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NONLINEAR STABILITY OF THE BOUNDARY LAYER AND RAREFACTION WAVE FOR THE INFLOW PROBLEM GOVERNED BY THE HEAT-CONDUCTIVE IDEAL GAS WITHOUT VISCOSITY*

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Abstract. This paper is devoted to studying the inflow problem for an ideal polytropic model with non-viscous gas in one-dimensional half space. We show the existence of the boundary layer in different areas. By employing the energy method, we also prove the unique global-in-time existence of the solution and the asymptotic stability of both the boundary layers, the 3−rarefaction wave and their superposition wave under some smallness conditions. Series of simple but tricky operations on boundary need to be carefully done by taking good advantage of construction on the system and domain properties.

Keywords. non-viscous; inflow problem; boundary layer; rarefaction wave.

AMS subject classifications. 35B35; 35B40; 35M33; 35Q35; 76N10; 76N15.

1. Introduction

In this paper, we consider the system of heat-conductive ideal gas without viscosity in one dimension under Euler coordinates:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho(e + \frac{u^2}{2}))_t + (\rho u(e + \frac{u^2}{2}) + pu)_x = \kappa\theta_{xx}, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}_+, t > 0$ and $\rho(t,x) > 0, u(t,x), \theta(t,x) > 0, e(t,x) > 0$ and $p(t,x) > 0$ are density, fluid velocity, absolute temperature, internal energy, and pressure respectively, while $\kappa > 0$ is the coefficient of the heat conduction. Here we study ideal and polytropic fluids so that p and e are given by the state equations

$$p = R\rho\theta = A\rho^\gamma \exp(\frac{\gamma-1}{R}s), \quad e = C_v\theta \quad (C_v = \frac{R}{\gamma-1}), \tag{1.2}$$

where s is the entropy, $\gamma > 1$ is the adiabatic exponent and A, R are both positive constants. The solution of (1.1) satisfies the following initial data and the far field states that

$$\begin{cases} (\rho, u, \theta)(0,x) = (\rho_0, u_0, \theta_0)(x) \to (\rho_+, u_+, \theta_+) =: z_+, \quad x \to +\infty, \\ \inf_{x \in \mathbb{R}_+} (\rho_0, \theta_0)(x) > 0, \end{cases} \tag{1.3}$$

where $\rho_+ > 0, u_+, \theta_+ > 0$ are given constants.

As far as we know, there are very few results on the well-posed problem for (1.1) due to the complexity and nonlinearity. Almost all the results are related to the analysis of the global-in-time stability of the viscous Riemann solutions. More precisely, if the

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heat effect is also neglected, the Riemann solution consists of elementary waves such as shock waves, rarefaction waves and contact discontinuities, which are dilation-invariant solutions of the Riemann problem (Euler system):

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p)_x=0, \\ \{\rho(e+\frac{u^2}{2})\}_t+\{\rho u(e+\frac{u^2}{2})+pu\}_x=0. \end{cases} \tag{1.4}$$

Let us introduce the sound speed and Mach number

$$c_s(\theta):=\sqrt{\frac{\gamma p}{\rho}}=\sqrt{\gamma R\theta}, \quad M(\rho,u,\theta):=\frac{|u|}{c_s(\theta)}. \tag{1.5}$$

Then the inviscid Euler system (1.4) has three characteristic speeds, they are

$$\lambda_1=u-c_s(\theta), \quad \lambda_2=u, \quad \lambda_3=u+c_s(\theta). \tag{1.6}$$

The system (1.4) is a typical example of the hyperbolic conservation laws. It is of great importance to study the corresponding viscous system, such as isentropic or non-isentropic case. There are many works on the large-time behavior of the solutions to the Cauchy problem of the compressible gas dynamic equations. We refer to ([2,5,6,10,12,18,28,34]) and some references therein.

Due to the appearance of the boundary layer in the initial boundary value problem of the gas dynamic equations, people pay more attention to this kind of problem, and the hottest equations studied are the Navier-Stokes equations:

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p)_x=\mu u_{xx}, \\ (\rho(e+\frac{u^2}{2}))_t+(\rho u(e+\frac{u^2}{2})+pu)_x=\kappa\theta_{xx}+(\mu u u_x)_x, \end{cases} \tag{1.7}$$

where $\mu>0$ stands for the coefficient of viscosity. For the system (1.7), we divide the phase space into following regions to study the initial and boundary value problem:

$$\begin{aligned} &\Omega^+_{sub}:=\{(\rho,u,\theta);\ 0<u<c_s(\theta)\}, \quad \Omega^-_{sub}:=\{(\rho,u,\theta);\ -c_s(\theta)<u<0\}, \\ &\Omega^+_{supper}:=\{(\rho,u,\theta);\ u>c_s(\theta)\}, \quad \Omega^-_{supper}:=\{(\rho,u,\theta);\ u<-c_s(\theta)\}, \\ &\Gamma^\pm_{trans}:=\{(\rho,u,\theta);\ |u|=c_s(\theta)\}, \quad \Gamma^0_{sub}:=\{(\rho,u,\theta);\ u=0\}. \end{aligned}$$

For the inflow problem of (1.7), Huang-Li-Shi [4] studied the asymptotic stability of boundary layer and its superposition with 3−rarefaction wave. Nakamura-Nishibata [23] proved the existence and stability of boundary layer solution of (1.7) in half space. Qin-Wang ([30,31]) proved the stability of the combination of BL-solution, rarefaction wave and viscous contact wave. For other interesting works, we refer to ([1,3,7–9,11,13,15,16,20–22,27,29,33]).

Therefore, a natural question arises that what are the large-time behaviors of the solutions for the initial boundary value problem of the non-viscous system (1.1)? Especially, how about the asymptotic stability of the boundary layer, 3−rarefaction wave or their composite wave? We will give a positive answer in this paper. To do so, we should firstly define some proper boundary conditions. Thus, changing the system (1.1) in an equivalent form as

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \dfrac{p}{\rho^2}\rho_x = -R\theta_x, \\ C_v\theta_t - \dfrac{\kappa}{\rho}\theta_{xx} = -C_v u\theta_x - R\theta u_x. \end{cases} \tag{1.8}$$

This is a hyperbolic-parabolic system, there are two eigenvalues of the hyperbolic part

$$\widetilde{\lambda}_1 = u - \widetilde{c}_s(\theta), \quad \widetilde{\lambda}_2 = u + \widetilde{c}_s(\theta), \tag{1.9}$$

where

$$\widetilde{c}_s(\theta) := \sqrt{\frac{p}{\rho}} = \sqrt{R\theta}. \tag{1.10}$$

Here we denote

$$M_+ = \frac{|u_+|}{\sqrt{\gamma R\theta_+}}, \quad \tilde{M}(\rho,u,\theta) = \frac{|u|}{\sqrt{R\theta}}, \quad \tilde{M}_+ = \frac{|u_+|}{\sqrt{R\theta_+}} \tag{1.11}$$

for clear expression later.

By [19], the boundary conditions of (1.1) depend closely on the sign of two eigenvalues $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$. Thus the global solution of (1.1) is considered in a small neighborhood $\Omega(z_+)$ of z_+, such that $\widetilde{\lambda}_i (i=1,2)$ at the boundary $x=0$ keeps the same sign as $\widetilde{\lambda}_i (i=1,2)$ at the far field $x=+\infty$, which are determined by the right state z_+. Hence, we divide the phase space into new sonic regions

$$\widetilde{\Omega}^+_{sub} := \{(\rho,u,\theta);\ 0<u<\widetilde{c}_s(\theta)\}, \quad \widetilde{\Omega}^-_{sub} := \{(\rho,u,\theta);\ -\widetilde{c}_s(\theta)<u<0\};$$

$$\widetilde{\Omega}^+_{supper} := \{(\rho,u,\theta);\ u>\widetilde{c}_s(\theta)\}, \quad \widetilde{\Omega}^-_{supper} := \{(\rho,u,\theta);\ u<-\widetilde{c}_s(\theta)\};$$

$$\widetilde{\Gamma}^+_{trans} := \{(\rho,u,\theta);\ u=\widetilde{c}_s(\theta)\}, \quad \widetilde{\Gamma}^-_{trans} := \{(\rho,u,\theta);\ u=-\widetilde{c}_s(\theta)\},$$

$$\widetilde{\Gamma}^0_{sub} := \{(\rho,u,\theta);\ u=0\}.$$

In different domain, the boundary conditions are listed as follows: (Figure 1.1 shows the division of the phase space)

Case (1): If $z_+ = (\rho_+,u_+,\theta_+) \in \widetilde{\Omega}^-_{supper}$, in the neighborhood of $U(z_+)$, $\widetilde{\lambda}_1<0$, $\widetilde{\lambda}_2<0$, the boundary condition of (1.1) should be

$$\theta(t,0) = \theta_-. \tag{1.12}$$

Case (2): If $z_+ = (\rho_+,u_+,\theta_+) \in \widetilde{\Omega}^-_{sub} \bigcup \widetilde{\Omega}^+_{sub} \bigcup \widetilde{\Gamma}^0_{sub}$, in the neighborhood of $U(z_+)$, $\widetilde{\lambda}_1<0$, $\widetilde{\lambda}_2>0$, the boundary condition of (1.1) should be

$$u(t,0) = u_-, \quad \theta(t,0) = \theta_-. \tag{1.13}$$

Case (3): If $z_+ = (\rho_+,u_+,\theta_+) \in \widetilde{\Omega}^+_{supper}$, in the neighborhood of $U(z_+)$, $\widetilde{\lambda}_1>0$, $\widetilde{\lambda}_2>0$, the boundary condition of (1.1) should be

$$\rho(t,0) = \rho_-, \quad u(t,0) = u_-, \quad \theta(t,0) = \theta_-. \tag{1.14}$$

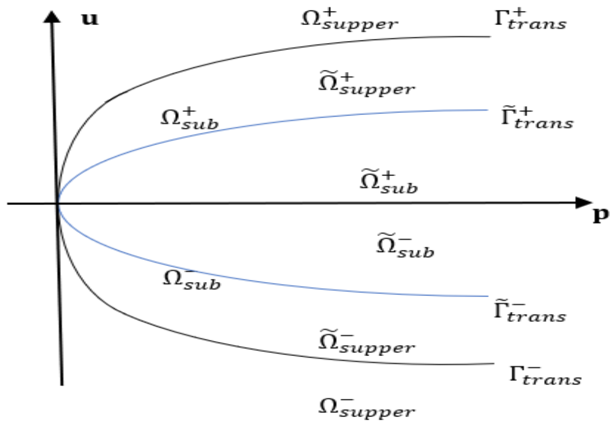

FIG. 1.1. *The division of the phase space*

Motivated by ([4, 23, 30, 31]), here we take our attention to the inflow problem of (1.1), (1.3) and (1.14). We firstly discuss the existence of boundary layer solution to system (1.1) for $u_+>0$. Precisely speaking, if $(\rho_+,u_+,\theta_+)\in\widetilde{\Omega}^+_{supper}\bigcap\Omega^+_{sub}$, the boundary layer solution is non-degenerate; if $(\rho_+,u_+,\theta_+)\in\Gamma^+_{trans}$, the boundary layer solution is degenerate. Then we prove the unique global-in-time existence of the solution and the asymptotic stability of both the boundary layer and its superposition with the 3−rarefaction wave in supersonic case, that is, $u_+>\sqrt{R\theta_+}$, under some smallness conditions. It should be mentioned that Nishibata and his group recently proved the existence and stability of boundary layer solution for a class of symmetric hyperbolic-parabolic systems, see [17]. There are also other interesting works for symmetric hyperbolic-parabolic system, see ([24–26]).

Our analysis is based on the energy method. Since the fact that system (1.1) is less dissipative, we need more subtle estimates to recover the regularity and dissipativity for the hyperbolic part. Precisely to say, similar as Cauchy problem of (1.1) in [2], we should ask the perturbed solution to be at least in $C(H^2)$.

The second main difficulty is how to control the higher order derivatives of boundary terms. For the first-order derivatives as in (4.19), we use the interior relations between functions and the character of the domain itself, that's very helpful. Moreover, for the second-order derivatives of boundary terms, estimates on the diameter direction besides the normal direction must be introduced. We take the advantages of the boundary condition adequately (as in Lemma 4.5-4.7) and avoid emerging the second normal derivatives on the boundary. As far as we know, few works use estimates on derivative of the diameter direction to study the asymptotic stability of the elementary waves. This method here maybe also helpful to other related problems with similar analytical difficulties. Just because of this, we must require the initial perturbed data $(\phi_0,\psi_0)(\xi)\in H^3(\mathbb{R}_+)$, $\zeta_0(\xi)\in H^4(\mathbb{R}_+)$ to let the computations make sense.

This manuscript is organized as follows. In Section 2, we obtain the existence of the boundary layer and list some properties of the boundary layer and rarefaction wave, then the main theorems are stated, see Theorems 2.1-2.3. In Section 3, we give the local existence of perturbed solution in proper function space and introduce a priori estimates to get the global solution, see Propositions 3.1-3.2. In Section 4, series of estimates are established and the main theorem is proved.

Notations. Throughout this paper, c and C denote some positive constants (generally large). $A \lesssim B$ means that there is a generic constant $C>0$ such that $A \le CB$ and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. For function spaces, $L^p(\mathbb{R}_+)(1 \le p \le \infty)$ denotes the usual Lebesgue space on $\mathbb{R}_+$ with norm $\|\cdot\|_{L^p}$ and $H^k(\mathbb{R}_+)$ the usual Sobolev space in the L^2 sense with norm $\|\cdot\|_k$. We note $\|\cdot\| = \|\cdot\|_{L^2}$ for simplicity and $C^k(I;H^p)$ is the space of k-times continuously differentiable functions on the interval I with values in $H^p(\mathbb{R}_+)$ and $L^2(I;H^p)$ is the space of L^2-functions on I with values in $H^p(\mathbb{R}_+)$.

2. Boundary layer, rarefaction wave and main results

Because of the coordinate transformation in Sections 2.2 and 2.3, the solution (ρ,u,θ) turns to (v,u,θ) and z_+ becomes (v_+,u_+,θ_+). So here we introduce new symbol for clearer clarification later, the solution (in Lagrange coordinates) we considered here is located in a small neighborhood Ω_+ of the right state z_+ as

$$\Omega_+ = \{(v,u,\theta) | |(v-v_+, u-u_+, \theta-\theta_+)| \le \delta\} \subseteq \Omega^+_{sub} \cap \widetilde{\Omega}^+_{supper}, \tag{2.1}$$

where δ is a positive constant depending only on z_+.

2.1. The existence of boundary layer.

At first, we discuss the existence of boundary layer solution to system (1.1) for $u_+>0$. The boundary layer solution $(\bar\rho,\bar u,\bar\theta)(x)$ to (1.1) should satisfy

$$\begin{cases} (\bar\rho\bar u)_x = 0, \quad x>0, \\ (\bar\rho\bar u^2 + \bar p)_x = 0, \\ (\bar\rho\bar u(\bar e + \dfrac{\bar u^2}{2}) + \bar p\bar u)_x = \kappa\bar\theta_{xx}, \end{cases} \tag{2.2}$$

and

$$\bar\theta(0) = \theta_-, \quad \lim_{x\to+\infty}(\bar\rho,\bar u,\bar\theta)(x) = (\rho_+,u_+,\theta_+) = z_+, \quad \inf_{x\in\mathbb{R}_+}(\bar\rho,\bar\theta)(x) > 0. \tag{2.3}$$

Integrating (2.2) over $[x,+\infty)$, we have

$$\begin{cases} \bar\rho\bar u = \rho_+u_+, \\ \bar\rho\bar u^2 + \bar p = \rho_+u_+^2 + p_+, \\ [\bar\rho\bar u(C_v\bar\theta + \dfrac{\bar u^2}{2}) - \rho_+u_+(C_v\theta_+ + \dfrac{u_+^2}{2})] + (\bar p\bar u - p_+u_+) = \kappa\bar\theta_x, \end{cases} \tag{2.4}$$

From $(2.4)_1$, we see that

$$\bar u(0) = \frac{\rho_+u_+}{\bar\rho(0)} > 0$$

is a necessary condition. Dividing both sides of $(2.4)_2$ by $\bar\rho\bar u(\rho_+u_+)$, we get

$$(u_+\bar u - R\theta_+)(\bar u - u_+) = Ru_+(\theta_+ - \bar\theta). \tag{2.5}$$

In order to analyze the relationship between $\bar u$ and $\bar\theta$ in (2.5) precisely, we introduce

$$\tilde w_1(x) = \frac{\bar u(x)}{u_+} = \frac{\rho_+}{\bar\rho(x)} > 0, \qquad \tilde w_2(x) = \frac{\bar\theta(x)}{\theta_+} > 0,$$

then (2.5) turns to

$$u_+^2\tilde{w}_1^2(x)-(R\theta_++u_+^2)\tilde{w}_1(x)+R\theta_+\tilde{w}_2(x)=0. \tag{2.6}$$

With the help of the boundary conditions in (2.3), i.e.,

$$\lim_{x\to+\infty}\tilde{w}_1(x)=\lim_{x\to+\infty}\tilde{w}_2(x)=1,$$

and (2.6), we deduce that

$$\begin{cases}\tilde{w}_1=\dfrac{(R\theta_++u_+^2)-\sqrt{(R\theta_++u_+^2)^2-4R\theta_+u_+^2\tilde{w}_2}}{2u_+^2} & 0<u_+<\sqrt{R\theta_+},\\ \tilde{w}_1=\dfrac{(R\theta_++u_+^2)+\sqrt{(R\theta_++u_+^2)^2-4R\theta_+u_+^2\tilde{w}_2}}{2u_+^2} & \sqrt{R\theta_+}<u_+,\\ \tilde{w}_1=1\pm\sqrt{1-\tilde{w}_2}, \qquad u_+=\sqrt{R\theta_+}. \end{cases} \tag{2.7}$$

Equation (2.7) also implies that $\tilde{w}_2(x)$ should satisfy

$$\tilde{w}_2(x)\le\frac{(R\theta_++u_+^2)^2}{4R\theta_+u_+^2}=\tilde{w}_{2\sup}, \tag{2.8}$$

where $\tilde{w}_{2\sup}\ge1$, and $\tilde{w}_{2\sup}=1$ if and only if $u_+^2=R\theta_+$.

Remembering the definition of $\tilde{w}_1,\tilde{w}_2$, (2.3)-(2.4) could be rearranged as

$$\begin{cases}\tilde{w}_{2x}=\dfrac{\rho_+u_+}{\kappa}\left[\dfrac{R\gamma}{(\gamma-1)}(\tilde{w}_2-1)+\dfrac{u_+^2}{2\theta_+}(\tilde{w}_1+1)(\tilde{w}_1-1)\right],\\ \tilde{w}_2(0)=\dfrac{\theta_-}{\theta_+}, \quad \tilde{w}_2(+\infty)=1, \quad \inf\limits_{x\in\mathbb{R}_+}\tilde{w}_2(x)>0,\end{cases} \tag{2.9}$$

where the relationship between $\tilde{w}_1$ and $\tilde{w}_2$ has been stated in (2.7).

Hence the existence of solution to (2.3)-(2.4) is equivalent to the existence of solution to (2.9). Now we mainly study the latter. Obviously, the range of $\tilde{w}_2(x)$ for which the boundary layer solution may exist should be $(0,\tilde{w}_{2sup}]$ and we seek for the nontrivial solution to (2.9), that is, $\tilde{w}_2(0)\neq1$. All the cases we considered below are under this premise.

We have following discussion:

(1) If $z_+\in\tilde{\Omega}_{sub}^+$, that is $0<u_+<\sqrt{R\theta_+}(\tilde{M}_+<1)$. In this case, substituting $(2.7)_1$ into (2.9), it becomes

$$\begin{cases}\tilde{w}_{2x}=L_1(\tilde{w}_2)(1-\tilde{w}_2)=g_1(\tilde{w}_2),\\ \tilde{w}_2(0)=\dfrac{\theta_-}{\theta_+}, \quad \tilde{w}_2(+\infty)=1, \quad \inf\limits_{x\in\mathbb{R}_+}\tilde{w}_2(x)>0,\end{cases} \tag{2.10}$$

where

$$L_1(\tilde{w}_2)=:\frac{R\rho_+u_+}{2\kappa}\left(\frac{R\theta_++3u_+^2-\sqrt{(R\theta_++u_+^2)^2-4R\theta_+u_+^2\tilde{w}_2}}{(u_+^2-R\theta_+)-\sqrt{(R\theta_++u_+^2)^2-4R\theta_+u_+^2\tilde{w}_2}}-\frac{2\gamma}{\gamma-1}\right). \tag{2.11}$$

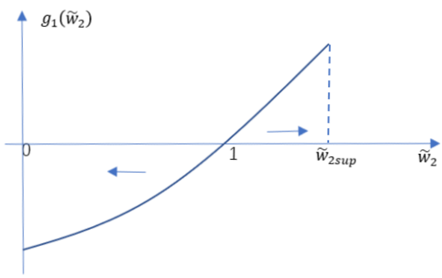

FIG. 2.1. *The graph of* $g_1(\tilde{w}_2)$ *in case (1)*

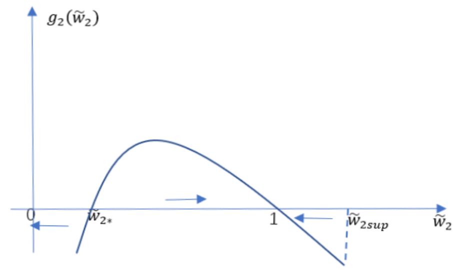

FIG. 2.2. *The graph of* $g_2(\tilde{w}_2)$ *in case {2.1}*

After tedious computation, the property of $g_1(\tilde{w}_2)$ is listed as follows

$$\begin{cases} g_1(\tilde{w}_2=1)=0, \quad \left.\dfrac{dg_1(\tilde{w}_2)}{d\tilde{w}_2}\right|_{\tilde{w}_2=1}>0, \\ \dfrac{d^2g_1(\tilde{w}_2)}{d\tilde{w}_2^2}>0, \quad \text{for } \tilde{w}_2\in(0,\tilde{w}_{2sup}). \end{cases} \tag{2.12}$$

Then (2.12) implies that there exists a small positive constant σ such that the following is true. If $\tilde{w}_2(x)\in[1-\sigma,1], g_1(\tilde{w}_2)\le 0$, i.e., $\tilde{w}_{2x}\le 0$. Thus, $\tilde{w}_2(x)$ is decreasing in $[1-\sigma,1]$. So when $\tilde{w}_2(0)<1$, $\tilde{w}_2(x)$ can not approach to 1 as $x\to+\infty$. If $\tilde{w}_2(x)\in[1,1+\sigma], g_1(\tilde{w}_2)\ge 0$, i.e., $\tilde{w}_{2x}\ge 0$. Thus, $\tilde{w}_2(x)$ is increasing in $[1,1+\sigma]$. Therefore when $\tilde{w}_2(0)>1$, $\tilde{w}_2(x)$ can not approach to 1 as $x\to+\infty$. Consequently, there does not exist a solution to (2.10) in this case. The graph of $g_1(\tilde{w}_2)$ is shown in Figure 2.1.

(2) If $z_+\in\widetilde{\Omega}^+_{supper}\bigcap\Omega^+_{sub}$, that is $\sqrt{R\theta_+}<u_+<\sqrt{\gamma R\theta_+}(\tilde{M}_+>1$ and $M_+<1)$. Using $(2.7)_2$, (2.9) becomes

$$\begin{cases} \tilde{w}_{2x}=L_2(\tilde{w}_2)(1-\tilde{w}_2)=g_2(\tilde{w}_2), \\ \tilde{w}_2(0)=\dfrac{\theta_-}{\theta_+}, \quad \tilde{w}_2(+\infty)=1, \quad \inf\limits_{x\in\mathbb{R}_+}\tilde{w}_2(x)>0, \end{cases} \tag{2.13}$$

where

$$L_2(\tilde{w}_2)=:\frac{R\rho_+u_+}{2\kappa}\left(\frac{R\theta_++3u_+^2+\sqrt{(R\theta_++u_+^2)^2-4R\theta_+u_+^2\tilde{w}_2}}{(u_+^2-R\theta_+)+\sqrt{(R\theta_++u_+^2)^2-4R\theta_+u_+^2\tilde{w}_2}}-\frac{2\gamma}{\gamma-1}\right). \tag{2.14}$$

After tedious computation, the property of $g_2(\tilde{w}_2)$ in this situation is listed as follows

$$\begin{cases} g_2(\tilde{w}_2=1)=0, \quad \left.\dfrac{dg_2(\tilde{w}_2)}{d\tilde{w}_2}\right|_{\tilde{w}_2=1}<0, \\ \dfrac{d^2g_2(\tilde{w}_2)}{d\tilde{w}_2^2}<0, \quad \text{for } \tilde{w}_2\in(0,\tilde{w}_{2sup}), \\ \lim\limits_{\tilde{w}_2\to 0}g_2(\tilde{w}_2)=\dfrac{R\rho_+}{2\kappa(\gamma-1)}\dfrac{(\gamma-1)R\theta_+-2u_+^2}{u_+}. \end{cases} \tag{2.15}$$

There are two subcases.

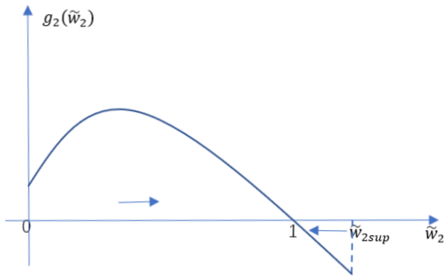

FIG. 2.3. *The graph of $g_2(\tilde{w}_2)$ in case (2.2)*

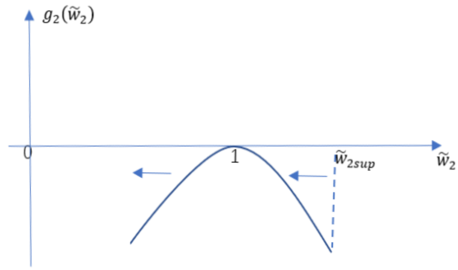

FIG. 2.4. *The graph of $g_2(\tilde{w}_2)$ in case (3)*

(2.1) When ($1<\gamma\le 3$ and $\sqrt{R\theta_+}<u_+<\sqrt{\gamma R\theta_+}$) or ($\gamma>3$ and $\sqrt{\frac{\gamma-1}{2}R\theta_+}<u_+<\sqrt{\gamma R\theta_+}$), (2.15) implies that

$$\lim_{\tilde{w}_2\to 0} g_2(\tilde{w}_2)<0. \tag{2.16}$$

Equations (2.15), (2.16) tell us that $g_2(\tilde{w}_2)$ has two positive zero points, one is 1, the other we denoted by $\tilde{w}_{2*}$ satisfies $L_2(\tilde{w}_{2*})=0$. Moreover, $\tilde{w}_{2*}$ is between 0 and 1. If $\tilde{w}_2(x)\le\tilde{w}_{2*}$, then $g_2(\tilde{w}_2)\le 0$, i.e., $\tilde{w}_{2x}\le 0$. That is, $\tilde{w}_2(x)$ is decreasing. So when $\tilde{w}_2(0)\le\tilde{w}_{2*}$, $\tilde{w}_2(x)$ can not approach to 1 as $x\to+\infty$. If $\tilde{w}_{2*}<\tilde{w}_2(x)<1$, then $g_2(\tilde{w}_2)>0$, i.e., $\tilde{w}_{2x}>0$. That is, $\tilde{w}_2(x)$ is increasing. Hence, when $\tilde{w}_{2*}<\tilde{w}_2(0)<1$, there exists a monotonically increasing solution $\tilde{w}_2(x)$ to (2.13). Lastly, if $1<\tilde{w}_2(x)<\tilde{w}_{2\text{sup}}$, then $g_2(\tilde{w}_2)<0$, i.e., $\tilde{w}_{2x}<0$. That is ,$\tilde{w}_2(x)$ is decreasing. Therefore when $1<\tilde{w}_2(0)<\tilde{w}_{2\text{sup}}$, there exists a monotonically decreasing solution $\tilde{w}_2(x)$ to (2.13). When $\tilde{w}_2(0)=\tilde{w}_{2sup}$, the solution $\tilde{w}_2(x)$ is not smooth (both $\frac{dg_2(\tilde{w}_2)}{d\tilde{w}_2}$ and $\frac{d^2g_2(\tilde{w}_2)}{d\tilde{w}_2^2}$ are infinity at this endpoint, then $\tilde{w}_2(x)$ is second-order non-differentiable at this endpoint), so we don't consider those types of solutions in following discussion for similar reasons. Thus, smooth solution $\tilde{w}_2(x)$ to (2.13) exists if and only if $\tilde{w}_2(0)\in(\tilde{w}_{2*},1)\cup(1,\tilde{w}_{2\text{sup}})$ and the decay estimates of the solution are obtained from (2.13),

$$\begin{aligned}&|\frac{d^n}{dx^n}(\tilde{w}_2(x)-1)|\lesssim|\tilde{w}_2(0)-1|e^{-c_0x}\\ &\qquad\qquad\text{for}\quad n=1,2,3,\ldots,\quad c_0=L_2(\tilde{w}_2=1),\end{aligned} \tag{2.17}$$

and the graph of $g_2(\tilde{w}_2)$ in this situation is shown in Figure 2.2.

(2.2) When ($\gamma>3$ and $\sqrt{R\theta_+}<u_+\le\sqrt{\frac{\gamma-1}{2}R\theta_+}$), (2.15) implies that

$$\begin{cases}\lim\limits_{\tilde{w}_2\to 0} g_2(\tilde{w}_2)>0, & \text{when } \sqrt{R\theta_+}<u_+<\sqrt{\dfrac{\gamma-1}{2}R\theta_+},\\ \lim\limits_{\tilde{w}_2\to 0} g_2(\tilde{w}_2)=0, & \text{when } u_+=\sqrt{\dfrac{\gamma-1}{2}R\theta_+}.\end{cases} \tag{2.18}$$

Combining (2.15) and (2.18), $g_2(\tilde{w}_2)$ has only one positive zero point 1. If $0<\tilde{w}_2(x)<1$, then $g_2(\tilde{w}_2)>0$, i.e., $\tilde{w}_{2x}>0$. Therefore, when $0<\tilde{w}_2(0)<1$,

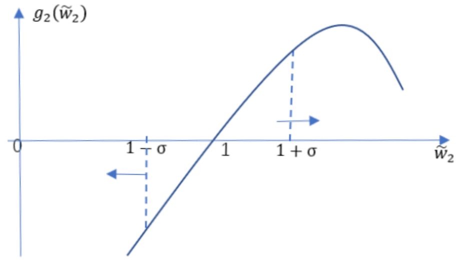

FIG. 2.5. *The graph of $g_2(\tilde{w}_2)$ in case (4)*

there exists a monotonically increasing solution $\tilde{w}_2(x)$ to (2.13). If $1<\tilde{w}_2(x)<\tilde{w}_{2sup}$, then $g_2(\tilde{w}_2)<0$, i.e., $\tilde{w}_{2x}<0$. So when $1<\tilde{w}_2(0)<\tilde{w}_{2sup}$, there exists a monotonically decreasing solution $\tilde{w}_2(x)$ to (2.13). Hence smooth solution $\tilde{w}_2(x)$ to (2.13) exists if and only if $\tilde{w}_2(0)\in(0,1)\cup(1,\tilde{w}_{2\mathrm{sup}})$. Moreover, the decay estimates of the solution are same as (2.17). The graph of $g_2(\tilde{w}_2)$ in this situation is shown in Figure 2.3.

(3) If $z_+\in\Gamma^+_{trans}$, that is $u_+=\sqrt{\gamma R\theta_+}(M_+=1)$. In this case, (2.9) still becomes (2.13). Moreover, the property of $g_2(\tilde{w}_2)$ under this situation is

$$\begin{cases} g_2(\tilde{w}_2=1)=0, \quad \left.\dfrac{dg_2(\tilde{w}_2)}{d\tilde{w}_2}\right|_{\tilde{w}_2=1}=0, \\ \dfrac{d^2g_2(\tilde{w}_2)}{d\tilde{w}_2^2}<0, \quad \text{for } \tilde{w}_2\in(0,\tilde{w}_{2sup}). \end{cases} \tag{2.19}$$

Using (2.19), $g_2(\tilde{w}_2)$ only has one zero point 1. Furthermore, $g_2(\tilde{w}_2)<0$, i.e, $\tilde{w}_{2x}<0$ whatever $0<\tilde{w}_2(x)<1$ or $1<\tilde{w}_2(x)<\tilde{w}_{2sup}$. Similar to the above discussion, smooth solution $\tilde{w}_2(x)$ to (2.13) exists if and only if $\tilde{w}_2(0)\in(1,\tilde{w}_{2sup})$. The decay estimates of the solution are obtained by (2.13) and (2.19)

$$|\frac{d^n}{dx^n}(\tilde{w}_2(x)-1)|\lesssim\frac{(\tilde{w}_2(0)-1)^{n+1}}{(1+\tilde{c}_0(\tilde{w}_2(0)-1)x)^{n+1}} \quad \text{for} \quad n=1,2,3,..., \tag{2.20}$$

where

$$\tilde{c}_0=-\frac{\left.\frac{d^2g_2(\tilde{w}_2)}{d\tilde{w}_2^2}\right|_{\tilde{w}_2=1}}{2}>0. \tag{2.21}$$

The graph of $g_2(\tilde{w}_2)$ in this situation is shown in Figure 2.4.

(4) If $z_+\in\Omega^+_{supper}$, that is $u_+>\sqrt{\gamma R\theta_+}(M_+>1)$. In this case, (2.9) still becomes (2.13). Moreover, the property of $g_2(\tilde{w}_2)$ under this situation is

$$\begin{cases} g_2(\tilde{w}_2=1)=0, \quad \left.\dfrac{dg_2(\tilde{w}_2)}{d\tilde{w}_2}\right|_{\tilde{w}_2=1}>0, \\ \dfrac{d^2g_2(\tilde{w}_2)}{d\tilde{w}_2^2}<0, \quad \text{for } \tilde{w}_2\in(0,\tilde{w}_{2sup}). \end{cases} \tag{2.22}$$

Similar as (1), there exists a small positive constant σ such that when $\tilde{w}_2(x)\in[1-\sigma,1], \tilde{w}_{2x}\le 0$, when $\tilde{w}_2(x)\in[1,1+\sigma], \tilde{w}_{2x}\ge 0$. Therefore, there does not exist a solution to (2.13) in this case. The graph of $g_2(\tilde{w}_2)$ in this situation is shown in Figure 2.5.

(5) If $z_+\in\widetilde{\Gamma}^+_{trans}$, that is $u_+=\sqrt{R\theta_+}(\tilde{M}_+=1)$. In this case, $\tilde{w}_{2sup}=1=\tilde{w}_2(+\infty)$. After verification, the solution is second-order non-differentiable at $\tilde{w}_{2sup}=1$ even if it exists. So we don't consider those types of solutions.

Summarizing (1) – (5), we have the following existence theorem of BL solution.

Proposition 2.1. *For $\gamma\in(1,+\infty)$, the boundary value problem (2.9) has a unique smooth solution $\tilde{w}_2(x)$ if and only if $\tilde{M}_+>1$ and $M_+\le 1$. Precisely to say,*

(1) For $\tilde{M}_+>1$ and $M_+<1$, there are two subcases: ($\tilde{w}_{2sup}=\frac{(R\theta_++u_+^2)^2}{4R\theta_+u_+^2}$)

(i) If $1<\gamma\le 3$ and $\sqrt{R\theta_+}<u_+<\sqrt{\gamma R\theta_+}$ or $\gamma>3$ and $\sqrt{\frac{\gamma-1}{2}R\theta_+}<u_+<\sqrt{\gamma R\theta_+}$, there exists a unique smooth solution to (2.13) when $\tilde{w}_2(0)\in(\tilde{w}_{2*},1)\cup(1,\tilde{w}_{2\sup})$, where $\tilde{w}_{2*}\in(0,1)$ satisfies $L_2(\tilde{w}_{2*})=0$. Moreover, if $\tilde{w}_2(0)\lesseqgtr 1$, then $\tilde{w}_{2x}\gtreqless 0$.

(ii) If ($\gamma>3$ and $\sqrt{R\theta_+}<u_+\le\sqrt{\frac{\gamma-1}{2}R\theta_+}$), there exists a unique smooth solution to (2.13) when $\tilde{w}_2\in(0,1)\cup(1,\tilde{w}_{2sup})$. Moreover, if $\tilde{w}_2(0)\lesseqgtr 1$, then $\tilde{w}_{2x}\gtreqless 0$.

The decay estimates of the solution to both (i) and (ii) satisfy (2.17).

(2) For $M_+=1$, that is $u_+=\sqrt{\gamma R\theta_+}$, there exists a unique decreasing solution to (2.13) when $\tilde{w}_2(0)\in(1,\tilde{w}_{2sup})$. And the decay estimates of this solution satisfy (2.20).

2.2. The properties of boundary layer solution and the stability. In this subsection, we construct the boundary layer for the initial boundary value problem (1.1),(1.3) and (1.14) and then state our first main result. At first, we reform the system (1.1), (1.3) and (1.14) in Lagrange coordinates as

$$\begin{cases} v_t-s_-v_\xi-u_\xi=0, \qquad \xi>0, \quad t>0,\\ u_t-s_-u_\xi+p_\xi=0,\\ (C_v\theta+\frac{u^2}{2})_t-s_-(C_v\theta+\frac{u^2}{2})_\xi+(pu)_\xi=k(\frac{\theta_\xi}{v})_\xi,\\ (v,u,\theta)(t,0)=(v_-,u_-,\theta_-)=:z_-,\\ (v,u,\theta)(0,\xi)=(v_0,u_0,\theta_0)(\xi)\to(v_+,u_+,\theta_+)=z_+(\xi\to+\infty). \end{cases} \tag{2.23}$$

where $v=\frac{1}{\rho}$ is the specific volume of gas, the pressure $p=\frac{R\theta}{v}$ and the new variable $\xi=x-s_-t$ containing the moving boundary speed $s_-=-\frac{u_-}{v_-}$. In this new coordinates, the boundary layer solution $\bar{z}:=(\bar{v},\bar{u},\bar{\theta})(\xi)$ satisfies

$$\begin{cases} -s_-\bar{v}_\xi-\bar{u}_\xi=0, \quad \xi>0,\\ -s_-\bar{u}_\xi+\bar{p}_\xi=0,\\ -s_-(C_v\bar{\theta}+\frac{\bar{u}^2}{2})_\xi+(\bar{p}\bar{u})_\xi=k(\frac{\bar{\theta}_\xi}{\bar{v}})_\xi,\\ (\bar{v},\bar{u},\bar{\theta})(0)=(v_-,u_-,\theta_-), \quad u_->0,\\ (\bar{v},\bar{u},\bar{\theta})(+\infty)=(v_+,u_+,\theta_+). \end{cases} \tag{2.24}$$

Denote the strength of boundary layer solution as

$$\bar{\delta} := |\theta_+ - \theta_-|, \tag{2.25}$$

by the analysis in Section 2.1, we get the following lemma.

LEMMA 2.1 (Property of boundary layer). $(\bar{v},\bar{u},\bar{\theta})(\xi)$ *satisfies*

(1) *If* $z_+ \in \widetilde{\Omega}^+_{supper} \bigcap \Omega^+_{sub}$, *that is* $\sqrt{R\theta_+} < u_+ < \sqrt{\gamma R\theta_+}$, $\exists \bar{\delta}_0 > 0$, *such that if* $0 < \bar{\delta} \le \bar{\delta}_0$, *there exists a unique solution* $(\bar{v}_\xi \lesseqgtr 0, \bar{u}_\xi \lesseqgtr 0, \bar{\theta}_\xi \gtreqless 0)$ *for* (2.24) *which is non-degenerate and satisfies*

$$\left|\frac{d^n}{d\xi^n}(\bar{v}-v_+, \bar{u}-u_+, \bar{\theta}-\theta_+)(\xi)\right| \lesssim \bar{\delta} e^{-c_0 \xi}, \quad n = 1,2,3,\dots. \tag{2.26}$$

(2) *If* $z_+ \in \Gamma^+_{trans}$, *that is* $u_+ = \sqrt{\gamma R\theta_+}$, $\exists \bar{\delta}_1 > 0$ *such that if* $0 < \bar{\delta} \le \bar{\delta}_1$, *there exists a unique solution* $(\bar{v}_\xi > 0, \bar{u}_\xi > 0, \bar{\theta}_\xi < 0)$ *for* (2.24) *which is degenerate and satisfies*

$$\left|\frac{d^n}{d\xi^n}(\bar{v}-v_+, \bar{u}-u_+, \bar{\theta}-\theta_+)(\xi)\right| \lesssim \frac{\bar{\delta}^{n+1}}{1+(\bar{\delta}\xi)^{n+1}}. \quad n = 1,2,3,\dots. \tag{2.27}$$

This Lemma could be obtained immediately from our analysis in Proposition 2.1, we omit the proof for short. In order to express our theorems more convenient, the solution space is defined as:

$$\begin{aligned}
\mathbb{X}_{m_1,m_2,M}(0,t) = \{&(\phi,\psi,\zeta) | (\phi,\psi,\zeta) \in C([0,t]; H^2(\mathbb{R}_+)), (\phi,\psi,\zeta)_t \in C([0,t]; H^1(\mathbb{R}_+)),\\
&(\phi,\psi,\zeta)_{tt} \in C([0,t]; L^2(\mathbb{R}_+)), (\phi,\psi)_\xi \in L^2(0,t; H^1(\mathbb{R}_+)),\\
&(\zeta_\xi, \zeta_t) \in L^2(0,t; H^2(\mathbb{R}_+)), \zeta_{tt} \in L^2(0,t; H^1(\mathbb{R}_+)),\\
&\inf_{[0,t]\times\mathbb{R}_+} v(t,\xi) \ge m_1, \inf_{[0,t]\times\mathbb{R}_+} \theta(t,\xi) \ge m_2,\\
&N(t) := \sup_{[0,t]\times\mathbb{R}_+} \left(\|(\phi,\psi,\zeta)\|_2 + \|(\phi_t,\psi_t,\zeta_t)\|_1 + \|(\phi_{tt},\psi_{tt},\zeta_{tt})\| \right) \le M \}.
\end{aligned} \tag{2.28}$$

Then our first main result is as follows:

THEOREM 2.1. *Assume that* $z_+ \in \Omega^+_{sub} \cap \widetilde{\Omega}^+_{supper}$ *and* $z_- \in BL(z_+) \cap \Omega_+$, *that is,* z_-, z_+ *satisfy* (2.24) *and* $R\theta_- < u_-^2 < \gamma R\theta_- (\gamma > 1)$, *then there exist some small positive constants* δ_1 *and* η_1 *such that if* $\bar{\delta} \le \delta_1$ *and*

$$\|(v_0 - \bar{v}, u_0 - \bar{u})\|_3 + \|(\theta_0 - \bar{\theta})\|_4 \le \eta_1, \tag{2.29}$$

then the inflow problem (2.23) *has a unique solution* $(v,u,\theta)(t,\xi)$ *satisfying*

$$(v-\bar{v}, u-\bar{u}, \theta-\bar{\theta})(t,\xi) \in \mathbb{X}_{\frac{m_1}{2},\frac{m_2}{2},\eta_1}(0,+\infty). \tag{2.30}$$

Furthermore, it holds

$$\sup_{\xi\ge 0} |(v,u,\theta)(t,\xi) - (\bar{v},\bar{u},\bar{\theta})(\xi)| \to 0, \text{ as } t \to +\infty. \tag{2.31}$$

2.3. The properties of rarefaction wave and the stability. As in [4], if $z_+ \in R_3(z_-)$, that is, the 3-rarefaction wave $(v^r,u^r,\theta^r)(\frac{x}{t})$ connecting z_- and z_+ is the unique weak solution globally in time to the following Riemann problem:

$$\begin{cases} v^r_t - u^r_x = 0, \\ u^r_t + p^r_x = 0, \\ (e^r + \dfrac{{u^r}^2}{2})_t + (p^r u^r)_x = 0, \\ (v^r,u^r,\theta^r)(0,x) = \begin{cases} (v_-,u_-,\theta_-), & x<0, \\ (v_+,u_+,\theta_+), & x>0. \end{cases} \end{cases} \tag{2.32}$$

Here $\theta_- < \theta_+$ and $0 < u_- < u_+$, $p^r = \frac{R\theta^r}{v^r}, e^r = C_v\theta^r$. It is well known that the characteristic speeds of (2.32) are (see [4]),

$$\lambda_1(v^r,u^r,\theta^r) = -\sqrt{-p^r_v(v,s)}, \quad \lambda_2(v^r,u^r,\theta^r) = 0, \quad \lambda_3(v^r,u^r,\theta^r) = \sqrt{p^r_v(v,s)}. \tag{2.33}$$

To give the details of the large-time behavior of the solutions to the inflow problem (2.23), it is necessary to construct a smooth approximation $\tilde{z} := (\tilde{v},\tilde{u},\tilde{\theta})(t,x)$ of $(v^r,u^r,\theta^r)(\frac{x}{t})$. Let us consider the solution to the following Cauchy problem:

$$\begin{cases} w_t + ww_x = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R}, \\ w(0,x) = \begin{cases} w_-, & x<0, \\ w_- + C_q\tilde{\delta}\int_0^{\epsilon x} y^q e^{-y}dy, & x \geq 0. \end{cases} \end{cases} \tag{2.34}$$

Here $\tilde{\delta} = w_+ - w_- > 0$, $q > 16$ are two constants, C_q is a constant such that $C_q\int_0^{+\infty} y^q e^{-y}dy = 1$, $0 < \epsilon < 1$ is a small constant which will be determined later. Let $w_\pm = \lambda_3(v_\pm,u_\pm,\theta_\pm)$, we construct the approximated function $\tilde{z}(t,x)$ by

$$\begin{cases} S^r(\tilde{v},\tilde{u},\tilde{\theta})(t,x) = S^r(v_+,u_+,\theta_+), \\ \lambda_3(\tilde{v},\tilde{u},\tilde{\theta})(t,x) = w(t,x), \\ \tilde{u} = u_+ - \displaystyle\int_{v_+}^{\tilde{v}} \lambda_3(s,S^r)ds. \end{cases} \tag{2.35}$$

Remind that $\xi = x - s_-t$, $\tilde{z}(t,\xi)$ satisfy

$$\begin{cases} \tilde{v}_t - s_-\tilde{v}_\xi - \tilde{u}_\xi = 0, \\ \tilde{u}_t - s_-\tilde{u}_\xi + \tilde{p}_\xi = 0, \\ (\tilde{e} + \dfrac{\tilde{u}^2}{2})_t - s_-(\tilde{e} + \dfrac{\tilde{u}^2}{2})_\xi + (\tilde{p}\tilde{u})_\xi = 0, \\ (\tilde{v},\tilde{u},\tilde{\theta})(t,0) = (v_-,u_-,\theta_-), \quad u_- > 0, \\ (\tilde{v},\tilde{u},\tilde{\theta})(0,\xi) = (\tilde{v}_0,\tilde{u}_0,\tilde{\theta}_0)(\xi) \to (v_+,u_+,\theta_+)(\xi \to +\infty). \end{cases} \tag{2.36}$$

For the smooth rarefaction wave $\tilde{z}(t,\xi)$, we have the following lemma (see [4], Lemma 3.6).

LEMMA 2.2. *Smooth rarefaction wave $\tilde{z}(t,\xi)$ obtained via (2.36) satisfies*

(1) $\tilde{u}_\xi \geq 0, \quad$ *for* $\xi > 0, t > 0$.

(2) For any p $(1 \leq p \leq +\infty)$, *there exists a constant* C_{pq} *such that*

$$\left\| \left(\tilde{v}_\xi, \tilde{u}_\xi, \tilde{\theta}_\xi\right)(t) \right\|_{L^p} \leq C_{pq} \min\left\{ \tilde{\delta}\epsilon^{1-\frac{1}{p}}, \tilde{\delta}^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}} \right\},$$
$$\left\| \left(\tilde{v}_{\xi\xi}, \tilde{u}_{\xi\xi}, \tilde{\theta}_{\xi\xi}\right)(t) \right\|_{L^p} \leq C_{pq} \min\left\{ \tilde{\delta}\epsilon^{2-\frac{1}{p}}, \tilde{\delta}^{\frac{1}{p}}(1+t)^{-1+\frac{1}{q}} \right\}, \quad t \geq 0. \tag{2.37}$$

(3) $\lim_{t \to +\infty} \sup_{\xi \in \mathbb{R}_+} \left| \left(\tilde{v}, \tilde{u}, \tilde{\theta}\right)(t,\xi) - (v^r, u^r, \theta^r)(t,\xi) \right| = 0.$

Then our second main result is stated as follows:

THEOREM 2.2. *Assume that* $z_+ \in \Omega^+_{sub} \cap \widetilde{\Omega}^+_{supper}$ *and* $z_- \in R_3(z_+) \cap \Omega_+$, *that is,* z_-, z_+ *satisfy (2.36) and* $R\theta_- < u_-^2 < \gamma R\theta_- (\gamma > 1)$, *there exist some small positive constants* δ_2 *and* η_2 *such that if* $\epsilon \leq \delta_2$ *and*

$$\|(v_0 - \tilde{v}_0, u_0 - \tilde{u}_0)\|_3 + \|(\theta_0 - \tilde{\theta}_0)\|_4 \leq \eta_2, \tag{2.38}$$

then the inflow problem (2.23) has a unique solution $(v,u,\theta)(t,\xi)$ *satisfying*

$$(v - \tilde{v}, u - \tilde{u}, \theta - \tilde{\theta})(t,\xi) \in \mathbb{X}_{\frac{m_1}{2}, \frac{m_2}{2}, \eta_2}(0, +\infty) \tag{2.39}$$

Furthermore, it holds

$$\sup_{\xi \geq 0} |(v,u,\theta)(t,\xi) - (\tilde{v}, \tilde{u}, \tilde{\theta})(t,\xi)| \to 0, \text{ as } t \to +\infty. \tag{2.40}$$

REMARK 2.1. Note that the strength of 3−rarefaction wave does not need be small in this situation.

2.4. Composition waves and the stability. If left state $z_- \in BLR_3(z_+) \cap \Omega_+$, we see that, there exists a unique intermediate state $z_m := (v_m, u_m, \theta_m) \in R_3(z_+)$ such that z_m, z_+ are connected by the 3−rarefaction wave and z_m, z_- are connected by the BL-solution. Precisely, replacing z_- by z_m in (2.36), it holds that

$$S^r(v_m, u_m, \theta_m) = S^r(v_+, u_+, \theta_+), \quad u_m = u_+ - \int_{v_+}^{v_m} \lambda_3(\eta, S^r) d\eta. \tag{2.41}$$

For this z_m, instead z_+ by z_m in (2.24), we expect that the superposition of this boundary layer and this 3−rarefaction wave is stable. To do this, let

$$(\hat{v}, \hat{u}, \hat{\theta})(t,\xi) = (\bar{v}, \bar{u}, \bar{\theta})(\xi) + (\tilde{v}, \tilde{u}, \tilde{\theta})(t,\xi) - (v_m, u_m, \theta_m), \tag{2.42}$$

and it satisfies

$$\begin{cases} \hat{v}_t - s_- \hat{v}_\xi - \hat{u}_\xi = 0, \quad \xi > 0, \quad t > 0 \\ \hat{u}_t - s_- \hat{u}_\xi + \hat{p}_\xi = G_1, \\ C_v \hat{\theta}_t - s_- C_v \hat{\theta}_\xi + \hat{p}\hat{u}_\xi = k(\dfrac{\hat{\theta}_\xi}{\hat{v}})_\xi + G_2, \\ (\hat{v}, \hat{u}, \hat{\theta})(t,0) = (v_-, u_-, \theta_-), \quad u_- > 0, \\ (\hat{v}, \hat{u}, \hat{\theta})(0,\xi) = (\hat{v}_0, \hat{u}_0, \hat{\theta}_0)(\xi) \to (v_+, u_+, \theta_+), \quad \xi \to +\infty. \end{cases} \tag{2.43}$$

where

$$G_1 := (\hat{p} - \bar{p} - \tilde{p} + p_m)_\xi,$$

$$
\begin{aligned}
&= O(1)(|\bar{z}_\xi||\tilde{z}-z_m|+|\tilde{z}_\xi||\bar{z}-z_m|)\\
&= O(1)\bar{\delta}e^{-c(|\xi|+t)},\\
G_2 := &(\hat{p}\hat{u}_\xi - \bar{p}\bar{u}_\xi - \tilde{p}\tilde{u}_\xi) - k(\frac{\hat{\theta}_\xi}{\hat{v}} - \frac{\bar{\theta}_\xi}{\bar{v}})_\xi\\
&= O(1)(|\bar{z}_\xi||\tilde{z}-z_m|+|\tilde{z}_\xi||\bar{z}-z_m|)+O(1)(|\tilde{\theta}_{\xi\xi}|+|\tilde{\theta}_\xi|^2)\\
&= O(1)\bar{\delta}e^{-c(|\xi|+t)}+O(1)(|\tilde{\theta}_{\xi\xi}|+|\tilde{\theta}_\xi|^2).
\end{aligned}
\tag{2.44}
$$

The third main result is given below:

THEOREM 2.3. *Assume that $z_+ \in \Omega^+_{sub} \cap \widetilde{\Omega}^+_{supper}$, and $z_- \in BLR_3(z_+) \cap \Omega_+$, that is, there exists $z_m = (v_m, u_m, \theta_m)$ such that $z_m \in R_3(z_+), z_- \in BL(z_m)$, and z_-, z_m satisfy (2.24) just replace z_+ with z_m, z_m, z_+ satisfy (2.41), $R\theta_- < u_-^2 < \gamma R\theta_- (\gamma > 1)$. There exist some small positive constants δ_3 and η_3, such that if $\bar{\delta} + \epsilon \le \delta_3$ and*

$$
\|(v_0 - \hat{v}_0, u_0 - \hat{u}_0)\|_3 + \|(\theta_0 - \hat{\theta}_0)\|_4 \le \eta_3, \tag{2.45}
$$

then the inflow problem (2.23) has a unique solution $(v,u,\theta)(t,\xi)$ satisfying

$$
(v-\hat{v}, u-\hat{u}, \theta-\hat{\theta})(t,\xi) \in \mathbb{X}_{\frac{m_1}{2},\frac{m_2}{2},\eta_3}(0,+\infty), \tag{2.46}
$$

Furthermore, it holds

$$
\sup_{\xi\ge 0} |(v,u,\theta)(t,\xi) - (\hat{v},\hat{u},\hat{\theta})(t,\xi)| \to 0, \quad as \quad t \to +\infty. \tag{2.47}
$$

3. Local existence and stability analysis

In this section, we give the proofs of the main theorems. Since the result of Theorem 2.3 covers that of Theorem 2.1 and Theorem 2.2 if $(v_\pm, u_\pm, \theta_\pm) = (v_m, u_m, \theta_m)$, we only show the asymptotic stability of the composition wave, that is, Theorem 2.3.

Define the perturbation function

$$
(\phi,\psi,\zeta)(t,\xi) = (v,u,\theta)(t,\xi) - (\hat{v},\hat{u},\hat{\theta})(t,\xi), \tag{3.1}
$$

then the reformed equation is

$$
\begin{cases}
\phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi > 0, \quad t > 0,\\
\psi_t - s_-\psi_\xi + (\frac{R\zeta}{v})_\xi - (\frac{\hat{p}\phi}{v})_\xi = -G_1,\\
C_v\zeta_t - s_-C_v\zeta_\xi + p\psi_\xi + \hat{u}_\xi(p-\hat{p}) = \kappa(\frac{\zeta_\xi}{v} - \frac{\hat{\theta}_\xi\phi}{v\hat{v}})_\xi - G_2,\\
(\phi,\psi,\zeta)(t,0) = (0,0,0),\\
(\phi,\psi,\zeta)(0,\xi) = (\phi_0,\psi_0,\zeta_0)(\xi) \to (0,0,0), \quad as \quad \xi \to +\infty,
\end{cases}
\tag{3.2}
$$

and we order that the initial data satisfies the compatible condition

$$
(\phi_j, \psi_j, \zeta_j)(0) = (0,0,0), \; j = 0,1, \tag{3.3}
$$

where $(\phi_j,\psi_j,\zeta_j) := (\partial_t^j\phi, \partial_t^j\psi, \partial_t^j\zeta)\,|_{t=0}\,(j=1,2)$ are defined by the iterated sequence from (3.2),

$$
\begin{cases}
\partial_t^j\phi\,|_{t=0} = \partial_t^{j-1}(s_-\phi_\xi + \psi_\xi)\,|_{t=0},\\
\partial_t^j\psi\,|_{t=0} = \partial_t^{j-1}(s_-\psi_\xi - (\frac{R\zeta}{v})_\xi + (\frac{\hat{p}\phi}{v})_\xi - G_1)\,|_{t=0},\\
\partial_t^j\zeta\,|_{t=0} = \partial_t^{j-1}(s_-\zeta_\xi + \frac{1}{C_v}[\kappa(\frac{\zeta_\xi}{v} - \frac{\hat{\theta}_\xi\phi}{v\hat{v}})_\xi - G_2 - p\psi_\xi - \hat{u}_\xi(p-\hat{p})])\,|_{t=0}\,.
\end{cases}
\tag{3.4}
$$

The local existence of the solution to system (3.2) is stated as follows :

Proposition 3.1 (Local existence). *There exist positive constants $\overline{\delta}_1$, $\bar{\eta}_3$ and $\overline{C}$ such that the following statements hold. Under the assumption $\overline{\delta}+\epsilon\le\overline{\delta}_1$, there exists a positive constant $t_0=t_0(M)$ such that if $\|(\phi_0,\psi_0)\|_3+\|\zeta_0\|_4\le M(\overline{C}M\le\bar{\eta}_3)$, then the problem (3.2) has a unique solution $(\phi,\psi,\zeta)(t,\xi)\in\mathbb{X}_{\frac{m_1}{2},\frac{m_2}{2},\overline{C}M}(0,t_0)$.*

Proof. At first, similar to [14], we rewrite system (3.2) into the following form

$$\begin{cases}A^0(U,V)U_t+A^1(U,V)U_\xi=F_1(U,V,V_\xi),\\ V_t-(B(U,V)V_\xi)_\xi=F_2(U,V,U_\xi,V_\xi),\\ (U^T,V)(t,0)=(0,0,0),\quad (U^T,V)(0,\xi)=(\phi_0,\psi_0,\zeta_0)(\xi).\end{cases}\tag{3.5}$$

where $U=(\phi,\psi)^T, V=\zeta$, and

$$\begin{aligned}&A^0=\begin{pmatrix}\frac{R}{(\hat{v}+\phi)^2} & 0\\ 0 & \frac{1}{\hat{\theta}+\zeta}\end{pmatrix},\ A^1=\begin{pmatrix}\frac{-s_-R}{(\hat{v}+\phi)^2} & -\frac{R}{(\hat{v}+\phi)^2}\\ -\frac{R}{(\hat{v}+\phi)^2} & -\frac{s_-}{\hat{\theta}+\zeta}\end{pmatrix},\\ &F_1=\begin{pmatrix}0\\ f_1(U,V,V_\xi)\end{pmatrix},\ B=\left(\frac{\kappa}{C_v(\hat{v}+\phi)}\right),\end{aligned}\tag{3.6}$$

and

$$\begin{cases}f_1=\dfrac{\zeta}{\theta\widehat{\theta}}(\widehat{u}_t-s_-\widehat{u}_\xi)-\widehat{v}_\xi\dfrac{R(v+\widehat{v})\phi}{(v\widehat{v})^2}+\widehat{\theta}_\xi(\dfrac{Rv\zeta+R\widehat{\theta}\phi}{v\theta\widehat{v}\widehat{\theta}})-\dfrac{1}{\widehat{\theta}}G_1-\dfrac{R}{v\theta}\zeta_\xi,\\ F_2=s_-\zeta_\xi-\dfrac{1}{C_v}[p\psi_\xi+\hat{u}_\xi(p-\hat{p})-\kappa(\dfrac{\hat{\theta}_\xi\phi}{v\hat{v}})_\xi-G_2].\end{cases}\tag{3.7}$$

For $U^0=(\phi_0,\psi_0)^T, V^0=\zeta_0$, we define the iterated sequence $({U^k}^T,V^k)=(\phi^k,\psi^k,\zeta^k), k\ge 1$ as follows:

$$\begin{cases}A^0(U^{k-1},V^{k-1})U_t^k+A^1(U^{k-1},V^{k-1})U_\xi^k=F_1(U^{k-1},V^{k-1},V_\xi^{k-1}),\\ {U^k}^T(t,0)=(0,0),\quad {U^k}^T(0,\xi)=(\phi_0,\psi_0)(\xi),\end{cases}\tag{3.8}$$

and

$$\begin{cases}V_t^k-(B(U^{k-1},V^{k-1})V_\xi^k)_\xi=F_2(U^{k-1},V^{k-1},U_\xi^{k-1},V_\xi^{k-1}),\\ V^k(t,0)=0,\quad V^k(0,\xi)=\zeta_0(\xi).\end{cases}\tag{3.9}$$

Following the standard steps in [32], for each k, we could show that for the linear hyperbolic problem (3.8) there exists a unique solution U^k such that it satisfies

$$\begin{aligned}&||U^k(t)||_2^2+||U_t^k(t)||_1^2+||U_{tt}^k(t)||^2\\ &\le Ce^{CMt}\{\Sigma_{i=0}^2\|\partial_t^iU^k(0)\|_{2-i}^2+t\Sigma_{i=0}^2\int_0^t||\partial_t^if_1(\tau)||_{2-i}^2d\tau\},\end{aligned}\tag{3.10}$$

and for linear parabolic problem (3.9), by the standard energy estimates, there exists a unique solution V^k for each k such that it satisfies

$$||V^k(t)||_2^2+||V_t^k(t)||_1^2+||V_{tt}^k(t)||^2+\int_0^t(\|V_\xi^k(\tau)\|_2^2+\|V_t^k(\tau)\|_2^2+\|V_{tt}^k(\tau)\|_1^2)d\tau$$

$$\leq Ce^{CMt}\{\Sigma_{i=0}^{2}\|\partial_t^i V^k(0)\|_{2-i}^2+Mt\}. \tag{3.11}$$

Here from system (3.5), we see that

$$\begin{aligned}&\Sigma_{i=0}^{2}\|\partial_t^i U^k(0)\|_{2-i}^2+\Sigma_{i=0}^{2}\|\partial_t^i V^k(0)\|_{2-i}^2\\ \lesssim&\Sigma_{j=0}^{2}\|(\phi_j,\psi_j,\zeta_j)\|^2\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2.\end{aligned} \tag{3.12}$$

That's why we order that our initial data (ϕ_0,ψ_0) belong to H^3 and ζ_0 belongs to H^4. Combining results (3.10)-(3.11), choosing M and $t(\leq t_0)$ suitably small, we can show that the iterated sequence $\{(U^{k^T},V^k),k\geq 0\}=\{(\phi^k,\psi^k,\zeta^k),k\geq 0\}$ is uniformly bounded in $\mathbb{X}_{\frac{m_1}{2},\frac{m_2}{2},\overline{C}M}(0,t_0)$. Moreover, $\{(U^{k^T},V^k),k\geq 0\}$ is a Cauchy sequence in $C([0,t_0];H^1)\cap C^1([0,t_0];L^2)\times C([0,t_0];H^2)\cap C^1([0,t_0];L^2)\cap L^2([0,t_0];H^3)$ and $\lim_{k\to+\infty}(\phi^k,\psi^k,\zeta^k)=(\phi,\psi,\zeta)$. Finally, (ϕ,ψ,ζ) is the unique solution which belongs to $\mathbb{X}_{\frac{m_1}{2},\frac{m_2}{2},\overline{C}M}(0,t_0)$. Thus the local solution has been constructed. □

Suppose that $(\phi,\psi,\zeta)(t,\xi)$ obtained in Proposition 3.1 has been extended to some time $T>t$, we want to obtain the following priori estimates to get a global solution.

PROPOSITION 3.2 (A priori estimates). *Under the conditions listed in Theorem 2.3, $(\phi,\psi,\zeta)(t,\xi)\in\mathbb{X}_{\frac{m_1}{2},\frac{m_2}{2},\eta_3}(0,T)(\eta_3\leq\bar{\eta}_3)$ obtained in Proposition 3.1 is the solution of the problem (3.2) which has been extended to some $T>0$, then it holds that for $t\in[0,T]$,*

$$\begin{aligned}&||(\phi,\psi,\zeta)(t)||_2^2+||(\phi_t,\psi_t,\zeta_t)(t)||_1^2+||(\phi_{tt},\psi_{tt},\zeta_{tt})(t)||^2\\ &\quad+\int_0^t\big(\|(\phi_\xi,\psi_\xi,\zeta_\xi,\zeta_{\xi\xi})(\tau)\|_1^2+\|(\phi_t,\psi_t,\zeta_t,\zeta_{tt})(\tau)\|_1^2\big)d\tau\\ &\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar{\delta}+\epsilon^{\frac{1}{8}}.\end{aligned} \tag{3.13}$$

Once Proposition 3.2 is proved, we can extend the unique local solution $(\phi,\psi,\zeta)(t,\xi)$ obtained in Proposition 3.1 to $t=\infty$, moreover, estimate (3.13) implies that

$$\int_0^\infty\big(\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2+\big|\frac{d}{dt}\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2\big|\big)d\tau<+\infty, \tag{3.14}$$

which together with Sobolev inequality easily leads to the asymptotic behavior (2.47), this concludes the proof of Theorem 2.3. In the rest of this section, our main task is to show the priori estimates.

4. A priori estimates

In the following part of this section, we mainly prove Proposition 3.2.

4.1. Basic energy estimates. At first, we show the basic energy estimates.

LEMMA 4.1. *Under the same assumptions listed in Proposition 3.2, if $\bar{\delta},\epsilon,N(t)$ are suitably small, it holds that*

$$\begin{aligned}&\|(\phi,\psi,\zeta)(t)\|^2+\int_0^t(\|\sqrt{\tilde{u}_\xi}(\phi,\zeta)(\tau)\|^2+\|\zeta_\xi(\tau)\|^2)d\tau\\ \lesssim&\|(\phi_0,\psi_0,\zeta_0)\|^2+(\bar{\delta}+\epsilon^{\frac{1}{8}})\big(\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|^2d\tau+1\big).\end{aligned} \tag{4.1}$$

Proof. Define the energy form

$$E = R\hat{\theta}\Phi(\frac{v}{\hat{v}}) + \frac{\psi^2}{2} + C_v\hat{\theta}\Phi(\frac{\theta}{\hat{\theta}}), \tag{4.2}$$

where $\Phi(s) = s - 1 - \ln s$. Obviously, there exists a positive constant C(s) such that

$$C(s)^{-1}s^2 \le \Phi(s) \le C(s)s^2,$$

we can get the following estimate

$$\begin{aligned} &E_t - s_- E_\xi + \kappa\frac{\hat{\theta}\zeta_\xi^2}{v\theta^2} + \hat{p}\tilde{u}_\xi(\Phi(\frac{\theta\hat{v}}{v\hat{\theta}}) + \gamma\Phi(\frac{v}{\hat{v}})) \\ &+ \{(p-\hat{p})\psi - \kappa(\frac{\zeta_\xi}{v} - \frac{\hat{\theta}_\xi\phi}{v\hat{v}})\frac{\zeta}{\theta}\}_\xi = G_3 - G_1\psi - G_2\frac{\zeta}{\theta}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} G_3 = &-\hat{p}\bar{u}_\xi(\Phi(\frac{\theta\hat{v}}{v\hat{\theta}}) + \gamma\Phi(\frac{v}{\hat{v}})) + [\kappa(\frac{\hat{\theta}_\xi}{\hat{v}})_\xi + G_2][(\gamma-1)\Phi(\frac{v}{\hat{v}}) + \Phi(\frac{\theta}{\hat{\theta}}) - \frac{\zeta^2}{\theta\hat{\theta}}] \\ &+ \kappa\frac{\hat{\theta}_\xi\zeta_\xi\zeta}{v\theta^2} + \kappa(\frac{1}{v} - \frac{1}{\hat{v}})\frac{\zeta\hat{\theta}_\xi^2}{\theta^2} - \kappa(\frac{1}{v} - \frac{1}{\hat{v}})\frac{\hat{\theta}\hat{\theta}_\xi\zeta_\xi}{\theta^2}. \end{aligned} \tag{4.4}$$

It is easy to see that

$$|G_3| \lesssim \frac{1}{4}\frac{\kappa\hat{\theta}\zeta_\xi^2}{v\theta^2} + |(\bar{u}_\xi, (\bar{\theta} - \theta_m)\tilde{\theta}_\xi + \bar{\theta}_\xi(\tilde{\theta} - \theta_m), \tilde{\theta}_{\xi\xi} + \tilde{\theta}_\xi^2)|(\phi^2 + \zeta^2), \tag{4.5}$$

Since

$$|f(\xi)| = |f(0) + \int_0^\xi f_y dy| \le |f(0)| + \sqrt{\xi}\|f_\xi\|, \tag{4.6}$$

and by the fact that $(\phi,\psi,\zeta)(t,0) = (0,0,0)$, we get

$$\begin{aligned} &\int_0^t \int_{R_+} |\bar{u}_\xi|(\phi^2 + \zeta^2) d\xi d\tau \\ \lesssim &\bar{\delta}\int_0^t \int_{\mathbb{R}_+} e^{-c\xi}(\phi^2 + \zeta^2)(t,\xi) d\xi d\tau \\ \lesssim &\bar{\delta}\int_0^t \|(\phi_\xi, \zeta_\xi)(\tau)\|^2 \int_{\mathbb{R}_+} \xi e^{-c\xi} d\xi d\tau \\ \lesssim &\bar{\delta}\int_0^t \|(\phi_\xi, \zeta_\xi)(\tau)\|^2 d\tau. \end{aligned} \tag{4.7}$$

By the properties of rarefaction wave as

$$\begin{aligned} &\|(\tilde{v}_\xi, \tilde{u}_\xi, \tilde{\theta}_\xi)(t)\|^2 \lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}, \\ &\|(\tilde{v}_{\xi\xi}, \tilde{u}_{\xi\xi}, \tilde{\theta}_{\xi\xi})(t)\|_{L^1} \lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{13}{16}}, \end{aligned} \tag{4.8}$$

we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}(|\tilde\theta_{\xi\xi}|+|\tilde\theta_\xi|^2)(\phi^2+\zeta^2)(\tau,\xi)d\xi d\tau\\
\lesssim&\int_0^t(\|\phi(\tau)\|\|\phi_\xi(\tau)\|+\|\zeta(\tau)\|\|\zeta_\xi(\tau)\|)(\|\tilde\theta_\xi(\tau)\|^2+\|\tilde\theta_{\xi\xi}(\tau)\|_{L^1})d\tau\\
\lesssim&\epsilon^{\frac18}\int_0^t(1+\tau)^{-\frac{13}{8}}\|(\phi,\zeta)(\tau)\|^2+\|(\phi_\xi,\zeta_\xi)(\tau)\|^2d\tau\\
\lesssim&\epsilon^{\frac18}\{1+\int_0^t\|(\phi_\xi,\zeta_\xi)(\tau)\|^2d\tau\}.
\end{aligned}\tag{4.9}$$

And the remaining terms satisfy

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}((\bar\theta-\theta_m)\tilde\theta_\xi+\bar\theta_\xi(\tilde\theta-\theta_m))(\phi^2+\zeta^2)d\xi d\tau\\
\lesssim&\bar\delta\int_0^t\int_{\mathbb{R}_+}e^{-c(\xi+\tau)}(\phi^2+\zeta^2)(\tau,\xi)d\xi d\tau\\
\lesssim&\bar\delta\int_0^t(\|\phi(\tau)\|\|\phi_\xi(\tau)\|+\|\zeta(\tau)\|\|\zeta_\xi(\tau)\|)e^{-c\tau}d\tau\\
\lesssim&\bar\delta\{1+\int_0^t\|(\phi_\xi,\zeta_\xi)(\tau)\|^2d\tau\}.
\end{aligned}\tag{4.10}$$

Integrating (4.3) over $[0,t]\times\mathbb{R}_+$ and making use of the estimates (4.5)-(4.10), it holds

$$\begin{aligned}
&\|(\phi,\psi,\zeta)(t)\|^2+\int_0^t(\|\zeta_\xi(\tau)\|^2+\|\sqrt{\tilde u_\xi}(\phi,\zeta)(\tau)\|^2)d\tau\\
\lesssim&\|(\phi_0,\psi_0,\zeta_0)\|^2+(\bar\delta+\epsilon^{\frac18})\{1+\int_0^t\|(\phi_\xi,\zeta_\xi)(\tau)\|^2d\tau\}+\int_0^t\int_{\mathbb{R}_+}\big(|G_1\psi|+|G_2\zeta|\big)d\xi d\tau,
\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}\big(|G_1\psi|+|G_2\zeta|\big)d\xi d\tau\\
\lesssim&\int_0^t\int_{\mathbb{R}_+}\bar\delta e^{-c(\xi+\tau)}(\|\psi\|^{\frac12}\|\psi_\xi\|^{\frac12}+\|\zeta\|^{\frac12}\|\zeta_\xi\|^{\frac12})d\xi d\tau\\
&+\int_0^t(\|\zeta\|^{\frac12}\|\zeta_\xi\|^{\frac12}(\|\tilde\theta_\xi\|^2+\|\tilde\theta_{\xi\xi}\|_{L^1})d\tau\\
\lesssim&\bar\delta\int_0^te^{-c\tau}\|(\psi,\zeta)(\tau)\|^{\frac23}+\|(\psi_\xi,\zeta_\xi)(\tau)\|^2d\tau\\
&+\int_0^t\|\zeta(\tau)\|^{\frac12}\|\zeta_\xi(\tau)\|^{\frac12}[\epsilon^{\frac18}(1+\tau)^{-\frac78}+\epsilon^{\frac18}(1+\tau)^{-\frac{13}{16}}]d\tau\\
\lesssim&\bar\delta\int_0^te^{-c\tau}(\|(\psi,\zeta)(\tau)\|^2+1)d\tau+(\bar\delta+\epsilon^{\frac18})\int_0^t\|(\psi_\xi,\zeta_\xi)(\tau)\|^2d\tau+\epsilon^{\frac18}\int_0^t(1+\tau)^{-\frac{13}{12}}\|\zeta(\tau)\|^{\frac23}d\tau\\
\lesssim&(\bar\delta+\epsilon^{\frac18})\{1+\int_0^t\|(\psi_\xi,\zeta_\xi)(\tau)\|^2d\tau\}.
\end{aligned}\tag{4.12}$$

Inserting (4.12) into (4.11), we can get the estimate (4.1) and complete the proof of Lemma 4.1. ∎

Lemma 4.2. *Under the same assumptions listed in Proposition 3.2, if $\bar{\delta},\epsilon,N(t)$ are suitably small, then it holds that*

$$\begin{aligned}&\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2+\int_0^t\big(|(\phi_\xi,\psi_\xi,\zeta_\xi)|^2(\tau,0)+\|\zeta_{\xi\xi}(\tau)\|^2\big)d\tau\\ \lesssim&\|(\phi_0,\psi_0,\zeta_0)\|_1^2+\bar{\delta}+\epsilon^{\frac{1}{8}}+(\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))\int_0^t\|(\phi_\xi,\psi_\xi)(\tau)\|_1^2d\tau.\end{aligned}\tag{4.13}$$

Proof. Multiplying $(3.2)_1$ by $-\frac{p}{v}\phi_{\xi\xi}$ and $(3.2)_2$ by $-\psi_{\xi\xi}$, $(3.2)_3$ by $-\frac{\zeta_{\xi\xi}}{\theta}$ and adding the results, we can get

$$\begin{aligned}&\frac{1}{2}(\frac{p}{v}\phi_\xi^2+\psi_\xi^2+C_v\frac{\zeta_\xi^2}{\theta})_t+\kappa\frac{\zeta_{\xi\xi}^2}{v\theta}+\frac{s_-}{2}(\frac{p}{v}\phi_\xi^2+\psi_\xi^2+C_v\frac{\zeta_\xi^2}{\theta})_\xi\\ &+(\frac{p}{v}\phi_\xi\psi_\xi-\frac{R}{v}\zeta_\xi\psi_\xi)_\xi-(\frac{p}{v}\phi_\xi\phi_t+\psi_\xi\psi_t+C_v\frac{\zeta_\xi}{\theta}\zeta_t)_\xi=F_3.\end{aligned}\tag{4.14}$$

where

$$\begin{aligned}F_3=&\frac{1}{2}[(\frac{p}{v})_t+s_-(\frac{p}{v})_\xi]\phi_\xi^2+\frac{C_v}{2}[(\frac{1}{\theta})_t+s_-(\frac{1}{\theta})_\xi]\zeta_\xi^2\\ &-(\frac{p}{v})_\xi\phi_t\phi_\xi-C_v(\frac{1}{\theta})_\xi\zeta_t\zeta_\xi+\hat{u}_\xi(p-\hat{p})\frac{\zeta_{\xi\xi}}{\theta}\\ &-\kappa(\frac{1}{v})_\xi\zeta_\xi\frac{\zeta_{\xi\xi}}{\theta}+\kappa(\frac{\hat{\theta}_\xi\phi}{v\hat{v}})_\xi\frac{\zeta_{\xi\xi}}{\theta}+G_1\psi_{\xi\xi}+G_2\frac{\zeta_{\xi\xi}}{\theta}\\ &+(\frac{p}{v})_\xi\phi_\xi\psi_\xi-(\frac{R}{v})_\xi\psi_\xi\zeta_\xi+[(\frac{p-\hat{p}}{v})\hat{v}_\xi-(\frac{\hat{p}_\xi}{v})\phi]\psi_{\xi\xi}.\end{aligned}\tag{4.15}$$

It is easy to see that

$$\begin{aligned}|F_3|\lesssim&(|\bar{v}_\xi|+|\tilde{v}_\xi|+N(T))(\phi_\xi^2+\psi_\xi^2+\zeta_\xi^2+\psi_{\xi\xi}^2+\zeta_{\xi\xi}^2)\\ &+(|\tilde{u}_\xi|+|\bar{u}_\xi|)(\phi^2+\zeta^2)+|\hat{v}_\xi G_2^2|+|G_2\zeta_{\xi\xi}|+|G_1\psi_{\xi\xi}|,\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}|\hat{v}_\xi|G_2^2d\xi d\tau\lesssim&\int_0^t\int_{\mathbb{R}_+}|(\tilde{v}_\xi+\bar{v}_\xi)|(\bar{\delta}e^{-c(\xi+t)}+|\tilde{\theta}_{\xi\xi}|^2+|\tilde{\theta}_\xi|^4)d\xi d\tau\\ \lesssim&\bar{\delta}+\int_0^t(\|\tilde{\theta}_{\xi\xi}\|_{L^\infty}^2+\|\tilde{\theta}_\xi\|_{L^\infty}^4)(\|\bar{v}_\xi\|_{L^1}+\|\tilde{v}_\xi\|_{L^1})d\tau\\ \lesssim&\bar{\delta}+\epsilon^{\frac{1}{8}}\int_0^t(1+\tau)^{-\frac{39}{28}}d\tau\\ \lesssim&\bar{\delta}+\epsilon^{\frac{1}{8}},\end{aligned}\tag{4.17}$$

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+}\big(|G_2\zeta_{\xi\xi}|+|G_1\psi_{\xi\xi}|\big)d\xi d\tau\\ \lesssim&\bar{\delta}\int_0^t\int_{\mathbb{R}_+}e^{-c(\xi+t)}(\zeta_{\xi\xi}+\psi_{\xi\xi})d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}(|\tilde{\theta}_{\xi\xi}|+|\tilde{\theta}_\xi|^2)\zeta_{\xi\xi}d\xi d\tau\\ \lesssim&\bar{\delta}(1+\int_0^t\|(\zeta_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2d\tau)+\int_0^t(\|\tilde{\theta}_{\xi\xi}\|+\|\tilde{\theta}_\xi\|_{L^4}^2)\|\zeta_{\xi\xi}(\tau)\|d\tau\end{aligned}$$

$$
\begin{aligned}
&\lesssim \bar{\delta}(1+\int_0^t \|(\zeta_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2 d\tau)+\int_0^t \epsilon^{\frac{1}{8}}(1+\tau)^{-\frac{3}{4}}\|\zeta_{\xi\xi}(\tau)\|d\tau \\
&\lesssim \bar{\delta}(1+\int_0^t \|(\zeta_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2 d\tau)+\epsilon^{\frac{1}{8}}\int_0^t [(1+\tau)^{-\frac{3}{2}}+\|\zeta_{\xi\xi}(\tau)\|^2]d\tau \\
&\lesssim (\bar{\delta}+\epsilon^{\frac{1}{8}})\big(1+\int_0^t \|(\zeta_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2 d\tau\big). \qquad (4.18)
\end{aligned}
$$

Integrating (4.14) over $[0,t]\times\mathbb{R}_+$, and making use of (4.1), (4.7), (4.16)-(4.18), we get

$$
\begin{aligned}
&\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2+\int_0^t \|\frac{\zeta_{\xi\xi}}{\sqrt{v\theta}}(\tau)\|^2 d\tau \\
&\quad +\int_0^t \Big\{\frac{|s_-|}{2}(\frac{p_-}{v_-}\phi_\xi^2+\psi_\xi^2+C_v\frac{\zeta_\xi^2}{\theta_-})-\frac{p_-}{v_-}\phi_\xi\psi_\xi+\frac{R}{v_-}\zeta_\xi\psi_\xi\Big\}(\tau,0)d\tau \\
&\lesssim \|(\phi_0,\psi_0,\zeta_0)\|_1^2+\bar{\delta}+\epsilon^{\frac{1}{8}}+(\epsilon^{\frac{1}{8}}+\bar{\delta}+N(t))\int_0^t \|(\phi_\xi,\psi_\xi)(\tau)\|_1^2 d\tau. \qquad (4.19)
\end{aligned}
$$

Then we should deal with the boundary terms. Since $z_-\in\Omega_+$, see (2.1), that is, $R\theta_-<u_-^2<\gamma R\theta_-$, the discriminant of the quadratic form

$$
\frac{|s_-|}{2}(\frac{p_-}{v_-}\phi_\xi^2+\psi_\xi^2)-\frac{p_-}{v_-}\phi_\xi\psi_\xi
$$

is less than zero, i.e.

$$
\begin{aligned}
D=&(\frac{p_-}{v_-})^2-4\times\frac{|s_-|}{2}\frac{p_-}{v_-}\times\frac{|s_-|}{2} \\
=&(\frac{p_-}{v_-})(\frac{p_-}{v_-}-|s_-|^2)=(\frac{p_-}{v_-})(\frac{R\theta_-}{v_-^2}-\frac{u_-^2}{v_-^2})<0. \qquad (4.20)
\end{aligned}
$$

Thus, the binomial expression is positive, we get for some constant $c_1>0$ such that

$$
\begin{aligned}
&\int_0^t \Big\{\frac{|s_-|}{2}(\frac{p_-}{v_-}\phi_\xi^2+\psi_\xi^2+C_v\frac{\zeta_\xi^2}{\theta_-})-\frac{p_-}{v_-}\phi_\xi\psi_\xi\Big\}(\tau,0)d\tau \\
\geq& c_1\int_0^t (\phi_\xi^2+\psi_\xi^2+\zeta_\xi^2)(\tau,0)d\tau. \qquad (4.21)
\end{aligned}
$$

Secondly, by the Sobolev inequality, it holds that

$$
\begin{aligned}
&\int_0^t \frac{R}{v_-}(\zeta_\xi\psi_\xi)(\tau,0)d\tau \\
\lesssim& \frac{c_1}{4}\int_0^t \psi_\xi^2(\tau,0)d\tau+\int_0^t \|\zeta_\xi(\tau)\|_\infty^2 d\tau \\
\lesssim& \frac{c_1}{4}\int_0^t \psi_\xi^2(\tau,0)d\tau+\frac{1}{4}\int_0^t \|\frac{\zeta_{\xi\xi}}{\sqrt{v\theta}}(\tau)\|^2 d\tau+\int_0^t \|\zeta_\xi(\tau)\|^2 d\tau. \qquad (4.22)
\end{aligned}
$$

Inserting (4.21), (4.22) into (4.19) and using the result of (4.1), we get the estimate of (4.13) and complete the proof of Lemma 4.2. ∎

As for $\int_0^t \|(\phi_\xi,\psi_\xi)(\tau)\|^2 d\tau$, we have the following Lemma.

Lemma 4.3. *Under the same assumptions listed in Proposition 3.2, if $\bar{\delta},\epsilon,N(t)$ are suitably small, it holds that*

$$\begin{aligned}&\int_0^t \|(\phi_\xi,\psi_\xi)(\tau)\|^2 d\tau\\ \lesssim&\|(\phi_0,\psi_0,\zeta_0)\|_1^2+\bar{\delta}+\epsilon^{\frac{1}{8}}+(\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))\int_0^t \|(\phi_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2 d\tau.\end{aligned} \tag{4.23}$$

Proof. Multiplying $(3.2)_2$ by $-\frac{p}{2}\phi_\xi$, it holds that

$$\begin{aligned}&-(\frac{p}{2}\phi_\xi\psi)_t+(\frac{p}{2})_t\psi\phi_\xi+(\frac{p}{2}\phi_t\psi)_\xi-\frac{p_\xi}{2}\psi(s_-\phi_\xi+\psi_\xi)-\frac{p}{2}\psi_\xi^2\\ &+\frac{p^2}{2v}\phi_\xi^2-\frac{R}{v}\zeta_\xi\frac{p}{2}\phi_\xi=(G_1-\frac{\hat{p}_\xi\phi}{v}+\frac{(\hat{p}-p)}{v}\hat{v}_\xi)\frac{p}{2}\phi_\xi.\end{aligned} \tag{4.24}$$

Multiplying $(3.2)_3$ by ψ_ξ, it holds that

$$\begin{aligned}&(C_v\zeta\psi_\xi)_t-(C_v\zeta\psi_t)_\xi-C_v\zeta_\xi((p-\hat{p})_\xi+G_1)+p\psi_\xi^2\\ =&\kappa(\frac{\zeta_\xi}{v}-\frac{\hat{\theta}_\xi\phi}{\hat{v}v})_\xi\psi_\xi-(p-\hat{p})\hat{u}_\xi\psi_\xi-G_2\psi_\xi.\end{aligned} \tag{4.25}$$

Combining together, we get

$$\begin{aligned}&(C_v\zeta\psi_\xi-\frac{p}{2}\phi_\xi\psi)_t-(C_v\zeta\psi_t-\frac{p}{2}\phi_t\psi)_\xi+\frac{p}{2}\psi_\xi^2+\frac{p^2}{2v}\phi_\xi^2\\ =&\frac{p_\xi}{2}\psi(s_-\phi_\xi+\psi_\xi)-(\frac{p}{2})_t\psi\phi_\xi+\frac{R}{v}\zeta_\xi\frac{p}{2}\phi_\xi+C_v\zeta_\xi((p-\hat{p})_\xi+G_1)\\ &+(G_1-\frac{\hat{p}_\xi\phi}{v}+\frac{(\hat{p}-p)}{v}\hat{v}_\xi)\frac{p}{2}\phi_\xi+\kappa(\frac{\zeta_\xi}{v}-\frac{\hat{\theta}_\xi\phi}{\hat{v}v})_\xi\psi_\xi-(p-\hat{p})\hat{u}_\xi\psi_\xi-G_2\psi_\xi\\ \lesssim&O(1)|\hat{u}_\xi||(\phi,\psi,\xi)||(\phi_\xi,\psi_\xi,\zeta_\xi)|+O(1)(\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))|(\phi_\xi^2,\psi_\xi^2,\zeta_\xi^2)|\\ &+|\phi_\xi\zeta_\xi|+|\psi_\xi\zeta_{\xi\xi}|+|G_1\phi_\xi|+|G_2\psi_\xi|+|G_1\zeta_\xi|+|\zeta_\xi|^2.\end{aligned} \tag{4.26}$$

Integrating above equation over $[0,t]\times\mathbb{R}_+$, we obtain that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+}(\phi_\xi^2+\psi_\xi^2)d\xi d\tau\\ \lesssim&\|(\phi_\xi,\psi_\xi,\psi,\zeta)(t)\|^2+\|(\phi_{0\xi},\psi_{0\xi},\psi_0,\zeta_0)(t)\|^2+\bar{\delta}+\epsilon^{\frac{1}{8}}\\ &+\int_0^t\|(\zeta_\xi,\zeta_{\xi\xi})(\tau)\|^2 d\tau+\int_0^t\int_{\mathbb{R}_+}|\hat{u}_\xi||(\phi,\xi)|^2 d\xi dt\\ &+(\frac{1}{4}+\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))\int_0^t\int_{\mathbb{R}_+}(\phi_\xi^2+\psi_\xi^2)d\xi d\tau.\end{aligned} \tag{4.27}$$

Using the result of (4.1), (4.7) and (4.13) into (4.27), if $\bar{\delta},\epsilon,N(t)$ are suitably small, we could get (4.23) and complete the proof of Lemma 4.3. ◻

4.2. Higher order energy estimates. To get the higher order estimates, we need to get the estimates on the diameter direction as follows.

LEMMA 4.4. *Under the same assumptions listed in Proposition 3.2, if $\bar{\delta},\epsilon,N(t)$ are suitably small, it holds that*

$$\|(\phi_t,\psi_t,\zeta_t)(t)\|^2+\int_0^t\|\zeta_{t\xi}(\tau)\|^2d\tau$$
$$\lesssim\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\bar{\delta}+\epsilon^{\frac{1}{8}}+(\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))\int_0^t\|(\phi_{t\xi},\psi_{t\xi})(\tau)\|^2d\tau. \tag{4.28}$$

Proof. Let $(3.2)_{1t}\times\frac{p}{v}\phi_t,(3.2)_{2t}\times\psi_t,(3.2)_{3t}\times\frac{\zeta_t}{\theta}$, we get that

$$\frac{1}{2}(\frac{p}{v}\phi_t^2+\psi_t^2+C_v\frac{\zeta_t^2}{\theta})_t-\frac{s_-}{2}(\frac{p}{v}\phi_t^2+\psi_t^2+C_v\frac{\zeta_t^2}{\theta})_\xi$$
$$-\{\frac{p}{v}\phi_t\psi_t-\frac{R}{v}\zeta_t\psi_t+\kappa((\frac{\zeta_\xi}{v}-\frac{\theta_\xi\phi}{v\hat{v}})_t\frac{\zeta_t}{\theta})\}_\xi+\kappa\frac{\zeta_{t\xi}^2}{v\theta}=F_4, \tag{4.29}$$

where

$$|F_4|\lesssim(|\tilde{v}_\xi|+|\bar{v}_\xi|+N(t))(\phi_t^2+\phi_\xi^2+\psi_t^2+\psi_\xi^2+\zeta_\xi^2+\zeta_t^2+\phi_{t\xi}^2+\psi_{t\xi}^2+\zeta_{t\xi}^2)$$
$$+|G_{1t}\psi_t|+|G_{2t}\zeta_t|+|\hat{v}_\xi|(\phi^2+\zeta^2)$$
$$\lesssim(|\tilde{v}_\xi|+|\bar{v}_\xi|+N(t))(\phi_\xi^2+\psi_\xi^2+\zeta_\xi^2+\zeta_{\xi\xi}^2+\phi_{t\xi}^2+\psi_{t\xi}^2+\zeta_{t\xi}^2)$$
$$+|G_{1t}\psi_t|+|G_{2t}\zeta_t|+|\hat{v}_\xi|(\phi^2+\zeta^2), \tag{4.30}$$

Integrating (4.29) over $[0,t]\times\mathbb{R}_+$, and noticing that $(\phi_t,\psi_t,\zeta_t)(t,0)=(0,0,0)$, by the results of Lemma 4.1-Lemma 4.3 and similar computations as before, we obtain

$$\|(\phi_t,\psi_t,\zeta_t)(t)\|^2+\int_0^t\|\zeta_{t\xi}(\tau)\|^2d\tau$$
$$\lesssim\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\bar{\delta}+\epsilon^{\frac{1}{8}}+(\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))\int_0^t\|(\phi_{t\xi},\psi_{t\xi},\phi_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2d\tau. \tag{4.31}$$

Taking the derivative of both sides of $(3.2)_1$ and $(3.2)_2$ with the variable ξ, we see that

$$\|(\phi_{\xi\xi},\psi_{\xi\xi})(t)\|^2$$
$$\lesssim\|(\phi_{t\xi},\psi_{t\xi},\zeta_{\xi\xi})(t)\|^2+\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2+\|\sqrt{|\hat{u}_\xi|}(\phi,\zeta)(t)\|^2+\|G_{1\xi}(t)\|^2. \tag{4.32}$$

Using (4.32), we can get (4.28) and complete the proof of Lemma 4.4. ∎

LEMMA 4.5. *Under the same assumptions listed in Proposition 3.2, if $\bar{\delta},\epsilon,N(t)$ are suitably small, it holds that*

$$\|(\phi_{tt},\psi_{tt},\zeta_{tt})(t)\|^2+\int_0^t\|\zeta_{tt\xi}(\tau)\|^2d\tau$$
$$\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar{\delta}+\epsilon^{\frac{1}{8}}+(\bar{\delta}+\epsilon^{\frac{1}{8}}+N(t))\int_0^t\|(\phi_{t\xi},\psi_{t\xi},\zeta_{tt})(\tau)\|^2d\tau. \tag{4.33}$$

Proof. Just let $(3.2)_{1tt}\times\frac{p}{v}\phi_{tt},(3.2)_{2tt}\times\psi_{tt},(3.2)_{3tt}\times\frac{\zeta_{tt}}{\theta}$, we get that

$$\frac{1}{2}(\frac{p}{v}\phi_{tt}^2+\psi_{tt}^2+C_v\frac{\zeta_{tt}^2}{\theta})_t-\frac{s_-}{2}(\frac{p}{v}\phi_{tt}^2+\psi_{tt}^2+C_v\frac{\zeta_{tt}^2}{\theta})_\xi$$

$$-\{(\frac{p}{v}\phi_{tt}\psi_{tt}-\frac{R}{v}\zeta_{tt}\psi_{tt}+\kappa((\frac{\zeta_\xi}{v}-\frac{\hat\theta_\xi\phi}{v\hat v})_{tt}\frac{\zeta_{tt}}{\theta})\}_\xi+\kappa\frac{\zeta_{tt\xi}^2}{v\theta}=F_5, \tag{4.34}$$

where

$$\begin{aligned}F_5=&(\frac{R}{v})_\xi\psi_{tt}\zeta_{tt}-(\frac{p}{v})_\xi\phi_{tt}\psi_{tt}+(\frac{1}{2}(\frac{p}{v})_t-\frac{s_-}{2}(\frac{p}{v})_\xi)\phi_{tt}^2\\&-[(\frac{R}{v})_{tt}\zeta_\xi+2(\frac{R}{v})_t\zeta_{t\xi}-(\frac{p}{v})_{tt}\phi_\xi-2(\frac{p}{v})_t\phi_{t\xi}-G_{1tt}+(\frac{\hat p_\xi\phi}{v})_{tt}]\psi_{tt}\\&+\frac{C_v}{2}[(\frac{1}{\theta})_t-s_-(\frac{1}{\theta})_\xi]\zeta_{tt}^2-[p_{tt}\psi_\xi+2p_t\psi_{t\xi}+(\hat u_\xi(p-\hat p))_{tt}-G_{2tt}]\frac{\zeta_{tt}}{\theta}\\&+\kappa(\frac{\hat\theta_\xi\phi}{v\hat v})_{tt}(\frac{\zeta_{tt}}{\theta})_\xi+\kappa(2\frac{\zeta_{t\xi}v_t}{v^2}-(\frac{1}{v})_{tt}\zeta_\xi)(\frac{\zeta_{tt}}{\theta})_\xi+\kappa\frac{\zeta_{tt\xi}}{v}\frac{\zeta_{tt}\theta_\xi}{\theta^2}.\end{aligned} \tag{4.35}$$

Note that

$$\begin{aligned}|F_5|\lesssim&(|\tilde v_\xi|+|\bar v_\xi|+N(t))(\phi_{tt}^2+\psi_{tt}^2+\zeta_{tt}^2+\zeta_{t\xi}^2+\phi_\xi^2+\psi_\xi^2+\zeta_\xi^2\\&+\phi_{\xi\xi}^2+\psi_{\xi\xi}^2+\zeta_{\xi\xi}^2+\zeta_{tt\xi}^2)+|G_{1tt}\psi_{tt}|+|G_{2tt}\zeta_{tt}|\\&+(|\tilde v_\xi|+|\bar v_\xi|)(\phi^2+\zeta^2).\end{aligned} \tag{4.36}$$

Integrating (4.34) over $[0,t]\times\mathbb{R}_+$ and noticing that $(\phi_{tt},\psi_{tt},\zeta_{tt})(t,0)=(0,0,0)$, if $\bar\delta,\epsilon,N(t)$ suitably small, it yields,

$$\begin{aligned}&\|(\phi_{tt},\psi_{tt},\zeta_{tt})(t)\|^2+\int_0^t\|\zeta_{tt\xi}(\tau)\|^2d\tau\\&\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar\delta+\epsilon^{\frac18}\\&\quad+(\bar\delta+\epsilon^{\frac18}+N(t))\int_0^t\|(\phi_{tt},\psi_{tt},\zeta_{tt},\phi_{t\xi},\psi_{t\xi},\phi_{\xi\xi},\psi_{\xi\xi})(\tau)\|^2d\tau\\&\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar\delta+\epsilon^{\frac18}+(\bar\delta+\epsilon^{\frac18}+N(t))\int_0^t\|(\zeta_{tt},\phi_{t\xi},\psi_{t\xi})(\tau)\|^2d\tau.\end{aligned} \tag{4.37}$$

where we have used (4.32). Taking the derivative of both sides of $(3.2)_1$ and $(3.2)_2$ with the variable t,

$$\begin{aligned}&\|(\phi_{tt},\psi_{tt})(t)\|^2\\&\lesssim\|(\phi_{t\xi},\psi_{t\xi},\zeta_{t\xi})(t)\|^2+\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|_1^2+\|\sqrt{|\hat u_\xi|}(\phi,\zeta)(t)\|^2+\|G_{1t}(t)\|^2.\end{aligned} \tag{4.38}$$

Using (4.38), we obtain (4.37) and complete the proof of Lemma 4.5. □

LEMMA 4.6. *Under the same assumptions listed in Proposition 3.2, if $\bar\delta,\epsilon,N(t)$ are suitably small, it holds that*

$$\begin{aligned}&\|\zeta_{t\xi}(t)\|^2+\int_0^t\|\zeta_{tt}(\tau)\|^2d\tau\\&\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar\delta+\epsilon^{\frac18}+(\bar\delta+\epsilon^{\frac18}+N(t))\int_0^t\|(\phi_{t\xi},\psi_{t\xi})(\tau)\|^2d\tau.\end{aligned} \tag{4.39}$$

Proof. Let $(3.2)_{3t}\times\zeta_{tt}$, we have

$$C_v\zeta_{tt}^2+\frac{\kappa}{2}(\frac{\zeta_{t\xi}^2}{v})_t-s_-C_v(\zeta_{t\xi}\zeta_t)_\xi+s_-C_v\zeta_{tt\xi}\zeta_t+(p\psi_t\zeta_{tt})_\xi$$

$$
\begin{aligned}
=&\kappa((\frac{\zeta_\xi}{v}-\frac{\hat{\theta}_\xi\phi}{v\hat{v}})_t\zeta_{tt})_\xi+p_\xi\psi_t\zeta_{tt}-p_t\psi_\xi\zeta_{tt}+p\psi_t\zeta_{tt\xi}-G_{2t}\zeta_{tt}\\
&-(\hat{u}_\xi(p-\hat{p}))_t\zeta_{tt}+\kappa\frac{v_t}{v^2}\zeta_\xi\zeta_{tt\xi}+\kappa(\frac{\hat{\theta}_\xi\phi}{v\hat{v}})_t\zeta_{tt\xi}-\frac{\kappa}{2v^2}v_t\zeta_{t\xi}^2.
\end{aligned}
\tag{4.40}
$$

Integrating (4.40) over $[0,t]\times\mathbb{R}_+$ and noticing that $(\phi_t,\psi_t,\zeta_t,\zeta_{tt})(t,0)=(0,0,0,0)$. Using previous lemmas, we could get (4.39) and complete the proof of Lemma 4.6. □

We remark that by using the relationship (4.32), (4.38) which is derived from system (3.2) and

$$
\begin{aligned}
&\|(\phi_{t\xi},\psi_{t\xi})(t)\|^2\\
\lesssim&\|(\phi_{tt},\psi_{tt},\zeta_{t\xi},\zeta_t)(t)\|^2+\|(\phi_\xi,\psi_\xi,\zeta_\xi)(t)\|^2+\|\sqrt{|\hat{u}_\xi|}(\phi,\zeta)(t)\|^2+\|G_{1t}(t)\|^2,\\
&\|\zeta_{\xi\xi}(t)\|^2\lesssim\|(\phi,\psi,\zeta)(t)\|_1^2+\|\zeta_t(t)\|^2+\|G_2(t)\|^2,
\end{aligned}
\tag{4.41}
$$

then from the results of Lemma 4.1-Lemma 4.6, one can verify that

$$
\begin{aligned}
&||(\phi,\psi,\zeta)||_2^2+||(\phi_t,\psi_t,\zeta_t)||_1^2+||(\phi_{tt},\psi_{tt},\zeta_{tt})||^2+\int_0^t|(\phi_\xi,\psi_\xi)|^2(\tau,0)d\tau\\
&+\int_0^t\left(\|(\phi_\xi,\psi_\xi)(\tau)\|^2+\|\zeta_\xi(\tau)\|_1^2+\|\zeta_t(\tau)\|_1^2+\|\zeta_{tt}(\tau)\|_1^2\right)d\tau\\
\lesssim&\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar{\delta}+\epsilon^{\frac18}+(\bar{\delta}+\epsilon^{\frac18}+N(t))\int_0^t\|(\phi_{t\xi},\psi_{t\xi})(\tau)\|^2.
\end{aligned}
\tag{4.42}
$$

Therefore, we just need to estimate $\int_0^t\|(\phi_{t\xi},\psi_{t\xi})(\tau)\|^2d\tau$ at last.

LEMMA 4.7. *Under the same assumptions listed in Proposition 3.2, if $\bar{\delta},\epsilon,N(t)$ are suitably small, it holds that*

$$
\int_0^t\|(\phi_{t\xi},\psi_{t\xi})(\tau)\|^2d\tau\lesssim\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar{\delta}+\epsilon^{\frac18}
\tag{4.43}
$$

Proof. Multiplying $(3.2)_{2t}$ by $-\frac{p}{2}\phi_{t\xi}$, it holds that

$$
\begin{aligned}
&-(\frac{p}{2}\phi_t\psi_{tt})_\xi+(\frac{p}{2})_\xi\psi_{tt}\phi_t+(\frac{p}{2}\phi_t\psi_{t\xi})_t-(\frac{p}{2})_t\psi_{t\xi}\phi_t-\frac{p}{2}\psi_{t\xi}(\phi_{tt}-s_-\phi_{t\xi})\\
&-(\frac{R}{v}\zeta_\xi)_t\frac{p}{2}\phi_{t\xi}+\frac{p^2}{2v}\phi_{t\xi}^2=(G_{1t}-(\frac{p}{v})_t\phi_\xi-(\frac{\hat{p}_\xi\phi}{v})_t)\frac{p}{2}\phi_{t\xi}.
\end{aligned}
\tag{4.44}
$$

Multiplying $(3.2)_{3t}$ by $\psi_{t\xi}$, it holds that

$$
\begin{aligned}
&(C_v\zeta_{tt}\psi_t)_\xi-(C_v\zeta_{t\xi}\psi_t)_t+C_v\zeta_{t\xi}(\psi_{tt}-s_-\psi_{t\xi})+p\psi_{t\xi}^2\\
=&\kappa(\frac{\zeta_\xi}{v}-\frac{\hat{\theta}_\xi\phi}{\hat{v}v})_{t\xi}\psi_{t\xi}-p_t\psi_\xi\psi_{t\xi}-((p-\hat{p})\hat{u}_\xi)_t\psi_{t\xi}-G_{2t}\psi_{t\xi}.
\end{aligned}
\tag{4.45}
$$

Combining (4.44)-(4.45) together, we get

$$
\begin{aligned}
&(\frac{p}{2}\phi_t\psi_{t\xi}-C_v\zeta_{t\xi}\psi_t)_t+(C_v\zeta_{tt}\psi_t-\frac{p}{2}\phi_t\psi_{tt})_\xi+\frac{p^2}{2v}\phi_{t\xi}^2+\frac{p}{2}\psi_{t\xi}^2\\
=&(\frac{R}{v}\zeta_\xi)_t\frac{p}{2}\phi_{t\xi}+(G_{1t}-(\frac{p}{v})_t\phi_\xi-(\frac{\hat{p}_\xi\phi}{v})_t)\frac{p}{2}\phi_{t\xi}-C_v\zeta_{t\xi}(\psi_{tt}-s_-\psi_{t\xi})
\end{aligned}
$$

$$
\begin{aligned}
&+\kappa(\frac{\zeta_\xi}{v}-\frac{\hat\theta_\xi\phi}{\hat v v})_{t\xi}\psi_{t\xi}-p_t\psi_\xi\psi_{t\xi}+(\frac{p}{2})_t\psi_{t\xi}\phi_t-(\frac{p}{2})_\xi\psi_{tt}\phi_t\\
&-((p-\hat p)\hat u_\xi)_t\psi_{t\xi}-G_{2t}\psi_{t\xi}\\
\lesssim&\frac{p^2}{9v}\phi_{t\xi}^2+\frac{p}{9}\psi_{t\xi}^2+O(1)|\hat u_\xi||(\phi,\xi)|^2+O(1)(\bar\delta+\epsilon^{\frac18}+N(t))|(\phi_\xi^2,\psi_\xi^2,\zeta_\xi^2,\zeta_{\xi\xi}^2)|\\
&+|G_{1t}(\phi_{t\xi},\zeta_{t\xi})|+|G_{2t}\psi_{t\xi}|+|G_{1t}(\phi_\xi,\psi_\xi)|+|\zeta_{t\xi}|^2+\kappa(\frac{\zeta_\xi}{v})_{t\xi}\psi_{t\xi}.
\end{aligned}\tag{4.46}
$$

Differentiating both sides of $(3.2)_{1,2}$ with respect to time t, we deduce that

$$
\begin{aligned}
\phi_{t\xi}&=\frac{1}{s_-}(\phi_{tt}-\psi_{t\xi}),\\
\psi_{t\xi}&=\frac{1}{s_-}(\psi_{tt}+(\frac{R}{v}\zeta_\xi)_t-(\frac{p}{v})_t\phi_\xi-\frac{p}{v}\phi_{t\xi}+(\frac{\hat p-p}{v}\hat v_\xi-\frac{\hat p_\xi\phi}{v})_t+G_{1t}).
\end{aligned}\tag{4.47}
$$

Using (4.47), it yields

$$
\begin{aligned}
\kappa(\frac{\zeta_\xi}{v})_{t\xi}\psi_{t\xi}=&\kappa(\frac{\zeta_\xi}{v})_{t\xi}\frac{s_-}{s_-^2-\frac{p}{v}}(\psi_{tt}+\frac{R}{v}\zeta_{t\xi}-\frac{p}{v}\frac{1}{s_-}\phi_{tt})\\
&+\kappa(\frac{\zeta_\xi}{v})_{t\xi}\frac{s_-}{s_-^2-\frac{p}{v}}[G_{1t}+(\frac{R}{v})_t\zeta_\xi-(\frac{p}{v})_t\phi_\xi+(\frac{\hat p-p}{v}\hat v_\xi-\frac{\hat p_\xi\phi}{v})_t]\\
=&\kappa[(\frac{\zeta_\xi}{v})_t\frac{s_-}{s_-^2-\frac{p}{v}}(\psi_{tt}-\frac{p}{v}\frac{1}{s_-}\phi_{tt})]_\xi-\kappa(\frac{\zeta_\xi}{v})_t(\frac{s_-}{s_-^2-\frac{p}{v}})_\xi(\psi_{tt}-\frac{p}{v}\frac{1}{s_-}\phi_{tt})\\
&+\kappa(\frac{\zeta_\xi}{v})_t\frac{1}{s_-^2-\frac{p}{v}}(\frac{p}{v})_\xi\phi_{tt}-\kappa[(\frac{\zeta_\xi}{v})_t\frac{s_-}{s_-^2-\frac{p}{v}}(\psi_{t\xi}-\frac{p}{v}\frac{1}{s_-}\phi_{t\xi})]_t\\
&+\kappa((\frac{\zeta_\xi}{v})_t\frac{s_-}{s_-^2-\frac{p}{v}})_t\psi_{t\xi}-\kappa((\frac{\zeta_\xi}{v})_t\frac{1}{s_-^2-\frac{p}{v}}\frac{p}{v})_t\phi_{t\xi}\\
&+\kappa[(\frac{\zeta_\xi}{v})_\xi\frac{s_-}{s_-^2-\frac{p}{v}}\frac{R}{v}\zeta_{t\xi}]_t-\kappa(\frac{\zeta_\xi}{v})_\xi(\frac{s_-}{s_-^2-\frac{p}{v}}\frac{R}{v}\zeta_{t\xi})_t\\
&+\{\kappa(\frac{\zeta_\xi}{v})_\xi\frac{s_-}{s_-^2-\frac{p}{v}}[G_{1t}+(\frac{R}{v})_t\zeta_\xi-(\frac{p}{v})_t\phi_\xi+(\frac{\hat p-p}{v}\hat v_\xi-\frac{\hat p_\xi\phi}{v})_t]\}_t\\
&-\kappa(\frac{\zeta_\xi}{v})_\xi(\frac{s_-}{s_-^2-\frac{p}{v}}[G_{1t}+(\frac{R}{v})_t\zeta_\xi-(\frac{p}{v})_t\phi_\xi+(\frac{\hat p-p}{v}\hat v_\xi-\frac{\hat p_\xi\phi}{v})_t])_t.
\end{aligned}\tag{4.48}
$$

Therefore, using (4.42), the following estimate holds,

$$
\begin{aligned}
&|\int_0^t\int_{\mathbb{R}_+}\kappa(\frac{\zeta_\xi}{v})_{t\xi}\psi_{t\xi}d\xi d\tau|\\
\lesssim&\|(\phi_0,\psi_0)\|_3^2+\|\zeta_0\|_4^2+\bar\delta+\epsilon^{\frac18}+\int_0^t\int_{\mathbb{R}_+}\frac{p^2}{9v}\phi_{t\xi}^2+\frac{p}{9}\psi_{t\xi}^2d\xi d\tau
\end{aligned}\tag{4.49}
$$

Integrating Equation (4.46) over $[0,t]\times\mathbb{R}_+$ and using (4.42), (4.49), we can get (4.43) after similar computations. This completes the proof of Lemma 4.7. □

Combining the results of Lemma 4.1-Lemma 4.7, from system (3.2), all the terms in the priori estimates (3.13) could be obtained and we complete the proof of Proposition 3.2.

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