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WAVE BREAKING FOR A MODEL EQUATION FOR SHALLOW WATER WAVES OF MODERATE AMPLITUDE*

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Abstract. This paper is devoted to studying wave breaking for a model equation for shallow water waves of moderate amplitude (also called the Constanin-Lannes equation), which was proposed by Constantin and Lannes. We first present a blow-up criterion and the precise blow-up scenario of strong solutions to the equation. Next, we show a sufficient condition on the initial data to guarantee wave breaking. Moreover, the estimate of life span is given. The key of the method is to refine the analysis on characteristics and conserved quantities to the Riccati-type differential inequality.

Keywords. Wave breaking; Shallow water waves of moderate amplitude; Blow-up.

AMS subject classifications. 35G25; 35Q53.

1. Introduction

In this paper, we consider a model equation for shallow water waves of moderate amplitude

$$\begin{cases} u_t - u_{txx} + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} + 14uu_{xxx} + 28u_xu_{xx} = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where the function $u(t,x)$ stands for the free surface elevation, was proposed by Constantin and Lannes in describing the surface water waves of moderate amplitude in the shallow water regime [19]. The model Equation (1.1) has the following two conservation laws

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx$$

and

$$F(u) = \int_{\mathbb{R}} \left(\frac{1}{2}u^2 + u^3 - \frac{1}{2}u^4 + \frac{3}{5}u^5 - \frac{1}{2}u_x^2 - 7uu_x^2 \right) dx.$$

The study of water waves is a fascinating subject because the phenomena are familiar and mathematical problems are various [41]. Since the exact governing equations for water waves have proven to be nearly intractable (Gerstner waves being the only known explicit solutions to the full equations [11, 25, 30, 33]), the quest for suitable simplified model equations was initiated at the earliest stages of the development of hydrodynamics. Until the early twentieth century, the study of water waves was confined almost exclusively to linear theory. Since linearization failed to explain some important aspects, several nonlinear models have been proposed in order to understand some important aspects of water waves, like wave breaking or solitary waves. One of the typical models is the Camassa-Holm (CH) equation:

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{1.2}$$

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The CH equation was originally implied in Fokas and Fuchssteiner in [23], but became well-known since 1993, when Camassa and Holm [6] derived it as a model for unidirectional propagation of shallow water flow over a flat bottom with a famous feature of peaked solitons (peakons). The CH equation has been studied extensively in the last two decades because of its many remarkable properties: infinity of conservation laws and complete integrability [6, 23, 24], with action angle variables constructed using inverse scattering [13, 16, 20, 40], peakons [6], which describes an essential feature of the travelling waves of largest amplitude [12, 17, 18], geometric formulations [31, 34], well-posedness [21, 22, 32], orbital stability [14, 15], global conservative solutions [1] and dissipative solutions [2]. It is shown in [7–10, 35] that the blow-up occurs in the form of breaking waves, namely, for certain initial data the solution remains bounded but its slope becomes unbounded in finite time.

Similarly to the CH equation, the model Equation (1.1) can also capture the phenomenon of wave breaking [19]. The local well-posedness for the Cauchy problem (1.1) was first established by Constantin and Lannes [19], and then improved using Katos semigroup approach for quasi-linear equations and an approach due to Kato by Duruk Mutlubas [37, 38]. The well-posedness in Besov spaces and persistence properties have been studied in [39]. The existence of weak solutions in lower order Sobolev spaces $H^s(\mathbb{R})$ with $1<s\leq 3/2$ was obtained in [42]. Zhou [43] established the existence of a semigroup of global solutions with nonincreasing $H^1(\mathbb{R})$ energy. Continuity and asymptotic behaviors for the model Equation (1.1) has been recently studied in [44]. The existence and symmetry of solitary waves were shown in [26–28]. The orbital stability of solitary waves has been proved in [36] using an approach proposed by Grillakis, Shatah and Strauss [29].

Recently, Brandolese and Cortez [3–5] introduced a local-in-space criteria for blow-up in the study of the CH-type equations which highlights how local structure of the solution affects the blow-ups. For the model Equation (1.1), the convolution contains cubic, even quartic nonlinearities which do not have a lower bound in terms of the local terms. For this reason, the main difficulty to obtain a sufficient condition on the initial data to guarantee wave breaking is that we here deal with higher order nonlinearities, and analysis on characteristics and conserved quantities to the Riccati-type differential inequality.

The rest of this paper is organized as follows. In Section 2, we present a blow-up criterion and the precise blow-up scenario of strong solutions to (1.1). In Section 3, we show a sufficient condition on the initial data to guarantee wave breaking.

2. Blow-up criterion and scenario

In this section, we present a blow-up criterion and the precise blow-up scenario of strong solutions to (1.1). First, we recall the local well-posedness.

THEOREM 2.1 ([37]). *Let $u_0\in H^s(\mathbb{R}), s>\frac{3}{2}$ be given. Then there exists $T>0$, depending on u_0, such that there is a unique solution u to the Cauchy problem* (1.1) *satisfying*

$$u\in C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);L^2(\mathbb{R})).$$

Moreover, the map $u_0\in H^s(\mathbb{R})\to u$ is continuous from $H^s(\mathbb{R})$ to $C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);L^2(\mathbb{R}))$.

Next, motivated by the method in Ref. [21], we can obtain the following blow-up criterion. The proof is similar to that of the method in Ref. [21], so we omit it.

THEOREM 2.2. *Let $u_0\in H^s(\mathbb{R})$ with $s>\frac{3}{2}$, and $T>0$ be the maximal existence time* *of Cauchy problem* (1.1). *If* $T<\infty$, *then*

$$\int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau = \infty.$$

Finally, using the the classical energy method, we can obtain the following precise blow-up scenario. It is shown that the solution to the model Equation (1.1) can only have singularities which correspond to wave breaking.

THEOREM 2.3. *Let* $u_0 \in H^s(\mathbb{R})$ *with* $s>\frac{3}{2}$, *and* $T>0$ *be the maximal existence time of Cauchy problem* (1.1). *Then the solution* u *blows up in finite time if and only if*

$$\lim_{t\uparrow T^-}\left\{\sup_{x\in\mathbb{R}} u_x(t,x)\right\}=\infty.$$

Proof. By a density argument, we just need to consider the case of $s\geq 3$, here assume $s=3$.

Differentiating (1.1) with respect to x, and multiplying the result by u_x, then integrating over $\mathbb{R}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|_{H^1}^2 &= -3\int_{\mathbb{R}} u_x^3 dx + 21\int_{\mathbb{R}} u_x u_{xx}^2 dx - 6\int_{\mathbb{R}} u^2 u_x u_{xx} dx + 12\int_{\mathbb{R}} u^3 u_x u_{xx} dx \\ &\leq 21\int_{\mathbb{R}} u_x(u_x^2+u_{xx}^2)dx + 3\int_{\mathbb{R}} u^2(u_x^2+u_{xx}^2)dx + 6\int_{\mathbb{R}} u^3(u_x^2+u_{xx}^2)dx. \end{aligned} \tag{2.1}$$

Let us assume that there exists $M>0$ such that

$$u_x(t,x)\leq M$$

for all $(t,x)\in[0,T)\times\mathbb{R}$. Note that $\|u\|_{H^1}=\|u_0\|_{H^1}$, then it follows from (2.1) that

$$\begin{aligned}\frac{d}{dt}\|u\|_{H^2}^2 &\leq 42\int_{\mathbb{R}} u_x(u_x^2+u_{xx}^2)dx + 6\int_{\mathbb{R}} u^2(u_x^2+u_{xx}^2)dx + 12\int_{\mathbb{R}} u^3(u_x^2+u_{xx}^2)dx \\ &\leq (42M + 6\|u\|_{L^\infty}^2 + 12\|u\|_{L^\infty}^3)\int_{\mathbb{R}} u_x^2+u_{xx}^2 dx \\ &\leq C(M+\|u\|_{H^1}^2+\|u\|_{H^1}^3)\|u\|_{H^2}^2 \\ &\leq C(M+\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^3)\|u\|_{H^2}^2. \end{aligned} \tag{2.2}$$

Taking advantage of Gronwall's inequality yields

$$\|u(t)\|_{H^2}^2 \leq \|u(0)\|_{H^2}^2 e^{C(M+\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^3)t}. \tag{2.3}$$

Differentiating (1.1) with respect to x, and multiplying the result by u_{xxx}, then integrating over $\mathbb{R}$, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(u_{xx}^2+u_{xxx}^2)dx \\ =& -15\int_{\mathbb{R}} u_x u_{xx}^2 dx + 2\int_{\mathbb{R}}(-6u^2u_x+12u^3u_x)_x u_{xxx}dx + 70\int_{\mathbb{R}} u_x u_{xxx}^2 dx \\ \leq& C(M+M^2+\|u\|_{L^\infty}^2+\|u\|_{L^\infty}^4)\int_{\mathbb{R}} u_{xx}^2+u_{xxx}^2 dx + CM\int_{\mathbb{R}} u_x^4 dx \\ \leq& C(M+M^2+\|u\|_{H^1}^2+\|u\|_{H^1}^4)\int_{\mathbb{R}} u_{xx}^2+u_{xxx}^2 dx + CM\|u_x\|_{L^2}^3\|u_{xx}\|_{L^2}\end{aligned}$$

$$\leq C(M+M^2+\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^3+\|u_0\|_{H^1}^4)\int_{\mathbb{R}} u_{xx}^2+u_{xxx}^2 dx, \tag{2.4}$$

where we used the Sobolev's embedding and the the interpolation inequality $\|f\|_{L^4}\leq C\|f\|_{L^2}^{\frac{3}{4}}\|f_x\|_{L^2}^{\frac{1}{4}}$. Taking advantage of Gronwall's inequality in (2.4) yields

$$\int_{\mathbb{R}}(u_{xx}^2+u_{xxx}^2)dx \leq \mathrm{e}^{C(M+M^2+\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^3+\|u_0\|_{H^1}^4)t}\int_{\mathbb{R}}(u_{0xx}^2+u_{0xxx}^2)dx,$$

which together with (2.3) yields

$$\|u(t)\|_{H^3}^2 \leq 3\|u(0)\|_{H^3}^2\mathrm{e}^{C(M+M^2+\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^3+\|u_0\|_{H^1}^4)t}.$$

Therefore, we have shown that the boundedness of $u_x(t,x)$ from up, for $(t,x)\in[0,T)\times\mathbb{R}$ ensure the boundedness of $\|u\|_{H^3}$ on finite time interval, which contradicts the assumption of the theorem.

On the other hand, by the Sobolev's embedding $H^s(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})(s>\frac{1}{2})$, we can see that if

$$\lim_{t\uparrow T^-}\left\{\sup_{x\in\mathbb{R}} u_x(t,x)\right\}=\infty,$$

then the solution blows up in finite time. This completes the proof of Theorem 2.3. ☐

3. Wave breaking

In this section, we show a sufficient condition on the initial data to guarantee wave breaking.

In order to derive condition of wave breaking, we consider the following ordinary differential equation:

$$\begin{cases} \dfrac{dq(t,x)}{dt}=-(1+14u)\big(t,q(t,x)\big), & (t,x)\in[0,T)\times\mathbb{R},\\ q(0,x)=x, & x\in\mathbb{R}. \end{cases} \tag{3.1}$$

By a direct calculation, we have

$$\frac{dq_x(t,x)}{dt}=-14u_x\big(t,q(t,x)\big)q_x(t,x).$$

Furthermore,

$$q_x(t,x)=\exp\left(-\int_0^t 14u_x\big(s,q(s,x)\big)ds\right)>0, \quad \text{for all } (t,x)\in[0,T)\times\mathbb{R},$$

which implies that the mapping $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$. Consequently, the L^∞-norm of any function $v(t,\cdot)\in L^\infty$ is preserved under the family of diffeomorphisms $q(t,\cdot)$, that is

$$\|v(t,\cdot)\|_{L^\infty}=\|v(t,q(t,\cdot))\|_{L^\infty}, \quad t\in[0,T).$$

Similarly,

$$\inf_{x\in\mathbb{R}} v(t,x)=\inf_{x\in\mathbb{R}} v(t,q(t,x)), \qquad \sup_{x\in\mathbb{R}} v(t,x)=\sup_{x\in\mathbb{R}} v(t,q(t,x))$$

We are now in a position to state our wave-breaking result.

THEOREM 3.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, and $T > 0$ be the maximal existence time of Cauchy problem (1.1). Assume there exists a point $x_0 \in \mathbb{R}$ such that*

$$u_{0,x}(x_0) > \sqrt{\frac{\alpha-1}{7}}\left|u_0(x_0)+1\right| + \sqrt{\frac{C_0}{7}},$$

where

$$C_0 = \frac{27}{2}\|u_0\|_{H^1}^2 + \frac{3\sqrt{2}}{2}\|u_0\|_{H^1}^3 + \frac{9}{4}\|u_0\|_{H^1}^4 > 0,\ \ \alpha = \frac{7}{2}\left(1-\sqrt{\frac{3}{7}}\right).$$

Then the solution $u(t,x)$ blows up at a time T with

$$T \le \frac{1}{7\sqrt{u_{0,x}^2(x_0) - \frac{\alpha-1}{7}(u_0(x_0)+1)^2 - \sqrt{7C_0}}}.$$

Proof. Using $p(x) \triangleq \frac{1}{2}\mathrm{e}^{-|x|}$, we can rewrite Equation (1.1) as the following form:

$$u_t - (1+14u)u_x + p_x * (2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2) = 0. \tag{3.2}$$

Taking the space derivative in (3.2), we get

$$u_{xt} - (1+14u)u_{xx} = 7u_x^2 + 2u + 10u^2 - 2u^3 + 3u^4 - p*(2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2). \tag{3.3}$$

By the definition of $q(t,x)$ in (3.1), we have

$$\frac{du\big(t,q(t,x)\big)}{dt} = -p_x * (2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2), \tag{3.4}$$

$$\frac{du_x\big(t,q(t,x)\big)}{dt} = 7u_x^2 + 2u + 10u^2 - 2u^3 + 3u^4 - p*(2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2). \tag{3.5}$$

Let us denote

$$A(t,x_0) = \left(\sqrt{\frac{\alpha-1}{7}}(u+1) - u_x\right)(t,q(t,x_0)),$$

$$B(t,x_0) = \left(\sqrt{\frac{\alpha-1}{7}}(u+1) + u_x\right)(t,q(t,x_0))$$

and define the two convolution operators p_+ and p_- as

$$p_+ * f(x) = \frac{\mathrm{e}^{-x}}{2}\int_{-\infty}^{x}\mathrm{e}^{y}f(y)dy, \quad p_- * f(x) = \frac{\mathrm{e}^{x}}{2}\int_{x}^{-\infty}\mathrm{e}^{-y}f(y)dy.$$

Then we have the relation

$$p = p_+ + p_-, \quad p_x = p_- - p_+.$$

Applying Lemma 3.1 in Ref. [4] with $M = 1, \gamma = 14$ and $K = \sqrt{\frac{1}{14}}$ we have the following convolution estimates

$$p_\pm * (7u_x^2 - u^2 - 2u) \ge -\frac{\alpha}{2}(u+1)^2 + \frac{1}{2}.$$

From (3.4) and (3.5), we have

$$\begin{aligned}
&\frac{dA(t,x_0)}{dt}\\
=&-(7u_x^2+2u+10u^2-2u^3+3u^4)-\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_-*(7u_x^2-u^2-2u)\\
&-\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_+*(7u_x^2-u^2-2u)+\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_-*(9u^2-2u^3+3u^4)\\
&+\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_+*(9u^2-2u^3+3u^4)\\
\leq&7A(t,x_0)B(t,x_0)-9u^2+2u^3-3u^4+\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_-*(9u^2-2u^3+3u^4)\\
&+\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_+*(9u^2-2u^3+3u^4) \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{dB(t,x_0)}{dt}\\
=&7u_x^2+2u+10u^2-2u^3+3u^4+\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_-*(7u_x^2-u^2-2u)\\
&+\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_+*(7u_x^2-u^2-2u)-\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_-*(9u^2-2u^3-3u^4)\\
&-\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_+*(9u^2-2u^3-3u^4)\\
\geq&-7A(t,x_0)B(t,x_0)+9u^2-2u^3+3u^4-\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_-*(9u^2-2u^3-3u^4)\\
&-\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_+*(9u^2-2u^3-3u^4). \qquad (3.7)
\end{aligned}$$

Taking advantage of Young's inequality, Sobolev's embedding inequality $\|u\|_{L^\infty}\leq \frac{\sqrt{2}}{2}\|u\|_{H^1}$ and conservation law $E=\int_{\mathbb{R}} u^2+u_x^2dx$, we obtain

$$\begin{aligned}
&\Big|9u^2-3u^3+3u^4-\Big(1+\sqrt{\frac{\alpha-1}{7}}\Big)p_\mp*(9u^2-2u^3-3u^4)\\
&\quad-\Big(1-\sqrt{\frac{\alpha-1}{7}}\Big)p_\pm*(9u^2-2u^3-3u^4)\Big|\\
\leq&3(9\|u\|_{L^\infty}^2+2\|u\|_{L^\infty}^3+3\|u\|_{L^\infty}^4)\\
\leq&\frac{27}{2}\|u\|_{H^1}^2+\frac{3\sqrt{2}}{2}\|u\|_{H^1}^3+\frac{9}{4}\|u\|_{H^1}^4\\
\leq&\frac{27}{2}\|u_0\|_{H^1}^2+\frac{3\sqrt{2}}{2}\|u_0\|_{H^1}^3+\frac{9}{4}\|u_0\|_{H^1}^4=C_0.
\end{aligned}$$

Therefore, we have

$$\frac{dA(t,x_0)}{dt}\leq 7AB+C_0,$$

and

$$\frac{dB(t,x_0)}{dt} \geq -7AB - C_0.$$

By our assumption on the initial data, it's obvious that

$$A(0,x_0) = \sqrt{\frac{\alpha-1}{7}}(u_0(x_0)+1) - u_{0,x}(x_0) < 0, \quad B(0,x_0) = \sqrt{\frac{\alpha-1}{7}}(u_0+1) + u_{0,x}(x_0) > 0.$$

Let us set

$$\tau = \sup\{t \in [0,T) : A(\cdot,x_0) < 0 \text{ and } B(\cdot,x_0) > 0 \text{ on } [0,t]\}$$

By continuity, $\tau > 0$. If $\tau < T$, then at least one of the inequalities $A(\tau,x_0) \geq 0$ and $B(\tau,x_0) \leq 0$ hold true. This contradicts the fact that on the interval $[0,\tau]$, we have $A(\tau,x_0)B(\tau,x_0) < 0$, hence $A(\tau,x_0) \leq A(0,x) < 0$ and $B(\tau,x_0) \geq B(0,x_0) > 0$. Thus $\tau = T$. This ensures that

$$\frac{dA(t,x_0)}{dt} < 0, \quad \frac{dB(t,x_0)}{dt} > 0,$$

thus,

$$A(t,x_0) < A(0,x_0) < 0, \quad B(t,x_0) > B(0,x_0) > 0.$$

Set

$$h(t) = \sqrt{-A(t,x_0)B(t,x_0)},$$

a direct computation of the derivative of $h(t)$ leads to

$$\begin{aligned} \frac{dh(t)}{dt} &= -\frac{A_tB + AB_t}{2h(t)}(t,x_0) \\ &\geq \frac{-(B-A)(7AB+C_0)}{2h(t)}(t,x_0). \end{aligned}$$

Using the inequality $\frac{B-A}{2h} = \frac{B+(-A)}{2\sqrt{-AB}} \geq 1$ and the fact that $-7AB - C_0 = 7h^2 - C_0 = \left(\sqrt{7}h - \sqrt{C_0}\right)\left(\sqrt{7}h + \sqrt{C_0}\right) \geq \left(\sqrt{7}h - \sqrt{C_0}\right)^2$, we have

$$\frac{dh}{dt} \geq \left(\sqrt{7}h - \sqrt{C_0}\right)^2.$$

Hence, the solution blows up in finite time T with

$$T \leq \frac{1}{7\sqrt{u_{0,x}^2(x_0) - \frac{\alpha-1}{7}(u_0(x_0)+1)^2} - \sqrt{7C_0}}.$$

This completes the proof of Theorem 3.1. ∎

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