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STABILITY OF A COMPOSITE WAVE OF VISCOUS CONTACT WAVE AND RAREFACTION WAVES FOR RADIATIVE AND REACTIVE GAS WITHOUT VISCOSITY*

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Abstract. The Cauchy problem of the 1D compressible radiative and reactive gas without viscosity is studied in this paper. When the radiation effect is under consideration, the equations present high nonlinearity, together with the lack of viscosity, which result in many more difficulties. When the solution to the corresponding Riemann problem of the Euler equation consists of a contact discontinuity and rarefaction waves, we proved that there exists a unique global-in-time solution and which tends to the combination of a viscous contact wave and rarefaction waves asymptotically with small initial data. The proof is given by the elementary energy method.

Keywords. contact wave; rarefaction wave; radiative and reactive gas; non-viscous; nonlinear stability.

AMS subject classifications. 35Q35; 35B40.

1. Introduction

In this article, we investigate the Cauchy problem of a 1D compressible radiative and reactive gas without viscosity:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ \left(e+\frac{u^2}{2}\right)_t + (up)_x &= \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x + \lambda\varphi z, \\ z_t &= \left(\frac{dz_x}{v^2}\right)_x - \varphi z, \end{aligned} \tag{1.1}$$

in which, the unknowns are the specific volume $v=v(t,x)$, the velocity $u=u(t,x)$, the absolute temperature $\theta=\theta(t,x)$, and the mass fraction of the reactant $z=z(t,x)$. While the specific internal energy e and the pressure p are the functions of v and θ. The constants $d>0$ and $\lambda>0$ are the species diffusion and the heat release coefficient, respectively. And the heat conduction coefficient takes the form (cf. [1])

$$\kappa(v,\theta) = \kappa_1 + \kappa_2 v\theta^b,$$

for some positive constants κ_1, κ_2 and b. The reaction rate function $\varphi=\varphi(\theta)$ is defined, from the Arrhenius law [28], by

$$\varphi(\theta) = K\theta^\beta \exp\left(-\frac{A}{\theta}\right), \tag{1.2}$$

where the constants $K>0$ and $A>0$ are the coefficients of the rates of the reactant and the activation energy, respectively, and β is a non-negative number.

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We treat the radiation as a continuous field and study both the wave and photonic effect, and assume that the high-temperature radiation is at thermal equilibrium with the fluid (cf. [3]). Then the pressure p and the internal energy e consist of a linear term in θ corresponding to the perfect polytropic contribution and a fourth-order radiative part due to the Stefan-Boltzmann radiative law [22,28]:

$$p(v,\theta)=\frac{R\theta}{v}+\frac{a\theta^4}{3},\quad e(v,\theta)=C_v\theta+av\theta^4, \tag{1.3}$$

where the constants $R>0$ and $C_v>0$ are the perfect gas constant and the specific heat, respectively. $a>0$ is the radiation constant which measures the amount of heat that is emitted by a black body, which absorbs all of the radiant energy that hits it, and will emit all the radiant energy. It is defined as (cf. [22])

$$a=\frac{4\sigma}{c}=\frac{8\pi^5k^4}{15c^3h^3}, \tag{1.4}$$

where σ is the Stefan-Boltzmann constant, c is the speed of light, k is Boltzmann constant, and h is Planck's constant. Numerically, $a=7.5657\times10^{-16}\mathrm{Jm^{-3}K^{-4}}$. In general, the radiation constant a is much smaller than the perfect gas constant R and the specific heat C_v.

In this article, we concern the system (1.1) with the following initial data and far-field condition:

$$\begin{cases}(v,u,\theta,z)(x,0)=(v_0,u_0,\theta_0,z_0)(x), & x\in\mathbb{R},\\ (v,u,\theta,z)(\pm\infty,t)=(v_\pm,u_\pm,\theta_\pm,z_\pm), & t>0,\end{cases} \tag{1.5}$$

where $v_\pm(>0)$, $\theta_\pm(>0)$, $u_\pm(\in\mathbb{R})$ and $z_\pm(\in\mathbb{R})$ are given constants and the initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ are assumed to satisfy $\inf_{x\in\mathbb{R}}v_0(x)>0$, $\inf_{x\in\mathbb{R}}\theta_0(x)>0$ and $(v_0,u_0,\theta_0,z_0)(\pm\infty)=(v_\pm,u_\pm,\theta_\pm,z_\pm)$ as compatibility conditions.

If the viscosity of the fluid is under consideration, the system (1.1) is written as:

$$\begin{aligned} v_t-u_x&=0,\\ u_t+p_x&=\mu\left(\frac{u_x}{v}\right)_x,\\ \left(e+\frac{u^2}{2}\right)_t+(up)_x&=\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x+\mu\left(\frac{uu_x}{v}\right)_x+\lambda\varphi z,\\ z_t&=\left(\frac{dz_x}{v^2}\right)_x-\varphi z.\end{aligned} \tag{1.6}$$

This model was established to describe the dynamic combustion of a radiative-reaction gas, which is closely related to the combustion theory (cf. [29]) and also the evolution of a stellar (cf. [2]). Recently, the problem on the global solvability of compressible viscous radiative reaction system (1.6) is a hot and interesting topic in the field of nonlinear partial differential equations, which has attracted many mathematicians and hobbyists to study this model and many results have been obtained. We will only focus on the Cauchy problem in 1D case, for the initial-boundary value problem please refer to [1,3,16,17,24,27,28] and references therein, and [19,25,27,30] and references therein for the multidimensional case.

For the Cauchy problem to the compressible viscous radiative reaction gas model (1.6), (1.5), if $(v_\pm,u_\pm,\theta_\pm,z_\pm)$, the far-field of initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$, is assumed to be (1,0,1,0), Liao and Zhao [20] established the time-asymptotic nonlinear stability of the solution, if the far fields $(v_\pm, u_\pm, \theta_\pm)$ are unequal, but $z_- = z_+$, Gong, He and Liao [7] proved the nonlinear stability of rarefaction waves, while recently, the time-asymptotic stability of viscous contact discontinuity was proved by Gong, Xu and Zhao [8]. Besides, He, Liao, Wang and Zhao [9] studied the compressible Navier-Stokes system for the viscous radiative gas. The proof is based on some analysis on uniform positive lower and upper bounds of the specific volume and absolute temperature.

For the non-viscous case (i.e. $\mu = 0$), if $a = 0, z = 0$ (i.e. a compressible heat conductive gas without viscosity), Fan and Matsumura [6] established the nonlinear stability of the composition of viscous shock waves to this problem, while the nonlinear stability of viscous contact waves was obtained by Ma and Wang [21]. Very recently, Fan, Gong and Tang [5] constructed the stability of the composite of a viscous contact wave and rarefaction waves.

Based on the above results, to the best of our knowledge, no result has been obtained for the nonlinear stability of solutions to the non-viscous radiative and reactive gas so far. So, in this paper, we will devote ourselves to this problem, precisely, we are concerned with the stability of a composite wave of viscous contact wave and rarefaction waves for the Cauchy problem (1.1)-(1.5) when the far-field states of the initial data are different.

Motivated by [5, 7, 8, 11–15, 20] and so on, we expect that the large-time asymptotic profiles of solutions to the Cauchy problem (1.1)-(1.5) are the same as the compressible Navier-Stokes system in the case of $z_+ = z_- = 0$. More precisely, we will show that the large-time behavior of the solution to the Cauchy problem (1.1)-(1.5) can be described by the corresponding compressible Euler system:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ \left(e + \frac{u^2}{2}\right)_t + (up)_x &= 0, \\ z_t &= 0, \end{aligned} \tag{1.7}$$

with Riemann initial data

$$(v(0,x), u(0,x), \theta(0,x), z(0,x)) = \begin{cases} (v_-, u_-, \theta_-, 0), & x < 0, \\ (v_+, u_+, \theta_+, 0), & x > 0. \end{cases} \tag{1.8}$$

The rest of this paper is arranged as follows. In Section 2, we will first construct the viscous contact wave and the rarefaction waves, and then some properties of the viscous contact wave and rarefaction wave will be stated, at last we will present the main results. Finally, in Section 3, we will focus on the main theorem, some a priori estimates will be proved which leads to the main theorem immediately.

Notations: Throughout this paper, the notation C denotes a generic positive constant, which may change from line to line. For two quantities A and B, $A \lesssim B$ means that there exists a constant C independent of δ, t and x such that $A \le CB$, while $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. And L^p, H^s denote the usual Lebesgue space and Sobolev space on R with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_s$, respectively. For simplicity, we take $\|\cdot\| := \|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty} := \|\cdot\|_\infty$.

2. Preliminaries and main results

In this section, at first, we will construct the viscous contact wave and the combination of viscous contact wave with two rarefaction waves for the Cauchy problem

(1.1)-(1.5). Then the main results will be presented. For each $(v_-,u_-,\theta_-,0)$, we denote the neighborhood of $(v_-,u_-,\theta_-,0)$ by Ω_- defined as the following:

$$\Omega_-=\{(v,u,\theta,z):|(v-v_-,u-u_-,\theta-\theta_-)|\le\bar{\delta},z=0\},$$

here $\bar{\delta}$ is a positive constant depending only on v_-, u_- and θ_-. We can see our situation takes place provided $(v_+,u_+,\theta_+,0)$ is located on a quarter of a curved surface in a small neighborhood of $(v_-,u_-,\theta_-,0)$.

2.1. Viscous contact wave. As in [13, 14], we firstly construct the viscous contact wave $(\widetilde{v},\widetilde{u},\widetilde{\theta},\widetilde{z})$ for the system (1.1). For the Riemann problem (1.7), (1.8), it is known that the contact discontinuity solution $(\widetilde{V},\widetilde{U},\widetilde{\Theta},\widetilde{Z})(x,t)$ takes the form (cf. [26])

$$(\widetilde{V},\widetilde{U},\widetilde{\Theta},\widetilde{Z})(x,t)=\begin{cases}(v_-,u_-,\theta_-,0), & x<0,\ t>0.\\ (v_+,u_+,\theta_+,0), & x>0,\ t>0.\end{cases} \tag{2.1}$$

provided that

$$u_-=u_+,\quad p_-=\frac{R\theta_-}{v_-}+\frac{a\theta_-^4}{3}=p_+=\frac{R\theta_+}{v_+}+\frac{a\theta_+^4}{3}. \tag{2.2}$$

In the setting of the system (1.1), the smooth approximate wave $(\widetilde{v},\widetilde{u},\widetilde{\theta},\widetilde{z})$ to the contact wave behaves as a diffusion wave due to the dissipation effect and we call this wave "viscous contact wave". Hence, we can construct viscous contact wave $(\widetilde{v},\widetilde{u},\widetilde{\theta},\widetilde{z})$ as follows (cf. [8, 13, 14]).

Since the pressure for the profile $(\widetilde{v},\widetilde{u},\widetilde{\theta},\widetilde{z})$ is expected to be constant asymptotically, we set

$$p_+=\frac{R\widetilde{\theta}}{\widetilde{v}}+\frac{a\widetilde{\theta}^4}{3}, \tag{2.3}$$

from (2.3) we can deduce that

$$\widetilde{v}=\frac{R\widetilde{\theta}}{p_+-\frac{1}{3}a\widetilde{\theta}^4},\quad \text{if}\quad p_+-\frac{1}{3}a\widetilde{\theta}^4>0. \tag{2.4}$$

Besides, (2.3) indicates that the leading part of the energy equation $(1.1)_3$ is

$$\widetilde{e}_t+p_+\widetilde{u}_x=\left(\kappa(\widetilde{v},\widetilde{\theta})\frac{\widetilde{\theta}_x}{\widetilde{v}}\right)_x, \tag{2.5}$$

where $\widetilde{e}=C_v\widetilde{\theta}+a\widetilde{v}\widetilde{\theta}^4$.

By the equation $\widetilde{v}_t=\widetilde{u}_x$, (2.4) and (2.5), one can obtain that

$$\left[\frac{\partial\widetilde{e}}{\partial\widetilde{\theta}}+\left(\frac{\partial\widetilde{e}}{\partial\widetilde{v}}+p_+\right)\frac{\partial\widetilde{v}(\widetilde{\theta})}{\partial\widetilde{\theta}}\right]\frac{\partial\widetilde{\theta}}{\partial t}=\left(\kappa(\widetilde{v},\widetilde{\theta})\frac{p_+-\frac{1}{3}a\widetilde{\theta}^4}{R\widetilde{\theta}}\widetilde{\theta}_x\right)_x. \tag{2.6}$$

If we note that

$$A(\widetilde{\theta})=\frac{\partial\widetilde{e}}{\partial\widetilde{\theta}}+\left(\frac{\partial\widetilde{e}}{\partial\widetilde{v}}+p_+\right)\frac{\partial\widetilde{v}(\widetilde{\theta})}{\partial\widetilde{\theta}},\quad B(\widetilde{\theta})=\kappa\left(\widetilde{v}(\widetilde{\theta}),\widetilde{\theta}\right)\frac{p_+-\frac{1}{3}a\widetilde{\theta}^4}{R\widetilde{\theta}},$$

and notice that $\kappa(\widetilde{v},\widetilde{\theta})=\kappa_1+\kappa_2\widetilde{v}\widetilde{\theta}^b>0$, then (2.6) can be written as

$$A(\widetilde{\theta})\widetilde{\theta}_t=\Big(B(\widetilde{\theta})\widetilde{\theta}_x\Big)_x,$$

furthermore, if we take

$$\Lambda=H(\widetilde{\theta}),\quad \frac{dH(\widetilde{\theta})}{d\widetilde{\theta}}=A(\widetilde{\theta}),$$

by virtue of the fact that

$$\frac{\partial\widetilde{e}}{\partial\widetilde{\theta}}=C_v+4a\widetilde{v}\widetilde{\theta}^3>0,\quad \frac{\partial\widetilde{e}}{\partial\widetilde{v}}=a\widetilde{\theta}^4>0,\quad \frac{\partial\widetilde{v}(\widetilde{\theta})}{\partial\widetilde{\theta}}=\frac{R\widetilde{v}+\frac{4}{3}a\widetilde{v}^2\widetilde{\theta}^3}{R\widetilde{\theta}}>0,$$

which implies $H'(\widetilde{\theta})>0$, thus (2.5) leads to a nonlinear diffusion equation

$$\Lambda_t=\left(\frac{B\big(H^{-1}(\Lambda)\big)}{H'(H^{-1}(\Lambda))}\Lambda_x\right)_x,\quad \Lambda(\pm\infty,t)=H(\theta_\pm). \tag{2.7}$$

If (2.4) holds true, we can deduce that $B(\widetilde{\theta})>0$. Together with $H'(\widetilde{\theta})>0$, according to [4, 10], the two-point boundary problem (2.7) has a unique self-similiar solution $\Lambda(x,t)=\Lambda(\zeta),\zeta=\frac{x}{\sqrt{1+t}}$. Furthermore, $\Lambda(\zeta)$ is monotone, increasing if $H(\theta_+)>H(\theta_-)$ and decreasing if $H(\theta_-)>H(\theta_+)$. The monotonicity of $\Lambda(\xi)$ and $H'(\widetilde{\theta})>0$ implies the monotonicity of $\widetilde{\theta}$, thus with the help of (2.2) one has

$$p_+-\frac{1}{3}a\widetilde{\theta}^4\geq\min\left\{\frac{R\theta_-}{v_-},\frac{R\theta_+}{v_+}\right\}, \tag{2.8}$$

which means that (2.4) is always true.

Moreover, there exists some positive constant δ, such that for $\delta=|\theta_+-\theta_-|$, Λ satisfies

$$(1+t)\left|\Lambda_{xx}\left(\frac{x}{\sqrt{1+t}}\right)\right|+(1+t)^{\frac{1}{2}}\left|\Lambda_x\left(\frac{x}{\sqrt{1+t}}\right)\right|+\left|\Lambda\left(\frac{x}{\sqrt{1+t}}\right)-H(\theta_\pm)\right|\lesssim\delta e^{-\frac{C_1x^2}{1+t}}, \tag{2.9}$$

where $C_1>0$ is constant and depends only on $\theta_\pm$. Since $\widetilde{\theta}$ has positive upper bound and lower bound and $H'(\widetilde{\theta})$ is continuous, (2.9) leads to

$$(1+t)\left|\widetilde{\theta}_{xx}\left(\frac{x}{\sqrt{1+t}}\right)\right|+(1+t)^{\frac{1}{2}}\left|\widetilde{\theta}_x\left(\frac{x}{\sqrt{1+t}}\right)\right|+\left|\widetilde{\theta}\left(\frac{x}{\sqrt{1+t}}\right)-\theta_\pm\right|\lesssim\delta e^{-\frac{C_2x^2}{1+t}}, \tag{2.10}$$

where $C_2>0$ is constant and depends only on $\theta_\pm$. Once $\widetilde{\theta}$ is determined, the contact wave profile $(V^c,U^c,\Theta^c,Z^c)(x,t)$ is defined as follows:

$$V^c=\frac{R}{p_+-\frac{a\widetilde{\theta}^4}{3}}\widetilde{\theta},\quad \Theta^c=\widetilde{\theta},\quad U^c_x=V^c_t,\quad Z^c=0. \tag{2.11}$$

It's easy to check that the contact wave $(V^c,U^c,\Theta^c,Z^c)(x,t)$ solves the viscous radiative and reactive gas system (1.1) time asymptotically, that is

$$\begin{cases} V^c_t-U^c_x=0,\\ U^c_t+P(V^c,\Theta^c)_x=U^c_t,\\ E^c_t+P(V^c,\Theta^c)U^c_x=\left(\dfrac{\kappa(V^c,\Theta^c)\Theta^c_x}{V^c}\right)_x,\\ Z^c=0, \end{cases} \tag{2.12}$$

where

$$E^c = C_v\Theta^c + aV^c(\Theta^c)^4. \tag{2.13}$$

Now we can present our first main result as follows:

THEOREM 2.1. *For any given left state $(v_-,u_-,\theta_-,0)$, suppose that the right state $(v_+,u_+,\theta_+,0)\in\Omega_-$ satisfies (2.2), let $(V^c,U^c,\Theta^c,Z^c)(x,t)$ is the viscous contact wave defined in (2.11) with strength $\delta=|\theta_+-\theta_-|$. There exist two positive constants ϵ_1 and δ_1 which are only depend on $(v_-,u_-,\theta_-,0)$, such that if $\delta<\delta_1$ and the initial data satisfying*

$$\|(v_0(\cdot)-V^c(\cdot,0),u_0(\cdot)-U^c(\cdot,0),\theta_0(\cdot)-\Theta^c(\cdot,0),z_0(\cdot))\|_2\le\epsilon_1, \tag{2.14}$$

then the Cauchy problem (1.1), (1.2) admits a unique global solution $(v,u,\theta,z)(t,x)$ satisfies

$$(v-V^c,u-U^c,\theta-\Theta^c,z)(t,x)\in X([0,+\infty)),$$

and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(v-V^c,u-U^c,\theta-\Theta^c,z)(x,t)|=0, \tag{2.15}$$

here the solution space $X(I)$ will be defined later in (3.5).

2.2. Composition waves. When the relation (2.2) fails, the basic theory of conservation laws (cf. [26]) shows that for any given constant state $(v_-,u_-,\theta_-,0)$, if $(v_+,u_+,\theta_+,0)\in\Omega_-$ and $\bar\delta$ is suitably small, the Riemann problem (1.5), (1.7) has a unique solution. Hence, our next aim is to study the stability of superposition of a viscous contact wave with rarefaction waves. Precisely, we suppose that

$$(v_+,u_+,\theta_+,0)\in R_1CR_3(v_-,u_-,\theta_-,0)\subseteq\Omega_-, \tag{2.16}$$

where

$$\begin{aligned} R_1CR_3(v_-,u_-,\theta_-,0):=\Big\{&(v,u,\theta,z)\in\Omega_-\Big|s\neq s_-,z=0,\\ &u\ge u_- -\int_{v_-}^{e^{c(s_--s)}v}\lambda_1(\eta,s_-)d\eta,\ u\ge u_- -\int_{e^{c(s-s_-)}v}^{v_-}\lambda_3(\eta,s)d\eta\Big\}, \end{aligned} \tag{2.17}$$

in which $\lambda_1(v,s)=-\sqrt{-\hat p_v(v,s)}$, $\lambda_3(v,s)=-\lambda_1(v,s)$ and $\hat p(v,s)=p(v,\theta(v,s))$, where s is entropy which is defined as follows:

$$s=C_v\ln\theta+4av\frac{\theta^3}{3}+R\ln v,\quad s_\pm=C_v\ln\theta_\pm+4av_\pm\frac{\theta_\pm^3}{3}+R\ln v_\pm. \tag{2.18}$$

It is known that if some sufficiently small $\delta_1>0$ such that for

$$|\theta_--\theta_+|\le\delta_1,$$

then there exists a unique pair of points $(v_-^m,u^m,\theta_-^m,0)$ and $(v_+^m,u^m,\theta_+^m,0)$ in Ω_- such that

$$\frac{R\theta_-^m}{v_-^m}+\frac{a(\theta_-^m)^4}{3}=\frac{R\theta_+^m}{v_+^m}+\frac{a(\theta_+^m)^4}{3}:=p_m, \tag{2.19}$$

and

$$|v^m_\pm - v_\pm| + |u^m - u_\pm| + |\theta^m_\pm - \theta_\pm| \lesssim |\theta_+ - \theta_-|. \tag{2.20}$$

Moreover, the states $(v^m_-, u^m, \theta^m_-, 0)$ and $(v^m_+, u^m, \theta^m_+, 0)$ belong to the 1-rarefaction wave curve $\mathbb{R}_-(v_-, u_-, \theta_-, 0)$ and the 3-rarefaction wave curve $\mathbb{R}_+(v_+, u_+, \theta_+, 0)$ respectively, where

$$\mathbb{R}_-(v_-, u_-, \theta_-, 0) = \left\{ (v, u, \theta, z) \middle| u = u_- - \int_{v_-}^{v} \lambda_1(\eta, s_-) d\eta, v > v_-, s = s_-, z = 0 \right\},$$
$$\mathbb{R}_+(v_+, u_+, \theta_+, 0) = \left\{ (v, u, \theta, z) \middle| u = u_+ - \int_{v_+}^{v} \lambda_3(\eta, s_+) d\eta, v > v_+, s = s_+, z = 0 \right\},$$

which means the state $(v_-, u_-, \theta_-, 0)$ connects with $(v^m_-, u^m, \theta^m_-, 0)$ by the 1-rarefaction wave $r_1 := (v^r_1, u^r_1, \theta^r_1, 0)(\frac{x}{t})$, and $(v^m_+, u^m, \theta^m_+, 0)$ connects with $(v_+, u_+, \theta_+, 0)$ by the 3-rarefaction wave $r_3 := (v^r_3, u^r_3, \theta^r_3, 0)(\frac{x}{t})$. In other words, the 1-rarefaction wave is the weak solution of Riemann problem of the Euler system (1.7)-(1.8) with the following Riemann data

$$r_1(x,0) = \begin{cases} (v_-, u_-, \theta_-, 0), & x < 0, \\ (v^m_-, u^m, \theta^m_-, 0), & x > 0, \end{cases}$$

and the 3-rarefaction wave with Riemann data as

$$r_3(x,0) = \begin{cases} (v^m_+, u^m, \theta^m_+, 0), & x < 0, \\ (v_+, u_+, \theta_+, 0), & x > 0. \end{cases}$$

To study the stability problem, we need to construct the smooth approximations of the rarefaction waves. Motivated by [18], we begin to recall the problem of the Burgers equation:

$$\begin{cases} w^r_t + w^r w^r_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ w^r(0,x) = w^r_0(x) := \frac{1}{2}\big(w_r + w_l\big) + \frac{1}{2}\big(w_r - w_l\big)\tanh(x). \end{cases} \tag{2.21}$$

Let $w_l = \lambda_1(v_-, s_-), w_r = \lambda_1(v^m_-, s_-)$ and $w(x,t)$ be the unique global solution of (2.21), then the smooth approximation of the 1-rarefaction wave can be defined by $R^r_1(x,t) := (V^r_1, U^r_1, \Theta^r_1, 0)(x,t)$ as

$$\begin{cases} \lambda_1(V^r_1, s_-) = w(x,t), \\ U^r_1 = u_- - \int_{v_-}^{V^r_1} \lambda_1(\eta, s_-) d\eta, \\ \Theta^r_1 = \hat{\theta}(V^r_1, s_-), \\ Z^r_1 = 0. \end{cases} \tag{2.22}$$

Meanwhile, if we take $w_l = \lambda_3(v^m_+, s_+), w_r = \lambda_3(v_+, s_+)$, the smooth approximation of the 3-rarefaction wave is given by $R^r_3(x,t) := (V^r_3, U^r_3, \Theta^r_3, 0)(x,t)$ constructed by the same way as (2.22)

$$\begin{cases} \lambda_3(V^r_3, s_+) = w(x,t), \\ U^r_3 = u_+ - \int_{v_+}^{V^r_3} \lambda_3(\eta, s_+) d\eta, \\ \Theta^r_3 = \hat{\theta}(V^r_3, s_+), \\ Z^r_3 = 0. \end{cases} \tag{2.23}$$

Due to the conditions (2.19), (2.20), $(v_-^m,u^m,\theta_-^m,0)$ is connected to $(v_+^m,u^m,\theta_+^m,0)$ by the viscous contact wave $(V^c,U^c,\Theta^c,0)(x,t)$ constructed in (2.11). Motivated by [11], we divide $\mathbb{R}\times[0,t]$ into three parts $\mathbb{R}\times[0,t]=\Omega_1\cup\Omega_c\cup\Omega_3$ with

$$\begin{aligned}
&\Omega_1=\left\{(x,t)|2x<\lambda_1(v_-^m,s_-)t\right\},\\
&\Omega_3=\left\{(x,t)|2x>\lambda_3(v_+^m,s_+)t\right\},\\
&\Omega_c=\left\{(x,t)|\lambda_1(v_-^m,s_-)t\le 2x\le\lambda_3(v_+^m,s_+)t\right\}.
\end{aligned}\tag{2.24}$$

Then, we show some properties of the rarefaction waves $R_i^r(x,t)(i=1,3)$ and viscous contact wave $(V^c,U^c,\Theta^c,0)(x,t)$ as follows:

Lemma 2.1 (cf. [5,11]). *For any given left state $(v_-,u_-,\theta_-,0)$, assume that the right state $(v_+,u_+,\theta_+,0)\in R_1CR_3(v_-,u_-,\theta_-,0)\subset\Omega_-$, then we have the smooth rarefaction wave $(V_i^r,U_i^r,\Theta_i^r,0)(i=1,3)$ and $(V^c,U^c,\Theta^c,0)(t,x)$ satisfying:*

(1) $(U_i^r)_x>0(i=1,3)$ for all $x\in\mathbb{R}$, $t>0$.

(2) For $1\le p\le\infty$, it holds that

$$\begin{aligned}
\|(V_i^r,U_i^r,\Theta_i^r)_x(t)\|_{L^p}&\lesssim\min\left\{\delta,\delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\right\},\quad i=1,3,\\
\|(V_i^r,U_i^r,\Theta_i^r)_{xx}(t)\|_{L^p}&\lesssim\min\left\{\delta,(1+t)^{-1}\right\},\quad i=1,3.
\end{aligned}$$

(3) In Ω_c, we have

$$|(V_i^r,U_i^r,\Theta_i^r)_x|+|V_i^r-v_i^m|+|\Theta_i^r-\theta_i^m|\lesssim\delta e^{-c(|x|+t)},\quad i=1,3,$$

and in Ω_i we have

$$\begin{aligned}
|V_x^c|+|\Theta_x^c|+|V^c-v_-^m|+|U_x^c|+|\Theta^c-\theta_-^m|&\lesssim\delta e^{-c(|x|+t)},\quad i=1,\\
|V_x^c|+|\Theta_x^c|+|V^c-v_+^m|+|U_x^c|+|\Theta^c-\theta_+^m|&\lesssim\delta e^{-c(|x|+t)},\quad i=3,\\
|(V_3^r)_x+(U_3^r)_x|+|V_3^r-v_+^m|+|(\Theta_3^r)_x|+|\Theta_3^r-\theta_+^m|&\lesssim\delta e^{-c(|x|+t)},\quad i=1,\\
|(V_1^r)_x+(U_1^r)_x|+|V_1^r-v_-^m|+|(\Theta_1^r)_x|+|\Theta_1^r-\theta_-^m|&\lesssim\delta e^{-c(|x|+t)},\quad i=3.
\end{aligned}$$

(4) For the rarefaction wave $(v_i^r,u_i^r,\theta_i^r)\left(\frac{x}{t}\right)(i=1,3)$, it holds

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}}\left|(V_i^r,U_i^r,\Theta_i^r)(x,t)-(v_i^r,u_i^r,\theta_i^r)\left(\tfrac{x}{t}\right)\right|=0,\quad i=1,3.$$

Set $(V,U,\Theta,Z)(x,t)$ as

$$\begin{cases}
V(x,t)=V_1^r(x,t)+V^c(x,t)+V_3^r(x,t)-v_-^m-v_+^m,\\
U(x,t)=U_1^r(x,t)+U^c(x,t)+U_3^r(x,t)-2u^m,\\
\Theta(x,t)=\Theta_1^r(x,t)+\Theta^c(x,t)+\Theta_3^r(x,t)-\theta_-^m-\theta_+^m,\\
Z(x,t)=0.
\end{cases}\tag{2.25}$$

Our second main result can be stated as follows:

Theorem 2.2. *For any given left state $(v_-,u_-,\theta_-,0)$, assume that the right state $(v_+,u_+,\theta_+,0)\in R_1CR_3(v_-,u_-,\theta_-,0)\subseteq\Omega_-$ with $|\theta_+-\theta_-|\le\delta_1$. There exist three positive* *constants* ϵ_2*,* a_0 *and* $\delta_2(\le \min\{\overline{\delta},\delta_1\})$*, such that* $0<a<a_0$ *and* $\delta<\delta_2$ *and the initial data satisfying*

$$\|(v_0(\cdot)-V(\cdot,0),u_0(\cdot)-U(\cdot,0),\theta_0(\cdot)-\Theta(\cdot,0),z_0(\cdot))\|_2\le\epsilon_2, \tag{2.26}$$

then the Cauchy problem (1.1), (1.2) admits a unique global solution $(v,u,\theta,z)(x,t)$ *satisfies*

$$(v-V,u-U,\theta-\Theta,z-Z)\in X([0,+\infty)),$$

and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(v-V,u-U,\theta-\Theta,z)(x,t)|=0, \tag{2.27}$$

here the solution space $X(I)$ *is defined in (3.5).*

Remark 2.1. Some remarks to the Theorem 2.2 can be listed as below:

- Compared to Theorem 2.1, there is a smallness condition imposed on the radiation constant a in Theorem 2.2, due to the appearance of the rarefaction wave (for details please refer to (3.20)). It's worth to point out that this condition is not needed in the stability analysis of the single contact wave (cf. [8]). That would be an interesting problem to consider the stability of rarefaction wave without the smallness of the radiative constant a.
- If $z=0,a=0$ and $\kappa_2=0$, our results degenerate to the results obtained in [5].
- This is the first result considering the stability of 1D compressible Navier-Stokes-type system for a radiative and reactive gas without viscosity, while the corresponding stability of viscous shock profile is still open. Another interesting problem is to study the case that $z_-\ne z_+$, however, this is still an open problem for both viscous and non-viscous cases.

We now introduce some difficulties we encountered and some main strategies we used in this paper. The first difficulty is that the absence of viscosity leads to the system (1.1) being less dissipative than the viscous ones considered in the literature before. In the case that $\mu>0$, Gong, He and Liao [7] observe some cancellations between the flux terms and viscosity terms for a viscous radiative and reactive gas. Then, by elementary energy method, they derive the dissipative mechanisms induced by the viscosity and conductivity which contribute to prove the nonlinear stability of rarefaction waves for a viscous radiative and reactive gas with large initial perturbation. However, if we neglect the viscosity, for compressible Navier-Stokes-type system for a compressible, radiative and reactive gas, we do not have a good estimate for the derivatives of u. Hence, the above argument can not be used anymore. This difficulty was first solved by Fan and Matsumura in [6], which shows that if the strengths of the viscous waves and the initial perturbation are suitably small, there exists a unique global-in-time solution and asymptotically tends toward the corresponding viscous contact wave or the composition of a viscous contact wave with rarefaction waves. Our result generalizes the corresponding results of the compressible Navier-Stokes obtained by Huang, Li and Matsumura in [11] for the case that the viscous coefficient $\mu>0$ and the heat conduction coefficient $\kappa>0$ and also the results obtained by Fan, Gong and Tang [5] for the case that $\mu=0,\kappa>0$, and extends the result of Ma and Wang [21] for the nonlinear stability of the viscous contact waves.

And the second difficulty is how to control the possible growth of its solutions caused by the nonlinearity and the interaction of waves from different families in the stability of composite waves. Motivated by Huang, Li and Masumura [11], similar to Gong, He and Liao [7], we make full use of the properties of the rarefaction wave that $(U_1^r)_x>0$, $(U_3^r)_x>0$, and furthermore we need the term $((U_-^r)_x+(U_+^r)_x)Q_1$ (see (3.17), (3.18), (3.19)) to be positive to control the possible growth of the solution, it's where the condition that the radiative constant a is small, is imposed.

3. Stability analysis

In this section, we show the asymptotic behavior of the solution for non-viscous compressible Navier-Stokes-type system for a radiative and reactive gas (1.1)-(1.5). If $(v_\pm^m,u^m,\theta_\pm^m,0)=(v_\pm,u_\pm,\theta_\pm,0)$, Theorem 2.2 will coincide with the result of Theorem 2.1, therefore we omit the proof of the Theorem 2.1 for brevity, and we will prove the stability of the composition wave only.

3.1. Reform the system. Note that the composition wave $(V,U,\Theta,Z)(x,t)$ defined in (2.25) satisfies

$$\begin{cases} V_t-U_x=0,\\ U_t+P_x=-R_1,\\ E_t+PU_x=(\frac{\kappa(V,\Theta)\Theta_x}{V})_x-R_2,\\ Z=0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &E:=C_v\Theta+aV\Theta^4,\\ &P:=\frac{R\Theta}{V}+\frac{a\Theta^4}{3},\quad P_i=\frac{R\Theta_i^r}{V_i^r}+\frac{a(\Theta_i^r)^4}{3}\quad (i=1,3),\\ &R_1:=-(P-P_1-P_3-p_m)_x+U_t^c:=R_1^1+U_t^c,\\ &R_2:=\{(p_m-P)U_x^c+(P_1-P)U_{1x}^r+(P_3-P)U_{3x}^r\}\\ &\qquad+\left\{\left(\frac{\kappa(V,\Theta)\Theta_x}{V}\right)_x-\left(\frac{\kappa(V^c,\Theta^c)\Theta_x^c}{V^c}\right)_x\right\}:=R_2^1+R_2^2. \end{aligned} \tag{3.2}$$

Let the perturbation is

$$(\phi,\psi,\xi,z):=(v,u,\theta,z)-(V,U,\Theta,0),$$

then the reformed equations are

$$\begin{cases} \phi_t-\psi_x=0,\\ \psi_t+(p-P)_x=R_1,\\ C_v\xi_t+a(v\theta^4-V\Theta^4)_t+p\psi_x+(p-P)U_x\\ \quad =\left(\frac{\kappa(v,\theta)\xi_x}{v}-\frac{\kappa(v,\theta)\Theta_x\phi}{vV}+\frac{\kappa(v,\theta)-\kappa(V,\Theta)}{V}\Theta_x\right)_x+\lambda\varphi z+R_2,\\ z_t=\left(\frac{dz_x}{v^2}\right)_x-\varphi z, \end{cases} \tag{3.3}$$

with the initial data

$$\begin{aligned} (\phi,\psi,\xi,z)(x,0)&=(\phi_0,\psi_0,\xi_0,z_0)(x)\\ &=(v_0(x)-V(x,0),u_0(x)-U(x,0),\theta_0(x)-\Theta(x,0),z_0(x)). \end{aligned} \tag{3.4}$$

The solution space is defined as

$$X([0,T]):=\left\{(\phi,\psi,\xi,z)(x,t)\ \middle|\ \begin{array}{c}(\phi,\psi,\xi)(x,t)\in C([0,T],H^1(\mathbb{R})),\\ (\phi_x,\psi_x)(x,t)\in L^2([0,t),H^1(\mathbb{R})),\\ \xi_x\in L^2([0,T],H^2(\mathbb{R})),\\ z(x,t)\in C([0,T],H^1(\mathbb{R})\cap L^1(\mathbb{R})),\\ 0\le z(x,t)\le 1,\ (x,t)\in\mathbb{R}\times[0,T],\\ z\in L^2([0,T],H^3(\mathbb{R})).\end{array}\right\} \tag{3.5}$$

The local existence is known in [23].

Proposition 3.1 (Local existence). *Under the assumptions stated in Theorem 2.1, the Cauchy problem (3.3), (3.4) admits a unique smooth solution $(\phi,\psi,\xi,z)(x,t)\in X([0,t_1])$ for some sufficiently small $t_1>0$, and $(\phi,\psi,\xi,z)(x,t)$ satisfies*

$$\sup_{0\le t\le t_1}\|(\phi,\psi,\xi,z)(t)\|_2^2\le 2\|(\phi_0,\psi_0,\xi_0,z_0)\|_2^2. \tag{3.6}$$

Suppose that $(\phi,\psi,\xi,z)(x,t)$ has been extended to the time $T>t_1$, we need to derive the following a priori estimates to get a global solution.

Proposition 3.2 (A prior estimates). *Under the assumptions listed in Theorem 2.2, there exist positive constants $\epsilon_2\le 1$, $\delta_2\le\min\{\delta_1,\overline{\delta},1\}$, a_2 and C, such that if $(\phi,\psi,\xi,z)\in X([0,T])$ for some $T>0$ is a solution of (3.3), (3.4) and satisfying*

$$N(T)=\sup_{0\le\tau\le T}\|(\phi,\psi,\xi,z)(\tau)\|_2\le\epsilon_2,\quad \delta=|\theta_--\theta_+|<\delta_2,\quad a<a_2, \tag{3.7}$$

then we have the following estimate

$$\sup_{0\le\tau\le T}\|(\phi,\psi,\xi,z)(\tau)\|_2^2+\int_0^T\left(\|(\phi_x,\psi_x)(\tau)\|_1^2+\|\xi_x(\tau)\|_2^2+\|z(\tau)\|_3^2\right)d\tau$$
$$\lesssim\|(\phi_0,\psi_0,\xi_0,z_0)\|_2^2+\delta^{\frac{1}{8}}. \tag{3.8}$$

Once Proposition 3.2 is proved, we can use the standard continuation argument to extend the unique local solution $(\phi,\psi,\xi,z)(x,t)$ obtained in Proposition 3.1 to be a global solution, that is $T=\infty$. Moreover, the estimate (3.8) implies that

$$\int_0^\infty\left(\|(\phi_x,\psi_x,\xi_x,z_x)(t)\|^2+\left|\frac{d}{dt}\|(\phi_x,\psi_x,\xi_x,z_x)(t)\|^2\right|\right)d\tau\lesssim+\infty, \tag{3.9}$$

which together with Sobolev inequality leads to the asymptotic behavior (2.27), this concludes the proof of Theorem 2.1. Therefore, in the rest of this section, our main work is to prove these a prior estimates.

3.2. A Priori estimates. Firstly, we prove the basic estimates.

Lemma 3.1. *Under the assumptions in Proposition 3.2, then*

$$\begin{gathered}\|z(t)\|_{L^1}+\int_0^t\int_{\mathbb{R}}\varphi z\,dxd\tau\lesssim\|z_0\|_{L^1},\\ \|z(t)\|^2+\int_0^t\int_{\mathbb{R}}\left(\frac{d}{v^2}z_x^2+\varphi z^2\right)dxd\tau\lesssim\|z_0\|^2,\end{gathered} \tag{3.10}$$

$$
\begin{aligned}
&\int_{\mathbb{R}} \eta(t,x)dx + \int_0^t \int_{\mathbb{R}} \left[(U_{1x}^r + U_{3x}^r)(\phi^2+\xi^2) + \xi_x^2\right] dx d\tau \\
&\lesssim \int_{\mathbb{R}} \eta_0 dx + \|z_0\|_{L^1} + \delta^{\frac{1}{8}} + \delta^{\frac{1}{8}} \int_0^t \|(\phi_x,\psi_x)(\tau)\|^2 d\tau \\
&\quad + \delta \int_0^t \frac{1}{1+\tau} \int_{\mathbb{R}} (\phi^2+\xi^2) e^{\frac{-cx^2}{1+\tau}} dx d\tau. \qquad (3.11)
\end{aligned}
$$

Proof. Inequalities $(3.10)_1$ and $(3.10)_2$ follows directly from $(3.3)_4$ and integration by parts, the specific proof process is omitted. Now we devote our efforts to the last inequality. Firstly, multiplying $(3.3)_1$ by $-R\Theta\left(\frac{1}{v}-\frac{1}{V}\right)$, $(3.3)_2$ by ψ and $(3.3)_3$ by $\frac{\xi}{\theta}$, respectively, and noticing that

$$
\begin{aligned}
-R\Theta\left(\frac{1}{v}-\frac{1}{V}\right)\phi_t &= \left\{R\Theta\Phi\left(\frac{v}{V}\right)\right\}_t + \frac{R\Theta U_x}{vV^2}\phi^2 - R\Theta_t\Phi\left(\frac{v}{V}\right), \\
C_v\frac{\xi}{\theta}\xi_t &= \left\{C_v\Phi\left(\frac{\theta}{\Theta}\right)\right\}_t + C_v\frac{\Theta_t}{\Theta\theta}\xi^2 - C_v\Theta_t\Phi\left(\frac{\theta}{\Theta}\right),
\end{aligned} \qquad (3.12)
$$

then adding the resultant equations together, by a tedious calculation, we can get that

$$
\eta_t + Q + N = H_{1x} + \psi R_1 + \frac{\xi}{\theta}R_2 + \frac{\lambda\varphi z\xi}{\theta}, \qquad (3.13)
$$

here

$$
\eta = \frac{1}{2}\psi^2 + R\Theta\Phi\left(\frac{v}{V}\right) + C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right) + \frac{a}{3}v\xi^2\left(3\theta^2+2\theta\Theta+\Theta^2\right), \qquad (3.14)
$$

in which $\Phi(y) = y-1-\ln y$, and

$$
\begin{aligned}
Q =& -R\Theta_t\Phi\left(\frac{v}{V}\right) + \frac{R\Theta}{V^2 v}U_x\phi^2 + C_v\frac{\Theta_t}{\Theta\theta}\xi^2 - C_v\Theta_t\Phi\left(\frac{\theta}{\Theta}\right) \\
&+ \frac{\xi}{\theta}\left(\frac{R\theta}{v} - \frac{R\Theta}{V}\right)U_x + \frac{a}{3\theta}(4\Theta^3+3\Theta^2\theta+2\Theta\theta^2+\theta^3)U_x\xi^2 \\
&+ \frac{4aV}{3\theta}\Theta_t\xi^2(3\Theta^2+2\theta+\theta^2) + \frac{4a\Theta_t}{3}(\theta^2+\theta\Theta+\Theta^2)\phi\xi + \frac{\kappa(v,\theta)\Theta}{v\theta^2}\xi_x^2, \\
N =& \frac{\kappa(v,\theta)\Theta_x^2}{v\theta^2 V}\xi\phi - \frac{\kappa(v,\theta)\Theta\Theta_x}{vV}\phi\xi_x - \frac{\kappa(v,\theta)\Theta_x}{v\theta^2}\xi\xi_x \\
&+ \frac{\Theta_x(\Theta_x\xi - \Theta\xi_x)(\kappa(v,\theta)-\kappa(V,\Theta))}{V\theta^2}, \\
H_1 =& \frac{\xi}{\theta}\left(\frac{\kappa(v,\theta)\theta_x}{v} - \frac{\kappa(V,\Theta)\Theta_x}{V}\right) - (p-P)\psi,
\end{aligned} \qquad (3.15)
$$

From (3.1), making use of the relation that $U_x = U_{1x}^r + U_x^c + U_{3x}^r$, we can deduce that

$$
\begin{aligned}
-\Theta_t =& \frac{1}{C_v + 4aV\Theta^3}\left[\left(a\Theta^4+P\right)U_x - \left(\frac{\kappa(V,\Theta)\Theta_x}{V}\right)_x + R_2\right] \\
=& \frac{a\Theta^4+P}{C_v+4aV\Theta^3}\left(U_{1x}^r + U_{3x}^r\right) \\
&+ \frac{1}{C_v+4aV\Theta^3}\left[\left(a\Theta^4+P\right)U_x^c - \left(\frac{\kappa(V,\Theta)\Theta_x}{V}\right)_x + R_2\right] \\
:=& D(V,\Theta)\left(U_{1x}^r+U_{3x}^r\right) + F(V,U,\Theta).
\end{aligned}
$$

A further calculation shows that

$$
\begin{aligned}
&F(V,U,\Theta)\\
=&\frac{1}{C_v+4aV\Theta^3}\left[\left(a\Theta^4+P\right)U_x^c-\left(\frac{\kappa(V,\Theta)\Theta_x}{V}\right)_x+R_2\right]\\
=&\frac{1}{C_v+4aV\Theta^3}\left[\left(a\Theta^4+P\right)U_x^c-\left(\frac{\kappa(V^c,\Theta^c)\Theta_x^c}{V^c}\right)_x-(p_m-P)U_x^c\right]\\
&-\frac{1}{C_v+4aV\Theta^3}\left[(P_1-P)(U_1^r)_x+(P_3-P)(U_3^r)_x\right]\\
\lesssim&|(U_x^c,V_x^c,\Theta_x^c,\Theta_{xx}^c)|+\delta|(U_{1x}^r,U_{3x}^r)|.
\end{aligned}
$$

Thus the term Q can be rewritten as

$$
Q=(U_{1x}^r+U_{3x}^r)Q_1+Q_2+\frac{\kappa(v,\theta)\Theta}{v\theta^2}\xi_x^2, \tag{3.16}
$$

where

$$
\begin{aligned}
Q_1=&C_vD(V,\Theta)\left\{\frac{R}{C_v}\Phi\left(\frac{v}{V}\right)-\Phi\left(\frac{\Theta}{\theta}\right)+\frac{\phi^2}{Vv}+\frac{V\xi}{R\Theta\theta}\left(\frac{R\theta}{v}-\frac{R\Theta}{v}\right)\right\}\\
&+\left(1-\frac{VC_vD(V,\Theta)}{R\Theta}\right)\left(\frac{R}{\theta v}\xi^2-\frac{R\Theta}{Vv\theta}\xi\phi+\frac{R\Theta}{V^2v}\phi^2\right)\\
\geq&C_vD(V,\Theta)\left\{\Phi\left(\frac{v}{V}\right)+\Phi\left(\frac{\theta V}{\Theta v}\right)\right\}\\
&+\left(1-\frac{VC_vD(V,\Theta)}{R\Theta}\right)\left(\frac{R}{\theta v}\xi^2-\frac{R\Theta}{Vv\theta}\xi\phi+\frac{R\Theta}{V^2v}\phi^2\right),
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
Q_2=&F(V,U,\Theta)\left(R\Phi\left(\frac{v}{V}\right)-C_v\Phi\left(\frac{\Theta}{\theta}\right)\right)+\frac{R\Theta U_x^c}{V^2v}\phi^2+\frac{\xi}{\theta}\left(\frac{R\theta}{v}-\frac{R\Theta}{V}\right)U_x^c\\
&+\frac{a}{3\theta}(4\Theta^3+3\Theta^2\theta+2\Theta\theta^2+\theta^3)U_x\xi^2+\frac{4aV}{3\theta}\Theta_t\xi^2(3\Theta^2+2\theta+\theta^2)+\frac{4a\Theta_t}{3}(\theta^2+\theta\Theta+\Theta^2)\phi\xi.
\end{aligned}
$$

For Q_1, we observe that

$$
\left|1-\frac{VC_vD(V,\Theta)}{R\Theta}\right|=\left|\frac{4aV\Theta^2(3R-C_v)}{3R^2(C_v+4aV\Theta^3)}\right|\sim a, \tag{3.18}
$$

and

$$
\Phi\left(\frac{v}{V}\right)+\Phi\left(\frac{\theta V}{\Theta v}\right)\gtrsim\xi^2+\phi^2,\quad \frac{R}{\theta v}\xi^2-\frac{R\Theta}{Vv\theta}\xi\phi+\frac{R\Theta}{V^2v}\phi^2\lesssim\xi^2+\phi^2, \tag{3.19}
$$

so, if we assume $a<a_1$ small enough, then we have that

$$
Q_1\gtrsim\phi^2+\xi^2. \tag{3.20}
$$

For Q_2, its easy to check that

$$
Q_2\lesssim|(U_x^c,V_x^c,\Theta_x^c,\Theta_{xx}^c)|(\phi^2+\xi^2)+(\delta+a)|(U_{1x}^r,U_{3x}^r)|(\phi^2+\xi^2).
$$

Meanwhile, for the term N

$$
|N|\lesssim\frac{1}{8}\xi_x^2+\Theta_x^2(\phi^2+\xi^2), \tag{3.21}
$$

where

$$\begin{aligned}\Theta_x^2 \lesssim&(\Theta_{1x}^r)^2+(\Theta_{3x}^r)^2+(\Theta_x^c)^2\\ \lesssim&(\Theta_{1x}^r)^2+(\Theta_{3x}^r)^2+\frac{\delta}{1+t}e^{\frac{-cx^2}{1+t}}.\end{aligned}\tag{3.22}$$

Lastly, we need to get some estimates on R_1 and R_2, which are defined in (3.2), by using Lemma 2.1. Since $p_m=\frac{R\Theta^c}{V^c}+\frac{a(\Theta^c)^4}{3}$, direct calculation yields that

$$\begin{aligned}R_1^1=&(P_1+P_3+p_m-P)_x\\ =&R\left(\frac{\Theta_1^r}{V_1^r}+\frac{\Theta_3^r}{V_3^r}+\frac{\Theta^c}{V^c}-\frac{\Theta}{V}\right)_x+\frac{a}{3}\left((\Theta_1^r)^4+(\Theta_3^r)^4+(\Theta^c)^4-\Theta^4\right)_x\\ :=&R_1^{11}+R_1^{12},\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}R_1^{11}=&R\left(\frac{\Theta_1^r}{V_1^r}+\frac{\Theta_3^r}{V_3^r}+\frac{\Theta^c}{V^c}-\frac{\Theta}{V}\right)_x\\ =&R\Theta_{1x}^r\left((V_1^r)^{-1}-V^{-1}\right)+R\Theta_{3x}^r\left((V_3^r)^{-1}-V^{-1}\right)\\ &+R\Theta_x^c((V^c)^{-1}-V^{-1})+RV_{1x}^r\left(\frac{\Theta}{V^2}-\frac{\Theta_1^r}{(V_1^r)^2}\right)\\ &+RV_{3x}^r\left(\frac{\Theta}{V^2}-\frac{\Theta_3^r}{(V_3^r)^2}\right)+RV_x^c\left(\frac{\Theta}{V^2}-\frac{\Theta^c}{(V^c)^2}\right).\end{aligned}\tag{3.24}$$

By virtue of Lemma 2.1 (3), it is easy to compute

$$\left|\Theta_{1x}^r((V_1^r)^{-1}-V^{-1})\right|\lesssim|\Theta_{1x}^r|\left(|V_3^r-v_+^m|+|V^c-v_-^m|\right)\lesssim\delta e^{-c(|x|+t)},$$

and we can treat the other terms on the right-hand side of (3.24) in the same way to obtain

$$|R_1^{11}|\lesssim\delta e^{-c(|x|+t)},\tag{3.25}$$

and similarly, for R^{12} we have

$$\begin{aligned}R_1^{12}=&\frac{a}{3}\Big((\Theta_1^r)^4+(\Theta_3^r)^4+(\Theta^c)^4-\Theta^4\Big)_x\\ =&\frac{4a}{3}\left[(\Theta_1^r)^3\Theta_{1x}^r+(\Theta_3^r)^3\Theta_{3x}^r+(\Theta^c)^3\Theta_x^c-\Theta^3\Theta_x\right]\\ =&\frac{4a}{3}\left[\Theta_{1x}^r((\Theta_1^r)^3-\Theta^3)+\Theta_{3x}^r((\Theta_3^r)^3-\Theta^3)+\Theta_x^c((\Theta^c)^3-\Theta^3)+\Theta^3(\theta_-^m+\theta_+^m)\right]\\ =&\frac{4a}{3}\Theta_{1x}^r(\Theta_1^r-\Theta)\left[(\Theta_1^r)^2+\Theta_1^r\Theta+\Theta^2\right]+\frac{4a}{3}\Theta_{3x}^r(\Theta_3^r-\Theta)\left[(\Theta_3^r)^2+\Theta_3^r\Theta+\Theta^2\right]\\ &+\frac{4a}{3}\Theta_x^c(\Theta^c-\Theta)\left[(\Theta^c)^2+\Theta^c\Theta+\Theta^2\right]\\ :=&R_1^{121}+R_1^{122}+R_1^{123},\end{aligned}\tag{3.26}$$

where

$$\begin{aligned}\left|R_1^{121}\right|\lesssim&|\Theta_{1x}^r|(|\Theta_3^r-\theta_+^m|+|\Theta^c-\theta_-^m|)\\ \lesssim&|\Theta_{1x}^r|_{\Omega_3\bigcup\Omega_c}+(|\Theta_3^r-\theta_+^m|+|\Theta^c-\theta_-^m|)_{\Omega_1}\\ \lesssim&\delta e^{-c(|x|+t)},\end{aligned}$$

same to R_1^{121}, we have

$$\left|R_1^{122}\right| \lesssim \delta e^{-c(|x|+t)}, \quad \left|R_1^{123}\right| \lesssim \delta e^{-c(|x|+t)},$$

consequently,

$$\left|R_1^{12}\right| = |R_1^{121}| + |R_1^{122}| + |R_1^{123}| \lesssim \delta e^{-c(|x|+t)},$$

then, we can obtain

$$\left|R_1^1\right| = \left|R_1^{11}\right| + |R_1^{12}| \lesssim \delta e^{-c(|x|+t)}.$$

So, it holds that

$$|R_1| \lesssim |R_1^1| + |U_t^c| \lesssim \delta e^{-c(|x|+t)} + \delta(1+t)^{-\frac{3}{2}} e^{\frac{-cx^2}{1+t}}. \tag{3.27}$$

Similarly, we derive from (3.2) and Lemma 2.1 that

$$\left|R_2^1\right| \lesssim \delta e^{-c(|x|+t)}.$$

Since

$$\begin{aligned}
R_2^2 =& \left(\kappa(V,\Theta)\frac{\Theta_x}{V} - \kappa(V^c,\Theta^c)\frac{\Theta_x^c}{V^c}\right)_x \\
=& \kappa(V,\Theta)\left(\frac{\Theta_x}{V} - \frac{\Theta_x^c}{V^c}\right)_x + \kappa_x(V,\Theta)\left(\frac{\Theta_x}{V} - \frac{\Theta_x^c}{V^c}\right) \\
&+ (\kappa(V,\Theta) - \kappa(V^c,\Theta^c))\left(\frac{\Theta_x^c}{V^c}\right)_x + (\kappa_x(V,\Theta) - \kappa_x(V^c,\Theta^c))\left(\frac{\Theta_x^c}{V^c}\right) \\
:=& R_2^{21} + R_2^{22} + R_2^{23} + R_2^{24},
\end{aligned}$$

where

$$\begin{aligned}
R_2^{21} + R_2^{22} =& \kappa(V,\Theta)\left(\frac{\Theta_x}{V} - \frac{\Theta_x^c}{V^c}\right)_x + \kappa_x(V,\Theta)\left(\frac{\Theta_x}{V} - \frac{\Theta_x^c}{V^c}\right) \\
=& \kappa(V,\Theta)\left(\frac{(\Theta_1^r)_x}{V} + \frac{(\Theta_3^r)_x}{V}\right)_x + \kappa(V,\Theta)\left(\frac{\Theta_x^c}{V} - \frac{\Theta_x^c}{V^c}\right)_x + \kappa_x(V,\Theta)\left(\frac{\Theta_x}{V} - \frac{\Theta_x^c}{V^c}\right) \\
:=& R_2^{211} + R_2^{212} + R_2^{213}.
\end{aligned}$$

For R_2^{211}, a direct calculation shows

$$\begin{aligned}
R_2^{211} \lesssim& (|(\Theta_1^r)_{xx}| + |(\Theta_3^r)_{xx}| + |(\Theta_1^r)_x(V_1^r)_x| + |(\Theta_3^r)_x(V_3^r)_x|) \\
&+ |(\Theta_1^r)_x|(|(V_3^r)_x| + |V_x^c|) + |(\Theta_3^r)_x(V_1^r)_x| + |V_x^c|),
\end{aligned}$$

it follows from (2.10) and Lemma 2.1 that

$$|R_2^{211}| \lesssim \delta^{\frac{1}{8}}(1+t)^{-\frac{7}{8}},$$

similarly, for R_2^{212} and R_2^{213} we have

$$\begin{aligned}
|R_2^{212}| \lesssim& \Big(|\Theta_{xx}^c| + |\Theta_x^c||V_x^c|\Big)\Big(|V_+^r - v_+^m| + |V_-^r - v_-^m|\Big) + |\Theta_x^c|\Big(|(V_-^r)_x| + |(V_+^r)_x|\Big) \\
\lesssim& \delta e^{-c(|x|+t)}, \\
|R_2^{213}| \lesssim& (|V_x| + |\Theta_x|)\left(\frac{|\Theta_x - \Theta_x^c|}{V} + |\Theta_x^c|\right) \lesssim \delta^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}.
\end{aligned}$$

Thus we derive that

$$|R_2^{21}+R_2^{22}|\lesssim \delta^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}.$$

By the same way, it holds that

$$|R_2^{23}|\lesssim \delta e^{-c(|x|+t)},\quad |R_2^{24}|\lesssim \delta e^{-c(|x|+t)},$$

therefore, we obtain

$$|R_2|\lesssim \delta^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}. \tag{3.28}$$

Similarly, we can also have

$$\begin{aligned} &|(R_{1x},R_{1xx})(x,t)|\lesssim \delta e^{-c(|x|+t)}+\frac{\delta}{(1+t)^{\frac{3}{2}}}e^{\frac{-cx^2}{1+t}},\\ &|(R_{2x},R_{2xx})(x,t)|\lesssim \delta^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}. \end{aligned} \tag{3.29}$$

Then integrating (3.13) on $[0,t]\times\mathbb{R}$ and using the Lemma 2.1 and the above estimates, and taking a and δ small enough, we have

$$\begin{aligned} &\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left[(|U_{1x}^r|+|U_{3x}^r|)(\phi^2+\xi^2)+\xi_x^2\right]dxd\tau\\ \lesssim&\int_{\mathbb{R}}\eta_0dx+\|z_0\|_{L^1}+\delta\int_0^t\frac{1}{1+\tau}\int_{\mathbb{R}}(\phi^2+\xi^2)e^{\frac{-cx^2}{1+\tau}}dxd\tau\\ &+\int_0^t\int_{\mathbb{R}}\left[(\Theta_{1x}^r)^2+(\Theta_{3x}^r)^2\right](\phi^2+\xi^2)dxd\tau+\int_0^t\int_{\mathbb{R}}(|\psi||R_1|+|\xi||R_2|)dxd\tau. \end{aligned} \tag{3.30}$$

Noticing that $|(\Theta_{1x}^r,\Theta_{3x}^r)|\lesssim \delta^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}$, one can easily get

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}}((\Theta_{1x}^r)^2+(\Theta_{3x}^r)^2)(\phi^2+\xi^2)dxd\tau\\ \lesssim&\delta^{\frac{1}{4}}\int_0^t\|(\phi,\xi)\|_\infty^2(1+\tau)^{-\frac{7}{4}}d\tau\lesssim\delta^{\frac{1}{4}}\|(\phi,\xi)\|_1^2\int_0^t(1+\tau)^{-\frac{7}{4}}d\tau\\ \lesssim&N(T)\cdot\delta^{\frac{1}{4}}. \end{aligned} \tag{3.31}$$

For the last term in (3.30), we can derive the following estimate

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}}(|\psi||R_1|+|\xi||R_2|)dxd\tau\\ \lesssim&\delta\int_0^t\|\psi\|_\infty\left(\int_{\mathbb{R}}e^{-c(|x|+\tau)}dx+\int_{\mathbb{R}}\frac{\delta}{(1+\tau)^{\frac{3}{2}}}e^{\frac{-cx^2}{1+\tau}}dx\right)d\tau+\delta^{\frac{1}{8}}\int_0^t\|\xi\|_\infty(1+\tau)^{-\frac{7}{8}}d\tau\\ \lesssim&\delta\int_0^t\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}(1+\tau)^{-1}d\tau+\delta^{\frac{1}{8}}\int_0^t\|\xi\|^{\frac{1}{2}}\|\xi_x\|^{\frac{1}{2}}(1+\tau)^{-\frac{7}{8}}d\tau\\ \lesssim&\delta^{\frac{1}{8}}\int_0^t\|(\psi_x,\xi_x)\|^2d\tau+\delta^{\frac{1}{8}}\int_0^t\|(\psi,\xi)\|^{\frac{2}{3}}(1+\tau)^{-\frac{7}{6}}d\tau\\ \lesssim&\delta^{\frac{1}{8}}\int_0^t\|(\psi_x,\xi_x)\|^2d\tau+\delta^{\frac{1}{8}}. \end{aligned} \tag{3.32}$$

Inserting (3.31) and (3.32) into (3.30), then we can derive (3.11) and complete the proof of Lemma 3.1. □

LEMMA 3.2 (cf. [8, 11]). *Under the assumptions in Proposition 3.2, we have*

$$\int_0^t \frac{1}{1+\tau}\int_{\mathbb{R}}(\phi^2+\psi^2+\xi^2)e^{-\frac{cx^2}{1+\tau}}dxd\tau$$
$$\lesssim 1+\int_0^t \|(\phi_x,\psi_x,\xi_x)\|^2 d\tau+\int_0^t\int_{\mathbb{R}}(|(U_1^r)_x|+|(U_3^r)_x|)(\phi^2+\xi^2)dxd\tau. \tag{3.33}$$

Same as in [11], for $\alpha>0$, we define the important heat kernel $\omega(x,t)$ as follows

$$\omega(x,t)=(1+t)^{-\frac{1}{2}}e^{-\frac{\alpha x^2}{1+t}},$$

if we take note that

$$\begin{aligned}F(x,t)&=v^2(p-P)^2+Pv\psi^2\\&=\left[R\xi+\frac{a}{3}(v\theta^4-V\Theta^4)-P\phi\right]^2+Pv\psi^2,\\G(x,t)&=\left[C_v\xi+a(v\theta^4-V\Theta^4)+P\phi\right]^2,\end{aligned}$$

and by a direct calculation, we have that

$$F(x,t)+G(x,t)\gtrsim \xi^2+\phi^2+\psi^2.$$

Then the proof of (3.33) can be divided into two parts:

$$\int_0^t\int_{\mathbb{R}}\omega^2F(x,t)dxd\tau\lesssim 1+\int_0^t\|(\phi_x,\psi_x,\xi_x)(\tau)\|^2d\tau+\delta\int_0^t\int_{\mathbb{R}}\omega^2(\phi^2+\psi^2+\xi^2)dxd\tau$$
$$+\int_0^t\int_{\mathbb{R}}(|U_{1x}^r|+|U_{3x}^r|)(\phi^2+\xi^2)dxd\tau+\int_0^t\int_{\mathbb{R}}\varphi z dxd\tau, \tag{3.34}$$

and for any $\tilde{\eta}>0$,

$$\int_0^t\int_{\mathbb{R}}\omega^2G(x,t)dxd\tau$$
$$\lesssim 1+\int_0^t\|(\phi_x,\psi_x,\xi_x)(\tau)\|^2d\tau+(\delta+\tilde{\eta})\int_0^t\int_{\mathbb{R}}\omega^2(\phi^2+\psi^2+\xi^2)dxd\tau$$
$$+\int_0^t\int_{\mathbb{R}}(|U_{1x}^r|+|U_{3x}^r|)(\phi^2+\xi^2)dxd\tau+\int_0^t\int_{\mathbb{R}}\varphi z dxd\tau. \tag{3.35}$$

The proof of (3.34) and (3.35) are same as that in [8, 11] but more tedious, we will omit the details for brevity. Now we turn to deal with the higher order estimates.

LEMMA 3.3. *Under the assumptions in Proposition 3.2, we derive that*

$$\|(\phi_x,\psi_x,\xi_x,z_x)(t)\|^2+\int_0^t\|\xi_{xx}(\tau)\|^2+\|z_{xx}(\tau)\|^2d\tau$$
$$\lesssim\|(\phi_{0x},\psi_{0x},\xi_{0x},z_{0x})\|^2+\delta+(\delta+N(t))\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau. \tag{3.36}$$

Proof. Multiplying $(3.3)_{1x}$ by $\frac{P}{v}\phi_x$, $(3.3)_{2x}$ by ψ_x, $(3.3)_{3x}$ by $\frac{\xi_x}{\theta}$ and $(3.3)_{4x}$ by z_x, respectively, and adding the resulting equations together, then

$$\begin{aligned}&\left\{\frac{P}{2v}\phi_x^2+\frac{\psi_x^2}{2}+\frac{C_v\xi_x^2}{2\theta}+\frac{z_x^2}{2}\right\}_t+\frac{\kappa}{v\theta}\xi_{xx}^2+\frac{d}{v^2}z_{xx}^2+\varphi z_x^2+H_{2x}+J\\ &=R_{1x}\psi_x+R_{2x}\frac{\xi_x}{\theta},\end{aligned}\tag{3.37}$$

where

$$\begin{aligned}H_2=&\frac{\xi_x}{\theta}\Big(\Big(\frac{\kappa(v,\theta)\xi_x}{v}-\frac{\kappa(v,\theta)\Theta_x\phi}{vV}\Big)_x-\big[(\kappa(v,\theta)-\kappa(V,\Theta))\frac{\Theta_x}{V}\big]_x\Big)+(\frac{dz_x}{v^2})_x z_x\\ &+\frac{\xi_x}{\theta}\Big((p-P)U_x+(av\theta^4-aV\Theta^4)_t\Big)+\big((p-P)_x+(av\theta^4-aV\Theta^4)_x\big)\psi_x,\\ J=&\Big(\frac{\xi_x}{\theta}\Big)_x\Big(\Big(\frac{\kappa(v,\theta)\xi_x}{v}-\frac{\kappa(v,\theta)\Theta_x\phi}{vV}\Big)_x+\Big[(\kappa(v,\theta)-\kappa(V,\Theta))\frac{\Theta_x}{V}\Big]_x\Big)\\ &-\Big(\frac{\xi_x}{\theta}\Big)_x\Big((p-P)U_x+(av\theta^4-aV\Theta^4)_t\Big)-\frac{\kappa(v,\theta)}{v\theta}\xi_{xx}^2-\Big(\frac{P}{2v}\Big)_t\phi_x^2\\ &-\frac{C_v}{2}\Big(\frac{1}{\theta}\Big)_t\xi_x^2-\Big(\frac{R}{v}\Big)_x\xi\psi_{xx}+\Big(\frac{P}{v}\Big)_x\phi\psi_{xx}+p_x\psi_x\frac{\xi_x}{\theta}\\ &-\frac{d}{v^3}V_xz_xz_{xx}+\varphi_xzz_x-\frac{d}{v^3}\phi_xz_xz_{xx}-\frac{\lambda\varphi z_x\xi_x}{\theta}\\ =&O(1)\big(N(t)+\delta+\eta\big)|(\phi_x,\xi_x,z_x,\psi_{xx},\xi_{xx},z_{xx})|^2+|(V_x,U_x,\Theta_{xx})|^2(\phi^2+\xi^2).\end{aligned}\tag{3.38}$$

Integrating (3.37) on $[0,t]\times\mathbb{R}$ leads to

$$\begin{aligned}&\|(\phi_x,\psi_x,\xi_x,z_x)(t)\|^2+\int_0^t\|\xi_{xx}(\tau)\|^2+\|z_{xx}(\tau)\|^2d\tau\\ \lesssim&\|(\phi_{0x},\psi_{0x},\xi_{0x},z_{0x})\|^2+(\delta+N(t)+\tilde\eta)\int_0^t\|(\phi_x,\psi_x,\xi_x,z_x)(\tau)\|_1^2dxd\tau\\ &+\int_0^t\int_{\mathbb{R}}(|\Theta_{xx}|+|\Theta_x|)^2(\phi^2+\xi^2)dxd\tau+\int_0^t\int_{\mathbb{R}}(|R_{1x}\psi_x|+|R_{2x}\xi_x|)dxd\tau,\end{aligned}\tag{3.39}$$

here $\tilde\eta>0$ is a constant suitably small, and the last two terms on the right-hand side of the last inequality can be treated similarly as (3.31) and (3.32), respectively. Then, with the help of the results of Lemma 3.1 and Lemma 3.2 we can complete the proof of Lemma 3.3. ∎

LEMMA 3.4. *Under the assumption in Proposition 3.2, we derive that*

$$\begin{aligned}&\|(\phi_{xx},\psi_{xx},\xi_{xx},z_{xx})(t)\|^2+\int_0^t\|\xi_{xxx}(\tau)\|^2+\|z_{xxx}(\tau)\|^2d\tau\\ \lesssim&\|(\phi_{0xx},\psi_{0xx},\xi_{0xx},z_{0xx})\|^2+\delta+(\delta+N(t))\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau.\end{aligned}\tag{3.40}$$

Proof. Multiplying $(3.3)_{1xx}$ by $\frac{P}{v}\phi_{xx}$, $(3.3)_{2xx}$ by ψ_{xx}, $(3.3)_{3xx}$ by $\frac{\xi_{xx}}{\theta}$ and $(3.3)_{4xx}$ by z_{xx}, respectively, and adding the results together, it is easy to obtain

$$\left\{\frac{P}{2v}\phi_{xx}^2+\frac{\psi_{xx}^2}{2}+\frac{C_v\xi_{xx}^2}{2\theta}+\frac{z_{xx}^2}{2}\right\}_t+\frac{\kappa(v,\theta)}{v\theta}\xi_{xxx}^2+\frac{d}{v_2}z_{xxx}^2+H_{3x}+J_3=R_{1xx}\psi_{xx}+R_{2xx}\frac{\xi_{xx}}{\theta},\tag{3.41}$$

where

$$
\begin{aligned}
H_3 =&(p-P)_{xx}\psi_{xx}+\frac{\xi_{xx}}{\theta}\bigg((p-P)U_x-\Big(\frac{\kappa(v,\theta)\xi_x}{v}-\frac{\kappa(v,\theta)\Theta_x\phi}{vV}\Big)_x\bigg)_x\\
&-(\frac{dz_x}{v^2})_{xx}-(\varphi z)_{xx}z_{xx},\\
J_3 =&\Big(\frac{\xi_{xx}}{\theta}\Big)_x\bigg(\Big(\frac{\kappa(v,\theta)\xi_x}{v}-\frac{\kappa(v,\theta)\Theta_x\phi}{vV}\Big)_x+\Big((\kappa(v,\theta)-\kappa(V,\Theta))\frac{\Theta_x}{V}\Big)_x\bigg)_x\\
&-\frac{\kappa(v,\theta)}{v\theta}\xi_{xxx}^2-\Big((p-P)U_x+(av\theta^4-aV\Theta^4)_t\Big)_x-\Big(\frac{P}{2v}\Big)_t\phi_{xx}^2-\Big(\frac{C_v}{2\theta}\Big)_t\xi_{xx}^2\\
&+(2p_x\psi_{xx}+p_{xx}\psi_x)\frac{\xi_{xx}}{\theta}+3\Big\{\Big(\frac{R}{v}\Big)_x\xi_x\Big\}_x\psi_{xx}-3\Big\{\Big(\frac{P}{v}\Big)_x\phi_x\Big\}_x\psi_{xx}\\
&+(\lambda\varphi z)_{xx}\cdot\frac{\xi_{xx}}{\theta}+2\frac{d}{v^3}z_{xx}v_x+\frac{d}{v^3}v_{xx}z_x+3\frac{d}{v^4}z_xv_x^2+2J_3^1+J_3^2.
\end{aligned}
\tag{3.42}
$$

Here J_3^1, J_3^2 are the following equations

$$
\begin{aligned}
J_3^1 :=&\psi_{xxx}\bigg(\Big(\frac{P}{v}\Big)_x\phi_x-\Big(\frac{R}{v}\Big)_x\xi_x\bigg)-z_{xxx}(\phi z)_x,\\
J_3^2 :=&\psi_{xxx}\bigg(\Big(\frac{P}{v}\Big)_{xx}\phi-\Big(\frac{R}{v}\Big)_{xx}\xi\bigg).
\end{aligned}
$$

Meanwhile, we can get

$$
\begin{aligned}
J_3^1 =&\bigg\{\psi_{xx}\bigg(\Big(\frac{P}{v}\Big)_x\phi_x-\Big(\frac{R}{v}\Big)_x\xi_x\bigg)\bigg\}_x-\psi_{xx}\bigg(\Big(\frac{P}{v}\Big)_x\phi_x-\Big(\frac{R}{v}\Big)_x\xi_x\bigg)_x\\
&-(z_{xx}(\varphi z)_x)_x+z_{xx}(\varphi z)_{xx}\\
=&\bigg\{\psi_{xx}\bigg(\Big(\frac{P}{v}\Big)_x\phi_x-\Big(\frac{R}{v}\Big)_x\xi_x\bigg)-z_{xx}(\varphi z)_x\bigg\}_x\\
&+O(1)(N(T)+\delta)\left|(\phi_x,\xi_x,\phi_{xx},\psi_{xx},\xi_{xx},z_{xx})\right|^2,
\end{aligned}
\tag{3.43}
$$

and

$$
\begin{aligned}
&\psi_{xxx}\Big(\frac{P}{v}\Big)_{xx}\phi\\
=&\psi_{xxx}\phi\bigg(\frac{P_{xx}}{v}+2P_x\Big(\frac{1}{v}\Big)_x+P\Big(\frac{2v_x^2}{v^3}-\frac{V_{xx}}{v^2}\Big)\bigg)-\frac{P\phi}{v^2}\phi_{xx}\psi_{xxx}\\
=&-\frac{P\phi}{v^2}\phi_{xx}\phi_{txx}+\bigg\{\psi_{xx}\phi\bigg(\frac{P_{xx}}{v}+2P_x\Big(\frac{1}{v}\Big)_x+P\Big(\frac{2v_x^2}{v^3}-\frac{V_{xx}}{v^2}\Big)\bigg)\bigg\}_x\\
&-\psi_{xx}\bigg\{\phi\bigg(\frac{P_{xx}}{v}+2P_x\Big(\frac{1}{v}\Big)_x+P\Big(\frac{2v_x^2}{v^3}-\frac{V_{xx}}{v^2}\Big)\bigg)\bigg\}_x\\
=&-\Big\{\frac{P\phi}{v^2}\frac{\phi_{xx}^2}{2}\Big\}_t+\bigg\{\psi_{xx}\phi\bigg(\frac{P_{xx}}{v}+2P_x\Big(\frac{1}{v}\Big)_x+P\Big(\frac{2v_x^2}{v^3}-\frac{V_{xx}}{v^2}\Big)\bigg)\bigg\}_x\\
&+O(1)(N(T)+\delta)|(\phi_x,\xi_x,\phi_{xx},\psi_{xx},\xi_{xx})|^2+|(\Theta_x,\Theta_{xx})|^2|(\phi,\xi)|^2.
\end{aligned}
\tag{3.44}
$$

Similar to the estimate (3.44), we have

$$\psi_{xxx}\left(\frac{R}{v}\right)_{xx}\xi=\left\{\frac{R\xi}{v^2}\frac{\phi_{xx}^2}{2}\right\}_t+\left\{R\psi_{xx}\xi\left(\frac{V_{xx}}{v^2}-\frac{2v_x^2}{v^3}\right)\right\}_x+|(\Theta_x,\Theta_{xx})|^2|(\phi,\xi)|^2$$
$$+O(1)(N(T)+\delta)|(\phi_x,\xi_x,\phi_{xx},\psi_{xx},\xi_{xx})|^2.$$

By the same way, we have the estimate

$$-\psi_{xxx}\left(av\theta^4-aV\Theta^4\right)_{xx}=-\left(\psi_{xx}\left(av\theta^4-aV\Theta^4\right)_{xx}\right)_x+|(\Theta_x,\Theta_{xx})|^2|(\phi,\xi)|^2$$
$$+O(1)(N(T)+\delta)|(\phi_x,\xi_x,\phi_{xx},\psi_{xx},\xi_{xx})|^2.$$

Therefore, it holds

$$\begin{aligned}J_3=&\left\{\frac{R\xi}{v^2}\frac{\phi_{xx}^2}{2}-\frac{P\phi}{v^2}\frac{\phi_{xx}^2}{2}\right\}_t+\left\{\psi_{xx}\phi\left(\frac{P_{xx}}{v}+2P_x\left(\frac{1}{v}\right)_x+P\left(\frac{2v_x^2}{v^3}-\frac{V_{xx}}{v^2}\right)\right)\right\}_x\\&+\left\{\psi_{xx}R\xi\left(\frac{V_{xx}}{v^2}-\frac{2v_x^2}{v^3}\right)\right\}_x+\left\{\psi_{xx}\left(\left(\frac{P}{v}\right)_x\phi_x-\left(\frac{R}{v}\right)_x\xi_x\right)\right\}_x\\&-\left\{z_{xx}(\varphi z)_x-\psi_{xx}(av\theta^4-aV\Theta^4)_{xx}\right\}_x\\&+O(1)(N(T)+\delta)|(\phi_x,\xi_x,\phi_{xx},\psi_{xx},\xi_{xx})|^2+|(\Theta_x,\Theta_{xx})|^2|(\phi,\xi)|^2.\end{aligned}\tag{3.45}$$

After integrating (3.41) on $[0,t]\times\mathbb{R}$, we get

$$\begin{aligned}&\|(\phi_{xx},\psi_{xx},\xi_{xx},z_{xx})(t)\|^2+\int_0^t\|\xi_{xxx}(\tau)\|^2+\|z_{xxx}(\tau)\|^2d\tau\\\lesssim&\|(\phi_0,\psi_0,\xi_0,z_0)\|_2^2+(\delta+N(t))\int_0^t\|(\phi_x,\psi_x,\xi_x,z_x)(\tau)\|_1^2d\tau\\&+\int_0^t\int_{\mathbb{R}}(|\Theta_{xx}|+|\Theta_x|)^2|(\phi,\xi)|^2dxd\tau+\int_0^t\int_{\mathbb{R}}|R_{1xx}\psi_{xx}+R_{2xx}\xi_{xx}|dxd\tau.\end{aligned}\tag{3.46}$$

For the estimate of the last term in (3.46), we get

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}|R_{1xx}\psi_{xx}|dxd\tau\\\lesssim&\delta\int_0^t\|\psi_{xx}(\tau)\|\left\{\left(\int_{\mathbb{R}}e^{-2c|x|}e^{-2c\tau}dx\right)^{\frac12}+\left(\int_{\mathbb{R}}\frac{1}{(1+\tau)^3}e^{\frac{-2cx^2}{1+\tau}}dx\right)^{\frac12}\right\}d\tau\\\lesssim&\delta\int_0^t(1+\tau)^{-\frac54}\|\psi_{xx}(\tau)\|d\tau\lesssim N(T)\cdot\delta,\end{aligned}\tag{3.47}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}|R_{2xx}\xi_{xx}|dxd\tau\lesssim&\delta^{\frac18}\int_0^t\|\xi_{xx}\|^{\frac12}\|\xi_{xxx}\|^{\frac12}(1+\tau)^{-\frac78}d\tau\\\lesssim&\delta^{\frac18}\int_0^t\|\xi_{xx}\|^2+\|\xi_{xxx}\|^2d\tau+\delta^{\frac18}\int_0^t(1+\tau)^{-\frac74}d\tau\\\lesssim&\delta^{\frac18}\int_0^t\|\xi_{xx}\|^2+\|\xi_{xxx}\|^2d\tau+\delta^{\frac18},\end{aligned}$$

then by virtue of Lemma 3.1-Lemma 3.3, we can complete the proof of Lemma 3.4. ∎

Combining the results of Lemma 3.1-Lemma 3.4, we know that

$$
\begin{aligned}
&\|(\phi,\psi,\xi,z)(t)\|_2^2+\int_0^t\|\xi_x(\tau)\|_2^2+\int_0^t\|z(\tau)\|_3^2d\tau\\
\lesssim&\|(\phi_0,\psi_0,\xi_0,z_0)\|_2^2+\delta+(\delta+N(T)+\eta)\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau. \qquad (3.48)
\end{aligned}
$$

Based on the above analysis, we need to deal with the last term on the right-hand side in (3.48).

LEMMA 3.5. *Under the assumption in Proposition 3.2, we derive that*

$$
\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau\lesssim\|(\phi_0,\psi_0,\xi_0)\|_2^2+\delta. \qquad (3.49)
$$

Proof. Multiplying $(3.3)_2$ by $-\frac{P}{2}\phi_x$, and $(3.3)_3$ by ψ_x, respectively, and adding all the resultant equations yields that

$$
\begin{aligned}
&\Big\{C_v\xi\psi_x-\frac{P}{2}\phi_x\psi+(av\theta^4-aV\Theta^4)\psi_x\Big\}_t\\
&+\Big\{\frac{P}{2}\phi_t\psi-C_v\xi\psi_t-(av\theta^4-aV\Theta^4)\psi_t-\big(\kappa(v,\theta)-\kappa(V,\Theta)\big)\frac{\Theta_x}{V}\psi_x\Big\}_x\\
=&\frac{P_x}{2}\psi\psi_x-\frac{P_t}{2}\phi_x\psi+\frac{P}{2}\phi_x\bigg(\Big(\frac{R\xi}{v}\Big)_x-\Big(\frac{P}{v}\Big)_x\phi-R_1\bigg)-(av\theta^4-aV\Theta^4)_x\psi_t\\
&-C_v\xi_x\psi_t+(av\theta^4-aV\Theta^4)\frac{P_x}{2}\phi_x+(av\theta^4-aV\Theta^4)\frac{P_x}{2}\phi_{xx}\\
&-\big(\kappa(v,\theta)-\kappa(V,\Theta)\big)\frac{\Theta_x}{V}\psi_{xx}+\Big(\kappa(v,\theta)\frac{\xi_x}{v}-\kappa(v,\theta)\frac{\Theta_x\phi}{vV}\Big)_x\psi_x\\
&-(p-P)(U_x+\psi_x)\psi_x+R_2\psi_x+\lambda\varphi z\psi_x. \qquad (3.50)
\end{aligned}
$$

Integrating (3.50) on $[0,t]\times\mathbb{R}$ and using the inequality (3.48), it holds that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}}(\phi_x^2+\psi_x^2)dxd\tau\\
\lesssim&\|(\phi,\psi,\xi)\|_1^2+\|(\phi_0,\psi_0,\xi_0)\|_1^2+\int_0^t\int_{\mathbb{R}}(|\Theta_{xx}|+|\Theta_x|)^2(\phi^2+\psi^2)dxd\tau\\
&+\int_0^t\|\xi_x(\tau)\|_1^2d\tau+\big(\tfrac14+\delta+N(T)\big)\int_0^t\int_{\mathbb{R}}(\phi_x^2+\psi_x^2)dxd\tau\\
&+\int_0^t\int_{\mathbb{R}}|R_1\phi_x+R_2\psi_x|dxd\tau. \qquad (3.51)
\end{aligned}
$$

Similar to the estimates of (3.31) and (3.32), we can easily derive that

$$
\int_0^t\int_{\mathbb{R}}(\phi_x^2+\psi_x^2)dxd\tau\lesssim\|(\phi_0,\psi_0,\xi_0)\|_2^2+\delta+(\delta+N(T))\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau. \quad (3.52)
$$

By the same way, multiplying $(3.3)_{2x}$ by $-\frac{P}{2}\phi_{xx}$, $(3.3)_{3x}$ by ψ_{xx}, respectively, and integrating the result on $[0,t]\times\mathbb{R}$, then by using (3.48), we can also obtain

$$
\int_0^t\int_{\mathbb{R}}(\phi_{xx}^2+\psi_{xx}^2)dxd\tau\lesssim\|(\phi_0,\psi_0,\xi_0)\|_2^2+\delta+(\delta+N(T))\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau. \quad (3.53)
$$

Then adding the results of (3.52) with (3.53) and taking δ small enough, we can complete the proof of Lemma 3.5. □

Inserting (3.49) into (3.48), then we can complete the proof of Proposition 3.2.

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