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ASYMPTOTIC ANALYSIS OF THE BOLTZMANN EQUATION WITH VERY SOFT POTENTIALS FROM ANGULAR CUTOFF TO NON-CUTOFF*

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Abstract. Our focus is the Boltzmann equation in a torus under very soft potentials around equilibrium. We analyze the asymptotics of the equation from angular cutoff to non-cutoff. We first prove a refined decay result of the semi-group stemming from the linearized Boltzmann operator. Then we prove the global well-posedness of the equations near equilibrium, refined decay patterns of the solutions. Finally, we rigorously give the asymptotic formula between the solutions to cutoff and non-cutoff equations with an explicit convergence rate.

Keywords. Boltzmann equation; very soft potential; asymptotic analysis; angular cutoff; angular non-cutoff; short-range interaction; long-range interaction.

AMS subject classifications. 35Q20; 35B40; 82C40.

1. Introduction

Grad's angular cutoff assumption plays an important role in the study of Boltzmann equation throughout the history. A relatively satisfactory mathematical theory has been established for the cutoff case. We review some relevant results in the near-equilibrium framework, i.e., a small perturbation around global Maxwellians. Independently, Caflisch [7] and Ukai-Asano [20] constructed global classical solutions near-equilibrium for the inhomogeneous Boltzmann equation with a soft potential $\gamma > -1$ (see below for the meaning of parameter γ). Guo [12] extended the result to the full range $\gamma > -3$.

Without Grad's cutoff assumption, Pao in [17, 18] studied the spectrum of the linearized Boltzmann operator. In the seminal work [1], the authors proved some entropy dissipation formula, which accelerates the study of non-cutoff Boltzmann equation. In the near-equilibrium framework, two groups independently built the well-posedness theory by introducing some implicit anisotropic norms, see Alexandre-Morimoto-Ukai-Xu-Yang [4] and Gressman-Strain [10].

Now that the well-posedness theory has been established for both cutoff and non-cutoff Boltzmann equations, it is natural to consider the relation between them. In the near-equilibrium framework, the analysis of linearized Boltzmann operator plays a central role. The asymptotic analysis of the linearized Boltzmann operator from cutoff to non-cutoff is given in [14]. Understandably, the analysis relies on keeping the angular cutoff threshold as a parameter and getting some estimates uniformly with respect to it. As an application of the analysis, in the moderately soft potential case $-2s \leq \gamma < 0$, different decay patterns are connected in [14] for the semi-groups generated by the cutoff and non-cutoff linearized Boltzmann operators. In this work, we consider the very soft potential case $-3 < \gamma < -2s$, and discover the role of cutoff threshold in the asymptotic

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process from cutoff to non-cutoff.

1.1. The Boltzmann equation. To go further, we introduce the Boltzmann equation and its linearized counterpart.

1.1.1. The Boltzmann collision operator. The Boltzmann collision operator Q is a bilinear operator acting only on the velocity variable v, given by,

$$Q(g,h)(v) := \int_{\mathbb{S}^2\times\mathbb{R}^3} B(v-v_*,\sigma)(g'_*h' - g_*h)d\sigma dv_*.$$

Here the standard shorthand $h=h(v)$, $g_*=g(v_*)$, $h'=h(v')$, $g'_*=g(v'_*)$ is used, where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \sigma\in\mathbb{S}^2. \tag{1.1}$$

The Boltzmann collision kernel $B(v-v_*,\sigma)$ is always assumed to depend only on $|v-v_*|$ and $\cos\theta := \frac{v-v_*}{|v-v_*|}\cdot\sigma$. By some symmetrization, we can assume that $B(v-v_*,\sigma)$ is supported in the set $0\le\theta\le\frac{\pi}{2}$. The collision kernels studied in this work satisfy the following conditions.

For the non-cutoff collision kernel, we assume that

- The cross-section $B(v-v_*,\sigma)$ takes a product form of

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta),$$

where $-3<\gamma<0$ and b is a nonnegative function satisfying that

$$K^{-1}\theta^{-1-2s} \le \sin\theta b(\cos\theta) \le K\theta^{-1-2s}, \quad \text{with } 0<s<1,\ K\ge 1.$$

The parameters γ and s verify that $\gamma+2s<0$.

Note that we impose $\gamma+2s<0$, which represents very soft potential. For moderately soft potential $-2s\le\gamma<0$, some asymptotic analysis has been done in [14].

The Cauchy problem of the non-cutoff Boltzmann equation in a periodic box reads:

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = Q(F,F), & t>0, x\in\mathbb{T}^3, v\in\mathbb{R}^3; \\ F|_{t=0} = F_0. \end{cases} \tag{1.2}$$

Here $F(t,x,v)\ge 0$ is the density function of particles which at time $t\ge 0$, position $x\in\mathbb{T}^3 := [-\pi,\pi]^3$, move with velocity $v\in\mathbb{R}^3$.

For the cutoff collision kernel, we assume that

- The cross-section $B^\epsilon(v-v_*,\sigma)$ takes a product form of

$$B^\epsilon(v-v_*,\sigma) = |v-v_*|^\gamma b^\epsilon(\cos\theta),$$

where $b^\epsilon = b\big(1-\phi(\sin\frac{\theta}{2}/\epsilon)\big)$, where $0<\epsilon\le\frac{\sqrt{2}}{2}$ and ϕ is a function defined by (1.18), which has support in $[0,4/3]$ and equals to 1 on $[0,3/4]$.

The angular cutoff Boltzmann collision operator and its associated equation are defined by

$$Q^\epsilon(g,h)(v) := \int_{\mathbb{S}^2\times\mathbb{R}^3} B^\epsilon(v-v_*,\sigma)(g'_*h' - g_*h)d\sigma dv_*,$$

and

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q^\epsilon(F,F), & t>0, x\in\mathbb{T}^3, v\in\mathbb{R}^3; \\ F|_{t=0} = F_0. \end{cases} \tag{1.3}$$

We mention that the solutions to (1.2) and (1.3) have the fundamental physical properties of conserving total mass, momentum and kinetic energy, that is, for all $t \geq 0$,

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F(t,x,v)\phi(v)dxdv = \int_{\mathbb{T}^3\times\mathbb{R}^3} F(0,x,v)\phi(v)dxdv, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1,2,3.$$

As a result, if initially $F_0(x,v)$ has the same mass, momentum and total energy as those of the global Maxwellian $\mu(v) := (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$, then for any $t \geq 0$,

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} (F-\mu)(t)\phi dxdv = 0, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1,2,3. \tag{1.4}$$

1.1.2. The linearized Boltzmann collision operator. In the non-cutoff case, the linearized Boltzmann operator $\mathcal{L}$ is defined by

$$\mathcal{L}g := -\Gamma(\mu^{1/2}, g) - \Gamma(g, \mu^{1/2}), \quad \text{where} \quad \Gamma(g,h) := \mu^{-1/2} Q(\mu^{1/2}g, \mu^{1/2}h). \tag{1.5}$$

In the cutoff case, the linearized Boltzmann operator $\mathcal{L}^\epsilon$ is defined by

$$\mathcal{L}^\epsilon g := -\Gamma^\epsilon(\mu^{1/2}, g) - \Gamma^\epsilon(g, \mu^{1/2}), \quad \text{where} \quad \Gamma^\epsilon(g,h) := \mu^{-1/2} Q^\epsilon(\mu^{1/2}g, \mu^{1/2}h). \tag{1.6}$$

The null spaces of the operators $\mathcal{L}^\epsilon$ and $\mathcal{L}$, $\mathcal{N}(\mathcal{L}^\epsilon)$ and $\mathcal{N}(\mathcal{L})$, verify

$$\mathcal{N}(\mathcal{L}^\epsilon) = \mathcal{N}(\mathcal{L}) = \mathcal{N} := \mathrm{span}\{\sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}|v|^2\}.$$

If we set $F = \mu + \mu^{1/2} f$, then (1.2) and (1.3) become

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L} f = \Gamma(f,f), & t>0; \\ f|_{t=0} = f_0. \end{cases} \tag{1.7}$$

and

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^\epsilon f = \Gamma^\epsilon(f,f), & t>0; \\ f|_{t=0} = f_0. \end{cases} \tag{1.8}$$

We can regard (1.8) when $\epsilon = 0$ as (1.7). Without loss of generality, we assume that f_0 verifies

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \sqrt{\mu} f_0 \phi dxdv = 0, \quad \phi(v) = 1, v_j, |v|^2, \quad j = 1,2,3. \tag{1.9}$$

By (1.4), the solutions to (1.7) and (1.8) also verify (1.9).

1.2. Problems and motivations. Our motivations originate from the following two problems.

1.2.1. Problem 1: longtime behavior. What is the longtime behavior of $e^{-\mathcal{L}^\epsilon t} f_0$ with $f_0 \in \mathcal{N}^\perp$ for $\gamma \in (-3, -2s)$ in the limit process that ϵ goes to 0?

Set $f^\epsilon(t) = e^{-\mathcal{L}^\epsilon t} f_0$ and $f(t) = e^{-\mathcal{L} t} f_0$. As we know, for $\gamma \in (-3, -2s)$, both f^ϵ and f enjoy polynomial decay rate if $f_0 \in L^2_l$ for some $l > 0$. However, their decay rates are different. To be precise, if $l = -p(\gamma/2 + s)$ for some $p > 0$, one has

$$|f(t)|^2_{L^2} \lesssim O(t^{-p}). \tag{1.10}$$

However, according to [19], one only has

$$|f^\epsilon(t)|^2_{L^2} \lesssim O(t^{-q}), \tag{1.11}$$

with $q = p(1 + 2s/\gamma) < p$. Let us explain a bit more how these results can be derived. Denote by $\langle f, g \rangle := \int_{\mathbb{R}^3} f(v) g(v) dv$ the inner product in L^2 space. Previous works [3, 4, 10, 11, 13] show that

$$\langle \mathcal{L} f, f \rangle + |f|^2_{L^2_{\gamma/2}} \sim |f|^2_{L^2_{s+\gamma/2}} + |f|^2_{H^s_{\gamma/2}} + |(-\Delta_{\mathbb{S}^2})^{s/2} f|^2_{L^2_{\gamma/2}}. \tag{1.12}$$

From which together with spectral gap estimate $\langle \mathcal{L} f, f \rangle \gtrsim |f - \mathbb{P} f|^2_{L^2_{\gamma/2}}$, for $f \in \mathcal{N}^\perp$ one further has

$$\langle \mathcal{L} f, f \rangle \gtrsim |f|^2_{L^2_{s+\gamma/2}}. \tag{1.13}$$

Here $\mathbb{P}$ is the projection (see (1.21) below) to the null space $\mathcal{N}$. In the cutoff case as in [12], one has for $f \in \mathcal{N}^\perp$,

$$\langle \mathcal{L}^\epsilon f, f \rangle \gtrsim |f|^2_{L^2_{\gamma/2}}. \tag{1.14}$$

By some interpolation techniques, one can get (1.10) and (1.11) from (1.13) and (1.14) respectively.

Recently, [14] shows that

$$\langle \mathcal{L}^\epsilon f, f \rangle + |f|^2_{L^2_{\gamma/2}} \sim |W^\epsilon f|^2_{L^2_{\gamma/2}} + |W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2}) f|^2_{L^2_{\gamma/2}} + |W^\epsilon(D) f|^2_{L^2_{\gamma/2}},$$

where W^ϵ is defined by

$$W^\epsilon(v) := (1 + |v|^2)^{s/2} \phi(\epsilon |v|) + \epsilon^{-s}(1 - \phi(\epsilon |v|)). \tag{1.15}$$

Then for $f \in \mathcal{N}^\perp$, there holds

$$\langle \mathcal{L}^\epsilon f, f \rangle \gtrsim |W^\epsilon f|^2_{L^2_{\gamma/2}}. \tag{1.16}$$

Sending ϵ to 0, (1.16) turns out to be (1.13). However, (1.11) does not lead to (1.10). The mismatch here indicates the result (1.11) is not good enough when ϵ is very small. This inconsistency is largely due to that like in [12] or other cutoff setting, a specified and fixed value of the parameter ϵ is considered. Therefore, it is meaningful and interesting to consider the limit process and to improve the estimate (1.11) in order to eradiate the mismatch.

1.2.2. Problem 2: asymptotic formula. Which kind of asymptotic formula connects the solutions of the nonlinear Equations (1.7) and (1.8) ?

Formally when the parameter ϵ goes to 0, the solution f^ϵ of (1.8) is expected to converge to the solution f of (1.7). The motivation here is to justify this convergence and look for an asymptotic formula to capture the error between them. Like in [14], it is natural to conjecture

$$f^\epsilon - f \sim O(\epsilon^{2-2s}).$$

Obviously some uniform estimates w.r.t. ϵ are needed in order to rigorously derive the above result.

1.3. Notations. We collect some function spaces and notations in this subsection. Most of them are standard. One may skip this part and come back when necessary.

1.3.1. Basic notations. We denote the multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ with $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. We write $a\lesssim b$ to indicate that there is a universal constant C which is independent of a,b but may depend on the parameters γ,s and be different across different lines, such that $a\leq Cb$. We use the notation $a\sim b$ whenever $a\lesssim b$ and $b\lesssim a$. The Japanese bracket $\langle\cdot\rangle$ is defined by $\langle v\rangle:=(1+|v|^2)^{\frac{1}{2}}$. The weight function W_l is defined by $W_l(v):=\langle v\rangle^l$. We denote $C(\lambda_1,\lambda_2,\cdots,\lambda_n)$ or $C_{\lambda_1,\lambda_2,\cdots,\lambda_n}$ by a constant depending on parameters $\lambda_1,\lambda_2,\cdots,\lambda_n$. The notations $\langle f,g\rangle:=\int_{\mathbb{R}^3} f(v)g(v)dv$ and $(f,g):=\int_{\mathbb{R}^3\times\mathbb{T}^3} fgdxdv$ are used to denote the inner products for v variable and for x,v variables respectively. As usual, 1_A is the characteristic function of a set A.

1.3.2. Function spaces. For simplicity, we set $\partial^\alpha:=\partial^\alpha_x,\partial_\beta:=\partial^\beta_v,\partial^\alpha_\beta:=\partial^\alpha_x\partial^\beta_v$.

(1) For $n\in\mathbb{N},l\in\mathbb{R}$, the weighted Sobolev space on $\mathbb{R}^3$ is defined by

$$H^n_l:=\left\{f(v)\big||f|^2_{H^n_l}:=\sum_{|\beta|\leq n}|\partial_\beta f|^2_{L^2_l}<\infty\right\},$$

where $|f|_{L^2_l}:=|W_l f|_{L^2}$ is the usual L^2 norm with weight W_l.

(2) For $n\in\mathbb{N},l\in\mathbb{R}$, we denote the weighted pure order-n space on $\mathbb{R}^3$ by

$$\dot{H}^n_l:=\left\{f(v)\big||f|^2_{\dot{H}^n_l}:=\sum_{|\beta|=n}|\partial_\beta f|^2_{L^2_l}<\infty\right\}. \tag{1.17}$$

(3) For $m\in\mathbb{N}$, we denote the Sobolev space on $\mathbb{T}^3$ by

$$H^m_x:=\left\{f(x)\big||f|^2_{H^m_x}:=\sum_{|\alpha|\leq m}|\partial^\alpha f|^2_{L^2_x}<\infty\right\}.$$

(4) For $m,n\in\mathbb{N},l\in\mathbb{R}$, the weighted Sobolev space on $\mathbb{T}^3\times\mathbb{R}^3$ is defined by

$$H^m_xH^n_l:=\left\{f(x,v)\big|\|f\|^2_{H^m_xH^n_l}:=\sum_{|\alpha|\leq m,|\beta|\leq n}||\partial^\alpha_\beta f|_{L^2_l}|^2_{L^2_x}<\infty\right\}.$$

For simplicity, we write $\|f\|_{H^m_xL^2_l}:=\|f\|_{H^m_xH^0_l}$ if $n=0$ and $\|f\|_{L^2_xL^2_l}:=\|f\|_{H^0_xH^0_l}$ if $m=n=0$. The space $H^m_x\dot{H}^n_l$ can be similarly defined.

1.3.3. Dyadic decomposition. Let us give a brief introduction to dyadic decomposition. Let $B_{4/3} := \{v \in \mathbb{R}^3 : |v| \leq 4/3\}$ and $C := \{v \in \mathbb{R}^3 : 3/4 \leq |v| \leq 8/3\}$. Then one may introduce two radial functions $\phi \in C_0^\infty(B_{4/3})$ and $\psi \in C_0^\infty(C)$ which satisfy

$$0 \leq \phi, \psi \leq 1, \text{ and } \phi(v) + \sum_{j \geq 0} \psi(2^{-j} v) = 1, \text{ for all } v \in \mathbb{R}^3. \tag{1.18}$$

Now define $\varphi_{-1}(v) := \phi(v)$ and $\varphi_j(v) := \psi(2^{-j} v)$ for any $v \in \mathbb{R}^3$ and $j \geq 0$. Let $\mathcal{P}_j$ be the projection operator on the region $|v| \sim 2^j$ defined by $(\mathcal{P}_j f)(v) := \varphi_j(v) f(v)$. Then one has the following dyadic decomposition

$$f = \sum_{j=-1}^{\infty} \mathcal{P}_j f, \tag{1.19}$$

for any function defined on $\mathbb{R}^3$. Let us further introduce

$$f^l = \phi(\epsilon \cdot) f, \quad f^h = (1 - \phi(\epsilon \cdot)) f, \tag{1.20}$$

which stand for low velocity part $|v| \lesssim 1/\epsilon$ and high velocity part $|v| \gtrsim 1/\epsilon$ of function f.

1.3.4. Macro-Micro decomposition. Recalling

$$\mathcal{N} = \text{span}\{\sqrt{\mu}, \sqrt{\mu} v_1, \sqrt{\mu} v_2, \sqrt{\mu} v_3, \sqrt{\mu} |v|^2\},$$

we introduce the projection operator $\mathbb{P}$ on $\mathcal{N}$ as follows:

$$\mathbb{P} f = (a + b \cdot v + c|v|^2) \sqrt{\mu}, \tag{1.21}$$

where for $1 \leq i \leq 3$,

$$a = \int_{\mathbb{R}^3} (2 - \frac{|v|^2}{2}) \sqrt{\mu} f dv; \quad b_i = \int_{\mathbb{R}^3} v_i \sqrt{\mu} f dv; \quad c = \int_{\mathbb{R}^3} (\frac{|v|^2}{6} - \frac{1}{2}) \sqrt{\mu} f dv. \tag{1.22}$$

Now $f = \mathbb{P} f + (f - \mathbb{P} f)$. Usually, $\mathbb{P} f$ is called the macro part, and $f - \mathbb{P} f$ is called the micro part.

1.3.5. Function spaces related to coercivity estimate. Recalling W^ϵ defined by (1.15), we naturally define some spaces resulting from the coercivity estimates of $\mathcal{L}^\epsilon$ in Theorem 2.1. For $l \geq 0, -l \leq m \leq l$, let Y_l^m be real spherical harmonics verifying $(-\triangle_{\mathbb{S}^2}) Y_l^m = l(l+1) Y_l^m$. Then the operator $W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})$ is defined by: if $v = r\sigma$, then

$$(W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2}) f)(v) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} W^\epsilon((l(l+1))^{1/2}) Y_l^m(\sigma) f_l^m(r), \tag{1.23}$$

where $f_l^m(r) = \int_{\mathbb{S}^2} Y_l^m(\sigma) f(r\sigma) d\sigma$. Now we introduce

(1) *Space* $L^2_{\epsilon,l}$. For functions defined on $\mathbb{R}^3$, the space $L^2_{\epsilon,l}$ with $l \in \mathbb{R}$ is defined by

$$L^2_{\epsilon,l} := \{f(v) \,|\, |f|^2_{L^2_{\epsilon,l}} < \infty\},$$

where

$$|f|^2_{L^2_{\epsilon,l}} := |W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2}) W_l f|^2_{L^2} + |W^\epsilon(D) W_l f|^2_{L^2} + |W^\epsilon W_l f|^2_{L^2}.$$

(2) *Space* $H^m_x H^n_{\epsilon,l}$. For functions defined on $\mathbb{T}^3 \times \mathbb{R}^3$, the space $H^m_x H^n_{\epsilon,l}$ with $m,n \in \mathbb{N}$ is defined by

$$H^m_x H^n_{\epsilon,l} := \left\{ f(x,v) \Big| \|f\|^2_{H^m_x H^n_{\epsilon,l}} := \sum_{|\alpha|\le m, |\beta|\le n} ||\partial^\alpha_\beta f|_{L^2_{\epsilon,l}}|^2_{L^2_x} < \infty \right\}.$$

For simplicity, we set $\|f\|_{H^m_x L^2_{\epsilon,l}} := \|f\|_{H^m_x H^0_{\epsilon,l}}$ if $n=0$ and $\|f\|_{L^2_x L^2_{\epsilon,l}} := \|f\|_{H^0_x H^0_{\epsilon,l}}$ if $m=n=0$. Again, the space $H^m_x \dot{H}^n_{\epsilon,l}$ can be defined accordingly.

1.4. Main results. Our first result is on the longtime behavior of $e^{-\mathcal{L}^\epsilon t} f_0$ with $f_0 \in \mathcal{N}^\perp$.

THEOREM 1.1. *Let* $\epsilon \ge 0$ *be small enough,* $\gamma \in (-3,-2s), N \in \mathbb{N}, l \ge 2, p > 0$ *and* $f_0 \in \mathcal{N}^\perp$. *Then* $f^\epsilon(t) := e^{-\mathcal{L}^\epsilon t} f_0$ *verifies the following statements.*

(1) **(Refined polynomial decay rates.)** *Assume* $f_0 \in H^N_{l-p(\gamma/2+s)}$*, let* $q = p(1+2s/\gamma)$. *For simplicity and clarity, denote* $c(f_0,\epsilon) := 2c(\epsilon^{2s})^{\frac{pq}{p-q}} C(p,q,N) |f_0|^2_{H^N_{l-p(\gamma/2+s)}} \sim (\epsilon^{2s})^{\frac{pq}{p-q}} |f_0|^2_{H^N_{l-p(\gamma/2+s)}}$, *where* $c > 1$ *is a constant depending only on* $N, l,$ *and* $C(p,q,N)$ *is an explicitly computable constant depending only on* p,q,N. *If* $|f_0|^2_{H^N_l} \ge c(f_0,\epsilon)$, *then there is a critical time* $t_* > 0$ *such that* $|f^\epsilon(t_*)|^2_{H^N_l} \le c(f_0,\epsilon) \le c|f^\epsilon(t_*)|^2_{H^N_l}$ *and*

$$|f^\epsilon(t)|^2_{H^N_l} \lesssim \frac{|f_0|^2_{H^N_l}}{(1+C_1 t)^p} 1_{t \le t_*} + \frac{|f^\epsilon(t_*)|^2_{H^N_l}}{(1+C_2(t-t_*))^q} 1_{t > t_*}. \tag{1.24}$$

Here $C_1 \sim |f_0|^{2/p}_{H^N_l} |f_0|^{-2/p}_{H^N_{l-p(\gamma/2+s)}}, C_2 \sim \epsilon^{2sq/(p-q)}$.
If $c|f_0|^2_{H^N_l} \le c(f_0,\epsilon)$, *then*

$$|f^\epsilon(t)|^2_{H^N_l} \lesssim \frac{|f_0|^2_{H^N_l}}{(1+C_2 t)^q}. \tag{1.25}$$

Here $C_2 \sim |f_0|^{2/q}_{H^N_l} |f_0|^{-2/q}_{H^N_{l-p(\gamma/2+s)}}$.

(2) **(Almost energy conservation in an arbitrarily large time span.)** *Assume* $|f_0|^2_{L^2} = 1$ *and* $|\mathcal{P}_j f_0|^2_{L^2} = 1-\eta$ *with* η *sufficiently small and* $2^j \ge \epsilon^{-1}$, *then for* $t \in [0, C^{-1}\epsilon^{2s} 2^{-j\gamma}\eta]$, *there holds*

$$|\mathcal{P}_j f^\epsilon(t)|^2_{L^2} \ge 1 - 2\eta - C\exp(-C_1 2^{2j}). \tag{1.26}$$

Here C, C_1 *are two universal constants.*

(3) **(Exponential decay in an arbitrarily large time span.)** *Assume* $|f_0|^2_{L^2} = 1$. *Fix* $\delta > 0$, *suppose* $j \in \mathbb{N}$ *verifies* $2^{-j\gamma}\exp(-C_1 2^{2j}) \le \delta$ *and* $2^j \ge \epsilon^{-1}$, *denote* $\Lambda = \lambda \epsilon^{-2s} 2^{j\gamma}$, $K = \frac{|\mathcal{P}_j f_0|^2_{L^2}}{\lambda^{-1} C(1+\delta)\epsilon^{2s}}$. *If* $K > 2$, *then for* $t \in [0, 2\lambda^{-1}\epsilon^{2s} 2^{-j\gamma} \ln(K-1)]$, *there holds*

$$|\mathcal{P}_j f^\epsilon(t)|^2_{L^2} \le \exp(-\Lambda t/2) |\mathcal{P}_j f_0|^2_{L^2}. \tag{1.27}$$

Here C, C_1 *are two universal constants and* λ *is the constant in* (2.2).

Some comments are in order:

Remark 1.1. If $f_0 \in H^N_{-p(\gamma/2+s)}$, then $\lim_{\epsilon\to 0} c(f_0,\epsilon)=0$, which means that $|f_0|^2_{H^N_l} \geq c(f_0,\epsilon)$ is valid when ϵ is sufficiently small. Therefore (1.24) invokes, and by sending ϵ to 0, we have $t_* \to \infty$ and thus recover the well-known polynomial decay (1.10) for solutions of the non-cutoff linearized Boltzmann equation,

$$|f^\epsilon(t)|^2_{H^N_l} \lesssim \frac{|f_0|^2_{H^N_l}}{(1+C_1 t)^p}. \tag{1.28}$$

Remark 1.2. Note that, the critical time t_* in (1.24) is a turning point of decay rates. That is, before the critical time t_*, the solution decays with the rate $O(t^{-p})$; after t_*, the decay rate becomes to $O(t^{-q})$. Note that t_* is the time when $|f^\epsilon(t_*)|^2_{H^N_l} \sim c(f_0,\epsilon)$, so it could be very large when ϵ is very small. In a word, the decay pattern is closely related to the cutoff parameter ϵ. In previous works, see [19] for instance, under the same assumption $f_0 \in H^N_{-p(\gamma/2+s)}$, since the parameter ϵ is fixed, in terms of large time behavior, one has

$$|f^\epsilon(t)|^2_{H^N_l} \lesssim \frac{|f_0|^2_{H^N_l}}{(1+C_2 t)^q}. \tag{1.29}$$

Because of the largeness of t_* and $p>q$, (1.24) is more refined than (1.29). More importantly, it reveals the role of the cut-off parameter ϵ, and discloses the difference between cutoff and non-cutoff. One may see the difference clearly in Figure 1.1, where

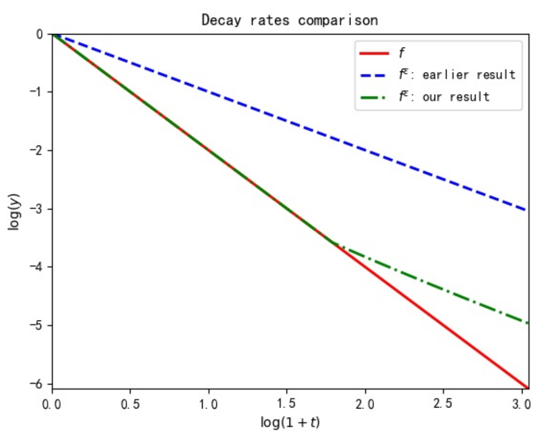

Fig. 1.1. *Comparison of different decay rates.*

we set $p=2$ and $q=1$ with $\gamma=-4s$. We choose $\epsilon=1/10$ in order to have a relatively visible difference. Note that the graph is drawn after taking the logarithm. Thus negative linear relation implies polynomial decay rates. In Figure 1.1, the red solid line represents decay of the solution f to the non-cutoff linearized Boltzmann equation. For the solution f^ϵ to the cutoff linearized Boltzmann equation, earlier result (1.29) is depicted by the blue dashed line, and our result (1.24) by the green dot-and-dashed line.

We have two comments. First, before the critical time t_*, both f and f^ϵ enjoy the same faster decay rate $O(t^{-2})$, while after t_*, the decay rate of f^ϵ shifts to $O(t^{-1})$. Second, the critical time t_* is extremely large when ϵ is extremely small, which demonstrates the superiority of the dot-and-dashed line over the dashed line, and thus that of (1.24) over (1.29).

REMARK 1.3. Equation (1.26) tells us that if the initial data f_0 is concentrated in some ring around $|v| \sim 2^j$ and far away from origin, the L^2 energy would conserve for a very long time. On the other hand, (1.27) insures that over the long time interval, the corresponding L^2 energy decays in an exponential pattern, albeit with a very slow rate $\lambda\epsilon^{-2s}2^{j\gamma}/2$.

Our second result is concerned with the global well-posedness and the global dynamics of Equation (1.8). As a direct consequence, we derive the asymptotic formula for the solutions to (1.7) and (1.8) , which solves **Problem 2**. We will use the following energy functional

$$\mathcal{E}^{N,l}(f) := \sum_{j=0}^{N} ||f||^2_{H^{N-j}_x \dot{H}^j_{l+j\gamma}}. \tag{1.30}$$

We assume $l \geq 2 - N\gamma$ in order to apply Lemma 2.1. For simplicity, set $\mathcal{E}^N(f) := \mathcal{E}^{N,2-N\gamma}(f)$.

THEOREM 1.2. *Let $\epsilon \geq 0$ be small enough, $-3 < \gamma < -2s$ and $\delta_0 > 0$ be a sufficiently small constant which is independent of ϵ. Let f_0 verify (1.9) and $\mathcal{E}^4(f_0) \leq \delta_0$.*

(1) **(Global well-posedness.)** *The Cauchy problem (1.8) (when $\epsilon = 0$, it is understood as (1.7)) admits a unique and global solution f^ϵ verifying*

$$\sup_{t \geq 0} \mathcal{E}^4(f^\epsilon(t)) \leq C_4 \mathcal{E}^4(f_0), \tag{1.31}$$

for some universal constant C_4.

(2) **(Propagation of regularity.)** *Fix $N \geq 4, l \geq 2 - N\gamma$, there is a sufficiently small constant $0 < \delta_{N,l} \leq \delta_0$ such that, if $\mathcal{E}^4(f_0) \leq \delta_{N,l}$ and $\mathcal{E}^{N,l}(f_0) < \infty$, then*

$$\sup_{t \geq 0} \mathcal{E}^{N,l}(f^\epsilon(t)) \leq P_{N,l}\left(\mathcal{E}^{N,l}(f_0)\right). \tag{1.32}$$

Here $\delta_{N,l}$ could depend on N, l but is independent of ϵ. $P_{N,l}$ is an increasing function verifying $P_{N,l}(0) = 0$.

(3) **(Global dynamics.)** *Fix $N \geq 4, l \geq 2 - N\gamma, p > 0, q = p(1 + 2s/\gamma)$, assume $\mathcal{E}^4(f_0) \leq \delta_{N,l-p(\gamma/2+s)}$ and $\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0) < \infty$. For simplicity and clarity, denote*

$$c(f_0,\epsilon) := 2c(\epsilon^{2s})^{\frac{pq}{p-q}} C(p,q,N) \mathcal{E}^{N,l-p(\gamma/2+s)}(f_0) \sim (\epsilon^{2s})^{\frac{pq}{p-q}} \mathcal{E}^{N,l-p(\gamma/2+s)}(f_0),$$

where $c > 1$ is a constant depending only on N, l, and $C(p,q,N)$ is an explicitly computable constant depending on p, q, N. If $\mathcal{E}^{N,l}(f_0) \geq c(f_0,\epsilon)$, there is critical time $t_* > 0$ such that $\mathcal{E}^{N,l}(f^\epsilon(t_*)) \leq c(f_0,\epsilon) \leq c\mathcal{E}^{N,l}(f^\epsilon(t_*))$ and

$$\mathcal{E}^{N,l}(f^\epsilon(t)) \lesssim \frac{\mathcal{E}^{N,l}(f_0)}{(1+C_1 t)^p} 1_{t \leq t_*} + \frac{\mathcal{E}^{N,l}(f^\epsilon(t_*))}{(1+C_2(t-t_*))^q} 1_{t > t_*}. \tag{1.33}$$

Here $C_1 \sim \mathcal{E}^{N,l}(f_0)^{1/p} \mathcal{E}^{N,l-p(\gamma/2+s)}(f_0)^{-1/p}, C_2 \sim \epsilon^{2sq/(p-q)}$.

If $c\mathcal{E}^{N,l}(f_0)\leq c(f_0,\epsilon)$*, then*

$$\mathcal{E}^{N,l}(f^\epsilon(t))\lesssim\frac{\mathcal{E}^{N,l}(f_0)}{(1+C_2t)^q},\tag{1.34}$$

where $C_2\sim\mathcal{E}^{N,l}(f_0)^{1/q}\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0)^{-1/q}$.

(4) **(Global asymptotic formula.)** *Fix* $N\geq 4, l\geq 2-N\gamma$*, assume that* $\mathcal{E}^4(f_0)\leq\delta_{N+2,l+2-2\gamma}$ *and* $\mathcal{E}^{N+2,l+2-2\gamma}(f_0)<\infty$*, then*

$$\sup_{t\geq 0}\mathcal{E}^{N,l}(f(t)-f^\epsilon(t))\leq C(\mathcal{E}^{N+2,l+2-2\gamma}(f_0))\epsilon^{4-4s},\tag{1.35}$$

where f *and* f^ϵ *are the solutions to* (1.7) *and* (1.8) *respectively.*

Some comments are in order:

REMARK 1.4. We study the Boltzmann equation with and without angular cutoff simultaneously in the near-equilibrium framework. As for the global well-posedness (1.31) and the propagation of regularity (1.32), we only require smallness of $\mathcal{E}^4(f_0)$, rather than smallness of $\mathcal{E}^{N,l}(f_0)$, which is different from and an improvement over the results in [12] for cutoff case and [4, 10] for non-cutoff case.

REMARK 1.5. As for the minimal regularity of initial datum, we only need $N\geq 4$. Actually, the minimal order can be improved to $-3-2\gamma+\delta$ for any $\delta>0$ if one needs to use the embedding $L^\infty\to H^{3/2+\delta}$ in dimension 3. Thus when γ is near -3, $N\geq 3+\delta$ is required, which means 4 is the smallest achievable integer. This issue is indicated in [10]. Note that [12] and [4] impose $N\geq 8$ and $N\geq 6$ respectively. Notably, very recently [9] establishes well-posedness of Boltzmann and Landau equation in a low regularity space containing $H^{3/2+\delta}_x L^2$ in our notation.

REMARK 1.6. Equations (1.24) and (1.33) together show that the behavior of the solution to the non-linear equations enjoy the same decay pattern as that for the semi-group generated by the linearized collision operator. The global error estimate (1.35) is also established for the solutions f^ϵ and f. To our best knowledge, these results are new for the very soft potentials.

1.5. Organization of the paper. In Section 2, we recall some known results on collision operators. Section 3 is devoted to the longtime behavior of the semi-group $e^{-\mathcal{L}^\epsilon t}$, that is, the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2. In the appendix, we list some useful results and an example of an ordinary differential equation.

2. Estimates of the collision operators

In this section, we recall from [14] some results, namely, coercivity estimate of $\mathcal{L}^\epsilon$, upper bound of $\Gamma^\epsilon(g,h)$, and commutator estimate between $\Gamma^\epsilon(g,\cdot)$ and W_l.

For explicit spectral gap and coercivity estimates of the linearized Boltzmann and Landau operators, one may refer to [6, 15, 16]. One may also refer to the recent work [2] for the sharp coercivity estimate of the linearized Boltzmann operator. The following is Theorem 1.1 in [14], which is a sharp coercivity estimate of $\mathcal{L}^\epsilon$. By "sharp" we mean the lower and upper bound share the same norm.

THEOREM 2.1. *There exists a constant* $\epsilon_0>0$ *such that for* $0\leq\epsilon\leq\epsilon_0$ *and any smooth function* f,

$$\langle\mathcal{L}^\epsilon f,f\rangle+|f|^2_{L^2_{\gamma/2}}\sim|f|^2_{\epsilon,\gamma/2}=|W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})f|^2_{L^2_{\gamma/2}}+|W^\epsilon(D)f|^2_{L^2_{\gamma/2}}+|W^\epsilon f|^2_{L^2_{\gamma/2}}.\tag{2.1}$$

Let λ be the largest number such that the following is valid for any smooth function f,

$$\langle \mathcal{L}^\epsilon f, f\rangle + |f|^2_{L^2_{\gamma/2}} \geq \lambda |f|^2_{\epsilon,\gamma/2}. \tag{2.2}$$

The following is Proposition 2.4 in [14], which says that $\langle \mathcal{L}^\epsilon f, f\rangle$ produces "strict" coercivity in the space $\mathcal{N}^\perp$. This type of estimate is usually referred to as "spectral gap" estimate.

PROPOSITION 2.1. *For any smooth function f, we have*

$$\langle \mathcal{L}^\epsilon f, f\rangle \geq \lambda |f - \mathbb{P}f|^2_{\epsilon,\gamma/2}.$$

In the near-equilibrium framework, a key step is to control the non-linear term $\Gamma(f,f)$ via the linear term $\mathcal{L}f$. That is, to establish $|\langle \Gamma(f,f), f\rangle| \lesssim \langle \mathcal{L}f, f\rangle$ under smallness assumption on f. For the estimate of the trilinear $\langle \Gamma(f,f), f\rangle$, one may refer to [4, 5, 10].

To study the non-linear Equation (1.8), we need to control $\langle \Gamma^\epsilon(g,h), f\rangle$ in terms of the norm $|\cdot|_{\epsilon,\gamma/2}$ of the coercivity estimate in Theorem 2.1. The following upper bound estimate of Γ^ϵ is from Theorem 2.2 in [14].

THEOREM 2.2. *For any $\eta > 0$ and smooth functions g, h and f, the following statements are valid.*

(1) If $\gamma > -3/2$, $|\langle \Gamma^\epsilon(g,h), f\rangle| \lesssim |g|_{L^2} |h|_{\epsilon,\gamma/2} |f|_{\epsilon,\gamma/2}$;

(2) If $\gamma = -3/2$, $|\langle \Gamma^\epsilon(g,h), f\rangle| \lesssim |g|_{L^2}(|W^\epsilon(D)\mu^{1/8}h|_{H^\eta} + |h|_{\epsilon,\gamma/2})|f|_{\epsilon,\gamma/2}$;

(3) If $-3 < \gamma \leq -3/2$,

$$|\langle \Gamma^\epsilon(g,h), f\rangle| \lesssim |\mu^{1/8} g|_{H^{s_1}} |W^\epsilon(D)\mu^{1/8}h|_{H^{s_2}} |W^\epsilon(D) f|_{H^{s_3}_{\gamma/2}} + |g|_{L^2}|h|_{\epsilon,\gamma/2}|f|_{\epsilon,\gamma/2},$$

where s_1, s_2 and s_3 verify that $s_1 + s_2 + s_3 = -\gamma - 3/2$ if $s_2 + s_3 \in (0, -\gamma - 3/2]$ and $s_1 = -\gamma - 3/2 + \eta$ if $s_2 = s_3 = 0$.

Note that the above result is little bit different from that in [14] in terms of the weight on function h. More precisely, in [14], the corresponding part is $|\mu^{1/8} g|_{H^{s_1}} |W^\epsilon(D)h|_{H^{s_2}_{\gamma/2}} |W^\epsilon(D)f|_{H^{s_3}_{\gamma/2}}$. We emphasize that the weight on h can be improved to $\mu^{1/8}$ by observing that the term comes from the velocity singularity $|v - v_*| \leq 1$, which implies $\mu_* \lesssim \mu_*^{1/2} \mu^{1/4}$. Thanks to the existence of μ_* in g_*, we can get some μ power for h.

Recall $\mathcal{L}^\epsilon g = -\Gamma^\epsilon(g, \mu^{1/2}) - \Gamma^\epsilon(\mu^{1/2}, g)$. Then as a direct consequence of Theorem 2.2, we have

COROLLARY 2.1. *If $\gamma > -3$, there holds*

$$|\langle \mathcal{L}^\epsilon g, f\rangle| \lesssim |g|_{\epsilon,\gamma/2} |f|_{\epsilon,\gamma/2}.$$

The following commutator estimate between $\Gamma^\epsilon(g,\cdot)$ and W_l is Lemma 2.11 in [14].

LEMMA 2.1. *Let $l \geq 2$. There hold*

(1) if $\gamma + 2 \geq 0$, $|\langle \Gamma^\epsilon(g, W_l h) - W_l \Gamma^\epsilon(g,h), f\rangle| \lesssim |g|_{L^2} |W_{l+\gamma/2} h|_{L^2} |f|_{\epsilon,\gamma/2}$;

(2) if $-3 < \gamma < -2$,

$$|\langle \Gamma^\epsilon(g, W_l h) - W_l \Gamma^\epsilon(g,h), f\rangle| \lesssim |g|_{L^2} |W_{l+\gamma/2} h|_{L^2} |f|_{\epsilon,\gamma/2} + |\mu^{1/32} g|_{H^{s_1}} |\mu^{1/32} h|_{H^{s_2}} |f|_{\epsilon,\gamma/2},$$

where $s_1, s_2 \in [0, -\gamma/2 - 1]$ with $s_1 + s_2 = -\gamma/2 - 1$.

As a result of Lemma 2.1, we have

COROLLARY 2.2. *If* $-3<\gamma<0, l\geq 2$, *there holds*

$$|\langle[\mathcal{L}^\epsilon, W_l]g, f\rangle| \lesssim |g|_{L^2_{l+\gamma/2}}|f|_{\epsilon,\gamma/2}.$$

3. Longtime behavior of $e^{-\mathcal{L}^\epsilon t}$

In this section, we will give the proof to Theorem 1.1. Throughout this section, we will set $f = e^{-\mathcal{L}^\epsilon t} f_0$ with $f_0 \in \mathcal{N}^\perp$. Then f verifies that $f \in \mathcal{N}^\perp$ and

$$\begin{cases} \partial_t f + \mathcal{L}^\epsilon f = 0; \\ f|_{t=0} = f_0. \end{cases} \tag{3.1}$$

To deal with derivatives w.r.t. the velocity variable v, let us deviate to introduce some notation. By binomial expansion, we have

$$\partial^\alpha_\beta \Gamma^\epsilon(g,h) = \sum_{\beta_0+\beta_1+\beta_2=\beta, \alpha_1+\alpha_2=\alpha} C^{\beta_0,\beta_1,\beta_2}_\beta C^{\alpha_1,\alpha_2}_\alpha \Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha_2}_{\beta_2} h; \beta_0), \tag{3.2}$$

where

$$\Gamma^\epsilon(g,h;\beta)(v) := \int_{\mathbb{S}^2\times\mathbb{R}^3} B^\epsilon(v-v_*,\sigma)(\partial_\beta \mu^{1/2})_*(g'_* h' - g_* h) d\sigma dv_*. \tag{3.3}$$

Here $C^{\beta_0,\beta_1,\beta_2}_\beta$ is the combination such that $\beta_0+\beta_1+\beta_2=\beta$. The notation $C^{\alpha_1,\alpha_2}_\alpha$ is similarly interpreted. We remark that $\Gamma^\epsilon(g,h;\beta)$ satisfies the upper bound in Theorem 2.2 and commutator estimate in Lemma 2.1. We define

$$\mathcal{L}^{\epsilon,\beta_0,\beta_1} g := -\Gamma^\epsilon(\partial_{\beta_1}\mu^{1/2}, g; \beta_0) - \Gamma^\epsilon(g, \partial_{\beta_1}\mu^{1/2}; \beta_0). \tag{3.4}$$

Therefore $\mathcal{L}^{\epsilon,\beta_0,\beta_1}$ enjoys the same commutator and upper bound as that of $\mathcal{L}^\epsilon$ in Corollary 2.2 and Corollary 2.1. Recalling $\mathcal{L}^\epsilon g = -\Gamma^\epsilon(\mu^{1/2}, g) - \Gamma^\epsilon(g, \mu^{1/2})$, (3.2), (3.3) and (3.4), we have

$$\begin{aligned} \partial_\beta \mathcal{L}^\epsilon g &= \mathcal{L}^\epsilon \partial_\beta g - \sum_{\beta_0+\beta_1+\beta_2=\beta, \beta_2<\beta} C^{\beta_0,\beta_1,\beta_2}_\beta [\Gamma^\epsilon(\partial_{\beta_1}\mu^{1/2}, \partial_{\beta_2} g; \beta_0) + \Gamma^\epsilon(\partial_{\beta_1} g, \partial_{\beta_2}\mu^{1/2}; \beta_0)] \\ &= \mathcal{L}^\epsilon \partial_\beta g + \sum_{\beta_0+\beta_1+\beta_2=\beta, \beta_2<\beta} C^{\beta_0,\beta_1,\beta_2}_\beta \mathcal{L}^{\epsilon,\beta_0,\beta_1} \partial_{\beta_2} g, \end{aligned} \tag{3.5}$$

where we use the fact $C^{\beta_0,\beta_1,\beta_2}_\beta = C^{\beta_0,\beta_2,\beta_1}_\beta$ in the last line.

We set to prove the following propagation result.

PROPOSITION 3.1. *Fix* $l\geq 2$, *suppose* f^ϵ *is the solution to* (3.1), *then*

$$\sup_t |f^\epsilon(t)|^2_{H^N_l} + \lambda \int_0^\infty |f^\epsilon(t)|^2_{H^N_{\epsilon,l+\gamma/2}} dt \lesssim |f_0|^2_{H^N_l}.$$

Proof. For simplicity, we omit the superscript ϵ in f^ϵ. Start from (3.1), take inner product with f, by the fact $f\in\mathcal{N}^\perp$ and Proposition 2.1, we get

$$\frac{d}{dt}|f|^2_{H^0} + 2\lambda |f|^2_{H^0_{\epsilon,\gamma/2}} \leq 0. \tag{3.6}$$

Fix an index β, apply $W_l\partial_\beta$ to both sides of $\partial_t f+\mathcal{L}^\epsilon f=0$, by (3.5), we have

$$\partial_t W_l\partial_\beta f+W_l\partial_\beta\mathcal{L}^\epsilon f=\partial_t W_l\partial_\beta f+W_l\sum_{\beta_0+\beta_1+\beta_2=\beta}C_\beta^{\beta_0,\beta_1,\beta_2}\mathcal{L}^{\epsilon,\beta_0,\beta_1}\partial_{\beta_2}f=0.$$

By introducing the commutator $[\mathcal{L}^{\epsilon,\beta_0,\beta_1},W_l]$ and rearranging, we get

$$\begin{aligned}\partial_t W_l\partial_\beta f+\mathcal{L}^\epsilon W_l\partial_\beta f=&\sum_{\beta_0+\beta_1+\beta_2=\beta}C_\beta^{\beta_0,\beta_1,\beta_2}[\mathcal{L}^{\epsilon,\beta_0,\beta_1},W_l]\partial_{\beta_2}f\\&-\sum_{\beta_0+\beta_1+\beta_2=\beta,\beta_2<\beta}C_\beta^{\beta_0,\beta_1,\beta_2}\mathcal{L}^{\epsilon,\beta_0,\beta_1}W_l\partial_{\beta_2}f.\end{aligned}\tag{3.7}$$

When $|\beta|=0$, we simply have $\partial_t W_l f+\mathcal{L}^\epsilon W_l f=[\mathcal{L}^\epsilon,W_l]f$. Taking inner product with $W_l f$, using (2.2), Corollary 2.2, by the basic inequality $2AB\le\eta A^2+\eta^{-1}B^2$, we have

$$\frac{d}{dt}|f|^2_{H^0_l}+\frac{3}{2}\lambda|f|^2_{H^0_{\epsilon,l+\gamma/2}}\lesssim|f|^2_{H^0_{l+\gamma/2}}\lesssim|f^l|^2_{H^0_{l+\gamma/2}}+|f^h|^2_{H^0_{l+\gamma/2}}.$$

By the definition of W^ϵ in (1.15) and interpolation, we have

$$|f^h|^2_{H^0_{l+\gamma/2}}\le\epsilon^{2s}|f|^2_{H^0_{\epsilon,l+\gamma/2}},\tag{3.8}$$

$$|f^l|^2_{H^0_{l+\gamma/2}}\le\eta|f^l|^2_{H^0_{l+\gamma/2+s}}+C_\eta|f^l|^2_{H^0_{\gamma/2+s}}\le\eta|f|^2_{H^0_{\epsilon,l+\gamma/2}}+C_\eta|f|^2_{H^0_{\epsilon,\gamma/2}}.\tag{3.9}$$

By taking η small enough such that $\eta\ll\lambda$, when ϵ is small such that $\epsilon^{2s}\ll\lambda$, we have

$$\frac{d}{dt}|f|^2_{H^0_l}+\lambda|f|^2_{H^0_{\epsilon,l+\gamma/2}}\lesssim|f|^2_{H^0_{\epsilon,\gamma/2}}.\tag{3.10}$$

Making a suitable combination of (3.10) and (3.6), we have

$$\frac{d}{dt}\left(M|f|^2_{H^0}+|f|^2_{H^0_l}\right)+\lambda\left(M|f|^2_{H^0_{\epsilon,\gamma/2}}+|f|^2_{H^0_{\epsilon,l+\gamma/2}}\right)\le0.\tag{3.11}$$

For $0\le i\le N$, set

$$\mathcal{V}^{i,l}(f):=M^i|f|^2_{H^0}+\sum_{j=0}^{i}K^i_j|f|^2_{\dot{H}^j_l},\mathcal{U}^{i,l}(f):=M^i|f|^2_{H^0_{\epsilon,\gamma/2}}+\sum_{j=0}^{i}K^i_j|f|^2_{\dot{H}^j_{\epsilon,l+\gamma/2}},$$

for some constants $M^i,K^i_j\ge1$ which will be determined later. For $0\le i\le N$, we proceed to establish

$$\frac{d}{dt}\mathcal{V}^{i,l}(f)+\lambda\mathcal{U}^{i,l}(f)\le0.\tag{3.12}$$

Note that when $i=0$, (3.12) reduces to (3.11), which has been proved. Also note that $\mathcal{V}^{N,l}(f)\sim|f|^2_{H^N_l}$ and $\mathcal{U}^{N,l}(f)\sim|f|^2_{H^N_{\epsilon,l+\gamma/2}}$. Moveover, we have $\mathcal{V}^{N,l}(f)\ge|f|^2_{H^N_l}$ and $\mathcal{U}^{N,l}(f)\ge|f|^2_{H^N_{\epsilon,l+\gamma/2}}$ since $M^i,K^i_j\ge1$. So when $i=N$, (3.12) yields the proposition immediately. In a word, it remains to derive (3.12).

We will prove (3.12) by mathematical induction. Suppose (3.12) is valid for $i=k$, that is,

$$\frac{d}{dt}\left(M^k|f|^2_{H^0}+\sum_{j=0}^{k}K^k_j|f|^2_{\dot{H}^j_l}\right)+\lambda\left(M^k|f|^2_{H^0_{\epsilon,\gamma/2}}+\sum_{j=0}^{k}K^k_j|f|^2_{\dot{H}^j_{\epsilon,l+\gamma/2}}\right)\le0,\tag{3.13}$$

we now go to prove that (3.12) is also valid for $i=k+1$. Start from (3.7) for $|\beta|=k+1$ and $l\geq 2$, take inner product with $W_l\partial_\beta f$, using (2.2), Corollary 2.2 and Corollary 2.1, by the basic inequality $2AB\leq \eta A^2+\eta^{-1}B^2$, we have

$$\frac{d}{dt}|\partial_\beta f|^2_{L^2_l}+\frac{3}{2}\lambda|\partial_\beta f|^2_{\epsilon,l+\gamma/2}\lesssim |f|^2_{H^{k+1}_{l+\gamma/2}}+|f|^2_{H^k_{\epsilon,l+\gamma/2}}.$$

Taking sum over $|\beta|=k+1$, we get

$$\frac{d}{dt}|f|^2_{\dot{H}^{k+1}_l}+\frac{3}{2}\lambda|f|^2_{\dot{H}^{k+1}_{\epsilon,l+\gamma/2}}\lesssim |f|^2_{H^{k+1}_{l+\gamma/2}}+|f|^2_{H^k_{\epsilon,l+\gamma/2}}.$$

By Proposition A.2, we have for any $\eta>0$,

$$|f|^2_{H^{k+1}_{l+\gamma/2}}\lesssim (\eta+\epsilon^{2s})|f|^2_{H^{k+1}_{\epsilon,l+\gamma/2}}+C_\eta|f|^2_{H^0_{l+\gamma/2}}\leq (\eta+\epsilon^{2s})|f|^2_{\dot{H}^{k+1}_{\epsilon,l+\gamma/2}}+C_\eta|f|^2_{H^k_{\epsilon,l+\gamma/2}}.$$

Taking η small enough such that $\eta\ll\lambda$, then when ϵ is small enough verifying $\epsilon^{2s}\ll\lambda$, we get

$$\frac{d}{dt}|f|^2_{\dot{H}^{k+1}_l}+\lambda|f|^2_{\dot{H}^{k+1}_{\epsilon,l+\gamma/2}}\lesssim |f|^2_{H^k_{\epsilon,l+\gamma/2}}. \tag{3.14}$$

Then a suitable combination of (3.13) and (3.14) will produce (3.12) for $i=k+1$. More precisely, we can multiply (3.13) by a large constant and then add the resulting inequality to (3.14) to cancel the term $|f|^2_{H^k_{\epsilon,l+\gamma/2}}$ on the right-hand side. □

We now prove a technical proposition regarding to the decay rate of a special type of ordinary differential inequality.

PROPOSITION 3.2. *Let $c\geq 1, c_1, c_2$ and $p>q$ be five universal and positive constants. Consider the ordinary differential inequality:*

$$\begin{cases}\frac{d}{dt}Y+c_1Y_1^{1+\frac{1}{p}}+c_2Y_2^{1+\frac{1}{q}}\leq 0;\\ Y|_{t=0}=Y_0,\end{cases} \tag{3.15}$$

where $c^{-1}(Y_1+Y_2)\leq Y\leq c(Y_1+Y_2)$ and $Y,Y_1,Y_2\geq 0$. If $Y_0>2c(c_1/c_2)^{\frac{pq}{p-q}}$, let t_* be the time such that $Y(t_*)=2c(c_1/c_2)^{\frac{pq}{p-q}}$, then for any $t\geq 0$,

$$Y(t)\leq \frac{Y_0}{(1+C_1t)^p}1_{t<t_*}+\frac{Y_*}{(1+C_2(t-t_*))^q}1_{t\geq t_*}, \tag{3.16}$$

where $Y_*=Y(t_*), C_1=\frac{c_1}{2cp}(\frac{Y_0}{2c})^{1/p}, C_2=\frac{c_2}{2cq}(c_1/c_2)^{\frac{p}{p-q}}$. Moreover, the critical time verifies $t_*\leq((Y_0/Y_*)^{1/p}-1)/C_1$.

If $Y_0\leq 2c(c_1/c_2)^{\frac{pq}{p-q}}$, then for any $t\geq 0$,

$$Y(t)\leq \frac{Y_0}{(1+C_2t)^q}, \tag{3.17}$$

where $C_2=\frac{c_2}{2cq}(\frac{Y_0}{2c})^{1/q}$.

Proof. It is easy to check that $Y(t)$ is a strictly decreasing function before it vanishes. When $Y_0>2c(c_1/c_2)^{\frac{pq}{p-q}}$, since $Y(t_*)=2c(c_1/c_2)^{\frac{pq}{p-q}}$, we have

$$c_1(\frac{Y(t_*)}{2c})^{1+\frac{1}{p}}=c_2(\frac{Y(t_*)}{2c})^{1+\frac{1}{q}}. \tag{3.18}$$

Since $Y \le c(Y_1+Y_2)$, one has $\max\{Y_1,Y_2\} \ge \frac{1}{2c}Y$. Then we deduce

$$\begin{aligned} c_1Y_1^{1+\frac{1}{p}}+c_2Y_2^{1+\frac{1}{q}} &\ge \max\{c_1Y_1^{1+\frac{1}{p}},c_2Y_2^{1+\frac{1}{q}}\} \\ &\ge \min\{c_1(\frac{Y}{2c})^{1+\frac{1}{p}},c_2(\frac{Y}{2c})^{1+\frac{1}{q}}\} = \begin{cases} c_1(\frac{Y}{2c})^{1+\frac{1}{p}}, & t<t_*; \\ c_2(\frac{Y}{2c})^{1+\frac{1}{q}}, & t\ge t_*. \end{cases} \end{aligned} \tag{3.19}$$

Note that the last equality employs (3.18), which is also the reason of our choice of t_*. When $t<t_*$, we have

$$\frac{d}{dt}Y+c_1(\frac{Y}{2c})^{1+\frac{1}{p}} \le 0,$$

from which we get

$$Y(t) \le \frac{Y_0}{(1+C_1t)^p}, \tag{3.20}$$

with $C_1 = \frac{c_1}{2cp}(\frac{Y_0}{2c})^{1/p}$. On the interval $[t_*,\infty)$, we have

$$\frac{d}{dt}Y+c_2(\frac{Y}{2c})^{1+\frac{1}{q}} \le 0,$$

from which we get

$$Y(t) \le \frac{Y(t_*)}{(1+C_2(t-t_*))^q}, \tag{3.21}$$

with $C_2 = \frac{c_2}{2cq}(\frac{Y(t_*)}{2c})^{1/q} = \frac{c_2}{2cq}(c_1/c_2)^{\frac{p}{p-q}}$. Patching together (3.20) and (3.21), we conclude (3.16). Since $Y(t)$ is a strictly decreasing function before it vanishes, we have

$$Y(t_*) \le \lim_{t\to t_*-} Y(t) \le \frac{Y_0}{(1+C_1t_*)^p},$$

which yields $t_* \le ((Y_0/Y_*)^{1/p}-1)/C_1$.

When $Y_0 \le 2c(\frac{c_1}{c_2})^{(1/q-1/p)^{-1}}$, we have

$$c_1(\frac{Y_0}{2c})^{1+\frac{1}{p}} \ge c_2(\frac{Y_0}{2c})^{1+\frac{1}{q}}. \tag{3.22}$$

Similar to (3.19), we have $c_1Y_1+c_2Y_2^{1+\frac{1}{q}} \ge c_2(\frac{Y}{2c})^{1+\frac{1}{q}}$ and thus on the interval $[0,\infty)$, we get

$$\frac{d}{dt}Y+c_2(\frac{Y}{2c})^{1+\frac{1}{q}} \le 0,$$

which yields (3.17). The proof is complete now. □

In the appendix, we give a special case of the inequality (3.15) as example A.1 which shows the decay structure (3.16) is optimal.

With the help of Proposition 3.2, we are ready to prove the first part of Theorem 1.1, namely the refined polynomial decay rates.

Proof. **(Proof of Theorem 1.1: Refined polynomial decay rates (1.24) and (1.25).)** Let $Y_1=|f^l|^2_{H^N_l}, Y_2=|f^h|^2_{H^N_l}$, then $\mathcal{V}^{N,l}(f)\sim|f|^2_{H^N_l}\sim Y_1+Y_2$. To be clear, we take a universal constant $c>2$ such that

$$c^{-1}(Y_1+Y_2)\le|f|^2_{H^N_l}\le\mathcal{V}^{N,l}(f)\le\frac{c}{2}|f|^2_{H^N_l}\le c(Y_1+Y_2).$$

We remark that the constant c could depend on N,l by the definition of $\mathcal{V}^{N,l}(f)$. We will use the following interpolation result:

$$|f|_{L^2}\le|f|^\theta_{L^2_\alpha}|f|^{1-\theta}_{L^2_\beta},\alpha<0<\beta,0<\theta<1,\theta\alpha+(1-\theta)\beta=0. \tag{3.23}$$

Fix $p>0$, set $\alpha=\gamma/2+s,\theta=\frac{p}{p+1},\beta=-p(\gamma/2+s)$ in (3.23), we get

$$\begin{aligned}|W_l\partial_\beta f^l|_{L^2}&\le|W_l\partial_\beta f^l|^{p/(p+1)}_{L^2_{\gamma/2+s}}|W_l\partial_\beta f^l|^{1/(p+1)}_{L^2_{-p(\gamma/2+s)}}\\&\le|W_l\partial_\beta f^l|^{p/(p+1)}_{L^2_{\gamma/2+s}}(C|f_0|^2_{H^N_{l-p(\gamma/2+s)}})^{1/2(p+1)}.\end{aligned} \tag{3.24}$$

where the last inequality comes from Proposition 3.1. More precisely, we used

$$|W_l\partial_\beta f^l(t)|_{L^2_{-p(\gamma/2+s)}}\le|f(t)|_{H^N_{l-p(\gamma/2+s)}}\le C|f_0|_{H^N_{l-p(\gamma/2+s)}}.$$

Rearranging (3.24), we have

$$|W_l\partial_\beta f^l|^2_{L^2_{\gamma/2+s}}\ge C^{-1/p}|f_0|^{-2/p}_{H^N_{l-p(\gamma/2+s)}}|W_l\partial_\beta f^l|^{2(1+1/p)}_{L^2}. \tag{3.25}$$

There is some constant $C(p,n)$, such that $(\sum_{i=1}^n a_i)^{1+1/p}\le C(p,n)\sum_{i=1}^n a_i^{1+1/p}$. Taking sum over $|\beta|\le N$, for some constant $C(p,N)$, we get

$$\sum_{|\beta|\le N}|W_l\partial_\beta f^l|^2_{L^2_{\gamma/2+s}}\ge C(p,N)|f_0|^{-2/p}_{H^N_{l-p(\gamma/2+s)}}|f^l|^{2(1+1/p)}_{H^N_l}.$$

Taking $\alpha=\gamma/2,q=p(1+2s/\gamma),\theta=\frac{q}{q+1}=\frac{-p(\gamma/2+s)}{-p(\gamma/2+s)-\gamma/2},\beta=-p(\gamma/2+s)$ in (3.23), we get

$$|W_l\partial_\beta f^h|_{L^2}\le|W_l\partial_\beta f^h|^{q/(q+1)}_{L^2_{\gamma/2}}|W_l\partial_\beta f^h|^{1/(q+1)}_{L^2_{-p(\gamma/2+s)}}. \tag{3.26}$$

By a similar argument, we have

$$\sum_{|\beta|\le N}|W_l\partial_\beta f^h|^2_{L^2_{\gamma/2}}\ge C(q,N)|f_0|^{-2/q}_{H^N_{l-p(\gamma/2+s)}}|f^h|^{2(1+1/q)}_{H^N_l}. \tag{3.27}$$

By the fact $|f|^2_{H^N_{\epsilon,l+\gamma/2}}\gtrsim|f^l|^2_{H^N_{\epsilon,l+\gamma/2+s}}+\epsilon^{-2s}|f^l|^2_{H^N_{\epsilon,l+\gamma/2}}$, and the estimates (3.25) and (3.27), we get

$$\begin{aligned}|f|^2_{H^N_{\epsilon,l+\gamma/2}}\ge{}&C(p,N)|f_0|^{-2/p}_{H^N_{l-p(\gamma/2+s)}}|f^l|^{2(1+1/p)}_{H^N_l}\\&+\epsilon^{-2s}C(q,N)|f_0|^{-2/q}_{H^N_{l-p(\gamma/2+s)}}|f^h|^{2(1+1/q)}_{H^N_l}.\end{aligned} \tag{3.28}$$

By (3.12) and the fact $\mathcal{U}^{N,l}(f)\ge|f|^2_{H^N_{\epsilon,l+\gamma/2}}$, we have

$$\frac{d}{dt}\mathcal{V}^{N,l}(f)+\lambda|f|^2_{H^N_{\epsilon,l+\gamma/2}}\le0. \tag{3.29}$$

Plugging (3.28) into (3.29), we have

$$\frac{d}{dt}\mathcal{V}^{N,l}(f)+c_1Y_1^{1+1/p}+c_2Y_2^{1+1/q}\leq 0,$$

where $c_1=\lambda C(p,N)|f_0|^{-2/p}_{H^N_{l-p(\gamma/2+s)}}, c_2=\lambda\epsilon^{-2s}C(q,N)|f_0|^{-2/q}_{H^N_{l-p(\gamma/2+s)}}$. Applying Proposition 3.2 with $Y=\mathcal{V}^{N,l}(f), Y_1=|f^l|^2_{H^N_l}, Y_2=|f^h|^2_{H^N_l}$, we get the following results. If $\mathcal{V}^{N,l}(f_0)>2c(c_1/c_2)^{\frac{pq}{p-q}}$, let t_* be the time such that $\mathcal{V}^{N,l}(f(t_*))=2c(c_1/c_2)^{\frac{pq}{p-q}}$, then for any $t\geq 0$,

$$\mathcal{V}^{N,l}(f(t))\leq\frac{\mathcal{V}^{N,l}(f_0)}{(1+C_1t)^p}1_{t<t_*}+\frac{\mathcal{V}^{N,l}(f(t_*))}{(1+C_2(t-t_*))^q}1_{t\geq t_*},\tag{3.30}$$

where $C_1=\frac{c_1}{2cp}(\frac{\mathcal{V}^{N,l}(f_0)}{2c})^{1/p}\sim|f_0|^{2/p}_{H^N_l}|f_0|^{-2/p}_{H^N_{l-p(\gamma/2+s)}}, C_2=\frac{c_2}{2cq}(c_1/c_2)^{\frac{p}{p-q}}\sim\epsilon^{2sq/(p-q)}$.

If $\mathcal{V}^{N,l}(f_0)\leq 2c(c_1/c_2)^{\frac{pq}{p-q}}$, then for any $t\geq 0$,

$$\mathcal{V}^{N,l}(f(t))\leq\frac{\mathcal{V}^{N,l}(f_0)}{(1+C_2t)^q},\tag{3.31}$$

where $C_2=\frac{c_2}{2cq}(\frac{\mathcal{V}^{N,l}(f_0)}{2c})^{1/q}\sim|f_0|^{2/q}_{H^N_l}|f_0|^{-2/q}_{H^N_{l-p(\gamma/2+s)}}$.

Note that

$$(c_1/c_2)^{\frac{pq}{p-q}}=(C(p,N)/C(q,N)\epsilon^{2s})^{\frac{pq}{p-q}}|f_0|^2_{H^N_{l-p(\gamma/2+s)}}:=(\epsilon^{2s})^{\frac{pq}{p-q}}C(p,q,N)|f_0|^2_{H^N_{l-p(\gamma/2+s)}}.$$

With the equivalence

$$|f|^2_{H^N_l}\leq\mathcal{V}^{N,l}(f)\leq c|f|^2_{H^N_l},$$

we get (1.24) and (1.25) from (3.30) and (3.31) respectively. □

Before going to prove the remaining part of Theorem 1.1, we prove the following technical lemma for a commutator estimate.

LEMMA 3.1. *Let $\gamma>-3$. Suppose $j\geq 3$, then for any $\eta>0$, there holds*

$$|\langle[\mathcal{L}^\epsilon,\mathcal{P}_j]f,\mathcal{P}_jf\rangle|\lesssim\eta^{-1}(\exp(-C_12^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}+\sum_{k=j-3}^{j+3}|\mathcal{P}_kf|^2_{L^2_{\gamma/2}})+\eta|\mathcal{P}_jf|^2_{\epsilon,\gamma/2},$$

where $C_1>0$ is a universal constant.

Proof. By the definition of $\mathcal{L}^\epsilon$(see (1.6)), it suffices to consider $\mathcal{I}(g,h):=\langle\Gamma^\epsilon(g,h\varphi_j)-\Gamma^\epsilon(g,h)\varphi_j,f\varphi_j\rangle$ where $(g,h)=(\mu^{\frac12},f)$ or $(g,h)=(f,\mu^{\frac12})$. Recalling $\varphi_j(\cdot)=\psi(2^{-j}\cdot)$ and ψ has support in $\{\frac34\leq|v|\leq\frac83\}$, so the support of φ_j is contained in $\{\frac34\times 2^j\leq|v|\leq\frac83\times 2^j\}$.

Direct calculation will give

$$\begin{aligned}\mathcal{I}(g,h)=\int B^\epsilon\big[&(g\mu^{\frac12})_*h(f\varphi_j)'\big(-(\varphi_j)'+\varphi_j\big)\\&+g_*\big((\mu^{\frac12})'_*-(\mu^{\frac12})_*\big)h(f\varphi_j)'\big(-(\varphi_j)'+\varphi_j\big)\big]d\sigma dv_*dv.\end{aligned}$$

Here and in the following $\sigma\in\mathbb{S}^2, v\in\mathbb{R}^3, v_*\in\mathbb{R}^3$, and we omit the integral region for notational brevity. By Cauchy-Schwartz inequality, we get

$$\begin{aligned}|\mathcal{I}(g,h)|\lesssim&\Big(\int B^\epsilon g_*^2h^2(\mu_*^{\frac12}+(\mu^{\frac12})'_*)\big((\varphi_j)'-\varphi_j\big)^2d\sigma dv_*dv\Big)^{\frac12}\\&\times\Big(\int B^\epsilon\big[\mu_*^{\frac12}\big((f\varphi_j)'-f\varphi_j\big)^2\\&+(f\varphi_j)^2\big((\mu^{\frac14})'_*-(\mu^{\frac14})_*\big)^2\big]d\sigma dv_*dv\Big)^{\frac12}\\&+\Big|\int B^\epsilon(g\mu^{\frac12})_*hf\varphi_j\big((\varphi_j)'-\varphi_j\big)d\sigma dv_*dv\Big|\\\lesssim&\,\eta|f\varphi_j|^2_{\epsilon,\gamma/2}+\eta^{-1}\mathcal{J}(g,h)+\mathcal{K}(g,h),\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}(g,h):=&\int B^\epsilon g_*^2h^2(\mu_*^{\frac12}+(\mu^{\frac12})'_*)\big((\varphi_j)'-\varphi_j\big)^2d\sigma dv_*dv,\\\mathcal{K}(g,h):=&\Big|\int B^\epsilon(g\mu^{\frac12})_*hf\varphi_j\big((\varphi_j)'-\varphi_j\big)d\sigma dv_*dv\Big|.\end{aligned}$$

It remains to analyze $\mathcal{J}(g,h)$ and $\mathcal{K}(g,h)$.

Step 1: Estimate of $\mathcal{J}(g,h)$. We separate $\mathcal{J}(g,h)=\mathcal{J}_1(g,h)+\mathcal{J}_2(g,h)+\mathcal{J}_3(g,h)$ corresponding to $\{|v_*|\le 2^j/10\}$, $\{|v_*|\ge 2^j/10, |v|\le|v_*|/4\}$ and $\{|v_*|\ge 2^j/10, |v|\ge|v_*|/4\}$ respectively.

Step 1.1: Estimate of $\mathcal{J}_1(g,h)$. In $\mathcal{J}_1(g,h)$, we have $\{|v_*|\le 2^j/10\}$ and then

$$|(\varphi_j)(v')-\varphi_j(v)|^2\mathrm{1}_{|v_*|\le 2^j/10}=|(\varphi_j)(v')-\varphi_j(v)|^2\mathrm{1}_{|v_*|\le 2^j/10,2^j/5\le|v|\le 10\times 2^j}.$$

By Taylor expansion and the fact $|\nabla\varphi_j|_{L^\infty}\lesssim 2^{-j}$, we get

$$\begin{aligned}|(\varphi_j)(v')-\varphi_j(v)|^2\mathrm{1}_{|v_*|\le 2^j/10}\lesssim&\,2^{-2j}|v-v_*|^2\theta^2\mathrm{1}_{|v_*|\le 2^j/10,2^j/5\le|v|\le 10\times 2^j}\\\lesssim&\,\theta^2\mathrm{1}_{|v_*|\le 2^j/10,2^j/5\le|v|\le 10\times 2^j}.\end{aligned}$$

From which we get

$$\begin{aligned}\int B^\epsilon(\varphi_j(v')-\varphi_j(v))^2d\sigma\lesssim&\,|v-v_*|^\gamma\mathrm{1}_{|v_*|\le 2^j/10,2^j/5\le|v|\le 10\times 2^j}\\\lesssim&\,\langle v\rangle^\gamma\mathrm{1}_{|v_*|\le 2^j/10,2^j/5\le|v|\le 10\times 2^j}.\end{aligned}$$

where we use $3|v|/2\ge|v-v_*|\ge|v|/2\gtrsim 1$, and thus $|v-v_*|^\gamma\sim\langle v-v_*\rangle^\gamma\sim\langle v\rangle^\gamma$. Also note that $|v_*|\le|v|$, we get for any $a\ge 0$,

$$\begin{aligned}\mathcal{J}_1(g,h)\lesssim&\int g_*^2h^2\langle v\rangle^{-a}\langle v\rangle^{\gamma+a}\mathrm{1}_{|v_*|\le 2^j/10,2^j/5\le|v|\le 10\times 2^j}dvdv_*\\\lesssim&\,|g\mathrm{1}_{|\cdot|\le 2^j/10}|^2_{L^2_{-a}}|h\mathrm{1}_{2^j/5\le|\cdot|\le 10\times 2^j}|^2_{L^2_{\gamma/2+a}}.\end{aligned}$$

When $(g,h)=(\mu^{1/2},f)$, take $a=0$, we get

$$\mathcal{J}_1(\mu^{1/2},f)\lesssim|\mu^{1/2}\mathrm{1}_{|\cdot|\le 2^j/10}|^2_{L^2}|f\mathrm{1}_{2^j/5\le|\cdot|\le 10\times 2^j}|^2_{L^2_{\gamma/2}}\lesssim|f\mathrm{1}_{2^j/5\le|\cdot|\le 10\times 2^j}|^2_{L^2_{\gamma/2}}.$$

When $(g,h)=(f,\mu^{1/2})$, take $a=0\vee(-\gamma/2):=\max\{0,-\gamma/2\}$, by the fact $\mu^{1/2}(v)\lesssim \exp(-C_1 2^{2j})$ if $|v|\in[2^j/5, 10\times 2^j]$, we get

$$\mathcal{J}_1(f,\mu^{1/2})\lesssim |f1_{|\cdot|\leq 2^j/10}|^2_{L^2_{\gamma/2}}|\mu^{1/2}1_{2^j/5\leq|\cdot|\leq 10\times 2^j}|^2_{L^2_{\gamma/2\vee 0}}\lesssim \exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}.$$

Step 1.2: Estimate of $\mathcal{J}_2(g,h)$. In $\mathcal{J}_2(g,h)$, we have $\{|v_*|\geq 2^j/10, |v|\leq|v_*|/4\}$, and so it is easy to check $\frac{3}{4}|v_*|\leq|v-v_*|\leq\frac{5}{4}|v_*|\sim|v-v'_*|\sim|v'_*|$. Together with $|v-v_*|/\sqrt{2}\leq|v-v'_*|\leq|v-v_*|$, we have $|v'_*|\geq|v-v'_*|-|v|\geq(\frac{3}{4\sqrt{2}}-1/4)|v_*|\geq|v_*|/4$. Therefore $(\mu_*^{\frac{1}{2}}+(\mu^{\frac{1}{2}})'_*)\lesssim\exp(-C_1 2^{2j})$. Thanks to $(\varphi_j(v')-\varphi_j(v))^2\lesssim\min\{|v-v_*|^2\theta^2,1\}$ and Proposition A.1, we get

$$\begin{aligned}&\int B^\epsilon(\mu_*^{\frac{1}{2}}+(\mu^{\frac{1}{2}})'_*)(\varphi_j(v')-\varphi_j(v))^2 d\sigma\\ &\lesssim \exp(-C_1 2^{2j})|v-v_*|^\gamma (W^\epsilon)^2(v-v_*)1_{|v_*|\geq 2^j/10,|v|\leq|v_*|/4}\\ &\lesssim \exp(-C_1 2^{2j})\langle v_*\rangle^\gamma (W^\epsilon)^2(v_*)1_{|v_*|\geq 2^j/10,|v|\leq|v_*|/4}.\end{aligned}$$

Also note that $|v|\leq|v_*|$, we get for any $a\geq 0$,

$$\begin{aligned}\mathcal{J}_2(g,h)&\lesssim \exp(-C_1 2^{2j})\int g_*^2 h^2\langle v\rangle^{-a}\langle v_*\rangle^{\gamma+a}1_{|v_*|\geq 2^j/10,|v|\leq|v_*|/4}dvdv_*\\ &\lesssim \exp(-C_1 2^{2j})|W^\epsilon g1_{|\cdot|\geq 2^j/10}|^2_{L^2_{\gamma/2+a}}|h|^2_{L^2_{-a}}.\end{aligned}$$

When $(g,h)=(\mu^{1/2},f)$, take $a=0\vee(-\gamma/2)$, and when $(g,h)=(f,\mu^{1/2})$, take $a=0$, we get

$$\mathcal{J}_2(\mu^{1/2},f)\lesssim\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}, \mathcal{J}_2(f,\mu^{1/2})\lesssim\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}.$$

Step 1.3: Estimate of $\mathcal{J}_3(g,h)$. In $\mathcal{J}_3(g,h)$, we have $\{|v_*|\geq 2^j/10, |v|\geq|v_*|/4\}$, and so there holds $|v|\geq 2^j/40$. Thanks to $(\varphi_j(v')-\varphi_j(v))^2\lesssim\min\{|v-v_*|^2\theta^2,1\}$, Proposition A.1 and $|v_*|\lesssim|v|$, we get

$$\begin{aligned}\int B^\epsilon(\varphi_j(v')-\varphi_j(v))^2 d\sigma&\lesssim|v-v_*|^\gamma(W^\epsilon)^2(v-v_*)1_{|v_*|\geq 2^j/10,|v|\geq|v_*|/4}\\ &\lesssim|v-v_*|^\gamma(W^\epsilon)^2(v)1_{|v_*|\geq 2^j/10,|v|\geq 2^j/40},\end{aligned}$$

which gives

$$\mathcal{J}_3(g,h)\lesssim\int g_*^2h^2|v-v_*|^\gamma(W^\epsilon)^2(v)1_{|v_*|\geq 2^j/10,|v|\geq 2^j/40}dvdv_*.$$

When $(g,h)=(\mu^{1/2},f)$, thanks to the fact $\int\mu_*|v-v_*|^\gamma 1_{|v_*|\geq 2^j/10}dv_*\lesssim\exp(-C_1 2^{2j})\langle v\rangle^\gamma$, we get

$$\mathcal{J}_3(\mu^{1/2},f)\lesssim\exp(-C_1 2^{2j})\int f^2(W^\epsilon)^2(v)\langle v\rangle^\gamma dv\lesssim\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}.$$

When $(g,h)=(f,\mu^{1/2})$, thanks to the fact $\int\mu|v-v_*|^\gamma(W^\epsilon)^2(v)1_{|v|\geq 2^j/40}dv\lesssim \exp(-C_1 2^{2j})\langle v_*\rangle^\gamma$, we get

$$\mathcal{J}_3(f,\mu^{1/2})\lesssim\exp(-C_1 2^{2j})\int f_*^2\langle v_*\rangle^\gamma dv_*\lesssim\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}.$$

Patch together the above estimates in *Step 1.1*, *Step 1.2* and *Step 1.3*, to get

$$\mathcal{J}(\mu^{1/2},f)+\mathcal{J}(f,\mu^{1/2})\lesssim\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}+|f1_{2^j/5\leq|\cdot|\leq 10\times 2^j}|^2_{L^2_{\gamma/2}}.$$

Step 2: Estimate of $\mathcal{K}(g,h)$. Recall $\mathcal{K}(g,h):=\big|\int B^\epsilon(g\mu^{\frac{1}{2}})_* hf\varphi_j\big((\varphi_j)'-\varphi_j\big)d\sigma dv_* dv\big|$. We separate $\mathcal{K}(g,h)\leq\mathcal{K}_1(g,h)+\mathcal{K}_2(g,h)$ corresponding to $\{|v_*|\leq 2^j/10\}$ and $\{|v_*|\geq 2^j/10\}$ respectively, where

$$\mathcal{K}_1(g,h):=\Big|\int B^\epsilon 1_{|v_*|\leq 2^j/10}(g\mu^{\frac{1}{2}})_* hf\varphi_j\big((\varphi_j)'-\varphi_j\big)d\sigma dv_* dv\Big|,$$
$$\mathcal{K}_2(g,h):=\Big|\int B^\epsilon 1_{|v_*|\geq 2^j/10}(g\mu^{\frac{1}{2}})_* hf\varphi_j\big((\varphi_j)'-\varphi_j\big)d\sigma dv_* dv\Big|.$$

Step 2.1: Estimate of $\mathcal{K}_1(g,h)$. In $\mathcal{K}_1(g,h)$, we have $\{|v_*|\leq 2^j/10\}$. By Taylor expansion

$$(\varphi_j)(v')-\varphi_j(v)=(\nabla\varphi_j)(v)\cdot(v-v')+\frac{1}{2}\int_0^1(\nabla^2\varphi_j)(v(\kappa)):(v'-v)\otimes(v'-v)d\kappa, \quad (3.32)$$

where $v(\kappa)=v+\kappa(v'-v)$. In this case, by previous arguments in *Step 1.1*, $|v_*|$ is relatively small and $|v|\sim|v-v_*|\sim|v(\kappa)|\sim 2^j$.

By the facts $|\nabla\varphi_j|_{L^\infty}\lesssim 2^{-j},|\nabla^2\varphi_j|_{L^\infty}\lesssim 2^{-2j}$, the symmetry property $\int B^\epsilon(v'-v)d\sigma=(v_*-v)\int B^\epsilon\sin^2(\theta/2)d\sigma$, we get

$$\Big|\int B^\epsilon(\varphi_j(v')-\varphi_j(v))d\sigma\Big|\lesssim\langle v\rangle^\gamma 1_{|v_*|\leq 2^j/10,2^j/5\leq|v|\leq 10\times 2^j},$$

which gives

$$\begin{aligned}|\mathcal{K}_1(g,h)|&\lesssim\int|(g\mu^{\frac{1}{2}})_* hf|\varphi_j\ \langle v\rangle^\gamma 1_{|v_*|\leq 2^j/10,2^j/5\leq|v|\leq 10\times 2^j}dvdv_*\\&\lesssim|g\mu^{\frac{1}{2}}1_{|\cdot|\leq 2^j/10}|_{L^1}|h1_{2^j/5\leq|\cdot|\leq 10\times 2^j}|_{L^2_{\gamma/2}}|f\varphi_j 1_{2^j/5\leq|\cdot|\leq 10\times 2^j}|_{L^2_{\gamma/2}}.\end{aligned}$$

When $(g,h)=(\mu^{1/2},f)$, we get

$$|\mathcal{K}_1(\mu^{1/2},f)|\lesssim|f1_{2^j/5\leq|\cdot|\leq 10\times 2^j}|^2_{L^2_{\gamma/2}}.$$

When $(g,h)=(f,\mu^{1/2})$, by the fact $\mu^{1/2}(v)\lesssim\exp(-C_1 2^{2j})$ if $|v|\in[2^j/5,10\times 2^j]$ and $|f\mu^{\frac{1}{2}}|_{L^1}\lesssim|f|_{L^2_{\gamma/2}}$, we get

$$|\mathcal{K}_1(f,\mu^{1/2})|\lesssim\exp(-C_1 2^{2j})|f|^2_{L^2_{\gamma/2}}.$$

Step 2.2: Estimate of $\mathcal{K}_2(g,h)$. In $\mathcal{K}_2(g,h)$, we have $\{|v_*|\geq 2^j/10\}$. Note that the support of $\nabla\varphi_j$ belongs to $[2^j/10,10\times 2^j]$, then by (3.32), we have

$$\begin{aligned}(\varphi_j)(v')-\varphi_j(v)=&1_{2^j/10\leq|v|\leq 10\times 2^j}(\nabla\varphi_j)(v)\cdot(v-v')\\&+\frac{1}{2}\int_0^1(\nabla^2\varphi_j)(v(\kappa)):(v'-v)\otimes(v'-v)d\kappa,\end{aligned}$$

which gives when $2^j/10\leq|v|\leq 10\times 2^j$,

$$\Big|\int b^\epsilon(\cos\theta)(\varphi_j)(v')-\varphi_j(v)d\sigma\Big|\leq\Big|\int b^\epsilon(\cos\theta)(\nabla\varphi_j)(v)\cdot(v-v')d\sigma\Big|$$

$$+|\frac{1}{2}\int b^{\epsilon}(\cos\theta)(\nabla^2\varphi_j)(v(\kappa)):(v'-v)\otimes(v'-v)d\sigma d\kappa|$$
$$\lesssim 2^{-j}|v-v_*|+2^{-2j}|v-v_*|^2.$$

While when $|v|<2^j/10$ or $|v|>10\times 2^j$, only the second order is left and we get $|(\varphi_j)(v')-\varphi_j(v)|\lesssim\min\{|v-v_*|^2\theta^2,1\}$ and thus

$$|\int b^{\epsilon}(\cos\theta)(\varphi_j)(v')-\varphi_j(v)d\sigma|\lesssim (W^{\epsilon})^2(|v-v_*|)\lesssim (W^{\epsilon})^2(v_*)(W^{\epsilon})^2(v),$$

which yields

$$\begin{aligned}\mathcal{K}_2(g,h) &= |\int B^{\epsilon}(g\mu^{\frac{1}{2}}1_{|\cdot|\geq 2^j/10})_*hf\varphi_j\big((\varphi_j)'-\varphi_j\big)d\sigma dv_*dv| \\ &\leq \int |v-v_*|^{\gamma}|(g\mu^{\frac{1}{2}}1_{|\cdot|\geq 2^j/10})_*|1_{2^j/10\leq|v|\leq 10\times 2^j}(2^{-j}|v-v_*|+2^{-2j}|v-v_*|^2) \\ &\quad\times|hf|\varphi_j dv_*dv+\int|v-v_*|^{\gamma}|(g\mu^{\frac{1}{2}}1_{|\cdot|\geq 2^j/10})_*|(1_{|v|<2^j/10}+1_{|v|>10\times 2^j}) \\ &\quad\times(W^{\epsilon})^2(v_*)(W^{\epsilon})^2(v)|hf|\varphi_j dv_*dv \\ &:=\mathcal{K}_{2,1}(g,h)+\mathcal{K}_{2,2}(g,h).\end{aligned}$$

When $(g,h)=(\mu^{1/2},f)$, by the facts $\mu^{1/2}(v)\lesssim\exp(-C_1 2^{2j})$ if $|v|\geq 2^j/10$, and $\int|v-v_*|^n(\mu^{1/2})_*dv_*\lesssim\langle v\rangle^n$ for $n>-3$, we get

$$\begin{aligned}\mathcal{K}_{2,1}(\mu^{1/2},f) &= \int(\mu 1_{|\cdot|\geq 2^j/10})_*1_{2^j/10\leq|v|\leq 10\times 2^j}a_j(v,v_*)f^2\varphi_j dv_*dv \\ &\lesssim\exp(-C_1 2^{2j})\int 1_{2^j/10\leq|v|\leq 10\times 2^j}(2^{-j}\langle v\rangle^{\gamma+1}+2^{-2j}\langle v\rangle^{\gamma+2})f^2\varphi_j dv \\ &\lesssim\exp(-C_1 2^{2j})|f|^2_{L^2_{\gamma/2}},\end{aligned}$$

where for notational brevity, we set $a_j(v,v_*):=2^{-j}|v-v_*|^{\gamma+1}+2^{-2j}|v-v_*|^{\gamma+2}$. Similar argument yields

$$\begin{aligned}\mathcal{K}_{2,2}(\mu^{1/2},f) &= \int|v-v_*|^{\gamma}(\mu 1_{|\cdot|\geq 2^j/10})_*(1_{|v|<2^j/10}+1_{|v|>10\times 2^j})(W^{\epsilon})^2(v_*)(W^{\epsilon})^2(v) \\ &\quad\times f^2\varphi_j dv_*dv \\ &\lesssim\exp(-C_1 2^{2j})|W^{\epsilon}f|^2_{L^2_{\gamma/2}}.\end{aligned}$$

When $(g,h)=(f,\mu^{1/2})$, we get by Cauchy-Schwartz inequality and similar arguments as before,

$$\begin{aligned}&\mathcal{K}_{2,1}(f,\mu^{1/2}) \\ &= \int|(f\mu^{1/2}1_{|\cdot|\geq 2^j/10})_*|1_{2^j/10\leq|v|\leq 10\times 2^j}a_j(v,v_*)\mu^{1/2}|f|\varphi_j dv_*dv \\ &\leq\Big(\int(f^2\mu^{1/2}1_{|\cdot|\geq 2^j/10})_*1_{2^j/10\leq|v|\leq 10\times 2^j}a_j(v,v_*)\mu^{1/2}dv_*dv\Big)^{1/2} \\ &\quad\times\Big(\int(\mu^{1/2}1_{|\cdot|\geq 2^j/10})_*1_{2^j/10\leq|v|\leq 10\times 2^j}a_j(v,v_*)\mu^{1/2}(f\varphi_j)^2dv_*dv\Big)^{1/2}\end{aligned}$$

$$\lesssim \exp(-C_1 2^{2j})|f|^2_{L^2_{\gamma/2}}.$$

Similar argument yields

$$\begin{aligned}
&\mathcal{K}_{2,2}(f,\mu^{1/2})\\
&=\int |v-v_*|^\gamma |(f\mu^{1/2}1_{|\cdot|\geq 2^j/10})_*|(1_{|v|<2^j/10}+1_{|v|>10\times 2^j})\\
&\quad\times (W^\epsilon)^2(v_*)(W^\epsilon)^2(v)\mu^{1/2}|f|\varphi_j dv_* dv\\
&\lesssim \left(\int (f^2\mu^{1/4}1_{|\cdot|\geq 2^j/10})_*|v-v_*|^\gamma \mu^{1/4} dv_* dv\right)^{1/2}\\
&\quad\times\left(\int (\mu^{1/4}1_{|\cdot|\geq 2^j/10})_*|v-v_*|^\gamma \mu^{1/4}(f\varphi_j)^2 dv_* dv\right)^{1/2}\lesssim \exp(-C_1 2^{2j})|f|^2_{L^2_{\gamma/2}}.
\end{aligned}$$

Patch together the above estimates in *Step 2.1* and *Step 2.2*, to get

$$\mathcal{K}(\mu^{1/2},f)+\mathcal{K}(f,\mu^{1/2})\lesssim \exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}.$$

Patching together all the above estimates, we get the lemma by the fact $\psi(x)=1$ if $|x|\in[4/3,3/2]$ and thus $|f1_{2^j/5\leq|\cdot|\leq 10\times 2^j}|^2_{L^2_{\gamma/2}}\leq \sum_{k=j-3}^{j+3}|\mathcal{P}_k f|^2_{L^2_{\gamma/2}}$. ◻

Now we set to prove the second part of Theorem 1.1, namely (1.26) and (1.27).

Proof. **(Proof of Theorem 1.1: (1.26) and (1.27).)** Applying $\mathcal{P}_j$ to both sides of $\partial_t f+\mathcal{L}^\epsilon f=0$, we have

$$\partial_t \mathcal{P}_j f+\mathcal{L}^\epsilon \mathcal{P}_j f=[\mathcal{L}^\epsilon,\mathcal{P}_j]f.$$

Take inner product with $\mathcal{P}_j f$, thanks to Theorem 2.1 and Lemma 3.1, we have

$$\frac{d}{dt}|\mathcal{P}_j f(t)|^2_{L^2}+C|\mathcal{P}_j f|^2_{\epsilon,\gamma/2}\gtrsim -\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}-\sum_{k=j-3}^{j+3}|\mathcal{P}_k f|^2_{L^2_{\gamma/2}}, \quad (3.33)$$

$$\frac{d}{dt}|\mathcal{P}_j f(t)|^2_{L^2}+\lambda|\mathcal{P}_j f|^2_{\epsilon,\gamma/2}\lesssim \exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}+\sum_{k=j-3}^{j+3}|\mathcal{P}_k f|^2_{L^2_{\gamma/2}}. \quad (3.34)$$

Since $2^j\geq 1/\epsilon$, we observe $|W^\epsilon \mathcal{P}_j f|^2_{L^2_{\gamma/2}}\sim \epsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f|^2_{L^2}$ and

$$|W^\epsilon(D)W_{\gamma/2}\mathcal{P}_j f|^2_{L^2}+|W^\epsilon((-\Delta_{\mathbb{S}^2})^{1/2})W_{\gamma/2}\mathcal{P}_j f|^2\lesssim \epsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f|^2_{L^2}.$$

It is obvious to see $\sum_{k=j-3}^{j+3}|\mathcal{P}_k f|^2_{L^2_{\gamma/2}}\lesssim 2^{j\gamma}|f|^2_{L^2}$. Plugging these facts into (3.33), we get

$$\frac{d}{dt}|\mathcal{P}_j f(t)|^2_{L^2}\gtrsim -\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}}-\epsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f|^2_{L^2}-2^{j\gamma}|f|^2_{L^2}.$$

By (3.6), we have $\sup_{t\geq 0}|f(t)|^2_{L^2}\leq |f_0|^2_{L^2}, \int_0^\infty |f(t)|^2_{\epsilon,\gamma/2}dt\leq (2\lambda)^{-1}|f_0|^2_{L^2}$. By the assumption $|f_0|_{L^2}=1$, we get $|\mathcal{P}_j f(t)|^2_{L^2}\geq |\mathcal{P}_j f_0|^2_{L^2}-C\epsilon^{-2s}2^{j\gamma}t-C\exp(-C_1 2^{2j})$. From this, we conclude the result (1.26).

We now set to prove (1.27). From (3.34) and the fact $\sum_{k=j-3}^{j+3}|\mathcal{P}_k f|^2_{L^2_{\gamma/2}} \lesssim 2^{j\gamma}|f|^2_{L^2}$, we have

$$\frac{d}{dt}|\mathcal{P}_j f(t)|^2_{L^2} + \lambda\epsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f(t)|^2_{L^2} \leq C(\exp(-C_1 2^{2j})|W^\epsilon f|^2_{L^2_{\gamma/2}} + 2^{j\gamma}|f|^2_{L^2}) := a(t).$$

Recall $\Lambda = \lambda\epsilon^{-2s}2^{j\gamma}$. By Grönwall's inequality, we have

$$|\mathcal{P}_j f(t)|^2_{L^2} \leq \exp(-\Lambda t)|\mathcal{P}_j f_0|^2_{L^2} + \int_0^t \exp(\Lambda(s-t))a(s)ds.$$

Since $\gamma/2+2s<0$, we have $|W^\epsilon f|^2_{L^2_{\gamma/2}} \leq |f|^2_{L^2_{\gamma/2+s}} \leq |f|^2_{L^2} \leq |f_0|^2_{L^2} = 1$, which gives

$$a(t) \leq C(\exp(-C_1 2^{2j}) + 2^{j\gamma}),$$

and thus

$$\begin{aligned}
|\mathcal{P}_j f(t)|^2_{L^2} &\leq \exp(-\Lambda t)|\mathcal{P}_j f_0|^2_{L^2} + \frac{1-\exp(-\Lambda t)}{\Lambda}C(\exp(-C_1 2^{2j}) + 2^{j\gamma}) \\
&= \exp(-\Lambda t)|\mathcal{P}_j f_0|^2_{L^2} + (1-\exp(-\Lambda t))C(\lambda^{-1}\epsilon^{2s}2^{-j\gamma}\exp(-C_1 2^{2j}) + \lambda^{-1}\epsilon^{2s}) \\
&\leq \exp(-\Lambda t)|\mathcal{P}_j f_0|^2_{L^2} + (1-\exp(-\Lambda t))\lambda^{-1}C(1+\delta)\epsilon^{2s},
\end{aligned} \tag{3.35}$$

where we use the assumption $2^{-j\gamma}\exp(-C_1 2^{2j}) \leq \delta$. When $K>2, t \leq 2\ln(K-1)/\Lambda$, it is easy to check

$$K\exp(-\Lambda t) + (1-\exp(-\Lambda t)) \leq K\exp(-\Lambda t/2).$$

Then set $K = \frac{|\mathcal{P}_j f_0|^2_{L^2}}{\lambda^{-1}C(1+\delta)\epsilon^{2s}}$, for $t \in [0, 2\lambda^{-1}\epsilon^{2s}2^{-j\gamma}\ln(K-1)]$, revisit (3.35), we have

$$|\mathcal{P}_j f(t)|^2_{L^2} \leq \exp(-\Lambda t/2)|\mathcal{P}_j f_0|^2_{L^2},$$

which is exactly (1.27). □

4. Boltzmann equation near equilibrium

This section is devoted to the proof to Theorem 1.2, which includes three subsections. In subsection 4.1, we prove global well-posedenss and propagation of regularity. Global dynamics is derived in subsection 4.2 by employing Proposition 3.2 once again. The global asymptotic formula is established in the last subsection.

4.1. Global well-posedenss and propagation of regularity.

We only provide the *a priori* estimates for the equation, which is Theorem 4.1, from which together with local existence result in [12], the first part **(global well-posedness)** in Theorem 1.2 can be established. The second part **(propagation of regularity)** in Theorem 1.2 follows directly from Theorem 4.1.

4.1.1. Estimate for the linear equation.

Fix a small $\epsilon>0$ and a general function g, suppose f^ϵ is a solution to

$$\partial_t f + v\cdot\nabla_x f + \mathcal{L}^\epsilon f = g. \tag{4.1}$$

For simplicity, we omit the superscript ϵ in f^ϵ. We set $f_1 := \mathbb{P}f$ and $f_2 := f - \mathbb{P}f$.

By the Definition (1.21) of the projection operator $\mathbb{P}$, one has

$$f_1(t,x,v) = \{a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2\}\mu^{1/2}, \tag{4.2}$$

which satisfies

$$\partial_t f_1 + v\cdot\nabla_x f_1 = r + l + g, \tag{4.3}$$

where $r = -\partial_t f_2$ and $l = -v\cdot\nabla_x f_2 - \mathcal{L}^\epsilon f_2$.

We recall that $\{e_j\}_{1\le j\le 13}$ is defined explicitly as

$$\begin{aligned}
&e_1 = \mu^{1/2}, e_2 = v_1\mu^{1/2}, e_3 = v_2\mu^{1/2}, e_4 = v_3\mu^{1/2},\\
&e_5 = v_1^2\mu^{1/2}, e_6 = v_2^2\mu^{1/2}, e_7 = v_3^2\mu^{1/2}, e_8 = v_1v_2\mu^{1/2}, e_9 = v_2v_3\mu^{1/2}, e_{10} = v_3v_1\mu^{1/2},\\
&e_{11} = |v|^2v_1\mu^{1/2}, e_{12} = |v|^2v_2\mu^{1/2}, e_{13} = |v|^2v_3\mu^{1/2}.
\end{aligned}$$

Let $A = (a_{ij})_{1\le i,j\le 13}$ be the real matrix given by $a_{ij} := \langle e_i, e_j\rangle$ and y be the 13-dimensional vector with components $\partial_t a, \{\partial_t b_i + \partial_i a\}_{1\le i\le 3}, \{\partial_t c + \partial_i b_i\}_{1\le i\le 3}, \{\partial_i b_j + \partial_j b_i\}_{1\le i<j\le 3}, \{\partial_i c\}_{1\le i\le 3}$. Set $z := (z_i)_{i=1}^{13} := (\langle r+l+g, e_i\rangle)_{i=1}^{13}$. Taking inner product between (4.3) and $\{e_j\}_{1\le j\le 13}$, one has $Ay = z$, which gives $y = A^{-1}z$. For notational simplicity, we denote $z^r := (z_i^r)_{i=1}^{13} := (\langle r, e_i\rangle)_{i=1}^{13}$. We also use z^l, z^g, z^{f_2} in a similar way. Further, we set

$$\begin{aligned}
\tilde r &= (r^{(0)}, \{r_i^{(1)}\}_{1\le i\le 3}, \{r_i^{(2)}\}_{1\le i\le 3}, \{r_{ij}^{(2)}\}_{1\le i<j\le 3}, \{r_i^{(3)}\}_{1\le i\le 3})^T = A^{-1}z^r,\\
\tilde l &= (l^{(0)}, \{l_i^{(1)}\}_{1\le i\le 3}, \{l_i^{(2)}\}_{1\le i\le 3}, \{l_{ij}^{(2)}\}_{1\le i<j\le 3}, \{l_i^{(3)}\}_{1\le i\le 3})^T = A^{-1}z^l,\\
\tilde g &= (g^{(0)}, \{g_i^{(1)}\}_{1\le i\le 3}, \{g_i^{(2)}\}_{1\le i\le 3}, \{g_{ij}^{(2)}\}_{1\le i<j\le 3}, \{g_i^{(3)}\}_{1\le i\le 3})^T = A^{-1}z^g.
\end{aligned}$$

With a little abuse of notation, we set $\tilde f := A^{-1}z^{f_2}$. That is,

$$\tilde f = (\tilde f^{(0)}, \{\tilde f_i^{(1)}\}_{1\le i\le 3}, \{\tilde f_i^{(2)}\}_{1\le i\le 3}, \{\tilde f_{ij}^{(2)}\}_{1\le i<j\le 3}, \{\tilde f_i^{(3)}\}_{1\le i\le 3})^T = A^{-1}(\langle f_2, e_i\rangle)_{i=1}^{13}.$$

With these notations, one has $\tilde r = -\partial_t \tilde f$, and thus

$$y = -\partial_t \tilde f + \tilde l + \tilde g. \tag{4.4}$$

Following the notations in [8], let us define the temporal energy functional $\mathcal{I}^N(f)$ as

$$\mathcal{I}^N(f) := \sum_{|\alpha|\le N-1}\sum_{i=1}^{3}(\mathcal{I}^a_{\alpha,i}(f) + \mathcal{I}^b_{\alpha,i}(f) + \mathcal{I}^c_{\alpha,i}(f) + \mathcal{I}^{ab}_{\alpha,i}(f)), \tag{4.5}$$

where
$\mathcal{I}^a_{\alpha,i}(f) = \langle\partial^\alpha \tilde f_i^{(1)}, \partial_i\partial^\alpha a\rangle$, $\mathcal{I}^b_{\alpha,i}(f) = -\sum_{j\ne i}\langle\partial^\alpha\tilde f_j^{(2)}, \partial_i\partial^\alpha b_i\rangle + \sum_{j\ne i}\langle\partial^\alpha \tilde f_{ji}^{(2)}, \partial_j\partial^\alpha b_i\rangle + 2\langle\partial^\alpha\tilde f_i^{(2)}, \partial_i\partial^\alpha b_i\rangle$, $\mathcal{I}^c_{\alpha,i}(f) = \langle\partial^\alpha\tilde f_i^{(3)}, \partial_i\partial^\alpha c\rangle$ and $\mathcal{I}^{ab}_{\alpha,i}(f) = \langle\partial_i\partial^\alpha a, \partial^\alpha b_i\rangle$. There is some universal constant M such that

$$|\mathcal{I}^N(f)| \le M\|f\|^2_{H^N_x L^2}. \tag{4.6}$$

We recall a result on the dissipation of (a,b,c).

LEMMA 4.1. *There exists a constant $C>0$ such that*

$$\frac{d}{dt}\mathcal{I}^N(f) + \frac12|\nabla_x(a,b,c)|^2_{H^{N-1}_x} \le C(\|f_2\|^2_{H^N_x L^2_{\epsilon,\gamma/2}} + \sum_{|\alpha|\le N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g, e_j\rangle|^2dx). \tag{4.7}$$

The proof of Lemma 4.1 can be found in the end of the Appendix in [14].

For non-negative integers n,m, we recall

$$||f||^2_{H^n_x\dot{H}^m_l}=\sum_{|\alpha|\le n,|\beta|=m}||W_l\partial^\alpha_\beta f||^2_{L^2},||f||^2_{H^n_x\dot{H}^m_{\epsilon,l+\gamma/2}}=\sum_{|\alpha|\le n,|\beta|=m}||W_l\partial^\alpha_\beta f||^2_{L^2_{\epsilon,\gamma/2}}.$$

For some constants $K_j,-2\le j\le N$, which can be explicitly determined later, we define the energy functional

$$\Xi^{N,l}(f):=K_{-2}\mathcal{I}^N(f)+K_{-1}||f||^2_{H^N_xL^2}+\sum_{j=0}^N K_j||f||^2_{H^{N-j}_x\dot{H}^j_{l+j\gamma}},$$

and the corresponding dissipation functional

$$\mathcal{D}^{N,l}_\epsilon(f):=|\mathcal{MA}|^2_{H^N_x}+||f_2||^2_{H^N_xL^2_{\epsilon,\gamma/2}}+\sum_{j=0}^N||f_2||^2_{H^{N-j}_x\dot{H}^j_{\epsilon,l+j\gamma+\gamma/2}}\gtrsim\sum_{j=0}^N||f||^2_{H^{N-j}_x\dot{H}^j_{\epsilon,l+j\gamma+\gamma/2}},$$

where $\mathcal{MA}:=(a(t,x),b(t,x),c(t,x))$ which stands for the macro-part of a solution f.

With these notations in hand, we derive the following *a priori* estimate of (4.1).

PROPOSITION 4.1. *Let $N\ge 2,l\ge 2-\gamma N$, suppose f is a smooth solution to (4.1). Then there holds*

$$\begin{aligned}\frac{d}{dt}\Xi^{N,l}(f)+\lambda\mathcal{D}^{N,l}_\epsilon(f)\lesssim&\sum_{j=0}^N\sum_{|\alpha|\le N-j,|\beta|=j}|(W_{l+j\gamma}\partial^\alpha_\beta g,W_{l+j\gamma}\partial^\alpha_\beta f)|\\&+\sum_{|\alpha|\le N}|(\partial^\alpha g,\partial^\alpha f)|+\sum_{|\alpha|\le N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx.\end{aligned}\tag{4.8}$$

Here the $\lesssim$ could result in a constant $C_{N,l}$ on the right-hand side.

Proof. Note that $\Xi^{N,l}(f)$ contains many items. We already have the term $\mathcal{I}^N(f)$ from Lemma 4.1. We add the rest step by step.

Step 1: $||f||^2_{H^N_xL^2}$. Applying ∂^α to Equation (4.1), taking inner product with $\partial^\alpha f$, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{L^2}+(\mathcal{L}^\epsilon\partial^\alpha f,\partial^\alpha f)=(\partial^\alpha g,\partial^\alpha f).$$

Thanks to $(\partial^\alpha f)_2=\partial^\alpha f_2$ and Proposition 2.1, taking sum over $|\alpha|\le N$, we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{H^N_xL^2}+\lambda\|f_2\|^2_{H^N_xL^2_{\epsilon,\gamma/2}}\le\sum_{|\alpha|\le N}|(\partial^\alpha g,\partial^\alpha f)|.\tag{4.9}$$

Multiplying (4.9) by a large constant M_1 and adding the resulting inequality to (4.7), we get

$$\begin{aligned}&\frac{d}{dt}(\mathcal{I}^N(f)+M_1\|f\|^2_{H^N_xL^2})+\frac{1}{2}(|\nabla_x\mathcal{MA}|^2_{H^{N-1}_x}+\|f_2\|^2_{H^N_xL^2_{\epsilon,\gamma/2}})\\&\lesssim\sum_{|\alpha|\le N}|(\partial^\alpha g,\partial^\alpha f)|+\sum_{|\alpha|\le N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx.\end{aligned}\tag{4.10}$$

Here M_1 is also chosen large enough such that $M_1\|f\|^2_{H^N_x L^2}+\mathcal{I}^N(f)\sim\|f\|^2_{H^N_x L^2}$ thanks to (4.6).

Step 2: $||f||^2_{H^N_x L^2_l}$. Applying $W_l\partial^\alpha$ to Equation (4.1), taking inner product with $W_l\partial^\alpha f$, taking sum over $|\alpha|\le N$, we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{H^N_x L^2_l}+\sum_{|\alpha|\le N}(W_l\mathcal{L}^\epsilon\partial^\alpha f_2,W_l\partial^\alpha f)=\sum_{|\alpha|\le N}(W_l\partial^\alpha g,W_l\partial^\alpha f).$$

By splitting $f=f_1+f_2$, we have

$$(W_l\mathcal{L}^\epsilon\partial^\alpha f_2,W_l\partial^\alpha f)=(W_l\mathcal{L}^\epsilon\partial^\alpha f_2,W_l\partial^\alpha f_1)+(W_l\mathcal{L}^\epsilon\partial^\alpha f_2,W_l\partial^\alpha f_2):=A_1+A_2.$$

By Corollary 2.1 on the upper bound estimate of $\mathcal{L}^\epsilon$, moving all the weights to f_1, we have

$$|A_1|\lesssim\int|\partial^\alpha f_2|_{\epsilon,\gamma/2}|\partial^\alpha\mathcal{M}\mathcal{A}|dx\lesssim|\mathcal{M}\mathcal{A}|_{H^N_x}||f_2||_{H^N_x L^2_{\epsilon,\gamma/2}}.$$

Rearrange A_2 by introducing the commutator operator $[W_l,\mathcal{L}^\epsilon]$:

$$A_2=(\mathcal{L}^\epsilon W_l\partial^\alpha f_2,W_l\partial^\alpha f_2)+([W_l,\mathcal{L}^\epsilon]\partial^\alpha f_2,W_l\partial^\alpha f_2):=A_{2,1}+A_{2,2}.$$

By the coercivity result (2.2), we have

$$A_{2,1}\ge\lambda||W_l\partial^\alpha f_2||^2_{L^2_{\epsilon,\gamma/2}}-C\|f_2\|^2_{H^N_x L^2_{l+\gamma/2}}.$$

By the commutator estimate in Corollary 2.2, we have

$$|A_{2,2}|\lesssim\|f_2\|_{H^N_x L^2_{l+\gamma/2}}||W_l\partial^\alpha f_2||_{L^2_{\epsilon,\gamma/2}}.$$

Taking sum over $|\alpha|\le N$, together with $2AB\le\eta A^2+\eta^{-1}B^2$ for any $\eta>0$, we get

$$\sum_{|\alpha|\le N}(W_l\mathcal{L}^\epsilon\partial^\alpha f_2,W_l\partial^\alpha f)\ge\frac{3}{4}\lambda||f_2||^2_{H^N_x L^2_{\epsilon,l+\gamma/2}}-C(\|f_2\|^2_{H^N_x L^2_{l+\gamma/2}}+|\mathcal{M}\mathcal{A}|^2_{H^N_x}).$$

By (3.8) and (3.9), we have

$$\|f_2\|^2_{H^N_x L^2_{l+\gamma/2}}\le(\eta+\epsilon^{2s})||f_2||^2_{H^N_x L^2_{\epsilon,l+\gamma/2}}+C_\eta||f_2||^2_{H^N_x L^2_{\epsilon,\gamma/2}}.$$

Taking η small enough such that $C\eta\le\lambda/8$, then when ϵ is small such that $C\epsilon^{2s}\le\lambda/8$, we have

$$\frac{d}{dt}\|f\|^2_{H^N_x L^2_l}+\lambda||f_2||^2_{H^N_x L^2_{\epsilon,l+\gamma/2}}\le C(\|f_2\|^2_{H^N_x L^2_{\epsilon,\gamma/2}}+|\mathcal{M}\mathcal{A}|^2_{H^N_x})+2\sum_{|\alpha|\le N}(W_l\partial^\alpha g,W_l\partial^\alpha f).$$

Thanks to (1.4) and (1.9), Poincare inequality gives $|\mathcal{M}\mathcal{A}|_{H^N_x}\sim|\nabla_x\mathcal{M}\mathcal{A}|_{H^{N-1}_x}$. Multiply (4.10) by a large constant M_2 and add it to the previous inequality, to get

$$\begin{aligned}&\frac{d}{dt}(M_2\mathcal{I}^N(f)+M_1M_2\|f\|^2_{H^N_x L^2}+\|f\|^2_{H^N_x L^2_l})\\&\quad+\lambda(|\mathcal{M}\mathcal{A}|^2_{H^N_x}+\|f_2\|^2_{H^N_x L^2_{\epsilon,\gamma/2}}+\|f_2\|^2_{H^N_x L^2_{\epsilon,l+\gamma/2}})\end{aligned}$$

$$\lesssim \sum_{|\alpha|\leq N}|(\partial^\alpha g,\partial^\alpha f)|+\sum_{|\alpha|\leq N}|(W_l\partial^\alpha g,W_l\partial^\alpha f)|+\sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx. \quad (4.11)$$

Step 3: $\sum_{j=1}^N K_j||f||^2_{H^{N-j}_x\dot{H}^j_{l+j\gamma}}$. (Mathematical Induction)

We prove, for any $0\leq i\leq N$, there exist some constants $K^i_j,-2\leq j\leq i$, such that

$$\begin{aligned}&\frac{d}{dt}(K^i_{-2}\mathcal{I}^N(f)+K^i_{-1}\|f\|^2_{H^N_xL^2}+\sum_{0\leq j\leq i}K^i_j\|f\|^2_{H^{N-j}_x\dot{H}^j_{l+j\gamma}})\\&+\lambda(|\mathcal{MA}|^2_{H^N_x}+||f_2||^2_{H^N_xL^2_{\epsilon,\gamma/2}}+\sum_{j=0}^{i}||f_2||^2_{H^{N-j}_x\dot{H}^j_{\epsilon,l+j\gamma+\gamma/2}})\\&\lesssim\sum_{|\alpha|\leq N}|(\partial^\alpha g,\partial^\alpha f)|+\sum_{j=0}^{i}\sum_{|\alpha|\leq N-j,|\beta|=j}|(W_{l+j\gamma}\partial^\alpha_\beta g,W_{l+j\gamma}\partial^\alpha_\beta f)|\\&+\sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx. \end{aligned}\quad (4.12)$$

Our final goal (4.8) is actually (4.12) with $i=N$.

Note that (4.12) is true when $i=0$, which is given by (4.11). More precisely, we can take $K^0_{-2}=M_2,K^0_{-1}=M_1M_2,K^0_0=1$.

We prove (4.12) by induction on i. Suppose (4.12) is true when $i=k$ for some $0\leq k\leq N-1$, we prove it is also valid when $i=k+1$.

Take two indexes α and β such that $|\alpha|\leq N-(k+1)$ and $|\beta|=k+1\geq 1$, set $q=l+(k+1)\gamma$. Applying $W_q\partial^\alpha_\beta$ to both sides of (4.1), we have

$$\partial_tW_q\partial^\alpha_\beta f+v\cdot\nabla_xW_q\partial^\alpha_\beta f+\sum_{\beta_1\leq\beta,|\beta_1|=1}W_q\partial^{\alpha+\beta_1}_{\beta-\beta_1}f+W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2=W_q\partial^\alpha_\beta g. \quad (4.13)$$

Taking inner product with $W_q\partial^\alpha_\beta f$ over (x,v), one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{L^2_q}+\sum_{\beta_1\leq\beta,|\beta_1|=1}(W_q\partial^{\alpha+\beta_1}_{\beta-\beta_1}f,W_q\partial^\alpha_\beta f)+(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2,W_q\partial^\alpha_\beta f)\\&=(W_q\partial^\alpha_\beta g,W_q\partial^\alpha_\beta f).\end{aligned} \quad (4.14)$$

We first go to deal with $(W_q\partial^{\alpha+\beta_1}_{\beta-\beta_1}f,W_q\partial^\alpha_\beta f)$. By Cauchy-Schwartz inequality and using $f=f_1+f_2$, we get

$$\begin{aligned}|(W_q\partial^{\alpha+\beta_1}_{\beta-\beta_1}f,W_q\partial^\alpha_\beta f)|&\leq\|\partial^{\alpha+\beta_1}_{\beta-\beta_1}f\|_{L^2_xL^2_{q-\gamma/2}}\|\partial^\alpha_\beta f\|_{L^2_xL^2_{q+\gamma/2}}\\&\lesssim\|f_2\|^2_{H^{N-k}_x\dot{H}^k_{\epsilon,q-\gamma/2}}+\|f_2\|^2_{H^{N-k-1}_x\dot{H}^{k+1}_{q+\gamma/2}}+|\mathcal{MA}|^2_{H^{N-k}_x}.\end{aligned} \quad (4.15)$$

We now go to deal with $(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2,W_q\partial^\alpha_\beta f)$. Observe

$$(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2,W_q\partial^\alpha_\beta f)=(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2,W_q\partial^\alpha_\beta f_1)+(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2,W_q\partial^\alpha_\beta f_2). \quad (4.16)$$

Recalling (3.5), we have

$$W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2=\mathcal{L}^\epsilon W_q\partial^\alpha_\beta f_2+\sum_{\beta_2\leq\beta}C^{\beta_0,\beta_1,\beta_2}_\beta[W_q,\mathcal{L}^{\epsilon,\beta_0,\beta_1}]\partial^\alpha_{\beta_2}f_2+\sum_{\beta_2<\beta}C^{\beta_0,\beta_1,\beta_2}_\beta\mathcal{L}^{\epsilon,\beta_0,\beta_1}W_q\partial^\alpha_{\beta_2}f_2.$$

We remark that $\mathcal{L}^{\epsilon,\beta_0,\beta_1}$ satisfies the upper bound in Corollary 2.1 and commutator estimate in Corollary 2.2. By Corollary 2.1 and Corollary 2.2, we have

$$\begin{aligned}&|(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2, W_q\partial^\alpha_\beta f_1)|\\ \leq&\eta||\partial^\alpha_\beta f_2||^2_{L^2_{\epsilon,q+\gamma/2}}+C_\eta(|\mathcal{MA}|^2_{H^N_x}+\|f_2\|^2_{H^{N-k-1}_xH^{k+1}_{q+\gamma/2}}+||f_2||^2_{H^{N-k-1}_xH^k_{\epsilon,q+\gamma/2}}).\end{aligned}$$

By coercivity (2.2), upper bound in Corollary 2.1 and commutator estimate in Corollary 2.2, we have for any $\eta>0$,

$$\begin{aligned}(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2, W_q\partial^\alpha_\beta f_2)\geq&(\lambda-\eta)||\partial^\alpha_\beta f_2||^2_{L^2_{\epsilon,q+\gamma/2}}\\ &-C_\eta(\|f_2\|^2_{H^{N-k-1}_xH^{k+1}_{q+\gamma/2}}+||f_2||^2_{H^{N-k-1}_xH^k_{\epsilon,q+\gamma/2}}).\end{aligned}$$

Taking $\eta=\lambda/8$, and plugging the previous two results into (4.16), we get

$$\begin{aligned}(W_q\partial^\alpha_\beta\mathcal{L}^\epsilon f_2, W_q\partial^\alpha_\beta f)\geq&(3\lambda/4)||\partial^\alpha_\beta f_2||^2_{L^2_{\epsilon,q+\gamma/2}}\\ &-C_\eta(|\mathcal{MA}|^2_{H^N_x}+\|f_2\|^2_{H^{N-k-1}_xH^{k+1}_{q+\gamma/2}}+||f_2||^2_{H^{N-k-1}_xH^k_{\epsilon,q+\gamma/2}}),\end{aligned}$$

from which together with (4.15), back to (4.14), taking sum over $|\alpha|\leq N-(k+1),|\beta|=k+1$, we have

$$\begin{aligned}&\frac{d}{dt}\|f\|^2_{H^{N-k-1}_x\dot{H}^{k+1}_q}+\frac{3}{2}\lambda||f_2||^2_{H^{N-k-1}_x\dot{H}^{k+1}_{\epsilon,q+\gamma/2}}\\ \lesssim&\sum_{|\alpha|\leq N-k-1,|\beta|=k+1}|(W_q\partial^\alpha_\beta g, W_q\partial^\alpha_\beta f)|+|\mathcal{MA}|^2_{H^N_x}+\|f_2\|^2_{H^{N-k-1}_xH^{k+1}_{q+\gamma/2}}\\ &+||f_2||^2_{H^{N-k-1}_xH^k_{\epsilon,q+\gamma/2}}+\|f_2\|^2_{H^{N-k}_x\dot{H}^k_{\epsilon,q-\gamma/2}}.\end{aligned}\tag{4.17}$$

Recalling $q=l+(k+1)\gamma$ and by Proposition A.2, we have

$$\begin{aligned}\|f_2\|^2_{H^{N-k-1}_xH^{k+1}_{q+\gamma/2}}&\leq(\eta+\epsilon^{2s})\|f_2\|^2_{H^{N-k-1}_xH^{k+1}_{\epsilon,q+\gamma/2}}+C_\eta\|f_2\|^2_{H^{N-k-1}_xH^0_{q+\gamma/2}}\\ &\leq(\eta+\epsilon^{2s})\|f_2\|^2_{H^{N-k-1}_x\dot{H}^{k+1}_{\epsilon,q+\gamma/2}}+C_\eta\|f_2\|^2_{H^{N-k-1}_xH^k_{\epsilon,l+k\gamma+\gamma/2}},\\ \|f_2\|^2_{H^{N-k}_x\dot{H}^k_{q-\gamma/2}}&\leq\|f_2\|^2_{H^{N-k}_x\dot{H}^k_{\epsilon,l+k\gamma+\gamma/2}}.\end{aligned}$$

Plugging which into (4.17), taking η small enough such that $\eta\ll\lambda/8$, then when ϵ is small such that $\epsilon^{2s}\ll\lambda/8$, we have

$$\begin{aligned}&\frac{d}{dt}\|f\|^2_{H^{N-k-1}_x\dot{H}^{k+1}_q}+\lambda||f_2||^2_{H^{N-k-1}_x\dot{H}^{k+1}_{\epsilon,q+\gamma/2}}\\ \lesssim&\sum_{|\alpha|\leq N-k-1,|\beta|=k+1}|(W_q\partial^\alpha_\beta g, W_q\partial^\alpha_\beta f)|+|\mathcal{MA}|^2_{H^N_x}+\|f_2\|^2_{H^{N-k}_xH^k_{\epsilon,l+k\gamma+\gamma/2}}.\end{aligned}\tag{4.18}$$

By our induction assumption, (4.12) is true when $i=k$, that is,

$$\begin{aligned}&\frac{d}{dt}(K^k_{-2}\mathcal{I}^N(f)+K^k_{-1}\|f\|^2_{H^N_xL^2}+\sum_{j=0}^{k}K^k_j\|f\|^2_{H^{N-j}_x\dot{H}^j_{l+j\gamma}})\\ &+\lambda(|\mathcal{MA}|^2_{H^N_x}+||f_2||^2_{H^N_xL^2_{\epsilon,\gamma/2}}+\sum_{j=0}^{k}||f_2||^2_{H^{N-j}_x\dot{H}^j_{\epsilon,l+j\gamma+\gamma/2}})\end{aligned}$$

$$\lesssim \sum_{|\alpha|\leq N}|(\partial^\alpha g,\partial^\alpha f)|+\sum_{j=0}^{k}\sum_{|\alpha|\leq N-j,|\beta|=j}|(W_{l+j\gamma}\partial^\alpha_\beta g,W_{l+j\gamma}\partial^\alpha_\beta f)|$$
$$+\sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx. \tag{4.19}$$

Multiplying (4.19) by a large constant M, and adding the resulting inequality to (4.18) to cancel the two terms $|\mathcal{M}\mathcal{A}|^2_{H^N_x}$ and $\|f_2\|^2_{H^{N-k}_xH^k_{\epsilon,l+k\gamma+\gamma/2}}$, we get

$$\frac{d}{dt}(M(K^k_{-2}\mathcal{I}^N(f)+K^k_{-1}\|f\|^2_{H^N_xL^2}+\sum_{j=0}^{k}K^k_j\|f\|^2_{H^{N-j}_x\dot{H}^j_{l+j\gamma}})+\|f\|^2_{H^{N-k-1}_x\dot{H}^{k+1}_q})$$
$$+\lambda(|\mathcal{M}\mathcal{A}|^2_{H^N_x}+||f_2||^2_{H^N_xL^2_{\epsilon,\gamma/2}}+\sum_{j=0}^{k+1}||f_2||^2_{H^{N-j}_x\dot{H}^j_{\epsilon,l+j\gamma+\gamma/2}})$$
$$\lesssim \sum_{|\alpha|\leq N}|(\partial^\alpha g,\partial^\alpha f)|+\sum_{j=0}^{k+1}\sum_{|\alpha|\leq N-j,|\beta|=j}|(W_{l+j\gamma}\partial^\alpha_\beta g,W_{l+j\gamma}\partial^\alpha_\beta f)|$$
$$+\sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx. \tag{4.20}$$

Thus (4.12) is proved when $i=k+1$. In detail, we set $K^{k+1}_j=MK^{k+1}_j$ for $-2\leq j\leq k$ and $K^{k+1}_{k+1}=1$. ∎

4.1.2. Global well-posedness of the Boltzmann Equation (1.8). In this subsection, we derive some *a priori* estimates for solutions to the Cauchy problem (1.8). To this end, we employ Proposition 4.1 by taking $g=\Gamma^\epsilon(f,f)$. The *a priori* result can be concluded as follows:

THEOREM 4.1. *Let $\gamma<0,N\geq 4,l\geq 2-\gamma N$. There exists a sufficiently small constant $\delta>0$ which is independent of ϵ, such that if a solution f^ϵ to the Cauchy problem (1.8) satisfies $\sup_{0\leq t\leq T}\mathcal{E}^4(f^\epsilon(t))\leq\delta$, then for any $t\in[0,T]$, it verifies*

$$\mathcal{E}^{N,l}(f^\epsilon(t))+\int_0^t\mathcal{D}^{N,l}_\epsilon(f^\epsilon(s))ds\leq P_{N,l}\left(\mathcal{E}^{N,l}(f_0)\right), \tag{4.21}$$

where $P_{N,l}$ is a function with $P_{N,l}(0)=0$. Here $P_{4,l}(x)=C_lx$ for some constant C_l and $P_{k+1,l}(x)=C_{k+1,l}x\exp(C_{k+1,l}P_{k,l}(x))$ for some constants $C_{k+1,l}$ when $k\geq 4$.

We first prove a lemma to deal with some inner products regarding to the nonlinear term Γ^ϵ.

LEMMA 4.2. *Let $N\geq 4,l\geq 2-\gamma N$. Set*

$$\mathcal{A}_{N,l}(g,h,f):=\sum_{|\alpha|+|\beta|\leq N}|(W_{l+|\beta|\gamma}\partial^\alpha_\beta\Gamma^\epsilon(g,h),W_{l+|\beta|\gamma}\partial^\alpha_\beta f)|,$$

then

$$\mathcal{A}_{N,l}(g,h,f)\lesssim||g||_{H^4_{x,v}}\sqrt{\mathcal{D}^{N,l}_\epsilon(h)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)}+1_{N\geq 5}||g||_{H^N_{x,v}}\sqrt{\mathcal{D}^{N-1,l}_\epsilon(h)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)},$$

where $\|f\|^2_{H^N_{x,v}} := \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta f\|^2_{L^2}$. *Here the* $\lesssim$ *could result in a constant* $C_{N,l}$ *on the right-hand side.*

Proof. A typical term in $\mathcal{A}_{N,l}(g,h,f)$ is $|(W_{l+|\beta|\gamma}\partial^\alpha_\beta\Gamma^\epsilon(f,f), W_{l+|\beta|\gamma}\partial^\alpha_\beta f)|$ for some fixed α,β such that $|\alpha|+|\beta|\le N$. By the expansion (3.2) and the fact that $\Gamma^\epsilon(g,h;\beta)$ satisfies the upper bound in Theorem 2.2 and commutator estimate in Lemma 2.1, it suffices to consider the following term for $\alpha_1+\alpha_2=\alpha$ and $\beta_1+\beta_2\le\beta$,

$$\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta) := |(W_{l+|\beta|\gamma}\Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1}g, \partial^{\alpha_2}_{\beta_2}h), W_{l+|\beta|\gamma}\partial^\alpha_\beta f)|.$$

To utilize upper bound and commutator estimates, we make the following decomposition

$$\begin{aligned}\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta) &\le |(\Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1}g, W_{l+|\beta|\gamma}\partial^{\alpha_2}_{\beta_2}h), W_{l+|\beta|\gamma}\partial^\alpha_\beta f)| \\ &\quad + |([\Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1}g,\cdot), W_{l+|\beta|\gamma}]\partial^{\alpha_2}_{\beta_2}h, W_{l+|\beta|\gamma}\partial^\alpha_\beta f)| \\ &:= \mathcal{I}_u(\alpha_1,\beta_1,\alpha,\beta) + \mathcal{I}_c(\alpha_1,\beta_1,\alpha,\beta).\end{aligned}$$

We use upper bound to deal with $\mathcal{I}_u$ and commutator estimate to deal with $\mathcal{I}_c$. However, one can easily see that the commutator estimate in Lemma 2.1 can be controlled by the upper bound in Theorem 2.2, thus it is sufficient to consider $\mathcal{I}_u$ only.

For any $b_1,b_2\ge 0$ with $b_1+b_2>3/2$, according to Theorem 2.2, we have

$$|\langle\Gamma^\epsilon(g,h),f\rangle| \lesssim |\mu^{1/8}g|_{H^{b_1}}|\mu^{1/8}h|_{H^{b_2}_{\epsilon,\gamma/2}}|f|_{L^2_{\epsilon,\gamma/2}} + |g|_{H^0}|h|_{H^0_{\epsilon,\gamma/2}}|f|_{L^2_{\epsilon,\gamma/2}}.$$

If we denote the Fourier transform of f with respect to x variable by $\hat{f}$, then we have

$$(\Gamma^\epsilon(g,h),f) = \sum_{k,m\in\mathbb{Z}^3}\langle\Gamma^\epsilon(\hat{g}(k),\hat{h}(m-k)),\hat{f}(m)\rangle,$$

from which together with Theorem 2.2, we get

$$\begin{aligned}&|(\Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha_2}_{\beta_2}h),f)| \\ &\lesssim \sum_{k,m\in\mathbb{Z}^3}|k|^{|\alpha_1|}|m-k|^{|\alpha_2|}|\widehat{\mu^{1/8}\partial_{\beta_1}g}(k)|_{H^{b_1}}|\widehat{\mu^{1/8}\partial_{\beta_2}h}(m-k)|_{H^{b_2}_{\epsilon,\gamma/2}}|\hat{f}(m)|_{L^2_{\epsilon,\gamma/2}} \\ &\quad + \sum_{k,m\in\mathbb{Z}^3}|k|^{|\alpha_1|}|m-k|^{|\alpha_2|}|\widehat{\mu^{1/8}\partial_{\beta_1}g}(k)|_{H^0}|\widehat{\partial_{\beta_2}h}(m-k)|_{H^0_{\epsilon,\gamma/2}}|\hat{f}(m)|_{L^2_{\epsilon,\gamma/2}}.\end{aligned}$$

From this, we derive that for $a_1,a_2\ge 0$ with $a_1+a_2>\frac{3}{2}$ and $b_1,b_2\ge 0$ with $b_1+b_2>\frac{3}{2}$,

$$\begin{aligned}|(\Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1}g,\partial^{\alpha_2}_{\beta_2}h),f)| &\lesssim \|\mu^{1/8}g\|_{H^{|\alpha_1|+a_1}_x H^{|\beta_1|+b_1}}\|\mu^{1/8}h\|_{H^{|\alpha_2|+a_2}_x H^{|\beta_2|+b_2}_{\epsilon,\gamma/2}}\|f\|_{L^2_{\epsilon,\gamma/2}} \\ &\quad + \|g\|_{H^{|\alpha_1|+a_1}_x H^{|\beta_1|}}\|h\|_{H^{|\alpha_2|+a_2}_x H^{|\beta_2|}_{\epsilon,\gamma/2}}\|f\|_{L^2_{\epsilon,\gamma/2}}. \end{aligned} \tag{4.22}$$

Recalling $\mathcal{I}_u(\alpha_1,\beta_1,\alpha,\beta) = |(\Gamma^\epsilon(\partial^{\alpha_1}_{\beta_1}g, W_{l+|\beta|\gamma}\partial^{\alpha_2}_{\beta_2}h), W_{l+|\beta|\gamma}\partial^\alpha_\beta f)|$, then by (4.22) we get

$$\begin{aligned}\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta) &\lesssim \|\mu^{1/8}g\|_{H^{|\alpha_1|+a_1}_x H^{|\beta_1|+b_1}}\|\mu^{1/16}h\|_{H^{|\alpha_2|+a_2}_x H^{|\beta_2|+b_2}_{\epsilon,\gamma/2}}\|f\|_{H^{|\alpha|}_x H^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}} \\ &\quad + \|g\|_{H^{|\alpha_1|+a_1}_x H^{|\beta_1|}}\|h\|_{H^{|\alpha_2|+a_2}_x H^{|\beta_2|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}\|f\|_{H^{|\alpha|}_x H^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}.\end{aligned}$$

For simplicity, we always choose $a_1,a_2,b_1,b_2\in\{0,1,2\}$ with $a_1+a_2=2, b_1+b_2=2$. Fix $N\ge 4, |\alpha|+|\beta|\le N$, we consider all the combinations of $\alpha_1,\alpha_2,\beta_1,\beta_2$ such that $\alpha_1+\alpha_2=\alpha, \beta_1+\beta_2\le\beta$ as follows.

If $|\alpha_1|+|\beta_1|\leq 2$, we choose $a_1=2-|\alpha_1|, a_2=|\alpha_1|, b_1=2-|\beta_1|, b_2=|\beta_1|$, which gives $|\alpha_1|+a_1=2, |\beta_1|+b_1=2, |\alpha_2|+a_2\leq|\alpha|, |\beta_2|+b_2\leq|\beta|$ and

$$\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta)\lesssim\|g\|_{H^2_xH^2}\|h\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$\lesssim||g||_{H^4_{x,v}}\sqrt{\mathcal{D}^{N,l}_\epsilon(h)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)}.$$

If $|\alpha_1|+|\beta_1|=3$, which implies $|\alpha_2|+|\beta_2|\leq N-3$, we choose $a_1=a_2=1, b_1=0, b_2=2$, which gives $|\alpha_1|+a_1+|\beta_1|=4, |\alpha_2|+a_2+|\beta_2|+b_2\leq N$ and

$$\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta)\lesssim\|\mu^{1/8}g\|_{H^{|\alpha_1|+1}_xH^{|\beta_1|}}\|\mu^{1/16}h\|_{H^{|\alpha_2|+1}_xH^{|\beta_2|+2}_{\epsilon,\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$+\|g\|_{H^{|\alpha_1|+1}_xH^{|\beta_1|}}\|h\|_{H^{|\alpha_2|+1}_xH^{|\beta_2|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$\lesssim||g||_{H^4_{x,v}}\sqrt{\mathcal{D}^{N,l}_\epsilon(h)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)}.$$

If $|\alpha_1|+|\beta_1|=4$, which implies $|\alpha_2|+|\beta_2|\leq N-4$, we choose $a_1=b_1=0, a_2=b_2=2$, which gives $|\alpha_2|+2+|\beta_2|+2\leq N$ and

$$\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta)\lesssim\|\mu^{1/8}g\|_{H^{|\alpha_1|}_xH^{|\beta_1|}}\|\mu^{1/16}h\|_{H^{|\alpha_2|+2}_xH^{|\beta_2|+2}_{\epsilon,\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$+\|g\|_{H^{|\alpha_1|}_xH^{|\beta_1|}}\|h\|_{H^{|\alpha_2|+2}_xH^{|\beta_2|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$\lesssim||g||_{H^4_{x,v}}\sqrt{\mathcal{D}^{N,l}_\epsilon(h)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)}.$$

If $|\alpha_1|+|\beta_1|\geq 5$, which occurs only when $N\geq 5$ and implies $|\beta_2|+|\alpha_2|\leq N-5$, we choose $a_1=b_1=0, a_2=b_2=2$, which gives $|\alpha_1|+|\beta_1|\leq N, |\alpha_2|+2+|\beta_2|+2\leq N-1$ and

$$\mathcal{I}(\alpha_1,\beta_1,\alpha,\beta)\lesssim\|\mu^{1/8}g\|_{H^{|\alpha_1|}_xH^{|\beta_1|}}\|\mu^{1/16}h\|_{H^{|\alpha_2|+2}_xH^{|\beta_2|+2}_{\epsilon,\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$+\|g\|_{H^{|\alpha_1|}_xH^{|\beta_1|}}\|h\|_{H^{|\alpha_2|+2}_xH^{|\beta_2|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}\|f\|_{H^{|\alpha|}_xH^{|\beta|}_{\epsilon,l+|\beta|\gamma+\gamma/2}}$$
$$\lesssim||g||_{H^N_{x,v}}\sqrt{\mathcal{D}^{N-1,l}_\epsilon(h)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)}.$$

The lemma then follows by patching all the above estimates. □

Now we are ready to prove Theorem 4.1.

Proof. **(Proof of Theorem 4.1.)** To apply Proposition 4.1, we need to analyze $A_1+A_2+A_3$, where

$$A_1=\sum_{|\alpha|\leq N}|(\partial^\alpha\Gamma^\epsilon(f,f),\partial^\alpha f)|,$$
$$A_2=\sum_{|\alpha|+|\beta|\leq N}|(W_{l+|\beta|\gamma}\partial^\alpha_\beta\Gamma^\epsilon(f,f),W_{l+|\beta|\gamma}\partial^\alpha_\beta f)|,$$
$$A_3=\sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha\Gamma^\epsilon(f,f),e_j\rangle|^2dx.$$

It is not necessary to estimate A_1, since the upper bound of A_2 controls A_1 naturally. By Lemma 4.2,

$$A_2\lesssim||f||_{H^4_{x,v}}\mathcal{D}^{N,l}_\epsilon(f)+1_{N\geq 5}||f||_{H^N_{x,v}}\sqrt{\mathcal{D}^{N-1,l}_\epsilon(f)}\sqrt{\mathcal{D}^{N,l}_\epsilon(f)}$$

$$\lesssim ||f||_{H^4_{x,v}}\mathcal{D}^{N,l}_\epsilon(f)+1_{N\geq 5}(\eta\mathcal{D}^{N,l}_\epsilon(f)+\eta^{-1}||f||^2_{H^N_{x,v}}\mathcal{D}^{N-1,l}_\epsilon(f)).$$

In view of the proof of Lemma 4.2, it is much easier to check

$$\sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha\Gamma^\epsilon(f,f),e_j\rangle|^2dx\lesssim ||f||^2_{H^4_{x,v}}\mathcal{D}^{N,l}_\epsilon(f)+1_{N\geq 5}||f||^2_{H^N_{x,v}}\mathcal{D}^{N-1,l}_\epsilon(f).$$

Thus by Proposition 4.1, we get

$$\begin{aligned}\frac{d}{dt}\Xi^{N,l}(f)+\lambda\mathcal{D}^{N,l}_\epsilon(f)\lesssim&(||f||_{H^4_{x,v}}+||f||^2_{H^4_{x,v}})\mathcal{D}^{N,l}_\epsilon(f)\\&+1_{N\geq 5}(\eta\mathcal{D}^{N,l}_\epsilon(f)+\eta^{-1}||f||^2_{H^N_{x,v}}\mathcal{D}^{N-1,l}_\epsilon(f)).\end{aligned}\tag{4.23}$$

We take δ small enough such that $\delta+\sqrt{\delta}\ll\lambda/2$. Then under the assumption $\sup_{t\geq 0}\|f(t)\|^2_{H^4_{x,v}}\leq\sup_{t\geq 0}\mathcal{E}^4(f(t))\leq\delta$, we have

$$\frac{d}{dt}\Xi^{N,l}(f)+\frac{\lambda}{2}\mathcal{D}^{N,l}_\epsilon(f)\lesssim 1_{N\geq 5}(\eta\mathcal{D}^{N,l}_\epsilon(f)+\eta^{-1}\mathcal{E}^N(f)\mathcal{D}^{N-1,l}_\epsilon(f)).\tag{4.24}$$

When $N=4$, (4.24) reduces to

$$\frac{d}{dt}\Xi^{4,l}(f)+\frac{\lambda}{2}\mathcal{D}^{4,l}_\epsilon(f)\leq 0.\tag{4.25}$$

Recalling $\mathcal{E}^{4,l}(f)\leq\Xi^{4,l}(f)\leq C_l\mathcal{E}^{4,l}(f)$, we get (4.21) for the case $N=4$ directly from (4.25). Suppose for some $k\geq 4$, (4.21) is valid for $N=k$, that is,

$$\mathcal{E}^{k,l}(f^\epsilon(t))+\int_0^t\mathcal{D}^{k,l}_\epsilon(f^\epsilon(s))ds\leq P_{k,l}\left(\mathcal{E}^{k,l}(f_0)\right).\tag{4.26}$$

Then for $N=k+1\geq 5$, by (4.24), we get

$$\frac{d}{dt}\Xi^{k+1,l}(f)+\frac{\lambda}{2}\mathcal{D}^{k+1,l}_\epsilon(f)\lesssim\eta\mathcal{D}^{N,l}_\epsilon(f)+\eta^{-1}\mathcal{E}^{k+1,l}(f)\mathcal{D}^{k,l}_\epsilon(f).$$

Choosing $\eta\ll\frac{\lambda}{4}$, we have

$$\frac{d}{dt}\Xi^{k+1,l}(f)+\frac{\lambda}{4}\mathcal{D}^{k+1,l}_\epsilon(f)\leq C_{k+1,l}\mathcal{E}^{k+1,l}(f)\mathcal{D}^{k,l}_\epsilon(f).$$

Observing $\int_0^t\mathcal{D}^{k,l}_\epsilon(f^\epsilon(s))ds\leq P_{k,l}\left(\mathcal{E}^{k,l}(f_0)\right)$ given by (4.26), together with Grönwall's inequality, we arrive at

$$\begin{aligned}\Xi^{k+1,l}(f(t))+\frac{\lambda}{4}\int_0^t\mathcal{D}^{k+1,l}_\epsilon(f(t))dt&\leq\Xi^{k+1,l}(f_0)\exp(C_{k+1,l}\int_0^t\mathcal{D}^{k,l}_\epsilon(f^\epsilon(s))ds)\\&\leq\Xi^{k+1,l}(f_0)\exp(C_{k+1,l}P_{k,l}\left(\mathcal{E}^{k,l}(f_0)\right)).\end{aligned}$$

Recalling $\mathcal{E}^{k+1,l}(f)\leq\Xi^{k+1,l}(f)\leq C_{k+1,l}\mathcal{E}^{k+1,l}(f),\mathcal{E}^{k,l}(f_0)\leq\mathcal{E}^{k+1,l}(f_0)$, we have

$$\mathcal{E}^{k+1,l}(f^\epsilon(t))+\int_0^t\mathcal{D}^{k+1,l}_\epsilon(f^\epsilon(s))ds\leq C_{k+1,l}\mathcal{E}^{k+1,l}(f_0)\exp(C_{k+1,l}P_{k,l}\left(\mathcal{E}^{k+1,l}(f_0)\right)).$$

We define $P_{k+1,l}(x)=C_{k+1,l}x\exp(C_{k+1,l}P_{k,l}(x))$ to end the proof. ∎

4.2. Global dynamics. We will give the proof to the third part **(global dynamics)** of Theorem 1.2.

Proof. **(Proof of Theorem 1.2 (the third part: global dynamics).)** Let $Y_1=\mathcal{E}^{N,l}(f^l), Y_2=\mathcal{E}^{N,l}(f^h)$, then $\Xi^{N,l}(f)\sim\mathcal{E}^{N,l}(f)\sim Y_1+Y_2$, and

$$\begin{aligned}
\mathcal{D}_\epsilon^{N,l}(f) &\gtrsim \sum_{j=0}^{N}||f||^2_{H_x^{N-j}\dot{H}^j_{\epsilon,l+j\gamma+\gamma/2}} \geq \sum_{|\alpha|+|\beta|\leq N}||W^\epsilon W_{\gamma/2}W_{l+|\beta|\gamma}\partial^\alpha_\beta f||^2_{L^2} \\
&\sim \sum_{|\alpha|+|\beta|\leq N}||W^\epsilon W_{\gamma/2}W_{l+|\beta|\gamma}\partial^\alpha_\beta f^l||^2_{L^2} + \sum_{|\alpha|+|\beta|\leq N}||W^\epsilon W_{\gamma/2}W_{l+|\beta|\gamma}\partial^\alpha_\beta f^h||^2_{L^2} \\
&\sim \sum_{|\alpha|+|\beta|\leq N}||W_{s+\gamma/2}W_{l+|\beta|\gamma}\partial^\alpha_\beta f^l||^2_{L^2} + \sum_{|\alpha|+|\beta|\leq N}\epsilon^{-2s}||W_{\gamma/2}W_{l+|\beta|\gamma}\partial^\alpha_\beta f^h||^2_{L^2} \\
&\geq C(p,N)(\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0))^{-1/p}Y_1^{1+1/p} \\
&\quad +\epsilon^{-2s}C(q,N)(\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0))^{-1/q}Y_2^{1+1/q},
\end{aligned}$$

where the last inequality is obtained in the same manner as deriving (3.28). Therefore

$$\frac{d}{dt}\Xi^{N,l}(f)+c_1Y_1^{1+1/p}+c_2Y_2^{1+1/q}\leq 0, \tag{4.27}$$

where $c_1=C(p,N)(\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0))^{-1/p}$ and $c_2=\epsilon^{-2s}C(q,N)(\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0))^{-1/q}$. Here, we have

$$\begin{aligned}
(c_1/c_2)^{\frac{pq}{p-q}} &= (C(p,N)/C(q,N)\epsilon^{2s})^{\frac{pq}{p-q}}\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0) \\
&:= (\epsilon^{2s})^{\frac{pq}{p-q}}C(p,q,N)\mathcal{E}^{N,l-p(\gamma/2+s)}(f_0).
\end{aligned}$$

Applying Proposition 3.2, we get (1.33) and (1.34) by the equivalence $\mathcal{E}^{N,l}(f)\leq \Xi^{N,l}(f)\leq c\mathcal{E}^{N,l}(f)$ for some constant c depending only on N,l. ∎

4.3. Asymptotic formula. We will give the proof to the fourth part **(global asymptotic formula)** of Theorem 1.2. Let f and f^ϵ be the solutions to (1.7) and (1.8) respectively with the initial data f_0. Set $F^\epsilon_R:=\epsilon^{2s-2}(f^\epsilon-f)$, then it solves

$$\partial_t F^\epsilon_R+v\cdot\nabla_x F^\epsilon_R+\mathcal{L}F^\epsilon_R=\epsilon^{2s-2}[(\mathcal{L}-\mathcal{L}^\epsilon)f^\epsilon+(\Gamma^\epsilon-\Gamma)(f^\epsilon,f)]+\Gamma^\epsilon(f^\epsilon,F^\epsilon_R)+\Gamma(F^\epsilon_R,f).$$

We recall an estimate on the operator $\Gamma-\Gamma^\epsilon$, which is Lemma 4.2 in [14].

LEMMA 4.3. *If $\gamma>-3$, there holds*

$$|\langle(\Gamma-\Gamma^\epsilon)(g,h),f\rangle|\lesssim\epsilon^{2-2s}|g|_{L^2_{\gamma/2}}|h|_{H^2_{\gamma/2+2}}|f|_{L^2_{\gamma/2}}.$$

We set to establish the global asymptotic formula (1.35).

Proof. **(Proof of Theorem 1.2 (the fourth part: global asymptotic formula).)**

For simplicity, we set $g=g^1+g^2+g^3$, where $g^1:=\epsilon^{2s-2}[(\mathcal{L}-\mathcal{L}^\epsilon)f^\epsilon+(\Gamma^\epsilon-\Gamma)(f^\epsilon,f)], g^2:=\Gamma^\epsilon(f^\epsilon,F^\epsilon_R), g^3:=\Gamma(F^\epsilon_R,f)$. By applying Proposition 4.1, we have

$$\frac{d}{dt}\Xi^{N,l}(F^\epsilon_R)+\lambda\mathcal{D}_0^{N,l}(F^\epsilon_R)\lesssim\sum_{|\alpha|\leq N}|(\partial^\alpha g,\partial^\alpha F^\epsilon_R)|$$

$$+\sum_{|\alpha|+|\beta|\le N}|(W_{l+|\beta|\gamma}\partial^\alpha_\beta g,W_{l+|\beta|\gamma}\partial^\alpha_\beta F^\epsilon_R)|$$
$$+\sum_{|\alpha|\le N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g,e_j\rangle|^2dx:=A_1+A_2+A_3. \quad (4.28)$$

We remark that the non-cutoff linearized Boltzmann operator $\mathcal{L}$ produces $\mathcal{D}_0^{N,l}$. As before, we ignore A_1 since it can be controlled by the upper bound of A_2. Noting $A_2\le A_{2,1}+A_{2,2}+A_{2,3}$, where

$$A_{2,i}:=\sum_{|\alpha|+|\beta|\le N}|(W_{l+|\beta|\gamma}\partial^\alpha_\beta g^i,W_{l+|\beta|\gamma}\partial^\alpha_\beta F^\epsilon_R)|.$$

Let $q=l+|\beta|\gamma$. By the expansion (3.2) and Lemma 4.3, we have

$$|\langle W_q\partial^\alpha_\beta g^1,W_q\partial^\alpha_\beta F^\epsilon_R\rangle|\lesssim\sum_{\beta_1\le\beta}|\partial^\alpha_{\beta_1}f^\epsilon|_{H^2_{q+2+\gamma/2}}|\partial^\alpha_\beta F^\epsilon_R|_{L^2_{q+\gamma/2}}$$
$$+\sum_{\alpha_1+\alpha_2=\alpha,\beta_1+\beta_2\le\beta}|\partial^{\alpha_1}_{\beta_1}f^\epsilon|_{L^2_{\gamma/2}}|\partial^{\alpha_2}_{\beta_2}f|_{H^2_{q+2+\gamma/2}}|\partial^\alpha_\beta F^\epsilon_R|_{L^2_{q+\gamma/2}},$$

which yields

$$\begin{aligned}A_{2,1}&=\sum_{|\alpha|+|\beta|\le N}|(W_{l+|\beta|\gamma}\partial^\alpha_\beta g^1,W_{l+|\beta|\gamma}\partial^\alpha_\beta F^\epsilon_R)|\\&\lesssim\sqrt{\mathcal{D}_\epsilon^{N+2,l+2-2\gamma}(f^\epsilon)}\sqrt{\mathcal{D}_0^{N,l}(F^\epsilon_R)}\\&\quad+\sqrt{\mathcal{E}^{N+2,l+2-2\gamma}(f^\epsilon)}\sqrt{\mathcal{D}_0^{N+2,l+2-2\gamma}(f)}\sqrt{\mathcal{D}_0^{N,l}(F^\epsilon_R)}\\&\lesssim\eta\mathcal{D}_0^{N,l}(F^\epsilon_R)+C_\eta(\mathcal{D}_\epsilon^{N+2,l+2-2\gamma}(f^\epsilon)+\mathcal{E}^{N+2,l+2-2\gamma}(f^\epsilon)\mathcal{D}_0^{N+2,l+2-2\gamma}(f)).\end{aligned}$$

By Lemma 4.2, we get

$$\begin{aligned}A_{2,2}+A_{2,3}&\lesssim\|f^\epsilon\|_{H^4_{x,v}}\mathcal{D}_\epsilon^{N,l}(F^\epsilon_R)+1_{N\ge5}\|f^\epsilon\|_{H^N_{x,v}}\sqrt{\mathcal{D}_\epsilon^{N-1,l}(F^\epsilon_R)}\sqrt{\mathcal{D}_\epsilon^{N,l}(F^\epsilon_R)}\\&\quad+\sqrt{\mathcal{E}^{N,l}(F^\epsilon_R)}\sqrt{\mathcal{D}_0^{N,l}(f)}\sqrt{\mathcal{D}_0^{N,l}(F^\epsilon_R)}\\&\lesssim(\eta+\sqrt{\mathcal{E}^{4,14}(f^\epsilon)})\mathcal{D}_0^{N,l}(F^\epsilon_R)+1_{N\ge5}\eta^{-1}\mathcal{E}^{N,l}(f^\epsilon)\mathcal{D}_\epsilon^{N-1,l}(F^\epsilon_R)\\&\quad+\eta^{-1}\mathcal{D}_0^{N,l}(f)\mathcal{E}^{N,l}(F^\epsilon_R).\end{aligned}$$

Now we set to analyze A_3. Observe $A_3\lesssim A_{3,1}+A_{3,2}+A_{3,3}$, where

$$A_{3,i}:=\sum_{|\alpha|\le N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g^i,e_j\rangle|^2dx.$$

By Lemma 4.3, we have

$$|\langle\partial^\alpha g^1,e_j\rangle|\lesssim|\partial^\alpha f^\epsilon|_{H^2_{2+\gamma/2}}+\sum_{\alpha_1+\alpha_2=\alpha}|\partial^{\alpha_1}f^\epsilon|_{L^2_{\gamma/2}}|\partial^{\alpha_2}f|_{H^2_{2+\gamma/2}},$$

which gives

$$A_{3,1}\lesssim\|f^\epsilon\|^2_{H^N_xH^2_{2+\gamma/2}}+\|f^\epsilon\|^2_{H^N_xL^2_{\gamma/2}}\|f\|^2_{H^N_xH^2_{2+\gamma/2}}$$

$$\lesssim \mathcal{D}_\epsilon^{N+2,l+2-2\gamma}(f^\epsilon)+\mathcal{E}^{N+2,l+2-2\gamma}(f^\epsilon)\mathcal{D}_0^{N+2,l+2-2\gamma}(f).$$

Thanks to $|\langle\Gamma^\epsilon(g,h),e_j\rangle|\lesssim|g|_{L^2_{\gamma/2}}|h|_{L^2_{\epsilon,\gamma/2}}$ and $|\langle\Gamma(g,h),e_j\rangle|\lesssim|g|_{L^2_{\gamma/2}}|h|_{L^2_{0,\gamma/2}}$, by the fact $|\alpha|\leq N-1$ in the sum of A_3, we get

$$\begin{aligned}
A_{3,2}+A_{3,3}\lesssim&\|f^\epsilon\|^2_{H^4_xL^2_{\gamma/2}}\|F^\epsilon_R\|^2_{H^N_xL^2_{\epsilon,\gamma/2}}+1_{N\geq5}\|f^\epsilon\|^2_{H^N_xL^2_{\gamma/2}}\|F^\epsilon_R\|^2_{H^{N-1}_xL^2_{\epsilon,\gamma/2}}\\
&+\|f\|^2_{H^N_xL^2_{0,\gamma/2}}\|F^\epsilon_R\|^2_{H^N_xL^2_{\gamma/2}}\\
&\lesssim\mathcal{E}^{4,14}(f^\epsilon)\mathcal{D}_0^{N,l}(F^\epsilon_R)+1_{N\geq5}\mathcal{E}^{N,l}(f^\epsilon)\mathcal{D}_\epsilon^{N-1,l}(F^\epsilon_R)+\mathcal{D}_0^{N,l}(f)\mathcal{E}^{N,l}(F^\epsilon_R).
\end{aligned}$$

Patching together all the above estimates, and plugging them into (4.28), we have

$$\begin{aligned}
&\frac{d}{dt}\Xi^{N,l}(F^\epsilon_R)+\lambda\mathcal{D}_0^{N,l}(F^\epsilon_R)\\
\lesssim&(\eta+\sqrt{\mathcal{E}^{4,14}(f^\epsilon)}+\mathcal{E}^{4,14}(f^\epsilon))\mathcal{D}_0^{N,l}(F^\epsilon_R)\\
&+1_{N\geq5}C_\eta\mathcal{E}^{N,l}(f^\epsilon)\mathcal{D}_0^{N-1,l}(F^\epsilon_R)+C_\eta\mathcal{D}_0^{N,l}(f)\mathcal{E}^{N,l}(F^\epsilon_R)\\
&+C_\eta(\mathcal{D}_\epsilon^{N+2,l+2-2\gamma}(f^\epsilon)+\mathcal{E}^{N+2,l+2-2\gamma}(f^\epsilon)\mathcal{D}_0^{N+2,l+2-2\gamma}(f)).
\end{aligned}\tag{4.29}$$

Choosing η small enough, thanks to the smallness of $\mathcal{E}^{4,14}(f_0)$, we get

$$\begin{aligned}
&\frac{d}{dt}\Xi^{N,l}(F^\epsilon_R)+\frac{\lambda}{2}\mathcal{D}_0^{N,l}(F^\epsilon_R)\\
\lesssim&1_{N\geq5}\mathcal{E}^{N,l}(f^\epsilon)\mathcal{D}_0^{N-1,l}(F^\epsilon_R)+\mathcal{D}_0^{N,l}(f)\mathcal{E}^{N,l}(F^\epsilon_R)\\
&+\mathcal{D}_\epsilon^{N+2,l+2-2\gamma}(f^\epsilon)+\mathcal{E}^{N+2,l+2-2\gamma}(f^\epsilon)\mathcal{D}_0^{N+2,l+2-2\gamma}(f).
\end{aligned}\tag{4.30}$$

Since $\mathcal{E}^{N+2,l+2-2\gamma}(f_0)<\infty$, we have

$$\begin{aligned}
&\int_0^\infty[\mathcal{D}_0^{N,l}(f(t))+\mathcal{D}_\epsilon^{N+2,l+2-2\gamma}(f^\epsilon(t))+\mathcal{E}^{N+2,l+2-2\gamma}(f^\epsilon(t))\mathcal{D}_0^{N+2,l+2-2\gamma}(f(t))]dt\\
\lesssim&C(\mathcal{E}^{N+2,l+2-2\gamma}(f_0)).
\end{aligned}$$

By Grönwall's inequality, when $N=4$, we arrive at

$$\sup_{t\geq0}\Xi^{N,l}(F^\epsilon_R(t))+\frac{\lambda}{2}\int_0^\infty\mathcal{D}_\epsilon^{N,l}(F^\epsilon_R(t))dt\lesssim C(\mathcal{E}^{N+2,l+2-2\gamma}(f_0)).\tag{4.31}$$

When $N\geq5$, we can prove (4.31) through mathematical induction by observing that $\mathcal{D}_0^{N-1,l}(F^\epsilon_R)$ is integrable over $(0,\infty)$ in a previous step. This is similar to the arguments in the proof of Theorem 4.1, so we omit the details. Since $\Xi^{N,l}(F^\epsilon_R)\sim\mathcal{E}^{N,l}(F^\epsilon_R)$ and recalling $F^\epsilon_R=\epsilon^{2s-2}(f^\epsilon-f)$, we get (1.35). ◻

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Appendix. The following is Proposition 5.1 of [14].

PROPOSITION A.1. *Suppose* $A^\epsilon(\xi):=\int_{\mathbb{S}^2}b^\epsilon(\frac{\xi}{|\xi|}\cdot\sigma)\min\{|\xi|^2\sin^2(\theta/2),1\}d\sigma$. *Then we have* $A^\epsilon(\xi)\sim|\xi|^21_{|\xi|\leq2}+1_{|\xi|\geq2}(W^\epsilon(\xi))^2$.

The following is Proposition 4.3 of [14].

PROPOSITION A.2. *Let $l_1 \le l_2$. Suppose f is a smooth function. For any $\eta > 0$, we have*

$$|f|^2_{H^m_l} \lesssim (\eta + \epsilon^{2s})|W^\epsilon(D)f|^2_{H^m_l} + C(\eta)|f|^2_{L^2_l}, \quad |f|_{L^2_{\epsilon,l_1}} \lesssim |f|_{L^2_{\epsilon,l_2}}.$$

The following example is to show that the decay structure (3.16) is optimal for (3.15).

EXAMPLE A.1. *Let $\epsilon > 0$ be small enough. Let $c_1 = p = Y_0 = 1, q = 1/3, c_2 = \epsilon^{-2s}$, assume additionally $Y_1 + Y_2 = Y$, then we consider the following case of (3.15):*

$$\begin{cases} \dfrac{d}{dt}Y + Y_1^2 + \epsilon^{-2s}Y_2^4 = 0; \\ Y|_{t=0} = 1. \end{cases} \tag{A.1}$$

Assume further $Y_1 = \epsilon^{-s}Y_2^2$. Then there exists a critical time $t_* \sim \epsilon^{-s}$ such that $Y(t_*) = \epsilon^s/8$ and

$$\begin{aligned} &\frac{1}{(1+C_\beta t)^p}1_{t<t_*} + \frac{Y(t_*)}{(1+C_1(t-t_*))^q}1_{t\ge t_*} \\ &\le Y(t) \le \frac{1}{(1+C_\alpha t)^p}1_{t<t_*} + \frac{Y(t_*)}{(1+C_2(t-t_*))^q}1_{t\ge t_*}, \end{aligned} \tag{A.2}$$

where C_α, C_β are some universal constants and $C_1, C_2 \sim \epsilon^s$.

Proof. Since $Y_1 + Y_2 = Y$ and $Y_1 = \epsilon^{-s}Y_2^2$, we get

$$Y_2 = \frac{-1+\sqrt{1+4\epsilon^{-s}Y}}{2\epsilon^{-s}},$$

which gives

$$Y_1^2 + \epsilon^{-2s}Y_2^4 = 2\epsilon^{-2s}Y_2^4 = \frac{(-1+\sqrt{1+4\epsilon^{-s}Y})^4}{8\epsilon^{-2s}}.$$

Set $X := \epsilon^{-s}Y$, then we have the following ODE

$$\begin{cases} \dfrac{d}{dt}X + \dfrac{(-1+\sqrt{1+4X})^4}{8\epsilon^{-s}} = 0; \\ X|_{t=0} = \epsilon^{-s}. \end{cases} \tag{A.3}$$

Set $f(x) := (-1+\sqrt{1+4x})^4 = (1+4x)^2 - 4(1+4x)^{3/2} + 6(1+4x) - 4(1+4x)^{1/2} + 1$, then one has

$$\begin{aligned} f'(x) &= 8(1+4x) - 24(1+4x)^{1/2} + 24 - (1+4x)^{-1/2}, \\ f''(x) &= 32 - 48(1+4x)^{-1/2} + 16(1+4x)^{-3/2}, \\ f^{(3)}(x) &= 96(1+4x)^{-3/2} - 96(1+4x)^{-5/2}, \\ f^{(4)}(x) &= -576(1+4x)^{-5/2} - 96(1+4x)^{-7/2}, \\ f^{(5)}(x) &= 5760(1+4x)^{-7/2} - 13440(1+4x)^{-9/2}. \end{aligned}$$

By Taylor expansion, one has

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{1}{24}\int_0^x (x-t)^4 f^{(5)}(t)dt$$

$$= 16x^4 + \frac{1}{24}\int_0^x (x-t)^4 f^{(5)}(t)dt.$$

It is elementary to check $-7680 \le f^{(5)}(x) \le \frac{45840}{27\sqrt{3}}$, thus we have

$$16x^4 - 64x^5 \le f(x) \le 16x^4 + \frac{896}{81\sqrt{3}}x^5.$$

If $x \le 1/8$, then $16x^4 - 64x^5 = 16x^4(1-4x) \ge 8x^4$ and $\frac{896}{81\sqrt{3}}x^5 \le x^4$, which gives

$$8x^4 \le f(x) \le 17x^4, \quad x \le 1/8. \tag{A.4}$$

Set $\alpha = \min\{f''(1/8)/2, 4f'(1/8), 64f(1/8)\}$ and $g(x) := f(x) - \alpha x^2$. We now prove $g(x) \ge 0$ if $x \ge 1/8$. Observe that $f^{(3)}(x) \ge 0$ if $x > 0$. Then $f''(x)$ is an increasing function on $[0,\infty)$. Then when $x > 1/8$, we get $g''(x) = f''(x) - 2\alpha > f''(1/8) - 2\alpha \ge 0$, with $g''(1/8) = f''(1/8) - 2\alpha \ge 0$, we have $g'(x)$ is an increasing function on $[1/8,\infty)$. Thus $g'(x) \ge g'(1/8) = f'(1/8) - \alpha/4 \ge 0$. With the same argument, $g(x) \ge g(1/8) = f(1/8) - \alpha/64 \ge 0$. To summarize, we proved

$$f(x) \ge \alpha x^2, \qquad x \ge 1/8.$$

On the other hand, it is easy to find a $\beta > 0$ such that $f(x) \le \beta x^2$ for $x \ge 1/8$. Patching together, we get

$$\alpha x^2 \le f(x) \le \beta x^2, \qquad x \ge 1/8. \tag{A.5}$$

Suppose t_* is the critical time such that $X(t_*) = 1/8$, then by (A.5), we get

$$\frac{d}{dt}X + \epsilon^s \alpha X^2/8 \le \frac{d}{dt}X + \epsilon^s f(X)/8 = 0 \le \frac{d}{dt}X + \epsilon^s \beta X^2/8, \quad t \le t_*,$$

which gives

$$\frac{\epsilon^s \alpha}{8} \le \frac{d}{dt}(\frac{1}{X}) \le \frac{\epsilon^s \beta}{8}, \quad t \le t_*.$$

From which we have

$$\frac{X(0)}{1 + C_\beta t} \le X(t) \le \frac{X(0)}{1 + C_\alpha t}, \quad t \le t_*, \tag{A.6}$$

where $C_\alpha = \frac{\epsilon^s \alpha X(0)}{8} = \alpha/8$ and $C_\beta = \frac{\epsilon^s \beta X(0)}{8} = \beta/8$. By (A.4), we get

$$\frac{d}{dt}X + \epsilon^s X^4 \le \frac{d}{dt}X + \epsilon^s f(X)/8 = 0 \le \frac{d}{dt}X + \frac{17}{8}\epsilon^s X^4, \quad t \ge t_*,$$

which gives

$$3\epsilon^s \le \frac{d}{dt}(\frac{1}{X^3}) \le \frac{51}{8}\epsilon^s, \quad t \ge t_*.$$

Integrating over $[t_*, t]$, we have

$$\frac{X(t_*)}{(1 + C_1(t - t_*))^{1/3}} \le X(t) \le \frac{X(t_*)}{(1 + C_2(t - t_*))^{1/3}}, \quad t \ge t_*. \tag{A.7}$$

where $C_1 = \frac{51}{8}\epsilon^s X^3(t_*)$ and $C_2 = 3\epsilon^s X^3(t_*)$. By (A.6), recalling $X(0) = \epsilon^{-s}, X(t_*) = 1/8$, we have

$$\frac{8\epsilon^{-s} - 1}{C_\beta} \le t_* \le \frac{8\epsilon^{-s} - 1}{C_\alpha},$$

which implies $t_* \sim \epsilon^{-s}$. Recalling $X = \epsilon^{-s} Y$, we have (A.2). □

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