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ON THE EXISTENCE OF WEAK SOLUTIONS TO NON-LOCAL CAHN-HILLIARD/NAVIER-STOKES EQUATIONS AND ITS LOCAL ASYMPTOTICS*

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Abstract. Cahn-Hilliard/Navier-Stokes system is the combination of the Cahn-Hilliard equation with the Navier-Stokes equations. It describes the motion of unsteady mixing fluids and has a wide range of applications ranging from turbulent two-phase flows to microfluidics. In this paper we consider the non-local Cahn-Hilliard equation (the gradient term of the order parameter in the free energy is replaced with its spatial convolution) coupled with the Navier-Stokes equations. Assuming that the densities of the incompressible fluids are constant and the double-well potential is singular, we establish the existence of global weak solutions to the non-local system in three dimensional torus. In addition, we show that, under suitable initial assumptions, the solutions are asymptotic to those of the local Cahn-Hilliard/Navier-Stokes equations.

Keywords. Weak solutions; Cahn-Hilliard/Navier-Stokes; Non-local model; Local asymptotics.

AMS subject classifications. 76D05; 76D10; 45K05; 35K25.

1. Introduction

Cahn-Hilliard/Navier-Stokes (CH/NS) system is a diffuse-interface model (cf. [3,8]) describing the evolution of mixing fluids. The mixture is assumed to be macroscopically immiscible, but a partial mixing in a small interfacial region, in which the sharp interface is replaced by the Cahn-Hilliard (CH) equation in terms of the order parameter (the difference between two concentrations). CH equation is for modeling the loss of mixture homogeneity and the formation of pure phase regions. The Navier-Stokes (NS) equations are for the hydrodynamics of the mixing fluids, and it is influenced by the order parameter, due to the surface tension and its variations, through an extra capillarity force term. In the case of incompressible fluids with matched constant-densities ($i.e.,\rho=1$), we have the following CH/NS equations

$$\begin{cases} \mathrm{div}\mathbf{u}=0, \\ \partial_t\mathbf{u}+\mathrm{div}(\mathbf{u}\otimes\mathbf{u})+\nabla\pi=\mathrm{div}(\nu\nabla\mathbf{u})+\mu\nabla c, \\ \partial_t c+\mathrm{div}(\mathbf{u}c)=\mathrm{div}(m\nabla\mu), \\ \mu=\kappa^{-1}\Phi'(c)+\kappa\int_\Omega \mathbb{J}(x,y)(c(x,t)-c(y,t))dy, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$, the unknown functions $\mathbf{u},\pi,c,\mu$ are the average velocity, the pressure, the order parameter, the chemical potential, respectively. The viscosity ν, the mobility m, and the interface thickness κ are assumed to be positive constants. The $\Phi(s)$ is a double-well potential function, and $\mathbb{J}(x,y)$ is a positive and symmetric convolution kernel.

System (1.1) is called a non-local CN/NS equations. The total free energy is the sum of the kinetic energy $\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)}$ and the free energy functional of the form

$$\mathcal{E}[c]=\frac{\kappa}{4}\int_\Omega\int_\Omega \mathbb{J}(x,y)|c(x,t)-c(y,t)|^2dydx+\kappa^{-1}\int_\Omega\Phi(c)dx, \tag{1.2}$$

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where (1.2) is originally proposed in the papers [26, 27] by Giacomin-Lebowitz, where the authors considered the hydrodynamic limit of a microscopic model describing a multi-dimensional lattice gas evolving via a Poisson nearest-neighbor process. In this connection, $\mathcal{E}[c]$ is a non-local free energy functional, and the chemical potential function μ is defined as the functional derivative, namely, $\mu = \frac{\delta \mathcal{E}}{\delta c}$.

As a counterpart of (1.2), we have the standard local free energy functional of CH equation

$$\mathcal{E}_{loc}[c] = \frac{\kappa}{2} \int_\Omega |\nabla c|^2 dx + \kappa^{-1} \int_\Omega \Phi(c) dx, \tag{1.3}$$

where the free energy density Φ drives the system towards the segregation of the two phases, and the square of the gradient energetically penalizes the formations of the interface and restrains the segregation. The corresponding CH/NS equations are the local version

$$\begin{cases} \operatorname{div} \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi = \operatorname{div}(\nu \nabla \mathbf{u}) + \mu \nabla c, \\ \partial_t c + \operatorname{div}(\mathbf{u} c) = \operatorname{div}(m \nabla \mu), \\ \mu = \kappa^{-1} \Phi'(c) - \kappa \triangle c. \end{cases} \tag{1.4}$$

The existence of weak solutions of (1.4) (with suitable boundary information) has been established by Abels [1]. The uniqueness and regularity, as well as existence of strong solution, in both two and three dimensions are surveyed by Giorgini-Miranville-Temam [28]. There is abundant other literature about the local CH/NS equations such as [2, 7, 9, 23, 25, 30, 34, 41, 43], and the references therein.

The interest in the non-local diffusion model lies not only in the fundamental physical relevance (cf. [18, 40]) but also its rigorous justification as a macroscopic limit of microscopic phase segregation and the generality (cf. [10, 26]). Particularly, when the convolution kernel $\mathbb{J}(x,y)$ is symmetrical and concentrates around the origin, the behavior of the non-local interface evolution problem approaches to, at least formally, that of the standard local CH equation. Therefore, there is a close connection between the local and non-local energy functionals.

Roughly speaking, if there exists a sequence $\mathbb{J}_\lambda(x,y) = \mathbb{J}_\lambda(|x-y|)$ that approximates Dirac delta as λ goes to zero, the following asymptotic is valid, upon to a constant-multiplicator,

$$\int_\Omega \int_\Omega \mathbb{J}_\lambda(x,y) |c(x,t) - c(y,t)|^2 dy dx \to \int_\Omega |\nabla c|^2 dx \quad (\lambda \downarrow 0).$$

This can be rigorously justified and deeply understood from the seminal papers by Bourgain-Brezis-Mironescu [5, 6] and by Ponce in [37, 38] respectively. From the mathematical standpoint, although the local CH equation in (1.4) is a fourth order differential equation, and the non-local one in (1.1) is an integro-differential second order parabolic equation, they share a lot of fundamental features such as the gradient flow structure (in H^{-1} metric) and the lack of comparison principles.

Let us briefly review some previous results in this direction. We first focus on the single CH equation with non-local diffusion. For the existence, uniqueness, regularity, strict separation property, long-time behavior, and stationary state of the solutions, we refer to [4, 22, 24, 29], etc. Assume that the mobility is a positive constant, and the double-well potential is regular with bounded concavity from below, Melchionna-Ranetbauer [35] surveyed the nonlocal-to-local convergence of the solutions, as the convolution kernel $\mathbb{J}$ tends to a standard Dirac delta, by exploiting the Γ-convergence analysis. In case of periodic domain, Davoli-Ranetbauer-Scarpa-Trussardi [13] established the existence, uniqueness and regularity of solutions with degenerate potential, and showed the local asymptotics from non-local model in a stronger topology. Moreover, they [13] relaxed the restriction imposed in [35] and decompose the potential function as two parts: one is a proper, convex and lower semi-continuous function, another is an integral of Lipschitz-continuous function. Less restrictions allow more physically relevant double-well potentials (including the singular *logarithmic* and *double-obstacle* types). Later, Davoli-Scarpa-Trussardi [14] considered similar questions in a general bounded domain with reasonable boundary conditions. For other related analytical results, we refer readers to the papers [12, 16, 31, 35, 39], and so on.

The coupled system of the NS equations with the non-local CH equation has recently been an active research subject. If the potential function is regular enough which allows any polynomial growth, Colli-Frigeri-Grasselli [11] proved the global existence of a weak solution to (1.1), and discussed the energy identity and dissipative estimates in dimension two. In case when the potential function is singular, Frigeri-Grasselli [20] obtained the existence of global weak solution in a general bounded domain, by using delicate approximations and limit process, additionally, the existence of the global attractor for the generalized semi-flow is surveyed in dimension two. Frigeri-Grasselli-Rocca [21] prove the existence of a global weak solution for both degenerate mobility and non-degenerate mobility cases. Recently, Frigeri [19] obtained the global dissipative weak solutions for unmatched-densities fluids, but the mobility is required to be non-degenerate. See also the papers [15, 17] for the relevant research. We remark that all the aforementioned results on coupled CH/NS system require that the spatial kernel function $\mathbb{J}$ belongs to a regular class $W^{1,1}$, which are usually met by checking the following condition(cf. [35]):

$$\mathbb{J}(x,y) \sim |x-y|^{-\alpha}, \quad \alpha \in (0,3/2).$$

In this paper, we want to establish existence theory of global weak solutions for the non-local CH/NS Equations (1.1) under some structural assumptions. Built upon this we show that the weak solution of (1.1) converges to that of the corresponding local model. Our approach is to adopt and modify some ideas developed in the papers [13, 14, 35] for single non-local CH equations, and in the papers [11, 19, 20] for coupled CH/NS equations.

The rest of this paper is arranged as follows: Section 2 collects some known results and useful lemmas. Our main results are stated in Section 3, and in the final Sections 4-5 we are devoted to proving the Theorem 3.1 and Theorem 3.2 respectively.

2. Preliminaries

First, we denote by $L^2_\sigma(\Omega)$ and $H^1_\sigma(\Omega)$ the completion in $L^2(\Omega)$ and $H^1(\Omega)$ of the space $\{v \in C^\infty(\Omega;\mathbb{R}^3) : \operatorname{div} v = 0\}$ respectively. We refer to [42] and introduce the Stokes operator with null-mean condition. Let

$$A = -P\triangle : \quad H^2(\Omega) \cap L^2_\sigma(\Omega) \mapsto L^2_\sigma(\Omega),$$

where $P : L^2(\Omega) \mapsto L^2_\sigma(\Omega)$ is the Leray projector. The operator $A^{-1} : L^2_\sigma(\Omega) \mapsto L^2_\sigma(\Omega)$ is self-adjoint and compact, and therefore, there exists an increasing sequence of eigenvalues $\{\lambda_i\}_{i\geq 1}$ and corresponding eigenfunctions $\{w_i\}_{i\geq 1}$ in $H^2(\Omega) \cap L^2_\sigma(\Omega)$ which produce an orthonormal basis in $L^2_\sigma(\Omega)$.

Next introduce the operator with null mean

$$B=-\triangle+\mathbb{I}:\quad H^2(\Omega)\mapsto L^2(\Omega).$$

It is clear that B is linear and unbounded, the inverse $B^{-1}:L^2(\Omega)\mapsto L^2(\Omega)$ is self-adjoint and compact. Therefore, there is a sequence of eigenvalues and associated eigenfunctions $\{\psi_i\}_{i\geq 1}$ which form an orthonormal basis in $L^2(\Omega)$.

The following various estimates on the integral quantities are collected from the papers [5,6,37,38].

LEMMA 2.1 ([5, Theorem 1]). *Let the domain $\Omega\subset\mathbb{R}^N$ is bounded and Lipschitz continuous. Assume that $0\leq K_\lambda\subset L^1(\mathbb{R}^N)$ for fixed constant $\lambda>0$. Then, there is some constant C which depends on Ω,p,N such that the inequality*

$$\int_\Omega\int_\Omega K_\lambda(|x-y|)\frac{|f(x)-f(y)|^p}{|x-y|^p}dydx\leq C\|\nabla f\|^p_{L^p(\Omega)} \tag{2.1}$$

holds true for every $f\in W^{1,p}(\Omega)$ with $p\in[1,\infty)$.

LEMMA 2.2 ([37, Theorem 1.1]). *Let $p\in[1,\infty)$, $N\geq 2$, and let $K_\lambda\subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying*

$$\begin{cases} K_\lambda(|x|)\geq 0, \quad \operatorname{supp} K_\lambda\subset\Omega,\\ \displaystyle\int_{\mathbb{R}^N} K_\lambda dx=1,\\ \displaystyle\lim_{\lambda\downarrow 0}\int_\delta^\infty K_\lambda(r)r^2dr=0, \quad \forall\,\delta>0. \end{cases} \tag{2.2}$$

Then, for all $f\in L^p(\Omega)$,

$$\|f-(f)_\Omega\|^p_{L^p(\Omega)}\leq C\int_\Omega\int_\Omega K_\lambda(|x-y|)\frac{|f(x)-f(y)|^p}{|x-y|^p}dydx, \tag{2.3}$$

where $(f)_\Omega$ is the average of $f(x)$ in Ω, and the constant C may depend on Ω,p,N.

We remake that inequality (2.3) improves the standard Poincaré inequality. However, the following lemma shows that the quantities on the right-hand side of (2.3) and (2.1) are in fact identical, as λ tends to zero.

LEMMA 2.3 ([6,38]). *Under the same assumption made in Lemma 2.2. The following inequality holds true*

$$\lim_{\lambda\downarrow 0}\int_\Omega\int_\Omega K_\lambda(|x-y|)\frac{|f(x)-f(y)|^p}{|x-y|^p}dydx=k_{N,p}\int_\Omega|\nabla f|^p dx, \tag{2.4}$$

where $k_{N,p}=\pi^{-\frac12}\Gamma(\frac{N}{2})\Gamma(\frac{1+p}{2})\Gamma^{-1}(\frac{N+p}{2})$, and Γ is the Euler-Gamma function.

Next, we define

$$E_\lambda[f]=\frac14\int_\Omega\mathbb{J}_\lambda(x,y)|f(x)-f(y)|^2dy \quad\text{and}\quad \mathbb{J}_\lambda(x,y)=\frac{K_\lambda(|x-y|)}{|x-y|^2}, \tag{2.5}$$

where K_λ satisfies all hypotheses listed in Lemma 2.2. By scaling K_λ, we deduce from (2.5) and Lemmas 2.2-2.3 that

$$\lim_{\lambda\downarrow 0}\int_\Omega E_\lambda[f]dx=\frac12\int_\Omega|\nabla f|^2dx. \tag{2.6}$$

By Lemma 2.1, $E_\lambda[f]$ is convex and well defined in $H^1(\Omega)$. The variation formula implies that the sub-differential $E_\lambda[f]$ takes

$$\partial E_\lambda[f] = \mathfrak{B}_\lambda[f] := \int_\Omega \mathbb{J}_\lambda(x,y)(f(x)-f(y))dy. \tag{2.7}$$

We remark that $\mathfrak{B}_\lambda[f]$ is linear and meaningful provided that f is regular. In fact, the definition of $\mathfrak{B}_\lambda[f]$ can be extended to $H^1(\Omega)$. We have the following lemma

LEMMA 2.1. *The linear operator $\mathfrak{B}_\lambda$ from $H^1(\Omega)$ to $H^{-1}(\Omega)$ is bounded, that is,*

$$\|\mathfrak{B}_\lambda[f]\|_{H^{-1}(\Omega)} \le C\|f\|_{H^1(\Omega)}, \quad \forall\, f \in H^1(\Omega), \tag{2.8}$$

where the constant C is independent of λ.

In addition, for some sequence $\{f_n\}$ uniformly bounded in $H^1(\Omega)$, there exists respectively a function $f \in H^1(\Omega)$ and a linear operator $\mathfrak{B}$ in $H^{-1}(\Omega)$, such that

$$\lim_{n\uparrow\infty} \langle \mathfrak{B}_\lambda[f_n], g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} = \langle \mathfrak{B}_\lambda[f], g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} \tag{2.9}$$

and

$$\lim_{\lambda\downarrow 0} \langle \mathfrak{B}_\lambda[f], g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} = \langle \mathfrak{B}[f], g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} \tag{2.10}$$

are fulfilled for all $g \in H^1(\Omega)$.

Proof. The proof of (2.8) and (2.10) follows directly from [13, Lemma 2]. By (2.8) and the compactness theory of weak topology, there is some f belonging to $H^1(\Omega)$ such that f_n converges weakly to f in $H^1(\Omega)$. Then, (2.9) follows because the operator $\mathfrak{B}$ is linear. ◻

COROLLARY 2.1. *In (2.10), we claim that the operator $\mathfrak{B} = -\triangle$. Indeed, it follows from (2.7) and the convexity of E_λ that*

$$E_\lambda[f] \ge \langle \mathfrak{B}_\lambda[g], f-g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} + E_\lambda[g], \quad \forall\, f,g \in H^1(\Omega). \tag{2.11}$$

Thanks to (2.6) and (2.10), one deduces by passing $\lambda\downarrow 0$ in (2.11)

$$\frac{1}{2}\int_\Omega |\nabla f|^2 \ge \langle \mathfrak{B}[g], f-g\rangle_{H^{-1}(\Omega)\times H^1(\Omega)} + \int_\Omega |\nabla g|^2, \quad \forall\, f,g \in H^1(\Omega).$$

This implies that (see, e.g., [2, 13]) $\mathfrak{B} \in \partial \frac{1}{2}\|\nabla f\|^2_{L^2(\Omega)} \subset H^{-1}(\Omega)$, and hence $\mathfrak{B} = -\triangle$ in $H^{-1}(\Omega)$.

The final lemma is for the compactness inequalities involving the family of operators $\mathfrak{B}_\lambda$. It can be regarded as a variant of interpolation inequalities.

LEMMA 2.4. *Let $\lambda > 0$ and E_λ be taken from (2.5). For any fixed $\zeta > 0$, there is some constant C which depends on ζ but not on λ, such that the following two inequalities*

$$\|f\|^2_{L^2(\Omega)} \le \zeta\|E_\lambda[f]\|_{L^1(\Omega)} + C\|f\|^2_{H^{-1}(\Omega)}, \quad \forall\, f \in L^2(\Omega)$$

and

$$\|\nabla f\|^2_{L^2(\Omega)} \le \zeta\|E_\lambda[\nabla f]\|_{L^1(\Omega)} + C\|f\|^2_{L^2(\Omega)}, \quad \forall\, f \in H^1(\Omega)$$

are fulfilled.

Proof. The proof can be done by contradiction argument. The detailed process is available in [13, 35]. ◻

3. Main results

In this paper we assume that Ω is a bounded domain with periodic boundary conditions in $\mathbb{R}^3$, namely $\Omega=\mathbb{T}^3$. Equations (1.1) are supplemented with the initial functions

$$(\mathbf{u},c)(x,t=0)=(\mathbf{u}_0,c_0). \tag{3.1}$$

We will first develop an existence result of global weak solution to the non-local CH/NS Equations (1.1) with given initial data (3.1), and then study the asymptotic of the solution (1.1) to that of the local system (1.4) as the interaction kernel $\mathbb{J}_\lambda(x,y)$ concentrates near the origin. Before stating the main results, we need some restriction on the potential function.

HYPOTHESIS 3.1. *The potential function $\Phi(s)$ has the form*

$$\Phi(s)=F(s)+\Pi(s), \quad \forall\, s\in(-1,1),$$

where the functions F and Π satisfy the following properties:

(1) The derivative of $\Pi(s)$ is Lipschitz continuous over $[-1,1]$. Thus, the following growth conditions holds true

$$|\Pi'(s)|\le C(1+|s|) \quad |\Pi(s)|\le C(1+s^2), \quad \forall\, s\in[-1,1].$$

(2) $F\in C^2((-1,1))$ is nonnegative and satisfies

- $$\lim_{|s|\to 1} F(s)=+\infty,$$
- $$\lim_{s\to -1} F'(s)=-\infty \quad \text{and} \quad \lim_{s\to 1} F'(s)=+\infty,$$
- $$\lim_{|s|\to 1} F''(s)=+\infty \quad \text{and} \quad F''(s)\ge 0.$$

Finally, the definition of $\Phi(s)$ is extended to be $+\infty$ outside of $(-1,1)$.

Hypothesis 3.1 is motivated by the physically interesting logarithmic potential suggested by Cahn-Hilliard [8]:

$$\Phi_{log}(s)=\begin{cases}\dfrac{\theta}{2}((1+s)\ln(1+s)+(1-s)\ln(1-s))-\dfrac{\theta_c}{2}s^2, & s\in(-1,1),\\ +\infty, & s\notin(-1,1),\end{cases}$$

where the constants satisfy $0<\theta<\theta_c$. Besides, we include in our analysis other typical examples such as the polynomial potential of degree four and the double-obstacle type.

We state the definition of the weak solutions as follows.

DEFINITION 3.1. *Let $\mathbb{J}(x,y)=\mathbb{J}_\lambda(x,y)$ be as defined in (2.5), and $T<\infty$ be arbitrarily given. The function pair $(\mathbf{u},c)$ is called a weak solution to (1.1) and (3.1) over $(0,T)$, if the following properties hold true:*

- $$\begin{aligned}&\mathbf{u}\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega)),\\&\mu\in L^2(0,T;H^1(\Omega)),\\&c\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\&\partial_t\mathbf{u}\in L^{\frac{4}{3}}(0,T;H^{-1}_\sigma(\Omega));\quad \partial_t c\in L^2(0,T;H^{-1}(\Omega)).\end{aligned}$$

- *For every $\psi\in H^1_\sigma(\Omega)$, $\phi\in H^1(\Omega)$, it holds for a.e. $0\le t\le T$,*

$$\langle \mathbf{u}',\psi\rangle-(\mathbf{u}\otimes\mathbf{u},\nabla\psi)=-(\nu\nabla\mathbf{u},\nabla\psi)+(\mu\nabla c,\psi)\quad ('=\frac{d}{dt})$$

$$\langle c',\phi\rangle-(c\mathbf{u},\nabla\psi)=-(m\nabla\mu,\nabla\phi)$$

and

$$\mu=\mathfrak{B}_\lambda[c]+\Phi'(c)\quad a.e.\ \Omega\times(0,T).$$

- *The initial data (3.1) are meaningful in the sense of weak topologies.*

THEOREM 3.1 (Existence of weak solutions). *Let E_λ and $\mathfrak{B}_\lambda$ be as defined in (2.5) and (2.7) with fixed constant $\lambda>0$, and let the potential Φ satisfy Hypothesis 3.1. Assume that the initial functions in (3.1) satisfy*

$$\mathbf{u}_0\in L^2_\sigma(\Omega),\quad E_\lambda[c_0]\in L^1(\Omega),\quad \Phi(c_0)\in L^1(\Omega). \tag{3.2}$$

Then the problem (1.1) and (3.1) admits a weak solution in the sense of Definition 3.1, satisfying

$$\begin{cases}c\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)),\\E_\lambda[c],\Phi(c)\in L^\infty(0,T;L^1(\Omega)),\\E_\lambda[\nabla c]\in L^1(\Omega\times(0,T)),\\\mathfrak{B}_\lambda[c],\Phi'(c)\in L^2(0,T;L^2(\Omega)),\\-1<c(x,t)<1,\quad a.e.\ \Omega\times(0,T).\end{cases} \tag{3.3}$$

Additionally, the following energy inequality holds true

$$\int_\Omega \mathbb{E}[c](x,t)dx+\int_0^t\int_\Omega\left(|\nabla\mathbf{u}|^2+|\nabla\mu|^2\right)dxds\le\int_\Omega\mathbb{E}[c_0](x)dx,\quad a.e.\ t\in(0,T), \tag{3.4}$$

where

$$\mathbb{E}[c]=E_\lambda[c]+\Phi(c)+\frac{1}{2}|\mathbf{u}|^2. \tag{3.5}$$

REMARK 3.1. *By Hypothesis 3.1, the $L^1(\Omega)$ bound of Φ implies, for constants C and C_1,*

$$C\ge\int_\Omega\Phi(c_0(x))dx=\left(\int_{\{|c_0|<1\}}+\int_{\{|c_0|\ge1\}}\right)\Phi(c_0(x))dx\ge\int_{\{|c_0|\ge1\}}\Phi(c_0(x))dx-C_1.$$

Thus, $c_0\in(-1,1)$ *a.e. in* Ω *since* $\Phi=+\infty$ *outside of* $(-1,1)$. *Consequently, the mean value of* c_0 *lies in* $(-1,1)$, *that is,*

$$\frac{1}{|\Omega|}\int_\Omega c_0(x)dx=(c_0)_\Omega\in(-1,1).$$

REMARK 3.2. We remark that, in case of regular potentials, the lack of maximum principle makes the bound of $c(x,t)$ out of reach, even if the initial $c_0(x)$ is bounded.

REMARK 3.3. Theorem 3.1 relaxed the restriction imposed on $\mathbb{J}_\lambda$ in [11,19,20], where $\mathbb{J}_\lambda(x,y)\in W^{1,1}(\Omega)$ is required to guarantee the validity of gradient operation $\nabla(\mathbb{J}_\lambda * c_\lambda)$.

REMARK 3.4. We compare the regularity $c\in L^\infty(0,T;H^1(\Omega))$ in (3.3) with that in [11,20]. In case of either the regular potentials with polynominal growth (cf. [11]), or the singular potentials with additional $c_0\in L^\infty$ (cf. [20]), it has $c\in L^\infty(0,T;L^{2+2q})$ for some integer $q>0$.

The second theorem shows that, as $\lambda\downarrow 0$, the weak solution in Theorem 3.1 converges to that of the corresponding local CH/NS Equations (1.4).

THEOREM 3.2 (Local asymptotics). *For given small* $\lambda_0>0$, *assume that the functions* $(\mathbf{u}_{0\lambda},c_{0\lambda})$ *are uniformly bounded, i.e.,*

$$\sup_{\lambda\in(0,\lambda_0)}\left(\|\mathbf{u}_{0\lambda}\|_{L^2_\sigma(\Omega)}+\|E_\lambda[c_{0\lambda}]\|_{L^1(\Omega)}+\|\Phi(c_{0\lambda})\|_{L^1(\Omega)}\right)\le C, \tag{3.6}$$

and additionally, for some $(\mathbf{u}_0,c_0)\in L^2_\sigma(\Omega)\times H^1(\Omega)$,

$$\mathbf{u}_{0\lambda}\rightharpoonup\mathbf{u}_0\in L^2_\sigma(\Omega),\quad c_{0\lambda}\rightharpoonup c_0\in H^1(\Omega). \tag{3.7}$$

If $(\mathbf{u}_\lambda,c_\lambda)$ *is a weak solution to Equations* (1.1) *associated to the initial* $(\mathbf{u}_{0\lambda},c_{0\lambda})$, *as stated in Theorem 3.1. Then* $(\mathbf{u}_\lambda,c_\lambda)$ *converges to limit functions* $(\mathbf{u},c)$ *such that*

$$\begin{cases}\mathbf{u}\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega)),\\ \mu\in L^2(0,T;H^1(\Omega)),\\ c\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ \partial_t\mathbf{u}\in L^{\frac43}(0,T;H^{-1}_\sigma(\Omega)),\quad \partial_t c\in L^2(0,T;H^{-1}(\Omega)).\end{cases} \tag{3.8}$$

Furthermore, $(\mathbf{u},c)$ *solves Equations* (1.4) *in distributional sense, and agrees with* $(\mathbf{u}_0,c_0)$ *at initial time. The following energy inequality*

$$\int_\Omega \mathbb{E}_{loc}[c](x,t)dx+\int_s^t\int_\Omega\left(|\nabla\mathbf{u}|^2+|\nabla\mu|^2\right)dxd\tau\le\int_\Omega\mathbb{E}_{loc}[c](x,s)dx \tag{3.9}$$

is fulfilled for a.e. $0<s<t<T$, *and*

$$\mathbb{E}_{loc}[c]=\frac12|\mathbf{u}|^2+\frac12|\nabla c|^2+\Phi(c).$$

REMARK 3.5. The assumption of $c_0\in H^1(\Omega)$ imposed in (3.7) is reasonable. In fact, the uniform bound of $\|E_\lambda[c_{0\lambda}]\|_{L^1(\Omega)}$ in (3.6) guarantees that, by [37, Theorem 1.2], the sequence $\{c_{0\lambda}\}_\lambda$ is relatively compact in $L^2(\Omega)$, and hence $c_0\in H^1(\Omega)$ if $c_{0\lambda}\to c_0$ in $L^2(\Omega)$.

We comment on the analysis in the following two paragraphs.

The proof of Theorem 3.1 adopts and borrows some ideas developed in [5,11,13,19–21,32]. Our main contribution lies in the following several aspects. First, in the light of [5], we relaxed the regularity assumption on the $\mathbb{J}$ through transfer of the singular of $\mathbb{J}$ to c by partial integrations. Precisely, instead of analyzing the isolated term $\mathbb{J}_\lambda * c_\lambda$, we treat $\mathbb{J}_\lambda c_\lambda - \mathbb{J}_\lambda * c_\lambda$ as a whole so that less restriction is required on $\mathbb{J}_\lambda$, by means of integration by parts (because the domain is periodic) and higher regularity on c_λ. With the advantage of $\mathbb{J}$, our initial assumptions (especially on c_0) are from a different angle of view, and hence, a higher regularity on c is achieved, see Remark 3.4. Next, following the ideas in [20], we construct a family of approximating potentials which are regular and defined on the whole of $\mathbb{R}$, and then pass to a limit once the uniform estimates on the approximations are obtained. Another difficulty comes from the weaker compactness caused by the non-local chemical potential. This could be solved by adding an artificial perturbation $\delta\triangle c$, like in [13,19]. The proof of Theorem 3.1 is via three-fold approximation: (i) We begin with the Equations (4.4), in which the potential is assumed to be regular and an extra artificial diffusion term $\delta\triangle c$ is introduced. The existence of solution to (4.4) is based on the Faedo-Galerkin method. (ii) Adopting some ideas in [19–21], we build the uniform estimates on the approximations of (4.4) such that the singular potential is allowed by a limit procedure. (iii) By constructing appropriate initial approximations, utilizing the ideas developed in [13,32], we disappear the artificial diffusion to derive the weak solutions to our original problem.

The proof of Theorem 3.2 is in the spirit of the papers [13, 35] concerning the CH equation, as well as the integral estimates developed in [5,37] (i.e., Γ-convergence analysis). On the basis of results in Theorem 3.1, we show that the solutions of non-local Equations (1.1) converge to those of corresponding local model. However, the coupled Navier-Stokes equations make the regularity wilder, and hence the analysis is much more complicated than the single CH equation case. To our best knowledge, it is the first local asymptotics result from the non-local CH/NS Equations (1.1).

The proof of Theorem 3.1 and Theorem 3.2 will be carried out in the next Section 4 and Section 5, respectively. In what follows, the parameters ν, κ, m in Equations (1.1) are taken to be unity for simplicity reasons, i.e., $\nu = \kappa = m = 1$.

4. Proof of Theorem 3.1

Without causing the confusion, we will drop the subscript in (2.5), (2.7), and use $E = E_\lambda, \mathfrak{B} = \mathfrak{B}_\lambda$ for simplicity.

4.1. Faedo-Galerkin approximation. Introduce the regular version of the potential and consider the function of the form

$$F_\epsilon(s) = \begin{cases} F(1-\epsilon) + F'(1-\epsilon)(s+\epsilon-1) + \dfrac{1}{2}F''(1-\epsilon)(s+\epsilon-1)^2, & s \geq 1-\epsilon, \\ F(s), & |s| \leq 1-\epsilon, \\ F(\epsilon-1) + F'(\epsilon-1)(s+1-\epsilon) + \dfrac{1}{2}F''(\epsilon-1)(s+1-\epsilon)^2, & s \leq \epsilon - 1. \end{cases} \tag{4.1}$$

By Hypothesis 3.1, there is a small constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$,

$$|F'_\epsilon(s)|^2 \leq C_1(1+s^2), \quad C_2 s^2 - C_3 \leq F_\epsilon(s) \leq C_4 s^2 + C_5, \quad \forall\, s \in \mathbb{R}, \tag{4.2}$$

where the constants may depend on ϵ.

In addition, we assume that the function Π_ϵ is a C^1 extension of Π to $\mathbb{R}$, with Π'_ϵ being uniform in ϵ bounded in $\mathbb{R}$.

LEMMA 4.1. *For fixed constant $\epsilon>0$ and $\delta>0$, assume that the initial functions*

$$\mathbf{u}_0\in L^2_\sigma(\Omega) \quad \text{and} \quad c_0\in H^1(\Omega). \tag{4.3}$$

Then the following equations admit a weak solution in distributional sense

$$\begin{cases} \operatorname{div}\mathbf{u}=0, \\ \partial_t\mathbf{u}+\operatorname{div}(\mathbf{u}\otimes\mathbf{u})+\nabla\pi=\triangle\mathbf{u}+\mu\nabla c, \\ \partial_t c+\operatorname{div}(\mathbf{u}c)=\triangle\mu, \\ \mu=-\delta\triangle c+F'_\epsilon(c)+\Pi'_\epsilon(c)+\mathfrak{B}[c], \end{cases} \tag{4.4}$$

where $\mathfrak{B}[c]=\mathfrak{B}_\lambda[c]$, $F_\epsilon(s)$, and Π_ϵ are the same as mentioned earlier.

Proof. The proof is in the framework of Faedo-Galerkin approximation scheme. As mentioned in Section 2, let the family $\{w_i\}_{i\geq 1}$ of the eigenfunctions of the Stokes operator A be a Galerkin base in $L^2_\sigma(\Omega)$, and the family $\{\psi_i\}_{i\geq 1}$ of the eigenfunctions of the operator $B=-\triangle+\mathbb{I}$ be a Galerkin base in $H^1(\Omega)$, respectively. Consider the approximation $(\mathbf{u}_{0n},c_{0n})$ of the initial data in (4.3)

$$\sum_{i=1}^n a_i^n(0)w_i(x)=\mathbf{u}_{0n}\to\mathbf{u}_0 \text{ in } L^2_\sigma(\Omega), \quad \sum_{i=1}^n b_i^n(0)\psi_i(x)=c_{0n}\to c_0 \text{ in } H^1(\Omega). \tag{4.5}$$

We want to look for functions of the form

$$\mathbf{u}_n:=\sum_{i=1}^n a_i^n(t)w_i(x) \quad \text{and} \quad c_n:=\sum_{i=1}^n b_i^n(t)\psi_i(x), \tag{4.6}$$

which satisfy, for each $k=1,2...n$,

$$\begin{cases} \int_\Omega \mathbf{u}'_n w_k+\int_\Omega \nabla\mathbf{u}_n\nabla w_k=\int_\Omega \mathbf{u}_n\otimes\mathbf{u}_n\cdot\nabla w_k+\int_\Omega w_k\nabla c_n\mu_n, \\ \int_\Omega c'_n\psi_k+\int_\Omega \nabla\mu_n\nabla\psi_k=\int_\Omega \mathbf{u}_n\cdot\nabla\psi_k c_n, \\ \mu_n=\mathcal{P}_n\left(\mathfrak{B}[c_n]+F'_\epsilon(c_n)+\Pi'_\epsilon(c_n)-\delta\triangle c_n\right):=\sum_{i=1}^n d_i^n(t)\psi_i(x), \end{cases} \tag{4.7}$$

where $\mathcal{P}_n$ denotes the orthogonal projector of element in $H^1(\Omega)$ in the sub-space generated by $\{\psi_1,\psi_2,......,\psi_n\}$. Thus, for each $k=1,2,...,n$,

$$\begin{aligned} \int_\Omega \mu_n\psi_k &= \int_\Omega \mathcal{P}_n\left(\mathfrak{B}[c_n]+F'_\epsilon(c_n)+\Pi'_\epsilon(c_n)-\delta\triangle c_n\right)\psi_k \\ &= \int_\Omega \left(\mathfrak{B}[c_n]+F'_\epsilon(c_n)+\Pi'_\epsilon(c_n)-\delta\triangle c_n\right)\psi_k. \end{aligned} \tag{4.8}$$

The solvability of system (4.7) can be transformed into the ordinary differential equations in terms of $\mathbf{a}_n=(a_1^n,a_2^n,\cdots,a_n^n)$ and $\mathbf{b}_n=(b_1^n,b_2^n,\cdots,b_n^n)$ with initial values given in (4.5). By Cauchy-Lipschitz theorem, it has a unique continuous solution over $[0,T_n)$ for some $T_n>0$. We prove that

$$T_n=T=\infty. \tag{4.9}$$

Multiply $(4.7)_1$ by $\mathbf{u}_n$, $(4.7)_2$ by μ_n, utilize the fact that $\mathbf{u}_n$ is divergence-free, to deduce

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\mathbf{u}_n|^2\right)+\int_\Omega (c_n)'\mu_n+\int_\Omega |\nabla \mathbf{u}_n|^2+\int_\Omega |\nabla\mu_n|^2=0.$$

Observe from (2.7) and (4.8) that

$$\begin{aligned}\int_\Omega (c_n)'\mu_n &= \int_\Omega (c_n)'(\mathfrak{B}[c_n]+F_\epsilon'(c_n)+\Pi_\epsilon'(c_n)-\delta\triangle c_n)\\ &= \frac{d}{dt}\int_\Omega\left(E[c_n]+F_\epsilon(c_n)+\Pi_\epsilon(c_n)+\frac{\delta}{2}|\nabla c_n|^2\right),\end{aligned}$$

we obtain

$$\frac{d}{dt}\int_\Omega \mathbb{E}_{n,\epsilon}(x,t)+\int_\Omega\left(|\mathbf{u}_n|^2+|\nabla\mu_n|^2\right)=0, \tag{4.10}$$

where

$$\mathbb{E}_{n,\epsilon}:=\left(E[c_n]+F_\epsilon(c_n)+\Pi_\epsilon(c_n)+\frac{\delta}{2}|\nabla c_n|^2+\frac{1}{2}|\mathbf{u}_n|^2\right). \tag{4.11}$$

Thanks to (4.2)-(4.3), (4.5), and the fact $\|E[c_{0n}]\|_{L^1}\le\|\nabla c_{0n}\|_{L^2}^2$ (see (2.1) and (2.5)), we have

$$\int_\Omega \mathbb{E}_{n,\epsilon}(x,0)\le C,$$

where, during this subsection, the generic constant C depends on ϵ,δ but is uniform in n; additionally, the dependence on a specific parameter of C will be marked through a subscript. Therefore, integrating (4.10) over $(0,T_n)$ gives

$$\begin{aligned}&\|\mathbf{u}_n\|_{L^\infty(0,T_n;L^2_\sigma(\Omega))\cap L^2(0,T_n;H^1_\sigma(\Omega))}\le C,\\ &\|\nabla\mu_n\|_{L^2(0,T_n;L^2(\Omega))}\le C,\\ &\|c_n\|_{L^\infty(0,T_n;H^1(\Omega))}\le C,\\ &\|F_\epsilon(c_n)\|_{L^\infty(0,T_n;L^1(\Omega))}+\|\Pi_\epsilon(c_n)\|_{L^\infty(0,T_n;L^1(\Omega))}\le C.\end{aligned} \tag{4.12}$$

Suppose (4.9) is false and assume the opposite: $T_n<\infty$, then the uniform bounds in (4.12) enable us to define $(\hat{\mathbf{u}}_n,\hat{c}_n)(x,T_n^*)=\limsup_{t\uparrow T_n}(\mathbf{u}_n,c_n)(x,t)$. Taking T_n^* as a new initial time, we can extend the existence of $(\mathbf{u}_n,c_n)$ beyond T_n^*. A contradiction arises. Thus, (4.9) follows.

Observe that $-\triangle c_n$ still lies in the sub-space generated by $\{\psi_i\}_{i=1}^n$, we multiply $(4.7)_3$ by $-\triangle c_n$, to receive

$$\begin{aligned}\|\nabla c_n\|_{L^2}^2+C\|\nabla\mu_n\|_{L^2}^2 &\ge \int_\Omega \nabla\mu_n\nabla c_n\\ &= \int_\Omega (\nabla\mathfrak{B}[c_n]+F_\epsilon''(c_n)\nabla c_n)\nabla c_n+\int_\Omega \Pi_\epsilon'(c_n)\triangle c_n+\delta\int_\Omega|\triangle c_n|^2\\ &\ge \int_\Omega \Pi_\epsilon'(c_n)\triangle c_n+\delta\int_\Omega|\triangle c_n|^2\\ &\ge \frac{\delta}{2}\|\triangle c_n\|_{L^2}^2-C_\delta,\end{aligned} \tag{4.13}$$

where in the last two inequalities we have used $F''\geq 0$, as well as

$$\int_\Omega \Pi'_\epsilon(c_n)\triangle c_n \geq -C\|\triangle c_n\|_{L^2} \geq -\frac{\delta}{2}\|\triangle c_n\|_{L^2}^2 - C_\delta, \tag{4.14}$$

and

$$\int_\Omega \nabla\mathfrak{B}[c_n]\nabla c_n = \frac{1}{2}\int_\Omega\int_\Omega \mathbb{J}_\lambda(x,y)|\nabla c_n(x,t)-\nabla c_n(y,t)|^2 dydx \geq 0,$$

which comes from (2.5), (2.7), and the integrations by parts. Inequalities (4.13) and (4.12) yield

$$\|c_n\|_{L^2(0,T;H^2(\Omega))} \leq C_T. \tag{4.15}$$

Next, it gives from (4.2) that

$$\int_\Omega |F'_\epsilon(c_n)| \leq C + \int_\Omega |F'_\epsilon(c_n)|^2 \leq C + C\int_\Omega |F_\epsilon(c_n)|,$$

which, along with (4.12) and the fact $\int_\Omega \mathfrak{B}[c_n]dx = 0$, implies

$$\int_\Omega \mu_n = \int_\Omega (\mathfrak{B}[c_n] + F'_\epsilon(c_n) + \Pi'_\epsilon(c_n) - \delta\triangle c_n) \leq C.$$

By this and (4.12), we use Pioncaré inequality and deduce

$$\mu_n \in L^2(0,T;H^1(\Omega)). \tag{4.16}$$

We next derive the estimates on the time derivatives for $\mathbf{u}_n$ and c_n such that weak limits could be taken from the sequences of $(\mathbf{u}_n, c_n)$. Taking $C^2 \ni h(x) \leftarrow h_j(x) = \sum_{k=1}^j e_k^j w_k(x)$ and $\phi \in C^1([0,T])$, one deduces from $(4.7)_1$ that

$$\int_\Omega \mathbf{u}'_n h_n\phi = -\int_\Omega \nabla\mathbf{u}_n\nabla(h_n\phi) + \int_\Omega \mathbf{u}_n\otimes\mathbf{u}_n\cdot\nabla(h_n\phi) + \int_\Omega h_n\phi\nabla c_n\mu_n. \tag{4.17}$$

By virtue of (4.12), (4.16), direct computation shows

$$\left|-\int_\Omega \nabla\mathbf{u}_n\nabla(h_n\phi)\right| \leq \|\nabla\mathbf{u}_n\|_{L^2_\sigma}\|h\phi\|_{H^1_\sigma},$$

$$\left|\int_\Omega \mathbf{u}_n\otimes\mathbf{u}_n\cdot\nabla(h_n\phi)\right| \leq \|\mathbf{u}_n\|_{L^2_\sigma}^{\frac{1}{2}}\|\nabla\mathbf{u}_n\|_{H^1_\sigma}^{\frac{3}{2}}\|\nabla(h\phi)\|_{L^2_\sigma} \leq C\|\nabla\mathbf{u}_n\|_{H^1_\sigma}^{\frac{3}{2}}\|h\phi\|_{H^1_\sigma}$$

and

$$\left|\int_\Omega h_n\phi\nabla c_n\mu_n\right| \leq \|\mu_n\|_{L^2}^{\frac{1}{2}}\|\nabla\mu_n\|_{L^2}^{\frac{1}{2}}\|\nabla c_n\|_{L^2}\|h\phi\|_{L^6_\sigma} \leq C\|\mu_n\|_{H^1}\|h\phi\|_{H^1_\sigma}.$$

With the above estimates, integrating (4.17) in time yields

$$\partial_t\mathbf{u}_n \in L^{\frac{4}{3}}(0,T;H^{-1}_\sigma(\Omega)). \tag{4.18}$$

In a similar method, we select $C^2 \ni \tilde{h}(x) \leftarrow \tilde{h}_j(x) = \sum_{k=1}^{j} \tilde{e}_k^j \psi_k(x)$, and deduce from $(4.7)_2$ that

$$\int_\Omega c_n' \tilde{h}_n \phi = -\int_\Omega \nabla \mu_n \nabla(\tilde{h}_n \phi) + \int_\Omega \mathbf{u}_n \cdot \nabla(\tilde{h}_n \phi) c_n.$$

Owing to (4.12), one has

$$\left| -\int_\Omega \nabla \mu_n \nabla(\tilde{h}_n \phi) \right| \le \|\nabla \mu_n\|_{L^2} \|\tilde{h}\phi\|_{H^1},$$

$$\left| -\int_\Omega \mathbf{u}_n c_n \nabla(\tilde{h}_n \phi) \right| \le \|\mathbf{u}_n\|_{H^1_\sigma} \|c_n\|_{H^1} \|\nabla(\tilde{h}\phi)\|_{L^2} \le C \|\mathbf{u}_n\|_{H^1_\sigma} \|\tilde{h}\phi\|_{H^1},$$

and whence,

$$\partial_t c_n \in L^2(0,T;H^{-1}(\Omega)). \tag{4.19}$$

Therefore, by (4.12), (4.15)-(4.16), (4.18)-(4.19), and Aubin-Lions lemma, we deduce

$$\begin{cases} \mathbf{u}_n \rightharpoonup \mathbf{u}, \quad L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)); \\ \mu_n \rightharpoonup \mu, \quad L^2(0,T;H^1(\Omega)); \\ c_n \rightharpoonup c, \quad L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)); \\ \mathbf{u}_n \to \mathbf{u}, \quad L^2(0,T;L^q_\sigma(\Omega))\ (q<6); \\ \mathbf{u}_n \text{ is compact in } C_w([0,T];L^2_\sigma(\Omega)); \\ c_n \text{ is compact in } C_w([0,T];H^1(\Omega)). \end{cases} \tag{4.20}$$

Additionally, from (2.9), (4.20), the continuity of F_ϵ' and Π_ϵ', one has

$$\begin{aligned} \int_0^T \int_\Omega \tilde{h}\phi\mu \leftarrow \int_0^T \int_\Omega \tilde{h}\phi\mu_n &= \int_0^T \int_\Omega \tilde{h}\phi(\mathfrak{B}[c_n] + F_\epsilon'(c_n) + \Pi_\epsilon'(c_n) - \delta \triangle c_n) \\ &\to \int_0^T \int_\Omega \tilde{h}\phi(\mathfrak{B}[c] + F_\epsilon'(c) + \Pi_\epsilon'(c) - \delta \triangle c), \end{aligned}$$

and thus,

$$\mu = \mathfrak{B}[c] + F_\epsilon'(c) + \Pi_\epsilon'(c) - \delta \triangle c, \quad a.e.\ \Omega \times (0,T). \tag{4.21}$$

In terms of (4.20), we are allowed to take limits in (4.7) in weak sense such that the first three equations in (4.4) are satisfied by the limit functions, moreover, the initial conditions (4.3) are meaningful in the sense of weak topology. This together with (4.21) complete the proof of Lemma 4.1. ∎

4.2. Limit for singular potential. We want to pass $\epsilon \downarrow 0$ in $(\mathbf{u}_\epsilon, c_\epsilon)$, the solution sequence obtained in Lemma 4.1, so that the singular potential function is permitted.

Lemma 4.2. *For fixed $\delta > 0$, assume that the initial functions $(\mathbf{u}_0, c_0)$ satisfy the assumptions made in Lemma 4.1. Assume in addition that*

$$F(c_0) \in L^1(\Omega). \tag{4.22}$$

Then, the approximating solutions $(\mathbf{u}_\epsilon, c_\epsilon)$ obtained in Lemma 4.1 converge to some limit function $(\mathbf{u}, c)$ which solves the following equations in distributional sense

$$\begin{cases} \operatorname{div}\mathbf{u}=0, \\ \partial_t\mathbf{u}+\operatorname{div}(\mathbf{u}\otimes\mathbf{u})+\nabla\pi=\triangle\mathbf{u}+\mu\nabla c, \\ \partial_t c+\operatorname{div}(\mathbf{u}c)=\triangle\mu, \\ \mu=-\delta\triangle c+F'(c)+\Pi'(c)+\mathfrak{B}[c]. \end{cases} \tag{4.23}$$

Furthermore,

$$-1<c(x,t)<1 \quad a.e.\ \Omega\times(0,T), \tag{4.24}$$

and the following inequality holds true

$$\int_\Omega \mathbb{E}_\delta(x,t)dx+\int_0^t\int_\Omega \left(|\nabla\mathbf{u}|^2+|\nabla\mu|^2\right)dxdt\le\int_\Omega \mathbb{E}_\delta(x,0)dx, \quad a.e.\, t\in(0,T) \tag{4.25}$$

with

$$\mathbb{E}_\delta=\left(E[c]+\Phi(c)+\frac{\delta}{2}|\nabla c|^2+\frac{1}{2}|\mathbf{u}|^2\right). \tag{4.26}$$

REMARK 4.1. Inequality (2.1), Hypothesis 3.1, the initial assumption (4.3) in Lemma 4.1 guarantee that

$$E(c_0)\in L^1(\Omega) \quad \text{and} \quad \Pi(c_0)\in L^1(\Omega). \tag{4.27}$$

Proof. Multiplying (4.10) by a non-increasing $0\le\phi\in C_0^1([0,T))$, integrating it in time, we get

$$\int_0^T\int_\Omega \phi'\mathbb{E}_{n,\epsilon}(x,t)+\int_0^T\int_\Omega \phi\left(|\nabla\mathbf{u}_n|^2+|\nabla\mu_n|^2\right)=\int_\Omega \mathbb{E}_{n,\epsilon}(x,0), \tag{4.28}$$

where $\mathbb{E}_{n,\epsilon}$ is defined in (4.11). Straightforward calculation shows

$$\begin{aligned} &E[c_n]-E[c] \\ &=\int_\Omega \mathbb{J}_\lambda\left(|c_n(x,t)-c_n(y,t)|^2-|c(x,t)-c(y,t)|^2\right)dy \\ &=\int_\Omega \mathbb{J}_\lambda((c_n-c)(x,t)-(c_n-c)(y,t))((c_n+c)(x,t)-(c_n+c)(y,t))dy \\ &\le\left(\int_\Omega \mathbb{J}_\lambda|(c_n-c)(x,t)-(c_n-c)(y,t)|^2dy\right)^{\frac12}\left(\int_\Omega \mathbb{J}_\lambda|(c_n+c)(x,t)-(c_n+c)(y,t)|^2dy\right)^{\frac12}, \end{aligned}$$

which, along with (2.1), yields

$$\left|\int_0^T\int_\Omega \phi'(t)E[c_n]-\int_0^T\int_\Omega \phi'(t)E[c]\right|\le C\|(c_n-c)\|_{L^2(0,T;H^1)}\|(c_n+c)\|_{L^2(0,T;H^1)}. \tag{4.29}$$

By (4.12) and (4.15), it satisfies $c_n\in H^1(0,T;H^{-1})\cap L^2(0,T;H^2)\hookrightarrow\hookrightarrow L^2(0,T;H^1)$. Hence, (4.29) leads to

$$\lim_{n\to\infty}\int_0^T\int_\Omega \phi'(t)E[c_n]=\int_0^T\int_\Omega \phi'(t)E[c]. \tag{4.30}$$

In terms of (4.5), (4.12), (4.30), the continuity of Π_ϵ, we are able to pass $n\to\infty$ in (4.28) such that the limit function $(\mathbf{u}_\epsilon,c_\epsilon,\mu_\epsilon)$ satisfies

$$\int_0^T\phi'\int_\Omega \mathbb{E}_{\epsilon,\delta}(x,t)+\int_0^T\int_\Omega\phi\left(|\nabla\mathbf{u}_\epsilon|^2+|\nabla\mu_\epsilon|^2\right)\le\int_\Omega\mathbb{E}_{\epsilon,\delta}(x,0), \tag{4.31}$$

with

$$\mathbb{E}_{\epsilon,\delta}:=\left(E[c_\epsilon]+F_\epsilon(c_\epsilon)+\Pi_\epsilon(c_\epsilon)+\frac{\delta}{2}|\nabla c_\epsilon|^2+\frac{1}{2}|\mathbf{u}_\epsilon|^2\right).$$

For fixed $t\in(0,T)$, define

$$\phi_\eta(s)=\begin{cases}1, & 0\le s\le t,\\ \dfrac{t+\eta-s}{\eta}, & t\le s\le t+\eta.\end{cases} \tag{4.32}$$

Inserting (4.32) into (4.31) and sending $\eta\downarrow 0$, we obtain, for a.e. $t\in(0,T)$,

$$\int_\Omega\mathbb{E}_{\epsilon,\delta}(x,t)+\int_0^t\int_\Omega\left(|\nabla\mathbf{u}_\epsilon|^2+|\nabla\mu_\epsilon|^2\right)\le\int_\Omega\mathbb{E}_{\epsilon,\delta}(x,0). \tag{4.33}$$

Next, by the definitions of F_ϵ,Π_ϵ, and Hypothesis 3.1, we have

$$F_\epsilon(s)\le F(s),\quad |F'_\epsilon(s)|\le|F'(s)|,\quad \Pi_\epsilon(s)=\Pi(s),\quad \Pi'_\epsilon(s)=\Pi'(s),\quad s\in(-1,1). \tag{4.34}$$

Hence, (4.3), (4.22), (4.27), (4.34) and the fact $|c_0|<1$ (see Remark 3.1) ensure that

$$\begin{aligned}\int_\Omega\mathbb{E}_{\epsilon,\delta}(x,0)&=\int_\Omega\left(E[c_0]+F_\epsilon(c_0)+\Pi_\epsilon(c_0)+\frac{\delta}{2}|\nabla c_0|^2+\frac{1}{2}|\mathbf{u}_0|^2\right)\\&\le\int_\Omega\left(E[c_0]+F(c_0)+\Pi(c_0)+\frac{\delta}{2}|\nabla c_0|^2+\frac{1}{2}|\mathbf{u}_0|^2\right)\\&=\int_\Omega\mathbb{E}_\delta(x,0)\\&\le C.\end{aligned} \tag{4.35}$$

Combining (4.35) with (4.33), we conclude

$$\begin{aligned}&\|\mathbf{u}_\epsilon\|_{L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1_\sigma(\Omega))}\le C,\\&\|\nabla\mu_\epsilon\|_{L^2(0,T;L^2(\Omega))}\le C,\\&\|c_\epsilon\|_{L^\infty(0,T;H^1(\Omega))}\le C,\\&\|F_\epsilon(c_\epsilon)\|_{L^\infty(0,T;L^1(\Omega))}\le C.\end{aligned} \tag{4.36}$$

In the rest of this subsection, the constant C is independent of ϵ.

With the aid of (4.36), similar method as (4.15) and (4.19) runs that

$$\|c_\epsilon\|_{L^2(0,T;H^2(\Omega))}\le C_T,\quad \partial_t c_\epsilon\in L^2(0,T;H^{-1}(\Omega)). \tag{4.37}$$

Now let us prove (4.24). By Hypothesis 3.1, there is a small constant $\eta_0>0$ such that both $F(s)$ and $F_\epsilon(s)$ are increasing in $[1-\eta_0,1)$. Then, it gives from (4.1) and (4.36) that for any $\eta\in(0,\eta_0]$

$$\text{meas}\{x(t):c_\epsilon(x,t)\geq 1-\eta\}\,F_\epsilon(1-\eta)\leq\int_{\{x(t):c_\epsilon(x,t)\geq 1-\eta\}}F_\epsilon(c_\epsilon)\leq C. \tag{4.38}$$

Thanks to (4.37), we see that $c_\epsilon\to c$ a.e. in $\Omega\times(0,T)$. So, by (4.38) and the convergence of $F_\epsilon(1-\eta)$ to $F(1-\eta)$, we use Fatou's lemma and deduce

$$\begin{aligned}\text{meas}\{x(t):c(x,t)\geq 1-\eta\}&\leq\lim_{\epsilon\downarrow 0}\text{meas}\{x(t):c_\epsilon(x,t)\geq 1-\eta\}\\&\leq\lim_{\epsilon\downarrow 0}\frac{C}{F_\epsilon(1-\eta)}\\&=\frac{C}{F(1-\eta)}.\end{aligned}$$

By Hypothesis 3.1, it has $\lim_{s\to 1}F(s)=\infty$. Sending $\eta\to 0$ in above inequality yields

$$\text{meas}\{x(t):c(x,t)\geq 1\}=0,\quad a.e.\ t\in(0,T).$$

Similarly,

$$\text{meas}\{x(t):c(x,t)\leq -1\}=0,\quad a.e.\ t\in(0,T).$$

Hence, the desired (4.24) follows from the last two inequalities.

As a result of (4.24), the uniform convergence of $F_\epsilon'(s)$ to $F'(s)$ on any compact subset in $(-1,1)$, it takes

$$F_\epsilon'(c_\epsilon)\to F'(c)\quad a.e.\,\Omega\times(0,T). \tag{4.39}$$

Multiplying $(4.4)_2$ by $(-\triangle)^{-1}(c_\epsilon-(c_0)_\Omega)$, and $(4.4)_3$ by $(c_\epsilon-(c_0)_\Omega)$, we deduce (it has $(c_0)_\Omega\in(-1,1)$ owing to Remark 3.1 and (4.22))

$$\begin{aligned}&-\left\langle\partial_t c_\epsilon+\mathbf{u}_\epsilon\cdot\nabla c_\epsilon,\,(-\triangle)^{-1}(c_\epsilon-(c_0)_\Omega)\right\rangle_{H^{-1}\times H^1}\\&=\int_\Omega\mu_\epsilon(c_\epsilon-(c_0)_\Omega)\\&=\int_\Omega(\mathfrak{B}[c_\epsilon]+F_\epsilon'(c_\epsilon)+\Pi_\epsilon'(c_\epsilon)-\delta\triangle c_\epsilon)(c_\epsilon-(c_0)_\Omega)\\&\geq -C\|c_\epsilon\|_{H^1}+\int_\Omega F_\epsilon'(c_\epsilon)(c_\epsilon-(c_0)_\Omega),\end{aligned} \tag{4.40}$$

where in the last inequality we have used

$$\int_\Omega\mathfrak{B}[c_\epsilon](c_\epsilon-(c_0)_\Omega)=\int_\Omega\mathfrak{B}[c_\epsilon]c_\epsilon=2\int_\Omega E[c_\epsilon]\geq 0,\quad\int_\Omega\Pi_\epsilon'(c_\epsilon)(c_\epsilon-(c_0)_\Omega)\leq C\|c_\epsilon\|_{H^1},$$

which come from (2.5), (2.7), and the assumption on Π_ϵ. Hence, (4.36)-(4.37), (4.40) provide us

$$\begin{aligned}\int_\Omega F_\epsilon'(c_\epsilon)(c_\epsilon-(c_0)_\Omega)&\leq C\left(1+\|c_\epsilon\|_{H^1}+\|\partial_t c_\epsilon\|_{H^{-1}}\|(-\triangle)^{-1}(c_\epsilon-(c_0)_\Omega)\|_{H^1}\right)\\&\quad+C\|\mathbf{u}^\epsilon c_\epsilon\|_{L^2}\|\nabla(-\triangle)^{-1}(c_\epsilon-(c_0)_\Omega)\|_{L^2}\\&\leq C+C\|\partial_t c_\epsilon\|_{H^{-1}}+C\|\mathbf{u}_\epsilon\|_{H^1_\sigma}\\&\in L^2(0,T),\end{aligned}$$

which, along with (4.24) and the inequality $F'_\epsilon(c_\epsilon)(c_\epsilon-(c_0)_\Omega)\geq C_1|F'_\epsilon(c_\epsilon)|-C_2$ (cf. [36, Proposition A1]), leads to

$$\int_\Omega |F'_\epsilon(c_\epsilon)|dx\in L^2(0,T). \tag{4.41}$$

Consequently,

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega \mu_\epsilon &= \frac{1}{|\Omega|}\int_\Omega (\mathfrak{B}[c_\epsilon]+F'_\epsilon(c_\epsilon)+\Pi'_\epsilon(c_\epsilon)-\delta\triangle c_\epsilon)\\ &=\frac{1}{|\Omega|}\int_\Omega (F'_\epsilon(c_\epsilon)+\Pi'_\epsilon(c_\epsilon))\in L^2(0,T).\end{aligned}$$

By this, (4.36), Poincaré inequality, we find

$$\mu_\epsilon\in L^2(0,T;H^1(\Omega)). \tag{4.42}$$

The same deduction as that in (4.18) gives

$$\partial_t\mathbf{u}_\epsilon\in L^{\frac{4}{3}}(0,T;H^{-1}_\sigma(\Omega)). \tag{4.43}$$

We claim that (4.41) can be improved to

$$F'_\epsilon(c_\epsilon)\in L^2(0,T;L^2(\Omega)). \tag{4.44}$$

In fact, if we multiply (4.4) by $F'_\epsilon(c_\epsilon)$, we obtain

$$\begin{aligned}\|\mu_\epsilon\|^2_{L^2}+\frac{1}{4}\|F'_\epsilon(c_\epsilon)\|^2_{L^2}&\geq \int_\Omega \mu_\epsilon F'_\epsilon(c_\epsilon)\\ &=\int_\Omega (\mathfrak{B}[c_\epsilon]+F'_\epsilon(c_\epsilon)+\Pi'_\epsilon(c_\epsilon)-\delta\triangle c_\epsilon)F'_\epsilon(c_\epsilon)\\ &\geq \frac{1}{2}\|F'_\epsilon(c_\epsilon)\|^2_{L^2}-C,\end{aligned} \tag{4.45}$$

where the last inequality owes to $F''\geq 0$, as well as the following two inequalities

$$\int_\Omega \Pi'_\epsilon(c_\epsilon)F'_\epsilon(c_\epsilon)\geq -\frac{1}{4}\|F'_\epsilon(c_\epsilon)\|^2_{L^2}-C$$

and

$$\begin{aligned}\int_\Omega \mathfrak{B}[c_\epsilon]F'_\epsilon(c_\epsilon)&=\frac{1}{2}\int_\Omega\int_\Omega \mathbb{J}_\lambda(c_\epsilon(x,t)-c_\epsilon(x,t))(F'_\epsilon(c_\epsilon(x,t))-F'_\epsilon(c_\epsilon(y,t)))\\ &\geq \frac{1}{2}\int_\Omega\int_\Omega \mathbb{J}_\lambda|c_\epsilon(x,t)-c_\epsilon(y,t)|^2F''_\epsilon\geq 0.\end{aligned}$$

Hence, (4.44) follows directly from (4.45) and (4.42).

In conclusion, thanks to (2.9), (4.37), (4.39), (4.42), (4.44), the definition of Π'_ϵ, Hypothesis 3.1, we arrive at

$$\begin{aligned}\int_0^T\int_\Omega h\phi\mu \leftarrow \int_0^T\int_\Omega \tilde{h}\phi\mu_\epsilon &=\int_0^T\int_\Omega \tilde{h}\phi(\mathfrak{B}[c_\epsilon]+F'_\epsilon(c_\epsilon)+\Pi'_\epsilon(c_\epsilon)-\delta\triangle c_\epsilon)\\ &\to \int_0^T\int_\Omega \tilde{h}\phi(\mathfrak{B}[c]+F'(c)+\Pi'(c)-\delta\triangle c),\end{aligned}$$

that is,

$$\mu = \mathfrak{B}[c] + F'(c) + \Pi'(c) - \delta \triangle c, \quad a.e.\ \Omega \times (0,T). \tag{4.46}$$

In addition, the estimates (4.36)-(4.37) and (4.42)-(4.43) ensure that the approximations $(\mathbf{u}_\epsilon, c_\epsilon)$ converge weakly to some limit functions which satisfy Equations (4.23), and agree initially to $(\mathbf{u}_0, c_0)$ by taking limit in weak topologies.

The remaining task is to prove inequality (4.25). By (4.37), c_ϵ is relatively compact in $L^2(0,T;H^1(\Omega))$. The argument in (4.30) tells that

$$\lim_{\epsilon \downarrow 0} \int_0^T \int_\Omega \phi'(t) E[c_\epsilon] = \int_0^T \int_\Omega \phi'(t) E[c]. \tag{4.47}$$

By the continuity assumption on Π_ϵ, one easily deduces

$$\lim_{\epsilon \downarrow 0} \int_0^T \int_\Omega \phi'(t) \Pi_\epsilon(c_\epsilon) = \int_0^T \int_\Omega \phi'(t) \Pi(c). \tag{4.48}$$

Next, we adopt the strategy in [19, 20] to verify

$$\lim_{\epsilon \downarrow 0} \int_0^T \int_\Omega \phi'(t) F_\epsilon(c_\epsilon) = \int_0^T \int_\Omega \phi'(t) F(c). \tag{4.49}$$

Indeed, Hypothesis 3.1 and (4.1) guarantee that, for small ϵ, $F_\epsilon(s)$ is convex and satisfies $F_\epsilon(c_\epsilon) \le F_\epsilon(c) + F'_\epsilon(c_\epsilon)(c_\epsilon - c)$. Then,

$$\begin{aligned} \int_0^T \int_\Omega \phi'(t) F_\epsilon(c_\epsilon) &\ge \int_0^T \int_\Omega \phi'(t) F_\epsilon(c) + \phi'(t) F'_\epsilon(c_\epsilon)(c_\epsilon - c) \\ &\ge \int_0^T \int_\Omega \phi'(t) F_\epsilon(c) - C \|F'_\epsilon(c_\epsilon)\|_{L^2(0,T;L^2)} \|c_\epsilon - c\|_{L^2(0,T;L^2)} \\ &\ge \int_0^T \int_\Omega \phi'(t) F_\epsilon(c) - C \|c_\epsilon - c\|_{L^2(0,T;L^2)}, \end{aligned} \tag{4.50}$$

where we have used (4.44) and the fact $\phi' \le 0$ as ϕ is non-increasing. By the convergence of $F_\epsilon(c) \uparrow F(c)$ and (4.24), we take limit in (4.50) to receive

$$\lim_{\epsilon \downarrow 0} \int \int \phi'(t) F_\epsilon(c_\epsilon) \ge \lim_{\epsilon \downarrow 0} \int \int \phi'(t) F_\epsilon(c) = \int \int \phi'(t) F(c).$$

On the other hand, similar analysis as (4.39) yields $F_\epsilon(c_\epsilon) \to F(c)$ a.e. $\Omega \times (0,T)$. Hence, by the Fatou's Lemma,

$$\int \int \phi'(t) F(c) = \int \int \lim_{\epsilon \downarrow 0} \phi'(t) F_\epsilon(c_\epsilon) \ge \limsup_{\epsilon \downarrow 0} \int \int \phi'(t) F_\epsilon(c_\epsilon).$$

The last two inequalities give birth to the desired (4.49).

Having (4.47)-(4.49) obtained, as well as (4.36) and the lower semi-continuity of L^2 norm, we take $\epsilon \downarrow 0$ in (4.31) and conclude that the limit functions $(\mathbf{u}_\delta, \mu_\delta)$ satisfy

$$-\int_0^T \int_\Omega \phi' \mathbb{E}_\delta(x,t) + \int_0^T \int_\Omega \phi \left(|\nabla \mathbf{u}_\delta|^2 + |\nabla \mu_\delta|^2 \right) \le \int_\Omega \mathbb{E}_\delta(x,0). \tag{4.51}$$

Finally, choosing ϕ as in (4.32), we get (4.25) from (4.51) after sending η to zero. □

4.3. Vanishing artificial diffusion term. In this subsection we take the limit $\delta\downarrow 0$ to vanish the artificial diffusion $\delta\triangle c$ in problem (4.23), and complete the proof of Theorem 3.1. During this subsection, the generic constant C is assumed to be uniform in δ. Let us denote by $(\mathbf{u}_\delta, c_\delta)$ the solutions in Lemma 4.2 associated with initial $(\mathbf{u}_{0\delta}, c_{0\delta})$. Furthermore, we assume that

$$\begin{aligned}
&\mathbf{u}_{0\delta}\to\mathbf{u}_0 \text{ in } L^2_\sigma(\Omega), \quad H^1(\Omega)\ni c_{0\delta}\to c_0 \text{ in } L^2(\Omega),\\
&\|c_{0\delta}\|^2_{L^2(\Omega)}+\delta\|\nabla c_{0\delta}\|^2_{L^2(\Omega)}\le\|c_0\|^2_{L^2(\Omega)},\\
&\|E[c_{0\delta}]\|_{L^1(\Omega)}\le\|E[c_0]\|_{L^1(\Omega)}, \quad \|F(c_{0\delta})\|_{L^1(\Omega)}\le\|F(c_0)\|_{L^1(\Omega)}.
\end{aligned} \tag{4.52}$$

Here, the initial functions $(\mathbf{u}_0, c_0)$ satisfy the condition (3.2).

Remark 4.2. The initial functions $(\mathbf{u}_{0\delta}, c_{0\delta})$ satisfying (4.52) match all the requirements listed in Lemma 4.2. In other words, the conditions (4.3), (4.22), (4.27) are fulfilled.

We can construct the functions $(\mathbf{u}_{0\delta}, c_{0\delta})$ in (4.52) as follows: For given c_0, let $c_{0\delta}$ solve

$$c_{0\delta}-\delta\triangle c_{0\delta}=c_0. \tag{4.53}$$

If we multiply (4.53) by $c_{\delta 0}$ and $F'(c_{0\delta})$ respectively, we find

$$\|c_{0\delta}\|^2_{L^2}+2\delta\|\nabla c_{0\delta}\|^2_{L^2}\le\|c_0\|^2_{L^2} \tag{4.54}$$

and

$$\int_\Omega c_{0\delta}F'(c_{0\delta})+\delta\int_\Omega|\nabla c_{0\delta}|^2F''(c_{0\delta})=\int_\Omega c_0F'(c_{0\delta}). \tag{4.55}$$

The strong convergence of $c_{0\delta}$ in L^2 follows immediately from (4.54) and

$$\|c_0\|^2_{L^2}\le\liminf_{\delta\downarrow 0}\|c_{0\delta}\|^2_{L^2}\le\limsup_{\delta\downarrow 0}\left(\|c_{0\delta}\|^2_{L^2}+2\delta\|\nabla c_{0\delta}\|^2_{L^2}\right)\le\|c_0\|^2_{L^2}.$$

Since $F''\ge 0$, (4.55) implies $\int_\Omega F'(c_{0\delta})(c_{0\delta}-c_0)\le 0$. Thus, by the Taylor expansion formula,

$$\int_\Omega F(c_{0\delta})\le\int_\Omega F(c_0)+\int_\Omega F'(c_{0\delta})(c_{0\delta}-c_0)\le\int_\Omega F(c_0). \tag{4.56}$$

If we multiply (4.53) by $\mathfrak{B}[c_{0\delta}]$, we infer

$$\begin{aligned}
&\int_\Omega\int_\Omega\mathbb{J}_\lambda|c_{0\delta}(x)-c_{\delta 0}(y)|^2+\delta\int_\Omega\int_\Omega\mathbb{J}_\lambda|\nabla c_{0\delta}(x)-\nabla c_{0\delta}(y)|^2\\
&=\int_\Omega\int_\Omega\mathbb{J}_\lambda(c_{0\delta}(x)-c_{0\delta}(y))(c_0(x)-c_0(y))\\
&\le\frac{1}{2}\int_\Omega\int_\Omega\mathbb{J}_\lambda|c_{0\delta}(x)-c_{\delta 0}(y)|^2+\frac{1}{2}\int_\Omega\int_\Omega\mathbb{J}_\lambda|c_0(x)-c_0(y)|^2,
\end{aligned}$$

which, along with (2.5) and (3.2), implies

$$\int_\Omega E[c_{0\delta}]+2\delta\int_\Omega E[\nabla c_{0\delta}]\le\int_\Omega E[c_0]. \tag{4.57}$$

Therefore, (4.52) follows from (4.54) and (4.56)-(4.57).

In view of (4.52), (3.2), and Hypothesis 3.1, the inequality (4.25) in Lemma 4.2 guarantees the following uniform estimates

$$\begin{aligned}&\|\mathbf{u}_\delta\|_{L^\infty(0,T_n;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega))}\le C,\\&\|\nabla\mu_\delta\|_{L^2(0,T;L^2(\Omega))}\le C,\\&\sqrt{\delta}\|\nabla c_\delta\|_{L^\infty(0,T;L^2(\Omega))}\le C,\\&\|E[c_\delta]\|_{L^\infty(0,T;L^1(\Omega))}+\|F(c_\delta)\|_{L^\infty(0,T;L^1(\Omega))}\le C.\end{aligned}\tag{4.58}$$

In addition, the uniform bound of $\|E[c_\delta]\|_{L^\infty(0,T;L^1(\Omega))}$ in (4.58) tells that c_δ is relatively compact in $L^2(\Omega)$(cf. [37, Theorem 1.2]). Particularly, for a.e. $t\in(0,T)$,

$$c_\delta\to c \text{ in } L^2(\Omega),\quad c\in L^\infty(0,T;H^1(\Omega)).\tag{4.59}$$

With (4.58)-(4.59), the same argument as in (4.18)-(4.19), (4.42), (4.44) yields

$$\begin{aligned}&\partial_t c_\delta\in L^2(0,T;H^{-1}(\Omega)),\quad F'(c_\delta)\in L^2(0,T;L^2(\Omega)),\\&\mu_\delta\in L^2(0,T;H^1(\Omega)),\quad \partial_t\mathbf{u}_\delta\in L^{\frac43}(0,T;H^{-1}_\sigma(\Omega)).\end{aligned}\tag{4.60}$$

Next, multiplying the last equality in (4.23) by $-\triangle c_\delta$, utilizing (4.14) and the fact $F''\ge 0$, we have

$$\begin{aligned}&\frac12\|\nabla c_\delta\|^2_{L^2}+\frac12\|\nabla\mu_\delta\|^2_{L^2}\\&\ge\int_\Omega\nabla\mu_\delta\nabla c_\delta\\&=-\int_\Omega(\mathfrak{B}[c_\delta]+F'(c_\delta)+\Pi'(c_\delta)-\delta\triangle c_\delta)\triangle c_\delta\\&\ge\int_\Omega\left(2E[\nabla c_\delta]+F''(c_\delta)|\nabla c_\delta|^2+\Pi'(c_\delta)\triangle c_\delta\right)+\delta\|\triangle c_n\|^2_{L^2}\\&\ge\int_\Omega 2E[\nabla c_\delta]+\frac{\delta}{2}\|\triangle c_\delta\|^2_{L^2}-C_\delta.\end{aligned}\tag{4.61}$$

Recalling Lemma 2.4, one has

$$\|\nabla c_\delta\|^2_{L^2}\le\zeta\|E[\nabla c_\delta]\|_{L^1(\Omega\times(0,T))}+C_\zeta\|c_\delta\|^2_{L^2}.\tag{4.62}$$

Substituting (4.62) into (4.61) and choosing $\zeta>0$ so small such that

$$\begin{aligned}&\|c_\delta\|_{L^2(0,T;H^1)}+\sqrt{\delta}\|\triangle c_\delta\|_{L^2(0,T;L^2)}+\|E[\nabla c_\delta]\|_{L^1(\Omega\times(0,T))}\\&\le C\left(\|c_\delta\|_{L^2(\Omega\times(0,T)}+\|\nabla\mu_\delta\|_{L^2(0,T;L^2)}\right)\\&\le C.\end{aligned}\tag{4.63}$$

Similar to (4.59), the uniform bound $\|E[\nabla c_\delta]\|_{L^1(\Omega\times(0,T))}$ together with (4.60) provide that

$$c_\delta\to c \text{ in } L^2(0,T;H^1(\Omega)),\quad c\in L^2(0,T;H^2(\Omega)).\tag{4.64}$$

REMARK 4.3. Alternatively, the strong convergence in (4.64) could be achieved as follows: by (4.62) and (4.63),

$$\begin{aligned}\|c_{\delta_i}-c_{\delta_j}\|^2_{L^2(0,T;H^1)}&\le\zeta\|E[\nabla(c_{\delta_i}-c_{\delta_j})]\|_{L^1(\Omega\times(0,T)}+C_\zeta\|c_{\delta_i}-c_{\delta_j}\|^2_{L^2(0,T;L^2)}\\&\le\zeta C+C_\zeta\|c_{\delta_i}-c_{\delta_j}\|^2_{L^2(0,T;L^2)}.\end{aligned}\tag{4.65}$$

Also, (4.60) implies $\lim_{\delta_i,\delta_j\downarrow 0}\|c_{\delta_i}-c_{\delta_j}\|^2_{L^2(0,T;L^2)}=0$. Hence, we are done because $\zeta>0$ can be arbitrarily small. See, also, the paper [13]. Based on the estimates (4.58)-(4.60), (4.64), we conclude

$$\begin{cases}\mathbf{u}_\delta\rightharpoonup\mathbf{u}, & L^2(0,T;H^1_\sigma(\Omega));\\ \mu_\delta\rightharpoonup\mu, & L^2(0,T;H^1(\Omega));\\ c_\delta\to c, & L^2(0,T;H^1(\Omega));\\ \mathbf{u}_\delta\to\mathbf{u}, & L^2(0,T;L^q_\sigma)\ (q<6);\\ \mathbf{u}_\delta \text{ is compact in } C_w([0,T];L^2_\sigma(\Omega));\\ c_\delta \text{ is compact in } C_w([0,T];H^1(\Omega)).\end{cases} \tag{4.66}$$

Moreover, a similar but simpler method as that in (4.24) shows

$$-1<c_\delta(x,t)<1 \quad a.e.\ \Omega\times(0,T). \tag{4.67}$$

This and the continuity of $F'(s)$ in $(-1,1)$ implies $F'(c_\delta)\to F'(c)$ a.e. in $\Omega\times(0,T)$, and whence, by (4.60),

$$F'(c_\delta)\rightharpoonup F'(c) \quad L^2(0,T;L^2(\Omega)). \tag{4.68}$$

From (4.63) we have

$$\int_0^T\int_\Omega \tilde{h}\phi\delta\triangle c\le C\delta\|\triangle c_\delta\|_{L^2(0,T;L^2)}\le C\sqrt{\delta}\to 0 \quad \text{as} \quad \delta\to 0. \tag{4.69}$$

Finally, it follows from (2.9) and (4.66) that $\mathfrak{B}[c_\delta]\rightharpoonup\mathfrak{B}[c]$ in $L^2(0,T;H^{-1})$. However, (4.60), (4.63) and Lipschitz continuity of Π' guarantee

$$\mathfrak{B}[c_\delta]=\mu_\delta+\delta\triangle c_\delta-F'(c_\delta)-\Pi'(c_\delta)\in L^2(0,T;L^2(\Omega)). \tag{4.70}$$

So,

$$\mathfrak{B}[c_\delta]\rightharpoonup\mathfrak{B}[c] \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{4.71}$$

In terms of (4.66), (4.68)-(4.69), (4.71), the uniform continuity of Π', it takes

$$\mu=\mathfrak{B}[c]+F'(c)+\Pi'(c), \quad a.e.\ \Omega\times(0,T).$$

In conclusion, we have proved the existence of weak solutions to (1.1) and (3.1) in the sense of Definition 3.1. Moreover, the (3.3) is fulfilled due to (4.58)-(4.60), (4.64), (4.67), (4.70).

The only thing left is to take limit in (4.51) to justify the energy inequality (3.4). We compute the right-hand side of (4.51) as

$$\begin{aligned}\lim_{\delta\downarrow 0}\int_\Omega \mathbb{E}_\delta(x,0)&=\lim_{\delta\downarrow 0}\int_\Omega\left(E[c_{0\delta}]+F(c_{0\delta})+\Pi(c_{0\delta})+\frac{\delta}{2}|\nabla c_{0\delta}|^2+\frac{1}{2}|\mathbf{u}_{0\delta}|^2\right)(x,t)\\ &\le\lim_{\delta\downarrow 0}\int_\Omega\left(E[c_0]+F(c_0)+\Pi(c_{0\delta})+\frac{\delta}{2}|\nabla c_{0\delta}|^2+\frac{1}{2}|\mathbf{u}_{0\delta}|^2\right)(x,t)\\ &=\int_\Omega\left(E[c_0]+F(c_0)+\Pi(c_0)+\frac{1}{2}|\mathbf{u}_0|^2\right)(x,t)\end{aligned}$$

$$=\int_\Omega \mathbb{E}[c_0](x), \tag{4.72}$$

where we have used the uniform continuity of Π and the following

$$0\le \lim_{\delta\downarrow 0}\delta\|\nabla c_{0\delta}\|^2_{L^2}\le \lim_{\delta\downarrow 0}(\|c_0\|^2_{L^2}-\|c_{0\delta}\|^2_{L^2})=0,$$

owing to (4.52).

Next to deal with the term on the left-hand side of (4.51). By Hypothesis 3.1, the convexity of $F(s)$ implies $F(c_\delta)\le F(c)+F'(c_\delta)(c_\delta-c)$. This together with (4.60) and (4.66) guarantee that

$$\begin{aligned}\lim_{\delta\downarrow 0}\int_0^T\int_\Omega \phi'(t)F(c_\delta)&\ge \int_0^T\int_\Omega \phi'(t)F(c)+\lim_{\delta\downarrow 0}\int_0^T\int_\Omega \phi'(t)F'(c_\delta)(c_\delta-c)\\ &=\int_0^T\int_\Omega \phi'(t)F(c).\end{aligned} \tag{4.73}$$

On the other hand, since $F(c_\delta)$ converges to $F(c)$ a.e. $\Omega\times(0,T)$, the Fatou's Lemma gives

$$\int_0^T\int_\Omega \phi'(t)F(c)=\int_0^T\int_\Omega \lim_{\delta\downarrow 0}\phi'(t)F(c_\delta)\ge \limsup_{\delta\downarrow 0}\int_0^T\int_\Omega \phi'(t)F(c_\delta). \tag{4.74}$$

Inequalities (4.73) and (4.74) yield

$$\lim_{\delta\downarrow 0}\int_0^T\int_\Omega \phi'(t)F(c_\delta)=\int_0^T\int_\Omega \phi'(t)F(c). \tag{4.75}$$

Next, the same calculation as (4.29) gives

$$\begin{aligned}&\left|\int_0^T\int_\Omega \phi'(t)E[c_\delta]-\int_0^T\int_\Omega \phi'(t)E[c]\right|\\ &\le C\|(c_\delta-c)\|_{L^2(0,T;H^1)}\|(c_\delta+c)\|_{L^2(0,T;H^1)},\end{aligned}$$

which, along with (4.64), implies

$$\lim_{\delta\downarrow 0}\int_0^T\int_\Omega \phi'(t)E[c_\delta]=\int_0^T\int_\Omega \phi'(t)E[c]. \tag{4.76}$$

Therefore, with the help of (4.66), (4.72), (4.75)-(4.76), Hypothesis 3.1, we are allowed to pass δ to zero in (4.51), to deduce

$$-\int_0^T\phi'\int_\Omega \mathbb{E}[c](x,t)+\int_0^T\phi\int_\Omega\left(|\nabla \mathbf{u}|^2+|\nabla\mu|^2\right)\le\int_\Omega \mathbb{E}[c_0](x), \tag{4.77}$$

where the functional $\mathbb{E}$ is defined in (3.5). Choosing ϕ as in (4.32), we conclude (3.4) from (4.77). The proof of Theorem 3.1 is completed.

5. Proof of Theorem 3.2

Assume that the initial functions $(\mathbf{u}_{0\lambda}, c_{0\lambda})$ satisfy (3.6). Then, the basic energy inequality (3.4) obtained in Theorem 3.1 shows that the solutions $(\mathbf{u}_\lambda, c_\lambda)$ satisfy

$$\begin{aligned}
&\|\mathbf{u}_\lambda\|_{L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_\sigma(\Omega))} \le C,\\
&\|\nabla\mu_\lambda\|_{L^2(0,T;L^2(\Omega))} \le C,\\
&\|E_\lambda[c_\lambda]\|_{L^\infty(0,T;L^1(\Omega))} + \|F(c_\lambda)\|_{L^\infty(0,T;L^1(\Omega))} \le C.
\end{aligned} \tag{5.1}$$

Here and in what follows, the generic constant C is independent of λ.

With (5.1), the argument in (4.59) shows for a.e. $t\in(0,T)$

$$c_\lambda \to c \text{ in } L^2(\Omega), \quad c\in L^\infty(0,T;H^1(\Omega)). \tag{5.2}$$

Whence, similar to (4.19),

$$\partial_t c_\lambda \in L^2(0,T;H^{-1}(\Omega)). \tag{5.3}$$

Owing to (5.1)-(5.2), we multiply the last equality in (1.1) by $-\triangle c_\lambda$ and use the same deduction as (4.63), to deduce

$$\begin{aligned}
&\|c_\lambda\|_{L^2(0,T;H^1(\Omega))} + \|E_\lambda[\nabla c_\lambda]\|_{L^1(\Omega\times(0,T))}\\
&\le C\left(\|c_\lambda\|_{L^2(\Omega\times(0,T)} + \|\nabla\mu_\lambda\|_{L^2(0,T;L^2(\Omega))}\right)\\
&\le C + \left(\|E_\lambda[c_\lambda]\|_{L^1(\Omega\times(0,T))} + \|\nabla\mu_\lambda\|_{L^2(0,T;L^2(\Omega))}\right)\\
&\le C.
\end{aligned} \tag{5.4}$$

As a result of (5.1), (5.3)-(5.4), Hypothesis 3.2, we have (see (4.64))

$$c_\lambda \to c \text{ in } L^2(0,T;H^1(\Omega)), \quad c\in L^2(0,T;H^2(\Omega)). \tag{5.5}$$

Moreover, the same deduction as in (4.18), (4.24), (4.42), (4.44) concludes that

$$c_\lambda \to c\in(-1,1),\ a.e.\ \Omega\times(0,T) \tag{5.6}$$

and

$$F'(c_\lambda)\in L^2(0,T;L^2(\Omega)), \quad \mu_\lambda\in L^2(0,T;L^2(\Omega)), \quad \partial_t\mathbf{u}_\lambda \in L^{\frac{4}{3}}(0,T;H^{-1}(\Omega)). \tag{5.7}$$

So far, we are allowed to extract subsequence from $\{(\mathbf{u}_\lambda, c_\lambda, \mu_\lambda)\}_\lambda$ such that limit functions $(\mathbf{u}, c, \mu)$ satisfy the first three equalities in (1.4) in distributional sense. Next, let us prove

$$\mu = F'(c) + \Pi'(c) - \triangle c \quad a.e.\,\Omega\times(0,T). \tag{5.8}$$

From (5.4), (5.6)-(5.7), Hypothesis 3.1, we see that

$$\Pi'(c_\lambda)\rightharpoonup\Pi'(c) \quad \text{and} \quad F'(c_\lambda)\rightharpoonup F'(c) \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \tag{5.9}$$

and

$$\chi \leftharpoonup \mathfrak{B}_\lambda[c_\lambda] = \mu_\lambda - F'(c_\lambda) - \Pi'(c_\lambda) \in L^2(0,T;L^2(\Omega)). \tag{5.10}$$

Noting that $\mathfrak{B}_\lambda[c_\lambda]$ is the sub-differential of the convex functional $E_\lambda[c_\lambda]$, we have

$$\int_0^T\int_\Omega E_\lambda[c_\lambda] + \int_0^T\int_\Omega \mathfrak{B}_\lambda[c_\lambda](z-c_\lambda) \le \int_0^T\int_\Omega E_\lambda[z], \quad \forall\, z\in L^2(0,T;H^1(\Omega)). \tag{5.11}$$

By (2.6), one has

$$\lim_{\lambda\downarrow 0}\int_0^T\int_\Omega E_\lambda[z](x,t)dx=\frac{1}{2}\|\nabla z(\cdot,t)\|_{L^2(0,T;L^2(\Omega))}. \tag{5.12}$$

In terms of (5.5) and (4.29),

$$\begin{aligned}&\lim_{\lambda\downarrow 0}\left|\int_0^T\int_\Omega E_\lambda[c_\lambda]-\int_0^T\int_\Omega E_\lambda[c]\right|\\ &\leq C\lim_{\lambda\downarrow 0}\|(c_\lambda-c)\|_{L^2(0,T;H^1)}\|(c_\lambda+c)\|_{L^2(0,T;H^1)}=0.\end{aligned}$$

This and (5.12) bring us to

$$\lim_{\lambda\downarrow 0}\int_0^T\int_\Omega E_\lambda[c_\lambda]=\frac{1}{2}|\nabla c|^2. \tag{5.13}$$

Taking (5.12)-(5.13), and (5.10) into account, we take $\lambda\downarrow 0$ in (5.11) and conclude

$$\frac{1}{2}\int_0^T\int_\Omega|\nabla c|^2+\int_0^T\int_\Omega\chi(z-c)\leq\frac{1}{2}\int_0^T\int_\Omega|\nabla z|^2,\quad\forall\, z\in L^2(0,T;H^1(\Omega)), \tag{5.14}$$

which implies

$$-\triangle c=\chi\in L^2(0,T;L^2(\Omega)). \tag{5.15}$$

Therefore, (5.8) comes directly from (5.9)-(5.10) and (5.15).

REMARK 5.1. The regularity $c\in L^2(0,T;H^2(\Omega))$ could also be derived from the elliptic equation (5.15) (cf. [33]).

In the final part, let us prove the validity of (3.9). In fact, by (4.77), the following inequality is valid for every $(\mathbf{u}_\lambda,c_\lambda)$

$$-\int_0^T\tilde{\phi}'\int_\Omega\mathbb{E}_\lambda(x,t)+\int_0^T\tilde{\phi}\int_\Omega\left(|\nabla\mathbf{u}_\lambda|^2+|\nabla\mu_\lambda|^2\right)\leq 0, \tag{5.16}$$

where the cut-off $0\leq\tilde{\phi}\in C_0^1((0,T))$ and $\mathbb{E}_\lambda=\left(E_\lambda[c_\lambda]+\Phi(c_\lambda)+\frac{1}{2}|\mathbf{u}_\lambda|^2\right)$.

Thanks to (5.1), (5.5), (5.7), Hypothesis 3.1, we check

$$\lim_{\lambda\downarrow 0}\int_0^T\int_\Omega\tilde{\phi}'\left(E_\lambda[c_\lambda]+\Pi(c_\lambda)+\frac{1}{2}|\mathbf{u}_\lambda|^2\right)=\int_0^T\int_\Omega\tilde{\phi}'\left(\frac{1}{2}|\nabla c|^2+\Pi(c)+\frac{1}{2}|\mathbf{u}|^2\right). \tag{5.17}$$

Since $F(s)$ is convex, it has

$$F(c)+F'(c)(c_\lambda-c)\leq F(c_\lambda)\leq F(c)+F'(c_\lambda)(c_\lambda-c),$$

and thus,

$$\begin{aligned}&\lim_{\lambda\downarrow 0}\int_0^T\int_\Omega\tilde{\phi}'(t)F(c_\lambda)\\ &=\lim_{\lambda\downarrow 0}\left(\int_0^T\int_{\{\tilde{\phi}'\geq 0\}}+\int_0^T\int_{\{\tilde{\phi}'<0\}}\right)\tilde{\phi}'F(c_\lambda)\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T \int_\Omega \tilde{\phi}'(t)F(c) + \lim_{\lambda\downarrow 0}\left(\int_0^T \int_{\{\tilde{\phi}'\geq 0\}} \tilde{\phi}F'(c_\lambda)(c_\lambda - c) + \int_0^T \int_{\{\tilde{\phi}'<0\}} \tilde{\phi}F'(c)(c_\lambda - c)\right)\\
&\leq \int_0^T \int_\Omega \phi'(t)F(c) + \lim_{\lambda\downarrow 0}\| |F'(c_\lambda)| + |F'(c)| \|_{L^2(0,T;L^2)} \|c_\lambda - c\|_{L^2(0,T;L^2)}\\
&= \int_0^T \int_\Omega \phi'(t)F(c),
\end{aligned} \tag{5.18}
$$

where the last equality sign is due to (5.9) and (5.5).

In terms of (5.1), (5.17)-(5.18), we send $\lambda\downarrow 0$ in (5.16) and obtain

$$
-\int_0^T \tilde{\phi}' \int_\Omega \left(\frac{1}{2}|\nabla c|^2 + \Phi(c) + \frac{1}{2}|\mathbf{u}|^2\right) + \int_0^T \tilde{\phi} \int_\Omega \left(|\nabla \mathbf{u}|^2 + |\nabla \mu|^2\right) \leq 0. \tag{5.19}
$$

Finally, for fixed $0 < s < t < T$, let

$$
\tilde{\phi}_\eta(\tau) = \begin{cases} 0, & \tau \leq s,\\ \dfrac{\tau - s}{\eta}, & s \leq \tau \leq s+\eta,\\ 1, & s+\eta \leq \tau \leq t,\\ \dfrac{(t+\eta-\tau)}{\eta}, & t \leq \tau \leq t+\eta,\\ 0, & \tau \geq t+\eta. \end{cases}
$$

Inserting it back into (5.19) and passing η to zero give rise to the desired (3.9). The proof of Theorem 3.2 is finished.

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