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BOUNDARY LAYER AND ASYMPTOTIC STABILITY FOR THE NAVIER-STOKES-POISSON EQUATIONS WITH NONSLIP BOUNDARY CONDITIONS*

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Abstract. In this paper we consider the well-posedness of the compressible Prandtl boundary layer for the Navier-Stokes-Poisson equations with nonslip boundary condition and obtain the linear stability of the approximate solution constructed by boundary layer expansion.

Keywords. Isothermal plasma; Nonslip boundary; Prandtl boundary layer; Asymptotic stability.

AMS subject classifications. 76X05; 35Q35; 35B40.

1. Introduction

In this paper we consider a hydrodynamic model arising from the isothermal viscous plasma, which is governed by the following Navier-Stokes-Poisson (NSP) system

$$\begin{cases} \rho^\varepsilon_t + \nabla\cdot(\rho^\varepsilon u^\varepsilon) = 0, \\ (\rho^\varepsilon u^\varepsilon)_t + \nabla\cdot(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + T^i \rho^\varepsilon \mathbb{I}) = \rho^\varepsilon \nabla\phi^\varepsilon + \mu'_\varepsilon \Delta u^\varepsilon + (\mu'_\varepsilon + \nu'_\varepsilon)\nabla\nabla\cdot u^\varepsilon, \\ \lambda_\varepsilon \Delta\phi^\varepsilon + e^{-\phi^\varepsilon} = \rho^\varepsilon. \end{cases} \tag{1.1}$$

Notice that $\mathbb{T}\times\mathbb{R}^+ = \{(x,y)|x\in\mathbb{R}/\mathbb{Z}, 0\le y<+\infty\}$ denoting the periodic spatial domain and $u^\varepsilon(t,x,y) = (u_1^\varepsilon, u_2^\varepsilon)(t,x,y)$. The above unknowns $\rho^\varepsilon(t,x,y), u^\varepsilon(t,x,y), \phi^\varepsilon(t,x,y)$ with $(t,x,y)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}^+$ are the density of the flow, the velocity and electric potential, respectively. Let T^i denote the average temperature of the ions, and the small parameter λ_ε for the squared scaled Debye length. Another two small parameters μ'_ε and ν'_ε are the constant viscosity coefficient satisfying $\mu'_\varepsilon > 0$ and $\mu'_\varepsilon + \nu'_\varepsilon > 0$.

In this paper we intend to investigate a limit behaviour of the NSP system as the small parameters go to zero with the following Dirichlet boundary condition for both the velocity and the electric potential:

$$u^\varepsilon|_{y=0} = 0, \quad \phi^\varepsilon|_{y=0} = \phi_b(x), \tag{1.2}$$

where $\phi_b(x)$ is smooth and compactly supported. The initial data is denoted as

$$(\rho^\varepsilon, u^\varepsilon, \phi^\varepsilon)(0,x,y) = (\rho_0^\varepsilon, u_0^\varepsilon, \phi_0^\varepsilon)(x,y). \tag{1.3}$$

For our purpose, we assume the small parameters $\mu'_\varepsilon, \nu'_\varepsilon$ and λ_ε satisfy that

$$\mu'_\varepsilon = \mu\varepsilon^2, \quad \nu'_\varepsilon = \nu\varepsilon^2, \quad \lambda_\varepsilon = \varepsilon^2. \tag{1.4}$$

Formally, letting $\varepsilon = 0$, we immediately obtain the following Euler equations

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ u_t + u\cdot\nabla u + (T^i + 1)\nabla\ln\rho = 0, \end{cases} \tag{1.5}$$

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with the following relation

$$\rho = e^{-\phi}. \tag{1.6}$$

The Euler system (1.5) is well-posed only with the boundary condition

$$u_2|_{y=0} = 0. \tag{1.7}$$

There is a loss of boundary condition in the tangential velocity u_1 as ε tends to 0, which naturally leads to the appearance of boundary layer for the velocity near the boundary. Due to the boundary condition (1.2), we would reckon that the solution of (1.5) satisfies $\phi|_{y=0} = \phi_b$ through the relation (1.6). However, the solution to system (1.5) and (1.7) cannot hold $\rho|_{y=0} = e^{-\phi_b}$ in general. Hence we expect the formation of a boundary layer for the density to correct this boundary condition.

The asymptotic behaviour as the small parameters go to zero for this NSP system associated with the Navier-slip type boundary condition for the velocity u and the Dirichlet boundary condition for the electric potential have been well studied in [15, 16], in which the leading order for the asymptotic expansion of the velocity in the weak layer form:

$$u^\varepsilon = u^E + \varepsilon U(t, y, \frac{x_3}{\varepsilon}) + \cdots.$$

Compared with the boundary layer for the Navier-slip type boundary condition (see [22]), the boundary layer for the nonslip boundary condition is much stronger. In 1904, Prandtl introduced the boundary layer theory in [27] and the Prandtl boundary layer equation as following

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x P = \partial_y^2 u, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y\to\infty} u(t,x,y) = U(t,x), \end{cases} \tag{1.8}$$

which can be derived from the incompressible Navier-Stokes equation with nonslip boundary condition as the leading order of the asymptotic approximation near the boundary. Then much research interest has been focused on this crucial equation. Oleĭnik in [24–26] proved the local existence results under the monotonicity assumption through Crocco transform. Later, Alexandre et al. in [1] studied the local well-posedness of the Prandtl Equation (1.8) with the uniform outflow $U = 1$. More precisely, under the strictly monotonic assumption of the initial data in the normal direction, they established the local-in-time well posedness of the nonlinear Prandtl equation in Sobolev space by means of the weighted energy method and the Nash-Moser-Hörmander iteration scheme. Meanwhile, Masmoudi and Wong in [23] proved the local existence and uniqueness for the two dimensional Prandtl system (1.8) using a new nonlinear energy estimate under the Oleĭnik's monotonicity assumption without the Crocco transform or any change of variables. For some other research developments on the well-posedness of the Prandtl Equation (1.8) in different functional spaces the readers can also see [3, 7, 9, 18, 19, 28] and the references therein. In addition, some excellent instability results around the shear flow can be found in [6, 8, 10].

For the Prandtl boundary layer system corresponding to the compressible flow, recently, Wang et al. have also considered the local well posedness of the compressible Prandtl boundary layer equations derived from the compressible isentropic Navier-Stokes equations with nonslip boundary condition in [30] by expanding the methods in [1]. The compressible Prandtl layer system in [30] reads as following with

$(x,y)\in\mathbb{T}\times\mathbb{R}^+$.

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x P = \frac{1}{\bar{\rho}(t,x)}\partial_y^2 u, \\ \partial_x(\bar{\rho}u) + \partial_y(\bar{\rho}v) = -\bar{\rho}_t, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y\to\infty} u(t,x,y) = U(t,x), \end{cases} \tag{1.9}$$

with Bernoulli's law

$$U_t + UU_x + P_x = 0. \tag{1.10}$$

Compared with (1.9), the compressible Prandtl layer equations for NSP system have some new features since the boundary layer for the density in our paper is of order $O(1)$, which is much stronger than that in (1.9). Once the well-posedness of the Prandtl boundary layer equations is verified, the inviscid limit for the viscous flow associating with the Prandtl boundary layer is another interesting problem. Recently, there is some progress on the inviscid limit problem of incompressible Navier-Stokes equations with nonslip boundary for specific cases, for instance, in the analytic setting [28, 29] and the initial vorticity supported away from the boundary [20]. Guo and Nguyen also considered the inviscid limit of the steady Navier-Stokes flows over a moving plate using the Prandtl boundary layer expansion in [11] and one can also see [2, 12, 13, 22] and the references therein for the relevant inviscid limit problem involving nonslip or slip boundary effect.

For the limit behaviour of the NSP system in the whole space or specific domain without layer effect, one can see numerous relevant results, such as [4, 5, 14, 17]. However, there are few results on the zero-viscosity limit problem for the compressible NSP system with nonslip boundary condition, since the complicated model with the strong boundary layer effect and the undetermined high order boundary condition, which lead to the loss of derivatives. At present, we consider the linear stability of the linearized system for the compressible flow (NSP) with the nonslip boundary condition using approximate expansion. Compared with [16], the main difficulty here is that the boundary layer is a compressible Prandtl boundary layer system, which is much stronger. Hence we should first verify the well-posedness of the boundary profile under appropriate condition. Moreover, to avoid the singular effect when the normal derivatives act on the approximate solution, we also introduce the conormal Sobolev norm. And we can prove that the linear stability estimate for the NSP equations with Navier-slip boundary condition also holds for that of the NSP equations with nonslip boundary condition.

The arrangement of the paper is as following. We will construct the approximate solution in the second section, in which the Euler expansion and the Prandtl boundary layer expansion with the matched condition will be given. Meanwhile, the local well-posedness of the Prandtl boundary layer under the monotonicity assumption can be investigated through the scheme in [30]. In the final section, we will prove the linear stability in the conormal Sobolev space and remark some difficulties for the nonlinear system at present.

2. Construction of the approximate solution

In this section, we will construct an approximate solution by the matched asymptotic expansion. In the construction of the approximate solution, the leading order of the boundary layer expansion is a compressible Prandtl boundary layer system, which is more complicated than the weak layer for the Navier-slip type boundary condition and the compressible Prandtl boundary layer system in [30]. Hence the well-posedness of the compressible Prandtl Boundary layer in our paper is also a crucial question to be answered in this section.

2.1. Asymptotic expansion. Let (ρ_a,u_a,ϕ_a) denote the approximate solution of the density, the velocity and the electric potential, respectively and they are in the following form including the Euler terms for the outer flow and the boundary layer profiles for the inner layer:

$$\rho_a=\sum_{i=0}^{K}\varepsilon^i(\rho^i(t,x,y)+\Upsilon^i(t,x,\frac{y}{\varepsilon})), \tag{2.1}$$

$$u_a=\sum_{i=0}^{K}\varepsilon^i(u^i(t,x,y)+U^i(t,x,\frac{y}{\varepsilon})), \tag{2.2}$$

$$\phi_a=\sum_{i=0}^{K}\varepsilon^i(\phi^i(t,x,y)+\Phi^i(t,x,\frac{y}{\varepsilon})), \tag{2.3}$$

Where K is an arbitrary large integer. For simplicity, we will use $z=\frac{y}{\varepsilon}$ to denote the fast decay variable. For the boundary layer profiles (Υ^i,U^i,Φ^i) above, we shall assume that they satisfy the fast decay property with respective to z. That is,

$$(\Upsilon^i,U^i,\Phi^i)\to 0, \tag{2.4}$$

fast enough as $z\to\infty$.

In addition, according to the boundary condition (1.2), we require the following matched boundary condition for each order of the approximate solution on $\{y=z=0\}$.

$$u_1^i(t,x,0)+U_1^i(t,x,0)=0, \tag{2.5}$$
$$u_2^i(t,x,0)+U_2^i(t,x,0)=0, \tag{2.6}$$
$$\phi^0(t,x,0)+\Phi^0(t,x,0)=\phi_b(t,x), \tag{2.7}$$
$$\phi^k(t,x,0)+\Phi^k(t,x,0)=0, \tag{2.8}$$

for $i\geq 0$, and $k\geq 1$.

Now we substitute the approximate forms (2.1) (2.2) (2.3) into the NSP system (1.1) and collect the terms of $O(1)$ order. Taking limit $z\to\infty$, one has the compressible Euler system for the leading order of the outer flow

$$\begin{cases}\rho_t^0+\nabla\cdot(\rho^0u^0)=0,\\ u_t^0+u^0\cdot\nabla u^0+(T^i+1)\nabla\ln\rho^0=0,\end{cases} \tag{2.9}$$

with the relation $\phi^0=-\ln\rho^0$.

Collecting the coefficient of the order $O(\varepsilon)$ for the outer flow, we obtain that

$$\begin{cases}\rho_t^1+\nabla\cdot(\rho^0u^1)+\nabla\cdot(\rho^1u^0)=0,\\ u_t^1+u^0\cdot\nabla u^1+u^1\cdot\nabla u^0+T^i\nabla(\frac{\rho^1}{\rho^0})=\nabla\phi^1,\\ -e^{-\phi^0}\phi^1=\rho^1.\end{cases} \tag{2.10}$$

Similarly, we can obtain the outer flow system for the order $O(\varepsilon^j),j\geq 1$ in the following form

$$\begin{cases}\rho_t^j+\nabla\cdot(\rho^0u^j)+\nabla\cdot(\rho^ju^0)=f_\rho^j,\\ u_t^j+u^0\cdot\nabla u^j+u^j\cdot\nabla u^0+T^i\nabla(\frac{\rho^j}{\rho^0})\\ =\nabla\phi^j+\frac{1}{\rho^0}[\mu\Delta u^{j-2}+(\mu+\nu)\nabla\nabla\cdot u^{j-2}]+f_u^j,\\ -e^{-\phi^0}\phi^j=\rho^j+f_\phi^j,\end{cases} \tag{2.11}$$

where $(f_\rho^j, f_u^j, f_\phi^j)$ depend on (ρ^k, u^k, ϕ^k) and their derivatives with $k \le j-1$.

Before dealing with the inner layer terms near the boundary, we first give an overall notation that

$$\Gamma f = f(t,x,y)|_{y=0}.$$

We gather the $O(\varepsilon^{-1})$ terms and have the equations on the inner layer terms $(\Upsilon^0, U_2^0, \Phi^0)$:

$$\begin{cases} \partial_z[(\Gamma\rho^0 + \Upsilon^0)(\Gamma u_2^0 + U_2^0)] = 0, \\ (\Gamma u_2^0 + U_2^0)\partial_z U_2^0 + T^i \partial_z \ln(\Gamma\rho^0 + \Upsilon^0) = \partial_z \Phi^0. \end{cases} \tag{2.12}$$

Combining (2.12) with the matched boundary condition (2.6), one has

$$\begin{cases} (\Gamma\rho^0 + \Upsilon^0)(\Gamma u_2^0 + U_2^0) = 0, \\ T^i \partial_z \Upsilon^0 = (\Gamma\rho^0 + \Upsilon^0)\partial_z \Phi^0. \end{cases} \tag{2.13}$$

Hence we have

$$\Gamma u_2^0 + U_2^0 = 0, \tag{2.14}$$

since

$$\Gamma\rho^0 + \Upsilon^0 = \Gamma\rho^0 e^{\Phi^0/T^i} \tag{2.15}$$

from the second equation of (2.13). Then using the fast decay property (2.4), we have immediately

$$\Gamma u_2^0 = U_2^0 = 0. \tag{2.16}$$

The compressible Euler system (2.9) for the outer flow complemented with the boundary condition $\Gamma u_2^0 = 0$ is locally well-posed as long as the initial data satisfy the appropriate regularity, that is, $(\rho_0^0 - \bar\rho, u_0^0) \in H^{m+3+2K}, m \in \mathbb{N}, m \ge 3$, where the $\bar\rho$ is a strictly smooth function (see [21]).

From the $O(1)$ terms for the last equation of (1.1), one has

$$\partial_{zz}\Phi^0 + e^{-(\Gamma\phi^0 + \Phi^0)} = \Gamma\rho^0 + \Upsilon^0. \tag{2.17}$$

By virtue of the equality (2.15) and the relation $\phi^0 = -\ln\rho^0$, one can rewrite the above Equation (2.17) as following

$$\partial_{zz}\Phi^0 + \Gamma\rho^0(e^{-\Phi^0} - e^{\Phi/T^i}) = 0. \tag{2.18}$$

Obviously, it is a closed ODE for Φ^0 together with the following boundary conditions from the matched principle and the fast decay property

$$\Phi^0|_{z=0} = \phi_b - \phi^0|_{y=0}, \qquad \Phi^0|_{z=\infty} = 0. \tag{2.19}$$

It is proved in many references, such as [16], that (2.18) and (2.19) admit a unique exponentially decaying solution. Here we omit the repetition. Consequently, we have the result for Υ^0 from the relation (2.15). Until now, we have solved (ρ^0, u^0, ϕ^0) and $(\Upsilon^0, U_2^0, \Phi^0)$.

Now we turn to study the term U_1^0, which is a nontrivial solution compared with the leading order terms for the case of the Navier-slip type boundary condition. Collecting the $O(1)$ terms from the equation of the density and the velocity, we obtain that

$$\partial_t(\Gamma\rho^0+\Upsilon^0)+\partial_x((\Gamma\rho^0+\Upsilon^0)(\Gamma u_1^0+U_1^0))+\partial_z((\Gamma\rho^0+\Upsilon^0)(\Gamma u_2^1+U_2^1+z\Gamma\partial_y u_2^0))=0, \tag{2.20}$$

$$\begin{aligned}&\partial_t(\Gamma u_1^0+U_1^0)+(\Gamma u_1^0+U_1^0)\partial_x(\Gamma u_1^0+U_1^0)\\&\quad+(\Gamma u_2^1+U_2^1+z\Gamma\partial_y u_2^0)\partial_z U_1^0+(T^i+1)\partial_x\ln(\Gamma\rho^0)=\frac{\mu}{\Gamma\rho^0+\Upsilon^0}\partial_{zz}U_1^0,\end{aligned} \tag{2.21}$$

$$T^i\partial_z(\Gamma\rho^1+\Upsilon^1)=(\Gamma\rho^0+\Upsilon^0)\partial_z\Phi^1+(\Gamma u_2^1+U_2^1+z\Gamma\partial_y u_2^0)\partial_z\Phi^0. \tag{2.22}$$

Combining the Equations (2.20) (2.21) with the boundary condition $u_1^0(t,x,0)+U_1^0(t,x,0)=0$ and the matched initial condition $u_1^0(0,x,y)+U_1^0(0,x,\frac{y}{\varepsilon})=u_0(x,y)$, we obtain a closed initial-boundary system for the unknowns $(\Gamma u_1^0+U_1^0,\Gamma u_2^1+U_2^1+z\Gamma\partial_y u_2^0)$. We can ensure that there is no trivial solution $U_1^0=0$ for this system, but the well-posedness is left to be considered in next subsection.

Then we collect the higher order terms $O(\varepsilon^{j-1}),j\geq 2$ of the boundary layer from the equation of the density and the equation of the tangential velocity u_1, respectively.

$$\partial_z[(\Gamma\rho^0+\Upsilon^0)(\Gamma u_2^j+U_2^j+z\Gamma\partial_y u_2^{j-1})]+\partial_x[(\Gamma\rho^0+\Upsilon^0)(\Gamma u_1^{j-1}+U_1^{j-1})]=F_\rho^j. \tag{2.23}$$

$$\begin{aligned}&\partial_t(\Gamma u_1^{j-1}+U_1^{j-1})+(\Gamma u_2^j+U_2^j+z\Gamma\partial_y u_2^{j-1})\partial_z U_1^0\\&\quad+(\Gamma u_1^0+U_1^0)\partial_x(\Gamma u_1^{j-1}+U_1^{j-1})+(\Gamma u_1^{j-1}+U_1^{j-1})\partial_x(\Gamma u_1^0+U_1^0)\\&=\frac{\mu}{\Gamma\rho^0+\Upsilon^0}\partial_{zz}U_1^{j-1}+F_1^j.\end{aligned} \tag{2.24}$$

In addition, the terms of order $O(\varepsilon^j),j\geq 1$ from the equations of u_2 and ϕ in (1.1) give the relation between Υ^j and Φ^j:

$$T^i\partial_z\Big(\frac{\Gamma\rho^j+\Upsilon^j}{\Gamma\rho^0+\Upsilon^0}\Big)=\partial_z\Phi^j+F_2^j, \tag{2.25}$$

and

$$\partial_{zz}\Phi^j-\Gamma\rho^0e^{-\Phi^0}(\Gamma\phi^j+\Phi^j)=(\Gamma\rho^j+\Upsilon^j)+F_\phi^j, \tag{2.26}$$

where the terms $(F_\rho^j,F_1^j,F_2^j,F_\phi^j)$ only depend on the (ρ^k,u^k,ϕ^k) and $(\Upsilon^k,U_1^{k-1},U_2^k,\Phi^k)$ with $k\leq j-1$. The above (2.23) – (2.26) satisfy the matched boundary condition (2.5) and the following compatibility conditions

$$\begin{cases}\partial_x(\rho^0u^{j-1})|_{y=0}=F_\rho^j|_{z=\infty},\\(\partial_t u_1^{j-1}+u_1^0\partial_x u_1^{j-1}+u_1^{j-1}\partial_x u_1^0)|_{y=0}=F_1^j|_{z=\infty},\\F_2^j|_{z=\infty}=0,\\-\rho^0\phi^j|_{y=0}=\rho^j|_{y=0}+F_\phi^j|_{z=\infty}.\end{cases} \tag{2.27}$$

Actually we can derive the expansion terms ρ^k,u^k,ϕ^k and $\Upsilon^k,U_1^{k-1},U_2^k,\Phi^k$ order by order. First, let us assume they are known for $k\leq j-1$. Now we intend to obtain the profiles ρ^j,u^j,ϕ^j and $\Upsilon^j,U_1^{j-1},U_2^j,\Phi^j$. It follows from (2.23) that

$$\Gamma u_2^j+U_2^j=-\frac{1}{\Gamma\rho^0+\Upsilon^0}\int_0^z\{[\partial_z\Upsilon^0z\Gamma\partial_y u^{j-1}$$

$$+\Upsilon^0\Gamma\partial_y u_2^{j-1}+\partial_x[(\Gamma\rho^0+\Upsilon^0)(\Gamma u_1^{j-1}+U_1^{j-1})]-F_\rho^j]\}. \tag{2.28}$$

Therefore, we have

$$\Gamma u_2^j=-\frac{1}{\Gamma\rho^0}\int_0^\infty[\partial_z\Upsilon^0 z\Gamma\partial_y u^{j-1}+\Upsilon^0\Gamma\partial_y u_2^{j-1}+\partial_x[(\Gamma\rho^0+\Upsilon^0)(\Gamma u_1^{j-1}+U_1^{j-1})]-F_\rho^j] \tag{2.29}$$

as $z\to\infty$. Hence we can conclude that the system (2.11) with boundary condition (2.29) is locally well posed as long as the initial data $(\rho_0^j,u_0^j)\in H^{m+3+2K-2j}$. Based on (2.28), we can obtain U_2^j directly.

Moreover, substituting U_2^j and u_2^j into the Equation (2.23) and combining with the boundary condition (2.5), it forms a closed transport-diffusion equation on U_1^{j-1}, whose well-posedness can be verified by classical local existence theory.

Now it remains to determine (Υ^j,Φ^j). Replacing the term $\Gamma\rho^j+\Upsilon^j$ in (2.26) by the Equation (2.25), we have

$$\partial_{zz}\Phi^j-\Gamma\rho^0(e^{-\Phi^0}+\frac{1}{T^i}e^{\Phi^0/T^i})\Phi^j=\tilde{F}^j, \tag{2.30}$$

where $\tilde{F}^j$ is only depending on the lower order terms. Together with the following boundary condition

$$\Phi^j|_{z=0}=-\Gamma\phi^j,\quad \Phi^j|_{z=\infty}=0, \tag{2.31}$$

we can obtain the exponentially decaying solution Φ^j for the linear ODE (2.30). Back to the Equation (2.25), we have the term Υ^j.

Hence we get the high order asymptotic expansion once the local existence of the profile U_1^0 is verified in the next subsection.

2.2. Local well-posedness of the compressible Prandtl boundary layer. We devote this subsection to determining the leading order for the boundary layer profile. After a few changes to the equations of the leading order terms, we will find that it is equivalent to verifying the local well-posedness of the compressible Prandtl boundary layer. The main idea to study the well-posedness for the compressible Prandtl layer in this section is from the references [1,30]. That is, under the strict monotonicity assumption on the initial data, one can obtain the proper weighted energy estimate on the linearized equations and then apply the Nash-Moser-Hörmander iteration to get the local well-posedness of the nonlinear compressible Prandtl boundary layer equations.

In fact, there are quite a few differences between the model in [30] and in this paper, which is caused by the density occupying the strong boundary layer in our model. However, the corresponding appropriate linearization can help us apply the scheme proposed in [1,30] smoothly. Hence we will omit the repetition in the following writing. The readers who are interested in this scheme can refer to the references mentioned above or some other relevant works.

Now we introduce the following new notations

$$u(t,x,z):=\Gamma u_1^0+U_1^0(t,x,z),\quad v(t,x,z):=\Gamma u_2^1+U_2^1(t,x,z)+z\Gamma\partial_y u_2^0.$$

Then we can rewrite the Equations (2.20) and (2.21) as following:

$$\begin{cases}\partial_t u+u\partial_x u+v\partial_z u+\partial_x P=\frac{\mu}{\Gamma\rho^0+\Upsilon^0}\partial_z^2 u,\\ \partial_x((\Gamma\rho^0+\Upsilon^0)u)+\partial_z((\Gamma\rho^0+\Upsilon^0)v)=-(\Gamma\rho^0+\Upsilon^0)_t,\end{cases} \tag{2.32}$$

where $P(t,x) = (T^i+1)\ln\Gamma\rho^0$.

From the matched boundary condition and the fast decay property, we can easily verify that

$$u|_{z=0} = v|_{z=0} = 0, \quad \lim_{z\to\infty} u(t,x,z) = U(t,x) := \Gamma u_1^0(t,x). \tag{2.33}$$

The initial data for (2.32) is

$$u(0,x,z) = u_0^0(x,z). \tag{2.34}$$

Moreover, by virtue of the leading order of the Euler expansion, one can verify that the Bernoulli's law

$$U_t + UU_x + P_x = 0 \tag{2.35}$$

holds on the boundary $\{z=0\}$.

Hence the Equations (2.32) together with the boundary conditions (2.33) form a compressible Prandtl boundary layer system. Now the existence of U_1^0 in the equations (2.20) and (2.21) is turned into studying the well-posedness of the Prandtl boundary layer system.

Notice that the difference between (1.9) and the Equations (2.32) and (2.33) is caused by the strong boundary layer of the density and the electric potential for the NSP system with nonslip boundary condition. If the boundary layer for the electric potential is weak, then the boundary layer for the the density is also weak and the compressible Prandtl boundary layer for the NSP system (1.1) is just exactly same as the system (1.9). See that case in the following remark.

Remark 2.1. If the boundary conditions for the NSP system (1.1) are given by the following form

$$u^\varepsilon|_{y=0} = 0, \quad \frac{\partial\phi^\varepsilon}{\partial y}|_{y=0} = 0, \tag{2.36}$$

then the leading order profile for boundary layer expansion of the NSP system is also a compressible Prandtl boundary layer system, however, the boundary layer for electric potential is weak. The well posedness of that Prandtl boundary layer system in the Sobolev space can be investigated as in [30] under the monotonicity assumption on the initial data.

Before the discussion on the Prandtl boundary layer system (2.32) (2.33), we first list the assumption on the initial data as in [30].

(A1) For a fixed integer $k_0 \geq 9$, the initial data $u_0^0(x,z)$ satisfy the compatibility condition (2.33);

(A2) monotone condition $\partial_z u_0^0 \geq \frac{\sigma_0}{(1+z)^(\gamma+2)} > 0$ holds for all $x \in \mathbb{T}$ and $z \geq 0$ with some positive constant σ_0 and a positive integer $\gamma \geq 2$;

(A3) $\|(1+z)^{\gamma+\alpha_2} D^\alpha (u_0^0(x,z) - U(0,x))\|_{L^2(\mathbb{T}\times\mathbb{R}^+)} \leq C_0$, where $D^\alpha = \partial_x^{\alpha_1}\partial_z^{\alpha_2}$ with $|\alpha| = \alpha_1 + \alpha_2 \leq 4k_0 + 2$;

(A4) $\|(1+z)^{\gamma+2+\alpha_2} D^\alpha \partial_z u_0^0\|_{L^\infty(\mathbb{T}\times\mathbb{R}^+)} \leq \frac{1}{\sigma_0}$ for $|\alpha| \leq 3k_0$.

Now we introduce $(\tilde{u},\tilde{v})$ as a smooth background state satisfying the following conditions:

$$\begin{cases} \partial_z \tilde{u} > 0, \\ \partial_x((\Gamma\rho^0 + \Upsilon^0)\tilde{u}) + \partial_z((\Gamma\rho^0 + \Upsilon^0)\tilde{v}) = -(\Gamma\rho^0 + \Upsilon^0)_t, \\ \tilde{u}|_{z=0} = \tilde{v}|_{z=0} = 0, \quad \lim_{z\to\infty} \tilde{u} = U(t,x). \end{cases} \tag{2.37}$$

It follows from the first equation of the outer Euler system that, on the boundary $\{z=0\}$,

$$\Gamma\rho_t^0+\partial_x(\Gamma\rho^0U(t,x))+\Gamma\rho^0\Gamma\partial_z u_2^0=\Gamma\rho_t^0+\partial_x(\Gamma\rho^0U(t,x))+\Gamma\rho^0V(t,x)=0, \tag{2.38}$$

where $V(t,x)$ denotes $\Gamma\partial_z u_2^0$. Hence the $\tilde{v}$ is expressed as an integral form

$$\begin{aligned}\tilde{v}&=\frac{\Gamma\rho^0Vz}{\Gamma\rho^0+\Upsilon^0}-\frac{1}{\Gamma\rho^0+\Upsilon^0}\int_0^z\{\Upsilon_t^0+\partial_x((\Gamma\rho^0+\Upsilon^0)\tilde{u})-\partial_x(\Gamma\rho^0U(t,x))\}\\&=\frac{\Gamma\rho^0Vz}{\Gamma\rho^0+\Upsilon^0}+\bar{v}\end{aligned} \tag{2.39}$$

We linearize the Prandtl boundary layer system (2.32) around the smooth state $(\tilde{u},\tilde{v})$ and denote

$$u^R=u-\tilde{u},\quad v^R=v-\tilde{v}.$$

Then we have

$$\begin{cases}u_t^R+u^R\tilde{u}_x+\tilde{u}u_x^R+v^R\partial_z\tilde{u}+\tilde{v}\partial_z u^R-\frac{\mu}{\Gamma\rho^0+\Upsilon}\partial_{zz}^2u^R=f,\\ \partial_x((\Gamma\rho^0+\Upsilon^0)u^R)+\partial_z((\Gamma\rho^0+\Upsilon^0)v^R)=0,\\ u^R|_{z=0}=v^R|_{z=0}=0,\quad \lim_{z\to\infty}u^R=0,\quad u^R|_{t=0}=0.\end{cases} \tag{2.40}$$

Similar to [30], we introduce the transformation

$$\omega(t,x,z)=\Big(\frac{(\Gamma\rho^0+\Upsilon^0)u^R}{\mu\partial_z\tilde{u}}\Big)_z(t,x,z), \tag{2.41}$$

which also means

$$u^R=\frac{\mu\partial_z\tilde{u}}{\Gamma\rho^0+\Upsilon^0}\int_0^z\omega(t,x,z')dz'.$$

Then we write (2.40) into the following form

$$\begin{cases}\omega_t+(\omega\tilde{u})_x+\Big(\frac{\Gamma\rho^0Vz}{\Gamma\rho^0+\Upsilon^0}\omega\Big)_z+(\bar{v}\omega)_z-2\Big(\frac{1}{\partial_z\tilde{u}}\partial_z(\frac{\mu\partial_z\tilde{u}}{\Gamma\rho^0+\Upsilon^0})\omega\Big)_z\\ -\partial_z(\frac{\mu}{\Gamma\rho^0+\Upsilon^0})\omega_z+[\xi\int_0^z\omega(t,x,z')dz']_z-\frac{\mu}{\Gamma\rho^0+\Upsilon^0}\omega_{zz}=\tilde{f}_z,\\ -\Big(\frac{2}{\partial_z\tilde{u}}\partial_z(\frac{\mu\partial_z\tilde{u}}{\Gamma\rho^0+\Upsilon^0})\omega+\frac{\mu}{\Gamma\rho^0+\Upsilon^0}\omega_z\Big)|_{z=0}=\tilde{f}|_{z=0},\\ \omega|_{t=0}=0,\end{cases} \tag{2.42}$$

where $\xi=[-(\partial_t+\tilde{v}\partial_z+\tilde{u}\partial_x)(\Gamma\rho^0+\Upsilon^0)+\frac{1}{\partial_z\tilde{u}}(\partial_t+\tilde{v}\partial_z+\tilde{u}\partial_x)\partial_z\tilde{u}-\frac{1}{\partial_z\tilde{u}}\partial_{zz}^2(\frac{\mu\partial_z\tilde{u}}{\Gamma\rho^0+\Upsilon^0})]$ and $\tilde{f}=\frac{\Gamma\rho^0+\Upsilon^0}{\mu\partial_z\tilde{u}}f$.

Then comparing (2.40) with the corresponding transformed linear equation in [30], we can conclude that the computation for dealing with (2.40) is more complicated but it will not cause any essential difficulties to follow the programme in [30]. Consequently, applying the similar weighted energy estimate and Nash-Moser-Hörmander iteration as in [30], we can obtain the local existence result for the compressible Prandtl boundary layer Equations (2.32) – (2.34).

Now we state the main result of the well-posedness of the compressible Prandtl boundary layer system to end this subsection.

THEOREM 2.1. *Given the appropriate initial data (ρ_0^0,u_0^0) for the outer Euler flow (2.9) such that it has smooth solution for $0\le t\le T_0$, the density $\Gamma\rho^0$ has both positive lower and upper bounds and so does $\Gamma\rho^0+\Upsilon^0$ naturally, the Sobolev norm $H^s([0,T_0]\times\mathbb{T})$ of $(\Gamma\rho^0,U,V)$ is bounded for a suitably large integer s. Moreover, the assumption $(A1-A4)$ on the initial data $u_0^0(x,z)$ also hold. Then there exists $0<T\le T_0$, such that the initial boundary value problem (2.32)$-$(2.34) has a unique classical solution (u,v) satisfying*

$$\sum_{|m_1|+[(m_2+1)/2]\le k_0}\|\langle z\rangle^l\partial_{(t,x)}^{m_1}\partial_z^{m_2}(u-U)\|_{L^2([0,T]\times\mathbb{T}\times\mathbb{R}^+)}<+\infty \tag{2.43}$$

for a fixed $l>\frac{1}{2}$ depending only on γ given in $(A1)-(A4)$ with $\langle z\rangle=(1+z)$, and

$$\sum_{|m_1|+[(m_2+1)/2]\le k_0-1}\sup_{z\in\mathbb{R}^+}\|\partial_{(t,x)}^{m_1}\partial_z^{m_2}(v-\frac{\Gamma\rho^0Vz}{\Gamma\rho^0+\Upsilon^0})(\cdot,z)\|_{L^2([0,T]\times\mathbb{T})}<+\infty. \tag{2.44}$$

Hence we have determined the term U_1^0 to complete the construction of the approximate solution. Now we summarize this section in the following theorem.

THEOREM 2.2. *Let $m\ge 3, K\in\mathbb{N}$, $\bar\rho$ is a strictly positive smooth function. Assume the component $(\rho_0^j,u_0^j), j\ge 0$ of the initial data satisfying the compatibility conditions with the boundary data and $(\rho_0^0-\bar\rho,u_0^0)\in H^{m+2K+3}(\mathbb{T}\times\mathbb{R}^+)$, $(\rho_0^j,u_0^j)\in H^{m+2K+3-2j}(\mathbb{T}\times\mathbb{R}^+), j\ge 1$. Moreover, the assumptions on the initial data in Theorem 2.1 also hold. Then there exists $T>0$ and a smooth approximation solution (ρ_a,u_a,ϕ_a) of order K in the form (2.1)$-$(2.3) for the NSP system (1.1) such that*

(1) (ρ^0,u^0) is the solution of the outer Euler system (2.9) on $[0,T]$ with initial data (ρ_0^0,u_0^0). Moreover, $(\rho^0,u^0)\in C^0([0,T],H^{m+2K+3}(\mathbb{T}\times\mathbb{R}^+))$, and $\phi^0=-\ln\rho^0$.

(2) $(\rho^j,u^j,\phi^j)\in C^0([0,T],H^{m+3}(\mathbb{T}\times\mathbb{R}^+))$ holds for any $1\le j\le K$.

(3) $\Upsilon^j,U^j,\Phi^j,0\le j\le K$ and their derivatives are smooth and exponentially decay with respect to the fast decay variable z.

3. Linear stability

Based on the construction of the approximate solution for the NSP system, we will derive the error equation, and analyze the stability of the approximate solution.

First, we denote the error terms between the solution $(\rho^\varepsilon,u^\varepsilon,\phi^\varepsilon)$ of the system (1.1) and the approximate solution (ρ_a,u_a,ϕ_a) as following

$$\rho=\rho^\varepsilon-\rho_a,\quad u=u^\varepsilon-u_a,\quad \phi=\phi^\varepsilon-\phi_a.$$

Also, we will use the norm $\|\cdot\|$ to denote $\|\cdot\|_{L^2(\mathbb{T}\times\mathbb{R}^+)}$ throughout the following writing. Then we have the equations of (ρ,u,ϕ)

$$\begin{cases}\partial_t\rho+(u_a+u)\cdot\nabla\rho+\rho\nabla\cdot(u+u_a)+\nabla\cdot(\rho_a u)=\varepsilon^K R_\rho,\\ \partial_t u+(u_a+u)\cdot\nabla u+u\cdot\nabla u_a+T^i(\frac{\nabla\rho}{\rho+\rho_a}-\frac{\nabla\rho_a}{\rho_a}(\frac{\rho}{\rho_a+\rho}))\\ \quad=\nabla\phi+\frac{\mu\varepsilon^2}{\rho_a+\rho}\Delta u+\frac{(\mu+\nu)\varepsilon^2}{\rho+\rho_a}\nabla\nabla\cdot u+g(\rho,\rho_a,u_a)+\varepsilon^K R_u,\\ \varepsilon^2\Delta\phi=\rho-e^{-\phi_a}(e^{-\phi}-1)+\varepsilon^{K+1}R_\phi,\end{cases} \tag{3.1}$$

where the reminder terms (R_ρ,R_u,R_ϕ) satisfy

$$\sup_{[0,T]}\|\nabla^\alpha(R_\rho,R_u,R_\phi)\|\le C\varepsilon^{-\alpha_2},$$

$$\forall\alpha=(\alpha_1,\alpha_2),\qquad |\alpha|\le m,$$

and $g(\rho,\rho_a,u_a)$ is given by

$$g(\rho,\rho_a,u_a) = -\frac{\mu\varepsilon^2}{\rho_a(\rho+\rho_a)}\Delta u_a - \frac{(\mu+\nu)\varepsilon^2\rho}{\rho_a(\rho+\rho_a)}\nabla\nabla\cdot u_a.$$

Unlike the Navier-Stokes-Poisson system with the Navier-slip type boundary condition, one can perform the nonlinear stability analysis by the energy estimate in conormal Sobolev space using a priori estimate. For the NSP system with nonslip boundary condition, the boundary layer is stronger and the high order boundary condition is also undetermined. Hence it is difficult to close the a priori estimate in conormal Sobolev space as in [15] because of the loss of derivatives. To overcome the loss of derivatives, one may seek to set this question into the analytic setting as in [29], which is reasonable, but we will meet at least two obstacles at present: (1) dealing with the nonlinear term $\rho u\cdot\nabla u$ and the linear term $\rho_a\nabla\cdot u$; (2) we can not eliminate the trace of the $t-$derivatives of velocity by using Biot-Sawart law as in [20].

Therefore, we will only consider the stability of the linearized system of (3.1) around the approximate solution (ρ_a,u_a,ϕ_a), which is as following, taking $T^i=1$ without loss of generality,

$$\begin{cases} \partial_t\rho + u_a\cdot\nabla\rho + \rho\nabla\cdot u_a + \nabla\cdot(\rho_a u) = \varepsilon^K R_\rho, \\ \rho_a(\partial_t u + u_a\cdot\nabla u + u\cdot\nabla u_a) + (\nabla\rho - \frac{\rho\nabla\rho_a}{\rho_a}) \\ \quad = \rho_a\nabla\phi + \mu\varepsilon^2\Delta u + (\mu+\nu)\varepsilon^2\nabla\nabla\cdot u + G(\rho,\rho_a,u_a) + \varepsilon^K R_u, \\ \varepsilon^2\Delta\phi = \rho + e^{-\phi_a}\phi + \varepsilon^{K+1}R_\phi, \end{cases} \tag{3.2}$$

where $G(\rho,\rho_a,u_a)$ is given by

$$G(\rho,\rho_a,u_a) = -\frac{\mu\varepsilon^2\rho}{\rho_a}\Delta u_a - \frac{(\mu+\nu)\varepsilon^2}{\rho_a}\nabla\nabla\cdot u_a.$$

The corresponding initial boundary conditions read

$$\rho|_{t=0} = \varepsilon^{K+1}\rho_0, \qquad u|_{t=0} = \varepsilon^{K+1}u_0, \tag{3.3}$$

and

$$u|_{y=0} = 0, \qquad \phi|_{y=0} = 0. \tag{3.4}$$

Now we state the main result on the stability of the linearized system (3.2)-(3.4). Notice that the constant C_a is a universal constant depending only on the approximate solution (ρ_a,u_a,ϕ_a).

THEOREM 3.1. *Let the initial data $(\rho_0,u_0)\in H^3(\mathbb{T}\times\mathbb{R}^+)$ satisfy some relevant compatibility conditions. Assume (ρ_a,u_a,ϕ_a) be the approximate solution of order K constructed in the Theorem 2.2 on $[0,T]$ with $K\in\mathbb{N}, K\geq 4$. Then the solution (ρ,u,ϕ) to the system (3.2)-(3.4) is defined on $[0,T]$ and satisfies the estimate*

$$\|(\rho,u,\phi)\|_{L^\infty([0,T]\times\mathbb{T}\times\mathbb{R}^+)} \leq C_a\varepsilon^{K-3}. \tag{3.5}$$

Before the proof of the Theorem 3.1, we introduce some necessary preliminaries. To deal with the boundary effect, we define the conormal functional space with the norm

$$\|f\|_{H^s_{co}} = \Big(\sum_{|\alpha|\leq s}\|Z^\alpha f\|^2\Big)^{\frac{1}{2}},$$

with $\alpha=(\alpha_0,\alpha_1,\alpha_2)$ and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, and $Z^\alpha=Z_0^{\alpha_0}Z_1^{\alpha_1}Z_2^{\alpha_2}$, where

$$Z_0=\partial_t,\quad Z_1=\partial_x,\quad Z_2=\varphi(y)\partial_y.$$

Here $\varphi(y)$ is a smooth bounded function satisfying $\varphi(0)=0,\varphi'(0)\neq 0$ and $\varphi(y)>0$ for $y>0$, for example,

$$\varphi(y)=\frac{y}{1+y}.$$

To derive the L^∞-estimate in the conormal Sobolev space, the following anisotropic Sobolev embedding inequality [22] will be used for $(t,x,y)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}^+$:

$$\|f\|_{L^\infty([0,T]\times\mathbb{T}\times\mathbb{R}^+)}\leq C(\|\partial_y f\|_{H_{co}^{m_0}}\|f\|_{H_{co}^{m_0}}+\|f\|_{H_{co}^{m_0}}^2),\tag{3.6}$$

where $m_0\geq 2$. For simplicity, we will also use $\|f\|_{m_0}$ to represent $\|f\|_{H_{co}^{m_0}}$, and denote the periodic space $\mathbb{T}\times\mathbb{R}^+$ as Ω.

Then we devote the remaining part to proving the Theorem 3.1. For the linear symmetrizable transport-diffusion Equations (3.2), the local well-posedness on $[0,T]$ is a naturally classical result as long as the approximate solution (ρ_a,u_a,ϕ_a) exists in the same time interval. Now we state the following series of lemmas to conclude the proof of the Theorem 3.1.

LEMMA 3.1. *Under the assumption of Theorem 3.1, assume (ρ,u,ϕ) is the solution to the problem (3.2)−(3.4) on $[0,T]$. Then we have*

$$\sup_{[0,T]}\|\rho,u,\phi,\varepsilon\nabla\phi\|^2+\varepsilon^2\int_0^T\|\nabla u\|^2\leq C_a\varepsilon^{2K}.\tag{3.7}$$

Proof. Calculating $\frac{\rho}{\rho_a}\times(3.2)_1+u\times(3.2)_2$, and then integrating over Ω, one has, using integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\frac{1}{\rho_a}\rho^2+\rho_a u^2)+\mu\varepsilon^2\int_\Omega|\nabla u|^2+(\mu+\nu)\varepsilon^2\int_\Omega|\nabla\cdot u|^2\\&\quad-\frac{1}{2}\int_\Omega\Big(\rho^2(\partial_t+u_a\cdot\nabla)\frac{1}{\rho_a}+u^2(\partial_t+u_a\cdot\nabla)\rho_a\Big)+\frac{1}{2}\int_\Omega(\frac{\rho^2}{\rho_a}\nabla\cdot u_a+\rho_a u^2\nabla\cdot u_a)\\&=\int_\Omega\rho_a u\cdot\nabla\phi+\int_\Omega G(\rho,\rho_a,u_a)u+\int_\Omega\frac{\rho}{\rho_a}\varepsilon^K R_\rho+\varepsilon^K uR_u,\end{aligned}\tag{3.8}$$

where we used the boundary condition (3.4).

From the construction of the approximate solution, $U_2^0=0,U_1^0\neq 0$, one can obtain

$$\begin{aligned}&|(\partial_t+u_a\cdot\nabla)\frac{1}{\rho_a}|+|(\partial_t+u_a\cdot\nabla)\rho_a|+|\rho_a\nabla\cdot u_a|+|\frac{1}{\rho_a}\nabla\cdot u_a|\leq C_a,\\&|\frac{1}{\rho_a}\Delta u_a|\leq\frac{C_a}{\varepsilon^2},\quad|\frac{1}{\rho_a}\nabla\nabla\cdot u_a|\leq\frac{C_a}{\varepsilon}.\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\frac{1}{\rho_a}\rho^2+\rho_a u^2)+\mu\varepsilon^2\int_\Omega|\nabla u|^2+(\mu+\nu)\varepsilon^2\int_\Omega|\nabla\cdot u|^2\\&\leq\int_\Omega\rho_a u\cdot\nabla\phi+C_a\int_\Omega(\rho^2+u^2)+\varepsilon^{2K}.\end{aligned}\tag{3.9}$$

From the first equation of (3.2), one has

$$\begin{aligned}&\int_\Omega \rho_a u\cdot\nabla\phi=-\int_\Omega \nabla\cdot(\rho_a u)\phi\\&=\int_\Omega \partial_t\rho\phi+\nabla\cdot(\rho u_a)\phi-\varepsilon^K R_\rho\phi\\&=\int_\Omega \Big(\varepsilon^2\Delta\partial_t\phi-\partial_t(e^{-\phi_a}\phi)-\varepsilon^{K+1}\partial_t R_\phi\Big)\phi\\&\quad-\int_\Omega\Big(\varepsilon^2\Delta\phi-(e^{-\phi_a}\phi)-\varepsilon^{K+1}R_\phi\Big)u_a\cdot\nabla\phi-\int_\Omega\varepsilon^K R_\rho\phi\\&\le-\frac{\varepsilon^2}{2}\frac{d}{dt}\int_\Omega|\nabla\phi|^2-\frac12\frac{d}{dt}\int_\Omega e^{-\phi_a}\phi^2+C_a\int_\Omega\phi^2+\varepsilon^2|\nabla\phi|^2+\varepsilon^{2K}.\end{aligned}\tag{3.10}$$

Substituting (3.10) into (3.9), one has

$$\begin{aligned}&\frac12\frac{d}{dt}\int_\Omega(\rho^2+u^2+\phi^2+\varepsilon^2|\nabla\phi|^2)+\mu\varepsilon^2\int_\Omega|\nabla u|^2+(\mu+\nu)\varepsilon^2\int_\Omega|\nabla\cdot u|^2\\&\le C_a\int_\Omega(\rho^2+u^2+\phi^2+\varepsilon^2|\nabla\phi|^2)+\varepsilon^{2K}.\end{aligned}\tag{3.11}$$

Hence we complete the proof of the Lemma 3.1 by using Grönwall's inequality. □

For the tangential estimate, we have the following result.

LEMMA 3.2. *Under the assumption of Theorem 3.1, assume* (ρ,u,ϕ) *is the solution to the problem* (3.2)–(3.4) *on* $[0,T]$. *Then we have, for* $j=1,2,3$,

$$\sup_{[0,T]}\|Z^j\rho,Z^ju,Z^j\phi,\varepsilon\nabla Z^j\phi\|^2+\varepsilon^2\int_0^T\|\nabla Z^ju\|^2\le C_a\varepsilon^{2K-2j}.\tag{3.12}$$

Proof. Based on the analysis in the last section, we know that the component u_1 of the velocity occupies the strong boundary layer here, which will cause a singularity of order $O(\frac{1}{\varepsilon})$. To overcome this difficulty in the estimate, one may use, for example, $Z=\varphi(y)\partial_y$,

$$\begin{aligned}|Z^ju_{a1}|&\le C_a(1+|\varphi^j(y)\frac{1}{\varepsilon^j}\partial_z^ju_{a1}|)\\&\le C_a(1+|z^j\partial_z^ju_{a1}|)\\&\le C_a,\end{aligned}\tag{3.13}$$

where we used the fast decay property of the boundary layer profile. Then we can prove this lemma as in [16]. □

Similarly, we can obtain the following mixed normal estimate of the density ρ.

LEMMA 3.3. *Under the assumption of Theorem 3.1, assume* (ρ,u,ϕ) *is the solution to the problem* (3.2)–(3.4) *on* $[0,T]$. *Then we have, for* $j=0,1,2$

$$\varepsilon^2\sup_{[0,T]}\|Z^j\partial_y\rho\|^2+\int_0^T\|Z^j\partial_y\rho\|^2\le C_a\varepsilon^{2K-2-2j}.\tag{3.14}$$

Proof. One can refer to [16] for the process of proof, but notice the difference that $|\partial_yu_{a1}|\le\frac{C_a}{\varepsilon}$, where the results in Lemma 3.2 are also used. □

Consequently, by virtue of the anisotropic Sobolev embedding inequality (3.6) and the facts

$$\|(\rho,u,\phi)\|_2 \leq \varepsilon^{K-2}, \quad \|\partial_y(\rho,u,\phi)\|_2 \leq \varepsilon^{K-4}$$

from Lemma 3.1 − 3.3, we can prove the Theorem 3.1 immediately.

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