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GLOBAL DYNAMICS BELOW THE GROUND STATES FOR NLS UNDER PARTIAL HARMONIC CONFINEMENT*

ALEX H. ARDILA† AND RÉMI CARLES‡

Abstract. We are concerned with the global behavior of the solutions of the focusing mass supercritical nonlinear Schrödinger equation under partial harmonic confinement. We establish a necessary and sufficient condition on the initial data below the ground states to determine the global behavior (blow-up/scattering) of the solution. Our proof of scattering is based on the variational characterization of the ground states, localized virial estimates, linear profile decomposition and nonlinear profiles.

Keywords. Nonlinear Schrödinger equation; ground states; global existence; blow-up; scattering.

AMS subject classifications. Primary 35Q55; Secondary 37K45; 35P25.

1. Introduction

In this paper we study the initial-value problem for the nonlinear Schrödinger equation under partial harmonic confinement

$$\begin{cases} i\partial_t u = Hu + \lambda |u|^{2\sigma} u, \quad x\in\mathbb{R}^d, \quad t\in\mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$, $\lambda\in\{-1,+1\}$, $d\geq 2$ and $0<\sigma<\frac{2}{d-2}$. The operator H is defined as

$$H := -\Delta_y + |y|^2 - \Delta_z, \quad x=(y,z)\in\mathbb{R}^n\times\mathbb{R}^{d-n},$$

where $1\leq n\leq d-1$. Nonlinear Schrödinger equations in the presence of a harmonic potential arise in various branches of physics, such as the Bose-Einstein condensates or the propagation of mutually incoherent wave packets in nonlinear optics. For more details we refer to [29]. In this context, anisotropy of the potential is often considered. Strong confinement in special directions leads to dimension reduction phenomena (see e.g. [4, 12]), while, as proven initially in [2], the case of partial confinement may lead to dispersive phenomena and asymptotically linear behavior (scattering).

As recalled briefly in Section 2, the Cauchy problem for (1.1) is locally well-posed in the energy space[1]

$$B_1 = \left\{ u\in H^1(\mathbb{R}^d;\mathbb{C}) : \|yu\|_{L^2}^2 = \int_{\mathbb{R}^d} |y|^2 |u(x)|^2 dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{B_1}^2 = \langle u, Hu\rangle = \|\nabla_x u\|_{L^2}^2 + \|yu\|_{L^2}^2 + \|u\|_{L^2}^2.$$

In particular, the linear propagator e^{-itH} preserves the B_1-norm. We can use a contraction mapping technique based on Strichartz estimates to show that (1.1) is locally

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†Department of Mathematics, Instituto de Ciências Exata (ICEx), Universidade Federal de Minas Gerais, CEP 30123-970, Belo Horizonte-MG, Brazil (ardila@impa.br).

‡Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France (Remi.Carles@math.cnrs.fr).

[1] The notation B_1 is borrowed from [6], for consistency in future references.

well-posed in B_1 (see Lemma 2.1): For any $u_0\in B_1$ there exists a unique maximal solution $u\in C((-T_-,T_+);B_1)$ of (1.1), $T_\pm\in(0,\infty]$. Furthermore, the solution u enjoys the conservation of energy, momentum and mass,

$$E(u(t))=E(u_0),\quad G(u(t))=G(u_0),\quad M(u(t))=M(u_0),\quad \forall t\in(-T_-,T_+), \tag{1.2}$$

where E, M and G are defined as

$$E(u)=\frac{1}{2}\int_{\mathbb{R}^d}|\nabla_x u|^2dx+\frac{1}{2}\int_{\mathbb{R}^d}|y|^2|u|^2dx+\frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d}|u|^{2\sigma+2}dx,$$

and

$$G(u)=\operatorname{Im}\int_{\mathbb{R}^d}\overline{u}\nabla_z u dx,\quad M(u)=\int_{\mathbb{R}^d}|u|^2dx. \tag{1.3}$$

We recall the definitions of scattering and blow-up in the framework of the energy space B_1.

Definition 1.1. *Let u be a solution of the Cauchy problem (1.1) on the maximal existence time interval $(-T_-,T_+)$. We say that the solution u scatters in B_1 (both forward and backward time) if $T_\pm=\infty$ and there exist $\psi^\pm\in B_1$ such that*

$$\|u(t)-e^{-itH}\psi^\pm\|_{B_1}=\|e^{itH}u(t)-\psi^\pm\|_{B_1}\to 0\quad \text{as } t\to\pm\infty.$$

On the other hand, if $T_+<\infty$ (resp. $T_-<\infty$), we say that the solution u blows up in positive time (resp. negative time). In the case $T_+<\infty$, this corresponds to the property

$$\|\nabla_x u(t)\|_{L^2(\mathbb{R}^d)}\underset{t\to T_+}{\longrightarrow}\infty.$$

We refer to the proof of Lemma 2.1 below to see why the momentum does not appear in the blow-up characterization. In [2], scattering was considered in the conformal space

$$\Sigma=B_1\cap\{f;\ x\mapsto|z|f(x)\in L^2(\mathbb{R}^d)\}=H^1(\mathbb{R}^d)\cap\{f;\ x\mapsto|x|f(x)\in L^2(\mathbb{R}^d)\},$$

which is of course smaller than B_1. In the present paper, we investigate the large-time behavior of the solution to (1.1) in B_1, both in the focusing ($\lambda=-1$) and in the defocusing ($\lambda=1$) case. As a preliminary, we state a result concerning the small data case.

Proposition 1.1. *Suppose $\frac{2}{d-n}\le\sigma<\frac{2}{d-2}$ and $\lambda\in\{-1,+1\}$. There exists $\nu>0$ such that if $\|u_0\|_{B_1}\le\nu$, then the solution to (1.1) is global in time ($T_\pm=\infty$) and scatters in B_1.*

This proposition follows directly from Lemma 5.1 below. We note that in [2], for the similar statement in the smaller space Σ, the lower bound on σ was $\sigma>\frac{d}{d+2}\frac{2}{d-n}$ (see [2, Theorem 1.5]). In terms of the variable $y\in\mathbb{R}^n$, confinement prevents complete dispersion. On the other hand, in the variable $z\in\mathbb{R}^{d-n}$, we benefit from the usual dispersion for the Schrödinger equation posed on $\mathbb{R}^{d-n}$. In other words, scattering is expected *somehow* as if we considered

$$i\partial_t v=-\Delta_z v+\lambda|v|^{2\sigma}v,\quad z\in\mathbb{R}^{d-n},$$

and the above lemma is the counterpart of small data scattering in $H^1(\mathbb{R}^{d-n})$ for L^2-critical or supercritical nonlinearities, and the presence of the extra variable y reads in the upper bound $\sigma<\frac{2}{d-2}$, to make the nonlinearity energy-subcritical. For large data, global existence and some blow-up results have been considered in [7]. Moreover, scattering for (1.1), for some σ, d and n, was studied in [2, 9, 23].

Consider the focusing case $\lambda=-1$, which is the core of this paper. In the case $0<\sigma<2/d$ the Cauchy problem (1.1) is globally well-posed, regardless of the sign of λ. Moreover, for small initial data the solution can be extended to a global one in the case $2/d<\sigma<2/(d-2)$. The issue of existence, stability and instability of standing waves has been studied in [4, 21, 34].

Introduce the following nonlinear elliptic problem

$$H\varphi+\varphi-|\varphi|^{2\sigma}\varphi=0, \quad \varphi\in B_1\setminus\{0\}. \tag{1.4}$$

We recall that a non-trivial solution Q to (1.4) is said to be the ground state solution, if it has some minimal action among all solutions of the elliptic problem (1.4), i.e.

$$S(Q)=\inf\{S(\varphi):\varphi \text{ is a solution of } (1.4)\}, \tag{1.5}$$

where the action functional S is defined by

$$S(u):=\frac{1}{2}\|\nabla_x u\|^2_{L^2}+\frac{1}{2}\|yu\|^2_{L^2}+\frac{1}{2}\|u\|^2_{L^2}-\frac{1}{2\sigma+2}\|u\|^{2\sigma+2}_{L^{2\sigma+2}}.$$

In Lemma 3.2 we obtain the existence of at least one ground state solution (see also Remark 3.1).

Remark 1.1. We could also consider, for any $\omega>0$,

$$H\varphi+\omega\varphi-|\varphi|^{2\sigma}\varphi=0, \quad \varphi\in B_1\setminus\{0\},$$

up to adapting the notations throughout the paper. We consider the case $\omega=1$ for simplicity.

Our main result consists in establishing a necessary and sufficient condition on the initial data below the ground state Q to determine the global behavior (blow-up/scattering) of the solution. As recalled above, when scattering occurs, it is reminiscent of the nonlinear Schrödinger equation without potential, posed on $\mathbb{R}^{d-n}$. With this in mind, we define the following functional of class C^2 on B_1,

$$P(u)=\frac{2}{d-n}\|\nabla_z u\|^2_{L^2}-\frac{\sigma}{\sigma+1}\|u\|^{2\sigma+2}_{L^{\sigma+2}}, \tag{1.6}$$

and we define the following subsets in B^1,

$$\mathcal{K}^+=\{\varphi\in B_1:S(\varphi)<S(Q), \quad P(\varphi)\geq 0\},$$
$$\mathcal{K}^-=\{\varphi\in B_1:S(\varphi)<S(Q), \quad P(\varphi)<0\}.$$

By a scaling argument, it is not difficult to show that $\mathcal{K}^\pm\neq\emptyset$. In our main result, we will show that the sets $\mathcal{K}^+$ and $\mathcal{K}^-$ are invariant under the flow generated by the Equation (1.1). Moreover, we obtain a sharp criterion between blow-up and scattering for (1.1) in terms of the functional P given by (1.6). In the case of a full confinement ($n=d$), such results were initiated in [36, 40]. Of course, in the absence of fully dispersive direction, the dichotomy concerns global existence *vs.* blow-up, and scattering cannot hold. The proof of scattering properties represents a large part of the present paper.

The assumption $\sigma > \frac{2}{d-n}$ is needed to prove the Lemmas 3.2 and 3.4 (existence and characterization of the ground states) and the profile decomposition result (see Proposition 5.1). Thus, in the case $\lambda = -1$, we assume

$$\frac{2}{d-n} < \sigma < \frac{2}{d-2}.$$

This condition implies that $n=1$ in the statement below, a condition which is reminiscent of [38], where a partial one-dimensional *geometrical* confinement is considered ($y \in \mathbb{T}$). Also, a step of our proof requires the extra property $\sigma \geq \frac{1}{2}$, and so we restrict to dimensions $2 \leq d \leq 5$.

THEOREM 1.1. *Let $\lambda = -1$, $n=1$, $\sigma \geq \frac{1}{2}$ with $\frac{2}{d-1} < \sigma < \frac{2}{d-2}$, and $u_0 \in B_1$. Let $u \in C(I;B_1)$ be the corresponding solution of (1.1) with initial data u_0 and lifespan $I = (T_-, T_+)$.*

(i) If $u_0 \in \mathcal{K}^+$, then the corresponding solution $u(t)$ exists globally and scatters.

(ii) If $u_0 \in \mathcal{K}^-$, then one of the following two cases occurs:

(1) The solution blows up in positive time, i.e., $T_+ < \infty$ and

$$\lim_{t \to T_+} \|\nabla_x u(t)\|_{L^2}^2 = \infty.$$

(2) The solution blows up at infinite positive time, i.e., $T_+ = \infty$ and there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ and $\lim_{t_k \to \infty} \|\nabla_x u(t_k)\|_{L^2}^2 = \infty$.

An analogous statement holds for negative time.

REMARK 1.2. We note that if the initial datum satisfies $u_0 \in \mathcal{K}^-$ and $xu_0 \in L^2(\mathbb{R}^d)$ (that is, $u_0 \in \Sigma$), then the corresponding solution blows up in finite time (see (4.8) below for more details, with $R = \infty$). In particular, the condition $P(u) \geq 0$ in Theorem 1.1 is sharp for global existence.

The proof of the scattering result is based on the concentration/compactness and rigidity argument of Kenig-Merle [30]. In [15], Duyckaerts-Holmer-Roudenko studied (1.1) with $d=3$, $\sigma=1$, without harmonic potential, and proved that if $u_0 \in H^1(\mathbb{R}^3)$ satisfies (see also [24] in the radial case)

$$M(u_0)E(u_0) < M(Q)E(Q), \quad \|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2},$$

then the corresponding solution exists globally and scatters in $H^1(\mathbb{R}^3)$, where Q is the ground state of the Equation (1.4). However, it seems that the method developed in [15, 24] cannot be applied to (1.1) with harmonic potential. The main difficulty concerning (1.1) is clearly the presence of the partial harmonic confinement. In particular, we cannot apply scaling techniques to obtain the critical element (see the proof of Proposition 5.4 in [24]). To overcome this problem, we use a variational approach based on the work of Ibrahim-Masmoudi-Nakanishi [27] (see also [28]). We mention the works of Ikea-Inu [28] and Guo-Wang-Yao [39] who also obtained an analogous result to Theorem 1.1 for the focusing NLS equation with a potential. The proof of the blow-up result is based on the techniques developed by Du-Wu-Zhang [14].

It is worth mentioning that Fang-Xie-Cazenave [16] and Akahor-Nawa [1] extended the results in Holmer-Roudenko [24] and Duyckaerts-Holmer-Roudenko [15] in terms of dimension and power. Concerning the scattering theory with a smooth short range potential in the energy-subcritical case, we refer to [8, 10, 25, 33]; see also [3, 32] for scattering theory with a singular potential in the energy-subcritical case. For other results, see e.g. [5, 13, 17], and [23] in the case of a partial confinement leading to long range scattering for small data.

REMARK 1.3. The tools that we use also yield scattering results in the defocusing case $\lambda=+1$. For $d\geq 2$, $n=1$, and $\sigma\geq\frac{1}{2}$ with $\frac{2}{d-1}<\sigma<\frac{2}{d-2}$, consider $u_0\in B_1$ and $u\in C(\mathbb{R};B_1)$ the solution to

$$i\partial_t u = Hu + |u|^{2\sigma}u; \qquad u_{|t=0}=u_0.$$

Then u scatters in B_1. As pointed out in [15, Section 7] in the case of the 3D cubic Schrödinger equation without potential, the proof is essentially the same as for scattering in the focusing case (Theorem 1.1). Also, in this defocusing case, we simply recover [9, Theorem 1.5], based on Morawetz estimates, where the assumption $\sigma\geq\frac{1}{2}$ was not needed.

REMARK 1.4. In the case of a partial geometric confinement ($x\in\mathbb{R}^d\times\mathbb{T}$, like in [37, 38]), the Kenig-Merle route map was used in [18] to prove scattering for the defocusing Klein-Gordon equation.

Organization of the paper. In the next section we introduce Strichartz estimates specific to the present context, and show that a specific norm suffices to ensure scattering. In Section 3, we show variational estimates, which will be key to obtain blow-up and scattering results in the focusing case. In Section 4, we show the blow-up results and the global part of Theorem 1.1 (i). Finally, in Section 5 we prove the scattering part of Theorem 1.1.

Notations. We summarize the notation used throughout the paper: $\mathbb{Z}$ denotes the set of all integers. We will use $A\lesssim B$ (resp. $A\gtrsim B$) for inequalities of type $A\leq CB$ (resp. $A\geq CB$), where C is a positive constant. If both the relations hold true, we write $A\sim B$. We denote by $\mathrm{NLS}(t)u_0$ the solution of the IVP (1.1) with initial data u_0.

For $1\leq p\leq\infty$, we denote its conjugate by $p'=\frac{p}{p-1}$. Moreover, $L^p=L^p(\mathbb{R}^d;\mathbb{C})$ are the classical Lebesgue spaces. The scale of harmonic (partial) Sobolev spaces is defined as follows, see [6]: For $s\geq 0$

$$B_s=B_s(\mathbb{R}^d)=\left\{u\in L^2(\mathbb{R}^d): H^{s/2}u\in L^2(\mathbb{R}^d)\right\}$$

endowed with the natural norm denoted by $\|\cdot\|_{B_s}$, and up to equivalence of norms we have (see [6, Theorem 2.1])

$$\|u\|_{B_s}^2=\|u\|_{H^s}^2+\||y|^s u\|_{L^2}^2.$$

For $\gamma\in\mathbb{Z}$, we set $I_\gamma=\pi[\gamma-1,\gamma+1)$. Let $\ell_\gamma^p L_t^q(I_\gamma;L_x^r(\mathbb{R}^d))$ be the space of measurable functions $u:\mathbb{R}\to L_x^r(\mathbb{R}^d)$ such that the norm $\|u\|_{\ell_\gamma^p L^q(I_\gamma;L_x^r(\mathbb{R}^d))}$ is finite, with

$$\|u\|^p_{\ell_\gamma^p L_t^q(I_\gamma;L_x^r(\mathbb{R}^d))}=\sum_{\gamma\in\mathbb{Z}}\|u\|^p_{L_t^q(I_\gamma;L_x^r(\mathbb{R}^d))}.$$

To simplify the notation, we will use $\|u\|_{\ell_\gamma^p L^q L^r}$ when it is not ambiguous. Finally, we write $\|u\|_{\ell^p_{\gamma_0\leq\gamma\leq\gamma_1}L^q(I_\gamma;L_x^r)}$ to signify

$$\|u\|^p_{\ell^p_{\gamma_0\leq\gamma\leq\gamma_1}L^q(I_\gamma;L_x^r)}=\sum_{\gamma_0\leq\gamma\leq\gamma_1}\|u\|^p_{L_t^q(I_\gamma;L_x^r(\mathbb{R}^d))}.$$

2. Strichartz estimates and scattering

2.1. Local Strichartz estimates and local well-posedness. Denote the (partial) harmonic potential by $V(x)=|y|^2$ (recall that $x=(y,z)\in\mathbb{R}^n\times\mathbb{R}^{d-n}$). As V is quadratic, it enters the general framework of at most quadratic smooth potentials considered in [20]. In particular, the propagator associated to H enjoys local dispersive estimates (as can be seen also from generalized Mehler formula, see e.g. [26])

$$\|e^{-itH}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)}\lesssim\frac{1}{|t|^{d/2}},\quad |t|\le 1,$$

which in turn imply *local-in-time* Strichartz estimates,

$$\|e^{-itH}u_0\|_{L^q(I;L^r(\mathbb{R}^d))}\le C_q(I)\|u_0\|_{L^2(\mathbb{R}^d)},\quad \frac{2}{q}=d\left(\frac{1}{2}-\frac{1}{r}\right),\quad 2\le r<\tfrac{2d}{d-2},$$

where the constant $C_q(I)$ actually depends on $|I|$. Indeed, we compute for instance

$$e^{-itH}\left(e^{-|y|^2/2}v_0(z)\right)=e^{-|y|^2/2+int}\left(e^{it\Delta_{\mathbb{R}^{d-n}}}v_0\right)(z).$$

Local-in-time Strichartz estimates suffice to establish local well-posedness in the energy space, as proved in [7]. We give some elements of proof which introduce some useful vector fields.

LEMMA 2.1. *Let $d\ge 2$, $1\le n\le d-1$, $0<\sigma<\frac{2}{d-2}$, and $u_0\in B_1$. There exists $T=T(\|u_0\|_{B_1})$ and a unique solution $u\in C([-T,T];B_1)\cap L^{\frac{4\sigma+4}{d\sigma}}([-T,T];L^{2\sigma+2}(\mathbb{R}^d))$ to (1.1). In addition, the conservations (1.2) hold.*

Either the solution is global in positive time, $u\in C(\mathbb{R}_+;B_1)\cap L^{\frac{4\sigma+4}{d\sigma}}_{\rm loc}(\mathbb{R}_+;L^{2\sigma+2}(\mathbb{R}^d))$, or there exists $T_+>0$ such that

$$\|\nabla_x u(t)\|_{L^2(\mathbb{R}^d)}\underset{t\to T_+}{\longrightarrow}\infty.$$

If $\lambda=+1$, then the solution is global in time, $u\in C(\mathbb{R};B_1)\cap L^{\frac{4\sigma+4}{d\sigma}}_{\rm loc}(\mathbb{R};L^{2\sigma+2}(\mathbb{R}^d))$.

Proof. (Sketch of the proof). The proof relies on a classical fixed-point argument applied to Duhamel's formula

$$u(t)=e^{-itH}u_0-i\lambda\int_0^t e^{-i(t-s)H}\left(|u|^{2\sigma}u\right)(s)ds,$$

using (local in time) Strichartz estimates. The gradient ∇_z commutes with e^{-itH}, since there is no potential in the z variable. On the other hand, in the y variable, the presence of the harmonic potential ruins this commutation property. It is recovered by considering the vector fields

$$A_1(t)=y\sin(2t)-i\cos(2t)\nabla_y,\quad A_2(t)=-y\cos(2t)-i\sin(2t)\nabla_y.$$

We recall from, for example [2, Lemma 4.1], the main properties that we will use:

$$\begin{pmatrix}A_1(t)\\A_2(t)\end{pmatrix}=\begin{pmatrix}\sin(2t)&\cos(2t)\\-\cos(2t)&\sin(2t)\end{pmatrix}\begin{pmatrix}y\\-i\nabla_y\end{pmatrix},$$

they correspond to the conjugation of momentum and position by the free flow,

$$A_1(t) = e^{-itH}(-i\nabla_y)e^{itH}, \quad A_2(t) = -e^{-itH} y e^{itH},$$

and therefore, they commute with the linear part of (1.1): $[i\partial_t - H, A_j(t)] = 0$. These vector fields act on gauge-invariant nonlinearities like derivatives, and we have the pointwise estimate

$$\left|A_j(t)\left(|u|^{2\sigma}u\right)\right| \lesssim |u|^{2\sigma}|A_j(t)u|.$$

Once all of this is noticed, we can just mimic the standard proof of local well-posedness of NLS in $H^1(\mathbb{R}^d)$ (see e.g. [11]), by considering $(A_1(t), A_2(t), \nabla_z)$ instead of (∇_y, ∇_z) (see also [2,7]). The conservations (1.2) follow from classical arguments (see e.g. [11]).

From the construction, either the solution is global, or the B_1-norm becomes unbounded in finite time. Like in the statement of the lemma, we consider positive time only, the case of negative time being similar. The obstruction to global existence reads

$$\|u(t)\|_{B_1} \underset{t\to T_+}{\longrightarrow} \infty,$$

for some $T_+ > 0$. But a standard virial computation yields

$$\frac{d}{dt}\|yu(t)\|_{L^2}^2 = 4\operatorname{Im}\int_{\mathbb{R}^d} \bar{u}(t,x) y \cdot \nabla_y u(t,x) dy.$$

Cauchy-Schwarz inequality shows that if $\|\nabla_y u(t)\|_{L^2}$ remains bounded locally in time, then so does $\|yu(t)\|_{L^2}$, hence the blow-up criterion. Global existence in the case $\lambda = +1$ is straightforward. □

For future reference, we note that

$$\|e^{itH}u(t)\|_{B_1}^2 \sim \sum_{A\in\{\mathrm{Id}, A_1, A_2, \nabla_z\}} \|A(t)u(t)\|_{L^2(\mathbb{R}^d)}^2. \tag{2.1}$$

2.2. Global Strichartz estimates. To prove scattering results, we use global-in-time Strichartz estimates, taking advantage of the full dispersion in the z variable, and of the local dispersion in the total variable $x = (y,z)$.

Lemma 2.2 (Global Strichartz estimates, Theorem 3.4 from [2]). *Let $d \geq 2$, $1 \leq n \leq d-1$ and $2 \leq r < \frac{2d}{d-2}$. Then the solution u to $(i\partial_t - H)u = F$ with initial data u_0 obeys*

$$\|u\|_{\ell_\gamma^{p_1} L^{q_1} L^{r_1}} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + \|F\|_{\ell_\gamma^{p_2'} L^{q_2'} L^{r_2'}}, \tag{2.2}$$

provided that the following conditions hold:

$$\frac{2}{q_k} = d\left(\frac{1}{2} - \frac{1}{r_k}\right), \quad \frac{2}{p_k} = (d-n)\left(\frac{1}{2} - \frac{1}{r_k}\right), \quad k = 1,2. \tag{2.3}$$

Moreover, as in e.g. [24] or [38], we will need the following inhomogeneous Strichartz estimates.

Lemma 2.3 (Inhomogeneous Strichartz estimates). *Let $d \geq 2$, $1 \leq n \leq d-1$. Then we have*

$$\left\|\int_0^t e^{-i(t-s)H} u(s) ds\right\|_{\ell_\gamma^p L^q L^r} \lesssim \|u\|_{\ell_\gamma^{\tilde{p}'} L^{\tilde{q}'} L^{r'}},$$

provided that $q,\tilde{q}\in[1,\infty]$ *and:*

$$\frac{2}{p}+\frac{2}{\tilde{p}}=(d-n)\left(1-\frac{2}{r}\right),$$
$$\frac{1}{p}+\frac{d-n}{r}<\frac{d-n}{2},\quad \frac{1}{\tilde{p}}+\frac{d-n}{r}<\frac{d-n}{2},\quad \textit{(acceptable pairs)}$$
$$\frac{1}{p}+\frac{1}{\tilde{p}}<1.$$

Proof. The proof of the inhomogeneous Strichartz estimates for non-admissible pairs is a direct adaptation of the proof of Theorem 1.4 in [19]. We emphasize that we consider the same Lebesgue index in space on the left- and right-hand sides in the above inequality, which makes the adaptation of [19, Theorem 1.4] easier. □

We will also need a weaker dispersive property:

LEMMA 2.4. *Let* $1\leq n\leq d-1$ *and* $2<r<\frac{2d}{d-2}$*. For any* $\varphi\in B_1$*,*

$$\|e^{-itH}\varphi\|_{L^r(\mathbb{R}^d)}\underset{t\to\pm\infty}{\longrightarrow}0.$$

This result is actually valid more generally if the harmonic potential $|y|^2$ is replaced by a potential bounded from below, as shown by the proof.

Proof. When φ belongs to the conformal space, $\varphi\in\Sigma$, we consider the Galilean operator in z (see e.g. [11, 22]),

$$J_z(t)=z+2it\nabla_z=2it e^{i|z|^2/(4t)}\nabla_z\left(\cdot e^{-i|z|^2/(4t)}\right).$$

Gagliardo-Nirenberg inequality yields

$$\|e^{-itH}\varphi\|_{L^r(\mathbb{R}^d)}\lesssim|t|^{-\delta}\|e^{-itH}\varphi\|^{1-\delta}_{L^2(\mathbb{R}^d)}\|(\nabla_y,J_z(t))e^{-itH}\varphi\|^{\delta}_{L^2(\mathbb{R}^d)},$$

where $\delta=(d-n)\left(\frac{1}{2}-\frac{1}{r}\right)$. Since the harmonic potential is non-negative,

$$\|(\nabla_y,J_z(t))e^{-itH}\varphi\|_{L^2(\mathbb{R}^d)}\lesssim\|\left(\left(-\Delta_y+|y|^2\right)^{1/2},J_z(t)\right)e^{-itH}\varphi\|_{L^2(\mathbb{R}^d)},$$

and since the operator $\left(-\Delta_y+|y|^2\right)^{1/2}$ commutes with e^{-itH}, which is unitary on $L^2(\mathbb{R}^d)$, and

$$J_z(t)=e^{it\Delta_z}ze^{-it\Delta_z}=e^{-itH}ze^{itH},$$

we infer

$$\|e^{-itH}\varphi\|_{L^r(\mathbb{R}^d)}\lesssim|t|^{-\delta}\|\varphi\|_{\Sigma}.$$

In view of Sobolev embedding and the fact that e^{-itH} preserves the B_1-norm,

$$\|e^{-itH}\varphi\|_{L^r(\mathbb{R}^d)}\lesssim\|e^{-itH}\varphi\|_{H^1(\mathbb{R}^d)}\lesssim\|e^{-itH}\varphi\|_{B_1}=\|\varphi\|_{B_1},$$

the result follows by a density argument. □

2.3. Fixing Lebesgue indices for the scattering analysis. From now on, we fix the exponents $\tilde{q}$, $\tilde{p}$, p, q, p_0, q_0, r as follows.

LEMMA 2.5. *Let* $\frac{2}{d-n}\le\sigma<\frac{2}{d-2}$, *and set*

$$\tilde{q}=\frac{4\sigma(\sigma+1)}{2d\sigma^2+\sigma(d-2)-2},\quad \tilde{p}=\frac{4\sigma(\sigma+1)}{2d\sigma^2+\sigma(d-2-n)-2(n\sigma^2+1)},$$
$$p=\frac{4\sigma(\sigma+1)}{2\sigma+2-(d-n)\sigma},\quad q=\frac{4\sigma(\sigma+1)}{2\sigma+2-d\sigma},\quad r=2\sigma+2,$$
$$p_0=\frac{4\sigma+4}{(d-n)\sigma},\quad q_0=\frac{4\sigma+4}{d\sigma}.$$

Then the triplet (p_0,q_0,r) *satisfies the condition* (2.3). *Moreover, the triplets* (p,q,r) *and* $(\tilde{p},\tilde{q},r)$ *satisfy the conditions in Lemma 2.3.*

Proof. That the triplet (p_0,q_0,r) satisfies the condition (2.3) is readily checked.

We note that $\tilde{q}\in[1,\infty]$ iff $\tilde{q}'\in[1,\infty]$. Thus we must check that $q\ge 2\sigma+1$. In turn this inequality follows provided that $4\sigma(\sigma+1)\ge(2\sigma+1)(2\sigma+2-d\sigma)$ and it is equivalent to $\sigma\ge\sigma_c(d)=2-d+\sqrt{d^2-12d+4}/4d$, a threshold which is classical in scattering theory for NLS (see e.g. [11]). Since $\sigma_c(d)<2/d<2/(d-n)$, the condition is fulfilled. Now we focus on the exponent $\tilde{p}$. We compute

$$\frac{1}{\tilde{p}}=(d-n)\frac{2\sigma+1}{4\sigma+4}-\frac{1}{2\sigma},$$

and thus

$$\frac{2}{p}+\frac{2}{\tilde{p}}=(d-n)\frac{2\sigma}{2\sigma+2}=(d-n)\left(1-\frac{1}{r}-\frac{1}{r}\right).$$

We also have, from the above formula,

$$\frac{1}{p}+\frac{1}{\tilde{p}}=(d-n)\frac{\sigma}{2\sigma+2}<1,\quad\text{since}\quad \sigma<\frac{2}{d-2}<\frac{2}{(d-n-2)_+}.$$

All that remains is to check that we have acceptable pairs:

$$\frac{1}{p}+\frac{d-n}{r}<\frac{d-n}{2}\Longleftrightarrow\frac{1}{2\sigma}+\frac{d-n}{4\sigma+4}<\frac{d-n}{2}.$$

Since $\sigma\ge 2/(d-n)$, we infer that

$$\frac{1}{2\sigma}+\frac{d-n}{4\sigma+4}\le\frac{d-n}{4}+\frac{d-n}{4\sigma+4},$$

and the above inequality is satisfied as soon as

$$\frac{d-n}{4\sigma+4}<\frac{d-n}{4},$$

which is trivially the case. Last, we check

$$\frac{1}{\tilde{p}}+\frac{d-n}{r}<\frac{d-n}{2}\Longleftrightarrow\frac{d-n}{4\sigma+4}<\frac{1}{2\sigma},$$

which is again the case since

$$\sigma<\frac{2}{d-2}<\frac{2}{(d-n-2)_+}.$$

□

We note that (q_0,r) corresponds to the admissible pair appearing in Lemma 2.1.

2.4. Scattering. The interest of the specific choice for (p,q,r) appears in the following lemma.

LEMMA 2.6. *Let $u_0\in B_1$ and u be the corresponding solution of Cauchy problem (1.1) with $u(0)=u_0$. If u is global, $u\in C(\mathbb{R};B_1)\cap L_{\rm loc}^{\frac{4\sigma+4}{d\sigma}}(\mathbb{R};L^{2\sigma+2}(\mathbb{R}^d))$, and satisfies*

$$\|u\|_{\ell_\gamma^p L^q L^r}<\infty,$$

then the solution u scatters in B_1 as $t\to\pm\infty$.

Proof. We first show that $\|Au\|_{\ell_\gamma^{p_0}L^{q_0}L^r}<\infty$ for all $A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}$. As $Au\in L^{q_0}_{\rm loc}(\mathbb{R};L^r)$, we need to show that for $\gamma_0\gg 1$, $\|Au\|_{\ell^{p_0}_{\gamma\ge\gamma_0}L^{q_0}(I_\gamma,L^r)}<\infty$, the case of negative times being similar. We consider the integral equation

$$u(t)=e^{-i(t-\pi\gamma_0)H}u(\pi\gamma_0)-i\lambda\int_{\pi\gamma_0}^t e^{-i(t-s)H}(|u|^{2\sigma}u)(s)ds.$$

Notice the algebraic identities,

$$\frac{1}{p_0'}=\frac{1}{p_0}+\frac{2\sigma}{p},\quad \frac{1}{q_0'}=\frac{1}{q_0}+\frac{2\sigma}{q}.$$

For $\gamma_1>\gamma_0>0$, Strichartz estimate (Lemma 2.2) and Hölder inequality yield

$$\begin{aligned}\|Au\|_{\ell^{p_0}_{\gamma_0\le\gamma\le\gamma_1}L^{q_0}L^r}&\lesssim\|Ae^{-itH}u_0\|_{\ell^{p_0}_{\gamma_0\le\gamma\le\gamma_1}L^{q_0}L^r}+\||u|^{2\sigma}Au\|_{\ell^{p_0'}_{\gamma_0\le\gamma\le\gamma_1}L^{q_0'}L^{r'}}\\&\lesssim\|Au_0\|_{L^2}+\|u\|^{2\sigma}_{\ell^p_{\gamma_0\le\gamma\le\gamma_1}L^qL^r}\|Au\|_{\ell^{p_0}_{\gamma_0\le\gamma\le\gamma_1}L^{q_0}L^r}.\end{aligned}$$

For $\gamma_0\gg 1$ so that $\|u\|_{\ell^p_{\gamma\ge\gamma_0}L^qL^r}$ is sufficiently small, a bootstrap argument yields

$$\|Au\|_{\ell^{p_0}_{\gamma_0\le\gamma\le\gamma_1}L^{q_0}L^r}\lesssim\|Au_0\|_{L^2}\lesssim\|u_0\|_{B_1},$$

uniformly in $\gamma_1>\gamma_0$, hence $Au\in\ell^{p_0}_\gamma L^{q_0}L^r$.

Using Strichartz estimates again, we have, for $t_2>t_1>0$,

$$\begin{aligned}\|A(t_2)u(t_2)-A(t_1)u(t_1)\|_{L^2}&=\left\|\int_{t_1}^{t_2}e^{isH}A(s)(|u|^{2\sigma}u)(s)ds\right\|_{L^2}\\&\lesssim\left\|A\left(|u|^{2\sigma}u\right)\right\|_{\ell^{p_0'}_{\gamma\gtrsim t_1}L^{q_0'}L^{r'}}\\&\lesssim\|u\|^{2\sigma}_{\ell^p_{\gamma\gtrsim t_1}L^qL^r}\|Au\|_{\ell^{p_0}_{\gamma\gtrsim t_1}L^{q_0}L^r}\underset{t_1\to\infty}{\longrightarrow}0,\end{aligned}$$

and so, in view of (2.1), $e^{itH}u(t)$ converges strongly in B_1 as $t\to\infty$. ∎

With Duhamel's formula in mind, we show that the homogeneous part always belongs to the scattering space considered in Lemma 2.6.

LEMMA 2.7. *Let $\psi\in B_1$. Then*

$$\|e^{-itH}\psi\|_{\ell^p_\gamma L^qL^r}\lesssim\|\psi\|_{B_1}.\tag{2.4}$$

Proof. We recall some details of the proof of [2, Theorem 3.4]. Consider a partition of unity

$$\sum_{\gamma\in\mathbb{Z}}\chi(t-\pi\gamma)=1,\quad\forall t\in\mathbb{R}\qquad\text{with}\quad\operatorname{supp}\chi\subset[-\pi,\pi].$$

Lemma 2.2 is actually proven by considering

$$\|\psi\|^p_{\ell^p_\gamma L^q L^r} = \sum_{\gamma\in\mathbb{Z}} \|\chi(\cdot-\gamma\pi)\psi\|^p_{L^q(\mathbb{R};L^r(\mathbb{R}^d))}.$$

By Sobolev embedding,

$$\|\chi(\cdot-\gamma\pi)e^{-itH}\psi\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}\psi\|_{W^{s,k}(\mathbb{R};L^r(\mathbb{R}^d))}, \quad \frac{1}{q}=\frac{1}{k}-s.$$

We note the relations

$$\frac{2}{p_0}=(d-n)\left(\frac{1}{2}-\frac{1}{r}\right), \quad \frac{2}{p}=(d-n)\left(\frac{1}{2}-\frac{1}{r}\right)-\left(\frac{d-n}{2}-\frac{1}{\sigma}\right),$$

hence $p\geq p_0$ since $\sigma\geq \frac{2}{d-n}$. Therefore,

$$\|e^{-itH}\psi\|_{\ell^p_\gamma L^q L^r} \lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}\psi\|_{\ell^{p_0}_\gamma W^{s,k}(\mathbb{R};L^r(\mathbb{R}^d))}. \tag{2.5}$$

If we set $k=q_0$ (in order to recover our initial triplet), we find

$$\frac{1}{q}=\frac{1}{2\sigma}-\frac{d}{4\sigma+4}=\underbrace{\frac{d}{2}\left(\frac{1}{2}-\frac{1}{2\sigma+2}\right)}_{=1/q_0}-s, \quad \text{hence} \quad s:=\frac{1}{2}\left(\frac{d}{2}-\frac{1}{\sigma}\right).$$

Using

$$\begin{aligned}\|\chi(\cdot-\gamma\pi)e^{-itH}\psi\|_{W^{s,q_0}(\mathbb{R};L^r(\mathbb{R}^d))} &\lesssim \|H^s\chi(\cdot-\gamma\pi)e^{-itH}\psi\|_{L^{q_0}(\mathbb{R};L^r(\mathbb{R}^d))}\\ &\lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}H^s\psi\|_{L^{q_0}(\mathbb{R};L^r(\mathbb{R}^d))},\end{aligned}$$

the homogeneous Strichartz estimate yields

$$\|e^{-itH}\psi\|_{\ell^p_\gamma L^q L^r} \lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}H^s\psi\|_{\ell^{p_0}_\gamma L^{q_0}(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|\psi\|_{B_{2s}} \lesssim \|\psi\|_{B_1},$$

since $0<s<\frac{1}{2}$, as $\frac{2}{d}<\frac{2}{d-n}\leq\sigma<\frac{2}{d-2}$. □

3. Variational estimates

From now on, we assume $\lambda=-1$.

We define on B_1 the Nehari functional

$$I(u)=\|\nabla_x u\|^2_{L^2}+\|yu\|^2_{L^2}+\|u\|^2_{L^2}-\|u\|^{2\sigma+2}_{L^{\sigma+2}}.$$

In this section we show that the set of ground states is not empty. Moreover, we prove that $I(u)$ and $P(u)$ have the same sign under the condition $S(Q)<S(u)$, which plays a vital role in the proof of Theorem 1.1. Here Q is a ground state. To prove this, we introduce the scaling quantity $\varphi^{a,b}_\lambda$ by

$$\varphi^{a,b}_\lambda(x)=e^{a\lambda}\varphi(y,e^{-b\lambda}z), \quad x=(y,z)\in\mathbb{R}^n\times\mathbb{R}^{d-n}, \tag{3.1}$$

where (a,b) satisfies the following conditions

$$a>0, \quad b\leq 0, \quad 2a+b(d-n)\geq 0, \quad \sigma a+b>0, \quad (a,b)\neq(0,0). \tag{3.2}$$

A simple calculation shows that

$$\|\nabla_y\varphi_\lambda^{a,b}\|_{L^2}^2=e^{\lambda(2a+b(d-n))}\|\nabla_y\varphi\|_{L^2}^2,\quad \|\nabla_z\varphi_\lambda^{a,b}\|_{L^2}^2=e^{\lambda(2a+b(d-n-2))}\|\nabla_z\varphi\|_{L^2}^2,$$
$$\|\varphi_\lambda^{a,b}\|_{L^2}^2=e^{\lambda(2a+b(d-n))}\|\varphi\|_{L^2}^2,\quad \|\varphi_\lambda^{a,b}\|_{L^{2\sigma+2}}^{2\sigma+2}=e^{\lambda(a(2\sigma+2)+b(d-n))}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2},$$
$$\|y\varphi_\lambda^{a,b}\|_{L^2}^2=e^{\lambda(2a+b(d-n))}\|y\varphi\|_{L^2}^2.$$

We define the functionals $J^{a,b}$ by

$$\begin{aligned}J^{a,b}(\varphi)&=\partial_\lambda S(\varphi_\lambda^{a,b})\Big|_{\lambda=0}\\&=\frac{2a+b(d-n)}{2}\|\nabla_y\varphi\|_{L^2}^2+\frac{2a+b(d-n-2)}{2}\|\nabla_z\varphi\|_{L^2}^2\\&\quad+\frac{2a+b(d-n)}{2}\|y\varphi\|_{L^2}^2\\&\quad+\frac{2a+b(d-n)}{2}\|\varphi\|_{L^2}^2-\frac{a(2\sigma+2)+b(d-n)}{2\sigma+2}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}.\end{aligned}$$

In particular, when $(a,b)=(1,0)$ and $(a,b)=(1,-2/(d-n))$ we obtain the functionals I and P respectively. In the next result, we see that $J^{a,b}$ is positive near the origin in the space B_1.

As a technical preliminary, denote

$$\|u\|_{\dot B_1}^2=\|\nabla_x u\|_{L^2}^2+\|yu\|_{L^2}^2$$

the homogeneous counterpart of the B_1-norm. From the uncertainty principle in y, and Cauchy-Schwarz inequality in z,

$$\|\varphi\|_{L^2}^2\le\frac{2}{n}\|y\varphi\|_{L^2}\|\nabla_y\varphi\|_{L^2}.$$

In particular, $\|u\|_{B_1}\sim\|u\|_{\dot B_1}$.

LEMMA 3.1. *Let (a,b) satisfy (3.2), with in addition $2a+b(d-n)>0$. Let $\{v_k\}_{k=1}^\infty\subset B_1\setminus\{0\}$ be bounded in B_1 such that $\lim_{k\to\infty}\|v_k\|_{\dot B_1}=0$. Then for sufficiently large k, we have $J^{a,b}(v_k)>0$.*

Proof. Gagliardo-Nirenberg inequality yields

$$\begin{aligned}J^{a,b}(v_k)&\ge\frac{2a+b(d-n)}{2}\|v_k\|_{\dot B_1}^2-\frac{a(2\sigma+2)+b(d-n)}{2\sigma+2}\|v_k\|_{L^{2\sigma+2}}^{2\sigma+2}\\&\ge\frac{2a+b(d-n)}{2}\|v_k\|_{\dot B_1}^2-\frac{a(2\sigma+2)+b(d-n)}{2\sigma+2}C\|v_k\|_{\dot B_1}^{2\sigma+2},\end{aligned}$$

where C is a positive constant. Since $2a+b(d-n)>0$, we infer that for sufficiently large k, $J^{a,b}(v_k)>0$. This proves the lemma. □

Next, we consider the minimization problem

$$d^{a,b}:=\inf\left\{S(u):u\in B_1\setminus\{0\},J^{a,b}(u)=0\right\},\tag{3.3}$$
$$U^{a,b}=\left\{\varphi\in B_1:S(\varphi)=d^{a,b}\quad\text{and}\quad J^{a,b}(u)=0\right\}.\tag{3.4}$$

Lemma 3.2. *Let (a,b) satisfy (3.2), with in addition $2a+b(d-n)>0$. Then the set $U^{a,b}$ is not empty. That is, there exists $Q\in B_1$ such that $S(Q)=d^{a,b}$ and $J^{a,b}(Q)=0$.*

Proof. We introduce the functional

$$B^{a,b}(u)=S(u)-\frac{1}{a(2\sigma+2)+b(d-n)}J^{a,b}(u)$$
$$=\alpha_1\|\nabla_y u\|_{L^2}^2+\alpha_2\|\nabla_z u\|_{L^2}^2+\alpha_1\|yu\|_{L^2}^2+\alpha_1\|u\|_{L^2}^2, \tag{3.5}$$

where

$$\alpha_1:=\frac{1}{2}\left(1-\frac{2a+b(d-n)}{a(2\sigma+2)+b(d-n)}\right)>0, \quad \alpha_2:=\frac{1}{2}\left(1-\frac{2a+b(d-n-2)}{a(2\sigma+2)+b(d-n)}\right)>0.$$

To claim that $\alpha_2>0$, we have used $\sigma a+b>0$. From (3.5), it is clear that there exist constants C_1, $C_2>0$ such that for all $u\in B_1$,

$$C_1\|u\|_{B_1}^2\le B^{a,b}(u)\le C_2\|u\|_{B_1}^2. \tag{3.6}$$

Notice that

$$d^{a,b}=\inf\left\{B^{a,b}(u):u\in B_1\setminus\{0\},J^{a,b}(u)=0\right\}. \tag{3.7}$$

Step 1. We claim that $d^{a,b}>0$. Indeed, let $u\neq 0$ such that $J^{a,b}(u)=0$. Then we have, in view of (3.2) and since $2a+b(d-n)>0$,

$$\|u\|_{B_1}^2\lesssim\|u\|_{L^{2\sigma+2}}^{2\sigma+2}\lesssim\|u\|_{B_1}^{2\sigma+2},$$

where we have used Gagliardo-Nirenberg inequality and the uncertainty principle like in the previous proof. This implies $\|u\|_{B_1}\gtrsim 1$, hence $B^{a,b}(u)\gtrsim 1$ from (3.6).

Step 2. If $u\in B_1$ satisfies $J^{a,b}(u)<0$, then $d^{a,b}<B^{a,b}(u)$. Indeed, as $J^{a,b}(u)<0$, a simple calculation shows that there exists $\lambda\in(0,1)$ such that $J^{a,b}(\lambda u)=0$. Thus, by definition of $d^{a,b}$, we obtain

$$d^{a,b}\le B^{a,b}(\lambda u)=\lambda^2 B^{a,b}(u)<B^{a,b}(u).$$

Step 3. We will need the following result that was proved in [4, Lemma 3.4] (see also [34]): Assume that the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in B_1 and satisfies

$$\limsup_{k\to\infty}\|u_k\|_{L^{2\sigma+2}}^{2\sigma+2}\ge C>0.$$

Then, there exists a sequence $\{z_k\}_{k=1}^{\infty}\subset\mathbb{R}^{d-n}$ and $u\neq 0$ such that, passing to a subsequence if necessary

$$\tau_{z_k}u_k(y,z):=u_k(y,z-z_k)\rightharpoonup u \quad \text{weakly in } B_1.$$

Step 4. We claim that $U^{a,b}$ is not empty. Let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence of $d^{a,b}$. Since $B^{a,b}(u_k)\to d^{a,b}$ as k goes to ∞, by (3.6) we infer that the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in B_1. Moreover, as $J^{a,b}(u_k)=0$ we have

$$\|u_k\|_{L^{2\sigma+2}}^{2\sigma+2}\gtrsim\|u_k\|_{B_1}^2\gtrsim B^{a,b}(u_k)\to d^{a,b}>0,$$

as $k\to\infty$. Therefore, $\limsup_{k\to\infty}\|u_k\|_{L^{2\sigma+2}}^{2\sigma+2}\ge C>0$. Thus, by Step 3 there exists a sequence $\{z_k\}\subset\mathbb{R}^{d-n}$ and $u\neq 0$ such that $\tau_{z_k}u_k\rightharpoonup u$ weakly in B_1. We set $v_k(x):=$

$\tau_{z_k} u_k(x)$. Now, we prove that $J^{a,b}(u)=0$. Suppose that $J^{a,b}(u)<0$. By the weakly lower semicontinuity of $B^{a,b}$ and Step 2 we see that

$$d^{a,b} < B^{a,b}(u) \le \liminf_{k\to\infty} B^{a,b}(u_k) = d^{a,b},$$

which is impossible. Now we assume that $J^{a,b}(u)>0$. From Brezis-Lieb lemma we get

$$\lim_{n\to\infty} J^{a,b}(u_n-u) = \lim_{n\to\infty} \left\{J^{a,b}(u_n) - J^{a,b}(u)\right\} = -J^{a,b}(u) < 0.$$

This implies that $J^{a,b}(u_n-u)<0$ for sufficiently large n. Thus, applying the same argument as above, we see that

$$d^{a,b} \le \lim_{n\to\infty} B^{a,b}(u_n-u) = \lim_{n\to\infty} \left\{B^{a,b}(u_n) - B^{a,b}(u)\right\} = d^{a,b} - B^{a,b}(u) < d^{a,b},$$

because $B^{a,b}(u)>0$. Therefore $J^{a,b}(u)=0$ and

$$d^{a,b} \le S(u) = B^{a,b}(u) \le \liminf_{n\to\infty} B^{a,b}(u_n) = d^{a,b}.$$

In particular, $S(u)=d^{a,b}$ and $u\in U^{a,b}$. This concludes the proof of lemma. ∎

Remark 3.1. Lemma 3.2 shows that the set of ground states is not empty. Indeed, in the case $(a,b)=(1,0)$, from Lemma 3.2 we have that there exists $Q\in B_1$ such that $S(Q)=\inf\{S(\varphi): I(\varphi)=0\}$. This implies that (see [11, Chapter 8])

$$S(Q) = \inf\{S(\varphi):\ \varphi \text{ is a solution of (1.4)}\}.$$

Now we define the mountain pass level β by setting

$$\beta := \inf_{\sigma\in\Gamma} \max_{s\in[0,1]} S(\sigma(s)), \tag{3.8}$$

where Γ is the set

$$\Gamma := \left\{\sigma \in C([0,1];B_1) : \sigma(0)=0, S(\sigma(1))<0\right\}.$$

Lemma 3.3. *Let (a,b) satisfy (3.2), with in addition $2a+b(d-n)>0$. We have the following properties.*

(i) The functional S has a mountain pass geometry, that is $\Gamma\neq\emptyset$ and $\beta>0$.

(ii) The identity $\beta=d^{a,b}$ holds. In particular, if Q is a ground state, then $S(Q)=\beta$.

Proof.
(i) Let $v\in B_1\setminus\{0\}$. For $s>0$ we obtain

$$S(sv) = s^2\|v\|_{B_1}^2 - \frac{s^{2\sigma+2}}{2\sigma+2}\|v\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

Let $L>0$ such that $S(Lv)<0$. We define $\sigma(s):=Lsv$. Then $\sigma\in C([0,1];B_1)$, $\sigma(0)=0$ and $S(\sigma(1))<0$; this implies that Γ is nonempty. On the other hand, notice that, by the embedding of $B_1\hookrightarrow L^{2\sigma+2}$ we have

$$S(v) \ge \frac{1}{2}\|v\|_{B_1}^2 - \frac{C}{2\sigma+2}\|v\|_{B_1}^{2\sigma+2}.$$

Taking $\varepsilon>0$ small enough we have

$$\delta:=\frac{1}{2}\varepsilon^2-\frac{C}{2\sigma+2}\varepsilon^{2\sigma+2}>0.$$

Thus, if $\|v\|_{B_1}^2<\varepsilon$, then $S(v)>0$. Therefore, for any $\sigma\in\Gamma$ we have $\|\sigma(1)\|_{B_1}^2>\varepsilon$, and by continuity of σ, there exists $s_0\in[0,1]$ such that $\sigma(s_0)=\varepsilon$. This implies that

$$\max_{s\in[0,1]} S(\sigma(s))\geq S(\sigma(s_0))\geq\delta>0.$$

By definition of β, we see that $\beta\geq\delta>0$.

(ii) Let $\sigma\in\Gamma$. Since $\sigma(0)=0$, by Lemma 3.1 we infer that there exists $s_0>0$ such that $J^{a,b}(\sigma(s_0))>0$. Also we note that from (3.5) we have

$$\begin{aligned} J^{a,b}(\sigma(1))&=(a(2\sigma+2)+b(d-n))\left\{S(\sigma(1))-B^{a,b}(\sigma(1))\right\}\\ &<(a(2\sigma+2)+b(d-n))S(\sigma(1))<0. \end{aligned}$$

By continuity of $s\mapsto J^{a,b}(\sigma(s))$, we infer that there exists $s^*\in(0,1)$ such that $J^{a,b}(\sigma(s^*))=0$. This implies that

$$\max_{s\in[0,1]} S(\sigma(s))\geq S(\sigma(s^*))\geq d^{a,b}.$$

Taking the infimum on Γ, we obtain $\beta\geq d^{a,b}$. Now we prove $\beta\leq d^{a,b}$. Let $\varphi\in B_1\setminus\{0\}$ be such that $J^{a,b}(\varphi)=0$. We put $f(s):=\varphi_s^{a,b}(y,z)$ for $s\in\mathbb{R}$, where $\varphi_s^{a,b}$ is defined in (3.1). Notice that as $a\sigma+b>0$, it follows that $S(f(s))<0$ for sufficiently large $s>0$. Since $\partial_s S(f(s))|_{s=0}=J^{a,b}(\varphi)=0$, it follows that $\max_{s\in\mathbb{R}} S(f(s))=S(f(0))=S(\varphi)$. Let $L>0$ be such that $S(f(L))<0$. We define

$$h(s):=\begin{cases} f(s) & \text{if } -\frac{L}{2}\leq s\leq L,\\ \frac{2}{L}(s+L)f(-\frac{L}{2}) & \text{if } -L\leq s\leq-\frac{L}{2}.\end{cases}$$

Then $s\mapsto h(s)$ is continuous in B_1, $S(h(L))<0$, $S(h(-L))=0$ and

$$\max_{s\in[-L,L]} S(h(s))=S(h(0))=S(\varphi).$$

By changing variables, we infer that there exists $\sigma\in\Gamma$ such that $\max_{s\in[0,1]} S(\sigma(s))=S(\varphi)$. Thus,

$$\beta\leq\max_{s\in[0,1]} S(\sigma(s))=S(\varphi)$$

for all $\varphi\in B_1\setminus\{0\}$ such that $J^{a,b}(\varphi)=0$. This implies that $\beta\leq d^{a,b}$. □

Now we introduce the sets $\mathcal{K}^{a,b,\pm}$ defined by

$$\begin{aligned} \mathcal{K}^{a,b,+}&=\left\{\varphi\in B_1: S(\varphi)<\beta,\quad J^{a,b}(\varphi)\geq0\right\},\\ \mathcal{K}^{a,b,-}&=\left\{\varphi\in B_1: S(\varphi)<\beta,\quad J^{a,b}(\varphi)<0\right\}. \end{aligned}$$

Lemma 3.4. *The sets $\mathcal{K}^{a,b,\pm}$ are independent of (a,b) satisfying (3.2).*

Proof. Suppose first that in addition to (3.2), we have $2a+b(d-n)>0$.

It is clear that $\mathcal{K}^{a,b,-}$ is open in B_1. Now we prove that $\mathcal{K}^{a,b,+}$ is open. First, notice that by Lemma 3.2, if $S(\varphi)<\beta$ and $J^{a,b}(\varphi)=0$ then $\varphi=0$. Moreover, using the fact that a neighborhood of 0 is contained in $\mathcal{K}^{a,b,+}$ by Lemma 3.1, this implies that $\mathcal{K}^{a,b,+}$ is open in B_1. On the other hand, since $2a+b(d-n)>0$ (notice that this implies that $\|\varphi_\lambda^{a,b}\|_{B_1}\to 0$ as $\lambda\to-\infty$), using the same argument developed in the proof of [27, Lemma 2.9] it is not difficult to show that $\mathcal{K}^{a,b,+}$ is connected. Thus, since $0\in\mathcal{K}^{a,b,+}$ and $\mathcal{K}^{a,b,+}\cup\mathcal{K}^{a,b,-}$ is independent of (a,b) (see Lemma 3.3 (ii)), we infer that $\mathcal{K}^{a,b,+}=\mathcal{K}^{a',b',+}$ for $(a,b)\neq(a',b')$ such that $2a+b(d-n)>0$ and $2a'+b'(d-n)>0$. In particular we have $\mathcal{K}^{a,b,-}=\mathcal{K}^{a',b',-}$.

Now assume that $2a+b(d-n)=0$. We choose a sequence $\{(a_j,b_j)\}_{j=1}^\infty$ such that (a_j,b_j) satisfies (3.2), converges to (a,b), and $2a_j+b_j(d-n)>0$ for all j. Then $J^{a_j,b_j}\to J^{a,b}$ and we have

$$\mathcal{K}^{a,b,\pm}\subset\bigcup_{j\geq 1}\mathcal{K}^{a_j,b_j,\pm}.$$

By using the fact that the right side is independent of the parameter, so is the left, which finishes the proof. □

The following remark will be used in the sequel.

REMARK 3.2. If $\varphi\neq 0$ satisfies $P(\varphi)=0$, then $S(\varphi)\geq\beta$. Indeed, we put $\varphi^r(x):=r^{\frac{d-n}{2}}\varphi(y,rz)$ for $r>0$. Then

$$I(\varphi^r)=r^2\|\nabla_z\varphi\|_{L^2}^2-r^{\sigma(d-n)}\|\varphi\|_{2\sigma+2}^{2\sigma+2}+K_\varphi,$$

where

$$K_\varphi=\|\nabla_y\varphi\|_{L^2}^2+\|\varphi\|_{L^2}^2+\|y\varphi\|_{L^2}^2>0.$$

From $P(\varphi)=0$, we see that

$$I(\varphi^r)=\left(\frac{(d-n)\sigma}{2(\sigma+1)}r^2-r^{\sigma(d-n)}\right)\|\varphi\|_{2\sigma+2}^{2\sigma+2}+K_\varphi.$$

Since $\sigma(d-n)>2$, there exists $r_0\in(0,\infty)$ such that $I(\varphi^{r_0})=0$. This implies that $S(\varphi^{r_0})\geq\beta$. Moreover, since $\sigma(d-n)>2$ and $\partial_r S(\varphi^r)|_{r=1}=\left(\frac{d-n}{2}\right)P(\varphi)=0$, it is not difficult to show that the function $r\mapsto S(\varphi^r)$, $r\in(0,\infty)$, attains its maximum at $r=1$. Therefore,

$$S(\varphi)\geq S(\varphi^{r_0})\geq\beta.$$

The next two lemmas will play an important role to get blow-up and global existence results.

LEMMA 3.5. *Let $\varphi\in\mathcal{K}^+$, then*

$$\frac{\sigma}{\sigma+1}\|\varphi\|_{B_1}^2\leq S(\varphi)\leq\frac{1}{2}\|\varphi\|_{B_1}^2$$

Proof. From Lemma 3.4 we see that $I(\varphi)$ and $P(\varphi)$ have the same sign under the condition $S(\varphi)<\beta$. Since $\varphi\in\mathcal{K}^+$, we obtain $I(\varphi)\geq 0$, which implies that

$$\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}\leq\|\varphi\|_{B_1}^2.$$

Therefore,

$$\frac{1}{2}\|\varphi\|_{B_1}^2 \geq S(\varphi) = \frac{1}{2}\|\varphi\|_{B_1}^2 - \frac{1}{2\sigma+2}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2} \geq \frac{\sigma}{\sigma+1}\|\varphi\|_{B_1}^2,$$

and the proof is complete. ∎

LEMMA 3.6. *If $\varphi \in \mathcal{K}^-$, then*

$$P(\varphi) \leq -\frac{4}{d-n}(\beta - S(\varphi)).$$

Proof. We consider $\varphi \in \mathcal{K}^-$. We put $s(\lambda) := S(\varphi_\lambda^{1,-2/(d-n)})$ (see (3.1)). Then

$$s(\lambda) = \frac{1}{2}\|\nabla_y \varphi\|_{L^2}^2 + \frac{e^{4\lambda/(d-n)}}{2}\|\nabla_z \varphi\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2 + \frac{1}{2}\|y\varphi\|_{L^2}^2 - \frac{e^{2\sigma\lambda}}{2\sigma+2}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2},$$

$$s'(\lambda) = \frac{2}{d-n}e^{4\lambda/(d-n)}\|\nabla_z \varphi\|_{L^2}^2 - \frac{\sigma}{\sigma+1}e^{2\sigma\lambda}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}, \tag{3.9}$$

$$s''(\lambda) = \frac{8}{(d-n)^2}e^{4\lambda/(d-n)}\|\nabla_z \varphi\|_{L^2}^2 - \frac{2\sigma^2}{\sigma+1}e^{2\sigma\lambda}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{3.10}$$

Thus, we infer

$$s''(\lambda) = \frac{2\sigma}{\sigma+1}\left(\frac{2}{d-n} - \sigma\right)e^{2\sigma\lambda}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2} + \frac{4}{d-n}s'(\lambda) \leq \frac{4}{d-n}s'(\lambda), \tag{3.11}$$

where we have used that $\sigma > 2/(d-n)$. Since $P(\varphi) < 0$ and $s'(\lambda) > 0$ for small $\lambda < 0$, then by continuity, there exists $\lambda_0 < 0$ such that $s'(\lambda) < 0$ for any $\lambda \in (\lambda_0, 0]$ and $s'(\lambda_0) = 0$. Since $s(\lambda_0) \geq \beta$ (see Remark 3.2), integrating (3.11) over $(\lambda_0, 0]$, we obtain

$$P(\varphi) = s'(0) = s'(0) - s'(\lambda_0) \leq \frac{4}{d-n}(s(0) - s(\lambda_0)) \leq \frac{4}{d-n}(S(\varphi) - \beta),$$

hence the result. ∎

4. Criteria for Global well-posedness and blow-up

In this section we prove our global well-posedness and blow-up result, that is, Theorem 1.1 up to the scattering part.

Proof. **(Proof of Theorem 1.1.)**
(i) Let $u_0 \in \mathcal{K}^+$. Since the energy and the mass are conserved, we have

$$u(t) \in \mathcal{K}^+ \cup \mathcal{K}^-, \quad \text{for every } t \text{ in the existence interval.} \tag{4.1}$$

Here $u(t)$ is the corresponding solution of (1.1) with $u(0) = u_0$. Assume that there exists $t_0 > 0$ such that $u(t_0) \in \mathcal{K}^-$. Since the map $t \mapsto P(u(t))$ is continuous, there exists $t_1 \in (0, t_0)$ such that $P(u(t)) < 0$ for all $t \in (t_1, t_0)$ and $P(u(t_1)) = 0$. Thus, by Remark 3.2 we see that if $u(t_1) \neq 0$, then $S(u(t_1)) \geq \beta$. However, by (4.1) we have $S(u(t_1)) < \beta$, which is absurd. Therefore, $u(t) \in \mathcal{K}^+$ for every t in the existence interval. Now, by Lemma 3.5 we obtain that $\|u(t)\|_{B_1} \sim S(u(t)) < \beta$ for every t. By the local theory (Lemma 2.1), this implies that u is global and $u(t) \in \mathcal{K}^+$ for every $t \in \mathbb{R}$. The scattering result will be shown in Section 5.

(ii) Similarly as above, we can show that if $u_0 \in \mathcal{K}^-$, then $u(t) \in \mathcal{K}^-$ for every t in the interval $[0, T_+)$. If $T_+ < +\infty$, by the local theory (Lemma 2.1), we have

$\lim_{t\to T_+}\|\nabla_x u(t)\|_{L^2}^2=+\infty$. On the other hand, if $T_+=+\infty$ we prove that there exists $t_k\to\infty$ such that $\lim_{t_k\to\infty}\|\nabla_x u(t_k)\|_{L^2}^2=+\infty$ by contradiction: suppose

$$k_0:=\sup_{t\geq 0}\|\nabla_x u(t)\|_{L^2}<+\infty.$$

Now we consider the localized virial identity and define

$$V(t):=\int_{\mathbb{R}^d}\phi(z)|u(t,x)|^2dx,\quad x=(y,z)\in\mathbb{R}^n\times\mathbb{R}^{d-n}. \tag{4.2}$$

Let $\phi\in C^4(\mathbb{R}^{d-n})$. If ϕ is a radial function (that is, $\phi(z)=\phi(|z|)$), by direct computations we have

$$V'(t)=2\mathrm{Im}\int_{\mathbb{R}^d}\nabla_z\phi\cdot\nabla_z u\bar{u}, \tag{4.3}$$

$$V''(t)=4\int_{\mathbb{R}^d}\mathrm{Re}\left\langle\nabla_z\bar{u},\nabla_z^2\phi\nabla_z u\right\rangle-\frac{2\sigma}{\sigma+1}\int_{\mathbb{R}^d}\Delta_z\phi|u|^{2\sigma+2}-\int_{\mathbb{R}^d}\Delta_z^2\phi|u|^2. \tag{4.4}$$

Before continuing the proof of Theorem 1.1 we first state the following result:

Lemma 4.1. *Let $\eta>0$. Then for all $t\leq\eta R/(4k_0\|u_0\|_{L^2})$ we have*

$$\int_{|z|\geq R}|u(t,x)|^2dx\leq\eta+o_R(1). \tag{4.5}$$

Proof. Fix $R>0$, and take ϕ in (4.2) such that

$$\phi(r)=\begin{cases}0, & 0\leq|z|\leq\frac{R}{2};\\ 1, & |z|\geq R,\end{cases}$$

where $r=|z|$ and

$$0\leq\phi\leq 1,\quad 0\leq\phi'\leq\frac{4}{R}.$$

From (4.3) we infer that

$$\begin{aligned}V(t)&=V(0)+\int_0^t V'(s)ds\leq V(0)+t\|\phi'\|_{L^\infty}\|u_0\|_{L^2}k_0\\&\leq\int_{|z|\geq R/2}|u_0(x)|^2dx+\frac{4\|u_0\|_{L^2}k_0}{R}t.\end{aligned}$$

Moreover, Lebesgue's dominated convergence theorem yields

$$\int_{|z|\geq R/2}|u_0(x)|^2dx=o_R(1),$$

and

$$\int_{|z|\geq R}|u(t,x)|^2dx\leq V(t).$$

Therefore for given $\eta>0$, if

$$t\leq\frac{\eta R}{4k_0\|u_0\|_{L^2}},$$

then we see that

$$\int_{|z|\geq R} |u(t,x)|^2 dx \leq \eta + o_R(1).$$

This concludes the proof of the lemma. ∎

Next we choose another function ϕ in (4.2) such that

$$\phi(r) = \begin{cases} r^2, & 0 \leq r \leq R; \\ 0, & r \geq 2R, \end{cases}$$

with

$$0 \leq \phi \leq r^2, \quad \phi'' \leq 2, \quad \phi^{(4)} \leq \frac{4}{R^2}.$$

By (4.3), $V'(t)$ and $V''(t)$ can be rewritten as

$$V'(t) = 2\mathrm{Im} \int_{\mathbb{R}^d} \frac{\phi'(r)}{r} z \cdot \nabla_z u \bar{u}, \tag{4.6}$$

$$\begin{aligned} V''(t) =& 4\int_{\mathbb{R}^d} \frac{\phi'}{r} |\nabla_z u|^2 + 4\int_{\mathbb{R}^d} \left(\frac{\phi''}{r^2} - \frac{\phi'}{r^3}\right) |z \cdot \nabla_z u|^2 \\ & - \frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^d} \left(\phi'' + (d-n-1)\frac{\phi'}{r}\right) |u|^{2\sigma+2} - \int_{\mathbb{R}^d} \Delta_z^2 \phi |u|^2 \qquad (4.7) \\ =& 4(d-n)P(u) + R_1 + R_2 + R_3, \qquad (4.8) \end{aligned}$$

where

$$\begin{aligned} R_1 &= 4\int_{\mathbb{R}^d} \left(\frac{\phi''}{r} - 2\right) |\nabla_z u|^2 + 4\int_{\mathbb{R}^d} \left(\frac{\phi''}{r^2} - \frac{\phi'}{r^3}\right) |z \cdot \nabla_z u|^2 \\ R_2 &= -\frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^d} \left(\phi'' + (d-n-1)\frac{\phi'}{r} - 2(d-n)\right) |u|^{2\sigma+2}, \qquad (4.9) \\ R_3 &= -\int_{\mathbb{R}^d} \Delta_z^2 \phi |u|^2. \end{aligned}$$

First we show that $R_1 \leq 0$. Indeed, we can decompose $\mathbb{R}^d$ into

$$\mathbb{R}^d = \underbrace{\{\phi''/r^2 - \phi'/r^3 \leq 0\}}_{=:\Omega_1} \cup \underbrace{\{\phi''/r^2 - \phi'/r^3 > 0\}}_{=:\Omega_2}.$$

On Ω_1, since $\phi' \leq 2r$,

$$4\int_{\Omega_1} \left(\frac{\phi''}{r} - 2\right) |\nabla_z u|^2 + 4\int_{\Omega_1} \left(\frac{\phi''}{r^2} - \frac{\phi'}{r^3}\right) |z \cdot \nabla_z u|^2 \leq 0.$$

On Ω_2,

$$\int_{\Omega_2} \left(\frac{\phi''}{r} - 2\right) |\nabla_z u|^2 + \int_{\Omega_2} \left(\frac{\phi''}{r^2} - \frac{\phi'}{r^3}\right) |z \cdot \nabla_z u|^2 \leq \int_{\mathbb{R}^d} (\phi'' - 2) |\nabla_z u|^2 dx \leq 0.$$

Secondly, notice that $\mathrm{supp}\chi \subset [R, \infty)$, where

$$\chi(r) = \left|\phi''(r) + (d-n-1)\frac{\phi'(r)}{r} - 2(d-n)\right|.$$

For $2\sigma+2<q<\frac{2d}{d-2}$, there exists $0<\theta<1$ such that $\frac{1}{2\sigma+2}=\frac{1-\theta}{q}+\frac{\theta}{2}$, and

$$R_2^{\frac{1}{2\sigma+2}} \lesssim \|u\|_{L^{2\sigma+2}(|z|>R)} \leq \|u\|_{L^q(|z|>R)}^{1-\theta}\|u\|_{L^2(|z|>R)}^{\theta} \lesssim k_0^{1-\theta}\|u\|_{L^2(|z|>R)}^{\theta}. \tag{4.10}$$

Finally,

$$R_3 \leq CR^{-2}\|u\|_{L^2(|z|>R)}^2. \tag{4.11}$$

Combining (4.8), (4.10) and (4.11) we obtain

$$V''(t) \leq 4(d-n)P(u(t)) + C\|u\|_{L^2(|z|>R)}^{(2\sigma+2)\theta} + CR^{-2}\|u\|_{L^2(|z|>R)}^2, \tag{4.12}$$

where $C>0$ depends only on $\|u_0\|_{L^2}$, k_0 and σ. By Lemma 4.1 we obtain that for all $t\leq T:=\eta R/(4k_0\|u_0\|_{L^2}^2)$,

$$V''(t) \leq 4(d-n)P(u(t)) + C\left(\eta^{(2\sigma+2)\theta}+\eta^2+o_R(1)\right),$$

and since $u(t)\in\mathcal{K}^-$, Lemma 3.6 yields $P(u(t))\leq -\frac{4}{d-n}(\beta-S(u_0))<0$. Thus,

$$V''(t) \leq -16(\beta-S(u_0)) + C\left(\eta^{(2\sigma+2)\theta}+\eta^2+o_R(1)\right). \tag{4.13}$$

Integrating (4.13) from 0 to T we infer

$$V(T) \leq V(0)+V'(0)T+\left(-16(\beta-S(u_0))+C\left(\eta^{(2\sigma+2)\theta}+\eta^2+o_R(1)\right)\right)T^2.$$

Choosing η sufficiently small and taking R large enough, it follows that for $T=\eta R/(4k_0\|u_0\|_{L^2})$ we have

$$-16(\beta-S(u_0))+C\left(\eta^{(2\sigma+2)\theta}+\eta^2+o_R(1)\right) < -8(\beta-S(u_0)),$$

and

$$V(T) \leq V(0)+V'(0)\frac{\eta R}{4k_0\|u_0\|_{L^2}}+\mu_0 R^2,$$

where

$$\mu_0 = -\frac{(\beta-S(u_0))\eta^2}{2k_0^2\|u_0\|_{L^2}^2} < 0.$$

Next notice that we have $V(0)\leq o_R(1)R^2$ and $V'(0)\leq o_R(1)R$. Indeed,

$$\begin{aligned} V(0) &\leq \int_{|z|<\sqrt{R}} |z|^2|u_0(x)|^2dx + \int_{\sqrt{R}<|z|<2R} |z|^2|u_0(x)|^2dx \\ &\leq R\|u_0\|_{L^2}^2 + 4R^2\int_{|z|>\sqrt{R}} |u_0(x)|^2dx \\ &= o_R(1)R^2. \end{aligned}$$

Moreover,

$$V'(0) \leq \int_{|z|<\sqrt{R}} |z||u_0||\nabla_z u_0|dx + \int_{\sqrt{R}<|z|<2R} |z||u_0||\nabla_z u_0|dx$$

$$\leq \sqrt{R}\|u_0\|_{H^1}^2 + 2R\int_{|z|>\sqrt{R}} |u_0||\nabla_z u_0|dx$$
$$= o_R(1)R.$$

Thus we get

$$V(T) \leq (o_R(1)+\mu_0)R^2,$$

and for R sufficiently large, $o_R(1)+\mu_0 < 0$, which is a contradiction since $V(T) > 0$. The proof of Theorem 1.1 is now complete. □

5. Proof of the scattering result

In Section 4 we showed that if $u_0 \in \mathcal{K}^+$, then the solution is global and belongs to $\mathcal{K}^+$ for all $t \in \mathbb{R}$. In this section we show that under this condition, the solution scatters in B_1.

5.1. Small data scattering. We begin with some lemmas complementing the results of Section 2.4. Recall that the indices considered here were introduced in Section 2.3. The first lemma covers both the Cauchy problem ($t_0 \in \mathbb{R}$) and the existence of wave operators ($|t_0| = \infty$).

Lemma 5.1 (Small data scattering). *Suppose $\frac{2}{d-n} \leq \sigma < \frac{2}{d-2}$, $\lambda \in \{-1,1\}$. Let $\varphi \in B_1$. There exists $\delta > 0$ such that if $\|e^{-itH}\varphi\|_{\ell_\gamma^p L^q L^r} \leq \delta$, then for all $t_0 \in [-\infty,\infty]$, the solution u to*

$$u(t) = e^{-itH}\varphi - i\lambda \int_{t_0}^{t} e^{-i(t-s)H}\left(|u|^{2\sigma}u\right)(s)ds \tag{5.1}$$

is global for both positive and negative times, and satisfies

$$\|u\|_{\ell_\gamma^p L^q L^r} \leq 2\|e^{-itH}\varphi\|_{\ell_\gamma^p L^q L^r}.$$

There exists $\nu > 0$ such that if $\|\varphi\|_{B_1} \leq \nu$, then $\|e^{-itH}\varphi\|_{\ell_\gamma^p L^q L^r} \leq \delta$, and for all $t_0 \in [-\infty,\infty]$, the solution u to (5.1) is global for both positive and negative times, and satisfies

$$\|u\|_{B_1} \leq 2\|\varphi\|_{B_1}.$$

Proof. Denote by

$$\Phi(u)(t) := e^{-itH}\varphi - i\lambda \int_{t_0}^{t} e^{-i(t-s)H}\left(|u|^{2\sigma}u\right)(s)ds.$$

First, consider

$$X = \Big\{u \in C(\mathbb{R};B_1);\ \|u\|_{\ell_\gamma^p L^q L^r} \leq 2\|e^{-itH}\varphi\|_{\ell_\gamma^p L^q L^r},$$
$$\forall A \in \{\mathrm{Id}, A_1, A_2, \nabla_z\},\ \|Au\|_{\ell_\gamma^{p_0} L^{q_0} L^{p_0}} \leq 2C_0\|A\varphi\|_{L^2}\Big\},$$

where C_0 is the constant associated to the homogeneous Strichartz estimate (2.2) ($F=0$) in the case $(p_1,q_1,r_1) = (p_0,q_0,r)$. Let $u \in X$. In view of the inhomogeneous Strichartz estimates (Lemmas 2.3 and 2.5), and since

$$p = (2\sigma+1)\tilde{p}', \quad q = (2\sigma+1)\tilde{q}', \quad r = (2\sigma+1)r',$$

we have

$$\begin{aligned}\|\Phi(u)\|_{\ell^p_\gamma L^q L^r} &\le \|e^{-itH}\varphi\|_{\ell^p_\gamma L^q L^r} + C\left\| |u|^{2\sigma}u\right\|_{\ell^{\tilde p'}_\gamma L^{\tilde q'} L^{r'}}\\ &\le \|e^{-itH}\varphi\|_{\ell^p_\gamma L^q L^r} + C\|u\|^{2\sigma+1}_{\ell^p_\gamma L^q L^r}.\end{aligned}$$

For $\delta>0$ sufficiently small, the right-hand side does not exceed 2δ.

Reproducing the estimates of the proof of Lemma 2.6, for $A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}$,

$$\|A\Phi(u)\|_{\ell^{p_0}_\gamma L^{q_0} L^{p_0}} \le C_0\|A\varphi\|_{L^2} + C_1\|u\|^{2\sigma}_{\ell^p_\gamma L^q L^r}\|Au\|_{\ell^{p_0}_\gamma L^{q_0} L^{p_0}}.$$

Up to choosing $\delta>0$ smaller, we infer

$$\|A\Phi(u)\|_{\ell^{p_0}_\gamma L^{q_0} L^{p_0}} \le 2C_0\|A\varphi\|_{L^2},$$

and so Φ maps X to itself. We equip X with the metric

$$d(u,v)=\|u-v\|_{\ell^p_\gamma L^q L^r}$$

which makes it a complete space (see e.g. [11]). We then have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{\ell^p_\gamma L^q L^r} &\lesssim \left\| |u|^{2\sigma}u-|v|^{2\sigma}v\right\|_{\ell^{\tilde p'}_\gamma L^{\tilde q'} L^{r'}}\\ &\lesssim \left\|\left(|u|^{2\sigma}+|v|^{2\sigma}\right)(u-v)\right\|_{\ell^{\tilde p'}_\gamma L^{\tilde q'} L^{r'}}\\ &\lesssim \left(\|u\|^{2\sigma}_{\ell^p_\gamma L^q L^r}+\|v\|^{2\sigma}_{\ell^p_\gamma L^q L^r}\right)\|u-v\|_{\ell^p_\gamma L^q L^r},\end{aligned}$$

so contraction follows, up to choosing $\delta>0$ smaller, hence the first part of the lemma.

For the second part, note that in view of Lemma 2.7, for $\nu>0$ sufficiently small, $\|e^{-itH}\varphi\|_{\ell^p_\gamma L^q L^r}\le\delta$, and we may use the first part of the lemma. Strichartz estimates also yield, for $A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}$,

$$\|Au\|_{L^\infty_t L^2} \le \|A\varphi\|_{L^2} + C_2\|u\|^{2\sigma}_{\ell^p_\gamma L^q L^r}\|Au\|_{\ell^{p_0}_\gamma L^{q_0} L^{p_0}}.$$

Up to choosing $\delta>0$ smaller, we infer

$$\|Au\|_{L^\infty_t L^2}\le 2\|A\varphi\|_{L^2},$$

hence the second part of the lemma, from (2.1). ∎

We now go back to the focusing case, $\lambda=-1$.

LEMMA 5.2 (Wave operators for not so small data). *Suppose $\frac{2}{d-n}\le\sigma<\frac{2}{d-2}$. Let $\psi\in B_1$ such that*

$$\frac12\|\nabla_x\psi\|^2_{L^2}+\frac12\|y\psi\|^2_{L^2}+\frac12\|\psi\|^2_{L^2}<\beta, \tag{5.2}$$

where β is given by (3.8). Then there exists $u_0\in\mathcal{K}^+$ such that the corresponding solution $u(t)$ of (1.1) with $u(0)=u_0$ satisfies

$$\|e^{itH}u(t)-\psi\|_{B_1}\underset{t\to\infty}{\longrightarrow}0.$$

Proof. Consider the integral equation

$$u(t)=e^{-itH}\psi-i\int_t^{+\infty}e^{-i(t-s)H}(|u|^{2\sigma}u)(s)ds=:\Phi(u)(t). \tag{5.3}$$

We first construct a solution defined on $[T,\infty)$ for $T\gg 1$ by a fixed-point argument similar to the one employed in the proof of Lemma 5.1. Introduce

$$X_T=\Big\{u\in C([\pi(T-1),\infty);B_1);\ \|u\|_{\ell^p_{\gamma\geq T}L^qL^r}\leq 2\|e^{-itH}\psi\|_{\ell^p_{\gamma\geq T}L^qL^r},\\ \sum_{A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}}\|Au\|_{\ell^{p_0}_{\gamma\geq T}L^{q_0}L^{p_0}}\leq 2C_0\|\psi\|_{B_1}\Big\},$$

where C_0 is the constant associated to the Strichartz estimate (2.2) in the case $(p_1,q_1,r_1)=(p_0,q_0,r)$. By Lemma 2.7, $\|e^{-itH}\psi\|_{\ell^p_{\gamma\geq T}L^qL^r}\to 0$ as $T\to\infty$. Therefore, choosing T sufficiently large is equivalent to requiring δ sufficiently small in the proof of Lemma 5.1. The proof is then the same, and we omit it. We must now prove that the solution u is defined for all time.

Since e^{-itH} conserves the linear energy and $\|e^{-itH}\psi\|^{2\sigma+2}_{L^{2\sigma+2}}\to 0$ as $t\to\infty$ (see Lemma 2.4), we have

$$S(u(t))=\lim_{t\to\infty}S(e^{-itH}\psi)=\frac{1}{2}\|\nabla\psi\|^2_{L^2}+\frac{1}{2}\|y\psi\|^2_{L^2}+\frac{1}{2}\|\psi\|^2_{L^2}<\beta,$$
$$\lim_{t\to\infty}I(u(t))=\lim_{t\to\infty}\left(\|e^{-itH}\psi\|^2_{B_1}-\|e^{-itH}\psi\|^{2\sigma+2}_{L^{2\sigma+2}}\right)=\|\psi\|^2_{B_1}>0.$$

Thus, there exists t^* sufficiently large such that $u(t^*)\in\mathcal{K}^+$. By using the fact that $\mathcal{K}^+$ is invariant by the flow of (1.1) we obtain that $u(0)=u_0\in\mathcal{K}^+$.

By Strichartz estimates, like in the proof of Lemma 2.6,

$$\begin{aligned}\|e^{itH}u(t)-\psi\|_{B_1}&\sim\sum_{A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}}\|A(t)u(t)-A(0)\psi\|_{L^2}\\&\lesssim\sum_{A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}}\left\|A(|u|^{2\sigma}u)\right\|_{\ell^{p_0'}_{\gamma\gtrsim t}L^{q_0'}L^{r'}}\\&\lesssim\sum_{A\in\{\mathrm{Id},A_1,A_2,\nabla_z\}}\|u\|^{2\sigma}_{\ell^p_{\gamma\gtrsim t}L^qL^r}\|Au\|_{\ell^{p_0}_{\gamma\gtrsim t}L^{q_0}L^r}\underset{t\to\infty}{\longrightarrow}0,\end{aligned}$$

hence the lemma. ∎

5.2. Perturbation lemma and linear profile decomposition. We begin with the following result

Lemma 5.3 (Perturbation lemma). *Suppose $\frac{2}{d-n}\leq\sigma<\frac{2}{d-2}$. Let $\tilde u\in C([0,\infty);B_1)$ be the solution of*

$$i\partial_t\tilde u-H\tilde u+|\tilde u|^{2\sigma}\tilde u=e, \tag{5.4}$$

where $e\in L^1_{\mathrm{loc}}([0,\infty);B_{-1})$. Given $A>0$, there exist $C(A)>0$ and $\varepsilon(A)>0$ such that if $u\in C([0,\infty);B_1)$ is a solution of (1.1), and if

$$\begin{aligned}&\|\tilde u\|_{\ell^p_\gamma L^qL^r}\leq A,\quad \|e\|_{\ell^{\tilde p'}_\gamma L^{\tilde q'}L^{r'}}\leq\varepsilon\leq\varepsilon(A),\\&\|e^{-itH}(u(0)-\tilde u(0))\|_{\ell^p_\gamma L^qL^r}\leq\varepsilon\leq\varepsilon(A),\end{aligned}\tag{5.5}$$

then $\|u\|_{\ell^p_\gamma L^qL^r}\leq C(A)<\infty$.

Proof. We omit the proof, which can be obtained by suitably adapting the argument of [16, Proposition 4.7], thanks to the same Strichartz estimates as in the proof of Lemma 5.1. ∎

We need the following linear profile decomposition, which is crucial in the construction of a minimal blow-up solution. This is where the assumption $\sigma \geq \frac{2}{d-n}$ becomes $\sigma > \frac{2}{d-n}$, in order to prove (5.13) below.

PROPOSITION 5.1 (Linear profile decomposition). *Suppose* $\frac{2}{d-n} < \sigma < \frac{2}{d-2}$. *Let* $\{\phi_k\}_{k=1}^{\infty}$ *be a uniformly bounded sequence in* B_1. *Then, up to subsequence, the following decomposition holds.*

$$\phi_k(x) = \sum_{j=1}^{M} e^{it_k^j H} \psi^j\left(y, z - z_k^j\right) + W_k^M(x) \quad \textit{for all } M \geq 1,$$

where $t_k^j \in \mathbb{R}$, $z_k^j \in \mathbb{R}^{d-n}$, $\psi^j \in B_1$ *are such that:*

- *Orthogonality of the parameters*

$$|t_k^j - t_k^\ell| + |z_k^j - z_k^\ell| \underset{k\to\infty}{\longrightarrow} \infty, \quad \textit{for } j \neq \ell, \tag{5.6}$$

- *Asymptotic smallness property:*

$$\lim_{M\to\infty} \left(\lim_{k\to\infty} \|e^{-itH} W_k^M\|_{\ell_\gamma^p L^q L^r} \right) = 0. \tag{5.7}$$

- *Orthogonality in norms: for any fixed* M *we have*

$$\|\phi_k\|_{L^2}^2 = \sum_{j=1}^{M} \|\psi^j\|_{L^2}^2 + \|W_k^M\|_{L^2}^2 + o_k(1), \tag{5.8}$$

$$\|\phi_k\|_{\dot{B}_1}^2 = \sum_{j=1}^{M} \|\psi^j\|_{\dot{B}_1}^2 + \|W_k^M\|_{\dot{B}_1}^2 + o_k(1). \tag{5.9}$$

Furthermore, we have

$$\|\phi_k\|_{L^{2\sigma+2}}^{2\sigma+2} = \sum_{j=1}^{M} \|e^{it_k^j H}\psi^j\|_{L^{2\sigma+2}}^{2\sigma+2} + \|W_k^M\|_{L^{2\sigma+2}}^{2\sigma+2} + o_k(1) \quad \textit{for all } M \geq 1. \tag{5.10}$$

In particular, for all $M \geq 1$

$$S(\phi_k) = \sum_{j=1}^{M} S\left(e^{it_k^j H}\psi^j\right) + S(W_k^M) + o_k(1) \tag{5.11}$$

$$I(\phi_k) = \sum_{j=1}^{M} I\left(e^{it_k^j H}\psi^j\right) + I(W_k^M) + o_k(1). \tag{5.12}$$

We note that cores are present only in the z-variable, not in the y-variable. This is so because the partial harmonic potential has a confining effect, hence in y, the situation is similar to the radial setting (as in [24, 30]).

Proof. First, we show that there exist $\theta \in (0,1)$ such that

$$\|e^{-itH} f\|_{\ell_\gamma^p L^q L^r} \lesssim \|f\|_{B_1}^{1-\theta} \|e^{-itH} f\|_{L_t^\infty L_x^r}^{\theta}, \quad \forall f \in B_1. \tag{5.13}$$

Indeed, from (2.5) we have

$$\|e^{-itH}f\|_{\ell^p_\gamma L^q L^r} \lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}f\|_{\ell^p_\gamma W^{s,q_0}(\mathbb{R};L^r(\mathbb{R}^d))}.$$

Since $\sigma > \frac{2}{d-n}$, we have $p_0 < p$ and thus there exists $\alpha \in (0,1)$ such that

$$\begin{aligned}\|e^{-itH}f\|_{\ell^p_\gamma L^q L^r} \lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}f\|^{\alpha}_{\ell^{p_0}_\gamma W^{s,q_0}(\mathbb{R};L^r(\mathbb{R}^d))} \times \\ \|\chi(\cdot-\gamma\pi)e^{-itH}f\|^{1-\alpha}_{\ell^{\infty}_\gamma W^{s,q_0}(\mathbb{R};L^r(\mathbb{R}^d))}.\end{aligned} \tag{5.14}$$

By the homogeneous Strichartz estimate we get, like in the proof of Lemma 2.7,

$$\begin{aligned}\|\chi(\cdot-\gamma\pi)e^{-itH}f\|_{\ell^{p_0}_\gamma W^{s,q_0}(\mathbb{R};L^r(\mathbb{R}^d))} &\lesssim \|\chi(\cdot-\gamma\pi)e^{-itH}H^s f\|_{\ell^{p_0}_\gamma L^{q_0}L^r} \\ &\lesssim \|f\|_{B_{2s}} \lesssim \|f\|_{B_1}.\end{aligned} \tag{5.15}$$

Next we interpolate between Sobolev spaces in time, there is $\eta \in (0,1)$ such that

$$\begin{aligned}&\|\chi(\cdot-\gamma\pi)e^{-itH}f\|_{W^{s,q_0}(\mathbb{R};L^r(\mathbb{R}^d))} \\ &\le \|\chi(\cdot-\gamma\pi)e^{-itH}f\|^{1-\eta}_{W^{1/2,q_0}(\mathbb{R};L^r(\mathbb{R}^d))}\|\chi(\cdot-\gamma\pi)e^{-itH}f\|^{\eta}_{L^{q_0}(\mathbb{R};L^r(\mathbb{R}^d))}.\end{aligned} \tag{5.16}$$

Moreover, we have

$$\begin{aligned}\|\chi(\cdot-\gamma\pi)e^{-itH}f\|_{\ell^\infty_\gamma W^{1/2,q_0}(\mathbb{R};L^r(\mathbb{R}^d))} &\lesssim \|e^{-itH}H^{1/2}f\|_{\ell^\infty_\gamma L^{q_0}L^r} \\ &\lesssim \|e^{-itH}H^{1/2}f\|_{\ell^{p_0}_\gamma L^{q_0}L^r} \\ &\lesssim \|H^{1/2}f\|_{L^2} = \|f\|_{B_1},\end{aligned} \tag{5.17}$$

and

$$\|\chi(\cdot-\gamma\pi)e^{-itH}f\|_{\ell^\infty_\gamma L^{q_0}(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|e^{-itH}f\|_{L^\infty_t L^r_x}. \tag{5.18}$$

Combining (5.14), (5.15), (5.16), (5.17) and (5.18) we obtain (5.13).

Since we will know that $\|W^M_k\|_{B_1}$ is uniformly bounded, then to prove (5.7), it will suffice to show that

$$\lim_{M\to\infty}\left(\lim_{k\to\infty}\|e^{-itH}W^M_k\|_{L^\infty_t L^r_x}\right) = 0.$$

We can then essentially repeat the proof of [16, Theorem 5.1], which generalized [15, Lemma 2.1]. Note that in the confined variable y, the situation is similar to the radial setting without potential (see e.g. [24, Lemma 5.2]), this is why no core in y will appear, only cores in z (denoted by z^j_k), due to the translation invariance in z. Another technical difference is that Sobolev spaces H^s have to be replaced with the spaces B_s defined in the introduction. Unlike in the case without potential, e^{-itH} does not commute with the convolution with Fourier multipliers, nor is unitary on $\dot{H}^s$, and this imposes some extra modification in the analysis.

Step 1. First we construct t^1_k, z^1_k, ψ^1 and W^1_k. This is done by adapting [16, Lemma 5.2]. By assumption, there exists a positive constant $\Lambda > 0$ such that $\|\phi_k\|_{B_1} \le \Lambda$. We infer $\|e^{-itH}\phi_k\|_{L^\infty_t L^r_x} \lesssim \|e^{-itH}\phi_k\|_{L^\infty_t B_1} = \|\phi_k\|_{B^1} \le \Lambda$. Passing to a subsequence, we define

$$A_1 := \lim_{k\to\infty}\|e^{-itH}\phi_k\|_{L^\infty_t L^r_x}. \tag{5.19}$$

If $A_1=0$, we set $\psi^j=0$ and $W_k^1=\phi_k$ for all $k\geq 1$. We now suppose that $A_1>0$. We introduce a real-valued, radially symmetric function $\varphi\in C_0^\infty(\mathbb{R}^d)$ supported in $\{\xi\in\mathbb{R}^d;\ |\xi|\leq 2\}$, such that $\varphi(\xi)=1$ for $|\xi|\leq 1$. For $N>1$ (to be chosen later), in the same fashion as in [23], define the operator

$$P_{\leq N}=\varphi\left(\frac{-\Delta_y+|y|^2}{N^2}\right)\varphi\left(\frac{-\Delta_z}{N^2}\right),$$

where the first operator is to be understood as a spectral cut-off, since the harmonic oscillator possesses an eigenbasis consisting of Hermite functions, and the second operator is a Fourier (in z) cut-off. By considering this operator instead of a Fourier cut-off in x (presented as a convolution in [16, 24]), we gain the commutation property

$$[e^{-itH},P_{\leq N}]=0.$$

Also, since $-\Delta_y+|y|^2$ and $-\Delta_z$ commute and are positive operators, we have for $s\in(0,1)$ and $f\in B_1$,

$$\|f-P_{\leq N}f\|_{B_s}=\|(1-P_{\leq N})H^{\frac{s-1}{2}}H^{\frac{1-s}{2}}f\|_{B_s}\leq\frac{1}{N^{1-s}}\|f\|_{B_1}.$$

In view of the Sobolev embedding $\dot H^s(\mathbb{R}^d)\hookrightarrow L^{2\sigma+2}(\mathbb{R}^d)$ with $s=\frac{d\sigma}{2\sigma+2}$, and of the fact that e^{-itH} is bounded on B_s,

$$\begin{aligned}\|e^{-itH}\phi_k-e^{-itH}P_{\leq N}\phi_k\|_{L_t^\infty L_x^r}&\lesssim\|e^{-itH}\phi_k-e^{-itH}P_{\leq N}\phi_k\|_{L_t^\infty \dot H_x^s}\\&\lesssim\|e^{-itH}\phi_k-e^{-itH}P_{\leq N}\phi_k\|_{L_t^\infty B_s}\\&\lesssim\|\phi_k-P_{\leq N}\phi_k\|_{L_t^\infty B_s}\leq C_0\frac{\Lambda}{N^{1-s}}\leq\frac{A_1}{2},\end{aligned}\tag{5.20}$$

with $N=\left(\frac{2C_0\Lambda}{A_1}\right)^{1/(1-s)}+1$. It follows by (5.20) that for k large,

$$\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^r}\geq\frac{1}{4}A_1.\tag{5.21}$$

Moreover, by interpolation we have

$$\begin{aligned}\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^r}&\leq\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^2}^{(r-2)/r}\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^\infty}^{2/r}\\&\leq\|\phi_k\|_{L^2}^{(r-2)/r}\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^\infty}^{2/r}\\&\leq\Lambda^{(r-2)/r}\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^\infty}^{2/r}.\end{aligned}$$

Thus by (5.21) we obtain, for k large enough,

$$\|P_{\leq N}e^{-itH}\phi_k\|_{L_t^\infty L_x^\infty}\geq\left(\frac{A_1}{4}\right)^{r/2}\Lambda^{1-r/2}.\tag{5.22}$$

In view of Lemmas 3.1 and 3.2 from [35], there exists $c>0$ independent of ϕ_k and t such that for all $x\in\mathbb{R}^d$,

$$|P_{\leq N}e^{-itH}\phi_k(x)|\lesssim N^{n/2}e^{-c|y|^2/N^2}\left(\int_{\mathbb{R}^n}\left|P_{\leq N}e^{-itH}\phi_k(y,z)\right|^2dy\right)^{1/2}.$$

Since $P_{\leq N}$ localizes the frequencies in z, Bernstein inequality implies

$$\int_{\mathbb{R}^n} \left|P_{\leq N} e^{-itH}\phi_k(y,z)\right|^2 dy \lesssim N^{d-n}\int_{\mathbb{R}^d} \left|P_{\leq N} e^{-itH}\phi_k(y,z)\right|^2 dydz,$$

and so

$$|P_{\leq N} e^{-itH}\phi_k(x)| \lesssim N^{d/2} e^{-c|y|^2/N^2}\Lambda.$$

We deduce from (5.22) that for R sufficiently large,

$$\|P_{\leq N} e^{-itH}\phi_k\|_{L_t^\infty L^\infty_{|y|\leq R}} \geq \frac{1}{2\Lambda^{r/2-1}}\left(\frac{A_1}{4}\right)^{r/2}. \tag{5.23}$$

It follows that there exist $t_k^1 \in \mathbb{R}$, $z_k^1 \in \mathbb{R}^{d-n}$ and $y_k^1 \in \mathbb{R}^n$, $|y_k^1| \leq R$, such that

$$|P_{\leq N} e^{-it_k^1 H}\phi_k|(y_k^1,z_k^1) \geq \frac{1}{4\Lambda^{r/2-1}}\left(\frac{A_1}{4}\right)^{r/2}. \tag{5.24}$$

Since $|y_k^1| \leq R$, possibly after extracting a subsequence, we get $y_k^1 \to y^1$. Let

$$w_k(x) = e^{-it_k^1 H}\phi_k(y, z+z_k^1).$$

Then $\{w_k\}_{k=1}^\infty$ is uniformly bounded in B_1 and there exists $\psi^1 \in B_1$ such that, passing to a subsequence if necessary, $w_k \rightharpoonup \psi^1$ in B_1 as $k \to \infty$. In particular, $\|\psi^1\|_{B_1} \leq \Lambda$. As $|P_{\leq N} e^{-it_k^1 H}\phi_k|(y^1,z_k^1) = |P_{\leq N} w_k|(y^1,0)$, by (5.24) we get

$$|P_{\leq N}\psi^1|(y^1,0) \geq \frac{1}{4\Lambda^{r/2-1}}\left(\frac{A_1}{4}\right)^{r/2}.$$

We note that the previous computations yield

$$\begin{aligned}\|\psi^1\|_{L^2(\mathbb{R}^d)} \geq \|P_{\leq N}\psi^1\|_{L^2(\mathbb{R}^d)} \gtrsim |P_{\leq N}\psi^1|(y^1,0) &\gtrsim \frac{1}{N^{d/2}}\frac{A_1^{r/2}}{\Lambda^{r/2-1}} \\ &\geq C_1\left(\frac{A_1}{\Lambda}\right)^{\frac{d}{2(1-s)}}\frac{A_1^{\sigma+1}}{\Lambda^\sigma},\end{aligned}$$

for a universal constant C_1. Set $W_k^1(x) := \phi_k(x) - e^{it_k^1 H}\psi^1(y, z - z_k^1)$: $W_k^1 \rightharpoonup 0$ in B_1. Furthermore, since

$$\|\psi^1\|^2_{\dot{B}^1} = \lim_{k\to\infty}\left\langle \psi^1, e^{-it_k^1 H}\phi_k(\cdot,\cdot+z_k^1)\right\rangle = \lim_{k\to\infty}\left\langle e^{-it_k^1 H}\psi^1, \phi_k(\cdot,\cdot+z_k^1)\right\rangle,$$

this implies that

$$\begin{aligned}\|\phi_k\|^2_{\dot{B}_1} &= \|\psi^1\|^2_{\dot{B}_1} + \|W_k^1\|^2_{\dot{B}_1} + o_k(1), \\ \|\phi_k\|^2_{L^2} &= \|\psi^1\|^2_{L^2} + \|W_k^1\|^2_{L^2} + o_k(1),\end{aligned}$$

as $k \to \infty$. Thus (5.8) and (5.9) hold. In particular we see that $\|W_k^1\|^2_{B_1} \leq \Lambda$.

We next replace $\{\phi_k\}_{k=1}^\infty$ by $\left\{W_k^1\right\}_{k=1}^\infty$ and repeat the same argument. If $A_2 := \limsup_{k\to\infty}\|e^{-itH}W_k^1\|_{L_t^\infty L_x^r} = 0$, we can take $\psi^j = 0$ for every $j \geq 2$ and the proof is over. Notice that the property (5.7) is an immediate consequence of (5.13). Otherwise there exist $\psi^2 \in B_1$, a sequence of time $\{t_k^2\}_{k=1}^\infty \subset \mathbb{R}$ and sequence $\{z_k^2\}_{n=1}^\infty \subset \mathbb{R}^{d-n}$ such that $e^{-it_k^2 H} W_k^1(\cdot,\cdot + z_k^2) \rightharpoonup \psi^2$ with

$$\|\psi^2\|_{L^2} \geq C_1 \left(\frac{A_2}{\Lambda}\right)^{\frac{d}{2(1-s)}} \frac{A_2^{\sigma+1}}{\Lambda^\sigma}.$$

We now show that

$$|t_k^2 - t_k^1| + |z_k^2 - z_k^1| \underset{k\to\infty}{\longrightarrow} \infty. \tag{5.25}$$

Let $g_k := e^{-it_k^1 H}\phi_k(\cdot,\cdot + z_k^1) - \psi^1 = e^{-it_k^1 H} W_k^1$. Notice that $g_k \rightharpoonup 0$ in B_1. Moreover, by definition $e^{-i(t_k^2 - t_k^1)H} g_k(\cdot,\cdot + (z_k^2 - z_k^1)) \rightharpoonup \psi^2 \neq 0$ weakly in B_1. Suppose by contradiction that $|t_k^2 - t_k^1| + |z_k^2 - z_k^1|$ is bounded. Then, after possible extraction, $t_k^2 - t_k^1 \to t^*$ and $z_k^2 - z_k^1 \to z^*$. However, since $g_k \rightharpoonup 0$, we infer that $e^{-i(t_k^2 - t_k^1)H} g_k(\cdot,\cdot + (z_k^2 - z_k^1)) \rightharpoonup 0$, which is impossible.

An argument of iteration and orthogonal extraction allows us to construct $\{t_k^j\}_{j\geq 1} \subset \mathbb{R}$, $\{z_k^j\}_{j\geq 1} \subset \mathbb{R}^{d-n}$ and the sequence of functions $\{\psi^j\}_{j\geq 1}$ in B_1 such that the properties (5.6), (5.7) and (5.8) hold and

$$\|\psi^M\|_{L^2} \geq C_1 \left(\frac{A_M}{\Lambda}\right)^{\frac{d}{2(1-s)}} \frac{A_M^{\sigma+1}}{\Lambda^\sigma}.$$

In view of (5.8), we obtain

$$\frac{1}{\Lambda^{2\sigma+2+\frac{d}{1-s}}} \sum_{M=1}^\infty A_M^{2\sigma+\frac{d}{1-s}} \lesssim \Lambda^2,$$

hence $A_M \to 0$ as $M \to \infty$. Finally, from (5.13) we infer that

$$\|e^{-itH} W_k^M\|_{\ell_\gamma^p L^q L^r} \lesssim \Lambda^{1-\theta} A_M^\theta,$$

and the property (5.7) holds.

Step 2. It remains to show (5.10). To this end, we show that for all $M \geq 1$,

$$\left\|\sum_{j=1}^M e^{it_k^j H}\psi^j(\cdot,\cdot - z_k)\right\|_{L^{2\sigma+2}}^{2\sigma+2} = \sum_{j=1}^M \|e^{it_k^j H}\psi^j\|_{L^{2\sigma+2}}^{2\sigma+2} + o_k(1). \tag{5.26}$$

We proceed as in [15, Lemma 2.3]. By reordering, we can choose $M^* \leq M$ such that
(i) For $1 \leq j \leq M^*$: The sequence $\{t_k^j\}_{k\geq 1}$ is bounded.
(ii) For $M^*+1 \leq j \leq M$: We have that $\lim_{k\to\infty} |t_k^j| = \infty$.
Consider the inequality

$$\left|\left|\sum_{j=1}^M z_j\right|^{2\sigma+2} - \sum_{j=1}^M |z_j|^{2\sigma+2}\right| \leq C_{\sigma,M} \sum_{j\neq j'} |z_j||z_j|^{2\sigma+1},$$

for $z_j \in \mathbb{C}$, $j=1, 2, \dots, M$. If $1 \le j \le \ell \le M^*$, the pairwise orthogonality (in space) (5.6) leads the cross terms in the sum of the left side of (5.26) to vanish as $k \to \infty$. Therefore,

$$\left\| \sum_{j=1}^{M^*} e^{it_k^j H} \psi^j(\cdot, \cdot - z_k) \right\|_{L^{2\sigma+2}}^{2\sigma+2} = \sum_{j=1}^{M^*} \left\| e^{it_k^j H} \psi^j \right\|_{L^{2\sigma+2}}^{2\sigma+2} + o_k(1). \tag{5.27}$$

On the other hand, if $M^*+1 \le j \le M$, then $|t_k^j| \to +\infty$ and, from Lemma 2.4,

$$\lim_{k\to\infty} \left\| e^{it_k^j H} \psi^j \right\|_{L^r}^r = 0. \tag{5.28}$$

Moreover, since (see proof of Step 1)

$$\lim_{M\to\infty} \left(\lim_{k\to\infty} \| e^{-itH} W_k^M \|_{L_t^\infty L_x^r} \right) = 0, \tag{5.29}$$

combining (5.27), (5.28) and (5.29), we obtain (5.26). This shows the last statement of the proposition and the proof is complete. □

Finally, we will show the following result related with the linear profile decomposition.

Lemma 5.4. *Let $M \in \mathbb{N}$ and let $\{\psi^j\}_{j=0}^M \subset B_1$ satisfy*

$$\sum_{j=0}^M S(\psi^j) - \varepsilon \le S\left(\sum_{j=0}^M \psi^j\right) \le \beta - \eta, \quad -\varepsilon \le I\left(\sum_{j=0}^M \psi^j\right) \le \sum_{j=0}^M I(\psi^j) + \varepsilon.$$

where $\varepsilon > 0$ and $2\varepsilon < \eta$. Then for all $0 \le j \le M$ we have $\psi^j \in \mathcal{K}^+$.

Proof. Assume by contradiction there exists $k \in \{0, 1, \dots, M\}$ such that $I(\psi^k) < 0$. Using the definition of $(\psi^k)_\lambda^{1,0}$ (see (3.1)) it is not difficult to show that there exists $\lambda < 0$ such that $I((\psi^k)_\lambda^{1,0}) > 0$. This implies that there exists $\lambda_0 < 0$ such that $I((\psi^k)_{\lambda_0}^{1,0}) = 0$. Moreover, a simple calculation shows that $\partial_\lambda B^{1,0}((\psi^k)_\lambda^{1,0}) \ge 0$ where $B^{1,0}$ is given by (3.5). Thus, by Lemma 3.2 we get

$$B^{1,0}(\psi^k) \ge B^{1,0}((\psi^k)_{\lambda_0}^{1,0}) = S((\psi^k)_{\lambda_0}^{1,0}) \ge \beta.$$

Notice that $B^{1,0}(\psi^j) \ge 0$ for $0 \le j \le M$, by Lemma 3.2. Since $2\varepsilon < \eta$, we obtain

$$\begin{aligned} \beta &\le \sum_{j=0}^M B^{1,0}(\psi^j) = \sum_{j=0}^M \left(S(\psi^j) - \frac{1}{4} I(\psi^j) \right) \\ &\le S\left(\sum_{j=0}^M \varphi^j\right) + \varepsilon - \frac{1}{4} I\left(\sum_{j=0}^M \varphi^j\right) + \frac{1}{4}\varepsilon \le \beta - \eta + 2\varepsilon < \beta, \end{aligned}$$

This is absurd. Therefore, we infer that $I(\psi^j) \ge 0$ for all $0 \le j \le M$. In particular, $S(\psi^j) = B^{1,0}(\psi^j) + \frac{1}{2\sigma+2} I(\psi^j) \ge 0$ and

$$\sum_{j=0}^M S(\psi^j) \le S\left(\sum_{j=0}^M \psi^j\right) + \varepsilon < \beta,$$

which implies that $S(\psi^j) < \beta$. It follows (see Lemma 3.4) that $\psi^j \in \mathcal{K}^+$. This completes the proof. □

5.3. Construction of a critical element. We define the critical action level τ_c by

$$\tau_c := \sup\left\{\tau : S(\varphi) < \tau \text{ and } \varphi \in \mathcal{K}^+ \text{ implies } \|u\|_{\ell^p_\gamma L^q L^r} < \infty\right\}.$$

Here, $u(t)$ is the corresponding solution of (1.1) with $u(0)=\varphi$. We observe that τ_c is a strictly positive number. Indeed, if $\varphi\in\mathcal{K}^+$, by Lemmas 3.5 and 2.7 we see that $\|e^{-itH}\varphi\|_{\ell^p_\gamma L^q L^r} \lesssim \|\varphi\|_{B_1} \lesssim S(\varphi)$. Therefore, taking $\tau>0$ sufficiently small we obtain that $\|u\|_{\ell^p_\gamma L^q L^r}<\infty$ by Lemma 5.1. Hence $0<\tau_c\le\beta$. We prove that $\tau_c=\beta$ by contradiction.

We assume $\tau_c<\beta$. By the definition of τ_c, there exists a sequence of solutions u_k to (1.1) in B_1 with initial data $\phi_k\in\mathcal{K}^+$ such that $S(\phi_k)\to\tau_c$ and $\|u_k\|_{\ell^p_\gamma L^q L^r}=\infty$. In the next results, we construct a critical solution $u_c(t)\in\mathcal{K}^+$ of (1.1) such that $S(u_c(t))=\tau_c$ and $\|u_c\|_{\ell^p_\gamma L^q L^r}=\infty$. Moreover, we prove that there exists a continuous path $z(t)$ in $\mathbb{R}^{d-n}$ such that the critical solution u_c has the property that $K=\{u_c(\cdot,\cdot-z(t))\}$ is precompact in B_1. This is where the requirement $\sigma\ge\frac{1}{2}$ appears, in addition to the previous assumption $\frac{2}{d-n}<\sigma<\frac{2}{d-2}$.

Proposition 5.2 (Critical element). *Let $n=1$ and $\sigma\ge\frac12$ with $\frac{2}{d-1}<\sigma<\frac{2}{d-2}$. We assume that $\tau_c<\beta$. Then there exists $u_{c,0}\in B_1$ such that the corresponding solution u_c to (1.1) with initial data $u_c(0)=u_{c,0}$ satisfies $u_c(t)\in\mathcal{K}^+$, $S(u_c(t))=\tau_c$ and $\|u_c\|_{\ell^p_\gamma L^q L^r}=\infty$.*

Proof. Since $S(\phi_k)\to\tau_c$, from Lemma 3.5 we see that $\{\phi_k\}_{k=1}^\infty$ is bounded in B_1. Indeed, $\|\phi_k\|_{B_1}\lesssim S(\phi_k)$, and $S(\phi_k)\le\beta$. Thus, by Proposition 5.1, up to extracting to a subsequence, we get

$$\phi_k=\sum_{j=1}^M e^{it_k^j H}\psi^j(\cdot,\cdot-z_k)+W_k^M \quad \text{for all } M\in\mathbb{N}, \tag{5.30}$$

and the sequence satisfies

$$S(\phi_k)=\sum_{j=1}^M S\left(e^{it_k^jH}\psi^j\right)+S(W_k^M)+o_k(1),$$
$$I(\phi_k)=\sum_{j=1}^M I\left(e^{it_k^jH}\psi^j\right)+I(W_k^M)+o_k(1).$$

By using the fact that $\phi_k\in\mathcal{K}^+$, we infer that there exists ε, $\eta>0$ such that $2\varepsilon<\eta$ and

$$S(\phi_k)\le\beta-\eta,$$
$$S(\phi_k)\ge\sum_{j=1}^M S\left(e^{it_k^jH}\psi^j\right)+S(W_k^M)-\varepsilon,$$
$$I(\phi_k)\ge-\varepsilon,$$
$$I(\phi_k)\le\sum_{j=1}^M I\left(e^{it_k^jH}\psi^j\right)+I(W_k^M)+\varepsilon$$

for sufficiently large k. Thus, from Lemma 5.4 we obtain that

$$e^{it_k^jH}\psi^j\in\mathcal{K}^+,\quad W_k^M\in\mathcal{K}^+ \quad \text{for sufficiently large } k. \tag{5.31}$$

This implies that $S(e^{it_k^j H}\psi^j)\geq 0$, $S(W_k^M)\geq 0$ and for each $1\leq j\leq M$,

$$0\leq \limsup_{k\to\infty} S(e^{it_k^j H}\psi^j)\leq \limsup_{k\to\infty} S(\phi_k)=\tau_c. \tag{5.32}$$

Now we have two cases: (i) $\limsup_{k\to\infty} S(e^{it_k^j H}\psi^j)=\tau_c$ fails for all j, or (ii) equality holds in (5.32) for some j.

Case (i): In this case, for each $1\leq j\leq M$ there exists $\eta_j>0$ such that

$$\limsup_{k\to\infty} S(e^{it_k^j H}\psi^j)\leq \tau_c-\eta_j, \quad S(e^{it_k^j H}\psi^j)\geq 0, \quad I(e^{it_k^j H}\psi^j)\geq 0. \tag{5.33}$$

Suppose that $t_k^j\to t^*$. If $t^*<\infty$ for some j (at most one such j exists by the orthogonality of the parameters (5.6)), then from the continuity of the linear flow we infer that

$$e^{it_k^j H}\psi^j \underset{k\to\infty}{\longrightarrow} e^{it^* H}\psi^j \quad \text{strongly in } B_1. \tag{5.34}$$

We set $\psi_*^j=\mathrm{NLS}(t^*)(e^{it^*H}\psi^j)$, where we recall that $\mathrm{NLS}(t)\varphi$ denotes the solution to (1.1) with initial datum $u_0=\varphi$. Notice that $\mathrm{NLS}(-t^*)\psi_*^j=e^{it^*H}\psi^j$. Moreover, by (5.31) and (5.33) we have that $\psi_*^j\in\mathcal{K}^+$ and $S(\psi_*^j)<\tau_c$. Thus, by definition of τ_c we get $\|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r}<\infty$. Finally, by (5.34) we obtain

$$\|\mathrm{NLS}(-t_k^j)\psi_*^j-e^{it^*H}\psi^j\|_{B_1}\to 0 \quad \text{as } k\to\infty. \tag{5.35}$$

On the other hand, suppose that $|t_k^j|\to\infty$: $\|e^{it_k^j H}\psi^j\|_{L^{2\sigma+2}}\to 0$, and therefore

$$\lim_{k\to\infty} S\left(e^{it_k^j H}\psi^j\right)=\frac{1}{2}\|\psi^j\|_{B_1}^2<\tau_c<\beta. \tag{5.36}$$

By Lemma 5.2, there exists ψ_*^j such that $\psi_*^j\in\mathcal{K}^+$ and

$$\|\mathrm{NLS}(-t_k^j)\psi_*^j-e^{it_k^j H}\psi^j\|_{B_1}\underset{k\to\infty}{\longrightarrow} 0. \tag{5.37}$$

Moreover, by (5.36) we have $S(\psi_*^j)=\frac{1}{2}\|\psi^j\|_{B_1}^2<\tau_c$. Again, by definition of τ_c we see that $\|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r}<\infty$.

In either case, we obtain a new profile ψ_*^j for the given ψ^j such that (5.37) holds and $\|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r}<\infty$. We rewrite ϕ_k as follows (see (5.30)):

$$\phi_k=\sum_{j=1}^{M}\mathrm{NLS}(-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)+\tilde{W}_k^M,$$

where

$$\tilde{W}_k^M=\sum_{j=1}^{M}\left[e^{it_k^j H}\psi^j(\cdot,\cdot-z_k^j)-\mathrm{NLS}(-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)\right]+W_k^M. \tag{5.38}$$

We observe that by Lemma 2.7,

$$\|e^{-itH}\tilde{W}_k^M\|_{\ell_\gamma^p L^q L^r}\lesssim \sum_{j=1}^{M}\|e^{-it_k^j H}\psi^j-\mathrm{NLS}(-t_k^j)\psi_*^j\|_{B_1}+\|e^{-itH}W_k^M\|_{\ell_\gamma^p L^q L^r}.$$

Thus, we have

$$\lim_{M\to\infty}\left(\lim_{k\to\infty}\|e^{-itH}\tilde{W}_k^M\|_{\ell_\gamma^p L^q L^r}\right)=0. \tag{5.39}$$

The idea now is to approximate

$$\mathrm{NLS}(t)\phi_k \approx \sum_{j=1}^{M}\mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j),$$

and use the approximation theory from Lemma 5.3 to obtain $\|\mathrm{NLS}(\cdot)\phi_k\|_{\ell_\gamma^p L^q L^r}<\infty$, which is a contradiction. With this in mind, we define

$$u_k(t)=\mathrm{NLS}(t)\phi_k,\quad v_k^j(t)=\mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j),\quad u_k^M(t)=\sum_{j=1}^{M}v_k^j(t).$$

A simple calculation shows that $i\partial_t u_k^M - Hu_k^M + |u_k^M|^{2\sigma}u_k^M = e_k^M$, where

$$e_k^M = |u_k^M|^{2\sigma}u_k^M - \sum_{j=1}^{M}|v_k^j|^{2\sigma}v_k^j.$$

and

$$u_k(0)-u_k^M(0)=\tilde{W}_k^M. \tag{5.40}$$

We rely on the following two claims.

Claim 1. There exists $A>0$ (independent of M) such that for each M, there exists $k_1=k_1(M)$ with the following property: If $k>k_1$ then we have the following estimate

$$\|u_k^M\|_{\ell_\gamma^p L^q L^r}\le A. \tag{5.41}$$

Claim 2. There exists $k_2=k_2(M,\varepsilon(A))$ such that if $k>k_2$, then we have the following estimate

$$\|e_k^M\|_{\ell_\gamma^{\tilde{p}'} L^{\tilde{q}'} L^{r'}}\le \varepsilon(A), \tag{5.42}$$

where A is given by (5.41) and $\varepsilon(A)$ is the associate value provided by Lemma 5.3.

To prove Claim 1, we note that following the same strategy as in e.g. [16,24,30,31], relying on an interpolation of the norm involved in the asymptotic smallness of W_k^M ((5.7), in our case) by norms of the form $L_{t,x}^\gamma$ and $L^\infty H^1$, seems doomed. Indeed, since $q>p$, it does not seem easy to control the $\ell_\gamma^p L^q L^r$ in this fashion. However, as noticed in [3], it is possible to do without, by just using the fact that the Lebesgue exponents at stake are all finite. We therefore resume the main ideas from [3, Appendix A], to obtain

$$\limsup_{k\to\infty}\|u_k^M\|_{\ell_\gamma^p L^q L^r}^{2\sigma+1}\le 2\sum_{j=1}^{M}\|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r}^{2\sigma+1}. \tag{5.43}$$

Recall the identities $\tilde{p}'=(2\sigma+1)p$, $\tilde{q}'=(2\sigma+1)q$ and $r'=(2\sigma+1)r$. To prove (5.43), we first notice that if $f_1,f_2\in C(\mathbb{R};B_1)\cap\ell_\gamma^p L^q L^r$ and

$$|t_k-s_k|+|z_k-\eta_k|\underset{k\to\infty}{\longrightarrow}\infty,$$

then

$$\left\| |f_1(t-t_k,y,z-z_k)|^{2\sigma} f_2(t-s_k,y,z-\zeta_k)\right\|_{\ell_\gamma^{\tilde p'} L^{\tilde q'} L^{r'}} \underset{k\to\infty}{\longrightarrow} 0. \tag{5.44}$$

Indeed, Hölder inequality in space yields

$$\begin{aligned}&\left\| |f_1(t-t_k,y,z-z_k)|^{2\sigma} f_2(t-s_k,y,z-\zeta_k)\right\|_{\ell_\gamma^{\tilde p'} L^{\tilde q'} L^{r'}}\\ &\le \left\| \|f_1(t-t_k)\|_{L^r}^{2\sigma} \|f_2(t-s_k)\|_{L^r}\right\|_{\ell_\gamma^{\tilde p'} L^{\tilde q'}},\end{aligned}$$

and (5.44) follows in the case $|t_k - s_k| \underset{k\to\infty}{\longrightarrow} \infty$, since $\tilde p'$ and $\tilde q'$ are finite. In the case where this sequence is bounded, for $\gamma_0 \ge 1$, Hölder inequality in space and time yields

$$\begin{aligned}&\left\| |f_1(t,y,z-z_k)|^{2\sigma} f_2(t+t_k-s_k,y,z-\zeta_k)\right\|_{\ell_{|\gamma|\ge\gamma_0}^{\tilde p'} L^{\tilde q'} L^{r'}}\\ &\le \|f_1\|_{\ell_{|\gamma|\ge\gamma_0}^{p} L^q L^r}^{2\sigma} \|f_2(t+t_k-s_k)\|_{\ell_{|\gamma|\ge\gamma_0}^{p} L^q L^r} \underset{\gamma_0\to\infty}{\longrightarrow} 0.\end{aligned}$$

Now for t fixed,

$$\begin{aligned}&\left\| |f_1(t,y,z-z_k)|^{2\sigma} f_2(t+t_k-s_k,y,z-\zeta_k)\right\|_{L_x^{r'}}\\ &= \left\| |f_1(t,y,z)|^{2\sigma} f_2(t+t_k-s_k,y,z+z_k-\zeta_k)\right\|_{L_x^{r'}} \underset{k\to\infty}{\longrightarrow} 0,\end{aligned}$$

since $|z_k-\zeta_k| \to \infty$, $|f_1(t,\cdot)|^{2\sigma} \in L^{\frac{r}{2\sigma}}$ for *all* t, using the property $f_1 \in C_t H^1$ and Sobolev embedding, and, for the same reason,

$$\{f_2(t+t_k-s_k,y,z+z_k-\zeta_k),\ k\in\mathbb{N}\} \text{ is compact in } L^r, \quad \forall t.$$

Invoking Hölder inequality in space again,

$$\begin{aligned}&\left\| |f_1(t,y,z)|^{2\sigma} f_2(t+t_k-s_k,y,z+z_k-\zeta_k)\right\|_{L^{r'}}\\ &\le \|f_1(t)\|_{L^r}^{2\sigma} \|f_2(t+t_k-s_k)\|_{L^r},\end{aligned}$$

the Lebesgue dominated convergence theorem implies, for any given $\gamma_0 \ge 1$,

$$\left\| |f_1(t,y,z-z_k)|^{2\sigma} f_2(t+t_k-s_k,y,z-\zeta_k)\right\|_{\ell_{|\gamma|\le\gamma_0}^{\tilde p'} L^{\tilde q'} L^{r'}} \underset{k\to\infty}{\longrightarrow} 0,$$

hence (5.44). Now we observe that for $M \ge 2$, there exists a constant $C_M > 0$ such that

$$\left| \left|\sum_{j=1}^M z_j\right|^{2\sigma} \sum_{j=1}^M z_j - \sum_{j=1}^M |z_j|^{2\sigma} z_j \right| \le C_M \sum_{1\le j\neq \ell\le M} |z_j|^{2\sigma}|z_\ell|. \tag{5.45}$$

Writing

$$\begin{aligned}\|u_k^M\|_{\ell_\gamma^p L^q L^r}^{2\sigma+1} &= \left\| \sum_{j=1}^M \mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)\right\|_{\ell_\gamma^p L^q L^r}^{2\sigma+1}\\ &\le \left\| \Big(\sum_{j=1}^M \left|\mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)\right|\Big)^{2\sigma+1}\right\|_{\ell_\gamma^{\tilde p'} L^{\tilde q'} L^{r'}}\end{aligned}$$

$$\leq \Big\| \sum_{j=1}^{M} \Big|\mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)\Big|^{2\sigma+1} \Big\|_{\ell_\gamma^{\tilde p'} L^{\tilde q'} L^{r'}}$$

$$+\Big\| \Big(\sum_{j=1}^{M} \big|\mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)\big| \Big)^{2\sigma+1} - \sum_{j=1}^{M} \Big|\mathrm{NLS}(t-t_k^j)\psi_*^j(\cdot,\cdot-z_k^j)\Big|^{2\sigma+1} \Big\|_{\ell_\gamma^{\tilde p'} L^{\tilde q'} L^{r'}}.$$

The last term goes to zero as $k\to\infty$, from (5.44) and (5.45), hence (5.43) thanks to triangle inequality. Now using (5.9) and (5.35), there exists M_0 such that

$$\|\psi_*^j\|_{B_1} \leq \nu, \quad \forall j \geq M_0,$$

where ν is given by Lemma 5.1. Lemma 5.1 then implies, for all $j \geq M_0$,

$$\|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r} \leq 2\|e^{-itH}\psi_*^j\|_{\ell_\gamma^p L^q L^r} \lesssim \|\psi_*^j\|_{B_1},$$

where we have used Lemma 2.7. For $\sigma \geq \frac{1}{2}$, we infer

$$\sum_{j=M_0}^{\infty} \|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r}^{2\sigma+1} \lesssim \sum_{j=M_0}^{\infty} \|\psi_*^j\|_{B_1}^{2\sigma+1} \lesssim \sum_{j=M_0}^{\infty} \|\psi_*^j\|_{B_1}^2 < \infty.$$

Now for $j < M_0$, we have seen that

$$\|\mathrm{NLS}(\cdot)\psi_*^j\|_{\ell_\gamma^p L^q L^r} < \infty,$$

hence Claim 1. Claim 2 then follows from (5.44) and (5.45).

Next notice that combining (5.40) and (5.39) we infer that for $\varepsilon(A)$ there exists $M_1 = M_1(\varepsilon)$ such that for any $M > M_1$, then there exists $k_3 = k_3(M_1)$ such that if $k > k_3$ then we obtain

$$\|e^{-itH}(u_k(0) - u_k^M(0))\|_{\ell_\gamma^p L^q L^r} \leq \varepsilon(A). \tag{5.46}$$

Therefore, by (5.41), (5.42) and (5.46) we see that for $k \geq \max\{k_1,k_2,k_3\}$ we obtain that $\|u_k^M\|_{\ell_\gamma^p L^q L^r} \leq A$, $\|e_k^M\|_{\ell_\gamma^{p'} L^{q'} L^{r'}} \leq \varepsilon(A)$ and $\|e^{-itH}(u_k(0)-u_k^M(0))\|_{\ell_\gamma^p L^q L^r} \leq \varepsilon(A)$. Thus by Lemma 5.3 we get $\|\phi_k\|_{\ell_\gamma^p L^q L^r} < \infty$, which is absurd.

Case (ii): We note that if equality holds in (5.32) for some j (we may assume $j=1$ by reordering), then $M=1$. In particular, $\limsup_{k\to\infty} S(W_k^1) = 0$. Since $S(W_k^1) \sim \|W_k^1\|_{B_1}^2$ (see Lemma 3.5), we have that $W_k^1 \to 0$ in B_1. Thus $\{\phi_k\}_{k=1}^\infty$ has only one nonlinear profile

$$\phi_k = e^{it_k^1 H}\psi^1(\cdot,\cdot - z_k) + W_k^1 \quad \text{and } W_k^1 \to 0 \text{ in } B_1. \tag{5.47}$$

Suppose that $t_k^1 \to t^*$. If $|t^*| < \infty$ (we may then assume $t^* = 0$), we put $\psi^* = \psi^1$. Then as $k \to \infty$, $\|e^{it_k^1 H}\psi^1 - \mathrm{NLS}(-t_k^1)\psi^*\|_{B_1} \to 0$. Now if $|t^*| = \infty$, then $\|e^{it_k^1 H}\psi^1\|_{L^{2\sigma+2}} \to 0$. This implies that

$$\frac{1}{2}\|\psi^1\|_{B^1}^2 = \frac{1}{2}\|e^{it_k^1 H}\psi^1\|_{B^1}^2 = \lim_{k\to\infty} S\left(e^{it_k^1 H}\psi^1\right) = \tau_c < \beta.$$

Thus, by Lemma 5.2 there exists ψ^* such that the corresponding solution $\mathrm{NLS}(t)\psi^* \in \mathcal{K}^+$ for all $t \in \mathbb{R}$ and

$$\|e^{it_k^1 H}\psi^1 - \mathrm{NLS}(-t_k^1)\psi^*\|_{B_1} \to 0 \quad \text{as } k \to \infty.$$

In either case, we set $u_{c,0} := \psi^*$. We note that $u_{c,0} \in \mathcal{K}^+$ and $S(u_{c,0}) = S(\psi^*) = \tau_c$. By (5.47) we can rewrite ϕ_k as

$$\phi_k = \mathrm{NLS}(-t_k^1)\psi^* + \tilde{W}_k^1,$$

where $\tilde{W}_k^1 = W_k^1 + e^{it_k^1 H}\psi^1 - \mathrm{NLS}(-t_k^1)\psi^*$. Since $W_k^1 \to 0$ in B_1, it follows by Lemma 2.7

$$\lim_{M\to\infty}\left\{\lim_{k\to\infty}\|e^{-itH}\tilde{W}_k^M\|_{\ell_\gamma^p L^q L^r}\right\} = 0.$$

Therefore, by the same argument as above (Case (i)) we infer that $\|u_c\|_{\ell_\gamma^p L^q L^r} = \infty$, which proves the proposition. ∎

5.4. Extinction of the critical element. In this subsection, we assume that $\|u\|_{\ell^p_{\gamma\geq 1} L^q L^r} = \infty$; we call it a forward critical element. We remark that the same argument as below does work in the case $\|u\|_{\ell^p_{\gamma\leq 1} L^q L^r} = \infty$.

LEMMA 5.5. *Let u_c be the critical element given in Proposition 5.2. Then $u_c = 0$.*

To prove Lemma 5.5, we need the following result.

LEMMA 5.6. *Let u_c be the critical element given in Proposition 5.2. Then there exists a function $z \in C([0,\infty);\mathbb{R}^{d-n})$ such that $\{u_c(t,\cdot,\cdot - z(t)); t \geq 0\}$ is relatively compact in B_1. In particular, we have the uniform localization of u_c:*

$$\sup_{t\geq 0}\int_{|z+z(t)|>R}\left[|\nabla u(t,x)|^2 + |u(t,x)|^{2\sigma+2} + |u(t,x)|^2\right]dx \underset{R\to\infty}{\longrightarrow} 0. \tag{5.48}$$

Proof. By [15, Appendix A] (see also proof of Proposition 6.1 in [16]), it is enough to show that the following condition is satisfied: For every sequence $\{t_k\}_{k=1}^\infty$, $t_k \to \infty$, extracting a subsequence from $\{t_k\}_{k=1}^\infty$ if necessary, there exists $\{z_k\}_{k=1}^\infty \subset \mathbb{R}^{d-n}$ and $\varphi \in B_1$ such that $u_c(t_k,\cdot,\cdot - z_k) \to \varphi$ in B_1.

We set $\phi_k := u_c(t_k)$. We note that ϕ_k satisfies:

$$S(\phi_k) = \tau_c \quad \text{and} \quad \phi_k \in \mathcal{K}^+. \tag{5.49}$$

Since $\|\phi_k\|_{B_1}^2 \lesssim S(\phi_k)$, it follows that $\{\phi_k\}_{k=1}^\infty$ is bounded in B_1. Thus, using the same argument developed in the proof of Proposition 5.2, we obtain that $\{\phi_k\}_{k=1}^\infty$ has only one nonlinear profile

$$\phi_k = e^{it_k^1 H}\psi^*(\cdot,\cdot - z_k) + W_k^1,$$

with $W_k^1 \to 0$ in B_1 (see proof of Case (ii) above). Assume that $|t_k^1| \to \infty$. Then we have two cases to consider. We first assume that $t_k^1 \to -\infty$. By Lemma 2.7 we see that

$$\|e^{-itH}u_c(t_k)\|_{\ell^p_{\gamma\geq 1}L^q L^r} \lesssim \|e^{-i(t-t_k^1)H}\psi^*\|_{\ell^p_{\gamma\geq 1}L^q L^r} + \|W_k^1\|_{B_1}.$$

Since $W_k^1 \to 0$ in B_1 and

$$\lim_{k\to\infty}\|e^{-i(t-t_k^1)H}\psi^*\|_{\ell^p_{\gamma\geq 1}L^q L^r} = \lim_{k\to\infty}\|e^{-itH}\psi^*\|_{\ell^p_{\gamma\gtrsim -t_k^1}L^q L^r} = 0,$$

it follows that $\|e^{-itH}u_c(t_k)\|_{\ell^p_{\gamma\geq 1}L^q L^r} \to 0$ as $k \to \infty$. In particular, for k large, we have $\|e^{-itH}u_c(t_k)\|_{\ell^p_{\gamma\geq 1}L^q L^r} \leq \delta$, where δ is given in Lemma 5.1. Then from Lemma 5.1 we obtain that

$$\|\mathrm{NLS}(t)u_c(t_k)\|_{\ell^p_{\gamma\geq 1}L^q L^r} \lesssim \delta,$$

which is absurd. Next, if $t_k^1 \to \infty$, then a similar argument shows that

$$\|e^{-itH}u_c(t_k)\|_{\ell^p_{\gamma\le 1}L^qL^r} \le \delta, \quad \text{for } k \text{ large}.$$

Again from Lemma 5.1 we have $\|u_c\|_{\ell^p_{\gamma\lesssim t_k}L^qL^r} \lesssim \delta$. Since $t_k \to \infty$ we infer that $\|u_c\|_{\ell^p_\gamma L^qL^r} \lesssim \delta$, which is also absurd. Therefore $t_k^1 \to t^*$, $t^* \in \mathbb{R}$. Thus

$$u_c(t_k,\cdot,\cdot+z_k) \to e^{it^*H}\psi^* \text{ in } B_1,$$

and this completes the proof. ∎

Proof. **(Proof of Lemma 5.5.)** We proceed by a contradiction argument. Assume that $\varphi := u_{c,0} \neq 0$. We observe that $G(\varphi) = 0$ (G, we recall, is defined in (1.3)). Indeed, suppose that $G(\varphi) \neq 0$. We define

$$\psi(x) := e^{iz\cdot z_0}\varphi(y,z), \quad \text{where} \quad z_0 = -\frac{G(\varphi)}{\|\varphi\|_{L^2}^2}.$$

It is not difficult to show that $G(\psi) = 0$, $\|\nabla_x\psi\|_{L^2}^2 < \|\nabla_x\varphi\|_{L^2}^2$ and $\|\psi\|_{L^{2\sigma+2}} = \|\varphi\|_{L^{2\sigma+2}}$. Notice that $\psi \in \mathcal{K}^+$. Indeed, since $\varphi \in \mathcal{K}^+$ we see that $S(\psi) < S(\varphi) = \tau_c < \beta$. Moreover, $I(\psi) \ge 0$. Assume by contradiction that $I(\psi) < 0$. Then there exists $\lambda \in (0,1)$ such that $I(\lambda\psi) = 0$. By using the fact $S(\varphi) \ge \frac{\sigma}{\sigma+1}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}$ we have

$$S(\lambda\psi) = \frac{1}{2}I(\lambda\psi) + \frac{\sigma}{\sigma+1}\|\lambda\psi\|_{L^{2\sigma+2}}^{2\sigma+2} < \frac{\sigma}{\sigma+1}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} = \frac{\sigma}{\sigma+1}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2} < \beta,$$

which is absurd by Lemma 3.2. Therefore, $I(\psi) \ge 0$, $S(\psi) < \tau_c$ and $\psi \in \mathcal{K}^+$ (see Lemma 3.4). The corresponding solution $v \in C([0,\infty);B_1)$ of (1.1) with $v(0) = \psi$ is given by

$$v(t,y,z) = e^{i(z\cdot z_0 - t|z_0|^2)}u(t,y,z-2tz_0).$$

Since $\|u_c\|_{\ell^p_\gamma L^qL^r} = \infty$, it follows that $\|v\|_{\ell^p_\gamma L^qL^r} = \infty$, which is a contradiction with the definition of τ_c.

Step 1. We claim that

$$\lim_{t\to\infty}\frac{|z(t)|}{t} = 0, \tag{5.50}$$

where $z(t)$ is given in Lemma 5.6. The proof in [15, Lemma 5.1] can be easily adapted to our case by considering the truncated center of mass of the form

$$\Gamma_R(t) = \int_{\mathbb{R}^d}\phi_R(z)|u_c(t,x)|^2dx,$$

where $\phi_R(z) = R\phi(\frac{z}{R})$, $\phi(z) = (\theta(z_1),\theta(z_2),\dots,\theta(z_{d-n}))$, $z \in \mathbb{R}^{d-n}$ such that $\theta \in C_c^\infty(\mathbb{R})$, $\theta(s) = 1$ for $-1 \le s \le 1$, $\theta(s) = 0$ for $|s| \ge 2^{1/3}$, $|\theta(s)| \le |s|$, $\|\theta\|_{L^\infty} \le 2$ and $\|\theta'\|_{L^\infty} \le 4$. Assume that (5.50) is false. Then there exist a sequence $t_k \to \infty$ and $\alpha > 0$ such that $|z(t_k)| \ge \alpha t_k$. Without loss of generality we may assume $z(0) = 0$. For $R > 0$ we set

$$t_0(R) = \inf\{t \ge 0; |z(t)| \ge R\}.$$

We define $R_k=|z(t_k)|$. Notice that $R_k\geq \alpha t_0(R_k)$ and $t_0(R_k)\to\infty$ as $k\to\infty$. On the other hand, $\Gamma'_R(t)=([\Gamma'_R(t)]_1,[\Gamma'_R(t)]_2\dots,[\Gamma'_R(t)]_{d-n})$, with

$$[\Gamma'_R(t)]_j=2\mathrm{Im}\int_{\mathbb{R}^d}\theta'\left(\tfrac{z_j}{R}\right)\partial_j u_c\overline{u_c}dx,\quad j\in\{1,2,\dots,d-n\}.$$

Since $G(u_c(t))=0$ for all $t\in\mathbb{R}$, we infer that

$$\mathrm{Im}\int_{|z_j|\leq R}\partial_j u_c\overline{u_c}dx=-\mathrm{Im}\int_{|z_j|>R}\partial_j u_c\overline{u_c}dx.$$

By using the fact that $\theta'(\frac{z_j}{R})=1$ for $|z_j|\leq R$, we conclude

$$[\Gamma'_R(t)]_j=-2\mathrm{Im}\int_{|z_j|\geq R}\partial_j u_c\overline{u_c}dx+2\mathrm{Im}\int_{|z_j|\geq R}\theta'\left(\tfrac{z_j}{R}\right)\partial_j u_c\overline{u_c}dx.$$

This implies

$$|\Gamma'_R(t)|\leq 10\int_{|z|\geq R}|\nabla u_c||u_c|dx\leq 5\int_{|z|\geq R}\left[|\nabla u_c|^2+|u_c|^2\right]dx. \tag{5.51}$$

Combining Lemma 5.6 and (5.51), given $\varepsilon>0$ (to be chosen later) there exists $R_\varepsilon>0$ such that if $\tilde{R}_k:=R_k+R_\varepsilon$, then

$$|\Gamma'_{\tilde{R}_k}(t)|\leq 5\varepsilon. \tag{5.52}$$

Moreover, by following the same argument as in the proof of [15, Lemma 5.1] we get

$$|\Gamma_{\tilde{R}_k}(0)|\leq R_\varepsilon\|\varphi\|^2_{L^2}+2\tilde{R}_k\varepsilon, \tag{5.53}$$

$$|\Gamma_{\tilde{R}_k}(t_k^*)|\geq \tilde{R}_k(\|\varphi\|^2_{L^2}-3\varepsilon)-2R_\varepsilon\|\varphi\|^2_{L^2}, \tag{5.54}$$

where $t_k^*=t_0(R_k)$. Since $\tilde{R}_k\geq R_k\geq\alpha\tilde{t}_k$, combining the inequalities (5.52), (5.53) and (5.54) we infer that

$$\begin{aligned}5\varepsilon t_k^*\geq \int_0^{t_k^*}|\Gamma'_{\tilde{R}_k}(t)|\geq |\Gamma_{\tilde{R}_k}(t_k^*)-\Gamma_{\tilde{R}_k}(0)|\\ \geq t_k^*\alpha(\|\varphi\|^2_{L^2}-3\varepsilon)-2R_\varepsilon\|\varphi\|^2_{L^2},\end{aligned}$$

that is,

$$t_k^*\left[\alpha\|\varphi\|^2_{L^2}-\varepsilon(3\alpha+5)\right]\leq 2R_\varepsilon\|\varphi\|^2_{L^2}.$$

By taking $\varepsilon>0$ sufficiently small, letting $t_k^*\to\infty$ in the inequality above yields a contradiction. This proves the claim.

Step 2. There exists $\eta>0$ such that $P(u_c(t))\geq\eta$ for all $t\geq 0$. Indeed, if not, there exists a sequence of times t_k such that

$$P(u_c(t_k))<\frac{1}{k}\quad\text{for all } k.$$

Since $\{u_c(t,\cdot,\cdot-z(t));t\geq 0\}$ is precompact, there exists $f\in B_1$ such that, passing to a subsequence if necessary, $g_k:=u_c(t_k,\cdot,\cdot-z(t_k))\to f$ in B_1. Notice that $S(f)=$

$\lim_{k\to\infty} S(g_k) = \tau_c < \beta$ and since $P(u_c(t_k)) \geq 0$, it follows that $P(f) = \lim_{k\to\infty} P(g_k) = 0$. Thus, $S(f) < \beta$ and $P(f) = 0$. By Remark 3.2, we infer that $f = 0$, which is absurd because $S(f) = \tau_c > 0$.

Step 3. Conclusion. We use the virial identities (4.6) and (4.8) with u_c in place of u. We recall that

$$V''(t) = 4(d-n)P(u_c(t)) + R_1 + R_2 + R_3, \tag{5.55}$$

where R_1, R_2 and R_3 are given by (4.9). Notice that there exists a constant K independent of t such that

$$|R_1 + R_2 + R_3 + R_4| \leq K \int_{|z|\geq R} \left[|\nabla u_c(t)|^2 + |u_c(t)|^2 + |u_c(t)|^{2\sigma+2}\right] dx. \tag{5.56}$$

By (4.6) it is clear that there exists a constant $L > 0$ such that

$$|V'(t)| \leq LR. \tag{5.57}$$

From Lemma 5.6, there exists $\rho > 1$ such that

$$\int_{|z+z(t)|\geq\rho} \left[|\nabla u_c(t)|^2 + |u_c(t)|^2 + |u_c(t)|^{2\sigma+2}\right] dx \leq \frac{2\eta(d-n)}{K}, \tag{5.58}$$

for every $t \geq 0$, where η is given in Step 2. Moreover, by (5.50) we obtain that there exists $t_0 > 0$ such that

$$|z(t)| \leq \frac{2\eta(d-n)}{4L} t \quad \text{for every } t \geq t_0. \tag{5.59}$$

For $t^* > t_0$ we put

$$R_{t^*} = \rho + \frac{2\eta(d-n)}{4L} t^*. \tag{5.60}$$

It is clear that $\{|z| \geq R_{t^*}\} \subset \{|z + z(t)| \geq \rho\}$ for all $t \in [t_0, t^*]$. Therefore, by (5.56) and (5.58) we get

$$|R_1 + R_2 + R_3 + R_4| \leq 2\eta(d-n), \quad \text{for all } t \in [t_0, t^*]. \tag{5.61}$$

Thus, by (5.61) and Step 2 we have

$$V''(t) \geq 2\eta(d-n) \quad \text{for all } t \in [t_0, t^*]. \tag{5.62}$$

Integrating (5.62) on (t_0, t^*), it follows from (5.62) and (5.57)

$$\begin{aligned} 2\eta(d-n)(t^* - t_0) &\leq \int_{t_0}^{t^*} V''(t)dt \leq |V'(t^*) - V'(t_0)| \leq 2LR_{t^*} \\ &= 2L\rho + \eta(d-n)t^*. \end{aligned}$$

Choosing t^* large enough, we get a contradiction. The proof of lemma is now completed. ∎

Proof. **(Proof of Theorem 1.1 (i) (scattering result)).** The proof of scattering part of Theorem 1.1 is an immediate consequence of the Proposition 5.2 and Lemma 5.5. ∎

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