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TIME-PERIODIC SOLUTION FOR AN INCOMPRESSIBLE MAGNETOHYDRODYNAMIC SYSTEM WITH AN EXTERNAL FORCE IN $\mathbb{R}^N$*

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Abstract. In this paper, we study the existence and uniqueness of time-periodic solutions to incompressible magnetohydrodynamic equations. Our approach combines energy estimates with topological degree theory, and our result follows from a limiting process. The main difficulty is to prove the compactness and continuity for a key operator Λ given in Definition 2.1, the proof is based on parabolic regularization.

Keywords. Incompressible magnetohydrodynamic equations; time-periodic solutions; topological degree theory.

AMS subject classifications. 35Q55; 35L70; 35Q35; 76B15.

1. Introduction and main results

The purpose of this paper is to investigate the following incompressible magnetohydrodynamic equations (MHD):

$$\partial_t\rho+u\cdot\nabla\rho=0, \tag{1.1}$$

$$\rho(\partial_t u+u\cdot\nabla u)-\Delta u+\nabla P(\rho)-B\cdot\nabla B=\rho f(x,t), \tag{1.2}$$

$$\partial_t B+u\cdot\nabla B-\Delta B=B\cdot\nabla u, \tag{1.3}$$

$$divB=0,\ divu=0, \tag{1.4}$$

where $x\in\Omega=(-L,L)^n, n\geq 2$ is the space dimension, $\rho(x,t)$ represents the density, $u(x,t)=(u_1,u_2,\dots,u_n)(x,t)\in\mathbb{R}^n$, $H(x,t)=(H_1,H_2,\dots,H_n)(x,t)\in\mathbb{R}^n$ are the velocity and magnetic field. $P=P(\rho)$ stands for the pressure, $P(\rho)$ is a smooth function in a neighborhood of $\overline{\rho}$ satisfying $P'(\overline{\rho})>0$, $\overline{\rho}$ is a given positive constant. Let $f=f(x,t)$ be an external force function which is periodic in t and odd in x. Indeed,

$$f(x,t+T)=f(x,t)\quad(x\in\Omega,t\in\mathbb{R}^+),$$

$$f(-x,t)=-f(x,t)\quad(x\in\Omega,t\in\mathbb{R}^+),$$

for a constant $T>0$. The system (1.1)-(1.4) describes the interaction between the magnetic field and the conductive fluid which couples the Navier-Stokes equations with the Maxwell equations.

In recent decades, the time-periodic solution of the magnetohydrodynamic equations have been extensively studied, cf. [3–7, 10, 11, 14–16, 19, 20, 23–25]. In the case of compressibility, Tan and Wang [20] obtained the existence, uniqueness, and time-asymptotic stability of periodic solutions to the equations defined on the whole space $\mathbb{R}^n$, where $n\geq 5$. In [24], Yan and Li derived the periodic weak solutions to the equations in a finite domain $\Omega\subset\mathbb{R}^3$. Cai and Tan got periodic solutions of the equations

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in a periodic domain, cf. [26]. For the case of incompressibility, Notte, Rojas, and Rojas [19] established periodic strong solutions of incompressible magnetohydrodynamic equations in the bounded region $\Omega\subset\mathbb{R}^n$ by using the spectral Galerkin method and the compactness arguments for $n=3$ or 4.

Specially, the magnetohydrodynamic equations can be reduced to the Navier-Stokes equations without the electromagnetic field, cf. [1,2,8,12,13,17,18,22]. The existence of the periodic solutions to the incompressible Navier-Stokes equations on a bounded domain or an unbounded domain is well-known. In [22], the authors proved the existence of a strong periodic solution. Matsumura, Nishida and Feireisl [8,18] obtained the small periodic solution to the equations in a bounded domain $\Omega\subset\mathbb{R}^n$, $n=3$. Ma, Ukai, Yang [17] established the periodic solution in an unbounded domain $\Omega\subset\mathbb{R}^n$, where $n\geq 5$. In [19], the authors got the existence and uniqueness of the periodic strong solutions when $n=3,4$. In particular, Jin and Yang [12] studied time-periodic solutions to the isentropic compressible Navier-Stokes equations in a periodic domain. It's worth mentioning that they used the parabolic regularization method to prove the compactness of the operator. Jin and Yang [12] also considered the existence of time-periodic solutions to whole space $\mathbb{R}^3$ through the topological degree theory. By the spectral properties, Y. Kagei and K. Tsuda [13] obtained time-periodic solutions for sufficiently small and symmetry conditions on the time-periodic external force when the space dimension is larger than or equal to 3.

Inspired by the method introduced in [12], we defined an operator corresponding to the incompressible magnetohydrodynamic equations, and proved its compactness and continuity by using the parabolic regularization. Then, by exploiting the energy estimates and the topological degree method, we established the existence of the time-periodic solution of the regularized problem. By the spectral properties and a limiting process, we got the existence of time-periodic solutions of the system (1.1)-(1.4) around the constant state $(\overline{\rho},0,0)$ in a periodic domain, which extends the space dimension in [12]. Besides, we obtain the uniqueness of the solution under the assumption of the smallness of the period.

By substituting $\varrho=\rho-\overline{\rho}$, $\gamma=\frac{P'(\overline{\rho})}{\overline{\rho}^2}$, $h(\varrho)=\frac{P'(\overline{\rho}+\varrho)}{\overline{\rho}+\varrho}-\frac{P'(\overline{\rho})}{\overline{\rho}}>0$, we reformulate the system (1.1)-(1.4) by:

$$\partial_t(\overline{\rho}+\varrho)+u\cdot\nabla(\overline{\rho}+\varrho)=0, \tag{1.5}$$

$$(\overline{\rho}+\varrho)\partial_t u+(\overline{\rho}+\varrho)u\cdot\nabla u-\Delta u+\nabla P(\rho)-B\cdot\nabla B=(\overline{\rho}+\varrho)f(x,t), \tag{1.6}$$

$$\partial_t B+u\cdot\nabla B-\Delta B=B\cdot\nabla u, \tag{1.7}$$

$$divB=0,\ divu=0. \tag{1.8}$$

We also note that

$$\frac{\nabla P(\rho)}{\overline{\rho}+\varrho}=\frac{\nabla P(\overline{\rho}+\varrho)}{\overline{\rho}+\varrho}=h(\varrho)\nabla\varrho+\frac{P'(\overline{\rho})}{\overline{\rho}}\nabla\varrho=h(\varrho)\nabla\varrho+\gamma\overline{\rho}\nabla\varrho,$$

then the system (1.1)-(1.4) can be rewritten as follows:

$$\partial_t\varrho=-u\cdot\nabla\varrho, \tag{1.9}$$

$$\partial_t u-\frac{1}{\overline{\rho}+\varrho}\Delta u=-(u\cdot\nabla)u-h(\varrho)\nabla\varrho-\gamma\overline{\rho}\nabla\varrho+\frac{B\cdot\nabla B}{\overline{\rho}+\varrho}+f, \tag{1.10}$$

$$\partial_t B-\Delta B=B\cdot\nabla u-u\cdot\nabla B, \tag{1.11}$$

$$divB=0,\ divu=0. \tag{1.12}$$

We obtain our results on the existence and uniqueness of the time-periodic solution for the system (1.9)-(1.12), and equivalently, we solve the original problems. We state our main result in the following theorem:

THEOREM 1.1. *Let m be an integer satisfying $m \geq [\frac{n}{2}]+1$, assume that $f(x,t) \in L^2(0,T;H^{m+1}(\Omega))$, $f(-x,t) = -f(x,t)$ and $\int_0^T \|f(x,t)\|_{H^{m+1}} dt$ is suitably small. Then the system (1.9)-(1.12) has a time-periodic solution (ϱ,u,B) with period T and satisfies $\varrho \in L^\infty(0,T;H^{m+2}(\Omega))$, $(u,B) \in L^\infty(0,T;H^{m+2}(\Omega)) \bigcap L^2(0,T;H^{m+1}(\Omega))$ and $(\varrho,u,B) \in S_\delta$ where*

$$S_\delta = \{(\rho,\upsilon,H) \in S; \sup_{0\leq t\leq T} \|(\rho,\upsilon,H)(t)\|^2_{H^{m+1}} + \int_0^T \|(\rho,\upsilon,H)(t)\|^2_{H^{m+2}} dt < \delta^2\} \quad \text{and}$$

$$S = \{(\rho,\upsilon,H) \in L^\infty(0,T;H^{m+1}(\Omega)) \cap L^2(0,T;H^{m+2}(\Omega)); \rho,\upsilon,H$$
satisfy (1),(2),(3) which are defined as follows$\}$

(1) (ρ,υ,H) is periodic both in time and space.

(2) $\int_\Omega \rho(x,t)dx = 0, \int_\Omega \upsilon(x,t)dx = 0, \int_\Omega H(x,t)dx = 0$.

(3) $\rho(x,t) = \rho(-x,t), \upsilon(x,t) = -\upsilon(-x,t), H(x,t) = -H(-x,t)$

and the uniqueness of time-periodic solution of (1.9)-(1.12) holds with $\sup_{0\leq t\leq T} \|(\varrho,u,B)(t)\|_{H^{m+2}}$ being suitably small.

The rest of this paper is organized as follows: In Section 2, we introduce some notations and auxiliary lemmas for later in this paper. Specifically, we define the key operator Λ (see Def. 2.1) and prove its compactness and continuity by using the parabolic regularization. Section 3 aims to establish the energy estimates, and considers the existence of time-periodic solutions for (1.9)-(1.12) by using topological degree theory and a limiting process. The uniqueness of small periodic solutions of the equations is derived in Section 4.

As a remark, $'C'$ denotes positive constants which vary at different lines in this paper.

2. Preliminaries

In this section, we elaborate on several basic lemmas that will be needed to demonstrate our main results. Before that, we introduce some notations on functional settings which will be used later. For a given Banach space $X, \|\cdot\|_X$ stands for the norm of X. $L^P(\Omega)(1 \leq P \leq \infty)$ represents the usual L^P spaces with norm $\|\cdot\|_{L^P}$. For a non-negative integer K, H^K stands for the usual L^2-Sobolev space of order K with norm $\|\cdot\|_{H^K}$, $H^0 = L^2$. C represents positive constants which are variable in different estimates. We denote the gradient operator $\nabla = (\partial_1, \partial_2, \ldots, \partial_n), \partial_i = \partial_{x_i}, i = 1,2,\ldots,n$. For any integer $l \geq 0$, $\nabla^l f$ represents $x-$derivatives of order l of a function f. $Q_T = \Omega \times (0,T)$. We denote the commutator of operators L_1 and L_2 by $[L_1,L_2]$, then

$$[L_1,L_2]f = L_1(L_2 f) - L_2(L_1 f).$$

In order to find the time-periodic problem for the system (1.9)-(1.12), we first regularize it to:

$$\partial_t \varrho - \epsilon \Delta \varrho = -u \cdot \nabla \varrho, \tag{2.1}$$

$$\partial_t u - \frac{1}{\overline{\rho} + \varrho} \Delta u = -(u \cdot \nabla)u - h(\varrho)\nabla \varrho - \gamma \overline{\rho} \nabla \varrho + \frac{B \cdot \nabla B}{\overline{\rho} + \varrho} + f, \tag{2.2}$$

$$\partial_t B - \Delta B = B \cdot \nabla u - u \cdot \nabla B, \tag{2.3}$$

$$divB = 0,\ divu = 0. \tag{2.4}$$

To study this system, we introduce the space of the triples of functions consisting of $(\rho,\upsilon,H)\in L^\infty(0,T;H^{m+1}(\Omega))\cap L^2(0,T;H^{m+2}(\Omega))$, where ρ, υ, H satisfy the following conditions:

(1) (ρ,υ,H) is periodic for both the time and the space.

(2) $\int_\Omega \rho(x,t)dx = 0, \int_\Omega \upsilon(x,t)dx = 0, \int_\Omega H(x,t)dx = 0.$

(3) $\rho(x,t)=\rho(-x,t), \upsilon(x,t)=-\upsilon(-x,t), H(x,t)=-H(-x,t).$

We use S to represent this space, and for $\delta>0$, we define

$$S_\delta = \{(\rho,\upsilon,H)\in S;\ \sup_{0\le t\le T} \|(\rho,\upsilon,H)(t)\|^2_{H^{m+1}} + \int_0^T \|(\rho,\upsilon,H)(t)\|^2_{H^{m+2}} dt < \delta^2\}.$$

Based on the above function spaces, we define the operator as follows:

Definition 2.1. *We define the operator:*

$$\Lambda : S_\delta \times [0,1] \mapsto S$$
$$((\rho,\upsilon,H),\tau) \mapsto (\varrho,u,B)$$

where $\tau\in[0,1],\delta$ *is sufficiently small,* $m\ge[\frac{n}{2}]+1$*, and* (ϱ,u,B) *is the solution satisfying the following linear problem:*

$$\partial_t\varrho - \epsilon\Delta\varrho = Q_1(\rho,\upsilon,H,\tau), \tag{2.5}$$

$$\partial_t u - \frac{1}{\bar\rho+\tau\rho}\Delta u = Q_2(\rho,\upsilon,H,\tau)+\tau f, \tag{2.6}$$

$$\partial_t B - \Delta B = Q_3(\rho,\upsilon,H,\tau), \tag{2.7}$$

$$divB=0,\ divu=0, \tag{2.8}$$

where

$$Q_1(\rho,\upsilon,H,\tau) = -\tau\upsilon\cdot\nabla\rho, \tag{2.9}$$

$$Q_2(\rho,\upsilon,H,\tau) = -\tau(\upsilon\cdot\nabla)\upsilon - h(\tau\rho)\nabla\rho - \gamma\bar\rho\nabla\rho + \tau\frac{H\cdot\nabla H}{\bar\rho+\tau\rho} + \tau f, \tag{2.10}$$

$$Q_3(\rho,\upsilon,H,\tau) = \tau H\cdot\nabla\upsilon - \tau\upsilon\cdot\nabla H. \tag{2.11}$$

With this definition, the linear problem can be reformulated as:

$$U_t = AU + Q(W) + F$$

where

$$A = \begin{pmatrix} \epsilon\Delta & 0 & 0 \\ 0 & \frac{1}{\bar\rho+\tau\rho}\Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}, U=(\varrho,u,B), W=(\rho,\upsilon,H),$$

$$Q(W) = (Q_1,Q_2,Q_3), F=(0,\tau f,0)$$

Now, we investigate the properties of the operator Λ. Before that, we introduce the following lemmas: cf. [9,21]:

Lemma 2.1. *Let* K *be a positive integer and* $u\in H^{\frac{n}{2}+1}(\mathbb{R}^n)$*, then there holds:*

$$\|u\|^2_{L^\infty} \le C\|\nabla^{K+1}u\|\|\nabla^{K-1}u\|, \qquad (n=2K)$$

$$\|u\|_{L^\infty}^2 \leq C\|\nabla^{K+1}u\|\|\nabla^K u\|, \qquad (n=2K+1)$$

LEMMA 2.2. *Let K be a positive integer and $u\in H^{\frac{n}{2}+1}(\mathbb{R}^n)$, $m\geq[\frac{n}{2}]+1$, then,*

$$\|u\|_{L^\infty}\leq C\|u\|_{H^m}.$$

LEMMA 2.3. *Let K be an integer satisfying $K\geq 1$, we give the definition of the commutator as follows:*

$$[\nabla^K,f]g=\nabla^K(fg)-f\nabla^K g,$$

then there holds the estimate which is useful to estimate the high order term:

$$\|[\nabla^K,f]g\|_{L_2}\leq C(\|\nabla f\|_{L_\infty}\|\nabla^{K-1}g\|_{L_2}+\|\nabla^K f\|_{L_2}\|g\|_{L_\infty}).$$

LEMMA 2.4. *There uniquely exists a time-periodic solution $(\varrho,u,B)\in S$ which satisfies the linear parabolic problem (2.5)-(2.8) for any $(\rho,\upsilon,H)\in S_\delta$, and any $\tau\in[0,1]$ as δ suitably small.*

Proof. Let us consider the initial value problem with periodic boundary conditions of (2.5)-(2.8) as follows:

$$\partial_t\varrho-\epsilon\Delta\varrho=0, \tag{2.12}$$

$$\partial_t u-\frac{1}{\bar{\rho}+\tau\rho}\Delta u=0, \tag{2.13}$$

$$\partial_t B-\Delta B=0, \tag{2.14}$$

$$divB=0,\ divu=0, \tag{2.15}$$

$$(\varrho,u,B)(x,0)=(\varrho_0,u_0,B_0)(x), \tag{2.16}$$

with even function $\varrho_0(x)$ satisfying $\int_\Omega\varrho_0dx=0$ and odd functions u_0,H_0. Then the solution (ϱ,u,B) also satisfies the same properties.

We deduce the following identity by applying ∇^{m+1} to (2.12), and multiplying the equation by $\gamma\nabla^{m+1}\varrho$:

$$\nabla^{m+1}\varrho_t-\epsilon\nabla^{m+1}\Delta\varrho=0, \tag{2.17}$$

which follows

$$(\nabla^{m+1}\varrho_t,\gamma\nabla^{m+1}\varrho)-(\epsilon\nabla^{m+1}\Delta\varrho,\gamma\nabla^{m+1}\varrho)=0, \tag{2.18}$$

then, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\gamma|\nabla^{m+1}\varrho|^2dx+\epsilon\gamma\int_\Omega|\nabla^{m+2}\varrho|^2dx=0. \tag{2.19}$$

By applying ∇^{m+1} to (2.13), and multiplying the equation by $\nabla^{m+1}u$, we have the following identity:

$$\nabla^{m+1}\partial_t u-\nabla^{m+1}(\frac{1}{\bar{\rho}+\tau\rho}\Delta u)=0, \tag{2.20}$$

which infers

$$\nabla^{m+1}\partial_t u-\sum_{0\leq l\leq m+1}C_{m+1}^l\nabla^l(\frac{1}{\bar{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u=0, \tag{2.21}$$

then, we have

$$(\nabla^{m+1}\partial_t u,\nabla^{m+1}u)-(\sum_{0\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u,\nabla^{m+1}u)$$
$$=\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+1}u|^2dx-\int_\Omega\sum_{0\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+1}udx,$$

where

$$\begin{aligned}&\int_\Omega\sum_{0\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+1}udx\\
=&\int_\Omega\sum_{1\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+1}udx\\
&+\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+1}\Delta u\nabla^{m+1}udx\\
=&\int_\Omega\sum_{1\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+1}udx\\
&-\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+2}u|^2dx-\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2}u\nabla^{m+1}udx,\end{aligned}$$

then, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+1}u|^2dx+\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+2}u|^2dx\\
=&\int_\Omega(\sum_{1\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+1}u\\
&-\nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2}u\nabla^{m+1}u)dx,\end{aligned}$$

by applying Lemma 2.3, we see

$$\begin{aligned}&\int_\Omega\sum_{1\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+1}udx\\
\le&\|\sum_{1\le l\le m+1}C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\|_{L_2}\|\nabla^{m+1}u\|_{L_2}\\
\le&C(\|\nabla\rho\|_{L_\infty}\|\nabla^2u\|_{H^m}+\|\nabla^2u\|_{L_\infty}\|\rho\|_{H^{m+1}})\|\nabla^{m+1}u\|_{L_2},\end{aligned}$$

$$\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2}u\nabla^{m+1}udx\le C\|\nabla\rho\|_{L_\infty}\|\nabla^{m+2}u\|_{L_2}\|\nabla^{m+1}u\|_{L_2},$$

which infers:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+1}u|^2dx+\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+2}u|^2dx\\
\le&C(\|\nabla\rho\|_{L_\infty}\|\nabla^2u\|_{H^m}+\|\nabla^2u\|_{L_\infty}\|\rho\|_{H^{m+1}})\|\nabla^{m+1}u\|_{L_2}\\
&+C\|\nabla\rho\|_{L_\infty}\|\nabla^{m+2}u\|_{L_2}\|\nabla^{m+1}u\|_{L_2},\end{aligned}$$

Then, we deduce the following inequality by Lemma 2.1:

$$\|\rho\|_{L_\infty(Q_T)} \le C \sup_{0\le t\le T} \|\rho\|_{H^{m+1}(\Omega)} \le C\delta.$$

Noting that δ is assumed to be small enough, we have

$$\frac{\overline{\rho}}{2} \le \overline{\rho} + \tau\rho \le 2\overline{\rho},$$

and then

$$\frac{1}{2\overline{\rho}} \le \frac{1}{\overline{\rho}+\tau\rho} \le \frac{2}{\overline{\rho}}.$$

By applying the Poincaré inequality, we deduce

$$\frac{d}{dt}\int_\Omega |\nabla^{m+1}u|^2 dx + \int_\Omega \frac{1}{3\overline{\rho}} |\nabla^{m+2}u|^2 dx \le 0. \tag{2.22}$$

On the other hand, the following identity can be deduced by applying ∇^{m+1} to (2.14), and multiplying the equation by $\nabla^{m+1}B$:

$$\nabla^{m+1}\partial_t B - \nabla^{m+1}\Delta B = 0,$$

which follows

$$(\nabla^{m+1}\partial_t B, \nabla^{m+1}B) - (\nabla^{m+1}\Delta B, \nabla^{m+1}B) = 0,$$

then, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla^{m+1}B|^2 dx + \int_\Omega |\nabla^{m+2}B|^2 dx = 0, \tag{2.23}$$

by adding (2.19) to (2.23), we deduce

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\gamma|\nabla^{m+1}\varrho|^2 + |\nabla^{m+1}B|^2) dx + \int_\Omega (\epsilon\gamma|\nabla^{m+2}\varrho|^2 + |\nabla^{m+2}B|^2) dx = 0, \tag{2.24}$$

and by adding (2.24) to (2.22), we see

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (\gamma|\nabla^{m+1}\varrho|^2 + |\nabla^{m+1}u|^2 + |\nabla^{m+1}B|^2) dx + \int_\Omega \frac{1}{3\overline{\rho}}|\nabla^{m+2}u|^2 dx \\ &\quad + 2\int_\Omega (\epsilon\gamma|\nabla^{m+2}\varrho|^2 + |\nabla^{m+2}B|^2) dx \le 0, \end{aligned}$$

then, the Poincaré inequality implies

$$\|\nabla^{m+1}(\varrho, u, B)(x,t)\|_{L^2} \le \|\nabla^{m+1}(\varrho_0, u_0, B_0)\|_{L^2} e^{-C\epsilon t},$$

which follows

$$\|e^{tA}U_0\|_{H^{m+1}} \le \|U_0\|_{H^{m+1}} e^{-C\epsilon t}.$$

The solution of the system (2.5)-(2.8) can be obtained by using the Duhamel's principle

$$U(t) = \int_{-\infty}^t e^{(t-s)A}(Q(W)(s) + F(s)) ds,$$

Moreover,

$$\begin{aligned}
\|U(t)\|_{H^{m+1}} \leq & \int_{-\infty}^{t} \|e^{(t-s)A}(Q(W)(s)+F(s))\|_{H^{m+1}} ds \\
\leq & \sum_{i=0}^{\infty} \int_{t-(i+1)T}^{t-iT} e^{-C\epsilon(t-s)} \|(Q(W)(s)+F(s))\|_{H^{m+1}} ds \\
= & \sum_{i=0}^{\infty} \int_{0}^{T} e^{-C\epsilon((i+1)T-s)} \|(Q(W)(s+t)+F(s+t))\|_{H^{m+1}} ds \\
\leq & \sum_{i=0}^{\infty} (\int_{0}^{T} e^{-2C\epsilon((i+1)T-s)} ds)^{\frac{1}{2}} (\int_{T}^{0} \|(Q(W)(s)+F(s))\|_{H^{m+1}}^2 ds)^{\frac{1}{2}} \\
\leq & C(\int_{0}^{T} \|(Q(W)(s)+F(s))\|_{H^{m+1}}^2 ds)^{\frac{1}{2}}.
\end{aligned}$$

Since both W and F hold the time and space periodic properties, $U(t)$ is periodic with time and space, i.e.

$$\begin{aligned}
U(t+T) = & \int_{-\infty}^{t+T} e^{(t+T-s)A}(Q(W)(s)+F(s)) ds \\
= & \int_{-\infty}^{t+T} e^{(t-(s-T))A}(Q(W)(s-T)+F(s-T)) ds \\
= & \int_{-\infty}^{t} e^{(t-s)A}(Q(W)(s)+F(s)) ds \\
= & U(t),
\end{aligned}$$

this completes the proof of the existence of time-periodic solution (ϱ, u, B) with any $(\rho, v, H) \in S_\delta, \tau \in [0,1]$.

We give the uniqueness of the time-periodic solution next. We assume $U_1 = (\varrho_1, u_1, B_1) \in S, U_2 = (\varrho_2, u_2, B_2) \in S$ are two periodic solutions, and $(\rho, v, H) \in S_\delta$, $\tau \in [0,1]$. We denote

$$(U_1 - U_2)_t = A(U_1 - U_2),$$

then

$$\begin{aligned}
(\varrho_1 - \varrho_2)_t - \epsilon \Delta(\varrho_1 - \varrho_2) = 0, \\
(u_1 - u_2)_t - \frac{1}{\overline{\rho} + \tau\rho} \Delta(u_1 - u_2) = 0, \\
(B_1 - B_2)_t - \Delta(B_1 - B_2) = 0.
\end{aligned}$$

By applying ∇^{m+2} to (2.12), and multiplying the equation by $\gamma \nabla^{m+2}(\varrho_1 - \varrho_2)$, we deduce

$$\nabla^{m+2}(\varrho_1 - \varrho_2)_t - \epsilon \nabla^{m+2} \Delta(\varrho_1 - \varrho_2) = 0,$$

then, we see

$$(\nabla^{m+2}(\varrho_1 - \varrho_2)_t, \gamma \nabla^{m+2}(\varrho_1 - \varrho_2)) - (\epsilon \nabla^{m+2} \Delta(\varrho_1 - \varrho_2), \gamma \nabla^{m+2}(\varrho_1 - \varrho_2)) = 0,$$

we also have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \gamma|\nabla^{m+2}(\varrho_1-\varrho_2)|^2dx+\epsilon\gamma\int_\Omega|\nabla^{m+3}(\varrho_1-\varrho_2)|^2dx=0, \tag{2.25}$$

Next, by applying ∇^{m+2} to (2.13), and multiplying the equation by $\nabla^{m+2}(u_1-u_2)$, we deduce

$$\nabla^{m+2}\partial_t(u_1-u_2)-\nabla^{m+2}(\frac{1}{\overline{\rho}+\tau\rho}\Delta(u_1-u_2))=0,$$

then, we see

$$\nabla^{m+2}\partial_t(u_1-u_2)-\sum_{0\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)=0,$$

and then, we have

$$\begin{aligned}
&(\nabla^{m+2}\partial_t(u_1-u_2),\nabla^{m+2}(u_1-u_2))\\
&\quad-(\sum_{0\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2),\nabla^{m+2}(u_1-u_2))\\
=&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}(u_1-u_2)|^2dx\\
&\quad-\int_\Omega\sum_{0\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx,
\end{aligned}$$

where

$$\begin{aligned}
&\int_\Omega\sum_{0\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx\\
=&\int_\Omega\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx\\
&+\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+2}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx\\
=&\int_\Omega\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx\\
&-\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+3}(u_1-u_2)|^2dx-\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+3}(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}(u_1-u_2)|^2dx+\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+3}(u_1-u_2)|^2dx\\
=&\int_\Omega(\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)\\
&\quad-\nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+3}(u_1-u_2)\nabla^{m+2}(u_1-u_2))dx.
\end{aligned}$$

According to Lemma 2.3, we see

$$\begin{aligned}&\int_\Omega\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx\\ \le&\|\sum_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta(u_1-u_2)\|_{L_2}\|\nabla^{m+2}(u_1-u_2)\|_{L_2}\\ \le&C(\|\nabla\rho\|_{L_\infty}\|\nabla^2(u_1-u_2)\|_{H^{m+1}}+\|\nabla^2(u_1-u_2)\|_{L_\infty}\|\rho\|_{H^{m+2}})\|\nabla^{m+2}(u_1-u_2)\|_{L_2},\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+3}(u_1-u_2)\nabla^{m+2}(u_1-u_2)dx\\ \le&C\|\nabla\rho\|_{L_\infty}\|\nabla^{m+3}(u_1-u_2)\|_{L_2}\|\nabla^{m+2}(u_1-u_2)\|_{L_2},\end{aligned}$$

then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}(u_1-u_2)|^2dx+\int_\Omega\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+3}(u_1-u_2)|^2dx\\ \le&C(\|\nabla\rho\|_{L_\infty}\|\nabla^2(u_1-u_2)\|_{H^{m+1}}+\|\nabla^2(u_1-u_2)\|_{L_\infty}\|\rho\|_{H^{m+2}})\|\nabla^{m+2}(u_1-u_2)\|_{L_2}\\ &+C\|\nabla\rho\|_{L_\infty}\|\nabla^{m+3}(u_1-u_2)\|_{L_2}\|\nabla^{m+2}(u_1-u_2)\|_{L_2},\end{aligned}$$

by applying the Poincáre inequality, we deduce

$$\frac{d}{dt}\int_\Omega|\nabla^{m+2}(u_1-u_2)|^2dx+\int_\Omega\frac{1}{3\overline{\rho}}|\nabla^{m+3}(u_1-u_2)|^2dx\le 0. \tag{2.26}$$

By applying ∇^{m+2} to (2.14), and multiplying the equation by $\nabla^{m+2}(B_1-B_2)$, we have

$$\nabla^{m+2}\partial_t(B_1-B_2)-\nabla^{m+2}\Delta(B_1-B_2)=0,$$

which follows

$$(\nabla^{m+2}\partial_t(B_1-B_2),\nabla^{m+2}(B_1-B_2))-(\nabla^{m+2}\Delta(B_1-B_2),\nabla^{m+2}(B_1-B_2))=0,$$

then, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}(B_1-B_2)|^2dx+\int_\Omega|\nabla^{m+3}(B_1-B_2)|^2dx=0, \tag{2.27}$$

by adding (2.25) to (2.27), we see

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}(\varrho_1-\varrho_2)|^2+|\nabla^{m+2}(B_1-B_2)|^2)dx\\ &+\int_\Omega(\epsilon\gamma|\nabla^{m+3}(\varrho_1-\varrho_2)|^2+|\nabla^{m+3}(B_1-B_2)|^2)dx=0,\end{aligned} \tag{2.28}$$

by adding (2.28) to (2.26), we see

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}(\varrho_1-\varrho_2)|^2+|\nabla^{m+2}(u_1-u_2)|^2+|\nabla^{m+2}(B_1-B_2)|^2)dx\\ &+\int_\Omega\frac{1}{3\overline{\rho}}|\nabla^{m+3}(u_1-u_2)|^2dx+2\int_\Omega(\epsilon\gamma|\nabla^{m+3}(\varrho_1-\varrho_2)|^2+|\nabla^{m+3}(B_1-B_2)|^2)dx\le 0,\end{aligned}$$

then, by applying the Poincaré inequality, we obtain

$$(\varrho_1-\varrho_2,u_1-u_2,B_1-B_2)=(0,0,0),$$

which implies the uniqueness of the time-periodic solution. ∎

REMARK 2.1. One can verify that if $(\varrho(x,t),u(x,t),B(x,t))$ is the periodic solution of (2.5)-(2.8), then $(\varrho(-x,t),-u(-x,t),-B(-x,t))$ is also a solution of (2.5)-(2.8). Moreover, we have

$$(\varrho(x,t),u(x,t),B(x,t))=(\varrho(-x,t),-u(-x,t),-B(-x,t))$$

by the uniqueness of solution.

We will show some key properties of the operator Λ. Firstly, we derive the compactness of the operator Λ as follows:

LEMMA 2.5. *The operator Λ is compact as δ is sufficiently small.*

Proof. By applying ∇^{m+2} to (2.5), and multiplying the equation by $\gamma\nabla^{m+2}\varrho$, we deduce

$$\nabla^{m+2}\varrho_t-\epsilon\nabla^{m+2}\Delta\varrho=-\nabla^{m+2}\tau(\upsilon\cdot\nabla\rho), \tag{2.29}$$

which follows

$$(\nabla^{m+2}\varrho_t,\gamma\nabla^{m+2}\varrho)-(\epsilon\nabla^{m+2}\Delta\varrho,\gamma\nabla^{m+2}\varrho)=-(\nabla^{m+2}\tau(\upsilon\cdot\nabla\rho),\gamma\nabla^{m+2}\varrho), \tag{2.30}$$

then, we see

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\gamma|\nabla^{m+2}\varrho|^2dx+\int_\Omega\varepsilon\gamma|\nabla^{m+3}\varrho|^2dx=-\int_\Omega\tau\gamma\nabla^{m+2}(\upsilon\cdot\nabla\rho)\nabla^{m+2}\varrho dx. \tag{2.31}$$

By applying ∇^{m+2} to (2.6), and multiplying the equation by $\nabla^{m+2}u$, we obtain

$$\begin{aligned}&\nabla^{m+2}u_t-\nabla^{m+2}(\frac{1}{\overline{\rho}+\tau\rho}\Delta u)\\=&\nabla^{m+2}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\overline{\rho}\nabla\rho+\tau f)+\nabla^{m+2}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}),\end{aligned} \tag{2.32}$$

which follows

$$\begin{aligned}&(\nabla^{m+2}u_t,\nabla^{m+2}u)-(\Sigma_{0\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta u,\nabla^{m+2}u)\\=&(\nabla^{m+2}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\overline{\rho}\nabla\rho+\tau f),\nabla^{m+2}u)+(\nabla^{m+2}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}),\nabla^{m+2}u),\end{aligned} \tag{2.33}$$

then, we have

$$\begin{aligned}&(\nabla^{m+2}u_t,\nabla^{m+2}u)-(\Sigma_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+2-l}\Delta u,\nabla^{m+2}u)\\&-(\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+2}\Delta u,\nabla^{m+2}u)\\=&(\nabla^{m+2}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\overline{\rho}\nabla\rho+\tau f),\nabla^{m+2}u)\end{aligned}$$

$$+(\nabla^{m+2}(\frac{\tau H\cdot\nabla H}{\bar{\rho}+\tau\rho}),\nabla^{m+2}u), \tag{2.34}$$

and then, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}u|^2dx-\int_\Omega\Sigma_{1\le l\le m+2}C^l_{m+2}\nabla^l(\frac{1}{\bar{\rho}+\tau\rho})\nabla^{m+2-l}\Delta u\nabla^{m+2}udx\\&+\int_\Omega\frac{1}{\bar{\rho}+\tau\rho}|\nabla^{m+3}u|^2dx-\int_\Omega\nabla\frac{1}{\bar{\rho}+\tau\rho}\nabla^{m+3}u\nabla^{m+2}udx\\=&\int_\Omega\nabla^{m+2}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\bar{\rho}\nabla\rho+\tau f)\nabla^{m+2}udx\\&+\int_\Omega\nabla^{m+2}(\frac{\tau H\cdot\nabla H}{\bar{\rho}+\tau\rho})\nabla^{m+2}udx.\end{aligned} \tag{2.35}$$

By applying ∇^{m+2} to (2.7), and multiplying the equation by $\nabla^{m+2}B$, we deduce

$$\nabla^{m+2}B_t-\nabla^{m+2}\Delta B=\nabla^{m+2}(\tau H\cdot\upsilon-\tau\upsilon\cdot\nabla H), \tag{2.36}$$

which follows

$$(\nabla^{m+2}B_t,\nabla^{m+2}B)-(\nabla^{m+2}\Delta B,\nabla^{m+2}B)=(\nabla^{m+2}(\tau H\cdot\upsilon-\tau\upsilon\cdot\nabla H),\nabla^{m+2}B), \tag{2.37}$$

then, we see

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}B|^2dx+\int_\Omega|\nabla^{m+3}B|^2dx=\int_\Omega\nabla^{m+2}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H)\nabla^{m+2}Bdx, \tag{2.38}$$

and then, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\varrho|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx+\int_\Omega(\epsilon\gamma|\nabla^{m+3}\varrho|^2+\nu|\nabla^{m+3}B|^2)dx\\&+\int_\Omega\frac{1}{\bar{\rho}+\tau\rho}|\nabla^{m+3}u|^2dx\\=&-\int_\Omega\tau\gamma\nabla^{m+2}(\upsilon\cdot\nabla\rho)\nabla^{m+2}\varrho dx+\int_\Omega\Sigma_{1\le l\le m+2}C^l_{m+2}\nabla^l(\frac{1}{\bar{\rho}+\tau\rho})\nabla^{m+2-l}\Delta u\nabla^{m+2}udx\\&-\int_\Omega\nabla\frac{1}{\bar{\rho}+\tau\rho}\nabla^{m+3}u\nabla^{m+2}udx+\int_\Omega\nabla^{m+2}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\bar{\rho}\nabla\rho+\tau f)\nabla^{m+2}udx\\&+\int_\Omega\nabla^{m+2}(\frac{\tau H\cdot\nabla H}{\bar{\rho}+\tau\rho})\nabla^{m+2}udx+\int_\Omega\nabla^{m+2}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H)\nabla^{m+2}Bdx.\end{aligned} \tag{2.39}$$

We estimate the integral terms above as in the following:

$$\begin{aligned}&|\int_\Omega\tau\gamma\nabla^{m+2}(\upsilon\cdot\nabla\rho)\nabla^{m+2}\varrho dx|\\=&|\int_\Omega\tau\gamma(\Sigma_{0\le l\le m+1}C^l_{m+1}\nabla^l\upsilon\nabla^{m+1-l}\nabla\rho)\nabla^{m+3}\varrho dx|\\\le&C\|\nabla^{m+3}\varrho\|_{L^2}(\|\upsilon\|_{L_\infty}\|\nabla^{m+2}\rho\|_{L^2}+\|\nabla^{m+1}\upsilon\|_{L_2}\|\nabla\rho\|_{L_\infty})\\\le&\frac{\epsilon\gamma}{2}\|\nabla^{m+3}\varrho\|^2_{L^2}+C(\|\rho\|^2_{H^{m+1}}\|\upsilon\|^2_{H^{m+1}}+\|\rho\|^2_{H^{m+2}}\|\upsilon\|^2_{H^{m+1}}),\end{aligned} \tag{2.40}$$

$$|\int_\Omega\Sigma_{1\le l\le m+2}C^l_{m+2}\nabla^l(\frac{1}{\bar{\rho}+\tau\rho})\nabla^{m+2-l}\Delta u\nabla^{m+2}udx|$$

$$
\begin{aligned}
=&|\int_\Omega \nabla^{m+2}(\frac{1}{\overline{\rho}+\tau\rho}\Delta u)\nabla^{m+2}udx-\int_\Omega \frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+2}\Delta u\nabla^{m+2}udx|\\
\leq&|\int_\Omega \nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\rho}\Delta u)\nabla^{m+3}udx|+|\int_\Omega \nabla(\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+1}\Delta u)\nabla^{m+2}udx|\\
&+|\int_\Omega \nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1}\Delta u\nabla^{m+2}udx|\\
=&|\int_\Omega \sum_{0\leq l\leq m+1} C^l_{m+1}\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u\nabla^{m+3}udx|+|\int_\Omega \frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+1}\Delta u\nabla^{m+3}udx|\\
&+|\int_\Omega \nabla(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1}\Delta u\nabla^{m+2}udx|\\
\leq& C\|\nabla^{m+3}u\|_{L^2}(\|\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}+\|\rho\|_{H^{m+1}}\|\nabla^2 u\|_{L^\infty})\\
&+C\|\nabla^{m+3}u\|_{L^2}\|\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}+C\|\nabla\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}\|\nabla^{m+2}u\|_{L^2}\\
\leq& C\|\rho\|_{H^{m+1}}\|\nabla^{m+3}u\|_{L^2}\\
\leq& C\delta\|\nabla^{m+3}u\|_{L^2},
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
|\int_\Omega \nabla\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+3}u\nabla^{m+2}udx|\leq&\|\nabla\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}\|\nabla^{m+2}u\|_{L^2}\\
\leq& C\delta\|\nabla^{m+3}u\|_{L^2},
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(-\tau(\upsilon\cdot\nabla)\upsilon)\nabla^{m+2}udx|\\
=&|\int_\Omega \nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon)\nabla^{m+3}udx|\\
=&|\int_\Omega \tau\Sigma_{0\leq l\leq m+1}C^l_{m+1}\nabla^l\upsilon\nabla^{m+1-l}\nabla\upsilon\nabla^{m+3}udx|\\
\leq& C\|\nabla^{m+3}u\|_{L^2}(\|\upsilon\|_{L^\infty}\|\nabla^{m+2}\upsilon\|_{L^2}+\|\nabla^{m+1}\upsilon\|_{L^2}\|\nabla\upsilon\|_{L^\infty})\\
\leq& C\|\nabla^{m+3}u\|_{L^2}\|\upsilon\|_{H^{m+1}}\|\nabla^{m+2}\upsilon\|_{L^2},
\end{aligned}
\tag{2.43}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(h(\tau\rho)\nabla\rho)\nabla^{m+2}udx|\\
=&|\int_\Omega \nabla^{m+1}(h(\tau\rho)\nabla\rho)\nabla^{m+3}udx|\\
=&|\int_\Omega \Sigma_{0\leq l\leq m+1}C^l_{m+1}\nabla^l(h(\tau\rho))\nabla^{m+1-l}\nabla\rho\nabla^{m+3}udx|\\
\leq& C\|\nabla^{m+3}u\|_{L^2}(\|\rho\|_{L^\infty}\|\nabla^{m+2}\rho\|_{L^2}+\|\nabla^{m+1}\rho\|_{L^2}\|\nabla\rho\|_{L^\infty})\\
\leq& C\|\rho\|_{H^{m+1}}\|\nabla^{m+2}\rho\|_{L^2}\|\nabla^{m+3}u\|_{L^2},
\end{aligned}
\tag{2.44}
$$

$$
|\int_\Omega \nabla^{m+2}(\gamma\overline{\rho}\nabla\rho)\nabla^{m+2}udx|=|\int_\Omega \nabla^{m+1}(\gamma\overline{\rho}\nabla\rho)\nabla^{m+3}udx|\leq C\|\nabla^{m+2}\rho\|_{L^2}\|\nabla^{m+3}u\|_{L^2},
\tag{2.45}
$$

$$
|\int_\Omega \nabla^{m+2}(\tau f)\nabla^{m+2}udx|=|\int_\Omega \nabla^{m+1}(\tau f)\nabla^{m+3}udx|\leq C\|\nabla^{m+1}f\|_{L^2}\|\nabla^{m+3}u\|_{L^2},
\tag{2.46}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(\frac{\tau H\cdot\nabla H}{\bar\rho+\tau\rho})\nabla^{m+2}udx|\\
=&|\int_\Omega \nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\bar\rho+\tau\rho})\nabla^{m+3}udx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau H\cdot\nabla H)\nabla^{m+1-l}(\frac{1}{\bar\rho+\tau\rho})\nabla^{m+3}udx|\\
\le&C\|H\|_{L^\infty}\|\nabla H\|_{L^\infty}\|\rho\|_{H^{m+1}}\|\nabla^{m+3}u\|_{L^2}\\
&+C(\|H\|_{L^\infty}\|H\|_{H^{m+2}}+\|H\|_{H^{m+1}}\|\nabla H\|_{L^\infty})\|\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}\\
\le&C\delta\|\nabla^{m+3}u\|_{L^2}(\|\nabla H\|_{H^{m+1}}\|H\|_{H^{m+1}}+\|H\|_{H^{m+1}}\|H\|_{H^{m+2}}),
\end{aligned}\tag{2.47}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(\tau H\cdot\nabla\upsilon)\nabla^{m+2}Bdx|\\
=&|\int_\Omega \nabla^{m+1}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H)\nabla^{m+3}Bdx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau H)\nabla^{m+1-l}(\nabla\upsilon)\nabla^{m+3}Bdx|\\
\le&C(\|H\|_{L^\infty}\|\nabla^{m+2}\upsilon\|_{L^2}\|\nabla^{m+3}B\|_{L^2}+\|\nabla^{m+1}H\|_{L^2}\|\nabla\upsilon\|_{L^\infty}\|\nabla^{m+3}B\|_{L^2})\\
\le&C\|\nabla^{m+3}B\|_{L^2}\|H\|_{H^{m+1}}\|\upsilon\|_{H^{m+2}}\\
\le&\frac{\nu}{4}\|\nabla^{m+3}B\|_{L^2}^2+C\|\upsilon\|_{H^{m+2}}^2\|H\|_{H^{m+1}}^2,
\end{aligned}\tag{2.48}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(\tau\upsilon\cdot\nabla H)\nabla^{m+2}Bdx|\\
=&|\int_\Omega \nabla^{m+1}(\tau\upsilon\cdot\nabla H)\nabla^{m+3}Bdx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau\upsilon)\nabla^{m+1-l}(\nabla H)\nabla^{m+3}Bdx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau\upsilon)\nabla^{m+1-l}(\nabla H)\nabla^{m+3}Bdx|\\
\le&C(\|\upsilon\|_{L^\infty}\|\nabla^{m+2}H\|_{L^2}\|\nabla^{m+3}B\|_{L^2}+\|\nabla^{m+1}\upsilon\|_{L^2}\|\nabla H\|_{L^\infty}\|\nabla^{m+3}B\|_{L^2})\\
\le&C\|\nabla^{m+3}B\|_{L^2}\|\upsilon\|_{H^{m+1}}\|H\|_{H^{m+2}}\\
\le&\frac{\nu}{4}\|\nabla^{m+3}B\|_{L^2}^2+C\|H\|_{H^{m+2}}^2\|\upsilon\|_{H^{m+1}}^2.
\end{aligned}\tag{2.49}
$$

By summarizing up the estimates above, we deduce

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\varrho|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx\\
&\quad+\int_\Omega(\epsilon\gamma|\nabla^{m+3}\varrho|^2+\nu|\nabla^{m+3}B|^2)dx+\int_\Omega\frac{1}{2\bar\rho}|\nabla^{m+3}u|^2dx\\
\le&C\|(\rho,\upsilon,H)\|_{H^{m+1}}^2\|(\rho,\upsilon,H)\|_{H^{m+1}}^2+C\|\nabla^{m+1}f\|_{L^2}^2.
\end{aligned}\tag{2.50}
$$

Noting that δ is assumed to be sufficiently small, we have

$$
\int_0^T(\epsilon\gamma\|\nabla^{m+3}\varrho\|_{L^2}^2+\frac{1}{2\bar\rho}\|\nabla^{m+3}u\|_{L^2}^2+\nu\|\nabla^{m+3}B\|_{L^2}^2)dt
$$

$$\leq C\sup_{0\leq t\leq T}\|(\rho,v,H)(t)\|_{H^{m+1}}^2\int_0^T\|(\rho,v,H)(t)\|_{H^{m+2}}^2dt+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt$$
$$:=M. \tag{2.51}$$

Clearly, there exists $t^*\in(0,T)$ such that

$$\gamma\|\nabla^{m+3}\varrho(t^*)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+3}u(t^*)\|_{L^2}^2+\|\nabla^{m+3}B(t^*)\|_{L^2}^2\leq CM, \tag{2.52}$$

then, we see

$$\gamma\|\nabla^{m+2}\varrho(t^*)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(t^*)\|_{L^2}^2+\|\nabla^{m+2}B(t^*)\|_{L^2}^2\leq CM, \tag{2.53}$$

and then, by integrating (2.32) from t^* to t, $t\in(t^*,T]$, we see

$$\begin{aligned}
&\gamma\|\nabla^{m+2}\varrho(t)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(t)\|_{L^2}^2+\|\nabla^{m+2}B(t)\|_{L^2}^2\\
\leq&\gamma\|\nabla^{m+2}\varrho(t^*)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(t^*)\|_{L^2}^2+\|\nabla^{m+2}B(t^*)\|_{L^2}^2\\
&+\int_{t^*}^t(\epsilon\gamma\|\nabla^{m+3}\varrho\|_{L^2}^2+\nu\|\nabla^{m+3}B\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+3}u\|_{L^2}^2)\\
\leq&CM.
\end{aligned} \tag{2.54}$$

Therefore, we obtain

$$\gamma\|\nabla^{m+2}\varrho(T)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(T)\|_{L^2}^2+\|\nabla^{m+2}B(T)\|_{L^2}^2\leq CM \tag{2.55}$$

by the arbitrariness of t. This implies

$$\gamma\|\nabla^{m+2}\varrho(0)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(0)\|_{L^2}^2+\|\nabla^{m+2}B(0)\|_{L^2}^2\leq CM. \tag{2.56}$$

By integrating (2.32) from 0 to t, we deduce

$$\sup_{0\leq t\leq T}(\gamma\|\nabla^{m+2}\varrho(t)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(t)\|_{L^2}^2+\|\nabla^{m+2}B(t)\|_{L^2}^2)\leq CM, \tag{2.57}$$

then, we have

$$\begin{aligned}
&\sup_{0\leq t\leq T}(\gamma\|\varrho(t)\|_{H^{m+2}}^2+\|(u,B)(t)\|_{H^{m+2}}^2)+\int_0^T(\epsilon\gamma\|\varrho\|_{H^{m+3}}^2+\frac{1}{2\overline{\rho}}\|u\|_{H^{m+3}}^2+\nu\|B\|_{H^{m+3}}^2)dt\\
\leq&C\sup_{0\leq t\leq T}\|(\rho,v,H)(t)\|_{H^{m+1}}^2\int_0^T\|(\rho,v,H)(t)\|_{H^{m+2}}^2dt+C\int_0^T\|\nabla^{m+1}f(t)\|_{L^2}^2dt.
\end{aligned} \tag{2.58}$$

By integrating (2.32) from t to $t+h$, we see

$$\begin{aligned}
&\gamma\|\nabla^{m+2}\varrho(t+h)\|_{L^2}^2+\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(t+h)\|_{L^2}^2+\|\nabla^{m+2}B(t+h)\|_{L^2}^2\\
&-\gamma\|\nabla^{m+2}\varrho(t)\|_{L^2}^2-\frac{1}{2\overline{\rho}}\|\nabla^{m+2}u(t)\|_{L^2}^2-\|\nabla^{m+2}B(t)\|_{L^2}^2\\
\leq&Ch\sup_{0\leq t\leq T}\|(\rho,v,H)(t)\|_{H^{m+1}}^2\|(\rho,v,H)(t)\|_{H^{m+2}}^2+Ch\|\nabla^{m+1}f(t)\|_{L^2}^2.
\end{aligned} \tag{2.59}$$

By integrating (2.32)from 0 to T, we obtain

$$\int_0^T (\|\nabla^{m+2}(\varrho,u,B)(t+h)\|_{L^2}^2 - \|\nabla^{m+2}(\varrho,u,B)(t)\|_{L^2}^2)dt$$
$$\leq Ch \sup_{0\leq t\leq T} \|(\rho,\upsilon,H)(t)\|_{H^{m+1}}^2 \int_0^T \|(\rho,\upsilon,H)(t)\|_{H^{m+2}}^2 dt + Ch\int_0^T \|\nabla^{m+1} f(t)\|_{L^2}^2 dt. \tag{2.60}$$

By applying ∇^{m+1} to (2.5) and multiplying the equation by $(\nabla^{m+1}\varrho)_t$, we have

$$\nabla^{m+1}\varrho_t - \epsilon\nabla^{m+1}\Delta\varrho = -\nabla^{m+1}\tau(\upsilon\cdot\nabla\rho) \tag{2.61}$$

then, we see

$$(\nabla^{m+1}\varrho_t,(\nabla^{m+1}\varrho)_t) - (\epsilon\nabla^{m+1}\Delta\varrho,(\nabla^{m+1}\varrho)_t) = -(\nabla^{m+1}\tau(\upsilon\cdot\nabla\rho),(\nabla^{m+1}\varrho)_t), \tag{2.62}$$

which implies

$$\int_\Omega |\nabla^{m+1}\varrho_t|^2 dx + \frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla^{m+2}\varrho|^2 dx = -\int_\Omega \tau\nabla^{m+1}(\upsilon\cdot\nabla\rho)\nabla^{m+1}\varrho_t dx. \tag{2.63}$$

By applying ∇^{m+1} to (2.6) and then multiplying the equation by $(\nabla^{m+1}u)_t$, we deduce

$$\nabla^{m+1}u_t - \nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\rho}\Delta u)$$
$$= \nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon - h(\tau\rho)\nabla\rho - \gamma\overline{\rho}\nabla\rho + \tau f) + \nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}), \tag{2.64}$$

which follows

$$(\nabla^{m+1}u_t,(\nabla^{m+1}u)_t) - (\nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\rho}\Delta u),(\nabla^{m+1}u)_t)$$
$$= (\nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon - h(\tau\rho)\nabla\rho - \gamma\overline{\rho}\nabla\rho + \tau f),(\nabla^{m+1}u)_t)$$
$$+ (\nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}),(\nabla^{m+1}u)_t) \tag{2.65}$$

then, we have

$$(\nabla^{m+1}u_t,(\nabla^{m+1}u)_t) - (\sum_{1\leq l\leq m+1} C_{m+1}^l \nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u,(\nabla^{m+1}u)_t)$$
$$- ((\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1}\Delta u,(\nabla^{m+1}u)_t)$$
$$= (\nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon - h(\tau\rho)\nabla\rho - \gamma\overline{\rho}\nabla\rho + \tau f),(\nabla^{m+1}u)_t)$$
$$+ (\nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}),(\nabla^{m+1}u)_t), \tag{2.66}$$

and then, we obtain

$$\int_\Omega |\nabla^{m+1}u_t|^2 dx - \int_\Omega \sum_{1\leq l\leq m+1} C_{m+1}^l \nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u(\nabla^{m+1}u)_t dx$$

$$
\begin{aligned}
&+\frac{1}{2(\overline{\rho}+\tau\rho)}\frac{d}{dt}\int_\Omega|\nabla^{m+2}u|^2dx-\int_\Omega\nabla\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+2}u\nabla^{m+1}u_tdx\\
=&\int_\Omega\nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\overline{\rho}\nabla\rho+\tau f)(\nabla^{m+1}u)_tdx\\
&+\int_\Omega\nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho})(\nabla^{m+1}u)_tdx.
\end{aligned}\tag{2.67}
$$

By applying ∇^{m+1} to (2.7) and multiplying the equation by $(\nabla^{m+1}B)_t$, we deduce

$$
\nabla^{m+1}B_t-\nabla^{m+1}\Delta B=\nabla^{m+1}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H),\tag{2.68}
$$

which follows

$$
\begin{aligned}
&(\nabla^{m+1}B_t,(\nabla^{m+1}B)_t)-(\nabla^{m+1}\Delta B,(\nabla^{m+1}B)_t)\\
=&(\nabla^{m+1}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H),(\nabla^{m+1}B)_t),
\end{aligned}\tag{2.69}
$$

then, we have

$$
\begin{aligned}
&\int_\Omega|\nabla^{m+1}B_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}B|^2dx\\
=&\int_\Omega\nabla^{m+1}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H\nabla^{m+1}B)_tdx,
\end{aligned}\tag{2.70}
$$

and then, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\epsilon|\nabla^{m+2}\varrho|^2+\frac{1}{\overline{\rho}+\tau\rho}|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx\\
&\quad+\int_\Omega(|\nabla^{m+1}\varrho_t|^2+|\nabla^{m+1}u_t|^2+|\nabla^{m+1}B_t|^2)dx\\
=&-\int_\Omega\tau\nabla^{m+1}(\upsilon\cdot\nabla\rho)\nabla^{m+1}\varrho_tdx+\int_\Omega\sum_{1\le l\le m+1}C_{m+1}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u(\nabla^{m+1}u)_tdx\\
&\quad+\int_\Omega\nabla\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+2}u\nabla^{m+1}u_tdx\\
&\quad+\int_\Omega\nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\overline{\rho}\nabla\rho+\tau f)(\nabla^{m+1}u)_tdx\\
&\quad+\int_\Omega\nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho})(\nabla^{m+1}u)_tdx+\int_\Omega\nabla^{m+1}(\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H)(\nabla^{m+1}B)_tdx.
\end{aligned}\tag{2.71}
$$

We estimate the integral terms above as follows:

$$
\begin{aligned}
&|\int_\Omega\tau\nabla^{m+1}(\upsilon\cdot\nabla\rho)\nabla^{m+1}\varrho_tdx|\\
=&|\int_\Omega\tau(\Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l\upsilon\nabla^{m+1-l}\nabla\rho)\nabla^{m+1}\varrho_tdx|\\
\le&C\|\nabla^{m+1}\varrho_t\|_{L^2}(\|\upsilon\|_{L^\infty}\|\nabla^{m+2}\rho\|_{L^2}+\|\nabla^{m+1}\upsilon\|_{L^2}\|\nabla\rho\|_{L^\infty})\\
\le&\frac{1}{2}\|\nabla^{m+1}\varrho_t\|_{L^2}^2+C(\|\rho\|_{H^{m+1}}^2\|\upsilon\|_{H^{m+2}}^2+\|\rho\|_{H^{m+2}}^2\|\upsilon\|_{H^{m+1}}^2),
\end{aligned}\tag{2.72}
$$

$$\begin{aligned}
&|\int_\Omega \sum_{1\le l\le m+1} C_{m+1}^l \nabla^l(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1-l}\Delta u(\nabla^{m+1}u)_t dx|\\
\le & C\|\nabla^{m+1}\varrho_t\|_{L^2}(\|\rho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}+\|\rho\|_{H^{m+1}}\|\nabla^2 u\|_{L^\infty})\\
\le & C\|\rho\|_{H^{m+1}}\|\nabla^{m+1}u_t\|_{L^2}^2 \le C\delta\|\nabla^{m+1}u_t\|_{L^2}^2,
\end{aligned}\tag{2.73}$$

$$\begin{aligned}
|\int_\Omega \nabla\frac{1}{\overline{\rho}+\tau\rho}\nabla^{m+2}u\nabla^{m+1}u_t dx| \le & \|\nabla\rho\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}\|\nabla^{m+1}u_t\|_{L^2}\\
\le & C\delta\|\nabla^{m+1}u_t\|_{L^2}^2,
\end{aligned}\tag{2.74}$$

$$\begin{aligned}
&|\int_\Omega \nabla^{m+1}(-\tau(\upsilon\cdot\nabla)\upsilon)(\nabla^{m+1}u)_t dx|\\
=&|\int_\Omega \tau\Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l\upsilon\nabla^{m+1-l}\nabla\upsilon\nabla^{m+1}u_t dx|\\
\le & C\|\nabla^{m+1}u_t\|_{L^2}(\|\upsilon\|_{L^\infty}\|\nabla^{m+2}\upsilon\|_{L^2}+\|\nabla^{m+1}\upsilon\|_{L^2}\|\nabla\upsilon\|_{L^\infty})\\
\le & C\|\nabla^{m+1}u_t\|_{L^2}\|\upsilon\|_{H^{m+1}}\|\nabla^{m+2}\upsilon\|_{L^2},
\end{aligned}\tag{2.75}$$

$$\begin{aligned}
&|\int_\Omega \nabla^{m+1}(h(\tau\rho)\nabla\rho)(\nabla^{m+1}u)_t dx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l h(\tau\rho)\nabla^{m+1-l}\nabla\rho\nabla^{m+1}u_t dx|\\
\le & C\|\nabla^{m+1}u_t\|_{L^2}(\|\rho\|_{L^\infty}\|\nabla^{m+2}\rho\|_{L^2}+\|\nabla^{m+1}\rho\|_{L^2}\|\nabla\rho\|_{L^\infty})\\
\le & C\|\nabla^{m+1}u_t\|_{L^2}\|\rho\|_{H^{m+1}}\|\nabla^{m+2}\rho\|_{L^2},
\end{aligned}\tag{2.76}$$

$$|\int_\Omega \nabla^{m+1}(\gamma\overline{\rho}\nabla\rho)(\nabla^{m+1}u)_t dx| \le C\|\nabla^{m+1}u_t\|_{L^2}\|\nabla^{m+2}\rho\|_{L^2},\tag{2.77}$$

$$|\int_\Omega \nabla^{m+1}(\tau f)(\nabla^{m+1}u)_t dx| \le C\|\nabla^{m+1}u_t\|_{L^2}\|\nabla^{m+1}f\|_{L^2},\tag{2.78}$$

$$\begin{aligned}
&|\int_\Omega \nabla^{m+1}(\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho})(\nabla^{m+1}u)_t dx|\\
=&|\int_\Omega \sum_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau H\cdot\nabla H)\nabla^{m+1-l}(\frac{1}{\overline{\rho}+\tau\rho})\nabla^{m+1}u_t dx|\\
\le & C\|H\|_{L^\infty}\|\nabla H\|_{L^\infty}\|\rho\|_{H^{m+1}}\|\nabla^{m+1}u_t\|_{L^2}\\
&+C(\|H\|_{L^\infty}\|H\|_{H^{m+2}}+\|H\|_{H^{m+1}}\|\nabla H\|_{L^\infty})\|\rho\|_{L^\infty}\|\nabla^{m+1}u_t\|_{L^2}\\
\le & C\delta\|\nabla^{m+1}u_t\|_{L^2}(\|\nabla H\|_{H^{m+1}}\|H\|_{H^{m+1}}+\|H\|_{H^{m+1}}\|H\|_{H^{m+2}}),
\end{aligned}\tag{2.79}$$

$$\begin{aligned}
&|\int_\Omega \nabla^{m+1}(\tau H\cdot\nabla\upsilon)(\nabla^{m+1}B)_t dx|\\
=&|\int_\Omega \sum_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau H)\nabla^{m+1-l}\nabla\upsilon\nabla^{m+1}B_t dx|
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\nabla^{m+1}B_t\|_{L^2}\|H\|_{H^{m+1}}\|\upsilon\|_{H^{m+2}}\\
&\le \frac{1}{4}\|\nabla^{m+1}B_t\|_{L^2}^2+C\|H\|_{H^{m+1}}^2\|\upsilon\|_{H^{m+2}}^2, \qquad (2.80)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+1}(\tau\upsilon\cdot\nabla H)(\nabla^{m+1}B)_t dx|\\
=&|\int_\Omega \sum_{0\le l\le m+1} C_{m+1}^l \nabla^l(\tau\upsilon)\nabla^{m+1-l}\nabla H\nabla^{m+1}B_t dx|\\
\le& C\|\nabla^{m+1}B_t\|_{L^2}\|H\|_{H^{m+1}}\|\upsilon\|_{H^{m+2}}\\
\le& \frac{1}{8}\|\nabla^{m+1}B_t\|_{L^2}^2+C\|H\|_{H^{m+1}}^2\|\upsilon\|_{H^{m+2}}^2, \qquad (2.81)
\end{aligned}
$$

By summarizing these estimates, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (\epsilon|\nabla^{m+2}\varrho|^2+\frac{1}{\bar\rho+\tau\rho}|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx\\
&\quad+\int_\Omega (|\nabla^{m+1}\varrho_t|^2+|\nabla^{m+1}u_t|^2+|\nabla^{m+1}B_t|^2)dx\\
\le& C\|(\rho,\upsilon,H)\|_{H^{m+1}}^2\|(\rho,\upsilon,H)\|_{H^{m+2}}^2+C\|\nabla^{m+1}f\|_{L^2}^2. \qquad (2.82)
\end{aligned}
$$

Noting that δ is assumed to be sufficiently small, we have

$$
\begin{aligned}
&\int_0^T (|\nabla^{m+1}\varrho_t|^2+|\nabla^{m+1}u_t|^2+|\nabla^{m+1}B_t|^2)dt\\
\le& C\sup_{0\le t\le T}\|(\rho,\upsilon,H)\|_{H^{m+1}}^2\int_0^T \|(\rho,\upsilon,H)\|_{H^{m+2}}^2 dt+C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt. \qquad (2.83)
\end{aligned}
$$

Therefore, the compactness of the operator Λ is proved. □

LEMMA 2.6. *The operator Λ is continuous as δ is sufficiently small.*

Proof. We assume that $(\rho_n,\upsilon_n,H_n)\in S_\delta,\tau_n\in[0,1],(\rho,\upsilon,H)\in S_\delta,\tau\in[0,1]$ satisfy the following condition:

$$
\begin{aligned}
&\lim_{n\to\infty}\sup_{0\le t\le T}\|(\rho_n-\rho,\upsilon_n-\upsilon,H_n-H)\|_{H^{m+1}}^2+\int_0^T \|(\rho_n-\rho,\upsilon_n-\upsilon,H_n-H)\|_{H^{m+2}}^2 dt\\
=&\lim_{n\to\infty}(\tau_n-\tau)=0. \qquad (2.84)
\end{aligned}
$$

Let $(\varrho_n,u_n,B_n)=\Lambda((\rho_n,\upsilon_n,H_n),\tau_n),(\varrho,u,B)=\Lambda((\rho,\upsilon,H),\tau),\varrho^*=\varrho_n-\varrho,u^*=u_n-u,B^*=B_n-B$, we have

$$
\begin{aligned}
\varrho_{nt}-\epsilon\Delta\varrho_n&=-\tau_n\upsilon_n\cdot\nabla\rho_n,\\
\varrho_t-\epsilon\Delta\varrho&=-\tau\upsilon\cdot\nabla\rho,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\varrho_t^*-\epsilon\Delta\varrho^*=&-\tau_n\upsilon_n\cdot\nabla\rho_n+\tau\upsilon\cdot\nabla\rho\\
=&(\tau-\tau_n)\upsilon\cdot\nabla\rho-\tau_n\upsilon_n\cdot\nabla\rho_n+\tau_n\upsilon_n\cdot\nabla\rho_n+\tau_n\upsilon\cdot\nabla\rho\\
=&(\tau-\tau_n)\upsilon\cdot\nabla\rho-\tau_n(\upsilon\cdot(\nabla\rho_n-\nabla\rho)+(\upsilon_n-\upsilon)\nabla\rho_n). \qquad (2.85)
\end{aligned}
$$

We also have

$$u_{nt}-\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u_n=-\tau_n(\upsilon_n\cdot\nabla)\upsilon_n-h(\tau_n\rho_n)\nabla\rho_n-\gamma\overline{\rho}\nabla\rho_n+\frac{\tau_nH_n\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}+\tau_nf$$
$$u_t-\frac{1}{\overline{\rho}+\tau\rho}\Delta u=-\tau(\upsilon\cdot\nabla)\upsilon-h(\tau\rho)\nabla\rho-\gamma\overline{\rho}\nabla\rho+\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}+\tau f,$$

therefore, we see

$$\begin{aligned}&u_t^*-(\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u_n-\frac{1}{\overline{\rho}+\tau\rho}\Delta u)\\&=-(\tau_n(\upsilon_n\cdot\nabla)\upsilon_n-\tau(\upsilon\cdot\nabla)\upsilon)-(h(\tau_n\rho_n)\nabla\rho_n-h(\tau\rho)\nabla\rho)\\&\quad-(\gamma\overline{\rho}\nabla\rho_n-\gamma\overline{\rho}\nabla\rho)+(\frac{\tau_nH_n\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}-\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho})+((\tau_n-\tau)f).\end{aligned}\tag{2.86}$$

Let us expend some terms of the equation above as in the following:

$$\begin{aligned}&\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u_n-\frac{1}{\overline{\rho}+\tau\rho}\Delta u\\&=\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u_n-\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u+\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u-\frac{1}{\overline{\rho}+\tau\rho}\Delta u-\frac{1}{\overline{\rho}+\tau\rho_n}\Delta u+\frac{1}{\overline{\rho}+\tau\rho_n}\Delta u\\&=\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u^*+(\frac{1}{\overline{\rho}+\tau_n\rho_n}-\frac{1}{\overline{\rho}+\tau\rho_n})\Delta u+(\frac{1}{\overline{\rho}+\tau\rho_n}-\frac{1}{\overline{\rho}+\tau\rho})\Delta u,\end{aligned}\tag{2.87}$$

$$\begin{aligned}&-(\tau_n(\upsilon_n\cdot\nabla)\upsilon_n-\tau(\upsilon\cdot\nabla)\upsilon)\\&=-\tau_n(\upsilon_n\cdot\nabla)\upsilon_n+\tau(\upsilon\cdot\nabla)\upsilon-\tau_n(\upsilon_n\cdot\nabla)\upsilon+\tau_n(\upsilon_n\cdot\nabla)\upsilon+\tau(\upsilon_n\cdot\nabla)\upsilon\\&\quad-\tau(\upsilon_n\cdot\nabla)\upsilon-\tau(\upsilon_n\cdot\nabla)\upsilon_n+\tau(\upsilon_n\cdot\nabla)\upsilon_n-\tau(\upsilon\cdot\nabla)\upsilon_n+\tau(\upsilon\cdot\nabla)\upsilon_n\\&=-(\tau_n-\tau)(\upsilon_n\cdot\nabla)\upsilon-\tau(((\upsilon_n-\upsilon)\cdot\nabla)\upsilon_n+(\upsilon\cdot\nabla)(\upsilon_n-\upsilon))\\&\quad-\tau_n(\upsilon_n\cdot\nabla)(\upsilon_n-\upsilon)+\tau(\upsilon_n\cdot\nabla)(\upsilon_n-\upsilon),\end{aligned}\tag{2.88}$$

$$\begin{aligned}&-(h(\tau_n\rho_n)\nabla\rho_n-h(\tau\rho)\nabla\rho)\\&=-h(\tau_n\rho_n)\nabla\rho_n+h(\tau_n\rho_n)\nabla\rho+\frac{P'(\overline{\rho}+\tau\rho)}{\overline{\rho}+\tau\rho}\nabla\rho-\frac{P'(\overline{\rho}+\tau_n\rho_n)}{\overline{\rho}+\tau_n\rho_n}\nabla\rho\\&=-h(\tau_n\rho_n)(\nabla\rho_n-\nabla\rho)+\frac{1}{\overline{\rho}+\tau\rho}(P'(\overline{\rho}+\tau\rho)-P'(\overline{\rho}+\tau_n\rho))\nabla\rho\\&\quad+\frac{1}{\overline{\rho}+\tau\rho}(P'(\overline{\rho}+\tau_n\rho)-P'(\overline{\rho}+\tau_n\rho_n))\nabla\rho+P'(\overline{\rho}+\tau_n\rho_n)(\frac{1}{\overline{\rho}+\tau\rho}-\frac{1}{\overline{\rho}+\tau_n\rho})\nabla\rho\\&\quad+P'(\overline{\rho}+\tau_n\rho_n)(\frac{1}{\overline{\rho}+\tau_n\rho}-\frac{1}{\overline{\rho}+\tau_n\rho_n})\nabla\rho\end{aligned}\tag{2.89}$$

$$\begin{aligned}&\frac{\tau_nH_n\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}-\frac{\tau H\cdot\nabla H}{\overline{\rho}+\tau\rho}\\&=(\tau_n-\tau)\frac{H_n\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}+\frac{\tau(H_n-H)\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}+\frac{\tau H\cdot\nabla(H_n-H)}{\overline{\rho}+\tau_n\rho_n}\\&\quad+\tau H\cdot\nabla H(\frac{1}{\overline{\rho}+\tau_n\rho_n}-\frac{1}{\overline{\rho}+\tau_n\rho})+\tau H\cdot\nabla H(\frac{1}{\overline{\rho}+\tau_n\rho}-\frac{1}{\overline{\rho}+\tau\rho}).\end{aligned}\tag{2.90}$$

Basing on these expansions, we have

$$
\begin{aligned}
&u^*_t-\frac{1}{\overline{\rho}+\tau_n\rho_n}\Delta u^*\\
=&(\frac{1}{\overline{\rho}+\tau_n\rho_n}-\frac{1}{\overline{\rho}+\tau\rho_n})\Delta u+(\frac{1}{\overline{\rho}+\tau\rho_n}-\frac{1}{\overline{\rho}+\tau\rho})\Delta u-(\tau_n-\tau)(\upsilon_n\cdot\nabla)\upsilon\\
&-\tau(((\upsilon_n-\upsilon)\cdot\nabla)\upsilon_n+(\upsilon\cdot\nabla)(\upsilon_n-\upsilon))-\tau_n(\upsilon_n\cdot\nabla)(\upsilon_n-\upsilon)+\tau(\upsilon_n\cdot\nabla)(\upsilon_n-\upsilon)\\
&-h(\tau_n\rho_n)(\nabla\rho_n-\nabla\rho)+\frac{1}{\overline{\rho}+\tau\rho}(P'(\overline{\rho}+\tau\rho)-P'(\overline{\rho}+\tau_n\rho))\nabla\rho\\
&+\frac{1}{\overline{\rho}+\tau\rho}(P'(\overline{\rho}+\tau_n\rho)-P'(\overline{\rho}+\tau_n\rho_n))\nabla\rho+P'(\overline{\rho}+\tau_n\rho_n)(\frac{1}{\overline{\rho}+\tau\rho}-\frac{1}{\overline{\rho}+\tau_n\rho})\nabla\rho\\
&+P'(\overline{\rho}+\tau_n\rho_n)(\frac{1}{\overline{\rho}+\tau_n\rho}-\frac{1}{\overline{\rho}+\tau_n\rho_n})\nabla\rho(\tau_n-\tau)\frac{H_n\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}+\frac{\tau(H_n-H)\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}\\
&+\frac{\tau H\cdot\nabla(H_n-H)}{\overline{\rho}+\tau_n\rho_n}+\tau H\cdot\nabla H(\frac{1}{\overline{\rho}+\tau_n\rho_n}-\frac{1}{\overline{\rho}+\tau_n\rho})+\tau H\cdot\nabla H(\frac{1}{\overline{\rho}+\tau_n\rho}-\frac{1}{\overline{\rho}+\tau\rho})\\
&+((\tau_n-\tau)f.
\end{aligned}\tag{2.91}
$$

On the other hand, we have

$$
\begin{aligned}
B_{nt}-\Delta B_n=&\tau_nH_n\cdot\nabla\upsilon_n-\tau_n\upsilon_n\cdot\nabla H_n,\\
B_t-\Delta B=&\tau H\cdot\nabla\upsilon-\tau\upsilon\cdot\nabla H,
\end{aligned}
$$

then, we have

$$
\begin{aligned}
B^*_t-\Delta B^*=&(\tau_nH_n\cdot\nabla\upsilon_n-\tau H\cdot\nabla\upsilon)-(\tau_n\upsilon_n\cdot\nabla H_n-\tau\upsilon\cdot\nabla H)\\
=&(\tau_n-\tau)H_n\cdot\nabla\upsilon_n+\tau(H_n-H)\nabla\upsilon_n+\tau H\cdot\nabla(\upsilon_n-\upsilon)\\
&-(\tau_n-\tau)\upsilon_n\cdot\nabla H_n-\tau(\upsilon_n-\upsilon)\cdot\nabla H_n-\tau\upsilon\cdot\nabla(H_n-H).
\end{aligned}\tag{2.92}
$$

Having this, we get the following system:

$$
\begin{aligned}
&\varrho^*_t-\epsilon\Delta\varrho^*=G_1(\rho_n,\upsilon_n,H_n,\tau_n,\rho,\upsilon,H,\tau)\\
&u^*_t-\frac{1}{\overline{\rho}+\tau\rho}\Delta u^*=G_2(\rho_n,\upsilon_n,H_n,\tau_n,\rho,\upsilon,H,\tau)\\
&B^*_t-\Delta B^*=G_3(\rho_n,\upsilon_n,H_n,\tau_n,\rho,\upsilon,H,\tau)\\
&divB^*=divu^*=0
\end{aligned}
$$

where

$$
G_1(\rho_n,\upsilon_n,H_n,\tau_n,\rho,\upsilon,H,\tau)=(\tau-\tau_n)\upsilon\cdot\nabla\rho-\tau_n(\upsilon\cdot(\nabla\rho_n-\nabla\rho)+(\upsilon_n-\upsilon)\nabla\rho_n),
$$

$$
\begin{aligned}
&G_2(\rho_n,\upsilon_n,H_n,\tau_n,\rho,\upsilon,H,\tau)\\
=&(\frac{1}{\overline{\rho}+\tau_n\rho_n}-\frac{1}{\overline{\rho}+\tau\rho_n})\Delta u+(\frac{1}{\overline{\rho}+\tau\rho_n}-\frac{1}{\overline{\rho}+\tau\rho})\Delta u-(\tau_n-\tau)(\upsilon_n\cdot\nabla)\upsilon\\
&-\tau(((\upsilon_n-\upsilon)\cdot\nabla)\upsilon_n+(\upsilon\cdot\nabla)(\upsilon_n-\upsilon))-\tau_n(\upsilon_n\cdot\nabla)(\upsilon_n-\upsilon)+\tau(\upsilon_n\cdot\nabla)(\upsilon_n-\upsilon)\\
&-h(\tau_n\rho_n)(\nabla\rho_n-\nabla\rho)+\frac{1}{\overline{\rho}+\tau\rho}(P'(\overline{\rho}+\tau\rho)-P'(\overline{\rho}+\tau_n\rho))\nabla\rho\\
&+\frac{1}{\overline{\rho}+\tau\rho}(P'(\overline{\rho}+\tau_n\rho)-P'(\overline{\rho}+\tau_n\rho_n))\nabla\rho+P'(\overline{\rho}+\tau_n\rho_n)(\frac{1}{\overline{\rho}+\tau\rho}-\frac{1}{\overline{\rho}+\tau_n\rho})\nabla\rho
\end{aligned}
$$

$$+P'(\overline{\rho}+\tau_n\rho_n)(\frac{1}{\overline{\rho}+\tau_n\rho}-\frac{1}{\overline{\rho}+\tau_n\rho_n})\nabla\rho(\tau_n-\tau)\frac{H_n\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}+\frac{\tau(H_n-H)\cdot\nabla H_n}{\overline{\rho}+\tau_n\rho_n}$$
$$+\frac{\tau H\cdot\nabla(H_n-H)}{\overline{\rho}+\tau_n\rho_n}+\tau H\cdot\nabla H(\frac{1}{\overline{\rho}+\tau_n\rho_n}-\frac{1}{\overline{\rho}+\tau_n\rho})+\tau H\cdot\nabla H(\frac{1}{\overline{\rho}+\tau_n\rho}-\frac{1}{\overline{\rho}+\tau\rho}),$$

$$G_3(\rho_n,\upsilon_n,H_n,\tau_n,\rho,\upsilon,H,\tau)$$
$$=(\tau_n-\tau)H_n\cdot\nabla\upsilon_n+\tau(H_n-H)\nabla\upsilon_n+\tau H\cdot\nabla(\upsilon_n-\upsilon)$$
$$-(\tau_n-\tau)\upsilon_n\cdot\nabla H_n-\tau(\upsilon_n-\upsilon)\cdot\nabla H_n-\tau\upsilon\cdot\nabla(H_n-H).$$

Basing on this, the following identity is clear:

$$\lim_{n\to\infty}\sup_{0\le t\le T}\|(\varrho_n-\varrho,u_n-u,B_n-B)(t)\|^2_{H^{m+1}}+\int_0^T\|(\varrho_n-\varrho,u_n-u,B_n-B)(t)\|^2_{H^{m+2}}dt=0.$$

Therefore, we proved the continuity of the operator Λ, and complete the proof of the lemma. ∎

3. Existence of periodic solution

Lemma 3.1. *Let $f(x,t)\in L^2(0,T;H^{m+1}(\Omega)), f(-x,t)=-f(x,t), m\ge[\frac{n}{2}]+1$ is an integer, then there exists a solution $(\varrho,u,B)\in S_\delta$ of system (2.1)-(2.4), where $\int_0^T\|f(x,t)\|_{H^{m+1}}dt$ is suitably small.*

Proof. Considering system (2.5)-(2.8) and replacing (ρ,υ,H) with (ϱ,u,B) as follows:

$$\varrho_t+\epsilon\Delta\varrho=-\tau u\cdot\nabla\varrho \tag{3.1}$$
$$u_t-\frac{1}{\overline{\rho}+\tau\varrho}\Delta u=-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\overline{\rho}\nabla\varrho+\frac{\tau B\cdot\nabla B}{\overline{\rho}+\tau\varrho}+\tau f \tag{3.2}$$
$$B_t-\Delta B=\tau B\cdot\nabla u-\tau u\cdot\nabla B \tag{3.3}$$
$$divu=divB=0. \tag{3.4}$$

By applying ∇^{m+2} to (3.1), and multiplying the equation by $\gamma\nabla^{m+2}\varrho$, we deduce

$$\nabla^{m+2}\varrho_t-\epsilon\nabla^{m+2}\Delta\varrho=-\nabla^{m+2}\tau(u\cdot\nabla\varrho), \tag{3.5}$$

which follows:

$$(\nabla^{m+2}\varrho_t,\gamma\nabla^{m+2}\varrho)-(\epsilon\nabla^{m+2}\Delta\varrho,\gamma\nabla^{m+2}\varrho)=-(\nabla^{m+2}\tau(u\cdot\nabla\varrho),\gamma\nabla^{m+2}\varrho), \tag{3.6}$$

then, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\gamma|\nabla^{m+2}\varrho|^2dx+\int\Omega\varepsilon\gamma|\nabla^{m+3}\varrho|^2dx=-\int_\Omega\tau\gamma\nabla^{m+2}(u\cdot\nabla\varrho)\nabla^{m+2}\varrho dx. \tag{3.7}$$

By applying ∇^{m+2} to (3.2), and multiplying the equation by $\nabla^{m+2}u$, we deduce

$$\nabla^{m+2}u_t-\nabla^{m+2}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u)$$
$$=\nabla^{m+2}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\overline{\rho}\nabla\varrho+\tau f)+\nabla^{m+2}(\frac{\tau B\cdot\nabla B}{\overline{\rho}+\tau\varrho}), \tag{3.8}$$

which implies

$$(\nabla^{m+2}u_t,\nabla^{m+2}u)-(\Sigma_{0\le l\le m+2}C^l_{m+2}\nabla^l(\frac{1}{\overline{\rho}+\tau\varrho})\nabla^{m+2-l}\Delta u,\nabla^{m+2}u)$$

$$=(\nabla^{m+2}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\bar{\rho}\nabla\varrho+\tau f),\nabla^{m+2}u)$$
$$+(\nabla^{m+2}(\frac{\tau B\cdot\nabla B}{\bar{\rho}+\tau\varrho}),\nabla^{m+2}u), \tag{3.9}$$

then, we see

$$(\nabla^{m+2}u_t,\nabla^{m+2}u)-(\Sigma_{1\leq l\leq m+2}C^l_{m+2}\nabla^l(\frac{1}{\bar{\rho}+\tau\varrho})\nabla^{m+2-l}\Delta u,\nabla^{m+2}u)$$
$$-(\frac{1}{\bar{\rho}+\tau\varrho}\nabla^{m+2}\Delta u,\nabla^{m+2}u)$$
$$=(\nabla^{m+2}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\bar{\rho}\nabla\varrho+\tau f),\nabla^{m+2}u)$$
$$+(\nabla^{m+2}(\frac{\tau B\cdot\nabla B}{\bar{\rho}+\tau\varrho}),\nabla^{m+2}u), \tag{3.10}$$

which follows

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}u|^2dx-\int_\Omega\Sigma_{1\leq l\leq m+2}C^l_{m+2}\nabla^l(\frac{1}{\bar{\rho}+\tau\varrho})\nabla^{m+2-l}\Delta u\nabla^{m+2}udx$$
$$+\int_\Omega\frac{1}{\bar{\rho}+\tau\varrho}|\nabla^{m+3}u|^2dx-\int_\Omega\nabla\frac{1}{\bar{\rho}+\tau\varrho}\nabla^{m+3}u\nabla^{m+2}udx$$
$$=\int_\Omega\nabla^{m+2}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\bar{\rho}\nabla\varrho+\tau f)\nabla^{m+2}udx$$
$$+\int_\Omega\nabla^{m+2}(\frac{\tau B\cdot\nabla B}{\bar{\rho}+\tau\varrho})\nabla^{m+2}udx. \tag{3.11}$$

By applying ∇^{m+2} to (3.3), and multiplying the equation by $\nabla^{m+2}B$, we deduce

$$\nabla^{m+2}B_t-\nabla^{m+2}\Delta B=\nabla^{m+2}(\tau B\cdot u-\tau u\cdot\nabla B), \tag{3.12}$$

which follows

$$(\nabla^{m+2}B_t,\nabla^{m+2}B)-(\nabla^{m+2}\Delta B,\nabla^{m+2}B)=(\nabla^{m+2}(\tau B\cdot u-\tau u\cdot\nabla B),\nabla^{m+2}B), \tag{3.13}$$

then, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+2}B|^2dx+\int_\Omega|\nabla^{m+3}B|^2dx=\int_\Omega\nabla^{m+2}(\tau B\cdot\nabla u-\tau u\cdot\nabla B)\nabla^{m+2}Bdx, \tag{3.14}$$

and then, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\varrho|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx+\int_\Omega(\epsilon\gamma|\nabla^{m+3}\varrho|^2+\nu|\nabla^{m+3}B|^2)dx$$
$$+\int_\Omega\frac{1}{\bar{\rho}+\tau\varrho}|\nabla^{m+3}u|^2dx$$
$$=-\int_\Omega\tau\gamma\nabla^{m+2}(u\cdot\nabla\varrho)\nabla^{m+2}\varrho dx+\int_\Omega\Sigma_{1\leq l\leq m+2}C^l_{m+2}\nabla^l(\frac{1}{\bar{\rho}+\tau\varrho})\nabla^{m+2-l}\Delta u\nabla^{m+2}udx$$
$$-\int_\Omega\nabla\frac{1}{\bar{\rho}+\tau\varrho}\nabla^{m+3}u\nabla^{m+2}udx+\int_\Omega\nabla^{m+2}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\bar{\rho}\nabla\varrho+\tau f)\nabla^{m+2}udx$$
$$+\int_\Omega\nabla^{m+2}(\frac{\tau B\cdot\nabla B}{\bar{\rho}+\tau\varrho})\nabla^{m+2}udx+\int_\Omega\nabla^{m+2}(\tau B\cdot\nabla u-\tau u\cdot\nabla B)\nabla^{m+2}Bdx. \tag{3.15}$$

The estimates for the integral terms of the expansion above are as in the following:

$$
\begin{aligned}
&|\int_\Omega \tau\gamma\nabla^{m+2}(u\cdot\nabla\varrho)\nabla^{m+2}\varrho dx|\\
=&|-\tau\gamma\int_\Omega (u\cdot\nabla\nabla^{m+2}\varrho+[\nabla^{m+2},u]\cdot\nabla\varrho)\nabla^{m+2}\varrho dx|\\
=&|-\tau\gamma\int_\Omega u\cdot\nabla\frac{|\nabla^{m+2}\varrho|^2}{2}+([\nabla^{m+2},u]\cdot\nabla\varrho)\nabla^{m+2}\varrho dx|\\
\le&|\frac{\tau\gamma}{2}\int_\Omega divu|\nabla^{m+2}\varrho|^2dx|+C\|\nabla^{m+2}\varrho\|_{L^2}\big(\|\nabla^{m+2}\varrho\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla\varrho\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}\big)\\
\le&C\|\nabla^{m+2}\varrho\|_{L^2}^2\|\nabla u\|_{H^m}+C'\|\nabla^{m+2}\varrho\|_{L^2}^2\|\nabla\varrho\|_{H^m}^2+\eta\|\nabla^{m+3}u\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
&|\int_\Omega \Sigma_{1\le l\le m+2}C_{m+2}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\varrho})\nabla^{m+2-l}\Delta u\nabla^{m+2}udx|\\
=&|\int_\Omega \nabla^{m+2}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u)\nabla^{m+2}udx-\int_\Omega\frac{1}{\overline{\rho}+\tau\varrho}\nabla^{m+2}\Delta u\nabla^{m+2}udx|\\
\le&|\int_\Omega \nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u)\nabla^{m+3}udx|+|\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\varrho}\nabla^{m+1}\Delta u)\nabla^{m+2}udx|\\
&\quad+|\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\varrho})\nabla^{m+1}\Delta u\nabla^{m+2}udx|\\
=&|\int_\Omega\sum_{0\le l\le m+1}C_{m+1}^l\nabla^l(\frac{1}{\overline{\rho}+\tau\varrho})\nabla^{m+1-l}\Delta u\nabla^{m+3}udx|+|\int_\Omega\frac{1}{\overline{\rho}+\tau\varrho}\nabla^{m+1}\Delta u\nabla^{m+3}udx|\\
&\quad+|\int_\Omega\nabla(\frac{1}{\overline{\rho}+\tau\varrho})\nabla^{m+1}\Delta u\nabla^{m+2}udx|\\
\le&C\|\nabla^{m+3}u\|_{L^2}(\|\nabla^2u\|_{H^{m+1}}\|\nabla\varrho\|_{L^\infty}+\|\nabla^2u\|_{L^\infty}\|\varrho\|_{H^{m+1}})\\
\le&C\|\nabla^{m+3}u\|_{L^2}^2\|\nabla\varrho\|_{H^m}+C'\|\nabla^{m+3}u\|_{L^2}^2+C''\|u\|_{H^{m+2}}^2\|\varrho\|_{H^{m+1}}^2,
\end{aligned}
$$

$$
\begin{aligned}
&|\int_\Omega\nabla\frac{1}{\overline{\rho}+\tau\varrho}\nabla^{m+3}u\nabla^{m+2}udx|\\
\le&C\|\nabla\varrho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}\|\nabla^{m+2}u\|_{L^2}\\
\le&C\|\nabla\varrho\|_{H^m}\|\nabla^{m+2}u\|_{L^2}^2+\eta\|\nabla^{m+3}u\|_{L^2}^2,
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
&|\int_\Omega\nabla^{m+2}(-\tau(u\cdot\nabla)u)\nabla^{m+2}udx|\\
=&|\int_\Omega\nabla^{m+1}(-\tau(u\cdot\nabla)u)\nabla^{m+3}udx|\\
=&|\int_\Omega\tau\Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^lu\nabla^{m+1-l}\nabla u\nabla^{m+3}udx|\\
\le&C\|\nabla^{m+3}u\|_{L^2}(\|u\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}+\|\nabla^{m+1}u\|_{L^2}\|\nabla u\|_{L^\infty})\\
\le&C\|\nabla^{m+2}u\|_{L^2}^2\|u\|_{H^{m+1}}^2+\eta\|\nabla^{m+3}u\|_{L^2}^2,
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(h(\tau\varrho)\nabla\varrho)\nabla^{m+2}udx|\\
=&|\int_\Omega \nabla^{m+1}(h(\tau\varrho)\nabla\varrho)\nabla^{m+3}udx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(h(\tau\varrho))\nabla^{m+1-l}\nabla\varrho\nabla^{m+3}udx|\\
\le& C\|\nabla^{m+3}u\|_{L^2}(\|\varrho\|_{L^\infty}\|\nabla^{m+2}\varrho\|_{L^2}+\|\nabla^{m+1}\varrho\|_{L^2}\|\nabla\varrho\|_{L^\infty})\\
\le& C\|\varrho\|_{H^{m+1}}^2\|\nabla^{m+2}\varrho\|_{L^2}^2+\eta\|\nabla^{m+3}u\|_{L^2}^2
\end{aligned}\tag{3.18}
$$

$$
|\int_\Omega \nabla^{m+2}(\gamma\bar{\rho}\nabla\varrho)\nabla^{m+2}udx|=|\int_\Omega \nabla^{m+1}(\gamma\bar{\rho}\nabla\varrho)\nabla^{m+3}udx|\le C\|\nabla^{m+2}\varrho\|_{L^2}^2+\eta\|\nabla^{m+3}u\|_{L^2}^2
$$

$$
|\int_\Omega \nabla^{m+2}(\tau f)\nabla^{m+2}udx|=|\int_\Omega \nabla^{m+1}(\tau f)\nabla^{m+3}udx|\le C\|\nabla^{m+1}f\|_{L^2}^2+\eta\|\nabla^{m+3}u\|_{L^2}^2\tag{3.19}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(\frac{\tau B\cdot\nabla B}{\bar{\rho}+\tau\varrho})\nabla^{m+2}udx|\\
=&|\int_\Omega \nabla^{m+1}(\frac{\tau B\cdot\nabla B}{\bar{\rho}+\tau\varrho})\nabla^{m+3}udx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau B\cdot\nabla B)\nabla^{m+1-l}(\frac{1}{\bar{\rho}+\tau\varrho})\nabla^{m+3}udx|\\
\le& C\|B\|_{L^\infty}\|\nabla B\|_{L^\infty}\|\varrho\|_{H^{m+1}}\|\nabla^{m+3}u\|_{L^2}\\
&\quad+C(\|B\|_{L^\infty}\|B\|_{H^{m+2}}+\|B\|_{H^{m+1}}\|\nabla B\|_{L^\infty})\|\varrho\|_{L^\infty}\|\nabla^{m+3}u\|_{L^2}\\
\le&\eta\delta\|\nabla^{m+3}u\|_{L^2}^2+C(\|\nabla B\|_{H^{m+1}}^2\|B\|_{H^{m+1}}^2\|\varrho\|_{H^{m+1}}^2+\|B\|_{H^{m+1}}^2\|\nabla B\|_{H^{m+1}})
\end{aligned}\tag{3.20}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(\tau B\cdot\nabla u)\nabla^{m+2}Bdx|\\
=&|\int_\Omega \nabla^{m+1}(\tau B\cdot\nabla u)\nabla^{m+3}Bdx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau B)\nabla^{m+1-l}(\nabla u)\nabla^{m+3}Bdx|\\
\le& C(\|B\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}\|\nabla^{m+3}B\|_{L^2}+\|\nabla^{m+1}B\|_{L^2}\|\nabla u\|_{L^\infty}\|\nabla^{m+3}B\|_{L^2})\\
\le& C\|\nabla^{m+3}B\|_{L^2}\|B\|_{H^{m+1}}\|u\|_{H^{m+2}}\\
\le&\frac{\nu}{4}\|\nabla^{m+3}B\|_{L^2}^2+C\|u\|_{H^{m+2}}^2\|B\|_{H^{m+1}}^2
\end{aligned}\tag{3.21}
$$

$$
\begin{aligned}
&|\int_\Omega \nabla^{m+2}(\tau u\cdot\nabla B)\nabla^{m+2}Bdx|=|\int_\Omega \nabla^{m+1}(\tau u\cdot\nabla B)\nabla^{m+3}Bdx|\\
&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau u)\nabla^{m+1-l}(\nabla B)\nabla^{m+3}Bdx|\\
=&|\int_\Omega \Sigma_{0\le l\le m+1}C_{m+1}^l\nabla^l(\tau u)\nabla^{m+1-l}(\nabla B)\nabla^{m+3}Bdx|\\
\le& C(\|u\|_{L^\infty}\|\nabla^{m+2}B\|_{L^2}\|\nabla^{m+3}B\|_{L^2}+\|\nabla^{m+1}u\|_{L^2}\|\nabla B\|_{L^\infty}\|\nabla^{m+3}B\|_{L^2})
\end{aligned}
$$

$$\begin{aligned}
&\le C\|\nabla^{m+3}B\|_{L^2}\|u\|_{H^{m+1}}\|B\|_{H^{m+2}}\\
&\le \frac{\nu}{4}\|\nabla^{m+3}B\|_{L^2}^2+C\|B\|_{H^{m+2}}^2\|u\|_{H^{m+1}}^2.
\end{aligned}\tag{3.22}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\varrho|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx\\
&\quad+\int_\Omega(\epsilon\gamma|\nabla^{m+3}\varrho|^2+\nu|\nabla^{m+3}B|^2)dx+\int_\Omega\frac{1}{2\overline{\rho}}|\nabla^{m+3}u|^2dx\\
&\le C\|(\varrho,u,B)\|_{H^{m+1}}^2\|(\varrho,u,B)\|_{H^{m+1}}^2+C\|\nabla^{m+1}f\|_{L^2}^2\\
&\quad+C\|\nabla B\|_{H^{m+1}}^2\|B\|_{H^{m+1}}^2\|\varrho\|_{H^{m+1}}^2+C\|\nabla^{m+2}\varrho\|_{L^2}^2
\end{aligned}$$

By applying ∇^{m+1} to (3.2), and multiplying the equation by $\nabla^{m+1}\nabla\varrho$, we deduce

$$\begin{aligned}
&\nabla^{m+1}u_t-\nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u)\\
&=\nabla^{m+1}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\overline{\rho}\nabla\varrho+\tau f)+\nabla^{m+1}(\frac{\tau B\cdot\nabla B}{\overline{\rho}+\tau\varrho}),
\end{aligned}\tag{3.23}$$

then, we see

$$\begin{aligned}
&(\nabla^{m+1}u_t,\nabla^{m+1}\nabla\varrho)-(\nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u),\nabla^{m+1}\nabla\varrho)\\
&=(\nabla^{m+1}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho-\gamma\overline{\rho}\nabla\varrho+\tau f),\nabla^{m+1}\nabla\varrho)+(\nabla^{m+1}(\frac{\tau B\cdot\nabla B}{\overline{\rho}+\tau\varrho}),\nabla^{m+1}\nabla\varrho),
\end{aligned}\tag{3.24}$$

and then, we have

$$\begin{aligned}
\gamma\overline{\rho}\int_\Omega|\nabla^{m+2}\varrho|^2dx=&-\int_\Omega\nabla^{m+1}u_t\nabla^{m+1}\nabla\varrho dx-\int_\Omega\nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u)\nabla^{m+1}\nabla\varrho dx\\
&-\int_\Omega\nabla^{m+1}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho+\frac{\tau B\cdot\nabla B}{\overline{\rho}+\tau\varrho}+\tau f)\nabla^{m+1}\nabla\varrho dx\\
\le&-\int_\Omega\nabla^{m+1}u_t\nabla^{m+1}\nabla\varrho dx+\|\nabla^{m+1}(\frac{1}{\overline{\rho}+\tau\varrho}\Delta u)\|_{L^2}\|\nabla^{m+2}\varrho\|_{L^2}\\
&+\|\nabla^{m+1}(-\tau(u\cdot\nabla)u-h(\tau\varrho)\nabla\varrho+\frac{\tau B\cdot\nabla B}{\overline{\rho}+\tau\varrho}+\tau f)\|_{L^2}\|\nabla^{m+2}\varrho\|_{L^2},
\end{aligned}$$

where

$$\begin{aligned}
&-\int_\Omega\nabla^{m+1}u_t\nabla^{m+1}\nabla\varrho dx\\
&=-\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\varrho dx-\int_\Omega\nabla^{m+1}divu\cdot\nabla^{m+1}\varrho_t dx\\
&=-\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\varrho dx\\
&\quad+\epsilon\int_\Omega\nabla^{m+1}\nabla\varrho\nabla^{m+1}\Delta udx+\int_\Omega\tau\nabla^{m+1}(u\cdot\nabla\varrho)\nabla^{m+1}divudx\\
&\le-\int_\Omega\nabla^{m+1}u_t\nabla^{m+1}\nabla\varrho dx
\end{aligned}$$

$$+\epsilon\|\nabla^{m+2}\varrho\|_{L^2}\|\nabla^{m+3}u\|_{L^2}+\tau\|\nabla^m div(\varrho u)\|_{L^2}\|\nabla^{m+2}divu\|_{L^2},$$

and

$$\|\nabla^m(u\cdot\nabla\varrho)\|_{L^2}\leq C(\|u\|_{L^\infty}\|\nabla^{m+1}\varrho\|_{L^2}+\|\nabla^m u\|_{L^2}\|\nabla\varrho\|_{L^\infty}). \tag{3.25}$$

Therefore, we have the following estimate:

$$\begin{aligned}&\frac{\gamma\bar{\rho}}{2}\int_\Omega|\nabla^{m+2}\varrho|^2dx+\frac{d}{dt}\int_\Omega\nabla^{m+1}u\nabla^{m+1}\nabla\varrho dx\\ \leq& C\|\nabla^{m+3}u\|_{L^2}^2+C\|\varrho\|_{H^{m+1}}^2\|u\|_{H^{m+1}}^2+\|u\|_{H^{m+1}}^2\|u\|_{H^{m+2}}^2\\ &+\|\varrho\|_{H^{m+1}}^2\|\varrho\|_{H^{m+1}}^2+\|\nabla B\|_{H^{m+1}}^2\|B\|_{H^{m+1}}^2\|\varrho\|_{H^{m+1}}^2+C\|\nabla^{m+1}f\|_{L^2}^2.\end{aligned} \tag{3.26}$$

and then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+2}\varrho|^2+|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2+C\nabla^{m+1}u\cdot\nabla^{m+1}\nabla\varrho)dx\\ &+\int_\Omega(|\nabla^{m+2}\varrho|^2+|\nabla^{m+3}u|^2+|\nabla^{m+3}B|^2)dx\\ \leq& C\|(\varrho,u,B)\|_{H^{m+1}}^2\|(\varrho,u,B)\|_{H^{m+2}}^2+\|\nabla B\|_{H^{m+1}}^2\|B\|_{H^{m+1}}^2\|\varrho\|_{H^{m+1}}^2\\ &+C\|\nabla^{m+1}f\|_{L^2}^2.\end{aligned} \tag{3.27}$$

by integrating (3.27) from 0 to T, we deduce

$$\begin{aligned}&\int_0^T\|\nabla^{m+2}(\varrho,u,B)(t)\|_{L^2}^2dt\leq C\sup_{0\leq t\leq T}\|(\varrho,u,B)\|_{H^{m+1}}^2\int_0^T\|(\varrho,u,B)\|_{H^{m+2}}^2dt\\ &+C\sup_{0\leq t\leq T}\|B\|_{H^{m+1}}^2\sup_{0\leq t\leq T}\|\varrho\|_{H^{m+1}}^2\int_0^T\|\nabla B\|_{H^{m+1}}^2+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt\\ \leq& C\delta^4+C\delta^6+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.\end{aligned} \tag{3.28}$$

Clearly, there exists $t^*\in(0,T)$ such that

$$\|\nabla^{m+2}(\varrho,u,B)(t^*)\|_{L^2}^2\leq C\delta^4+C\delta^6+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.$$

By integrating (3.27) from t^* to t, we deduce

$$\begin{aligned}&\|\nabla^{m+2}(\varrho,u,B)(t)\|_{L^2}^2-\|\nabla^{m+2}(\varrho,u,B)(t^*)\|_{L^2}^2\\ \leq& C\delta^4+C\delta^6+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt.\end{aligned} \tag{3.29}$$

Then, we see

$$\|\nabla^{m+2}(\varrho,u,B)(t)\|_{L^2}^2\leq C\delta^4+C\delta^6+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt, \tag{3.30}$$

because of the arbitrariness of t, we have

$$\|\nabla^{m+2}(\varrho,u,B)(T)\|_{L^2}^2\leq C\delta^4+C\delta^6+C\int_0^T\|\nabla^{m+1}f\|_{L^2}^2dt. \tag{3.31}$$

Then, since ϱ, u, B are periodic, we have

$$\|\nabla^{m+2}(\varrho,u,B)(0)\|_{L^2}^2 \leq C\delta^4 + C\delta^6 + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt. \tag{3.32}$$

By integrating (3.27) from 0 to t, we obtain

$$\sup_{0\leq t\leq T} \|\nabla^{m+2}(\varrho,u,B)(t)\|_{L^2}^2 \leq C\delta^4 + C\delta^6 + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt. \tag{3.33}$$

Then, we have

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|(\varrho,u,B)(t)\|_{H^{m+1}}^2 + \int_0^T \|(\varrho,u,B)(t)\|_{H^{m+2}}^2 dt\\
\leq &\sup_{0\leq t\leq T} \|(\varrho,u,B)(t)\|_{H^{m+2}}^2 + \int_0^T \|(\varrho,u,B)(t)\|_{H^{m+2}}^2 dt\\
\leq &C\delta^4 + C\delta^6 + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt,
\end{aligned} \tag{3.34}$$

which implies

$$\delta^2 \leq C\delta^4 + C\delta^6 + C\int_0^T \|\nabla^{m+1}f\|_{L^2}^2 dt. \tag{3.35}$$

One can verify this inequality does not hold forever, i.e. we get a contradiction here. We have shown that the system (2.1)-(2.4) is equivalent to the equation

$$U - \Lambda(U,1) = 0, U = (\varrho,u,B) \in S_\delta,$$

so we are able to show the existence of solution (ϱ,u,B) of the system (2.1)-(2.4) by the topological degree theory. Since the contradiction, we infer

$$(I - \Lambda(\cdot,\tau))(\partial \hat{B_\delta}(0)) \neq 0,$$

where $\hat{B_\delta}(0)$ is a ball of radius of δ centered at the origin in S, and $\delta > 0$. We only need to show that

$$deg(I - \Lambda(\cdot,1), \hat{B_\delta}(0), 0) \neq 0.$$

If $\tau = 0$, by multiplying (2.5) by ϱ, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\varrho|^2 dx + \varepsilon\int_\Omega |\nabla\varrho|^2 dx = 0,$$

we integrate this equation from 0 to T, and deduce

$$\int_0^T \int_\Omega \varepsilon|\nabla\varrho|^2 dx = 0.$$

According to the $Poinc\acute{a}re's$ inequality, we obtain

$$\varrho \equiv 0,$$

likewise, we obtain

$$\varrho \equiv u \equiv B \equiv 0,$$

that is

$$\Lambda((\rho,v,H),0) \equiv 0.$$

Then, we see

$$deg(I-\Lambda(\cdot,1),B_{\delta}\hat{(}0),0) = deg(I-\Lambda(\cdot,0),B_{\delta}\hat{(}0),0) = deg(I,B_{\delta}\hat{(}0),0) = 1,$$

which shows that the system (2.1)-(2.4) admits a solution $(\varrho,u,B) \in S_\delta$, where δ is independent of ε. Therefore, Lemma 3.1 is proved. □

Let (ϱ_n,u_n,B_n) be the periodic solution of the system (2.1)-(2.4), then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (\gamma|\nabla^{m+2}\varrho_n|^2 + |\nabla^{m+2}u_n|^2 + |\nabla^{m+2}B_n|^2 + C\nabla^{m+1}u_n \cdot \nabla^{m+1}\nabla\varrho_n)dx \\
&\quad + \int_\Omega (|\nabla^{m+2}\varrho_n|^2 + |\nabla^{m+3}u_n|^2 + |\nabla^{m+3}B_n|^2)dx \\
&\leq C\|(\varrho_n,u_n,B_n)\|^2_{H^{m+1}}\|(\varrho_n,u_n,B_n)\|^2_{H^{m+2}} + \|\nabla B_n\|^2_{H^{m+1}}\|B_n\|^2_{H^{m+1}}\|\varrho_n\|^2_{H^{m+1}} \\
&\quad + C\|\nabla^{m+1}f\|^2_{L^2}. \qquad (3.36)
\end{aligned}$$

By integrating (3.36) from 0 to T, we have

$$\sup_{0\leq t\leq T}\|(\varrho_n,u_n,B_n)(t)\|^2_{H^{m+2}} + \int_0^T (\|\varrho_n(t)\|^2_{H^{m+2}} + \|(u_n,B_n)(t)\|^2_{H^{m+3}})dt \leq C\delta^2. \quad (3.37)$$

We have

$$\varrho_n \in L^\infty((0,T);H^{m+2}(\Omega)),$$

then, it is clear that

$$\varrho_n \in C^\alpha(\Omega), \alpha \in (0,1),$$

therefore, we obtain

$$\begin{aligned}
&\int_{B_R} |\varrho_n(y,t_1) - \varrho_n(y,t_2)|dy \\
&= \int_{B_R} |\int_{t_1}^{t_2} \frac{\partial \varrho_n(y,t)}{\partial t}dt|dy \\
&\leq C(\int_{t_1}^{t_2}\int_{B_R} |\frac{\partial \varrho_n(y,t)}{\partial t}|^2 dydt)^{\frac{1}{2}}(\int_{t_1}^{t_2}\int_{B_R} 1dydt)^{\frac{1}{2}} \\
&\leq C|t_1-t_2|^{\frac{1}{2}}R^{\frac{n}{2}}, \qquad (3.38)
\end{aligned}$$

where B_R is a ball of radius R, $R = |t_1-t_2|^\theta$, $\theta = \frac{1}{2\alpha+n}$. There exists a point $x^* \in B_R$ such that

$$||\varrho_n(x^*,t_1) - \varrho_n(x^*,t_2)|| \leq C|t_1-t_2|^{\frac{1}{2}}R^{-\frac{n}{2}} = C|t_1-t_2|^{\frac{1-n\theta}{2}}.$$

Then, we have

$$
\begin{aligned}
&|\varrho_n(x,t_1)-\varrho_n(x,t_2)|\\
\le&|\varrho_n(x,t_1)-\varrho_n(x^*,t_1)|+|\varrho_n(x^*,t_1)-\varrho_n(x^*,t_2)|+|\varrho_n(x^*,t_2)-\varrho_n(x,t_2)|\\
\le&C(|x-x^*|^{\theta\alpha}+|t_1-t_2|^{\frac{1-n\theta}{2}}).
\end{aligned}
\tag{3.39}
$$

Therefore, we obtain

$$
|\varrho_n(x_1,t_1)-\varrho_n(x_2,t_2)|\le c(|x_1-x_2|^{\alpha}+|t_1-t_2|^{\beta}),\beta=\frac{1-n\theta}{2}.
$$

Moreover, we have

$$
\varrho_n\in C^{\alpha,\beta}(\Omega\times(0,T)).
$$

On the other hand, we have

$$
u_n\in C^{\alpha_1,\beta_1}(\Omega\times(0,T)),
$$

$$
H_n\in C^{\alpha_2,\beta_2}(\Omega\times(0,T)),
$$

where $\alpha_i,\beta_i\in(0,1),i=1,2$. By integrating (3.36) from t to $t+h$, we have

$$
\|\nabla^{m+2}(\varrho_n,u_n,B_n)(t+h)\|_{L^2}^2-\|\nabla^{m+2}(\varrho_n,u_n,B_n)(t)\|_{L^2}^2\le Ch \tag{3.40}
$$

Then, we integrate (3.40) from 0 to T, and we deduce

$$
\int_0^T\|\nabla^{m+2}(\varrho_n,u_n,B_n)(t+h)\|_{L^2}^2-\|\nabla^{m+2}(\varrho_n,u_n,B_n)(t)\|_{L^2}^2dt\le Ch. \tag{3.41}
$$

Let $\varepsilon\to 0$, according to the Arzela-Ascoli theorem, we infer that there exists a subsequence denoted by (ϱ_n,u_n,B_n) such that

(1) $(\varrho_n,u_n,B_n)\to(\varrho,u,B)$ uniformly,
(2) $(\varrho_n,u_n,B_n)\rightharpoonup^*(\varrho,u,B)$ in $L^\infty((0,T);H^{m+2})$,
(3) $(u_n,B_n)\to(u,B)$ in $L^2((0,T);H^{m+3})$,
(4) $\varrho_n\to\varrho$ in $L^2((0,T);H^{m+1})$,
(5) $(u_n,B_n)\to(u,B)$ in $L^2((0,T);H^{m+2})$.

This completes the proof of the existence part of Theorem 1.1.

4. The uniqueness of periodic solutions

In this section, we give the uniqueness of the periodic solution to (1.9)-(1.12).

Proof. **(Proof of Theorem 1.1, Uniqueness)** Let $(\varrho_1,u_1,B_1)\in W\bigcap S_\delta,(\varrho_2,u_2,B_2)\in W\bigcap S_\delta$ be two periodic solutions of (1.9)-(1.12) such that

$$
\sup_{0\le t\le T}\|(\varrho_i,u_i,B_i)\|_{H^{m+2}}\le\theta,(i=1,2),
$$

where θ is sufficiently small. We set $\varrho=\varrho_1-\varrho_2,u=u_1-u_2,B=B_1-B_2$, then we have

$$
\begin{aligned}
\varrho_{1t}&=-u_1\cdot\nabla\varrho_1,\\
\varrho_{2t}&=-u_2\cdot\nabla\varrho_2.
\end{aligned}
\tag{4.1}
$$

Then, we see

$$\begin{aligned}\varrho_t &= -u_1\cdot\nabla\varrho_1 + u_2\cdot\nabla\varrho_2\\ &= -(u_1\cdot\nabla\varrho_1 - u_1\cdot\nabla\varrho_2) - u_1\cdot\nabla\varrho_2 + u_2\cdot\nabla\varrho_2\\ &= -u_1\cdot\nabla\varrho - u\cdot\nabla\varrho_2.\end{aligned}\tag{4.2}$$

We also have

$$\begin{aligned}u_{1t} - \frac{1}{\overline{\rho}+\varrho_1}\Delta u_1 &= -(u_1\cdot\nabla)u_1 - h(\varrho_1)\nabla\varrho_1 - \gamma\overline{\rho}\nabla\varrho_1 + \frac{B_1\cdot\nabla B_1}{\overline{\rho}+\varrho_1} + f\\ u_{2t} - \frac{1}{\overline{\rho}+\varrho_2}\Delta u_2 &= -(u_2\cdot\nabla)u_2 - h(\varrho_2)\nabla\varrho_2 - \gamma\overline{\rho}\nabla\varrho_2 + \frac{B_2\cdot\nabla B_2}{\overline{\rho}+\varrho_2} + f\end{aligned}\tag{4.3}$$

therefore, we have

$$\begin{aligned}&u_t - (\frac{1}{\overline{\rho}+\varrho_1}\Delta u_1 - \frac{1}{\overline{\rho}+\varrho_2}\Delta u_2)\\ =&-((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2) - (h(\varrho_1)\nabla\varrho_1 - h(\varrho_2)\nabla\varrho_2)\\ &-\gamma\overline{\rho}\nabla\varrho + (\frac{B_1\cdot\nabla B_1}{\overline{\rho}+\varrho_1} - \frac{B_2\cdot\nabla B_2}{\overline{\rho}+\varrho_2}).\end{aligned}\tag{4.4}$$

Two terms of the identity above can be expanded as in the following:

$$\begin{aligned}&-(\frac{1}{\overline{\rho}+\varrho_1}\Delta u_1 - \frac{1}{\overline{\rho}+\varrho_2}\Delta u_2)\\ =&-(\frac{1}{\overline{\rho}}\Delta u + \frac{1}{\overline{\rho}+\varrho_1}\Delta u_1 - \frac{1}{\overline{\rho}+\varrho_2}\Delta u_1 + \frac{1}{\overline{\rho}+\varrho_2}\Delta u_1 - \frac{1}{\overline{\rho}+\varrho_2}\Delta u_2 - \frac{1}{\overline{\rho}}\Delta u)\\ =&-\frac{1}{\overline{\rho}}\Delta u - (\frac{1}{\overline{\rho}+\varrho_1} - \frac{1}{\overline{\rho}+\varrho_2})\Delta u_1 - (\frac{1}{\overline{\rho}+\varrho_2} - \frac{1}{\overline{\rho}})\Delta u - ((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2)\\ =&-((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_1) + (u_2\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2\\ =&-(u\cdot\nabla)u_1 - (u_2\cdot\nabla)u - (h(\varrho_1)\nabla\varrho_1 - h(\varrho_2)\nabla\varrho_2)\\ =&\frac{P'(\overline{\rho})}{\overline{\rho}}\nabla\varrho_1 - \frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1}\nabla\varrho_1 + \frac{P'(\overline{\rho}+\varrho_2)}{\overline{\rho}+\varrho_2}\nabla\varrho_2 - \frac{P'(\overline{\rho})}{\overline{\rho}}\nabla\varrho_2 + \frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1}\nabla\varrho_2 - \frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1}\nabla\varrho_2\\ =&(\frac{P'(\overline{\rho})}{\overline{\rho}} - \frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho + (\frac{P'(\overline{\rho}+\varrho_2)}{\overline{\rho}+\varrho_2} - \frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho_2;\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\frac{B_1\cdot\nabla B_1}{\overline{\rho}+\varrho_1} - \frac{B_2\cdot\nabla B_2}{\overline{\rho}+\varrho_2}\\ =&\frac{B_1\cdot\nabla B_1}{\overline{\rho}+\varrho_1} - \frac{B_2\cdot\nabla B_1}{\overline{\rho}+\varrho_1} + \frac{B_2\cdot\nabla B_1}{\overline{\rho}+\varrho_1} - \frac{B_2\cdot\nabla B_2}{\overline{\rho}+\varrho_2} - \frac{B_2\cdot\nabla B_2}{\overline{\rho}+\varrho_1} + \frac{B_2\cdot\nabla B_2}{\overline{\rho}+\varrho_1}\\ =&\frac{B\cdot\nabla B_1}{\overline{\rho}+\varrho_1} + \frac{B_2\cdot\nabla B}{\overline{\rho}+\varrho_1} + B_2\cdot\nabla B_2(\frac{1}{\overline{\rho}+\varrho_1} - \frac{1}{\overline{\rho}+\varrho_2}).\end{aligned}\tag{4.6}$$

On the other hand, we have

$$B_{1t} - \Delta B_1 = B_1\cdot\nabla u_1 - u_1\cdot\nabla B_1,\tag{4.7}$$

$$B_{2t} - \Delta B_2 = B_2\cdot\nabla u_2 - u_2\cdot\nabla B_2,\tag{4.8}$$

therefore, we have

$$B_t - \Delta B = B_1\cdot\nabla u_1 - B_2\cdot\nabla u_2 - (u_1\cdot\nabla B_1 - u_2\cdot\nabla B_2).\tag{4.9}$$

We note that

$$\begin{aligned} B_1\cdot\nabla u_1 - B_2\cdot\nabla u_2 =& B_1\cdot\nabla u_1 - B_1\cdot\nabla u_2 + B_1\cdot\nabla u_2 - B_2\cdot\nabla u_2\\ =& B_1\cdot\nabla u + B\cdot\nabla u_2,\\ -(u_1\cdot\nabla B_1 - u_2\cdot\nabla B_2) =& -(u_1\cdot\nabla B_1 - u_1\cdot\nabla B_2 + u_1\cdot\nabla B_2 - u_2\cdot\nabla B_2)\\ =& -(u_1\cdot\nabla B + u\cdot\nabla B_2), \end{aligned} \tag{4.10}$$

then, we obtain the following equations:

$$\varrho_t = -u_1\cdot\nabla\varrho - u\cdot\nabla\varrho_2, \tag{4.11}$$

$$\begin{aligned} u_t - \frac{1}{\bar\rho}\Delta u =& \Big(\frac{1}{\bar\rho+\varrho_1} - \frac{1}{\bar\rho+\varrho_2}\Big)\Delta u_1 + \Big(\frac{1}{\bar\rho+\varrho_2} - \frac{1}{\bar\rho}\Big)\Delta u - (u\cdot\nabla)u_1 - (u_2\cdot\nabla)u\\ &+ \Big(\frac{P'(\bar\rho)}{\bar\rho} - \frac{P'(\bar\rho+\varrho_1)}{\bar\rho+\varrho_1}\Big)\nabla\varrho + \Big(\frac{P'(\bar\rho+\varrho_2)}{\bar\rho+\varrho_2} - \frac{P'(\bar\rho+\varrho_1)}{\bar\rho+\varrho_1}\Big)\nabla\varrho_2 - \gamma\bar\rho\nabla\varrho\\ &+ \frac{B\cdot\nabla B_1}{\bar\rho+\varrho_1} + \frac{B_2\cdot\nabla B}{\bar\rho+\varrho_1} + B_2\cdot\nabla B_2\Big(\frac{1}{\bar\rho+\varrho_1} - \frac{1}{\bar\rho+\varrho_2}\Big), \end{aligned} \tag{4.12}$$

$$B_t - \Delta B = B_1\cdot\nabla u + B\cdot\nabla u_2 - (u_1\cdot\nabla B + u\cdot\nabla B_2), \tag{4.13}$$

$$divB = divu = 0. \tag{4.14}$$

By applying ∇^{m+1} to (4.11) and then multiplying the equation by $\gamma\nabla^{m+1}\varrho$, we have

$$\nabla^{m+1}\varrho_t = -\nabla^{m+1}u_1\cdot\nabla\varrho - \nabla^{m+1}u\cdot\nabla\varrho_2, \tag{4.15}$$

which implies

$$(\nabla^{m+1}\varrho_t, \gamma\nabla^{m+1}) = -(\nabla^{m+1}u_1\cdot\nabla\varrho, \gamma\nabla^{m+1}) - (\nabla^{m+1}u\cdot\nabla\varrho_2, \gamma\nabla^{m+1}). \tag{4.16}$$

Then, we see

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \gamma|\nabla^{m+1}\varrho|^2 dx = -\int_\Omega (\gamma\nabla^{m+1}(u_1\cdot\nabla\varrho)\nabla^{m+1}\varrho + \gamma\nabla^{m+1}(u\cdot\nabla\varrho_2)\nabla^{m+1}\varrho)dx. \tag{4.17}$$

By applying ∇^{m+1} to (4.12) and then multiplying the equation by $\nabla^{m+1}u$, we deduce

$$\begin{aligned} &\nabla^{m+1}u_t - \frac{1}{\bar\rho}\nabla^{m+1}\Delta u\\ =& \nabla^{m+1}\Big(\Big(\frac{1}{\bar\rho+\varrho_1} - \frac{1}{\bar\rho+\varrho_2}\Big)\Delta u_1\Big) + \nabla^{m+1}\Big(\Big(\frac{1}{\bar\rho+\varrho_2} - \frac{1}{\bar\rho}\Big)\Delta u\Big)\\ &- \nabla^{m+1}((u\cdot\nabla)u_1) - \nabla^{m+1}((u_2\cdot\nabla)u) - \gamma\bar\rho\nabla^{m+1}\nabla\varrho\\ &+ \nabla^{m+1}\Big(\Big(\frac{P'(\bar\rho)}{\bar\rho} - \frac{P'(\bar\rho+\varrho_1)}{\bar\rho+\varrho_1}\Big)\nabla\varrho\Big) + \nabla^{m+1}\Big(\Big(\frac{P'(\bar\rho+\varrho_2)}{\bar\rho+\varrho_2} - \frac{P'(\bar\rho+\varrho_1)}{\bar\rho+\varrho_1}\Big)\nabla\varrho_2\Big)\\ &+ \nabla^{m+1}\frac{B\cdot\nabla B_1}{\bar\rho+\varrho_1} + \nabla^{m+1}\frac{B_2\cdot\nabla B}{\bar\rho+\varrho_1} + \nabla^{m+1}\Big(B_2\cdot\nabla B_2\Big(\frac{1}{\bar\rho+\varrho_1} - \frac{1}{\bar\rho+\varrho_2}\Big)\Big), \end{aligned} \tag{4.18}$$

which follows:

$$(\nabla^{m+1}u_t, \nabla^{m+1}u) - (\frac{1}{\bar\rho}\nabla^{m+1}\Delta u, \nabla^{m+1}u)$$

$$
\begin{aligned}
=&(\nabla^{m+1}((\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2})\Delta u_1),\nabla^{m+1}u)\\
&+(\nabla^{m+1}((\frac{1}{\overline{\rho}+\varrho_2}-\frac{1}{\overline{\rho}})\Delta u),\nabla^{m+1}u)\\
&-(\nabla^{m+1}((u\cdot\nabla)u_1),\nabla^{m+1}u)-(\nabla^{m+1}((u_2\cdot\nabla)u),\nabla^{m+1}u)\\
&-(\gamma\overline{\rho}\nabla^{m+1}\nabla\varrho,\nabla^{m+1}u)+(\nabla^{m+1}((\frac{P'(\overline{\rho})}{\overline{\rho}}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho),\nabla^{m+1}u)\\
&+(\nabla^{m+1}((\frac{P'(\overline{\rho}+\varrho_2)}{\overline{\rho}+\varrho_2}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho_2),\nabla^{m+1}u)+(\nabla^{m+1}\frac{B\cdot\nabla B_1}{\overline{\rho}+\varrho_1},\nabla^{m+1}u)\\
&+(\nabla^{m+1}\frac{B_2\cdot\nabla B}{\overline{\rho}+\varrho_1},\nabla^{m+1}u)+(\nabla^{m+1}(B_2\cdot\nabla B_2(\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2})),\nabla^{m+1}u). \qquad (4.19)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{m+1}u|^2dx+\int_\Omega\frac{1}{\overline{\rho}}|\nabla^{m+2}u|^2dx\\
=&\int_\Omega\nabla^{m+1}((\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2})\Delta u_1)\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}((\frac{1}{\overline{\rho}+\varrho_2}-\frac{1}{\overline{\rho}})\Delta u)\nabla^{m+1}udx\\
&-\int_\Omega\nabla^{m+1}((u\cdot\nabla)u_1)\nabla^{m+1}udx-\int_\Omega\nabla^{m+1}((u_2\cdot\nabla)u)\nabla^{m+1}udx\\
&-\int_\Omega\gamma\overline{\rho}\nabla^{m+1}\nabla\varrho\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}((\frac{P'(\overline{\rho})}{\overline{\rho}}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho)\nabla^{m+1}udx\\
&+\int_\Omega\nabla^{m+1}((\frac{P'(\overline{\rho}+\varrho_2)}{\overline{\rho}+\varrho_2}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho_2)\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}\frac{B\cdot\nabla B_1}{\overline{\rho}+\varrho_1}\nabla^{m+1}udx\\
&+\int_\Omega\nabla^{m+1}\frac{B_2\cdot\nabla B}{\overline{\rho}+\varrho_1}\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}(B_2\cdot\nabla B_2(\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2}))\nabla^{m+1}udx.
\end{aligned}
\qquad (4.20)
$$

By applying ∇^{m+1} to (4.13) and then multiplying the equation by $\nabla^{m+1}B$, we have

$$
\begin{aligned}
&\nabla^{m+1}B_t-\nabla^{m+1}\Delta B\\
=&\nabla^{m+1}(B_1\cdot\nabla u)+\nabla^{m+1}(B\cdot\nabla u_2)-\nabla^{m+1}(u_1\cdot\nabla B)-\nabla^{m+1}(u\cdot\nabla B_2),
\end{aligned}
\qquad (4.21)
$$

which follows:

$$
\begin{aligned}
&(\nabla^{m+1}B_t,\nabla^{m+1}B)-(\nabla^{m+1}\Delta B,\nabla^{m+1}B)\\
=&(\nabla^{m+1}(B_1\cdot\nabla u),\nabla^{m+1}B)+(\nabla^{m+1}(B\cdot\nabla u_2),\nabla^{m+1}B)\\
&-(\nabla^{m+1}(u_1\cdot\nabla B),\nabla^{m+1}B)-(\nabla^{m+1}(u\cdot\nabla B_2),\nabla^{m+1}B).
\end{aligned}
\qquad (4.22)
$$

Then, we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt\int_\Omega}|\nabla^{m+1}B|^2dx+\int_\Omega|\nabla^{m+2}B|^2dx\\
=&\int_\Omega(\nabla^{m+1}(B_1\cdot\nabla u)\nabla^{m+1}B+\nabla^{m+1}(B\cdot\nabla u_2)\nabla^{m+1}B)dx\\
&-\int_\Omega(\nabla^{m+1}(u_1\cdot\nabla B)\nabla^{m+1}B+\nabla^{m+1}(u\cdot\nabla B_2)\nabla^{m+1}B)dx.
\end{aligned}
\qquad (4.23)
$$

By summarizing the identities above, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+1}\varrho|^2+|\nabla^{m+1}u|^2+|\nabla^{m+1}B|^2)dx+\int_\Omega(\frac{1}{\overline{\rho}}|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx\\
=&-\int_\Omega(\gamma\nabla^{m+1}(u_1\cdot\nabla\varrho)\nabla^{m+1}\varrho+\gamma\nabla^{m+1}(u\cdot\nabla\varrho_2)\nabla^{m+1}\varrho)dx\\
&+\int_\Omega\nabla^{m+1}((\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2})\Delta u_1)\nabla^{m+1}udx-\int_\Omega\nabla^{m+1}((u_2\cdot\nabla)u)\nabla^{m+1}udx\\
&+\int_\Omega\nabla^{m+1}((\frac{1}{\overline{\rho}+\varrho_2}-\frac{1}{\overline{\rho}})\Delta u)\nabla^{m+1}udx-\int_\Omega\nabla^{m+1}((u\cdot\nabla)u_1)\nabla^{m+1}udx\\
&-\int_\Omega\gamma\overline{\rho}\nabla^{m+1}\nabla\varrho\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}((\frac{P'(\overline{\rho})}{\overline{\rho}}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho)\nabla^{m+1}udx\\
&+\int_\Omega\nabla^{m+1}((\frac{P'(\overline{\rho}+\varrho_2)}{\overline{\rho}+\varrho_2}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho_2)\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}\frac{B\cdot\nabla B_1}{\overline{\rho}+\varrho_1}\nabla^{m+1}udx\\
&+\int_\Omega\nabla^{m+1}\frac{B_2\cdot\nabla B}{\overline{\rho}+\varrho_1}\nabla^{m+1}udx+\int_\Omega\nabla^{m+1}(B_2\cdot\nabla B_2(\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2}))\nabla^{m+1}udx\\
&+\int_\Omega(\nabla^{m+1}(B_1\cdot\nabla u)\nabla^{m+1}B+\nabla^{m+1}(B\cdot\nabla u_2)\nabla^{m+1}B)dx\\
&-\int_\Omega(\nabla^{m+1}(u_1\cdot\nabla B)\nabla^{m+1}B+\nabla^{m+1}(u\cdot\nabla B_2)\nabla^{m+1}B)dx\\
:=&\sum_{i=1}^{16}W_i. \qquad (4.24)
\end{aligned}
$$

We estimate the integral terms above as in the following:

$$
\begin{aligned}
|W_1|=&|-\gamma\int_\Omega(u_1\cdot\nabla\nabla^{m+1}\varrho+[\nabla^{m+1},u_1]\cdot\nabla\varrho)\nabla^{m+1}\varrho dx|\\
=&|-\gamma\int_\Omega u_1\cdot\nabla\frac{|\nabla^{m+1}\varrho|^2}{2}+([\nabla^{m+1},u_1]\cdot\nabla\varrho)\nabla^{m+1}\varrho dx|\\
\leq&|\frac{\gamma}{2}\int_\Omega divu_1|\nabla^{m+1}\varrho|^2dx|+C\|\nabla^{m+1}\varrho\|_{L^2}\|\nabla^m\nabla\varrho\|_{L^2}\|\nabla u_1\|_{L^\infty}\\
&+C\|\nabla^{m+1}\varrho\|_{L^2}\|\nabla\varrho\|_{L^\infty}\|\nabla^{m+1}u_1\|_{L^2}\\
\leq&C\theta\|\nabla^{m+1}\varrho\|_{L^2}, \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
|W_2|=&|-\gamma\int_\Omega\sum_{0\leq l\leq m+1}C_{m+1}^l\nabla^l u\nabla^{m+1-l}\nabla\varrho_2\nabla^{m+1}\varrho dx|\\
\leq&C\|\nabla^{m+1}\varrho\|_{L^2}(\|u\|_{L^\infty}\|\nabla^{m+2}\varrho_2\|_{L^2}+\|\nabla^{m+1}u\|_{L^2}\|\nabla\varrho_2\|_{L^\infty})\\
\leq&C\theta(\|\nabla^{m+1}\varrho\|_{L^2}^2+\|\nabla^{m+1}u\|_{L^2}^2), \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
|W_3|\leq&C\|\nabla^{m+2}u\|_{L^2}(\|\varrho\|_{L^\infty}\|\nabla^{m+2}u_1\|_{L^2}+\|\nabla^m\varrho\|_{L^2}\|\nabla^2u_1\|_{L^\infty})\\
\leq&C\theta(\|\nabla^m\varrho\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2), \qquad (4.27)
\end{aligned}
$$

$$\begin{aligned}|W_4| \le & C\|\nabla^{m+2}u\|_{L^2}(\|\varrho_2\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}+\|\nabla^m\varrho_2\|_{L^2}\|\nabla^2 u\|_{L^\infty})\\ \le & C\theta\|\nabla^{m+2}u\|_{L^2}^2,\end{aligned} \tag{4.28}$$

$$\begin{aligned}|W_5| \le & C\|\nabla^{m+2}u\|_{L^2}(\|u\|_{L^\infty}\|\nabla^{m+2}u_1\|_{L^2}+\|\nabla^{m+1}u\|_{L^2}\|\nabla u_1\|_{L^\infty})\\ \le & C\theta\|\nabla^{m+1}u\|_{L^2}^2,\end{aligned} \tag{4.29}$$

$$\begin{aligned}|W_6| \le & C\|\nabla^{m+2}u\|_{L^2}(\|u_2\|_{L^\infty}\|\nabla^{m+2}u\|_{L^2}+\|\nabla^{m+1}u\|_{L^2}\|\nabla u\|_{L^\infty})\\ \le & C\theta\|\nabla^{m+1}u\|_{L^2}^2,\end{aligned} \tag{4.30}$$

$$\begin{aligned}|W_7| \le & C\|\nabla^{m+2}u\|_{L^2}(\|\nabla\varrho\|_{L^\infty}\|\nabla^m\varrho_1\|_{L^2}+\|\nabla^{m+1}\varrho\|_{L^2}\|\varrho_1\|_{L^\infty})\\ \le & C\theta(\|\nabla^m\varrho\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2),\end{aligned} \tag{4.31}$$

$$\begin{aligned}|W_8| \le & C\|\nabla^{m+2}u\|_{L^2}(\|\nabla\varrho_2\|_{L^\infty}\|\nabla^m\varrho\|_{L^2}+\|\nabla^{m+1}\varrho_2\|_{L^2}\|\varrho\|_{L^\infty})\\ \le & C\theta(\|\nabla^m\varrho\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2),\end{aligned} \tag{4.32}$$

$$\begin{aligned}|W_9| = & |\int_\Omega \nabla^m\frac{B\cdot B_1}{\bar\rho+\varrho_1}\nabla^{m+2}udx|\\ = & |\int_\Omega\sum_{0\le l\le m}C_m^l\nabla^l(B\cdot B_1)\nabla^{m-l}(\frac{1}{\bar\rho+\varrho_1})\nabla^{m+2}udx|\\ \le & C\|\nabla^{m+2}u\|_{L^2}(\|\nabla B\|_{L^\infty}\|B_1\|_{L^\infty}\|\varrho_1\|_{H^m}+\|\nabla B\|_{L^\infty}\|B_1\|_{H^m}\|\varrho_1\|_{L^\infty}\\ & +\|\nabla B\|_{H^m}\|B_1\|_{L^\infty}\|\varrho_1\|_{L^\infty})\le C\theta(\|\nabla^{m+1}B\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2),\end{aligned} \tag{4.33}$$

$$\begin{aligned}|W_{10}| = & |\int_\Omega \nabla^m\frac{B_2\cdot B}{\bar\rho+\varrho_1}\nabla^{m+2}udx|\\ = & |\int_\Omega\sum_{0\le l\le m}C_m^l\nabla^l(B_2\cdot B)\nabla^{m-l}(\frac{1}{\bar\rho+\varrho_1})\nabla^{m+2}udx|\\ \le & C\|\nabla^{m+2}u\|_{L^2}(\|\nabla B_2\|_{L^\infty}\|B\|_{L^\infty}\|\varrho_1\|_{H^m}+\|\nabla B_2\|_{L^\infty}\|B\|_{H^m}\|\varrho_1\|_{L^\infty}\\ & +\|\nabla B_2\|_{H^m}\|B\|_{L^\infty}\|\varrho_1\|_{L^\infty})\le C\theta(\|\nabla^m B\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2),\end{aligned} \tag{4.34}$$

$$\begin{aligned}|W_{11}| = & |\int_\Omega \nabla^m(B_2\cdot B_2(\frac{1}{\bar\rho+\varrho_1}-\frac{1}{\bar\rho+\varrho_2}))\nabla^{m+2}udx|\\ \le & C\|\nabla^{m+2}u\|_{L^2}(\|\nabla B_2\|_{L^\infty}\|B_2\|_{L^\infty}\|\nabla^m\varrho\|_{L^2}+\|\nabla B_2\|_{L^\infty}\|\nabla^m B_2\|_{L^\infty}\|\varrho\|_{L^\infty}\\ & +\|\nabla^{m+1}B_2\|_{L^2}\|B_2\|_{L^\infty}\|\varrho\|_{L^\infty})\le C\theta^2(\|\nabla^m\varrho\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2),\end{aligned} \tag{4.35}$$

$$|W_{12}| = |\int_\Omega\gamma\bar\rho\nabla^{m+1}\varrho\nabla^{m+2}udx|\le C(\|\nabla^{m+1}\varrho\|_{L^2}^2+\|\nabla^{m+2}u\|_{L^2}^2), \tag{4.36}$$

$$|W_{13}| = |\int_\Omega\nabla^m(B_1\cdot\nabla u)\nabla^{m+2}Bdx|$$

$$
\begin{aligned}
&=|\int_\Omega \sum_{0\le l\le m} C_m^l \nabla^l B_1 \nabla^{m-l}\nabla u \nabla^{m+2} B dx|\\
&\le C\|\nabla^{m+2}B\|_{L^2}(\|\nabla^{m+1}u\|_{L^2}\|B_1\|_{L^\infty}+\|\nabla u\|_{L^\infty}\|\nabla^m B_1\|_{L^2})\\
&\le C\theta(\|\nabla^{m+2}B\|_{L^2}^2+\|\nabla^{m+1}u\|_{L^2}^2),
\end{aligned}\tag{4.37}
$$

$$
\begin{aligned}
|W_{14}|&=|\int_\Omega \nabla^m(B\cdot\nabla u_2)\nabla^{m+2}Bdx|\\
&=|\int_\Omega \sum_{0\le l\le m} C_m^l \nabla^l B \nabla^{m-l}\nabla u_2 \nabla^{m+2} B dx|\\
&\le C\|\nabla^{m+2}B\|_{L^2}(\|\nabla^{m+1}u_2\|_{L^2}\|B\|_{L^\infty}+\|\nabla u_2\|_{L^\infty}\|\nabla^m B\|_{L^2})\\
&\le C\theta\|\nabla^{m+2}B\|_{L^2}^2,
\end{aligned}\tag{4.38}
$$

$$
\begin{aligned}
|W_{15}|&=|\int_\Omega \nabla^m(u_1\cdot\nabla B)\nabla^{m+2}Bdx|\\
&=|\int_\Omega \sum_{0\le l\le m} C_m^l \nabla^l u_1 \nabla^{m-l}\nabla B \nabla^{m+2} B dx|\\
&\le C\|\nabla^{m+2}B\|_{L^2}(\|\nabla^{m+1}B\|_{L^2}\|u_1\|_{L^\infty}+\|\nabla B\|_{L^\infty}\|\nabla^m u_1\|_{L^2})\\
&\le C\theta(\|\nabla^{m+2}B\|_{L^2}^2+\|\nabla^{m+1}B\|_{L^2}^2),
\end{aligned}\tag{4.39}
$$

$$
\begin{aligned}
|W_{16}|&=|\int_\Omega \nabla^m(u\cdot\nabla B_2)\nabla^{m+2}Bdx|\\
&=|\int_\Omega \sum_{0\le l\le m} C_m^l \nabla^l u \nabla^{m-l}\nabla B_2 \nabla^{m+2} B dx|\\
&\le C\|\nabla^{m+2}B\|_{L^2}(\|\nabla^{m+1}B_2\|_{L^2}\|u\|_{L^\infty}+\|\nabla B_2\|_{L^\infty}\|\nabla^m u\|_{L^2})\\
&\le C\theta(\|\nabla^{m+2}B\|_{L^2}^2+\|\nabla^{m+1}u\|_{L^2}^2).
\end{aligned}\tag{4.40}
$$

Basing on these estimates, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega(\gamma|\nabla^{m+1}\varrho|^2+|\nabla^{m+1}u|^2+|\nabla^{m+1}B|^2)dx+2\int_\Omega(\frac{1}{\overline{\rho}}|\nabla^{m+2}u|^2+|\nabla^{m+2}B|^2)dx\\
&\le C(\theta+\theta^2)(\|\nabla^{m+1}\varrho\|_{L^2}^2+\|\nabla^{m+2}(u,B)\|_{L^2}^2).
\end{aligned}\tag{4.41}
$$

By applying ∇^m to (4.12) and multiplying the equation by $\nabla^m\nabla\varrho$, we have

$$
\begin{aligned}
&\nabla^m u_t-\frac{1}{\overline{\rho}}\nabla^m\Delta u\\
=&\nabla^m((\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2})\Delta u_1)+\nabla^m((\frac{1}{\overline{\rho}+\varrho_2}-\frac{1}{\overline{\rho}})\Delta u)\\
&-\nabla^m((u\cdot\nabla)u_1)-\nabla^m((u_2\cdot\nabla)u)-\gamma\overline{\rho}\nabla^m\nabla\varrho\\
&+\nabla^m((\frac{P'(\overline{\rho})}{\overline{\rho}}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho)+\nabla^m((\frac{P'(\overline{\rho}+\varrho_2)}{\overline{\rho}+\varrho_2}-\frac{P'(\overline{\rho}+\varrho_1)}{\overline{\rho}+\varrho_1})\nabla\varrho_2)\\
&+\nabla^m\frac{B\cdot\nabla B_1}{\overline{\rho}+\varrho_1}+\nabla^m\frac{B_2\cdot\nabla B}{\overline{\rho}+\varrho_1}+\nabla^m(B_2\cdot\nabla B_2(\frac{1}{\overline{\rho}+\varrho_1}-\frac{1}{\overline{\rho}+\varrho_2})).
\end{aligned}\tag{4.42}
$$

Then, we see

$$
\begin{aligned}
&(\nabla^m u_t, \nabla^m \nabla \varrho) - (\frac{1}{\bar{\rho}} \nabla^m \Delta u, \nabla^m \nabla \varrho) \qquad (4.43)\\
=&(\nabla^m ((\frac{1}{\bar{\rho}+\varrho_1} - \frac{1}{\bar{\rho}+\varrho_2})\Delta u_1), \nabla^m \nabla \varrho)\\
&+(\nabla^m ((\frac{1}{\bar{\rho}+\varrho_2} - \frac{1}{\bar{\rho}})\Delta u), \nabla^m \nabla \varrho) - (\nabla^m ((u\cdot\nabla) u_1), \nabla^m \nabla \varrho) - (\nabla^m ((u_2\cdot\nabla) u), \nabla^m \nabla \varrho)\\
&-(\gamma\bar{\rho}\nabla^m \nabla \varrho, \nabla^m \nabla \varrho) + (\nabla^m ((\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\bar{\rho}+\varrho_1)}{\bar{\rho}+\varrho_1})\nabla \varrho), \nabla^m \nabla \varrho)\\
&+(\nabla^m ((\frac{P'(\bar{\rho}+\varrho_2)}{\bar{\rho}+\varrho_2} - \frac{P'(\bar{\rho}+\varrho_1)}{\bar{\rho}+\varrho_1})\nabla \varrho_2), \nabla^m \nabla \varrho) + (\nabla^m \frac{B\cdot\nabla B_1}{\bar{\rho}+\varrho_1}, \nabla^m \nabla \varrho)\\
&+(\nabla^m \frac{B_2\cdot\nabla B}{\bar{\rho}+\varrho_1}, \nabla^m \nabla \varrho) + (\nabla^m (B_2\cdot\nabla B_2(\frac{1}{\bar{\rho}+\varrho_1} - \frac{1}{\bar{\rho}+\varrho_2})), \nabla^m \nabla \varrho). \qquad (4.44)
\end{aligned}
$$

We note that

$$
\begin{aligned}
\gamma\bar{\rho}\int_\Omega |\nabla^{m+1}\varrho|^2 dx \le& -\int_\Omega \nabla^m u_t \cdot \nabla^m \nabla \varrho dx + \|\nabla^{m+2} u\|_{L^2} \|\nabla^{m+1}\varrho\|_{L^2}\\
&+C(\theta+\theta^2)(\|\nabla^{m+1}\varrho\|_{L^2}^2 + \|\nabla^{m+2}(u,B)\|_{L^2}^2), \qquad (4.45)
\end{aligned}
$$

where

$$
\begin{aligned}
-\int_\Omega \nabla^m u_t \cdot \nabla^m \nabla \varrho dx =& -\frac{d}{dt}\int_\Omega \nabla^m u \cdot \nabla^m \nabla \varrho dx - \int_\Omega \nabla^m divu \cdot \nabla^m \varrho_t dx\\
=& -\frac{d}{dt}\int_\Omega \nabla^m u \cdot \nabla^m \nabla \varrho dx + \int_\Omega \nabla^m (u_1\cdot\nabla\varrho + u\cdot\nabla\varrho_2)\nabla^m divu dx\\
\le& -\frac{d}{dt}\int_\Omega \nabla^m u \cdot \nabla^m \nabla \varrho dx + C\theta(\|\nabla^{m+1}\varrho\|_{L^2}^2 + \|\nabla^{m+2} u\|_{L^2}^2).
\end{aligned}
\qquad (4.46)
$$

Therefore, we have

$$
\begin{aligned}
&\frac{\gamma\bar{\rho}}{2}\int_\Omega |\nabla^{m+1}\varrho|^2 dx + \frac{d}{dt}\int_\Omega \nabla^m u \cdot \nabla^m \nabla \varrho dx\\
\le& C(\theta+\theta^2)(\|\nabla^{m+1}\varrho\|_{L^2}^2 + \|\nabla^{m+2}(u,B)\|_{L^2}^2) + C\|\nabla^{m+2} u\|_{L^2}^2, \qquad (4.47)
\end{aligned}
$$

then, we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (\gamma|\nabla^{m+1}\varrho|^2 + |\nabla^{m+1}u|^2 + |\nabla^{m+1}B|^2 + 2C\nabla^m u\cdot\nabla^m\nabla\varrho) dx\\
&+\int_\Omega (|\nabla^{m+1}\varrho|^2 + |\nabla^{m+2}u|^2 + |\nabla^{m+2}B|^2) dx \le 0. \qquad (4.48)
\end{aligned}
$$

By integrating (4.48) from 0 to T, we obtain

$$
\int_0^T (\|\nabla^{m+1}\varrho\|_{L^2}^2 + \|\nabla^{m+2}u\|_{L^2}^2 + \|\nabla^{m+2}B\|_{L^2}^2) dt \le 0, \qquad (4.49)
$$

which implies that

$$
\varrho = u = B = 0.
$$

Therefore, we obtain the uniqueness of periodic-solution to (1.2), and as well complete the proof of Theorem 1.1. ∎

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