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GLOBAL DISCONTINUOUS SOLUTION FOR $1D$ ISENTROPIC GAS DYNAMICS SYSTEM*

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Abstract. In this work, we study the global existence of one-dimensional isentropic gas dynamics system. We prove a global-in-time existence result of this system in the framework of discontinuous non-decreasing solutions considering bounded initial data. We remark that this result allows us to give a global meaning to the gas dynamics system in distributional sense, considering discontinuous solutions with vacuum.

Keywords. Isentropic gas dynamics system; Euler equations; non-decreasing solutions; discontinuous solutions; viscosity solutions.

AMS subject classifications. 35A01; 74G25; 35F20; 35Q31; 35L40; 35D40.

1. Introduction

1.1. Main results. In this paper, we present, relying on the existence result done in [6] and considering some monotonicity assumptions on the initial data, the global existence of a discontinuous solution of the following $1D$ isentropic gas dynamics system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + p(\rho)\right) = 0 \quad \text{with} \quad p(\rho) = \frac{(\gamma-1)^2}{4\gamma}\rho^\gamma & \text{in} \quad (0,T)\times\mathbb{R}, \\ u(0,x) = u_0(x) \quad \text{and} \quad \rho(0,x) = \rho_0(x) \geqslant 0 & \text{for} \quad x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where ρ is the density, u is the speed and $p(\rho)$ is the pressure given by a simple power law for an exponent $\gamma > 1$. First, we assume the following conditions

$$u_0, \rho_0 \in L^\infty(\mathbb{R}) \quad \text{and} \quad u_0 \pm \rho_0^\theta \quad \text{are non-decreasing functions with } \theta = \frac{\gamma-1}{2}. \tag{1.2}$$

Note that, the previous assumptions mean that the Riemann invariants $u \pm \rho^\theta$, corresponding to the isentropic gas dynamics equations, are initially supposed to be bounded and non-decreasing. This shows that $u_0 \pm \rho_0^\theta$ belong to $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ as well as u_0 and ρ_0^θ. Therefore, $\rho_0 \in L^\infty(\mathbb{R})$ but not necessarily in $BV(\mathbb{R})$ since it can be zero. Referring to these assumptions, we can also see that u_0 is non-decreasing. However, there is no reason for ρ_0 to be non-decreasing here. For example, we can choose the initial data as in Figure 1.1.
We will prove the following result.

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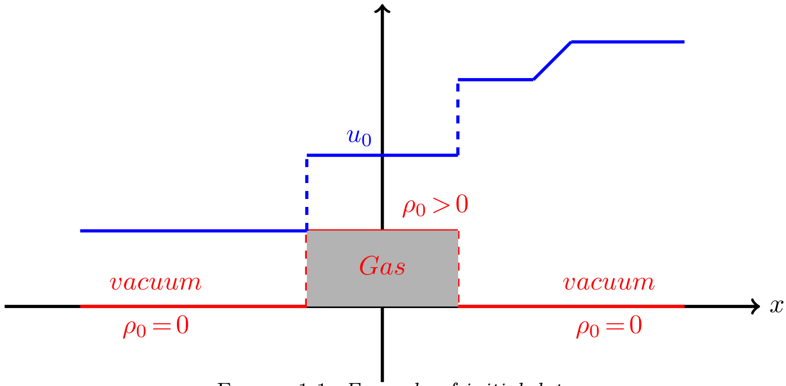

FIGURE 1.1. *Example of initial data.*

THEOREM 1.1. *Assume (1.2) is verified, with $\rho_0 \geqslant 0$ and $T > 0$. Then system (1.1) has a solution $(u,\rho) \in (L^\infty((0,T)\times\mathbb{R}))^2$ in distributional sense, namely*

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{in } \mathcal{D}'((0,T)\times\mathbb{R}), \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + \frac{(\gamma-1)^2}{4\gamma}\rho^\gamma\right) = 0 & \text{in } \mathcal{D}'((0,T)\times\mathbb{R}), \\ u(0,\cdot) = u_0(\cdot) \quad \text{and} \quad \rho(0,\cdot) = \rho_0(\cdot) & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases}$$

Moreover, we have $\rho \geqslant 0$,

$$u, \rho^\theta \in L^\infty((0,T)\times\mathbb{R}) \cap L^\infty((0,T); BV(\mathbb{R})) \cap C([0,T); L^1_{loc}(\mathbb{R})), \tag{1.3}$$

and the functions $u(t,\cdot) \pm \rho^\theta(t,\cdot)$ are non-decreasing for all $0 \le t < T$.

REMARK 1.1 (Vacuum case). System (1.1) is automatically satisfied whenever $\rho = 0$ on a subset $\omega \subset (0,T)\times\mathbb{R}$, and the function u can be chosen locally arbitrarily in ω. So, in vacuum state ($\rho = 0$), we do not have uniqueness of the solution.

The proof of this theorem arises directly from the global existence result of Lipschitz continuous viscosity solution proved in [6] and recalled below in Theorem 2.1. After regularizing the initial data, we prove, thanks to this result, that the diagonal system corresponding to system (1.2) has a unique Lipschitz continuous solution. This is achieved using the theory of viscosity solutions, initially introduced by Crandall and Lions in [2] to solve the first-order Hamilton-Jacobi equations and then extended by Ishii and Koike in [10, 11] to the case of systems. Then, we show, thanks to the maximum principle, that the Riemann invariants are increasing and uniformly bounded in L^∞. This allows us to obtain uniform *a priori* estimates on these quantities in $L^\infty \cap BV$. Finally, we go to the limit when the regularization vanishes, using some compactness arguments and relying on $L^\infty \cap BV$ bounds, to get our result.

Let us mention that, in the absence of vacuum, i.e. when the system is strictly hyperbolic everywhere, one can use the theory of symmetric hyperbolic systems developed by Friedrichs-Lax-Kato [8, 12, 13] to construct smooth solutions; for instance, see Majda [20]. Always in the case where $\rho_0 > 0$, Lax showed in [14] a global existence and uniqueness result of Lipschitz-continuous solution under some monotonicity assumptions on the initial data similar to those considered in (1.2). This result was then extended by T. T. Li, proved in [15, pp. 35-41] for C^1 solutions. In this paper we were able to establish a global existence result with vacuum, considering less regularity (namely L^∞ solution) which makes it possible to produce discontinuous solutions. In connection to Theorem 1.1, let us also mention the work of Lions et al. [19] where the existence of an entropy solution was obtained for $\rho_0 \geq 0$ with $u_0, \rho_0 \in L^\infty(\mathbb{R})$ and $\gamma > 1$, using a kinetic formulation developed in [17,18]. This extended previous results of DiPerna [3–5], showing the global existence of L^∞ entropy solution, in some special cases of behavior law concerning the pressure. The proof of DiPerna is inspired by the idea of Tartar [24] and relies on a compensated compactness argument based on the representation of the weak limit in terms of Young measures which must be reduced to a Dirac mass due to the presence of a large family of entropies. Contrary to these theories, Theorem 1.1 gives here the global existence of L^∞ solution without appealing to the entropy notion nor to the kinetic formulation.

Recently, El Hajj and Monneau [6] proved a global existence result considering continuous solutions with vacuum, relying on a new gradient entropy estimate previously introduced in [7]. We also refer the reader to Serre [21, 22] and Grassin [9], for some global existence results of classical solutions under some special conditions on the initial data, by extracting a dispersive effect after some invariant transformation.

REMARK 1.2.

(1) Due to the lack of uniqueness with vacuum, there is no reason that our solution as well as that obtained by Lions et al. in [19] satisfies the adequate physical properties. Nevertheless and even if this is not the objective of our work, it is important to point out that it is quite possible to prove, when $\rho_0 > 0$ (i.e. without vacuum) and considering small initial data in L^∞ norm, the uniqueness in a particular class of physical solutions of semi-group type, as was done in [1]. To do this, it suffices to combine our result with the theory of Bianchini-Bressan developed within the framework of vanishing viscosity solutions, taking small initial data in BV norm.

(2) Let us also point out that, the result presented in Theorem 1.1 makes it possible to give a global meaning in time to the solution of system (1.1) considering discontinuous initial data, included particularly in the Riemann problem with vacuum.

(3) A similar result to Theorem 1.1 can also be obtained in the case of the (3×3) gas dynamics system with ideal gas law, following the same procedures developed here.

1.2. Organization of the paper.

The paper is organized as follows: in Section 2 we recall some useful results. In particular, we recall two results: the first concerns the global existence of Lipschitz continuous viscosity solution and the second is the Simon compactness lemma. Then, we prove the desired result in Section 3.

2. Some useful results

First of all, we remark that system (1.1) is a diagonalizable hyperbolic system. Indeed, in the case where $\rho > 0$ and (ρ, u) is a smooth solution, we can check easily that the following two variables

$$r^1 = u + \rho^\theta \quad \text{and} \quad r^2 = u - \rho^\theta \quad \text{with} \quad \theta = \tfrac{(\gamma-1)}{2},$$

satisfy the following diagonal system

$$\begin{cases} \partial_t r^1 + \tilde{\lambda}^1(r^1, r^2)\partial_x r^1 = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ \partial_t r^2 + \tilde{\lambda}^2(r^1, r^2)\partial_x r^2 = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ r^1(0,x) = r_0^1(x),\ r^1(0,x) = r_0^2(x) & \text{for} \quad x\in\mathbb{R}, \end{cases} \tag{2.1}$$

where $\tilde{\lambda}^1$ and $\tilde{\lambda}^2$ are defined as follows

$$\begin{cases} \tilde{\lambda}^1(r^1,r^2) = \dfrac{r^1+r^2}{2} + \dfrac{\gamma-1}{4}(r^1-r^2) = u + \theta\rho^\theta, \\ \tilde{\lambda}^2(r^1,r^2) = \dfrac{r^1+r^2}{2} - \dfrac{\gamma-1}{4}(r^1-r^2) = u - \theta\rho^\theta. \end{cases}$$

To prove Theorem 1.1, we need to recall the existence result proved by El Hajj et al. in [6, Theorem 1.3] for the following parabolic regularization of system (2.1), defined for $0<\eta\le 1$, by

$$\begin{cases} \partial_t r_\eta^1 + \tilde{\lambda}^1(r_\eta^1, r_\eta^2)\partial_x r_\eta^1 - \eta\partial_{xx}^2 r_\eta^1 = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ \partial_t r_\eta^2 + \tilde{\lambda}^2(r_\eta^1, r_\eta^2)\partial_x r_\eta^2 - \eta\partial_{xx}^2 r_\eta^2 = 0 & \text{in} \quad (0,T)\times\mathbb{R}, \\ r_\eta^1(0,x) = r_{0,\eta}^1(x),\ r_\eta^1(0,x) = r_{0,\eta}^2(x) & \text{for} \quad x\in\mathbb{R}. \end{cases} \tag{2.2}$$

where $r_{0,\eta}^1$ and $r_{0,\eta}^2$ are the regularizations of the functions r_0^1 and r_0^2 respectively, given by convolution as follows

$$r_{0,\eta}^1 = \rho_\eta^1 \star r_0^1, \quad r_{0,\eta}^2 = \rho_\eta^1 \star r_0^2,$$

with ρ_ε^1 is the standard mollifier defined as

$$\rho_\eta^1(\cdot) = \frac{1}{\eta}\rho^1\left(\frac{\cdot}{\eta}\right), \quad \text{such that} \quad \left| \begin{array}{l} \rho^1 \in C_c^\infty(\mathbb{R}), \operatorname{supp}\{\rho^1\} \subseteq B(0,1), \\ \rho^1 \geqslant 0, \text{and} \displaystyle\int_{\mathbb{R}} \rho^1 = 1. \end{array} \right. \tag{2.3}$$

THEOREM 2.1 (**Lipschitz continuous solution for diagonal system**). *Let $T>0$ and ρ_η^1 be the mollifier defined in (2.3). Assume that $(r_0^1, r_0^2) \in (W^{1,\infty}(\mathbb{R}))^2$ and the functions r_0^i are non-decreasing for $i=1,2$. Then we have*

(i) **Existence, uniqueness and bounds.** *System (2.2) has a unique solution $(r_\eta^1, r_\eta^2) \in \left(W^{2,\infty}([0,T)\times\mathbb{R})\right)^2 \cap (C^\infty([0,T)\times\mathbb{R}))^2$, satisfying the following L^∞ estimates*

$$\|r_\eta^i(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|r_0^i\|_{L^\infty(\mathbb{R})} \quad \textit{for} \quad i=1,2, \quad \textit{and} \quad 0<t<T, \tag{2.4}$$

$$\max_{i=1,2} \|\partial_x r_\eta^i(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \max_{i=1,2} \|\partial_x r_0^i\|_{L^\infty(\mathbb{R})} \quad \textit{for} \quad i=1,2, \quad \textit{and} \quad 0<t<T. \tag{2.5}$$

Moreover, the functions $r_\eta^i(t,\cdot)$ are non-decreasing for $i=1,2$ and $0\le t<T$.

(ii) **Convergence.** *For an extracted sub-sequence, (r^1_η, r^2_η) converges locally uniformly, as η goes to zero, to a function $(r^1, r^2) \in \left(W^{1,\infty}([0,T)\times\mathbb{R})\right)^2$, where (r^1, r^2) is a continuous viscosity solution of (2.1). Moreover, for all $0 \le t < T$, the functions $r^i(t,\cdot)$ are non-decreasing for $i = 1,2$ and satisfy estimates (2.4) and (2.5).*

We also need to recall the following compactness lemma.

LEMMA 2.1 (**Simon's Lemma [23, Corollary 4]**). *Let X, B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a sequence uniformly bounded in $L^\infty((0,T);X)$ and $(\partial_t\theta_n)_n$ is uniformly bounded in $L^r((0,T);Y)$ where $r > 1$, then, $(\theta_n)_n$ is relatively compact in $C((0,T);B)$.*

The next section is reserved for the proof of Theorem 1.1.

3. Proof of Theorem 1.1

We present the proof in four steps.

Step 1. (Lipschitz continuous solution for diagonal system). For $0 < \varepsilon \le 1$, we consider system (2.1) with the following initial data

$$r^1_{0,\varepsilon} = \rho^1_\varepsilon \star (u_0 + \rho_0^\theta) + \varepsilon, \quad r^2_{0,\varepsilon} = \rho^1_\varepsilon \star (u_0 - \rho_0^\theta) - \varepsilon,$$

where $\theta = \frac{\gamma-1}{2}$ and ρ^1_ε is the mollifier defined in (2.3). Under assumption (1.2), we can check that these initial data satisfy the conditions of Theorem 2.1. This implies that there exists a continuous viscosity solution $(r^1_\varepsilon, r^2_\varepsilon) \in \left(W^{1,\infty}([0,T)\times\mathbb{R})\right)^2$ of (2.1), satisfying following L^∞ estimate

$$\|r^i_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|r^i_{0,\varepsilon}\|_{L^\infty(\mathbb{R})} \quad \text{for} \quad i = 1,2, \quad \text{and} \quad 0 < t < T.$$

Since $\gamma > 1$ and $\rho_0 \ge 0$, we deduce that

$$\|r^i_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})} + (\|\rho_0\|_{L^\infty(\mathbb{R})})^\theta + 1 \quad \text{for} \quad i = 1,2, \quad \text{and} \quad 0 < t < T. \tag{3.1}$$

Moreover, we know that the functions $r^i_\varepsilon(t,\cdot)$ are non-decreasing for $i = 1,2$ and $t > 0$. Therefore, according to (3.1) we can see that the Riemann invariants $r^1_\varepsilon(t,\cdot)$ and $r^2_\varepsilon(t,\cdot)$ are uniformly bounded in $BV(\mathbb{R})$, with

$$\left.\begin{aligned} \|\partial_x r^i_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} &\le 2\|r^i_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \\ &\le 2\left(\|u_0\|_{L^\infty(\mathbb{R})} + (\|\rho_0\|_{L^\infty(\mathbb{R})})^\theta + 1\right) \end{aligned}\right| \text{for } i = 1,2, \text{ and } 0 < t < T. \tag{3.2}$$

Since $(r^1_\varepsilon, r^2_\varepsilon)$ verify system (2.1) almost everywhere, we can also see that there exists a positive constant μ independent of ε, such that

$$\|\partial_t r^i_\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}))} \le \mu \quad \text{for} \quad i = 1,2. \tag{3.3}$$

Now, we will prove that $r^1_\varepsilon - r^2_\varepsilon > 0$. To this end, we recall, from Theorem 2.1, that $r^1_\varepsilon = \lim_{\eta\to 0} r^1_{\varepsilon,\eta}$ and $r^2_\varepsilon = \lim_{\eta\to 0} r^2_{\varepsilon,\eta}$, where $(r^1_{\varepsilon,\eta}, r^2_{\varepsilon,\eta})$ is the smooth solution of the following regularized parabolic system

$$\partial_t r^i_{\varepsilon,\eta} + \tilde{\lambda}^i(r^1_{\varepsilon,\eta}, r^2_{\varepsilon,\eta})\partial_x r^i_{\varepsilon,\eta} = \eta\partial^2_{xx} r^i_{\varepsilon,\eta} \quad \text{for} \quad i = 1,2,$$

with regular initial data $(r^1_{0,\varepsilon,\eta}, r^2_{0,\varepsilon,\eta})$ (see Theorem 2.1). To simplify, we set $r_{\varepsilon,\eta} = r^1_{\varepsilon,\eta} - r^2_{\varepsilon,\eta}$ and we can check that $r_{\varepsilon,\eta}$ satisfies the following equation

$$\partial_t r_{\varepsilon,\eta} = -\left(\frac{r^1_{\varepsilon,\eta} + r^2_{\varepsilon,\eta}}{2}\right)\partial_x r_{\varepsilon,\eta} - \frac{\gamma-1}{4} r_{\varepsilon,\eta}\partial_x(r^1_{\varepsilon,\eta} + r^2_{\varepsilon,\eta}) + \eta\partial^2_{xx} r_{\varepsilon,\eta}.$$

Using the maximum principle theorem for parabolic equations (see Lieberman [16, Th 2.10]), the η-uniform estimate (2.5) and the fact that $r^1_{0,\varepsilon,\eta} - r^2_{0,\varepsilon,\eta} \geq 2\varepsilon > 0$, we get

$$r^1_{\varepsilon,\eta}(t,\cdot) - r^1_{\varepsilon,\eta}(t,\cdot) \geq 2\varepsilon e^{-\alpha t} > 0,$$

for all $0 \leq t < T$, with $\alpha = \dfrac{\gamma-1}{2} \max_{i=1,2} ||\partial_x r^i_{0,\varepsilon}||_{L^\infty(\mathbb{R})}$. We pass to the limit as $\eta \to 0$ and we obtain, by uniform convergence, that

$$r^1_\varepsilon(t,\cdot) - r^1_\varepsilon(t,\cdot) \geq 2\varepsilon e^{-\alpha t} > 0. \tag{3.4}$$

Step 2. (From $(r^1_\varepsilon, r^2_\varepsilon)$ towards $(\rho_\varepsilon, u_\varepsilon)$). Let $(r^1_\varepsilon, r^2_\varepsilon)$ be the Lipschitz continuous solution of system (2.1), constructed in Step 1. For $\theta = \frac{\gamma-1}{2}$, we use the following variable change

$$u_\varepsilon = \frac{r^1_\varepsilon + r^2_\varepsilon}{2} \quad \text{and} \quad \rho^\theta_\varepsilon = \frac{r^1_\varepsilon - r^2_\varepsilon}{2}, \tag{3.5}$$

that is possible since $r^1_\varepsilon - r^2_\varepsilon > 0$. With a simple computation we can check that these new variables solve (almost everywhere) the following system

$$\begin{cases} \partial_t(\rho^\theta_\varepsilon) + u_\varepsilon \partial_x(\rho^\theta_\varepsilon) + \theta \rho^\theta_\varepsilon \partial_x u_\varepsilon = 0, \\ \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \theta \rho^\theta_\varepsilon \partial_x(\rho^\theta_\varepsilon) = 0. \end{cases}$$

According to (3.4), we know that $\rho_\varepsilon \geq \beta_\varepsilon > 0$ (for some positive constant β_ε), in addition to the fact that $(r^1_\varepsilon, r^2_\varepsilon) \in (W^{1,\infty}([0,T) \times \mathbb{R})^2$, we conclude that the functions u_ε and ρ_ε, defined above, belong to $W^{1,\infty}([0,T) \times \mathbb{R})$, and moreover solve the following system

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u^2_\varepsilon + p(\rho_\varepsilon)) = 0, \end{cases} \tag{3.6}$$

with the following initial data

$$u_\varepsilon(0,x) = \rho^1_\varepsilon \star u_0(x), \quad \text{and} \quad \rho^\theta_\varepsilon(0,x) = \rho^1_\varepsilon \star \rho^\theta_0(x) + \varepsilon, \quad \text{for} \quad x \in \mathbb{R}.$$

Step 3. (Convergence from $(\rho_\varepsilon, u_\varepsilon)$ toward (ρ, u)). In order to fulfill the proof of Theorem 1.1, we still have to pass to the limit in (3.6) as $\varepsilon \to 0$. Indeed, from ε-uniform estimates (3.1), (3.2), (3.3) and according to (3.5) we deduce that there exist three positive constants C_1, C_2 and C_3 independent of ε, such that

$$\begin{cases} \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|\rho^\theta_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C_1, \\ \|\partial_x u_\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}))} + \|\partial_x \rho^\theta_\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq C_2, \\ \|\partial_t u_\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}))} + \|\partial_t \rho^\theta_\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq C_3. \end{cases} \tag{3.7}$$

Using Simon's lemma (cf. Lemma 2.1) in the particular case $X = BV(K)$, $B = Y = L^1(K)$ associated to the following compact embedding $BV(K) \hookrightarrow L^p(K)$ for $1 \leq p < +\infty$ and for all compact $K \subset \mathbb{R}$, we will be able to extract a subsequence $(u_{\varepsilon_n}, \rho^\theta_{\varepsilon_n})$ that converges towards a limit $(u, \tilde{\rho}_\theta)$ strongly in $C([0,T); L^p(K))$, for all $1 \leq p < +\infty$, $T > 0$

and compact $K \subset \mathbb{R}$. Thanks to estimates (3.7), we can extract a subsequence, still denoted by $(u_{\varepsilon_n}, \rho^\theta_{\varepsilon_n})$, satisfying the following convergence estimates

$$\left| \begin{array}{llll} (u_{\varepsilon_n}, \rho^\theta_{\varepsilon_n}) \longrightarrow (u, \tilde{\rho}_\theta) & \text{strongly} & \text{in} & C([0,T);L^p(K)), \text{ for all compact } K \subset \mathbb{R}, \\ (u_{\varepsilon_n}, \rho^\theta_{\varepsilon_n}) \longrightarrow (u, \tilde{\rho}_\theta) & \text{weakly-}\star & \text{in} & L^\infty((0,T)\times\mathbb{R}), \\ (u_{\varepsilon_n}, \rho^\theta_{\varepsilon_n}) \longrightarrow (u, \tilde{\rho}_\theta) & \text{weakly-}\star & \text{in} & L^\infty((0,T);BV(\mathbb{R})). \end{array} \right.$$

Using the fact that $\rho_{\varepsilon_n} \geq 0$, the first estimate in (3.7) shows that ρ_{ε_n} is also uniformly bounded with respect to ε_n and then, for an extracted sub-sequence, we can verify that

$$\rho_{\varepsilon_n} \longrightarrow \rho \quad \text{weakly-}\star \quad \text{in} \quad L^\infty((0,T)\times\mathbb{R}),$$

for some function $\rho \in L^\infty((0,T)\times\mathbb{R})$. It remains to verify that (u,ρ) is solution of (1.1) in the distributional sense (i.e. in $\mathcal{D}'((0,T)\times\mathbb{R})$). Indeed, for all compact $K \subset \mathbb{R}$, we have on one hand that ρ_{ε_n} converges to ρ weakly-$\star$ in $L^\infty((0,T)\times K)$ and on the other hand u_{ε_n} converges to u strongly $L^1((0,T)\times K)$. This gives us a weak-$\star$ convergence in $L^\infty((0,T)\times K)$ times a strong convergence in $L^1((0,T)\times K)$ in the term $\rho_{\varepsilon_n} u_{\varepsilon_n}$. Hence the product $\rho_{\varepsilon_n} u_{\varepsilon_n}$ converges weakly in $L^1((0,T)\times K)$ to ρu and then in the distributional sense. Similarly, we can prove that $\rho_{\varepsilon_n} u^2_{\varepsilon_n}$ converges to ρu^2 in the distributional sense, using the weak-$\star$ convergence in $L^\infty((0,T)\times K)$ of ρ_{ε_n} toward ρ and the strong convergence in $L^2((0,T)\times K)$ of u_{ε_n} toward u, in other words the strong convergence in $L^1((0,T)\times K)$ of $u^2_{\varepsilon_n}$ toward u^2.
Moreover, since $\rho^\theta_{\varepsilon_n}$ converges to $\tilde{\rho}_\theta$ strongly in $L^1((0,T)\times K)$, then for an extracted sub-sequence $\rho^\theta_{\varepsilon_n}$ converges to $\tilde{\rho}_\theta$ almost everywhere in $(0,T)\times K$. Therefore ρ_{ε_n} converges to $(\tilde{\rho}_\theta)^{\frac{1}{\theta}}$ almost everywhere in $(0,T)\times K$. This implies, using the dominated convergence theorem and the fact that ρ_{ε_n} is uniformly bounded with respect to ε_n, that ρ_{ε_n} converges to $(\tilde{\rho}_\theta)^{\frac{1}{\theta}}$ strongly in $L^1((0,T)\times K)$ and in particular in $\mathcal{D}'((0,T)\times\mathbb{R})$. However, we know that ρ_{ε_n} converges also to ρ in $\mathcal{D}'((0,T)\times\mathbb{R})$, thanks to the weak-$\star$ convergence. Thus, $\tilde{\rho}_\theta = \rho^\theta$ in $\mathcal{D}'((0,T)\times\mathbb{R})$. In the same way, we can prove that $\rho^\gamma_{\varepsilon_n}$ converges to ρ^γ strongly in $L^1((0,T)\times K)$, for all $T>0$ and compact $K\subset\mathbb{R}$. This shows that (u,ρ) is a solution in $\mathcal{D}'((0,T)\times\mathbb{R})$, to the following system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0. \end{cases}$$

Taking the liminf in estimates (3.7) and using the lower semi-continuity of $\|\cdot\|_{L^\infty((0,T)\times\mathbb{R})}$ and $\|\cdot\|_{L^\infty((0,T);BV(\mathbb{R}))}$ with respect to weak-$\star$ topology, we can prove that u and ρ^θ satisfy (1.3).

Step 4. (Recovering the initial data). Now, we prove that the initial conditions (u_0,ρ_0) coincide with $(u(0,\cdot),\rho(0,\cdot))$. Indeed, by the ε-uniformly estimate given in (3.7), we can prove easily that

$$\|u_\varepsilon(t,\cdot) - \rho^1_\varepsilon \star u_0\|_{L^1(\mathbb{R})} + \|\rho^\theta_\varepsilon(t,\cdot) - \rho^1_\varepsilon \star \rho^\theta_0 - \varepsilon\|_{L^1(\mathbb{R})} \leq C_3 t.$$

Then, for all compact $K\subset\mathbb{R}$, we get

$$\|u(t,\cdot) - u_0\|_{L^1(K)} + \|\rho^\theta(t,\cdot) - \rho^\theta_0\|_{L^1(K)} \leq C_3 t.$$

where we have used the strong convergence in $C([0,T);L^1(K))$, This proves that $u(0,\cdot)=u_0(\cdot)$ and $\rho^\theta(0,\cdot)=\rho_0^\theta(\cdot)$ in $L^1_{loc}(\mathbb{R})$, therefore $u(0,\cdot)-u_0(\cdot)=\rho(0,\cdot)-\rho_0(\cdot)=0$ almost everywhere in $\mathbb{R}$. Since the functions $u(0,\cdot)$, $u_0(\cdot)$, $\rho(0,\cdot)$, $\rho_0(\cdot)$ belong to $L^1_{loc}(\mathbb{R})$ we deduce that $u(0,\cdot)=u_0(\cdot)$ and $\rho(0,\cdot)=\rho_0(\cdot)$ in $\mathcal{D}'(\mathbb{R})$.

Remark 3.1. Note that, from estimate (3.7) and using the strong convergence in $C([0,T);L^1_{loc}(\mathbb{R}))$ of u_ε and ρ_ε^θ we can show as before that u and ρ^θ belong to $C([0,T);L^1_{loc}(\mathbb{R}))$. This does not claim that ρ stays in the same space in general, except for $0<\theta\leq 1$, i.e. $1<\gamma\leq 3$. This allows us to find the result of Vasseur [25] proved when $\gamma=3$ in the framework of entropy solution, relying on the kinetic formulation developed in [18]. □

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