} \tag{3.9}$$

After integrating (3.5) for (x,t), taking the summation of the resulting equations and using (3.6)-(3.9) yields

$$\begin{aligned}
&\|(\varphi,\psi,\zeta,\chi)(t)\|^2+\int_0^t\|(\psi_x,\zeta_x)(\tau)\|^2d\tau+\int_0^t|(\varphi,\chi)(0,\tau)|^2d\tau\\
\lesssim&\|(\varphi,\psi,\zeta,\chi)(0)\|^2+\sum_{k=2}^3\int_0^t\int_0^\infty|\Delta_k|dxd\tau+\int_0^t\int_0^\infty|\hat{u}_x|\chi^2dxd\tau,
\end{aligned} \tag{3.10}$$

where it is essential that $u_-<0$. Using (1.9), $(\psi,\zeta)|_{x=0}=0$ and

$$|f(x)|\le|f(0)|+\sqrt{x}\|f_x\|,\ \forall f\in H^1(0,\infty) \tag{3.11}$$

yields

$$\begin{aligned}
|\Delta_2|&\lesssim\delta|(\psi_x,\zeta_x)|^2+\delta|(\varphi,\psi,\zeta)|^2\exp(-\hat{c}x)\\
&\lesssim\delta|(\psi_x,\zeta_x)|^2+\delta|\varphi(0,\tau)|^2+\delta\|(\varphi_x,\psi_x,\zeta_x)\|^2x\exp(-\hat{c}x),
\end{aligned}$$

which implies

$$\int_0^t\int_0^\infty |\Delta_2|dxd\tau \lesssim \delta\int_0^t \|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2 d\tau+\delta\int_0^t |\varphi(0,\tau)|^2 d\tau. \tag{3.12}$$

By the same lines as in (3.12), we have

$$\begin{aligned}\int_0^t\int_0^\infty |\Delta_3|dxd\tau &\lesssim \delta\int_0^t\int_0^\infty \left(|\varphi|^2+|\psi|^2+|\zeta|^2\right)\exp(-\hat{c}x)dxd\tau\\ &\lesssim \delta\int_0^t \|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2 d\tau+\delta\int_0^t |\varphi(0,\tau)|^2 d\tau.\end{aligned}$$

Also, using (1.9) and (3.11) yields

$$\int_0^\infty |\hat{u}_x|\chi^2 dx \lesssim \delta\int_0^\infty \chi^2\exp(-\hat{c}x)dx \lesssim \delta\|\chi_x(\tau)\|^2+\delta|\chi(0,t)|^2. \tag{3.13}$$

By the estimations for $\Delta_k (k=2,3)$ and (3.13), we get from (3.10)

$$\begin{aligned}&\|(\varphi,\psi,\zeta,\chi)(t)\|^2+\int_0^t \|(\psi_x,\zeta_x)(\tau)\|^2 d\tau+\int_0^t |(\varphi,\chi)(0,\tau)|^2 d\tau\\ \lesssim&\|(\varphi,\psi,\zeta,\chi)(0)\|^2+\delta\int_0^t \|(\varphi_x,\chi_x)(\tau)\|^2 d\tau.\end{aligned} \tag{3.14}$$

Step 2: Estimation for $\|\varphi_x(t)\|$.
By the same lines as in Lemma 3.2 of [5], we have

$$\left(\frac{\mu\varphi_x^2}{2\rho^3}+\frac{\varphi_x\psi}{\rho}\right)_t+\left(\frac{\mu u\varphi_x^2}{2\rho^3}-\frac{\varphi_t}{\rho}\psi\right)_x+\frac{p_\rho}{\rho^2}\varphi_x^2=(\rho E-\hat{\rho}\hat{E})\frac{\varphi_x}{\rho}+\sum_{k=1}^3 f_k, \tag{3.15}$$

where

$$\begin{aligned}f_1&=-\mu\frac{\varphi_x}{\rho^3}(\hat{\rho}_x\psi_x+\hat{u}_{xx}\varphi+\hat{\rho}_{xx}\psi),\\ f_2&=\frac{\varphi_x}{\rho^2}(\hat{u}\hat{\rho}-\rho u)\hat{u}_x-\frac{u}{\rho}\varphi_x\psi_x+\frac{(\rho u)_x\psi}{\rho^2}\hat{\rho}_x-\frac{(\rho u)_x}{\rho}\psi_x,\\ f_3&=-\frac{\varphi_x}{\rho^2}\left(p_\theta\zeta_x+\hat{\rho}_x(p_\rho-\hat{p}_\rho)+\hat{\theta}_x(p_\theta-\hat{p}_\theta)\right).\end{aligned}$$

By using $\psi|_{x=0}=0$, $u|_{x=0}=u_-<0$ and (1.9), we have

$$\begin{aligned}&\int_0^\infty\left(\frac{\mu u\varphi_x^2}{2\rho^3}-\frac{\varphi_t}{\rho}\psi\right)_x dx=-\frac{\mu u_-}{2\rho^3(0,t)}\varphi_x^2(0,t)\gtrsim\varphi_x^2(0,t),\\ &|f_1|\lesssim\delta\exp(-\hat{c}x)\left(\varphi_x^2+\psi_x^2+\varphi^2+\psi^2\right),\\ &|f_2|\lesssim\delta\exp(-\hat{c}x)\left(\varphi_x^2+\psi_x^2+\varphi^2+\psi^2\right)+\epsilon\varphi_x^2+\epsilon^{-1}\psi_x^2,\\ &|f_3|\lesssim\delta\exp(-\hat{c}x)\left(\varphi^2+\zeta^2\right)+\epsilon\varphi_x^2+\epsilon^{-1}\zeta_x^2,\ \forall\epsilon>0.\end{aligned} \tag{3.16}$$

After integrating (3.15) for (x,t) and using (3.16) and $p_\rho(\rho,\theta)>0$, by the same arguments as in step 1, we have

$$\|\varphi_x(t)\|^2+\int_0^t\|\varphi_x(\tau)\|^2 d\tau+\int_0^t|\varphi_x(0,\tau)|^2 d\tau\lesssim\|\varphi_x(0)\|^2$$

$$+\|\psi(t)\|^2+\int_0^t\|(\psi_x,\zeta_x)(\tau)\|^2d\tau+\int_0^t\int_0^\infty(\rho E-\hat\rho\hat E)\frac{\varphi_x}{\rho}dxd\tau. \tag{3.17}$$

Noticing that

$$(\rho E-\hat\rho\hat E)\frac{\varphi_x}{\rho}=\chi\varphi_x+\hat E\frac{\varphi\varphi_x}{\rho},\quad \chi\varphi_x=(\chi\varphi)_x-\chi_x^2$$

we have

$$\begin{aligned}&\int_0^\infty(\rho E-\hat\rho\hat E)\frac{\varphi_x}{\rho}dx\\&=-\int_0^\infty\chi_x^2dx-(\chi\varphi)(0,t)+\int_0^\infty\hat E\frac{\varphi\varphi_x}{\rho}dx.\end{aligned}$$

Therefore, using (1.9), we have

$$\int_0^\infty(\rho E-\hat\rho\hat E)\frac{\varphi_x}{\rho}dx\lesssim-\|\chi_x(t)\|^2+(\chi^2+\varphi^2)(0,t)+\delta\|\varphi_x(t)\|^2. \tag{3.18}$$

Substituting the estimations on (3.18) into (3.17) yields

$$\begin{aligned}&\|\varphi_x(t)\|^2+\int_0^t\|(\varphi_x,\chi_x)\|^2d\tau+\int_0^t|\varphi_x(0,\tau)|^2d\tau\lesssim\|\varphi_x(0)\|^2\\&+\left(\|\psi(t)\|^2+\int_0^t(\chi^2+\varphi^2)(0,\tau)d\tau+\int_0^t\|(\psi_x,\zeta_x)(\tau)\|^2d\tau\right).\end{aligned} \tag{3.19}$$

By (3.19) and (3.14), we get

$$\|\varphi_x(t)\|^2+\int_0^t\|(\varphi_x,\chi_x)\|^2d\tau+\int_0^t|\varphi_x(0,\tau)|^2d\tau\lesssim\|(\varphi,\psi,\zeta,\varphi_x,\chi)(0)\|^2. \tag{3.20}$$

Step 3: Estimation for $\|\psi_x(t)\|$.
Subtracting $(3.2)_2$ from $(3.1)_2$ and multiplying it by $-\frac{\psi_{xx}}{\rho}$ yields

$$\begin{aligned}&\left(\frac{\psi_x^2}{2}\right)_t-(\psi_t\psi_x)_x+\frac{\mu\psi_{xx}^2}{\rho}=-\chi\psi_{xx}-\hat E\frac{\varphi\psi_{xx}}{\rho}+f_4,\\&f_4=u\psi_x\psi_{xx}+\frac{(p-\hat p)_x}{\rho}\psi_{xx}+\frac{(\hat\rho\hat u-\rho u)\hat u_x}{\rho}\psi_{xx}.\end{aligned} \tag{3.21}$$

By using (1.9) and $\psi\,|_{x=0}=0$, we have

$$\begin{aligned}&\int_0^\infty(\psi_t\psi_x)_xdx=0,\\&|f_4|\lesssim(\epsilon+\delta)\psi_{xx}^2+\epsilon^{-1}|(\varphi_x,\psi_x,\zeta_x)|^2+\delta\exp(-\hat cx)|(\varphi,\psi,\zeta)|^2,\ \forall\epsilon>0.\end{aligned} \tag{3.22}$$

It is easy to check that

$$\begin{aligned}&-\int_0^\infty\chi\psi_{xx}dx=\int_0^\infty\chi_x\psi_xdx\lesssim\|(\chi_x,\psi_x)\|^2,\\&-\int_0^\infty\hat E\frac{\varphi\psi_{xx}}{\rho}dx\lesssim\delta\int_0^\infty\exp(-\hat cx)\left(\varphi^2+\psi_{xx}^2\right)dx\lesssim\delta\left(\varphi^2(0,t)+\|(\varphi_x,\psi_{xx})\|^2\right).\end{aligned} \tag{3.23}$$

After integrating (3.21) for (x,t), taking the summation of the resulting equations and using (3.22), (3.23), we have

$$\begin{aligned}\|\psi_x(t)\|^2+\int_0^t\|\psi_{xx}\|^2d\tau\lesssim\|\psi_x(0)\|^2\\+\int_0^t\|(\varphi_x,\psi_x,\zeta_x)\|^2d\tau+\delta\int_0^t\varphi^2(0,\tau)d\tau.\end{aligned}\tag{3.24}$$

Step 4: Estimation for $\|\zeta_x(t)\|$.
Subtracting $(3.2)_3$ from $(3.1)_3$ and using $e_t=e_\theta(\rho,\theta)\theta_t-e_\rho(\rho,\theta)(\rho u)_x$, we have

$$\begin{aligned}\rho e_\theta(\rho,\theta)\zeta_t+\rho u\zeta_x-\kappa\zeta_{xx}=\rho e_\rho(\rho,\theta)(\rho u-\hat\rho\hat u)_x\\+(\hat u\hat\rho-\rho u)\hat e_x-(pu_x-\hat p\hat u_x)+\mu(u_x^2-\hat u_x^2).\end{aligned}$$

Multiplying it by $-\frac{\zeta_{xx}}{\rho e_\theta(\rho,\theta)}$ yields

$$\left(\frac{\zeta_x^2}{2}\right)_t-(\zeta_t\zeta_x)_x+\frac{\kappa\zeta_{xx}^2}{\rho e_\theta(\rho,\theta)}=f_5,\tag{3.25}$$

where

$$\begin{aligned}f_5=\frac{\zeta_{xx}}{\rho e_\theta(\rho,\theta)}\left(\rho u\zeta_x+(pu_x-\hat p\hat u_x)-\rho e_\rho(\rho,\theta)(\rho u-\hat\rho\hat u)_x\right)\\-\frac{\zeta_{xx}}{\rho e_\theta(\rho,\theta)}\left((\hat u\hat\rho-\rho u)\hat e_x+\mu(u_x^2-\hat u_x^2)\right).\end{aligned}$$

By using (1.9) and $\zeta|_{x=0}=0$, we have

$$\begin{aligned}&\int_0^\infty(\zeta_t\zeta_x)_xdx=0,\\&|f_5|\lesssim(\lambda+\delta)\zeta_{xx}^2+\lambda^{-1}|(\varphi_x,\psi_x,\zeta_x)|^2+\delta\exp(-\hat cx)|(\varphi,\psi,\zeta)|^2,\ \forall\lambda>0.\end{aligned}\tag{3.26}$$

After integrating (3.25) for (x,t), using (3.26) and $e_\theta(\rho,\theta)>0$, we have

$$\|\zeta_x(t)\|^2+\int_0^t\|\zeta_{xx}\|^2d\tau\lesssim\|\zeta_x(0)\|^2+\int_0^t\|(\varphi_x,\psi_x,\zeta_x)\|^2d\tau+\delta\int_0^t\varphi^2(0,\tau)d\tau.\tag{3.27}$$

The proof of Proposition 3.1: By (3.14) and (3.20), we get

$$\begin{aligned}\|(\varphi,\psi,\zeta,\varphi_x,\chi)(t)\|^2+\int_0^t\|(\varphi_x,\psi_x,\zeta_x,\chi_x)\|^2d\tau\\+\int_0^t|(\varphi,\varphi_x,\chi)(0,\tau)|^2d\tau\lesssim\|(\varphi,\psi,\zeta,\varphi_x,\chi)(0)\|^2.\end{aligned}\tag{3.28}$$

By (3.28), (3.24) and (3.27), we get (3.3) which completes the proof of Proposition 3.1.

4. Convergence rate for supersonic stationary solution

In this section, we show the convergence rate stated in Theorem 1.3 by using a time- and space-weighted energy method.

The a priori estimate is obtained in the weighted Sobolev space $X_\omega(0,T)$ defined by

$$X_\omega(0,T):=\{(\varphi,\psi,\zeta,\chi)\in X([0,T])\mid\sqrt{\omega}(\varphi,\psi,\zeta,\chi)\in C([0,T];L^2(0,\infty))\}.$$

For a weight function $\omega(x):=(1+x)^\alpha$ or $\omega(x)=e^{\alpha x}$, we use the notation

$$|f|_{2,\omega}:=\left(\int_0^\infty \omega(x)f^2(x)dx\right)^{\frac{1}{2}},\ \|f\|_{a,\alpha}:=|f|_{2,(1+x)^\alpha},\ \|f\|_{e,\alpha}:=|f|_{2,e^{\alpha x}}.$$

To prove Theorem 1.3, it's enough to show the following the weighted norm estimates (see [17]).

Proposition 4.1. *Suppose that the same assumptions as in Theorem 1.3 hold.*

(1) exponential decay. *Suppose that (ρ,u,θ,E) is the solution to the outflow problem (1.1)-(1.4) satisfying $(\varphi,\psi,\zeta,\chi)\in X_{e^{\varsigma x}}(0,T)$ for certain positive constants $\varsigma>0$ and $T>0$. Then there exist positive constants $\varepsilon_1,\alpha(<\varsigma),\beta(\ll\alpha)$ such that*

$$\textit{if}\ \sup_{t\in[0,T]}(\|(\varphi,\psi,\zeta)(t)\|_1+\|\chi(t)\|)+\delta\le\varepsilon_1,$$

then the following weighted estimates are satisfied:

$$\begin{aligned}&e^{\beta t}\left(\|(\varphi,\psi,\zeta)(t)\|_1^2+\|(\varphi,\psi,\zeta,\chi)(t)\|_{e,\alpha}^2\right)\\&\qquad\le C\left(\|(\varphi,\psi,\zeta)(0)\|_1^2+\|(\varphi,\psi,\zeta,\chi)(0)\|_{e,\alpha}^2\right),\end{aligned}\tag{4.1}$$

where C is a positive constant independent of t,x,T,ε_1.

(2) algebraic decay. *Suppose that (ρ,u,θ,E) is the solution to the outflow problem (1.1)-(1.4) satisfying $(\varphi,\psi,\zeta,\chi)\in X_{(1+x)^\varsigma}(0,T)$ for certain positive constants $\varsigma>0$ and $T>0$. Then there exist positive constants ε_1 such that*

$$\textit{if}\ \sup_{t\in[0,T]}(\|(\varphi,\psi,\zeta)(t)\|_1+\|\chi(t)\|)+\delta\le\varepsilon_1,$$

then the following weighted estimates are satisfied:

$$(1+t)^\varsigma\|(\varphi,\psi,\zeta,\chi)(t)\|_1^2\le C\left(\|(\varphi,\psi,\zeta)(0)\|_1^2+\|(\varphi,\psi,\zeta,\chi)(0)\|_{a,\varsigma}^2\right),\tag{4.2}$$

where C is a positive constant independent of t,x,T,ε_1.

In the remainder of this section, we will prove Proposition 4.1. As in Section 3, we denote $A\lesssim B$ if $A\le C_0B$ holds uniformly on the constant C_0 independently of t,x,T,ε_1.

Step 1: Weighted energy estimates.

Suppose that $\eta(t)$ and $\omega(x)$ is the weight function like $(1+t)^\beta$ (or $e^{\beta t}$) and $(1+x)^\alpha$ (or $e^{\alpha x},\alpha\le\frac{\hat{c}}{2}$, where $\hat{c}$ is the positive number in (1.9)), respectively.

Setting $w(x,t)=\eta(t)\omega(x)$, we get from (3.5)

$$\begin{aligned}&(w\rho\mathcal{E})_t+\{w(\rho u\mathcal{E}-\Delta_1)\}_x-w_xG^1+w\left(\mu\frac{\hat{\theta}}{\theta}\psi_x^2+\kappa\frac{\hat{\theta}}{\theta^2}\zeta_x^2\right)\\=&w_t\rho\mathcal{E}-w_xG^2+w(\Delta_2+\Delta_3)+w\rho\psi\chi,\end{aligned}\tag{4.3}$$

where

$$G^1=\rho u\mathcal{E}+(p-\hat{p})\psi,\quad G^2=\mu\psi\psi_x+\kappa\frac{\zeta\zeta_x}{\theta}.$$

Using (3.7) yields

$$\int_0^\infty\{w(\rho u\mathcal{E}-\Delta_1)\}_xdx\gtrsim\eta(t)\varphi^2(0,t).\tag{4.4}$$

We estimate G^1.

Using $\widetilde{e}_v(v,s)=-p$ and $\widetilde{e}_s(v,s)=\theta$ (see (1.5) in [5]), we obtain from (3.4)

$$\mathcal{E}=\frac{1}{2}\left(\psi^2+\widetilde{e}_{vv}(\hat{v},\hat{s})\phi^2+2\widetilde{e}_{vs}(\hat{v},\hat{s})\phi\vartheta+\widetilde{e}_{ss}(\hat{v},\hat{s})\vartheta^2\right)+O(1)(|\phi|^3+|\vartheta|^3), \tag{4.5}$$

where $\vartheta=s-\hat{s}$, Using (4.5) and

$$p-\hat{p}=\widetilde{p}_v(\hat{v},\hat{s})\phi+\widetilde{p}_s(\hat{v},\hat{s})\vartheta+O(1)(|\phi|^2+|\vartheta|^2),$$

we can rewrite G^1 as

$$\begin{aligned}G^1=&\frac{\rho u}{2}\left(\psi^2+\widetilde{e}_{vv}(\hat{v},\hat{s})\phi^2+2\widetilde{e}_{vs}(\hat{v},\hat{s})\phi\vartheta+\widetilde{e}_{ss}(\hat{v},\hat{s})\vartheta^2\right)\\&+\psi\left(\widetilde{p}_v(\hat{v},\hat{s})\phi+\widetilde{p}_s(\hat{v},\hat{s})\vartheta\right)+O(1)(|\phi|^3+|\vartheta|^3)+O(1)(|\phi|^2+|\vartheta|^2)|\psi|\\=&G^1_1+G^1_2+G^1_3,\end{aligned}$$

and

$$\begin{aligned}G^1_1=&\frac{\rho_+u_+}{2}\left(\psi^2+\widetilde{e}_{vv}(v_+,s_+)\phi^2-2\widetilde{e}_{vs}(v_+,s_+)\phi\vartheta+\widetilde{e}_{ss}(v_+,s_+)\vartheta^2\right)\\&+\psi\left(\widetilde{p}_v(v_+,s_+)\phi+\widetilde{p}_s(v_+,s_+)\vartheta\right),\\G^1_2=&\frac{\rho u}{2}\left(\psi^2+\widetilde{e}_{vv}(\hat{v},\hat{s})\phi^2+2\widetilde{e}_{vs}(\hat{v},\hat{s})\phi\vartheta+\widetilde{e}_{ss}(\hat{v},\hat{s})\vartheta^2\right)\\&-\frac{\rho_+u_+}{2}\left(\psi^2+\widetilde{e}_{vv}(v_+,s_+)\phi^2+\widetilde{e}_{vs}(v_+,s_+)\phi\vartheta+\widetilde{e}_{ss}(v_+,s_+)\vartheta^2\right)\\&+\psi\left((\widetilde{p}_v(\hat{v},\hat{s})-\widetilde{p}_v(v_+,s_+))\phi+(\widetilde{p}_s(\hat{v},\hat{s})-\widetilde{p}_s(v_+,s_+))\vartheta\right),\\G^1_3=&O(1)(|\phi|^3+|\vartheta|^3)+O(1)(|\phi|^2+|\vartheta|^2)|\psi|.\end{aligned}$$

By using (1.6), (1.5) and $\widetilde{p}_s(v,s)=\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}$ (see (1.5) in [5]), we rewrite G^1_1 as

$$\begin{aligned}G^1_1=&\frac{\rho_+u_+}{2}f(\phi,\psi,\vartheta)\\f(\phi,\psi,\vartheta):=&\left(\psi^2+b_1\phi^2-2b_2\phi\vartheta+b_3\vartheta^2-\frac{2b_1}{\rho_+u_+}\phi\psi+\frac{2b_2}{\rho_+u_+}\vartheta\psi\right),\end{aligned} \tag{4.6}$$

where

$$b_1=-\widetilde{p}_v(v_+,s_+)>0,\ b_2=\frac{\theta_+p_\theta(v_+,\theta_+)}{e_\theta(v_+,\theta_+)},\ b_3=\frac{\theta_+}{e_\theta(v_+,\theta_+)}>0. \tag{4.7}$$

The A corresponding to the $f(\phi,\psi,\vartheta)$ is as the following:

$$A:=\begin{pmatrix}b_1 & -\frac{b_1}{\rho_+u_+} & -b_2\\ -\frac{b_1}{\rho_+u_+} & 1 & \frac{b_2}{\rho_+u_+}\\ -b_2 & \frac{b_2}{\rho_+u_+} & b_3\end{pmatrix}.$$

If all principal minors $\bar{\Delta}_i(i=1,2,3)$ of A are positive, the matrix A is positive. Noticing that

$$M^+=\frac{|u_+|}{\sqrt{-v_+^2\widetilde{p}_v(v_+,s_+)}}>1\rightleftarrows\frac{b_1}{(\rho_+u_+)^2}<1,$$

and by using (4.7) and (1.4), we compute $\bar{\Delta}_i(i=1,2,3)$ as follows:

$$\begin{aligned}&\bar{\Delta}_1=b_1=-\widetilde{p}_v(v_+,s_+)>0,\quad \bar{\Delta}_2=b_1-\frac{b_1^2}{(\rho_+u_+)^2}>0,\\&\bar{\Delta}_3=\det A=(b_1b_3-b_2^2)\left(1-\frac{b_1}{(\rho_+u_+)^2}\right)=-\frac{\theta_+p_v(v_+,\theta_+)}{e_\theta(v_+,\theta_+)}\left(1-\frac{b_1}{(\rho_+u_+)^2}\right)>0.\end{aligned}\tag{4.8}$$

By (4.6), (4.8), $\rho_+>0$ and $u_+<0$, we have

$$-G_1^1\gtrsim\left(\phi^2+\psi^2+\vartheta^2\right)\gtrsim\left(\phi^2+\psi^2+\zeta^2\right).\tag{4.9}$$

Noticing that

$$\begin{aligned}|G_2^1|&\lesssim(|\rho-\rho_+|+|u-u_+|)(\phi^2+\psi^2+\vartheta^2)+(|v-v_+|+|s-s_+|)(\phi^2+\vartheta^2)\\&\lesssim(|(\hat{\rho}-\rho_+,\hat{u}-u_+\hat{\theta}-\theta_+)|+|(\phi,\zeta)|)(\phi^2+\psi^2+\zeta^2).\end{aligned}$$

and using (1.9), we have

$$\int_0^\infty w_x|G_2^1|dx\lesssim(\varepsilon_1+\delta)\eta(t)|(\varphi,\psi,\zeta)|_{2,\omega_x}^2.\tag{4.10}$$

Similarly, we have

$$\int_0^\infty w_x|G_3^1|dx\lesssim\varepsilon_1\eta(t)|(\varphi,\psi,\zeta)|_{2,\omega_x}^2.\tag{4.11}$$

By (4.9)-(4.11), we obtain

$$-\int_0^\infty w_xG^1dx\gtrsim\eta(t)|(\phi,\psi,\zeta)|_{2,\omega_x}^2.\tag{4.12}$$

Similarly, it's easy to check that

$$\begin{aligned}&\int_0^\infty|w_xG^2|dx\lesssim\eta(t)\left(\epsilon|(\psi,\zeta)|_{2,\omega_x}^2+\epsilon^{-1}|(\psi_x,\zeta_x)|_{2,\omega_x}^2\right),\ \forall\epsilon>0,\\&\int_0^\infty|w_t\rho\mathcal{E}|dx\lesssim\eta'(t)|(\varphi,\psi,\zeta)|_{2,\omega}^2.\end{aligned}\tag{4.13}$$

By using (3.8) and integration by parts, we have

$$\begin{aligned}\int_0^\infty w\rho\psi\chi dx&=-\int_0^\infty w\chi\chi_tdx-\int_0^\infty w\chi\chi_x\hat{u}dx\\&=-\frac{1}{2}\frac{d}{dt}\left(\eta(t)\int_0^\infty\omega\chi^2dx\right)+\frac{1}{2}\eta'(t)\int_0^\infty\omega\chi^2dx\\&\quad+\frac{u_-}{2}\eta(t)\chi^2(0,t)+\frac{1}{2}\eta(t)\int_0^\infty(\hat{u}_x\omega+\hat{u}\omega_x)\chi^2dx.\end{aligned}\tag{4.14}$$

Noticing that $inf_{x\in[0,\infty)}(-\hat{u}(x))>0$, we have

$$-\int_0^\infty\hat{u}\omega_x\chi^2dx\gtrsim|\chi(t)|_{2,\omega_x}^2.\tag{4.15}$$

Integrating (4.3) for x,t and using (4.4), (4.12)-(4.15), we have

$$\begin{aligned}&\eta(t)|\Phi(t)|_{2,\omega}^2+\int_0^t\eta(\tau)\left(|\Phi(\tau)|_{2,\omega_x}^2+|(\psi_x,\zeta_x)(\tau)|_{2,\omega}^2+|(\varphi,\chi)(0,\tau)|^2\right)d\tau\\&\lesssim|\Phi(0)|_{2,\omega}^2+\int_0^t\left(\eta'(\tau)|\Phi(\tau)|_{2,\omega}^2+\eta(\tau)|(\psi_x,\zeta_x)(\tau)|_{2,\omega_x}^2\right)d\tau\\&\quad+\int_0^t\eta(\tau)\int_0^\infty\omega\left(|\hat{u}_x|\chi^2+|\Delta_2|+|\Delta_3|\right)dxd\tau,\end{aligned}\tag{4.16}$$

where $\Phi=(\varphi,\psi,\zeta,\chi)$. Using (1.9) and (3.11), it is easy to show that if $\omega(x)=(1+x)^\alpha$, then

$$\int_0^\infty\omega|\hat{u}_x|\chi^2dx\lesssim\delta\int_0^\infty\chi^2(1+x)^\alpha\exp(-\hat{c}x)dx\lesssim\delta\|\chi_x(t)\|^2+\delta|\chi(0,t)|^2$$

and if $\omega(x)=e^{\alpha x}$, $\alpha\le\frac{\hat{c}}{2}$, then

$$\int_0^\infty\omega|\hat{u}_x|\chi^2dx\lesssim\delta\int_0^\infty\chi^2\exp(\alpha x-\hat{c}x)dx\lesssim\delta\|\chi_x(t)\|^2+\delta|\chi(0,t)|^2,$$

which imply that

$$\int_0^\infty\omega|\hat{u}_x|\chi^2dx\lesssim\delta\|\chi_x(t)\|^2+\delta|\chi(0,t)|^2.\tag{4.17}$$

By similar arguments as in (4.17), we have

$$\int_0^\infty\omega(|\Delta_2|+|\Delta_3|)dx\lesssim\delta\|(\varphi_x,\psi_x,\zeta_x)(t)\|^2+\delta|\varphi(0,t)|^2.\tag{4.18}$$

By (4.17), (4.18) and $\|\cdot\|\lesssim|\cdot|_{2,\omega}$, we get from (4.16)

$$\begin{aligned}&\eta(t)|\Phi(t)|_{2,\omega}^2+\int_0^t\eta(\tau)\left(|\Phi(\tau)|_{2,\omega_x}^2+|(\psi_x,\zeta_x)(\tau)|_{2,\omega}^2+|(\varphi,\chi)(0,\tau)|^2\right)d\tau\lesssim|\Phi(0)|_{2,\omega}^2\\&+\int_0^t\left(\eta'(\tau)|\Phi(\tau)|_{2,\omega}^2+\eta(\tau)|(\psi_x,\zeta_x)(\tau)|_{2,\omega_x}^2\right)d\tau+\delta\int_0^t\eta(\tau)\|(\varphi_x,\chi_x)(\tau)\|^2d\tau.\end{aligned}\tag{4.19}$$

Setting $\omega=1$ in (4.19), we get

$$\begin{aligned}&\eta(t)\|\Phi(t)\|^2+\int_0^t\eta(\tau)\left(\|(\psi_x,\zeta_x)(\tau)\|^2+|(\varphi,\chi)(0,\tau)|^2\right)d\tau\\&\lesssim\|\Phi(0)\|^2+\delta\int_0^t\eta(\tau)\|(\varphi_x,\chi_x)(\tau)\|^2d\tau+\int_0^t\eta'(\tau)\|\Phi(\tau)\|^2d\tau,\end{aligned}\tag{4.20}$$

where $\Phi=(\varphi,\psi,\zeta,\chi)$.

Step 2: Weighted estimation for $\|\varphi_x(t)\|$.

By (3.15), we have

$$\left(\eta(\frac{\mu\varphi_x^2}{2\rho^3}+\frac{\varphi_x\psi}{\rho})\right)_t+\left(\eta(\frac{\mu u\varphi_x^2}{2\rho^3}-\frac{\varphi_t}{\rho}\psi)\right)_x+\frac{\eta p_\rho(\rho,\theta)}{\rho^2}\varphi_x^2=\eta(\rho E-\hat{\rho}\hat{E})\frac{\varphi_x}{\rho}+G^3,\tag{4.21}$$

where

$$G^3 = \eta(t)(f_1+f_2+f_3)+\eta'(t)\left(\frac{\mu\varphi_x^2}{2\rho^3}+\frac{\varphi_x\psi}{\rho}\right) =: G_1^3+G_2^3.$$

Using (3.16) and the assumptions of Proposition 4.1 yields

$$\begin{aligned}&\int_0^\infty |G_1^3|dx \lesssim (\delta+\epsilon)\eta(t)\|\varphi_x\|^2+\delta\eta(t)\varphi^2(0,t)+C_\epsilon\eta(t)\|(\psi_x,\zeta_x)\|^2,\\&\int_0^\infty |G_2^3|dx \lesssim \eta'(t)\|(\varphi_x,\psi)\|^2,\ \forall\epsilon>0.\end{aligned} \tag{4.22}$$

Integrating (4.21) for (x,t), and using $(3.16)_1$, (4.22) and (3.18), we obtain

$$\begin{aligned}&\eta(t)\|\varphi_x(t)\|^2+\int_0^t\eta(\tau)\|(\varphi_x,\chi_x)(\tau)\|^2d\tau+\int_0^t\eta(\tau)\varphi_x^2(0,\tau)d\tau\\&\lesssim\|\varphi_x(0)\|^2+\eta(t)\|\psi(t)\|^2+\int_0^t\eta(\tau)\|(\psi_x,\zeta_x)(\tau)\|^2d\tau\\&\quad+\int_0^t\eta(\tau)|(\varphi,\chi)(0,\tau)|^2d\tau+\int_0^t\eta'(\tau)\|(\varphi_x,\psi)(\tau)\|^2d\tau.\end{aligned} \tag{4.23}$$

By (4.20) and (4.23), we obtain

$$\begin{aligned}&\eta(t)\|\varphi_x(t)\|^2+\int_0^t\eta(\tau)\left(\|\varphi_x,\chi_x)(\tau)\|^2+\varphi_x^2(0,\tau)\right)d\tau\\&\lesssim\|(\Phi,\varphi_x)(0)\|^2+\int_0^t(\eta'(\tau)\left(\|\Phi(\tau)\|^2+\|\varphi_x(\tau)\|^2\right)d\tau,\end{aligned} \tag{4.24}$$

where $\Phi=(\varphi,\psi,\zeta,\chi)$.

Step 3: Weighted estimation for $\|\psi_x(t)\|$.
By (3.21), we have

$$\frac{1}{2}\left(\eta\psi_x^2\right)_t-\eta(\psi_t\psi_x)_x+\eta\frac{\mu\psi_{xx}^2}{\rho}=-\eta\chi\psi_{xx}-\eta\hat{E}\frac{\varphi\psi_{xx}}{\rho}+G^4, \tag{4.25}$$

where $G^4=\eta(t)f_4+\frac{1}{2}\eta'(t)\psi_x^2$. By (3.22), we have

$$\begin{aligned}\int_0^\infty |G^4|dx \lesssim&\,\eta(t)(\epsilon+\delta)\|\psi_{xx}\|^2+\eta(t)\delta|\varphi(0,t)|^2\\&+\eta(t)\epsilon^{-1}\|(\varphi_x,\psi_x,\zeta_x)\|^2+\eta'(t)\|\psi_x\|^2,\ \forall\epsilon>0.\end{aligned} \tag{4.26}$$

Integrating (4.25) for (x,t), and using (4.26), $(3.22)_1$ and (3.23), we obtain

$$\begin{aligned}\eta(t)\|\psi_x(t)\|^2+\int_0^t\eta(\tau)\|\psi_{xx}(\tau)\|^2d\tau\lesssim&\|\psi_x(0)\|^2+\int_0^t\eta'(t)\|\psi_x(\tau)\|^2d\tau\\&+\int_0^t\eta(\tau)\|(\varphi_x,\psi_x,\zeta_x,\chi_x)(\tau)\|^2d\tau+\delta\int_0^t\eta(\tau)|\varphi(0,\tau)|^2d\tau.\end{aligned} \tag{4.27}$$

Step 4: Weighted estimation for $\|\zeta_x(t)\|$.

By (3.25), we have

$$\left(\eta\frac{\zeta_x^2}{2}\right)_t - \eta(\zeta_t\zeta_x)_x + \eta\frac{\kappa\zeta_{xx}^2}{\rho e_\theta(\rho,\theta)} = G^5, \tag{4.28}$$

where $G^4 = \eta(t)f_5 + \frac{1}{2}\eta'(t)\zeta_x^2$. By (3.26), we have

$$\int_0^\infty |G^5|dx \lesssim \eta(t)(\epsilon+\delta)\|\zeta_{xx}\|^2 + \eta(t)\delta|\varphi(0,t)|^2 \\ + \eta(t)\epsilon^{-1}\|(\varphi_x,\psi_x,\zeta_x)\|^2 + \eta'(t)\|\zeta_x\|^2, \ \forall\epsilon>0. \tag{4.29}$$

Integrating (4.28) for (x,t), and using (4.29), $(3.26)_1$ and $e_\theta(\rho,\theta)>0$, we obtain

$$\eta(t)\|\zeta_x(t)\|^2 + \int_0^t \eta(\tau)\|\zeta_{xx}(\tau)\|^2 d\tau \lesssim \|\zeta_x(0)\|^2 + \int_0^t \eta'(t)\|\zeta_x(\tau)\|^2 d\tau \\ + \int_0^t \eta(\tau)\|(\varphi_x,\psi_x,\zeta_x,\chi_x)(\tau)\|^2 d\tau + \delta\int_0^t \eta(\tau)|\varphi(0,\tau)|^2 d\tau. \tag{4.30}$$

The proof of Proposition 4.1:
We first prove (4.1). Using (4.20), (4.24), (4.27) and (4.30), we have

$$\eta(t)\left(\|\Phi(t)\|^2 + \|(\varphi_x,\psi_x,\zeta_x)(t)\|^2\right) + \int_0^t \eta(\tau)\left(\|\Phi_x(\tau)\|^2 + \|(\psi_{xx},\zeta_{xx})(\tau)\|^2\right)d\tau \\ + \int_0^t \eta(\tau)|(\varphi,\varphi_x,\chi)(0,\tau)|^2 d\tau \lesssim \left(\|\Phi(0)\|^2 + \|(\varphi_x,\psi_x,\zeta_x)(0)\|^2\right) \\ + \int_0^t \eta'(\tau)\left(\|\Phi(t)\|^2 + \|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2\right)d\tau. \tag{4.31}$$

Also, by using (4.19), (4.31) and $\|\cdot\| \lesssim |\cdot|_{2,\omega}$, we get

$$\eta(t)\left(|\Phi(t)|_{2,\omega}^2 + \|(\varphi_x,\psi_x,\zeta_x)(t)\|^2\right) + \int_0^t \eta(\tau)\left(|\Phi(\tau)|_{2,\omega_x}^2 + |(\psi_x,\zeta_x)(\tau)|_{2,\omega}^2\right)d\tau \\ + \int_0^t \eta(\tau)\left(\|\Phi_x(\tau)\|^2 + \|(\psi_{xx},\zeta_{xx})(\tau)\|^2 + |(\varphi,\varphi_x,\chi)(0,\tau)|^2\right)d\tau \\ \le C_1\left(|\Phi(0)|_{2,\omega}^2 + \|(\varphi_x,\psi_x,\zeta_x)(0)\|^2\right) + C_2\int_0^t \eta'(\tau)|\Phi(\tau)|_{2,\omega}^2 d\tau \\ + C_3\int_0^t \eta(\tau)|(\psi_x,\zeta_x)(\tau)|_{2,\omega_x}^2 d\tau + C_4\int_0^t \eta'(\tau)\|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2 d\tau, \tag{4.32}$$

where $\Phi = (\varphi,\psi,\zeta,\chi)$ and $C_i(i=1,\cdots,4)$ are positive constants independent of t,x,T,ε_1. Setting $\omega(x) = e^{\alpha x}$ and $\eta(t) = e^{\beta t}$, we obtain from (4.32)

$$e^{\beta t}\left(\|\Phi(t)\|_{e,\alpha}^2 + \|(\varphi_x,\psi_x,\zeta_x)(t)\|^2\right) + (\alpha - C_2\beta)\int_0^t e^{\beta\tau}\|\Phi(\tau)\|_{e,\alpha}^2 d\tau \\ + (1-C_3\alpha)\int_0^t e^{\beta\tau}\|(\psi_x,\zeta_x)(\tau)\|_{e,\alpha}^2 d\tau + (1-C_4\beta)\int_0^t e^{\beta\tau}\|\Phi_x(\tau)\|^2 d\tau \\ + \int_0^t e^{\beta\tau}\left(\|(\psi_{xx},\zeta_{xx})(\tau)\|^2 + |(\varphi,\varphi_x,\chi)(0,\tau)|^2\right)d\tau$$

$$\leq C_1 \left(\|\Phi(0)\|_{e,\alpha}^2 + \|(\varphi_x, \psi_x, \zeta_x)(0)\|^2\right). \tag{4.33}$$

If we choose α and $\beta (0 < \beta < \alpha < \varsigma)$ satisfying

$$\alpha - C_2\beta \geq 0, \quad 1 - C_3\alpha \geq \frac{1}{2}, \quad 1 - C_4\beta \geq \frac{1}{2},$$

then (4.33) yields (4.1).

Next, we prove (4.2). Setting $\omega(x) = (1+x)^\alpha$ and $\eta(t) = (1+t)^\beta$, we obtain from (4.32)

$$\begin{aligned}
&(1+t)^\beta \left(\|\Phi(t)\|_{a,\alpha}^2 + \|(\varphi_x, \psi_x, \zeta_x)(t)\|^2\right) \\
&\quad + \int_0^t (1+\tau)^\beta \left(\alpha \|\Phi(\tau)\|_{a,\alpha-1}^2 + \|(\psi_x, \zeta_x)(\tau)\|_{a,\alpha}^2\right) d\tau \\
&\quad + \int_0^t (1+\tau)^\beta \left(\|\Phi_x(\tau)\|^2 + \|(\psi_{xx}, \zeta_{xx})(\tau)\|^2 + |(\varphi, \varphi_x, \chi)(0,\tau)|^2\right) d\tau \\
&\lesssim \left(\|\Phi(0)\|_{a,\alpha}^2 + \|(\varphi_x, \psi_x, \zeta_x)(0)\|^2\right) + \alpha \int_0^t (1+\tau)^\beta \|(\psi_x, \zeta_x)(\tau)\|_{a,\alpha-1}^2 d\tau \\
&\quad + \beta \int_0^t (1+\tau)^{\beta-1} \left(\|\Phi(\tau)\|_{a,\alpha}^2 + \|(\varphi_x, \psi_x, \zeta_x)(\tau)\|^2\right) d\tau.
\end{aligned} \tag{4.34}$$

Setting

$$\begin{aligned}
&E_\alpha(t)^2 := \|\Phi(t)\|_{a,\alpha}^2 + \|(\varphi_x, \psi_x, \zeta_x)(t)\|^2, \\
&D(t)^2 := \|\Phi_x(t)\|^2 + \|(\psi_{xx}, \zeta_{xx})(t)\|^2 + |(\varphi, \varphi_x, \chi)(0,t)|^2, \\
&D_\alpha(t)^2 := D(t)^2 + \alpha \|\Phi(t)\|_{a,\alpha-1}^2 + \|(\psi_x, \zeta_x)(t)\|_{a,\alpha}^2,
\end{aligned} \tag{4.35}$$

we rewrite (4.34) as

$$\begin{aligned}
&(1+t)^\beta E_\alpha(t)^2 + \int_0^t (1+\tau)^\beta D_\alpha(\tau)^2 d\tau \\
&\lesssim E_\alpha(0)^2 + \alpha \int_0^t (1+\tau)^\beta \|(\psi_x, \zeta_x)(\tau)\|_{a,\alpha-1}^2 d\tau + \beta \int_0^t (1+\tau)^{\beta-1} E_\alpha(\tau)^2 d\tau.
\end{aligned} \tag{4.36}$$

By the same induction argument as in Section 4 of [5], we obtain from (3.3) and (4.36)

$$(1+t)^k E_{\varsigma-k}(t)^2 + \int_0^t (1+\tau)^k D_{\varsigma-k}(\tau)^2 d\tau \lesssim E_\varsigma(0)^2 \tag{4.37}$$

and

$$(1+t)^k E_0(t)^2 + \int_0^t (1+\tau)^k D_0(\tau)^2 d\tau \lesssim E_\varsigma(0)^2 \tag{4.38}$$

for any $\varsigma > 0$ and integer $k = 0, 1, 2, \cdots, [\varsigma]$. If ς is an integer, we obtain (4.2) from (4.38) letting $k = \varsigma$.

In the case that ς is not an integer, we prove (4.2). Letting $\alpha = 0$ in (4.36), we have

$$(1+t)^\beta E_0(t)^2 \leq C E_0(0)^2 + C\beta \int_0^t (1+\tau)^{\beta-1} E_0(\tau)^2 d\tau. \tag{4.39}$$

We estimate the second term on the right-hand side of (4.39). Letting $k=[\varsigma]$ in (4.37) and noticing (4.35), we have

$$(1+t)^{[\varsigma]}E_{\varsigma-[\varsigma]}(t)^2+(\varsigma-[\varsigma])\int_0^t(1+\tau)^{[\varsigma]}E_{\varsigma-[\varsigma]-1}(\tau)^2d\tau\le CE_\varsigma(0)^2. \tag{4.40}$$

Noticing that

$$E_0(\tau)^2\le E_{\varsigma-[\varsigma]}(\tau)^{\frac{2}{q}}E_{\varsigma-[\varsigma]-1}(\tau)^{\frac{2}{p}},$$

where $p=(\varsigma-[\varsigma])^{-1}$, $q=(1-\varsigma+[\varsigma])^{-1}$ and using (4.40), we have

$$\begin{aligned}
&\int_0^t(1+\tau)^{\beta-1}E_0(\tau)^2d\tau\\
\le&\int_0^t(1+\tau)^{\beta-1-[\varsigma]}\left((1+\tau)^{[\varsigma]}E_{\varsigma-[\varsigma]}(\tau)^2\right)^{\frac{1}{q}}\left((1+\tau)^{[\varsigma]}E_{\varsigma-[\varsigma]-1}(\tau)^2\right)^{\frac{1}{p}}d\tau\\
\le&CE_\varsigma(0)^{\frac{2}{q}}\int_0^t(1+\tau)^{\beta-1-[\varsigma]}\left((1+\tau)^{[\varsigma]}E_{\varsigma-[\varsigma]-1}(\tau)^2\right)^{\frac{1}{p}}d\tau\\
\le&CE_\varsigma(0)^{\frac{2}{q}}\left(\int_0^t(1+\tau)^{(\beta-1-[\varsigma])q}d\tau\right)^{\frac{1}{q}}\left(\int_0^t(1+\tau)^{[\varsigma]}E_{\varsigma-[\varsigma]-1}(\tau)^2d\tau\right)^{\frac{1}{p}}\\
\le&CE_\varsigma(0)^2\left(\int_0^t(1+\tau)^{\frac{\beta-1-[\varsigma]}{1-\varsigma+[\varsigma]}}d\tau\right)^{1-\varsigma+[\varsigma]}.
\end{aligned}\tag{4.41}$$

Letting $\beta=\varsigma+\frac{1}{2}$ in (4.39) and (4.41), we can get (4.2).

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