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MODELS OF NONLINEAR ACOUSTICS VIEWED AS APPROXIMATIONS OF THE NAVIER-STOKES AND EULER COMPRESSIBLE ISENTROPIC SYSTEMS*

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Abstract. The derivation of different models of non linear acoustic in thermo-elastic media as the Kuznetsov equation, the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation and the nonlinear progressive wave equation (NPE) from an isentropic Navier-Stokes/Euler system is systematized using the Hilbert-type expansion in the corresponding perturbative and (for the KZK and NPE equations) paraxial *ansatz*. The use of small correctors, to compare to the constant state perturbations, allows to obtain the approximation results for the solutions of these models and to estimate the time during which they keep closed in the L^2 norm. In the aim to compare the solutions of the exact and approximate systems in found approximation domains a global well-posedness result for the Navier-Stokes system in a half-space with time periodic initial and boundary data was obtained.

Keywords. Non-linear acoustic; approximations of the Navier-Stokes system; Kuznetsov, KZK and NPE equations.

AMS subject classifications. 35L71; 35Q30; 35Q31; 35B51.

1. Introduction

There is a renewed interest in the study of nonlinear wave propagation, in particular because of recent applications to ultrasound imaging (*e.g.* HIFU) or technical and medical applications such as lithotripsy or thermotherapy. Such new techniques rely heavily on the ability to model accurately the nonlinear propagation of a finite-amplitude sound pulse in thermo-viscous elastic media. The most known nonlinear acoustic models, which we consider in this paper, are

(1) the Kuznetsov equation (see Equation (3.1) and Equation (3.11)), which is actually a quasi-linear (damped) wave equation, initially introduced by Kuznetsov [22] for the velocity potential, see also Refs. [12,17,19,25] for other different methods of its derivation;
(2) the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation (see Equation (4.11)), which can be written for the perturbations of the density or of the pressure (see the systematic physical studies in book [4]);
(3) the nonlinear progressive wave equation (NPE) (see Equation (5.10) and Equation (5.11)) originally derived in Ref. [31].

All these models were derived from a compressible nonlinear isentropic Navier-Stokes (for viscous media) and Euler (for the inviscid case) systems up to some small negligible terms. But all cited physical derivations of these models don't allow to say that their solutions approximate the solution of the Navier-Stokes or Euler system. The first work explaining it for the KZK equation is Ref. [35]. Starting in Section 2 to present the initial context of the isentropic Navier-Stokes system (actually, it is also an approximation of

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the compressible Navier-Stokes system (2.1)–(2.4)), which describes the acoustic wave motion in an homogeneous thermo-elastic medium [4, 12, 27], we systematize in this article the derivation of all these models using the ideas of Ref. [35], consisting of using correctors in the Hilbert-type expansions of corresponding physical *ansatzs*.

More precisely, we show that all these models are approximations of the isentropic Navier-Stokes or Euler system up to third-order terms of a small dimensionless parameter $\varepsilon > 0$ measuring the size of the perturbations of the pressure, the density and the velocity to compare with their constant state $(p_0, \rho_0, 0)$.

The Kuznetsov equation comes from the Navier-Stokes or Euler system only by small perturbations, but to obtain the KZK and the NPE equations we also need to perform, in addition to the small perturbations, a paraxial change of variables. We can notice that the Kuznetsov Equation (3.11) is a non-linear wave equation containing the terms of different order on ε. But the KZK- and NPE-paraxial approximations allow to have the approximate equations with all terms of the same order, *i.e.* the KZK and NPE equations [10]. The well-posedness results for boundary value problems for the Kuznetsov equation are given in Refs. [18,20,32] and for the Cauchy problem in Ref. [9].

The NPE equation is usually used to describe short-time pulses and a long-range propagation, for instance, in an ocean wave-guide, where the refraction phenomena are important [6, 30], while the KZK equation typically models the ultrasonic propagation with strong diffraction phenomena, combined with finite amplitude effects (see Ref. [35] and the references therein). Although the physical context and the physical use of the KZK and the NPE equations are different (see also Sections 4.1 and 5.1 respectively), there is a bijection (see Equation (5.12)) between the variables of these two models and they can be presented by the same type of differential operator with constant positive coefficients:

$$Lu = 0, \quad L = \partial^2_{tx} - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial^3_x \pm c_3 \Delta_y, \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}.$$

Therefore, the results on the solutions of the KZK equation from Ref. [34] are valid for the NPE equation. See also Ref. [15] for the exponential decay of the solutions of these models in the viscous case. The main hypothesis for the derivation of all these models are the following

- the motion is potential;
- the constant state of the medium given by $(p_0, \rho_0, 0)$ (0 for the velocity) is perturbed proportionally to a dimensionless parameter $\varepsilon > 0$ (for instance, equal to 10^{-5} in water with an initial power of the order of $0.3\,\mathrm{W/cm^2}$);
- all viscosities are small (of order ε).

Let us notice that *ansatz* (4.14)–(4.15), proposed initially in Ref. [4] and used in Ref. [35] to obtain the KZK equation from the Navier-Stokes or Euler systems, is different to *ansatz* (4.12)–(4.13) in Subsection 4.1: this time it is the composition of the Kuznetsov perturbative *ansatz* with the KZK paraxial change of variables [22] (see Figure 4.1). Moreover, this new approximation of the Navier-Stokes and the Euler systems is an improvement as compared to the derivation developed in Ref. [35] (see Subsection 4.1 for more details), as, in Ref. [35], the Navier-Stokes/Euler system could be only approximated up to $O(\varepsilon^{\frac{5}{2}})$-terms (instead of $O(\varepsilon^3)$ in our case).

The main result of the paper is the validation of the approximations of the compressible isentropic Navier-Stokes system by the different models: by the Kuznetsov (Section 3), the KZK (Section 4) and the NPE equations (Section 5). In Section 6 we do the same for the Euler system in the inviscid case.

The main difference between the viscous and the inviscid cases is the time existence and regularity of the solutions. Typically in the inviscid case, the solutions of the models and also of the Euler system itself (actually strong solutions), due to the non-linearity, can provide shock front formations at a finite time [2, 9, 34, 36, 40]. Thus, they are only locally well-posed, while in the viscous media all approximate models are globally well-posed for small enough initial data [9, 34]. These existence properties of solutions for the viscous and the inviscid cases may also imply the difference in the definition of the domain where the approximations hold: for example [35], for the approximation between the KZK equation and the Navier-Stokes system the approximation domain is a half-space, but for the analogous inviscid case of the KZK and the Euler system it is a cone (see also the concluding Table 7.1).

To keep a physical sense of the approximation problems, we consider especially the two or three dimensional cases, *i.e.* $\mathbb{R}^n$ with $n=2$ or 3, and in the following we use the notation $x=(x_1,x')\in\mathbb{R}^n$ with one (a propagative) axis $x_1\in\mathbb{R}$ and the traversal variable $x'\in\mathbb{R}^{n-1}$. In what follows we denote by $\mathbf{U}_\varepsilon$ a solution of the "exact" Navier-Stokes/Euler system

$$Exact(\mathbf{U}_\varepsilon)=0 \quad \text{(see Equation (3.19))}$$

and by $\overline{\mathbf{U}}_\varepsilon$ an approximate solution, constructed by the derivation *ansatz* from a regular solution of one of the approximate models (typically of the Kuznetsov, the KZK or the NPE equations), *i.e.* a function which solves the Navier-Stokes/Euler system up to ε^3 terms, denoted by $\varepsilon^3\mathbf{R}$:

$$Approx(\overline{\mathbf{U}}_\varepsilon)=Exact(\overline{\mathbf{U}}_\varepsilon)-\varepsilon^3\mathbf{R}=0 \quad \text{(see Equation (3.20))}.$$

To have the remainder term $\mathbf{R}\in C([0,T],L^2(\Omega))$ we ensure that

$$Exact(\overline{\mathbf{U}}_\varepsilon)\in C([0,T],L^2(\Omega)),$$

i.e. we need a sufficiently regular solution $\overline{\mathbf{U}}_\varepsilon$. The minimal regularity of the initial data to have such a $\overline{\mathbf{U}}_\varepsilon$ is given in Table 7.1.

Choosing for the exact system the same initial-boundary data found by the *ansatz* for $\overline{\mathbf{U}}_\varepsilon$ (the regular case) or the initial data taken in their small L^2-neighborhood, *i.e.*

$$\|\mathbf{U}_\varepsilon(0)-\overline{\mathbf{U}}_\varepsilon(0)\|_{L^2(\Omega)}\le\delta\le\varepsilon, \tag{1.1}$$

with $\mathbf{U}_\varepsilon(0)$ not necessarily smooth, but ensuring the existence of an admissible weak solution of a bounded energy (see Definition 3.1), we prove the existence of constants $C>0$ and $K>0$ independent of ε, δ and the time t such that

$$\text{for all } 0\le t\le\frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(\Omega)}\le K(\varepsilon^3 t+\delta^2)e^{K\varepsilon t}\le 9\varepsilon^2 \tag{1.2}$$

with Ω a domain where the both solutions $\mathbf{U}_\varepsilon$ and $\overline{\mathbf{U}}_\varepsilon$ exist (see Theorems 3.3, 4.3 and 5.4).

As we have mentioned, in the viscous case all approximate models have a global unique classical solution for small enough initial data in their corresponding approximate domains (Ω varies for different models, see Table 7.1: it is equal to $\mathbb{R}^n$, $\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}$ and $\mathbb{R}_+\times\mathbb{R}^{n-1}$ for the Kuznetsov equation, the NPE equation and the KZK equation respectively). If we take regular initial data $\mathbf{U}_\varepsilon(0)=\overline{\mathbf{U}}_\varepsilon(0)$, the same thing is true for the Navier-Stokes system with the same regularity for the solutions [29]. But in the case of the half-space for the approximation between the Navier-Stokes system and the KZK equation, firstly considered in Ref. [35], when, due to the periodic-in-time boundary conditions, coming from the initial conditions for the KZK equation, we prove the well-posedness for all finite time. To obtain it we use Ref. [35] Theorem 5.5. We updated it in the framework of the new *ansatz* (4.12)–(4.13) and corrected several misleading points in its proof (see Subsection 4.3 Theorem 4.2), which allows us in Theorem 4.3 of Subsection 4.4, to establish the approximation result between the KZK equation and the Navier-Stokes system by following Theorem 5.7 in Ref. [35] and just updating the stability approximation estimate.

For the inviscid case, given in Section 6, we verify that the existence time of (strong) solutions of all models is not less than $O(\frac{1}{\varepsilon})$ and estimate (1.2) still holds.

But to obtain estimate (1.2) we don't need the regularity of the classical solution of the Navier-Stokes (or Euler) system, it can be a weak solution (in the sense of Hoff [13] for the Navier-Stokes system or one of the solutions in the sense of Luo et al. [26] for the Euler system) satisfying the admissible conditions given in Definition 3.1 (see also Ref. [8] p.52 and Ref. [35] Definition 5.9).

2. Isentropic Navier-Stokes system for a subsonic potential motion

To describe the acoustic wave motion in a homogeneous thermo-elastic medium, we start from the Navier-Stokes system in $\mathbb{R}^n$

$$\partial_t\rho+\operatorname{div}(\rho\mathbf{v})=0, \tag{2.1}$$

$$\rho[\partial_t\mathbf{v}+(\mathbf{v}.\nabla)\mathbf{v}]=-\nabla p+\eta\Delta\mathbf{v}+\left(\zeta+\frac{\eta}{3}\right)\nabla.\operatorname{div}(\mathbf{v}), \tag{2.2}$$

$$\rho T[\partial_t S+(\mathbf{v}.\nabla)S]=\kappa\Delta T+\zeta(\operatorname{div}\mathbf{v})^2+\frac{\eta}{2}\left(\partial_{x_k}v_i+\partial_{x_i}v_k-\frac{2}{3}\delta_{ik}\partial_{x_i}v_i\right)^2, \tag{2.3}$$

$$p=p(\rho,S), \tag{2.4}$$

where the pressure p is given by the state law $p=p(\rho,S)$. The density ρ, the velocity $\mathbf{v}$, the temperature T and the entropy S are unknown functions in system (2.1)–(2.4). The coefficients β, κ and η are constant viscosity coefficients. The wave motion is supposed to be potential and the viscosity coefficients are supposed to be small in terms of a dimensionless small parameter $\varepsilon>0$:

$$\eta\Delta\mathbf{v}+\left(\zeta+\frac{\eta}{3}\right)\nabla.\operatorname{div}(\mathbf{v})=\left(\zeta+\frac{4}{3}\eta\right)\Delta\mathbf{v}:=\beta\Delta\mathbf{v}\quad\text{with }\beta=\varepsilon\tilde{\beta}.$$

Any constant state $(\rho_0,\mathbf{v}_0,S_0,T_0)$ is a stationary solution of system (2.1)–(2.4). Further we always take $\mathbf{v}_0=0$ using a Galilean transformation. Perturbation near this constant state $(\rho_0,0,S_0,T_0)$ introduces small increments in terms of the same dimensionless small parameter $\varepsilon>0$:

$$T(x,t)=T_0+\varepsilon\tilde{T}(x,t)\quad\text{and}\quad S(x,t)=S_0+\varepsilon^2\tilde{S}(x,t),$$
$$\rho_\varepsilon(x,t)=\rho_0+\varepsilon\tilde{\rho}_\varepsilon(x,t)\quad\text{and}\quad \mathbf{v}_\varepsilon(x,t)=\varepsilon\tilde{\mathbf{v}}_\varepsilon(x,t),$$

where the perturbation of the entropy is of order $O(\varepsilon^2)$, since it is the smallest size on ε of right-hand terms in Equation (2.3), due to the smallness of the viscosities (see Equation (2.5)).

Actually, ε is the Mach number, which is supposed to be small [4] ($\epsilon=10^{-5}$ for the propagation in water with an initial power of the order of $0.3\,\mathrm{W/cm^2}$):

$$\frac{\rho-\rho_0}{\rho_0}\sim\frac{T-T_0}{T_0}\sim\frac{|\mathbf{v}|}{c_0}\sim\epsilon,$$

where $c_0=\sqrt{p'(\rho_0)}$ is the speed of sound in the unperturbed media.

Using the transport heat Equation (2.3) up to the terms of the order of ε^3

$$\varepsilon^2\rho_0 T_0\partial_t\tilde{S}=\varepsilon^2\tilde{\kappa}\Delta\tilde{T}+\mathcal{O}(\varepsilon^3), \tag{2.5}$$

the approximate state equation

$$p=p_0+c^2\varepsilon\tilde{\rho}_\varepsilon+\frac{1}{2}(\partial^2_\rho p)_S\varepsilon^2\tilde{\rho}^2_\varepsilon+(\partial_S p)_\rho\varepsilon^2\tilde{S}+\mathcal{O}(\varepsilon^3)$$

(where the notation $(.)_S$ means that the expression in brackets is constant in S), can be replaced [4, 12, 27] by

$$p=p_0+c^2\varepsilon\tilde{\rho}_\varepsilon+\frac{(\gamma-1)c^2}{2\rho_0}\varepsilon^2\tilde{\rho}^2_\varepsilon-\varepsilon\tilde{\kappa}\left(\frac{1}{C_V}-\frac{1}{C_p}\right)\nabla.\mathbf{v}_\varepsilon+\mathcal{O}(\varepsilon^3),$$

using $T=\frac{p}{\rho R}$ from the theory of ideal gas and taking

$$p(\rho,S)=R\rho^\gamma e^{\dfrac{S-S_0}{C_V}}.$$

Here $\gamma=C_p/C_V$ denotes the ratio of the heat capacities at constant pressure and at constant volume respectively.

Hence, system (2.1)–(2.4) becomes an isentropic Navier-Stokes system

$$\partial_t\rho_\varepsilon+\operatorname{div}(\rho_\varepsilon\mathbf{v}_\varepsilon)=0, \tag{2.6}$$

$$\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon]=-\nabla p(\rho_\varepsilon)+\varepsilon\nu\Delta\mathbf{v}_\varepsilon, \tag{2.7}$$

with the approximate state equation $p(\rho,S)=p(\rho_\varepsilon)+O(\varepsilon^3)$:

$$p(\rho_\varepsilon)=p_0+c^2(\rho_\varepsilon-\rho_0)+\frac{(\gamma-1)c^2}{2\rho_0}(\rho_\varepsilon-\rho_0)^2, \tag{2.8}$$

and with a small enough and positive viscosity coefficient:

$$\varepsilon\nu=\beta+\kappa\left(\frac{1}{C_V}-\frac{1}{C_p}\right).$$

3. Navier-Stokes system and the Kuznetsov equation

We consider system (2.6)–(2.8) as the exact model. The state law (2.8) is a Taylor expansion of the pressure up to the terms of the third order on ε. Therefore an approximation of system (2.6)–(2.8) for $\mathbf{v}_\varepsilon$ and ρ_ε up to terms $O(\varepsilon^3)$ would be optimal. In the framework of the nonlinear acoustic between the known approximate models derived from system (2.6)–(2.8) are the Kuznetsov, the KZK and the NPE equations. In this section we focus on the first of these models, *i.e.* on the Kuznetsov equation.

Initially the Kuznetsov equation was derived by Kuznetsov [22] from the isentropic Navier-Stokes system (2.6)–(2.8) for the small velocity potential $\mathbf{v}_\varepsilon(\mathbf{x},t)=-\nabla\tilde{u}(\mathbf{x},t)$, $\mathbf{x}\in\mathbb{R}^n$, $t\in\mathbb{R}^+$:

$$\partial^2_t\tilde{u}-c^2\triangle\tilde{u}=\partial_t\left((\nabla\tilde{u})^2+\frac{\gamma-1}{2c^2}(\partial_t\tilde{u})^2+\frac{\varepsilon\nu}{\rho_0}\Delta\tilde{u}\right). \tag{3.1}$$

The derivation was latter discussed by a lot of authors [12, 17, 25].

Unlike in these physical derivations we introduce a Hilbert expansion type construction with a corrector $\varepsilon^2\rho_2(\mathbf{x},t)$ for the density perturbation, by considering the following *ansatz*

$$\rho_\varepsilon(\mathbf{x},t)=\rho_0+\varepsilon\rho_1(\mathbf{x},t)+\varepsilon^2\rho_2(\mathbf{x},t), \tag{3.2}$$

$$\mathbf{v}_\varepsilon(\mathbf{x},t)=-\varepsilon\nabla u(\mathbf{x},t). \tag{3.3}$$

The use of the second-order corrector in (3.2) allows to ensure the approximation of (2.7) up to terms of order ε^3 (see Subsection 3.1) and to open the question about the approximation between the exact solution of the isentropic Navier-Stokes system (2.6)–(2.8) and its approximation given by the solution of the Kuznetsov equation, as it was done for the KZK equation [35].

3.1. Derivation of the Kuznetsov equation from an isentropic Navier-Stokes system. Putting expressions for the density and velocity (3.2)–(3.3) into the isentropic Navier-Stokes system (2.6)–(2.8), we obtain for the momentum conservation (2.7)

$$\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon=\varepsilon\nabla(-\rho_0\partial_t u+c^2\rho_1)$$
$$+\varepsilon^2\left[-\rho_1\nabla(\partial_t u)+\frac{\rho_0}{2}\nabla((\nabla u)^2)+c^2\nabla\rho_2+\frac{(\gamma-1)c^2}{2\rho_0}\nabla(\rho_1^2)+\nu\nabla\Delta u\right]+O(\varepsilon^3). \tag{3.4}$$

In order to have an approximation up to the terms $O(\varepsilon^3)$ we put the terms of order one and two in ε equal to 0, which allows us to find the expressions for the density correctors:

$$\rho_1(\mathbf{x},t)=\frac{\rho_0}{c^2}\partial_t u(\mathbf{x},t), \tag{3.5}$$

$$\rho_2(\mathbf{x},t)=-\frac{\rho_0(\gamma-2)}{2c^4}(\partial_t u)^2-\frac{\rho_0}{2c^2}(\nabla u)^2-\frac{\nu}{c^2}\Delta u. \tag{3.6}$$

Indeed, we start by making $\varepsilon\nabla(-\rho_0\partial_t u+c^2\rho_1)=0$ and find the first-order perturbation of the density ρ_1 given by Equation (3.5). Consequently, if ρ_1 satisfies (3.5), then Equation (3.4) becomes

$$\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon=\varepsilon\nabla(-\rho_0\partial_t u+c^2\rho_1)$$
$$\varepsilon^2\nabla\left[-\frac{\rho_0}{2c^2}(\partial_t u)^2+\frac{\rho_0}{2}(\nabla u)^2+c^2\rho_2+\frac{(\gamma-1)\rho_0}{2c^2}(\partial_t u)^2+\nu\Delta u\right]+O(\varepsilon^3). \tag{3.7}$$

Thus, taking the corrector ρ_2 by formula (3.6), we ensure that

$$\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon=O(\varepsilon^3). \tag{3.8}$$

Now we put these expressions of ρ_1 from (3.5) and ρ_2 from (3.6) with *ansatz* (3.2)–(3.3) in Equation (2.6) of the mass conservation to obtain

$$\partial_t\rho_\varepsilon+\operatorname{div}(\rho_\varepsilon\mathbf{v}_\varepsilon)=\varepsilon\frac{\rho_0}{c^2}\left[\partial_t^2 u-c^2\Delta u\right.$$
$$\left.-\varepsilon\partial_t\left((\nabla u)^2+\frac{\gamma-2}{2c^2}(\partial_t u)^2+\frac{\nu}{\rho_0}\Delta u\right)-\varepsilon u_t\Delta u\right]+O(\varepsilon^3). \tag{3.9}$$

Then we notice that the right-hand term of the order ε in Equation (3.9) is actually the linear wave equation up to smaller on ε terms:

$$\partial_t^2 u - c^2\Delta u = O(\varepsilon).$$

Hence, we express

$$\varepsilon u_t \Delta u = \varepsilon \frac{1}{c^2} u_t u_{tt} + O(\varepsilon^2) = \varepsilon \frac{1}{2c^2}\partial_t((u_t)^2) + O(\varepsilon^2)$$

and, putting it in Equation (3.9), we finally have

$$\begin{aligned}\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = &\varepsilon \frac{\rho_0}{c^2}\left[\partial_t^2 u - c^2 \Delta u\right.\\ &\left.-\varepsilon\partial_t\left((\nabla u)^2 + \frac{\gamma-1}{2c^2}(\partial_t u)^2 + \frac{\nu}{\rho_0}\Delta u\right)\right] + O(\varepsilon^3).\end{aligned} \tag{3.10}$$

The right-hand side of Equation (3.10) gives us the Kuznetsov equation

$$\partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2}(\partial_t u)^2 + \frac{\nu}{\rho_0}\Delta u\right), \tag{3.11}$$

which is the first-order approximation of the isentropic Navier-Stokes system up to the terms $O(\varepsilon^3)$. Moreover, if u is a solution of the Kuznetsov equation, then with the relations for the density perturbations (3.5) and (3.6) and with *ansatz* (3.2)–(3.3) we have

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = O(\varepsilon^3), \tag{3.12}$$

$$\rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon \nu \Delta \mathbf{v}_\varepsilon = O(\varepsilon^3). \tag{3.13}$$

Hence, it is clear that the standard physical perturbative approach without the corrector ρ_2 (it is sufficient to take $\rho_2 = 0$ in our calculus) can't ensure (3.12)–(3.13).

Let us also notice, as it was originally mentioned by Kuznetsov, that the Kuznetsov Equation (3.11) contains terms of different orders, and hence, it is a wave equation with small size non-linear perturbations $\partial_t(\nabla u)^2$, $\partial_t(\partial_t u)^2$ and the viscosity term $\partial_t \Delta u$.

3.2. Approximation of the solutions of the isentropic Navier-Stokes system by the solutions of the Kuznetsov equation. Let us calculate the remainder terms in (3.12)–(3.13), which are denoted respectively by $\varepsilon^3 R_1^{NS-Kuz}$ and $\varepsilon^3 \mathbf{R}_2^{NS-Kuz}$:

$$\begin{aligned}\varepsilon^3 R_1^{NS-Kuz} = &\varepsilon^3\left[\frac{1}{c^2}\partial_t u\left(\frac{\rho_0(\gamma-2)}{2c^4}\partial_t[(\partial_t u)^2] + \frac{\rho_0}{c^2}\partial_t[(\nabla u)^2] + \frac{\nu}{c^2}\partial_t \Delta u\right)\right.\\ &\left.-\frac{\rho_0}{c^2}\partial_t u\,\Delta u - \nabla\rho_2.\nabla u - \rho_2 \Delta u\right] + \varepsilon^4 \frac{1}{c^2}\partial_t u(\nabla \rho_2.\nabla u + \rho_2 \Delta u),\end{aligned} \tag{3.14}$$

$$\varepsilon^3 \mathbf{R}_2^{NS-Kuz} = \varepsilon^3\left[\frac{\rho_1}{2}\nabla[(\nabla u)^2] - \rho_2 \nabla \partial_t u\right] + \varepsilon^4 \frac{\rho_2}{2}\nabla\left[(\nabla u)^2\right]. \tag{3.15}$$

If u is a sufficiently regular solution of the Cauchy problem for the Kuznetsov equation in $\mathbb{R}^n$

$$\begin{cases}\partial_t^2 u - c^2\Delta u = \varepsilon\partial_t\left((\nabla u)^2 + \frac{\gamma-1}{2c^2}(\partial_t u)^2 + \frac{\nu}{\rho_0}\Delta u\right),\\ u(0) = u_0,\ u_t(0) = u_1,\end{cases} \tag{3.16}$$

then, taking ρ_1 and ρ_2 according to formulas (3.5)-(3.6), we define $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ by Equations (3.2)-(3.3) and obtain a solution of the following approximate system

$$\partial_t\overline{\rho}_\varepsilon + \operatorname{div}(\overline{\rho}_\varepsilon\overline{v}_\varepsilon) = \varepsilon^3 R_1^{NS-Kuz}, \tag{3.17}$$

$$\overline{\rho}_\varepsilon[\partial_t\overline{\mathbf{v}}_\varepsilon + (\overline{\mathbf{v}}_\varepsilon.\nabla)\overline{\mathbf{v}}_\varepsilon] + \nabla p(\overline{\rho}_\varepsilon) - \varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon = \varepsilon^3\mathbf{R}_2^{NS-Kuz} \tag{3.18}$$

with $p(\overline{\rho}_\varepsilon)$ from the state law (2.8). With notations

$$\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon\mathbf{v}_\varepsilon)^t \text{ and } \overline{\mathbf{U}}_\varepsilon = (\overline{\rho}_\varepsilon, \overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)^t,$$

the exact (2.6)–(2.7) and the approximate (3.17)–(3.18) Navier-Stokes systems can be respectively rewritten in the following forms [8, 35]:

$$\partial_t\mathbf{U}_\varepsilon + \sum_{i=1}^n \partial_{x_i}\mathbf{G}_i(\mathbf{U}_\varepsilon) - \varepsilon\nu\begin{bmatrix}0\\ \Delta\mathbf{v}_\varepsilon\end{bmatrix} = 0, \tag{3.19}$$

$$\partial_t\overline{\mathbf{U}}_\varepsilon + \sum_{i=1}^n \partial_{x_i}\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) - \varepsilon\nu\begin{bmatrix}0\\ \Delta\overline{\mathbf{v}}_\varepsilon\end{bmatrix} = \varepsilon^3\mathbf{R}^{NS-Kuz} \tag{3.20}$$

with $\mathbf{R}^{NS-Kuz} = \begin{bmatrix}R_1^{NS-Kuz}\\ \mathbf{R}_2^{NS-Kuz}\end{bmatrix}$ from (3.14)–(3.15) and

$$\mathbf{G}_i(\mathbf{U}_\varepsilon) = \begin{bmatrix}\rho_\varepsilon v_i\\ \rho_\varepsilon v_i\mathbf{v}_\varepsilon + p(\rho_\varepsilon)\mathbf{e}_i\end{bmatrix}, \quad \partial_{x_i}\mathbf{G}_i(\mathbf{U}_\varepsilon) = D\mathbf{G}_i(\mathbf{U}_\varepsilon)\partial_{x_i}\mathbf{U}_\varepsilon. \tag{3.21}$$

The well-posedness results for the Cauchy problems (2.6)-(2.8) [29] and (3.16) [9] allow us to establish the global existence and the unicity of the classical solutions $\mathbf{U}_\varepsilon$ and $\overline{\mathbf{U}}_\varepsilon$, considered in the Kuznetsov approximation framework:

THEOREM 3.1. *There exists a constant $k>0$ such that if the initial data $u_0 \in H^5(\mathbb{R}^3)$ and $u_1 \in H^4(\mathbb{R}^3)$ for the Cauchy problem for the Kuznetsov Equation (3.16) are sufficiently small*

$$\|u_0\|_{H^5(\mathbb{R}^3)} + \|u_1\|_{H^4(\mathbb{R}^3)} < k,$$

then there exist global-in-time solutions $\overline{\mathbf{U}}_\varepsilon = (\overline{\rho}_\varepsilon, \overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)^t$ of the approximate Navier-Stokes system (3.20) and $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon\mathbf{v}_\varepsilon)^t$ of the exact Navier-Stokes system (3.19) respectively, with the same regularity corresponding to

$$\overline{\rho}_\varepsilon - \rho_0, \rho_\varepsilon - \rho_0 \in C([0,+\infty[;H^3(\mathbb{R}^3)) \cap C^1([0,+\infty[;H^2(\mathbb{R}^3)) \tag{3.22}$$

and

$$\overline{\mathbf{v}}_\varepsilon, \mathbf{v}_\varepsilon \in C([0,+\infty[;H^3(\mathbb{R}^3)) \cap C^1([0,+\infty[;H^1(\mathbb{R}^3)), \tag{3.23}$$

both considered with the state law (2.8) and with the same initial data

$$(\bar{\rho}_\varepsilon - \rho_\varepsilon)|_{t=0} = 0, \quad (\bar{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)|_{t=0} = 0, \tag{3.24}$$

where $\bar{\rho}_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial data for the Kuznetsov equation u_0 and u_1 according to formulas (3.2)–(3.3) and (3.5)–(3.6):

$$\bar{\rho}_\varepsilon|_{t=0} = \rho_0 + \varepsilon\frac{\rho_0}{c^2}u_1 - \varepsilon^2\left[\frac{\rho_0(\gamma-2)}{2c^4}u_1^2 + \frac{\rho_0}{2c^2}(\nabla u_0)^2 + \frac{\nu}{c^2}\Delta u_0\right], \tag{3.25}$$

$$\bar{\mathbf{v}}_\varepsilon|_{t=0} = -\varepsilon \nabla u_0. \tag{3.26}$$

Proof. On one hand, Theorem 1.2 in Ref. [9] applied for $n=3$ with $m=4$ ensures that for $u_0 \in H^5(\mathbb{R}^3)$ and $u_1 \in H^4(\mathbb{R}^3)$ there exists a constant $k_2 > 0$ such that if

$$\|u_0\|_{H^5(\mathbb{R}^3)} + \|u_1\|_{H^4(\mathbb{R}^3)} < k_2, \tag{3.27}$$

then the Cauchy problem for the Kuznetsov Equation (3.16) has a unique global-in-time solution

$$u \in C([0,+\infty[, H^5(\mathbb{R}^3)) \cap C^1([0,+\infty[, H^4(\mathbb{R}^3)) \cap C^2([0,+\infty[, H^2(\mathbb{R}^3)). \tag{3.28}$$

On the other hand, the Cauchy problem for the Navier-Stokes system is also globally well-posed in $\mathbb{R}^3$ for sufficiently small initial data (see Ref. [29] Theorem 7.1, p. 100): there exists a constant $k_1 > 0$ such that if the initial data

$$\rho_\varepsilon(0) - \rho_0 \in H^3(\mathbb{R}^3), \ \ \mathbf{v}_\varepsilon(0) \in H^3(\mathbb{R}^3) \tag{3.29}$$

satisfy

$$\|\rho_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{R}^3)} + \|\mathbf{v}_\varepsilon(0)\|_{H^3(\mathbb{R}^3)} < k_1,$$

then the Cauchy problem (2.6)-(2.8) with the initial data (3.29) has a unique solution $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ globally in time satisfying (3.22) and (3.23).

Thus, for the initial solutions of the Kuznetsov equation we need to impose $u_0 \in H^5(\mathbb{R}^3)$ to have $\Delta u_0 \in H^3(\mathbb{R}^3)$ to be able to ensure that $\rho_\varepsilon - \rho_0|_{t=0} \in H^3(\mathbb{R}^3)$. The regularity $u_1 \in H^4(\mathbb{R}^3)$ comes from the well-posedness of the Kuznetsov problem and obviously ensures $\mathbf{v}_\varepsilon|_{t=0} \in H^3(\mathbb{R}^3)$, which is necessary [29] to have a global solution of the exact Navier-Stokes system (3.19).

As $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ are defined by *ansatz* (3.2)-(3.3) with ρ_1 and ρ_2 given in (3.5) and (3.6) respectively, the regularity of u ensures for $\overline{\rho}_\varepsilon - \rho_0$ and $\overline{\mathbf{v}}_\varepsilon$ at least the same regularity as given in (3.22) and (3.23). To find it we use the following Sobolev embedding for the multiplication (see for example Ref. [5] or [21]):

$$H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n) \text{ for } s > \frac{n}{2}, \tag{3.30}$$
$$(u,v) \mapsto uv.$$

Moreover, considering formulas (3.14)–(3.15) with u as defined in (3.28), all terms in R_1^{NS-Kuz} and $\mathbf{R}_2^{NS-Kuz}$ are in $H^2(\mathbb{R}^3)$. Therefore, as $2 > \frac{3}{2}$, we use embedding (3.30) to find that

$$R_1^{NS-Kuz} \in C([0,+\infty[, H^2(\mathbb{R}^3)) \quad \text{and} \quad \mathbf{R}_2^{NS-Kuz} \in C([0,+\infty[, H^2(\mathbb{R}^3)).$$

Hence, the $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ norms of the remainder terms $R_1^{NS-Kuz}(t)$ and $\mathbf{R}_2^{NS-Kuz}(t)$ are bounded for $t \in [0,+\infty[$.

Finally, it is important to notice that, as $\mathbf{U}_\varepsilon(0) = \overline{\mathbf{U}}_\varepsilon(0)$,

$$\begin{aligned}\|\rho_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{R}^3)} + \|\mathbf{v}_\varepsilon(0)\|_{H^3(\mathbb{R}^3)} &= \|\overline{\rho}_\varepsilon(0) - \rho_0\|_{H^3(\mathbb{R}^3)} + \|\overline{\mathbf{v}}_\varepsilon(0)\|_{H^3(\mathbb{R}^3)} \\ &\le C(\|u_0\|_{H^5(\mathbb{R}^3)} + \|u_1\|_{H^4(\mathbb{R}^3)}).\end{aligned}$$

Thus, there exists $k > 0$ (necessarily $k \le k_2$) such that $\|u_0\|_{H^5} + \|u_1\|_{H^4} < k$ implies the global existence of $\mathbf{U}_\varepsilon$ and $\overline{\mathbf{U}}_\varepsilon$. ∎

The stability estimate which we obtain between the exact solution of the Navier-Stokes system $\mathbf{U}_\varepsilon$ and the solution of the Kuznetsov equation presented by $\overline{\mathbf{U}}_\varepsilon$ does not require for $\mathbf{U}_\varepsilon$ to have the regularity of a classical solution and allows to approximate less regular solutions of the Navier-Stokes system with initial data in a small L^2 neighborhood of $\overline{\mathbf{U}}_\varepsilon(0)$. To define the minimal regularity property of $\mathbf{U}_\varepsilon$ for which stability estimate (1.2) holds, we introduce admissible weak solutions of a bounded energy using the entropy of the Euler system (system (3.19) with $\nu=0$)

$$\eta(\mathbf{U}_\varepsilon)=\rho_\varepsilon h(\rho_\varepsilon)+\rho_\varepsilon\frac{\mathbf{v}_\varepsilon^2}{2}=H(\rho_\varepsilon)+\frac{1}{\rho_\varepsilon}\frac{\mathbf{m}^2}{2}, \tag{3.31}$$

which is convex [8] with $h'(\rho_\varepsilon)=\frac{p(\rho_\varepsilon)}{\rho_\varepsilon^2}$ and $\mathbf{v}_\varepsilon=\frac{\mathbf{m}}{\rho_\varepsilon}$. Thus, the first and second derivatives of η are [35]

$$\eta'(\mathbf{U}_\varepsilon)=\begin{bmatrix} H'(\rho_\varepsilon)-\frac{1}{\rho_\varepsilon^2}\frac{\mathbf{m}^2}{2} \\ \frac{\mathbf{m}}{\rho_\varepsilon}\end{bmatrix}^t=\begin{bmatrix} H'(\rho_\varepsilon)-\frac{\mathbf{v}_\varepsilon^2}{2} \\ \mathbf{v}_\varepsilon\end{bmatrix}^t, \tag{3.32}$$

$$\eta''(\mathbf{U}_\varepsilon)=\begin{bmatrix} H''(\rho_\varepsilon)+\frac{\mathbf{m}^2}{\rho_\varepsilon^3} & -\frac{\mathbf{m}}{\rho_\varepsilon^2} \\ -\frac{\mathbf{m}}{\rho_\varepsilon^2} & \frac{1}{\rho_\varepsilon}\end{bmatrix}=\begin{bmatrix} H''(\rho_\varepsilon)+\frac{\mathbf{v}_\varepsilon^2}{\rho_\varepsilon} & -\frac{\mathbf{v}_\varepsilon}{\rho_\varepsilon} \\ -\frac{\mathbf{v}_\varepsilon}{\rho_\varepsilon} & \frac{1}{\rho_\varepsilon}\end{bmatrix}, \tag{3.33}$$

knowing that $\eta''(\mathbf{U}_\varepsilon)$ is strictly positive-definite.

DEFINITION 3.1. *The function $\mathbf{U}_\varepsilon=(\rho_\varepsilon,\rho_\varepsilon\mathbf{v}_\varepsilon)$ is called an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system (2.6)–(2.8) if it satisfies the following properties:*

(1) The pair $(\rho_\varepsilon,\mathbf{v}_\varepsilon)$ is a weak solution of the Cauchy problem for the Navier-Stokes system (2.6)–(2.8) (in the distributional sense).

(2) The function $\mathbf{U}_\varepsilon$ satisfies in the sense of distributions (see Ref. [8, p.52])

$$\partial_t\eta(\mathbf{U}_\varepsilon)+\nabla.\boldsymbol{q}(\mathbf{U}_\varepsilon)-\varepsilon\nu\mathbf{v}_\varepsilon\triangle\mathbf{v}_\varepsilon\leq 0,\ \textit{where}\ \boldsymbol{q}(\mathbf{U}_\varepsilon)=\mathbf{v}_\varepsilon(\eta(\mathbf{U}_\varepsilon)+p(\rho_\varepsilon)), \tag{3.34}$$

or equivalently, for any positive test function ψ in $\mathcal{D}(\mathbb{R}^n\times[0,\infty[)$ the function $\mathbf{U}_\varepsilon$ satisfies

$$\int_0^T\int_{\mathbb{R}^n}\left(\partial_t\psi\eta(\mathbf{U}_\varepsilon)+\nabla\psi.\boldsymbol{q}(\mathbf{U}_\varepsilon)+\varepsilon\nu|\nabla.\mathbf{v}_\varepsilon|^2\psi+\varepsilon\nu\mathbf{v}_\varepsilon.[\nabla.\mathbf{v}_\varepsilon\nabla\psi]\right)dxdt$$
$$+\int_{\mathbb{R}^n}\psi(x,0)\eta(\mathbf{U}_\varepsilon(0))dx\geq 0.$$

(3) The function $\mathbf{U}_\varepsilon$ satisfies the equality (with the notation $\mathbf{v}_\varepsilon=(v_1,\dots,v_n)$)

$$-\int_{\mathbb{R}^n}\frac{\mathbf{U}_\epsilon^2(t)}{2}dx+\int_0^t\int_{\mathbb{R}^n}\left(\sum_{i=1}^n\mathbf{G}_i(\mathbf{U}_\varepsilon)\partial_{x_i}\mathbf{U}_\epsilon-\epsilon\nu\nabla(\rho_\varepsilon v_i).\nabla v_i\right)dxds$$
$$+\int_{\mathbb{R}^n}\frac{\mathbf{U}_\epsilon^2(0)}{2}dx=0.$$

Let us notice that any classical solution of (3.19), for instance the solution defined in Theorem 3.1, satisfies the entropy condition (3.34) by the equality and obviously it is sufficiently regular to perform the integration by parts resulting in the relation of point 3. For existence results of global weak solutions of the Cauchy problem for the Navier-Stokes system (3.19) with sufficiently small initial data around the constant state $(\rho_0, 0)$ (actually, $\rho_0 - \rho(0)$ is small in L^∞, $\mathbf{v}(0)$ is small in L^2 and bounded in L^{2^n}) and with the pressure $p(\rho) = K\rho^\gamma$ with $\gamma \geq 1$, we refer to results of D. Hoff [13, 14]. For fixing the idea of the regularity of a global weak solution we summarize the results of Hoff in the following theorem:

THEOREM 3.2 ([13]). *Let for $n=3$, $\beta=0$ and for $n=2$, β be arbitrarily small, N be a given arbitrarily large constant. There exists a constant $C_0>0$ such that if the initial data of (3.19) with $p(\rho)=K\rho^\gamma$ ($\gamma \geq 1$) satisfy the following smallness condition*

$$\|\rho_0 - \rho(0)\|^2_{L^\infty(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \left[(\rho_0 - \rho(0))^2 + |\mathbf{v}(0)|^2\right](1+|x|^2)^\beta \mathrm{d}x \leq C_0,$$
$$\|\mathbf{v}(0)\|_{L^{2^n}(\mathbb{R}^n)} \leq N,$$

then there exists a global weak solution $(\rho, \mathbf{v})$ (in the distributional sense) such that
(1) $\rho - \rho_0 \in L^\infty(\mathbb{R}^n \times [0,\infty[)$,
(2) $\mathbf{v} \in H^1(\mathbb{R}^n)$ for all $t>0$,
(3) for all $t \geq \tau > 0$ $\mathbf{v}(\cdot,t) \in L^\infty(\mathbb{R}^n)$,
(4) for all $\tau>0$ $\mathbf{v} \in C^{\alpha, \frac{\alpha}{2\alpha+2}}(\mathbb{R}^n \times [\tau,\infty[)$ for all $\alpha \in]0,1[$ when $n=2$ and $\mathbf{v} \in C^{\frac{1}{2},\frac{1}{8}}(\mathbb{R}^n \times [\tau,\infty[)$ when $n=3$,
(5) $\varepsilon\nu \operatorname{div}\mathbf{v} + p(\rho) - p(\rho_0) \in H^1(\mathbb{R}^n) \cap C^\alpha(\mathbb{R}^n)$ for almost all $t>0$ with $\alpha = \frac{1}{2}$ for $n=2$ and $\alpha = \frac{1}{10}$ when $n=3$.

In addition, $(\rho, \mathbf{v}) \to (\rho_0, 0)$ as $t \to +\infty$ in the sense that for all $q \in]2,+\infty[$

$$\lim_{T\to\infty}\left(\|\rho - \rho_0\|_{L^\infty(\mathbb{R}^n \times [T,\infty[)} + \|\mathbf{v}(\cdot,T)\|_{L^q(\mathbb{R}^n)}\right) = 0.$$

Therefore, from Theorem 3.2 it follows that a weak solution of the isentropic compressible Navier-Stokes system (2.6)–(2.8) is also an admissible weak solution of a bounded energy in the sense of Definition 3.1. But in the following we only consider the question of the validity of the stability estimate (1.2) for initial data close to $\overline{\mathbf{U}}_\varepsilon(0)$ in L^2 norm (thus for initial data not necessarily satisfying Theorem 3.2) and we don't consider the existence question of an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system. Thanks to Theorem 3.1 for classical solutions of two models and to Definition 3.1 containing the minimal conditions on $\mathbf{U}_\varepsilon$ necessary for saying that it is in a small L^2-neighborhood of the regular solution of the Kuznetsov equation, we validate the approximation of $\mathbf{U}_\varepsilon$ by $\overline{\mathbf{U}}_\varepsilon$ following the ideas of Ref. [35].

THEOREM 3.3. *Let $\nu>0$ and $\varepsilon>0$ be fixed and all assumptions of Theorem 3.1 hold. Then there exist constants $C>0$ and $K>0$, independent of ε and the time t, such that*
(1) for all $t \leq \frac{C}{\varepsilon}$

$$\|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(\mathbb{R}^3)} \leq K\varepsilon^3 t e^{K\varepsilon t} \leq 4\varepsilon^2;$$

(2) for all $b \in]0,1[$ during all time $t \leq \frac{C}{\varepsilon}\ln(\frac{1}{\varepsilon})$ it holds

$$\|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|_{L^2(\mathbb{R}^3)} \leq 2\varepsilon^b.$$

Moreover, if the initial conditions for the Kuznetsov equation are such that

$$u_0 \in H^{s+2}(\mathbb{R}^n), \quad u_1 \in H^{s+1}(\mathbb{R}^n) \text{ for } s > \frac{n}{2}, n \geq 2$$

and sufficiently small (in the sense of Ref. [9] Theorem 1.2), then there exists the unique global-in-time solution of the Cauchy problem for the Kuznetsov equation

$$\overline{\rho}_\varepsilon - \rho_0 \in C([0,+\infty[;H^s(\mathbb{R}^n))\cap C^1([0,+\infty[;H^{s-1}(\mathbb{R}^n)), \tag{3.35}$$

$$\overline{\boldsymbol{v}}_\varepsilon \in C([0,+\infty[;H^{s+1}(\mathbb{R}^n))\cap C^1([0,+\infty[;H^s(\mathbb{R}^n)) \tag{3.36}$$

and the remainder terms $(R_1^{NS-Kuz},\mathbf{R}_2^{NS-Kuz})$*, defined in Equations (3.14)-(3.15), belong to* $C([0,+\infty[,H^{s-1}(\mathbb{R}^n))$*.*

If in addition there exists an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system (3.19) (for instance if $\mathbf{U}_\varepsilon(0)$ *satisfies conditions of Theorem 3.2 there is such a global weak solution) on a time interval* $[0,T_{NS}[$ *for the initial data*

$$\|\mathbf{U}_\varepsilon(0)-\overline{\mathbf{U}}_\varepsilon(0)\|_{L^2(\mathbb{R}^n)}\le\delta\le\varepsilon,$$

then it holds for all $t<\min\{\frac{C}{\varepsilon},T_{NS}\}$ *the stability estimate (1.2):*

$$\|(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(\mathbb{R}^n)}\le K(\varepsilon^3 t+\delta^2)e^{K\varepsilon t}\le 9\varepsilon^2.$$

Proof. In terms of entropy, system (3.20), having, by the assumption, the unique classical solution $\overline{\mathbf{U}}_\varepsilon$, can be rewritten as follows

$$\partial_t\eta(\overline{\mathbf{U}}_\varepsilon)+\nabla.\mathbf{q}(\overline{\mathbf{U}}_\varepsilon)-\varepsilon\nu\overline{\mathbf{v}}_\varepsilon.\Delta\overline{\mathbf{v}}_\varepsilon=\varepsilon^3\left(\frac{\eta(\overline{\mathbf{U}}_\varepsilon)+p(\overline{\rho}_\varepsilon)}{\overline{\rho}_\varepsilon}R_1^{NS-Kuz}+\overline{\mathbf{v}}_\varepsilon.\mathbf{R}_2^{NS-Kuz}\right) \tag{3.37}$$

with

$$\mathbf{R}^{NS-Kuz}=(R_1^{NS-Kuz},\mathbf{R}_2^{NS-Kuz})$$

defined in Equation (3.14)-(3.15). To abbreviate the notations, we denote the remainder term of the entropy equation in system (3.37) by

$$\overline{R}^{NS-Kuz}=\left(\frac{\eta(\overline{\mathbf{U}}_\varepsilon)+p(\overline{\rho}_\varepsilon)}{\overline{\rho}_\varepsilon}R_1^{NS-Kuz}+\overline{\mathbf{v}}_\varepsilon.\mathbf{R}_2^{NS-Kuz}\right).$$

In the same time, it is assumed that for $\mathbf{U}_\varepsilon$ (3.34) holds in the sense of distributions.

Let us estimate in the sense of distributions

$$\frac{\partial}{\partial t}\Big(\eta(\mathbf{U}_\varepsilon)-\eta(\overline{\mathbf{U}}_\varepsilon)-\eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)\Big). \tag{3.38}$$

First we find from systems (3.34) and (3.37) that in the sense of distributions

$$\begin{aligned}\frac{\partial}{\partial t}(\eta(\mathbf{U}_\varepsilon)-\eta(\overline{\mathbf{U}}_\varepsilon))&\le-\nabla.(\mathbf{q}(\mathbf{U}_\varepsilon)-\mathbf{q}(\overline{\mathbf{U}}_\varepsilon))+\varepsilon\nu(\mathbf{v}_\varepsilon.\Delta\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon.\Delta\overline{\mathbf{v}}_\varepsilon)-\varepsilon^3\overline{R}^{NS-Kuz}\\&=-\nabla.(\mathbf{q}(\mathbf{U}_\varepsilon)-\mathbf{q}(\overline{\mathbf{U}}_\varepsilon))+\varepsilon\nu\sum_{i=1}^n\partial_{x_i}(\mathbf{v}_\varepsilon\partial_{x_i}\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon\partial_{x_i}\overline{\mathbf{v}}_\varepsilon)\\&\quad-\varepsilon\nu\sum_{i=1}^n(\partial_{x_i}\mathbf{v}_\varepsilon\partial_{x_i}\mathbf{v}_\varepsilon-\partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}\overline{\mathbf{v}}_\varepsilon)-\varepsilon^3\overline{R}^{NS-Kuz}.\end{aligned}$$

Then we notice that

$$-\frac{\partial}{\partial t}(\eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon))=-\partial_t\overline{\mathbf{U}}_\varepsilon^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)-\eta'(\overline{\mathbf{U}}_\varepsilon)(\partial_t\mathbf{U}_\varepsilon-\partial_t\overline{\mathbf{U}}_\varepsilon),$$

where in the sense of distributions

$$-\partial_t\overline{\mathbf{U}}_\varepsilon^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)=-\left[-\sum_{i=1}^n D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)\partial_{x_i}\overline{\mathbf{U}}_\varepsilon\right]^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)$$
$$-\left(\begin{bmatrix}0\\ \varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon\end{bmatrix}+\varepsilon^3\mathbf{R}^{NS-Kuz}\right)^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon),$$

and

$$\begin{aligned}-\eta'(\overline{\mathbf{U}}_\varepsilon)(\partial_t\mathbf{U}_\varepsilon-\partial_t\overline{\mathbf{U}}_\varepsilon)=&-\eta'(\overline{\mathbf{U}}_\varepsilon)(-\sum_{i=1}^n\partial_{x_i}(\mathbf{G}_i(\mathbf{U}_\varepsilon)-\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)))\\
&-\eta'(\overline{\mathbf{U}}_\varepsilon)\varepsilon\nu\begin{bmatrix}0\\ \Delta\mathbf{v}_\varepsilon-\Delta\overline{\mathbf{v}}_\varepsilon\end{bmatrix}+\varepsilon^3\eta'(\overline{\mathbf{U}}_\varepsilon)\mathbf{R}^{NS-Kuz}\\
=&\sum_{i=1}^n\partial_{x_i}(\eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{G}_i(\mathbf{U}_\varepsilon)-\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon))\\
&-\sum_{i=1}^n\partial_{x_i}\overline{U}^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{G}_i(\mathbf{U}_\varepsilon)-\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon))\\
&-\eta'(\overline{\mathbf{U}}_\varepsilon)\varepsilon\nu\begin{bmatrix}0\\ \Delta\mathbf{v}_\varepsilon-\Delta\overline{\mathbf{v}}_\varepsilon\end{bmatrix}+\varepsilon^3\eta'(\overline{\mathbf{U}}_\varepsilon)\mathbf{R}^{NS-Kuz}.\end{aligned}$$

Thanks to the convex property of the entropy we have

$$\eta''(\mathbf{U})D\mathbf{G}_i(\mathbf{U})=(D\mathbf{G}_i(\mathbf{U}))^t\eta''(\mathbf{U}),$$

and consequently

$$\begin{aligned}(D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)\partial_{x_i}\overline{\mathbf{U}}_\varepsilon)^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)=&\partial_{x_i}\overline{\mathbf{U}}_\varepsilon^t(D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon))^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)\\
=&\partial_{x_i}\overline{\mathbf{U}}_\varepsilon^t\eta''(\overline{\mathbf{U}}_\varepsilon)D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon).\end{aligned}$$

Moreover, we compute in the sense of distributions

$$-\begin{bmatrix}0\\ \varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon\end{bmatrix}^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)=-\varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon)-\varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon\frac{\rho_\varepsilon-\overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon}(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon)$$
$$=-\varepsilon\nu\sum_{i=1}^n\partial_{x_i}(\partial_{x_i}\overline{\mathbf{v}}_\varepsilon(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon))+\varepsilon\nu\sum_{i=1}^n\partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon)-\varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon\frac{\rho_\varepsilon-\overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon}(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon),$$

and

$$\begin{aligned}-\eta'(\overline{\mathbf{U}}_\varepsilon)\varepsilon\nu\begin{bmatrix}0\\ \Delta\mathbf{v}_\varepsilon-\Delta\overline{\mathbf{v}}_\varepsilon\end{bmatrix}=&-\varepsilon\nu\overline{\mathbf{v}}_\varepsilon.(\Delta\mathbf{v}_\varepsilon-\Delta\overline{\mathbf{v}}_\varepsilon)\\
=&-\varepsilon\nu\sum_{i=1}^n\partial_{x_i}(\overline{\mathbf{v}}_\varepsilon\partial_{x_i}(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon))+\varepsilon\nu\sum_{i=1}^n\partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon).\end{aligned}$$

We integrate expression (3.38) over $\mathbb{R}^n$ and notice that the integrals of the terms in divergence form in the development of (3.38) are equal to zero. For the regular case in the framework of Theorem 3.1 it is due to the regularity given by (3.22) and (3.23) and the following Sobolev embedding [1]

$$H^s(\mathbb{R}^n)\hookrightarrow C_0(\mathbb{R}^n):=\{f\in C(\mathbb{R}^n)|\ |f(x)|\to 0 \text{ as } \|x\|\to+\infty\}\text{ for } s>\frac{n}{2},\tag{3.39}$$

which allows us to use the fact that

$$\forall f \in C_0(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \nabla . f(x)\,dx = 0.$$

In the case of a weak admissible solution $\mathbf{U}_\varepsilon$ it follows from its bounded energy property (see Definition 3.1 point 3) which implies that $\rho_\varepsilon - \rho_0$ and $\mathbf{v}_\varepsilon$ tend to 0 for $|x| \to +\infty$ and also implies the existence of the integrals over $\mathbb{R}^n$. Therefore, we obtain the following estimate in which each term is well-defined in the sense of distributions on $[0,+\infty[\cap[0,T_{NS}]$

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \eta(\mathbf{U}_\varepsilon) - \eta(\overline{\mathbf{U}}_\varepsilon) - \eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)\mathrm{d}x \\
&\leq -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_{x_i}\overline{U}^t \eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) - D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon))\mathrm{d}x \\
&\quad - \varepsilon\nu\int_{\mathbb{R}^3}\sum_{i=1}^{3}(\partial_{x_i}\mathbf{v}_\varepsilon\partial_{x_i}\mathbf{v}_\varepsilon - \partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}\overline{\mathbf{v}}_\varepsilon)\mathrm{d}x \\
&\quad + 2\varepsilon\nu\int_{\mathbb{R}^3}\sum_{i=1}^{3}\partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}(\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon)\mathrm{d}x + \varepsilon\nu\int_{\mathbb{R}^3}\Delta\overline{\mathbf{v}}_\varepsilon\frac{\rho_\varepsilon - \overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon}(\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon)\mathrm{d}x \\
&\quad - \varepsilon^3\int_{\mathbb{R}^3}(\overline{R}^{NS-Kuz} - \eta'(\overline{\mathbf{U}}_\varepsilon)\mathbf{R}^{NS-Kuz})\mathrm{d}x - \varepsilon^3\int_{\mathbb{R}^3}[\mathbf{R}^{NS-Kuz}]^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)\mathrm{d}x.
\end{aligned} \tag{3.40}$$

Now we study lower bounds of the left-hand side and upper bounds of the right-hand side of (3.40) in order to obtain a suitable estimate. For the right-hand side of Equation (3.40) we notice that

$$\begin{aligned}
&-\varepsilon\nu\int_{\mathbb{R}^3}\sum_{i=1}^{3}(\partial_{x_i}\mathbf{v}_\varepsilon\partial_{x_i}\mathbf{v}_\varepsilon - \partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}\overline{\mathbf{v}}_\varepsilon)\mathrm{d}x + 2\varepsilon\nu\int_{\mathbb{R}^3}\sum_{i=1}^{3}\partial_{x_i}\overline{\mathbf{v}}_\varepsilon\partial_{x_i}(\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon)\mathrm{d}x \\
&= -\varepsilon\nu\int_{\mathbb{R}^3}\sum_{i=1}^{3}(\partial_{x_i}(\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon))^2\mathrm{d}x \leq 0,
\end{aligned}$$

hence this term can be passed in the left-hand side of Equation (3.40) and omitted in the estimation. As the entropy is convex it holds

$$\exists\delta_0 > 0: \quad \eta(\mathbf{U}_\varepsilon) - \eta(\overline{\mathbf{U}}_\varepsilon) - \eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) \geq \delta_0|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2.$$

Then using also its continuity, we find

$$\begin{aligned}
\delta_0\int_{\mathbb{R}^3}|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(t)\mathrm{d}x \leq \int_0^t \frac{d}{ds}\left(\int_{\mathbb{R}^3}\eta(\mathbf{U}_\varepsilon) - \eta(\overline{\mathbf{U}}_\varepsilon) - \eta'(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)\mathrm{d}x\right)ds \\
+ C_0\int_{\mathbb{R}^3}|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2(0)\mathrm{d}x.
\end{aligned}$$

On the right-hand side of (3.40), by the Taylor expansion we also have

$$\mathbf{G}_i(\mathbf{U}_\varepsilon) - \mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon) - D\mathbf{G}_i(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon) \leq C|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon|^2.$$

With the boundedness on $[0;+\infty[$ of $R_1(t)$ and $\mathbf{R}_2(t)$ in the L^2 and L^∞ norms, and thanks to the regularity of $\overline{\mathbf{U}}_\varepsilon$ defined in (3.35) and (3.36) (see also (3.22) and (3.23) for the case $\mathbf{U}_\varepsilon(0)=\overline{\mathbf{U}}_\varepsilon(0)$) and the energy boundedness of $\mathbf{U}_\varepsilon$, we estimate the other terms in Equation (3.40) in the following way

$$\begin{aligned}
&\varepsilon\nu\int_{\mathbb{R}^3}\Delta\overline{\mathbf{v}}_\varepsilon\frac{\rho_\varepsilon-\overline{\rho}_\varepsilon}{\overline{\rho}_\varepsilon}(\mathbf{v}_\varepsilon-\overline{\mathbf{v}}_\varepsilon)\mathrm{d}x\le K\varepsilon\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|^2_{L^2(\mathbb{R}^3)},\\
&-\varepsilon^3\int_{\mathbb{R}^3}(\overline{R}^{NS-Kuz}-\eta'(\overline{\mathbf{U}}_\varepsilon)\mathbf{R}^{NS-Kuz})\mathrm{d}x\le K\varepsilon^3,\\
&-\varepsilon^3\int_{\mathbb{R}^3}[\mathbf{R}^{NS-Kuz}]^t\eta''(\overline{\mathbf{U}}_\varepsilon)(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)\mathrm{d}x\\
\le&\varepsilon^3\|\eta''(\overline{\mathbf{U}}_\varepsilon)\|_{L^\infty(\mathbb{R}^3)}\|\mathbf{R}^{NS-Kuz}\|_{L^2(\mathbb{R}^3)}\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}\\
\le&K\varepsilon^3\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Now, by integrating on $[0,t]$, we obtain from (3.40) the following inequality

$$\begin{aligned}
\int_{\mathbb{R}^3}|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon|^2(t)\mathrm{d}x\le\int_0^t\Big[&(C\|\nabla\overline{\mathbf{U}}_\varepsilon\|_{L^\infty}+K\varepsilon)\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|^2_{L^2(\mathbb{R}^3)}\\
&+K\varepsilon^3+K\varepsilon^3\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}\Big]ds+C_1\int_{\mathbb{R}^3}|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon|^2(0)\mathrm{d}x.
\end{aligned}$$

Here K, C and C_1 are generic constants of order $O(\varepsilon^0)$ which do not depend on time. Using once more the regularity properties (3.22) and (3.23), we have the boundedness of $\|\nabla\overline{\mathbf{U}}_\varepsilon\|_{L^\infty}$. But knowing that $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ are defined by *ansatz* (3.2)–(3.3), we deduce that $\|\nabla\overline{\mathbf{U}}_\varepsilon\|_{L^\infty}\le C\varepsilon$. Therefore,

$$\begin{aligned}
\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|^2_{L^2}\le&\int_0^t K\Big(\varepsilon\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|^2_{L^2(\mathbb{R}^3)}+\varepsilon^3+\varepsilon^3\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}\Big)ds\\
&+C_1\int_{\mathbb{R}^3}|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon|^2(0)\mathrm{d}x.
\end{aligned}$$

Then applying the Grönwall lemma we have directly

$$\|(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(\mathbb{R}^3)}\le K(\varepsilon^3t+\delta^2)e^{K\varepsilon t},$$

since $K\varepsilon t$ is a non-decreasing function in time and $\varepsilon^3\sqrt{v}<K\varepsilon v$ for all $v\in\mathbb{R}^+$. In addition, to find the estimate of point 2 for the regular case $\mathbf{U}_\varepsilon(0)=\overline{\mathbf{U}}_\varepsilon(0)$, we notice that

$$\|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)}\le v,$$

where v is the solution of the following Cauchy problem

$$\begin{cases}(v^2)'=K(\varepsilon^3+\varepsilon^3v+\varepsilon v^2),\\ v(0)=0.\end{cases}$$

The study of this problem gives us

$$\frac{1}{K\varepsilon}\ln\left(1+v(t)+\frac{1}{\varepsilon^2}v(t)^2\right)$$

$$-\frac{1}{K}\frac{2}{\sqrt{4-\varepsilon^2}}\left[\arctan\left(\frac{2}{\sqrt{4\varepsilon^2-\varepsilon^4}}\left[v(t)+\frac{\varepsilon^2}{2}\right]\right)-\arctan\left(\frac{\varepsilon}{\sqrt{4-\varepsilon^2}}\right)\right]=t.$$

The boundedness of the function $\arctan x$ implies

$$1+v(t)+\frac{1}{\varepsilon^2}v(t)^2 \le e^{\frac{2\varepsilon}{\sqrt{4-\varepsilon^2}}} e^{\arctan\left[\frac{2}{\sqrt{4\varepsilon^2-\varepsilon^4}}\left(v(t)+\frac{\varepsilon^2}{2}\right)\right]-\arctan\left(\frac{\varepsilon}{\sqrt{4-\varepsilon^2}}\right)} e^{K\varepsilon t}$$
$$\le e^{\frac{2\varepsilon}{\sqrt{4-\varepsilon^2}}} e^{\frac{\pi}{2}} e^{K\varepsilon t} \le c_0^2 e^{K\varepsilon t}$$

with $c_0^2 = e^{\frac{2}{\sqrt{3}}} e^{\frac{\pi}{2}}$ which for instance is less than 3.5. Therefore, the estimate

$$\|\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon\|_{L^2(\mathbb{R}^3)} \le c_0 \varepsilon e^{K\varepsilon t}$$

gives the result as soon as $c_0\varepsilon e^{\varepsilon K t} \le 2\varepsilon^b$, with $b \le 1$, *i.e.* for $t \le \frac{C}{\varepsilon}$ when $b=1$, and for $t \le \frac{C}{\varepsilon}\ln(\frac{1}{\varepsilon})$ in the case $b<1$.

We finish the proof with the remark on the minimal regularity of the initial data for the Kuznetsov equation such that the approximation is possible, *i.e.* the remainder terms R_1^{NS-Kuz} and $\mathbf{R}_2^{NS-Kuz}$ keep bounded for a finite time interval. Indeed, if $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ with $s > \frac{n}{2}$ then $u \in C([0,+\infty[;H^{s+2}(\mathbb{R}^n))$ and

$$u_t \in C([0,+\infty[;H^{s+1}(\mathbb{R}^n)), \quad u_{tt} \in C([0,+\infty[;H^{s-1}(\mathbb{R}^n)).$$

Since $\overline{\rho}_\varepsilon$ is defined by (3.2) with (3.5) and (3.6) and $\overline{\mathbf{v}}_\varepsilon$ by (3.3) respectively, we exactly find regularity (3.35) and (3.36). Thus by the regularity of the left-hand side part for the approximate Navier-Stokes system (3.17)–(3.18) we obtain the desired regularity for the right-hand side. ∎

4. Navier-Stokes system and the KZK equation

4.1. Derivation of the KZK equation from an isentropic Navier-Stokes system. In the present section we focus on the derivation from the isentropic Navier-Stokes system of the Khoklov-Zabolotskaya-Kuznetsov equation (KZK) in non-linear media using the following acoustical properties of beam's propagation:

(1) the beams are concentrated near the x_1-axis;
(2) the beams propagate along the x_1-direction;
(3) the beams are generated either by an initial condition or by a forcing term on the boundary $x_1=0$.

The different type of derivations of the KZK equation are discussed in Ref. [35].

This time we perform it in two steps:

(1) We introduce small perturbations around a constant state of the compressible isentropic Navier-Stokes system according to the Kuznetsov *ansatz* (3.2)–(3.3):

$$\begin{aligned}\partial_t\rho_\varepsilon+\nabla.(\rho_\varepsilon\mathbf{v}_\varepsilon)=&\varepsilon[\partial_t\rho_1-\rho_0\Delta u]\\ &+\varepsilon^2[\partial_t\rho_2-\nabla\rho_1\nabla u-\rho_1\Delta u]+O(\varepsilon^3),\end{aligned} \tag{4.1}$$

and we have again (3.4) for the conservation of momentum.

(2) We perform the paraxial change of variable [35] (see Figure 4.1):

$$\tau = t-\frac{x_1}{c},\ z=\varepsilon x_1,\ y=\sqrt{\varepsilon}x'. \tag{4.2}$$

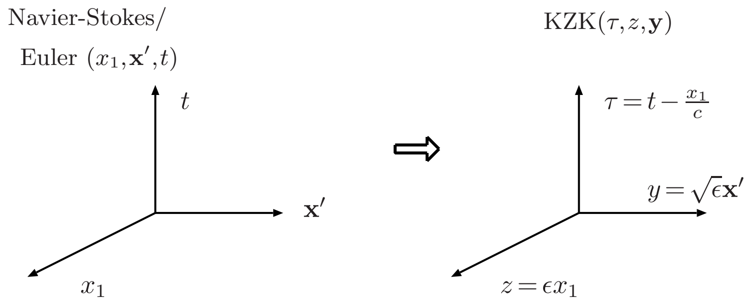

FIG. 4.1. *Paraxial change of variables for the profiles* $U(t-x_1/c,\epsilon x_1,\sqrt{\epsilon}\mathbf{x}')$.

Since the gradient ∇ in the coordinates (τ,z,y) becomes dependent on ε

$$\tilde{\nabla}=\left(\varepsilon\partial_z-\frac{1}{c}\partial_\tau,\sqrt{\varepsilon}\nabla_y\right)^t,$$

if we denote

$$u(x,t)=\Phi(t-x_1/c,\epsilon x_1,\sqrt{\epsilon}x')=\Phi(\tau,z,y), \tag{4.3}$$

we need to take attention to have the paraxial correctors of the order $O(1)$:

$$\rho_1(x,t)=I(\tau,z,y),\quad \rho_2(x,t)=H(\tau,z,y)=J(\tau,z,y)+O(\varepsilon),$$

where actually $H(\tau,z,y)$ is the profile function obtained from ρ_2 (see Equation (A.1) in the appendix) containing not only the terms of the order $O(1)$ but also terms up to ε^2. Hence, we denote by J all terms of H of order 0 on ε, which are significant in order to have an approximation up to the terms $O(\varepsilon^3)$.

In new variables (τ,z,y) Equation (3.4) becomes

$$\begin{aligned}&\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon.\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon\\&=\varepsilon\tilde{\nabla}[-\rho_0\partial_\tau\Phi+c^2I]+\varepsilon^2\left[-I\tilde{\nabla}(\partial_\tau\Phi)+\frac{\rho_0}{2}\tilde{\nabla}\left(\frac{1}{c^2}(\partial_\tau\Phi)^2\right)\right.\\&\qquad\left.+c^2\tilde{\nabla}J+\frac{\gamma-1}{2\rho_0}c^2\tilde{\nabla}(I^2)+\nu\tilde{\nabla}\left(\frac{1}{c^2}\partial_\tau^2\Phi\right)\right]+O(\varepsilon^3).\end{aligned} \tag{4.4}$$

Consequently, we find the correctors of the density as functions of Φ:

$$I(\tau,z,y)=\frac{\rho_0}{c^2}\partial_\tau\Phi(\tau,z,\mathbf{y}), \tag{4.5}$$

$$J(\tau,z,y)=-\frac{\rho_0(\gamma-1)}{2c^4}(\partial_\tau\Phi)^2-\frac{\nu}{c^4}\partial_\tau^2\Phi. \tag{4.6}$$

Indeed, we start by making

$$\varepsilon\tilde{\nabla}[-\rho_0\partial_\tau\Phi+c^2I]=0$$

and find the first-order perturbation of the density I given by Equation (4.5). Moreover, if ρ_1 satisfies (4.5), then Equation (4.4) becomes

$$\rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon=\varepsilon\tilde{\nabla}[-\rho_0\partial_\tau\Phi+c^2I]$$
$$\varepsilon^2\tilde{\nabla}\left[-\frac{\rho_0}{2c^2}(\partial_\tau\Phi)^2+\frac{\rho_0}{2c^2}(\partial_\tau\Phi)^2+c^2J+\frac{(\gamma-1)\rho_0}{2c^2}(\partial_\tau\Phi)^2+\frac{\nu}{c^2}\partial_\tau^2\Phi\right]+O(\varepsilon^3). \quad (4.7)$$

Thus, taking the corrector J in the expansion of ρ_ε

$$\rho_\varepsilon(\mathbf{x},t)=\rho_0+\varepsilon I(\tau,z,\mathbf{y})+\varepsilon^2 J(\tau,z,\mathbf{y}), \quad (4.8)$$

by formula (4.6), we ensure that

$$\rho_\varepsilon[\partial_t \mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon=O(\varepsilon^3). \quad (4.9)$$

Now we put these expressions of I from (4.5) and J from (4.6) with the paraxial approximation in Equation (4.1) of the mass conservation to obtain

$$\partial_t\rho_\varepsilon+\nabla.(\rho_\varepsilon\mathbf{v}_\varepsilon)=\varepsilon^2\left[\frac{\rho_0}{c^2}(2c\partial^2_{z\tau}\Phi-c^2\Delta_y\Phi)-\frac{\rho_0}{2c^4}(\gamma+1)\partial_\tau[(\partial_\tau\Phi)^2]-\frac{\nu}{c^4}\partial_\tau^3\Phi\right]+O(\varepsilon^3). \quad (4.10)$$

All terms of the second order on ε in relation (4.10) give us the equation on Φ, which is the KZK equation. If we use relation (4.5), we obtain the usual form of the KZK equation often written (see [4, 35]) for the first perturbation I of the density ρ_ϵ:

$$c\partial^2_{\tau z}I-\frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2-\frac{\nu}{2c^2\rho_0}\partial^3_\tau I-\frac{c^2}{2}\Delta_y I=0. \quad (4.11)$$

We notice that, as the Kuznetsov equation, this model still contains terms describing the wave propagation $\partial^2_{\tau z}I$, the non-linearity $\partial^2_\tau I^2$ and the viscosity effects $\partial^3_\tau I$ of the medium but also adds a diffraction effect by the transversal Laplacian $\Delta_y I$. This corresponds to the description of the quasi-one-dimensional propagation of a signal in a homogeneous nonlinear isentropic medium. By our derivation (see also Equations (4.33)–(4.34)) we obtain that the KZK equation is the second-order approximation of the isentropic Navier-Stokes system up to terms of $O(\varepsilon^3)$. In this sense, since the entropy and the pressure in Section 2 are approximated up to terms of the order of ε^3, *ansatz* (4.8)-(4.16) (for the KZK equation) is optimal, as the equations of the Navier-Stokes system are approximated up to $O(\varepsilon^3)$-terms.

Let us compare our *ansatz*

$$u(x_1,\mathbf{x}',t)=\Phi(t-x_1/c,\epsilon x_1,\sqrt{\epsilon}x'), \quad (4.12)$$
$$\rho_\varepsilon(x_1,\mathbf{x}',t)=\rho_0+\varepsilon I(t-x_1/c,\epsilon x_1,\sqrt{\epsilon}x')+\varepsilon^2 J(t-x_1/c,\epsilon x_1,\sqrt{\epsilon}x') \quad (4.13)$$

to the *ansatz* introduced in Ref. [35] by defining a corrector $\epsilon^2 v_2$ for the velocity perturbation along the propagation axis in the initial *ansatz*, proposed by Khokhlov and Zabolotskaya [4]:

$$\rho_\epsilon(x_1,\mathbf{x}',t)=\rho_0+\epsilon I(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}\mathbf{x}'), \quad (4.14)$$
$$\mathbf{v}_\epsilon(x_1,\mathbf{x}',t)=\epsilon(v_1+\epsilon v_2;\sqrt{\epsilon}\mathbf{w})(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}\mathbf{x}'). \quad (4.15)$$

This time, the assumption to work directly with the velocity potential (4.12) immediately implies the following velocity expansion

$$\mathbf{v}_\varepsilon(\mathbf{x},t)=-\varepsilon\left(-\frac{1}{c}\partial_\tau\Phi+\varepsilon\partial_z\Phi;\sqrt{\varepsilon}\nabla_y\Phi\right)(\tau,z,\mathbf{y}), \quad (4.16)$$

where we recognize the velocity *ansatz* of Ref. [35] with

$$v_1 = \frac{1}{c}\partial_\tau \Phi = \frac{c}{\rho_0} I, \quad \mathbf{w} = \nabla_y \Phi = \frac{c^2}{\rho_0}\partial_\tau^{-1}\nabla_y I,$$

but for the corrector v_2 this time

$$v_2 = -\partial_z \Phi = -\frac{c^2}{\rho_0}\partial_\tau^{-1}\partial_z I$$

instead of (see Ref. [35] and formula (4.19) for definition of the operator ∂_τ^{-1})

$$v_2^{Rozanova} = -\frac{c^2}{\rho_0}\partial_\tau^{-1}\partial_z I + \frac{(\gamma-1)}{2\rho_0^2} c I^2 + \frac{\nu}{c\rho_0^2}\partial_\tau I.$$

If we add the second-order correctors v_2 for the velocity to J for the density, we obtain exactly all terms of the corrector $v_2^{Rozanova}$. But the *ansatz* (4.14)–(4.15) is not optimal since the equation of momentum in transverse direction keeps the non-zero terms [35] of the order of $\epsilon^{\frac{5}{2}}$.

4.2. Well posedness of the KZK equation. We use Ref. [34] to give results on the well posedness of the Cauchy problem:

$$\begin{cases} c\partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}\Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ I(\tau,0,y) = I_0(\tau,y) \text{ on } \mathbb{T}_\tau \times \mathbb{R}^{n-1} \end{cases} \tag{4.17}$$

in the class of $L-$periodic functions with respect to the variable τ and with mean value zero

$$\int_0^L I(\tau,z,y)d\tau = 0. \tag{4.18}$$

The introduction of the operator ∂_τ^{-1}, defined by formula

$$\partial_\tau^{-1} I(\tau,z,y) := \int_0^\tau I(s,z,y)ds + \int_0^L \frac{s}{L} I(s,z,y)ds, \tag{4.19}$$

allows us to consider instead of Equation (4.11) the following equivalent equation

$$c\partial_z I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^2 I - \frac{c^2}{2}\partial_\tau^{-1}\Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \tag{4.20}$$

for which, in the viscous case $\nu > 0$, it holds a global in z well-posedness result [34] for sufficiently small by H^s norm ($s > \left[\frac{n}{2}\right] + 1$) initial data.

As it was mentioned in [23, 24, 33] for the KP-type equations in $\mathbb{R}^2$, the introduced operator ∂_τ^{-1} is singular in the sense that its Fourier transform gives a division [34] by a discrete variable m:

$$\mathcal{F}(\partial_\tau^{-1}\Delta_y I) = \frac{L\xi^2}{i2\pi m}\mathcal{F}(I)(m,\xi) \quad m \in \mathbb{Z}, \xi \in \mathbb{R}.$$

If we suppose that I has the mean value zero in τ, it implies that $\mathcal{F}(I)(0,\xi) = 0$ for all ξ, which makes disappear the singularity for $m = 0$. For the same reason this requires [34, Lemma 5.2] the additional constraint for the initial data $\partial_\tau^{-1}\triangle_y I_0 = \phi_0 \in H^{s-2}$ to be able to ensure that the solution $I\in C([0,T[,H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1}))$ can be also considered in $C^1([0,T[,H^{s-2}(\mathbb{T}_\tau\times\mathbb{R}^{n-1}))$ (see also a similar situation for the KP-type equations explained in [33]). At the same time, as it is discussed in [23, 24, 33], in the non-periodic case this regularity constraint is not physical. However, if we work in the class of periodic functions with the mean value zero this condition can be omitted.

Indeed, by definition (4.19) of the operator ∂_τ^{-1}, it preserves the property of a periodic function to have the mean value zero. Thus, if I_0 is a periodic function with the mean value zero on τ, the solution I belongs also in this class, where we find the equivalence between the Cauchy problem (4.17) and the analogous problem considered for Equation (4.20). Formula (4.19), as it is noticed in [34, p.796], allows to establish an analogue of the Poincaré inequality (which is false in the non-periodic case of $\mathbb{R}^n$):

$$\|I\|_{H^s(]0,L[\times\mathbb{R}^{n-1}_y)}\leq C\|\partial_\tau I\|_{H^s(]0,L[\times\mathbb{R}^{n-1}_y)},$$

coming from the following relation

$$I=\partial_\tau^{-1}\partial_\tau I=\int_0^\tau \partial_\tau I(s,y)ds+\int_0^L\frac{s}{L}\partial_\tau I(s,y)ds.$$

As, by (4.19), $\partial_\tau^{-1}I$ is L-periodic in τ and of mean value zero, this also gives us the following estimate

$$\|\partial_\tau^{-1}I\|_{H^s(\Omega_1)}\leq C\|\partial_\tau\partial_\tau^{-1}I\|_{H^s(\Omega_1)}=C\|I\|_{H^s(\Omega_1)}. \tag{4.21}$$

This means that in the class of periodic and of mean value zero functions as soon as $I_0\in H^s(\Omega_1)$, it implies that $\partial_\tau^{-1}I_0$ is also in $H^s(\Omega_1)$ and in the same class. Hence the condition $\partial_\tau^{-1}\Delta_y I_0\in H^{s-2}(\Omega_1)$ required in [34, Thm. 1.2, Point 4] is automatically verified for I_0 from H^s, periodic and of mean value zero in t ($\tau=t$ for $z=0$).

To be able to ensure the boundedness of the remainder terms in the KZK-type approximations we need to have very regular solutions of (4.17) corresponding to the propagation variable z, which exist according to the following theorem [34]:

THEOREM 4.1 ([34]). *Let $\nu\geq 0$, $s>\left[\frac{n}{2}\right]+1$, the operator ∂_τ^{-1} defined by formula (4.19) and $I_0\in H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1})$ be such that $\int_0^L I_0(\ell,y)d\ell=0$. Then the following results hold true for the Cauchy problem for the KZK equation*

$$\begin{cases} c\partial_z I-\frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2-\frac{\nu}{2c^2\rho_0}\partial_\tau^2 I-\frac{c^2}{2}\partial_\tau^{-1}\Delta_y I=0 \text{ on } \mathbb{T}_\tau\times\mathbb{R}_+\times\mathbb{R}^{n-1},\\ I(\tau,0,y)=I_0(\tau,y) \text{ on } \mathbb{T}_\tau\times\mathbb{R}^{n-1}.\end{cases} \tag{4.22}$$

(1) (Local existence.) There exists a constant $C(s,L)$ such that for any (previously defined) initial data I_0 on an interval $[0,T[$ with

$$T\geq\frac{1}{C(s,L)\|I_0\|_{H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1})}}$$

problem (4.22) has a unique solution I such that

$$I\in C([0,T[,H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1}))\cap C^1([0,T[,H^{s-2}(\mathbb{T}_\tau\times\mathbb{R}^{n-1})),$$

which satisfies the zero mean value condition (4.18).

(2) (Shock formation.) Let T^* be the largest time on which such a solution is defined, then we have

$$\int_0^{T^*} \sup_{\tau,y}(|\partial_\tau I(\tau,t,y)|+|\nabla_y I(\tau,t,y)|)\,dt=+\infty.$$

(3) (Global existence.) If $\nu>0$ we have the global existence for small enough data: there exists a constant $C_1>0$ such that

$$\|I_0\|_{H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1})}\leq C_1\Rightarrow T^*=+\infty.$$

(4) (Exponential decay.) [15, 34] If $\nu>0$, $s\in\mathbb{N}$ and $s\geq\left[\frac{n+1}{2}\right]$, then there exists a constant $C_2>0$ such that $\|I_0\|_{H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1})}\leq C_2$ implies for all $z\geq 0$

$$\|I(z)\|_{H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1})}\leq C\|I_0\|_{H^s(\mathbb{T}_\tau\times\mathbb{R}^{n-1})}e^{-\hat{c}z},$$

where $C>0$ and $\hat{c}\in]0,1[$ are constants.

Remark 4.1. We note that when $\nu=0$, all the corresponding statements of Theorem 4.1 remain valid for $0>t>-C$ with a suitable C [34].

Remark 4.2. In the study of the well-posedness of the KZK equation we invert the usual role of the time with the main space variable along the propagation axis z: for $\nu>0$ the solution $I(\tau,z,y)=I(t-\frac{x_1}{c},\varepsilon x_1,\sqrt{\varepsilon}x')$ is defined for $x_1>0$, as it is global on $z\in\mathbb{R}^+$. Hence if we want to compare the KZK equation to other models such as the Kuznetsov equation or the Navier-Stokes system we need the well posedness results for these models on the half-space

$$\{x_1>0,\quad t>0,\quad x'\in\mathbb{R}^{n-1}\},\tag{4.23}$$

taking into account the fact that the boundary conditions for the exact system come from the initial condition I_0 of the Cauchy problem (4.22) associated to the KZK equation.

4.3. Well posedness of the isentropic Navier-Stokes system on the half-space with inflow-outflow periodic boundary conditions. We follow now Section 5.2 in Ref. [35] updating it for the new *ansatz* and correct the proof of Theorem 5.5. See Ref. [35] for more details.

We consider the Cauchy problem for the KZK Equation (4.22) for the initial data

$$I(t,0,y)=I_0(t,y)\quad(\tau=t\ \text{ for } x_1=0),$$

which are L-periodic in time and of mean value zero. For $s>[\frac{n}{2}]+1$, Theorem 4.1 ensures that for all initial data I_0, defined in $\mathbb{T}_t\times\mathbb{R}^{n-1}$ with small enough H^s norm (with respect to ν), there exists a unique solution I of the KZK Equation (4.11), which as a function of (τ,z,y) is global on $z\in\mathbb{R}^+$, periodic in τ of period L and mean value zero, and decays for $z\to\infty$ [34].

Therefore, see Remark 4.2, we consider our approximation problem between the isentropic Navier-Stokes system (2.6)–(2.7) and the KZK equation in the half-space (4.23).

By I_0 we find I and thus also Φ and J, using Equations (4.5)–(4.6). This allows us to construct the density and velocities $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ in accordance with *ansatz* (4.8) and (4.16). Thus, by I we construct the function $\overline{\mathbf{U}}_\varepsilon=(\overline{\rho}_\varepsilon,\overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)^t$.

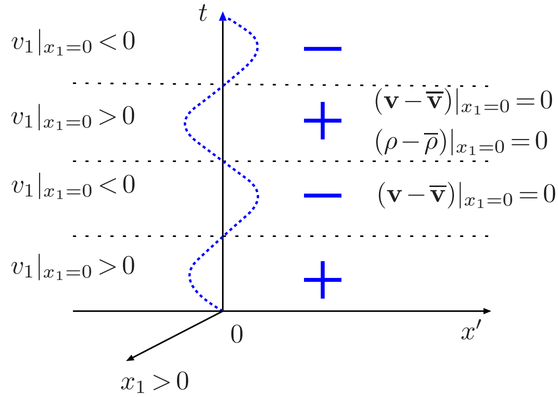

FIG. 4.2. *Periodic subsonic inflow-outflow boundary conditions for the Navier-Stokes system.*

In particular, for $t=0$ we have functions defined for $x_1>0$ because I is well-defined for any $z>0$

$$\overline{\rho}_\varepsilon(0,x_1,x')=\rho_0+\varepsilon I(-\frac{x_1}{c},\varepsilon x_1,\sqrt{\varepsilon}x')+\varepsilon^2 J(-\frac{x_1}{c},\varepsilon x_1,\sqrt{\varepsilon}x'),$$
$$\overline{\mathbf{v}}_\varepsilon(0,x_1,x')=(\overline{v}_1,\overline{\mathbf{v}}'_\varepsilon)(-\frac{x_1}{c},\varepsilon x_1,\sqrt{\varepsilon}x'),$$

where

$$\overline{v}_1=\varepsilon\frac{c}{\rho_0}I+\varepsilon^2\frac{c^2}{\rho_0}\partial_z\partial_\tau^{-1}I,\quad \overline{\mathbf{v}}'_\varepsilon=\sqrt{\varepsilon}\frac{c^2}{\rho_0}\nabla_y\partial_\tau^{-1}I$$

and for $x_1=0$ we have L-periodic functions with mean value zero

$$\overline{\rho}_\varepsilon(t,0,x')=\rho_0+\varepsilon I(t,0,\sqrt{\varepsilon}x')+\varepsilon^2 J(t,0,\sqrt{\varepsilon}x'),\tag{4.24}$$
$$\overline{\mathbf{v}}_\varepsilon(t,0,x')=(\overline{v}_1,\overline{\mathbf{v}}'_\varepsilon)(t,0,\sqrt{\varepsilon}x').\tag{4.25}$$

It is important to notice [34] that the solution $\overline{\mathbf{v}}_\varepsilon$ in system (2.6)–(2.7) is small on the boundary: $\overline{\mathbf{v}}_\varepsilon|_{x_1=0}=\varepsilon\tilde{\mathbf{v}}_\varepsilon|_{x_1=0}$. Therefore, we have $|\overline{\mathbf{v}}_\varepsilon|_{x_1=0}|<c$, which corresponds to the "subsonic" boundary case. More precisely, when the first velocity component is positive $\overline{v}_1|_{x_1=0}>0$, we have a subsonic inflow boundary condition, and when it is negative $\overline{v}_1|_{x_1=0}<0$, we have a subsonic outflow boundary condition, see Figure 4.2.

We also notice that, due to Equation (4.16), the first component of the velocity $\overline{\mathbf{v}}_1$ on the boundary has the following form

$$\overline{v}_1|_{x_1=0}=\left(\varepsilon\frac{c}{\rho_0}I+\varepsilon^2 G(I)\right)(t,0,\sqrt{\varepsilon}x')=\left(\varepsilon\frac{c}{\rho_0}I+\varepsilon^2 G(I)\right)\Big|_{z=0}$$
$$=\varepsilon\frac{c}{\rho_0}I_0(t,y)+\varepsilon^2 G(I_0)(t,y),$$

where

$$G(I)=\frac{c^2}{\rho_0}\partial_z\partial_\tau^{-1}I=\frac{c^2}{\rho_0}\partial_\tau^{-1}\left(\frac{(\gamma+1)}{4c\rho_0}\partial_\tau I^2+\frac{\nu}{2c^3\rho_0}\partial_\tau^2 I+\frac{c}{2}\partial_\tau^{-1}\Delta_y I\right).\tag{4.26}$$

Therefore, the boundary conditions for $\overline{\mathbf{v}}_1$ are defined by the initial conditions for KZK equation and are L-periodic in t and have mean value zero. In addition, the sign of

$\overline{\mathbf{v}}_1|_{x_1=0}$ is the same as the sign of I_0 (because the term $G(I_0)$ is of a higher order of smallness on ε). In addition, as the viscosity term $\varepsilon\nu\overline{\mathbf{v}}_\varepsilon$, where ε is a fixed small enough parameter, ν is a constant, and in our case $\overline{\mathbf{v}}_\varepsilon$ is of the order of ε, the boundary layer phenomenon is excluded.

THEOREM 4.2. *Let $n \leq 3$. Suppose that the initial data of the KZK Cauchy problem $I_0(t,y) = I_0(t,\sqrt{\epsilon}x')$ is such that*

(1) I_0 is L-periodic in t and with mean value zero,

(2) for fixed t, I_0 has the same sign for all $y \in \mathbb{R}^{n-1}$, and for $t \in]0,L[$ the sign changes, i.e. $I_0 = 0$, only for a finite number of times,

(3) $I_0(t,y) \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $s \geq 10$,

(4) I_0 is sufficiently small in the sense of Theorem 4.1 such that [34, p.20]

$$\|I_0\|_{H^s} < \frac{\nu}{2c^2\rho_0}\frac{C_1(L)}{C_2(s)}.$$

Consequently, there exists a unique global solution in time $I(\tau,z,y)$ of (4.22) for $z = \epsilon x_1 > 0$, moreover, the functions $\bar{\rho}_\epsilon$, $\overline{\boldsymbol{v}}_\varepsilon = (\overline{v}_1, \overline{\boldsymbol{v}}'_\varepsilon)$, defined by ansatz (4.8)-(4.16) and Equations (4.5)–(4.6) in the half-space (4.23) are smooth with $\Omega = \mathbb{T}_t \times \mathbb{R}^{n-1}_y$:

$$\bar{\rho}_\epsilon \in C\left([0,\infty[, H^{s-4}(\Omega)\right) \cap C^1\left([0,\infty[; H^{s-6}(\Omega)\right), \tag{4.27}$$

$$\bar{\boldsymbol{v}}_\epsilon \in C\left([0,\infty[; H^{s-4}(\Omega)\right) \cap C^1\left([0,\infty[; H^{s-6}(\Omega)\right). \tag{4.28}$$

The Navier-Stokes system (2.6)–(2.7) in the half-space with initial data (3.24) and following boundary conditions

$$(\bar{\boldsymbol{v}}_\epsilon - \mathbf{v}_\epsilon)|_{x_1=0} = 0,$$

with positive first component of the velocity $v_1|_{x_1=0} > 0$ (i.e. at points where the fluid enters the domain) has the additional boundary condition

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.$$

When $v_1|_{x_1=0} \leq 0$ there isn't any boundary condition for ρ_ϵ.

Then, for all finite times $T > 0$ there exists a unique solution $U_\epsilon = (\rho_\epsilon, \rho_\varepsilon u_\epsilon)$ of the Navier-Stokes system (2.6)–(2.7) with the following smoothness on $[0,T]$

$$\rho_\varepsilon \in C\left([0,T], H^3\left(\{x_1>0\}\times\mathbb{R}^{n-1}\right)\right) \cap C^1\left([0,T], H^2\left(\{x_1>0\}\times\mathbb{R}^{n-1}\right)\right), \tag{4.29}$$

$$u_\varepsilon \in C\left([0,T], H^3\left(\{x_1>0\}\times\mathbb{R}^{n-1}\right)\right) \cap C^1\left([0,T], H^1\left(\{x_1>0\}\times\mathbb{R}^{n-1}\right)\right). \tag{4.30}$$

REMARK 4.3 ([35]). The restriction to have the same sign for I_0 for all fixed times avoids a change in the type of the boundary condition applied to the tangential variables for the Navier-Stokes system. Moreover, Zabolotskaya [4] takes as the initial conditions for the KZK equation (which correspond to the boundary condition for v_1) the expression

$$I(\tau,0,y) = -F(y)\sin\tau$$

with an amplitude distribution $F(y) \geq 0$. Especially, for a Gaussian beam [4]

$$F(y) = e^{-y^2},$$

while for a beam with a polynomial amplitude [4]

$$F(y)=\begin{cases}(1-y^2)^2, & y\le 1,\\ 0, & y>1.\end{cases}$$

Proof. As previously, we use the fact that the entropy for the isentropic Euler system $\eta(\mathbf{U}_\varepsilon)$, defined by Equation (3.31) is a convex function [8].

Let us multiply the Navier-Stokes system (3.19), from the left, by $2\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)$

$$2\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\partial_t\mathbf{U}_\varepsilon+\sum_{i=1}^n 2\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)\partial_{x_i}\mathbf{U}_\varepsilon-\varepsilon\nu 2\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\begin{bmatrix}0\\ \triangle\mathbf{v}_\varepsilon\end{bmatrix}=0.$$

We notice that

$$\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\begin{bmatrix}0\\ \triangle\mathbf{v}_\varepsilon\end{bmatrix}=0,$$

and, therefore, we have

$$2\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\partial_t\mathbf{U}_\varepsilon=\partial_t[\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon]-\mathbf{U}_\varepsilon^T\partial_t\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon.$$

Moreover, by virtue of $\eta''(U)DG_i(U)=(DG_i(U))^T\eta''(U)$ we find

$$2\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)\partial_{x_i}\mathbf{U}_\varepsilon=\\ \partial_{x_i}[\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon]-\mathbf{U}_\varepsilon^T\partial_{x_i}[\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)]\mathbf{U}_\varepsilon.$$

Integrating over $[0,t]\times\{x_1>0\}$ $(x'\in\mathbb{R}^{n-1})$, we obtain

$$\int_0^t\int_{x_1>0}\partial_t[\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon]dxds+\int_0^t\int_{x_1>0}\sum_{i=1}^n\partial_{x_i}[\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon]dxds$$

$$-\int_0^t\int_{x_1>0}\mathbf{U}_\varepsilon^T\partial_t\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon dxds-\int_0^t\int_{x_1>0}\sum_{i=1}^n\mathbf{U}_\varepsilon^T\partial_{x_i}[\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)]\mathbf{U}_\varepsilon dxds=0.$$

Integrating by parts results in

$$\int_{x_1>0}\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon dx-\int_{x_1>0}\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon|_{t=0}dx\\ -\int_0^t\int_{x_1>0}\mathbf{U}_\varepsilon^T\left[\partial_t\eta''(\mathbf{U}_\varepsilon)+\sum_{i=1}^n\partial_{x_i}[\eta''(\mathbf{U}_\varepsilon)DG_i(\mathbf{U}_\varepsilon)]\right]\mathbf{U}_\varepsilon dxds\\ -\int_0^t\int_{\mathbb{R}^{n-1}}\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)DG_1(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon|_{x_1=0}dx'ds=0.$$

We recall that $\eta''(\mathbf{U}_\varepsilon)$ is positive definite, consequently for some $C>0$ and $\delta_0>0$

$$C|\mathbf{U}_\varepsilon|^2\ge\mathbf{U}_\varepsilon^T\eta''(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon\ge\delta_0|\mathbf{U}_\varepsilon|^2.$$

Therefore, we obtain for the initial data

$$\mathbf{U}_0=\begin{bmatrix}\rho_0+\varepsilon I+\varepsilon^2 J\\ \varepsilon\left(\rho_0+\varepsilon I+\varepsilon^2 J\right)\left(\frac{c}{\rho_0}I+\varepsilon G(I),\sqrt{\varepsilon}\vec{w}\right)\end{bmatrix}\left(-\frac{x_1}{c},\varepsilon x_1,\sqrt{\varepsilon}x'\right) \tag{4.31}$$

and the relation

$$\delta_0 \int_{x_1>0} \mathbf{U}_\varepsilon^2 dx - C\int_{x_1>0} \mathbf{U}_0^2 dx - \int_0^t \int_{\mathbb{R}^{n-1}} \mathbf{U}_\varepsilon^T \eta''(\mathbf{U}_\varepsilon) DG_1(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon|_{x_1=0} dx'ds$$
$$\leq C_1 \int_0^t \int_{x_1>0} \mathbf{U}_\varepsilon^2 dx\, ds.$$

As in Ref. [11], C_1 is an upper bound for the eigenvalues of the symmetric matrix

$$\partial_t \eta''(\mathbf{U}_\varepsilon) + \sum_{i=1}^n \partial_{x_i}[\eta''(\mathbf{U}_\varepsilon) DG_i(\mathbf{U}_\varepsilon)].$$

Let us now consider the integral on the boundary. With notation $\mathbf{v}_\varepsilon = (v_1, \mathbf{v}'_\varepsilon)^t$ for the velocity and $H''(\rho) = \frac{p'(\rho)}{\rho}$, we see with $DG_1(\mathbf{U}_\varepsilon)$ coming from Equation (3.21) that

$$\begin{aligned}
&\mathbf{U}_\varepsilon^T \eta''(\mathbf{U}_\varepsilon) DG_1(\mathbf{U}_\varepsilon)\mathbf{U}_\varepsilon \\
&= (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)^T \begin{pmatrix} H''(\rho_\varepsilon) + \frac{\mathbf{v}_\varepsilon^2}{\rho_\varepsilon} & -\frac{\mathbf{v}_\varepsilon}{\rho_\varepsilon} \\ -\frac{\mathbf{v}_\varepsilon}{\rho_\varepsilon} & \frac{1}{\rho_\varepsilon} Id_n \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -v_1^2 + p'(\rho_\varepsilon) & 2v_1 & 0 \\ -v_1 \mathbf{v}'_\varepsilon & \mathbf{v}'_\varepsilon & v_1 Id_{n-1} \end{pmatrix} \begin{pmatrix} \rho_\varepsilon \\ \rho_\varepsilon \mathbf{v}_\varepsilon \end{pmatrix} \\
&= (\rho_\varepsilon, \rho_\varepsilon v_1, \rho_\varepsilon \mathbf{v}'_\varepsilon)^T \begin{pmatrix} v_1\left(\frac{\mathbf{v}_\varepsilon^2}{\rho_\varepsilon} - \frac{p'(\rho_\varepsilon)}{\rho_\varepsilon}\right) & \frac{-v_1^2}{\rho_\varepsilon} + \frac{p'(\rho_\varepsilon)}{\rho_\varepsilon} & -v_1 \frac{\mathbf{v}'_\varepsilon}{\rho_\varepsilon} \\ \frac{-v_1^2}{\rho_\varepsilon} + \frac{p'(\rho_\varepsilon)}{\rho_\varepsilon} & \frac{v_1}{\rho_\varepsilon} & 0 \\ -v_1 \frac{\mathbf{v}'_\varepsilon}{\rho_\varepsilon} & 0 & \frac{v_1}{\rho_\varepsilon} Id_{n-1} \end{pmatrix} \begin{pmatrix} \rho_\varepsilon \\ \rho_\varepsilon v_1 \\ \rho_\varepsilon \mathbf{v}'_\varepsilon \end{pmatrix} \\
&= \rho_\varepsilon p'(\rho_\varepsilon) v_1.
\end{aligned}$$

Let us consider the initial condition $I_0(t,y)$ for the KZK equation of the type described in Remark 4.3. We suppose (without loss of generality) that $I_0 = 0$ for $t \in]0, L[$ only once. More precisely, we suppose that the sign of v_1 is changing in the following way:

- $v_1 \leq 0$ for $t \in [0 + (k-1)L, \frac{L}{2} + (k-1)L]$ $(k = 1, 2, 3, \ldots)$,
- $v_1 > 0$ for $t \in]\frac{L}{2} + (k-1)L, kL[$ $(k = 1, 2, 3, \ldots)$.

If $t \in [0, \frac{L}{2}]$ (for $k = 1$), the first component of the velocity $v_1|_{x_1=0} < 0$ is negative, and thus we have

$$\rho_\varepsilon p'(\rho_\varepsilon) v_1 < 0.$$

If $t \in]\frac{L}{2}, L[$, the first component of velocity is positive $v_1|_{x_1=0} > 0$, then we also impose $\rho_\varepsilon|_{x_1=0} = \rho_0 + \varepsilon I_0(t,y) + \varepsilon^2 J$, where $I_0(t,y)$ is the initial condition for the KZK equation and J coming from Equation (4.6). For the term

$$\rho_\varepsilon p'(\rho_\varepsilon) v_1 > 0$$

we see that on the boundary it has the form

$$\begin{aligned}
\rho_\varepsilon p'(\rho_\varepsilon) v_1 &= \varepsilon \left(\frac{c}{\rho_0} I_0 + \frac{c^2}{\rho_0} \partial_z \partial_\tau^{-1} I_0\right)(\rho_0 + \varepsilon I_0(t,y) + \varepsilon^2 J) p'(\rho_0 + \varepsilon I_0(t,y) + \varepsilon^2 J) \\
&\leq C_0 \varepsilon I_0
\end{aligned}$$

for some constant $C_0>0$ independent of ε. Consequently, for $k\geq 1$

$$\begin{aligned}\int_0^{kL}\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds &\leq \sum_{j=1}^{k}\int_{]\frac{L}{2}+(j-1)L,jL[}\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds\\ &\leq \sum_{j=1}^{k}\int_{]\frac{L}{2}+(j-1)L,jL[}\int_{\mathbb{R}^{n-1}}C_0\varepsilon I_0\leq Kk\varepsilon\|I_0\|_{H^s},\end{aligned}$$

where $K=O(1)$ is a positive constant independent of k.

However, for $t>0$ and $k\geq 1$ such that $t\in[(k-1)L,kL[$, it implies on one hand that if $t\in\left[(k-1)L,(k-1)L+\frac{L}{2}\right[$

$$\int_0^t\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds\leq\int_0^{(k-1)L}\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds,$$

and on the other hand, that if $t\in\left[(k-1)L+\frac{L}{2},kL\right[$

$$\int_0^t\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds\leq\int_0^{kL}\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds.$$

As a consequence, we obtain for all $t>0$

$$\int_0^t\int_{\mathbb{R}^{n-1}}\rho_\varepsilon p'(\rho_\varepsilon)v_1|_{x_1=0}dx'\,ds\leq K\left(\left[\frac{t}{L}\right]+1\right)\varepsilon\|I_0\|_{H^s}.$$

Therefore, we deduce the following estimate, as $\delta_0>0$

$$\int_{x_1>0}\mathbf{U}_\varepsilon^2dx\leq\frac{C}{\delta_0}\int_{x_1>0}\mathbf{U}_0^2dx+\varepsilon\frac{K}{\delta_0}\left(\left[\frac{t}{L}\right]+1\right)\|I_0\|_{H^s}+\frac{C_1}{\delta_0}\int_0^t\int_{x_1>0}\mathbf{U}_\varepsilon^2dx\,ds.$$

By the Grönwall lemma we find

$$\|\mathbf{U}_\varepsilon\|_{L^2}^2(t)\leq\frac{C}{\delta_0}\left(\|\mathbf{U}_0\|_{L^2}^2+\varepsilon\frac{K}{C}\left(\left[\frac{t}{L}\right]+1\right)\|I_0\|_{H^s}\right)e^{\frac{C_1}{\delta_0}t},$$

remaining bounded for all finite times.

Thus, for all $T<+\infty$ we obtain that

$$\mathbf{U}_\varepsilon\in L^\infty\left([0,T],L^2\left(\{x_1>0\}\times\mathbb{R}^{n-1}\right)\right).$$

If $I_0=0$ for $t\in]0,L[$ a finite number of times m, we obtain the same result.

Hence, by [11] we have proved that the chosen boundary conditions ensure the local well-posedness for the Navier-Stokes system in the half-space, which can be viewed as a symetrisable incompletely parabolic system. We apply now the theory of incompletely parabolic problems [11, p. 352] with the result of global well-posedness of the Navier-Stokes system in the half-space with the Dirichlet boundary conditions [28] for the velocity and with the initial data $\rho_\varepsilon(0)-\rho_0\in H^3(\{x_1>0\}\times\mathbb{R}^{n-1})$ and $\mathbf{v}_\varepsilon(0)\in H^3(\{x_1>0\}\times\mathbb{R}^{n-1})$ small enough. Hence, for sufficiently regular initial data $\mathbf{U}_0\in H^3(\{x_1>0\}\times\mathbb{R}^{n-1})$ ($n\leq 3$) for all finite time $T<\infty$, we obtain by the energy method that $\mathbf{U}_\varepsilon\in L^\infty([0,T],H^3(\{x_1>0\}\times\mathbb{R}^{n-1}))$.

To ensure that $\mathbf{U}_0$ defined in Equation (4.31) belongs to $H^3(\{x_1>0\}\times\mathbb{R}^{n-1})$, we need to take $I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1})$ such that

$$\overline{\rho}_\varepsilon\in C([0,+\infty[;H^3(\{x_1>0\}\times\mathbb{R}^{n-1}),\quad \overline{\mathbf{v}}_\varepsilon\in C([0,+\infty[;H^3(\{x_1>0\}\times\mathbb{R}^{n-1}).$$

By Theorem 4.1, $I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1})$ implies while $s-2\ell\geq 0$ that

$$I(\tau,z,y)\in C^\ell(\{x_1>0\};H^{s-2\ell}(\mathbb{T}_\tau\times\mathbb{R}^{n-1})),$$

but we can also say, thanks to point 4 of Theorem 4.1, that

$$\partial_z^\ell I(\tau,z,y)\in L^2(\{x_1>0\};H^{s-2\ell}(\mathbb{T}_\tau\times\mathbb{R}^{n-1})).$$

Considering the expressions of $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$

$$\overline{\rho}_\varepsilon=\rho_0+\varepsilon I-\frac{\varepsilon^2}{\rho_0}\left(\frac{\gamma-1}{2}I^2-\frac{\nu}{c^2}\partial_\tau I\right),\quad \overline{\mathbf{v}}_\varepsilon=\frac{c^2}{\rho_0}\left(\frac{\varepsilon}{c}I-\varepsilon^2\partial_\tau^{-1}\partial_z I;\varepsilon^{\frac{3}{2}}\partial_\tau^{-1}\nabla_y I\right),\quad (4.32)$$

the least regular term is $\partial_\tau^{-1}\partial_z I$. Thus we need to ensure

$$\partial_z I\in C([0,+\infty[;H^3(\{x_1>0\}\times\mathbb{R}^{n-1}),$$

which leads us to take $s\geq 10$ in order to have

$$\partial_z^\ell I(\tau,z,y)\in L^2(\{x_1>0\};H^{s-2\ell}(\mathbb{T}_\tau\times\mathbb{R}^{n-1}))$$

for $\ell\leq 4$ with $s-2\ell\geq 2$ as we want to have the continuity in time. This choice of the regularity for I_0 allows us to control the boundary terms appearing from the integration by parts in the energy method. Indeed, we can perform analogous computations as in Ref. [8, p. 103] to control the spatial derivative of $\mathbf{U}_\varepsilon$ of the order less than or equal to 3 and directly verify that all boundary terms are controlled by $\|I_0\|_{H^s}$, which is actually is a consequence of the well-posedness [28] in H^3.

Thus, for all finite times we obtain the existence of the unique solution of the Navier-Stokes system in the sense of (4.29) and (4.30). ∎

4.4. Approximation of the solutions of the isentropic Navier-Stokes system with the solutions of the KZK equation. Knowing from Subsection 4.1 that the KZK equation can be derived from the compressible isentropic Navier-Stokes system (2.6)–(2.7) using *ansatz* (4.12)–(4.13) with I and J given by (4.5) and (4.6) respectively, we obtain the following expansion of the Navier-Stokes equations

$$\partial_t\rho_\varepsilon+\nabla.(\rho_\varepsilon\mathbf{v}_\varepsilon)=\varepsilon^2[\frac{\rho_0}{c^2}(2c\partial^2_{z\tau}\Phi-c^2\Delta_y\Phi)-\frac{\rho_0}{2c^4}(\gamma+1)\partial_\tau[(\partial_\tau\Phi)^2]-\frac{\nu}{c^4}\partial_\tau^3\Phi]$$
$$+\varepsilon^3R_1^{NS-KZK}\quad (4.33)$$

and

$$\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon.\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)-\varepsilon\nu\Delta\mathbf{v}_\varepsilon=\varepsilon\tilde{\nabla}[-\rho_0\partial_\tau\Phi+c^2I]$$
$$+\varepsilon^2\tilde{\nabla}\left[c^2J+\frac{(\gamma-1)\rho_0}{2c^2}(\partial_\tau\Phi)^2+\frac{\nu}{c^2}\partial_\tau^2\Phi\right]+\varepsilon^3\mathbf{R}_2^{NS-KZK},\quad (4.34)$$

where R_1^{NS-KZK} and $\mathbf{R}_2^{NS-KZK}$ are the remainder terms given in the appendix. So, as it was previously explained for the approximation of the Navier-Stokes by the Kuznetsov equation in Subsection 3.2, if we consider a solution of the KZK equation I and define by it the functions Φ and J, then we define according to *ansatz* (4.12)–(4.13) $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ (see Equation (4.16)), which solve the approximate system (3.17)–(3.18) with the remainder terms R_1^{NS-KZK} and $\mathbf{R}_2^{NS-KZK}$ and, as previously, with $p(\overline{\rho}_\varepsilon)$ from the state law (2.8):

$$\partial_t\overline{\rho}_\varepsilon+\mathrm{div}(\overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)=\varepsilon^3R_1^{NS-KZK},\quad (4.35)$$

$$\overline{\rho}_\varepsilon[\partial_t\overline{\mathbf{v}}_\varepsilon+(\overline{\mathbf{v}}_\varepsilon.\nabla)\overline{\mathbf{v}}_\varepsilon]+\nabla p(\overline{\rho}_\varepsilon)-\varepsilon\nu\Delta\overline{\mathbf{v}}_\varepsilon=\varepsilon^3\mathbf{R}_2^{NS-KZK}. \tag{4.36}$$

As usual, we denote by $\mathbf{U}_\varepsilon=(\rho_\varepsilon\,,\rho_\varepsilon\mathbf{v}_\varepsilon)^t$ the solution of the exact Navier-Stokes system and by $\overline{\mathbf{U}}_\varepsilon=(\overline{\rho}_\varepsilon\,,\overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)^t$ the solution of (4.35)–(4.36).

We work on $\mathbb{R}_+\times\mathbb{R}^{n-1}$ ($n=2$ or 3) due to the domain of the well-posedness for the KZK equation. In this case the Navier-Stokes system is globally well-posed with non-homogeneous boundary conditions of $\overline{\mathbf{U}}_\varepsilon$, as they are directly determined by the initial condition I_0 of the KZK Equation (4.22) according to Theorem 4.2. Knowing the existence results for two problems, we validate the approximation of $\mathbf{U}_\varepsilon$ by $\overline{\mathbf{U}}_\varepsilon$ following Ref. [35] and Subsection 3.2:

THEOREM 4.3 ([35]). *Let $n=2$ or 3, $s\geq 10$ and Theorem 4.2 hold. Then there exist constants $C>0$ and $K>0$ such that if*

$$(\overline{\rho}_\varepsilon-\rho_\varepsilon)|_{t=0}=0 \quad \textit{and} \quad (\overline{\boldsymbol{v}}_\varepsilon-\boldsymbol{v}_\varepsilon)|_{t=0}=0,$$

we have the following stability estimate

$$0\leq t\leq\frac{C}{\varepsilon}\quad \|\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^{n-1})}(t)\leq K\varepsilon^3te^{K\varepsilon t}\leq 9\varepsilon^2.$$

Moreover, if the initial conditions for the KZK equation are such that

$$I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1}) \textit{ for } s\geq 8,$$

and sufficiently small (in the sense of Theorem 4.1), then there exists the unique global-in-time solution of the Cauchy problem for the KZK equation

$$\overline{\rho}_\varepsilon-\rho_0\in C([0,+\infty[;H^2(\{x_1>0\}\times\mathbb{R}^{n-1}))\cap C^1([0,+\infty[;H^1(\{x_1>0\}\times\mathbb{R}^{n-1})),$$
$$\overline{\boldsymbol{v}}_\varepsilon\in C([0,+\infty[;H^2(\{x_1>0\}\times\mathbb{R}^{n-1}))\cap C^1([0,+\infty[;H^1(\{x_1>0\}\times\mathbb{R}^{n-1}))$$

and the remainder terms $(R_1^{NS-KZK},\mathbf{R}_2^{NS-ZKZ})$ (see the appendix) belong to $C([0,+\infty[;L^2(\mathbb{R}_+\times\mathbb{R}^{n-1}))$.

If in addition there exists an admissible weak solution of a bounded energy of the Cauchy problem for the Navier-Stokes system (3.19) on a time interval $[0,T_{NS}[$ for the initial data

$$\|\mathbf{U}_\varepsilon(0)-\overline{\mathbf{U}}_\varepsilon(0)\|_{L^2(\mathbb{R}^n)}\leq\delta\leq\varepsilon,$$

then it holds for all $t<\min\{\frac{C}{\varepsilon},T_{NS}\}$ the stability estimate (1.2):

$$\|(\mathbf{U}_\varepsilon-\overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(\mathbb{R}^n)}\leq K(\varepsilon^3t+\delta^2)e^{K\varepsilon t}\leq 9\varepsilon^2.$$

Proof. We validate the approximation of $\mathbf{U}_\varepsilon$ by $\overline{\mathbf{U}}_\varepsilon$ following Ref. [35] and Subsection 3.2. For the regularity of the approximate solution, if $I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1})$ with $s>\max\{8,\frac{n}{2}\}$ then for $0\leq\ell\leq 4$

$$I(\tau,z,y)\in C^\ell(\{z>0\};H^{s-2\ell}(\mathbb{T}_\tau\times\mathbb{R}^{n-1})).$$

Let us denote $\Omega=\mathbb{T}_\tau\times\mathbb{R}^{n-1}$. Given the equations for $\overline{\rho}_\varepsilon$ by (4.8) with (4.5) and (4.6) and for $\overline{\mathbf{v}}_\varepsilon$ by (4.16) respectively, we have for $0\leq\ell\leq 2$

$$\partial_z^\ell\overline{\rho}_\varepsilon(\tau,z,y)\in C(\{z>0\};H^{s-1-2\ell}(\Omega)),\,\partial_z^\ell\overline{\mathbf{v}}_\varepsilon(\tau,z,y)\in C(\{z>0\};H^{s-2-2\ell}(\Omega)),$$

but we can also say thanks to point 4 of Theorem 4.1 that

$$\partial_z^\ell \overline{\rho}_\varepsilon(\tau,z,y)\in L^2(\{z>0\};H^{s-1-2\ell}(\Omega)),\quad \partial_z^\ell \overline{\mathbf{v}}_\varepsilon(\tau,z,y)\in L^2(\{z>0\};H^{s-2-2\ell}(\Omega)).$$

This implies for $0\leq \ell\leq 2$ (as $s\geq 8$) that $s-2-2\ell>2$ and

$$\partial_z^\ell \overline{\rho}_\varepsilon(\tau,z,y)\in C(\mathbb{T}_\tau;L^2(\{z>0\};H^{s-1-2\ell}(\mathbb{R}^{n-1}))),$$
$$\partial_z^\ell \overline{\mathbf{v}}_\varepsilon(\tau,z,y)\in C(\mathbb{T}_\tau;L^2(\{z>0\};H^{s-2-2\ell}(\mathbb{R}^{n-1}))).$$

Hence we find

$$\overline{\rho}_\varepsilon(t,x_1,x')\text{ and }\overline{\mathbf{v}}_\varepsilon(t,x_1,x')\in C([0,+\infty[;H^2(\{x_1>0\}\times\mathbb{R}^{n-1}).$$

As in addition for $0\leq\ell\leq 1$, considering $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ as functions of (τ,z,y),

$$\partial_z^\ell\partial_\tau\overline{\rho}_\varepsilon\in C(\{z>0\};H^{s-2-2\ell}(\Omega)),\,\partial_z^\ell\partial_\tau\overline{\mathbf{v}}_\varepsilon\in C(\{z>0\};H^{s-3-2\ell}(\Omega)),$$

we deduce in the same way that

$$\partial_t\overline{\rho}_\varepsilon(t,x_1,x')\text{ and }\partial_t\overline{\mathbf{v}}_\varepsilon(t,x_1,x')\in C([0,+\infty[;H^1(\{x_1>0\}\times\mathbb{R}^{n-1})).$$

These regularities of $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$, viewed as functions of (t,x_1,x'), allow to have all left-hand terms in the approximate Navier-Stokes system (4.35)–(4.36) of the regularity $C([0,T];L^2(\{x_1>0\}\times\mathbb{R}^{n-1}))$ and the remainder terms in the right-hand side inherit it.

The regularity of $I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1})$ with $s\geq 8$ (see Table 7.1), is minimal to ensure that R_1^{NS-KZK} and $\mathbf{R}_2^{NS-KZK}$ (see the appendix for their expressions), belongs to

$$C([0,+\infty[;L^2(\mathbb{R}_+\times\mathbb{R}^{n-1})).$$

It is due to the fact that the least regular term in R_1^{NS-KZK} and $\mathbf{R}_2^{NS-KZK}$ is of the form

$$\partial_z^3\Phi\in L^2(\{z>0\};H^{s-6}(\Omega))\cap C(\{z>0\};H^{s-6}(\Omega)).$$

□

5. Navier-Stokes system and the NPE equation

5.1. Derivation of the NPE equation. The NPE equation (nonlinear progressive wave equation), initially derived by McDonald and Kuperman [31], is an example of a paraxial approximation aiming to describe short-time pulses and a long-range propagation, for instance, in an ocean wave-guide, where the refraction phenomena are important. To compare to the KZK equation we use the following paraxial change of variables

$$u(t,x_1,x')=\Psi(\varepsilon t,x_1-ct,\sqrt{\varepsilon}x')=\Psi(\tau,z,y),\tag{5.1}$$

with

$$\tau=\varepsilon t,\quad z=x_1-ct,\quad y=\sqrt{\varepsilon}x'.\tag{5.2}$$

For the velocity we have

$$\mathbf{v}_\varepsilon(t,x_1,x')=-\varepsilon\nabla u(t,x_1,x')=-\varepsilon(\partial_z\Psi,\sqrt{\varepsilon}\nabla_y\Psi)(\tau,z,y).\tag{5.3}$$

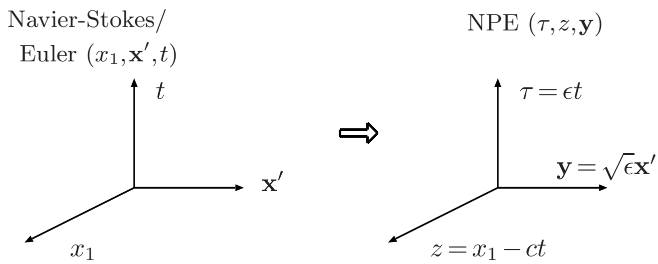

FIG. 5.1. *Paraxial change of variables for the profiles* $U(\epsilon t, x_1-ct, \sqrt{\epsilon}\mathbf{x}')$.

If we compare the NPE equation to the isentropic Navier-Stokes system this method of approximation does not allow to keep the Kuznetsov *ansatz* of perturbations (3.2)–(3.3) imposing (3.5)–(3.6), just by introducing the new paraxial profiles Ψ for u, ξ for ρ_1 and χ for ρ_2 and taking the term of order 0 in ε as it was done in the case of the KZK-approximation. This time the paraxial change of variables (5.2) for ρ_1 and ρ_2, defined in Equations (3.5)–(3.6), gives

$$\begin{aligned}
\rho_1 &= -\frac{\rho_0}{c}\partial_z\Psi + \varepsilon\frac{\rho_0}{c^2}\partial_\tau\Psi,\\
\rho_2 &= -\frac{\rho_0(\gamma-2)}{2c^2}(\partial_z\Psi)^2 - \frac{\rho_0}{2c^2}(\partial_z\Psi)^2 - \frac{\nu}{\rho_0}\partial_z^2\Psi\\
&\quad + \varepsilon\left[\frac{\rho_0(\gamma-2)}{2c^3}\partial_z\Psi\partial_\tau\Psi - \frac{\rho_0}{2c^2}(\nabla_y\Psi)^2 - \frac{\nu}{c^2}\Delta_y\Psi\right]\\
&\quad + \varepsilon^2\left(-\frac{\rho_0(\gamma-2)}{2c^4}\right)(\partial_\tau\Psi)^2.
\end{aligned}$$

Thus, one of the terms in the ρ_1-extension takes part of the second-order corrector of ρ_ε:

$$\rho_\varepsilon(t,x_1,x') = \rho_0 + \varepsilon\xi(\tau,z,y) + \varepsilon^2\chi(\tau,z,y), \tag{5.4}$$

with

$$\xi(\tau,z,y) = -\frac{\rho_0}{c}\partial_z\Psi, \tag{5.5}$$

$$\chi(\tau,z,y) = \frac{\rho_0}{c^2}\partial_\tau\Psi - \frac{\rho_0(\gamma-1)}{2c^2}(\partial_z\Psi)^2 - \frac{\nu}{c^2}\partial_z^2\Psi. \tag{5.6}$$

The obtained *ansatz* (5.3)–(5.4), applied to the Navier-Stokes system, gives

$$\begin{aligned}
\partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon\mathbf{v}_\varepsilon) =& \varepsilon^2\left(-\frac{2\rho_0}{c}\right)\left(\partial^2_{\tau z}\Psi - \frac{(\gamma+1)}{4}\partial_z(\partial_z\Psi)^2 - \frac{\nu}{2\rho_0}\partial_z^3\Psi + \frac{c}{2}\Delta_y\Psi\right)\\
&+\varepsilon^3 R_1^{NS-NPE},
\end{aligned}$$

and

$$\begin{aligned}
\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon.\nabla)\mathbf{v}_\varepsilon] + \nabla p(\rho_\varepsilon) - \varepsilon\nu\Delta\mathbf{v}_\varepsilon =& \varepsilon\nabla\left(\xi + \frac{\rho_0}{c}\partial_z\Psi\right)\\
&+ c^2\varepsilon^2\nabla\left[\chi - \frac{\rho_0}{c^2}\partial_\tau\Psi + \frac{\rho_0(\gamma-1)}{2c^2}(\partial_z\Psi)^2 + \frac{\nu}{c^2}\partial_z^2\Psi\right] + \varepsilon^3\mathbf{R}_2^{NS-NPE}.
\end{aligned}$$

The remainder term in the conservation of mass is given by

$$\begin{aligned}\varepsilon^3 R_1^{NS-NPE} =& \varepsilon^3\left(\partial_\tau\chi-\nabla_y\xi\,\nabla_y\Psi-\xi\,\Delta_y\Psi-\partial_z\chi\,\partial_z\Psi-\chi\,\partial_z^2\Psi\right)\\ &+\varepsilon^4(-\nabla_y\chi\,\nabla_y\Psi-\chi\,\Delta_y\Psi),\end{aligned} \tag{5.7}$$

while in the conservation of momentum along the x_1 axis it is given by

$$\begin{aligned}\varepsilon^3\mathbf{R}_2^{NS-NPE}.\vec{e}_1 =& \varepsilon^3\Big[-\frac{\rho_0}{c}\partial_z\Psi\,\partial_{\tau z}^2\Psi+\frac{\rho_0}{2}\partial_z(\nabla_y\Psi)^2+\nu\partial_z\Delta_y\Psi+\frac{\xi}{2}\partial_z(\partial_z\Psi)^2\\ &+c\chi\partial_z^2\Psi\Big]+\varepsilon^4\left(\frac{\xi}{2}\partial_z(\nabla_y\Psi)^2-\chi\partial_{\tau z}^2\Psi+\frac{\chi}{2}\partial_z(\partial_z\Psi)^2\right)+\varepsilon^5\frac{\chi}{2}\partial_z(\nabla_y\Psi)^2,\end{aligned} \tag{5.8}$$

and along all transversal direction x_j to the propagation x_1-axis

$$\begin{aligned}\varepsilon^3\mathbf{R}_2^{NS-NPE}.\vec{e}_j =& \varepsilon^{\frac{7}{2}}\Big[-\frac{\rho_0}{c}\partial_z\Psi\,\partial_{\tau y_j}^2\Psi+\frac{\rho_0}{2}\partial_{y_j}(\nabla_y\Psi)^2+\nu\partial_{y_j}\Delta_y\Psi+\frac{\xi}{2}\partial_{y_j}(\partial_z\Psi)^2\\ &+c\chi\partial_{zy_j}^2\Psi\Big]+\varepsilon^{\frac{9}{2}}\left(\frac{\xi}{2}\partial_{y_j}(\nabla_y\Psi)^2-\chi\partial_{\tau y_j}^2\Psi+\frac{\chi}{2}\partial_{y_j}(\partial_z\Psi)^2\right)+\varepsilon^{\frac{11}{2}}\frac{\chi}{2}\partial_{y_j}(\nabla_y\Psi)^2.\end{aligned} \tag{5.9}$$

As all previous models, for this *ansatz*, the NPE equation

$$\partial_{\tau z}^2\Psi-\frac{(\gamma+1)}{4}\partial_z(\partial_z\Psi)^2-\frac{\nu}{2\rho_0}\partial_z^3\Psi+\frac{c}{2}\Delta_y\Psi=0 \tag{5.10}$$

appears as the second-order approximation of the isentropic Navier-Stokes system up to the terms of the order of $\mathcal{O}(\varepsilon^3)$. In the sequel we work with the NPE equation satisfied by ξ (see Equation (5.5) for the definition)

$$\partial_{\tau z}^2\xi+\frac{(\gamma+1)c}{4\rho_0}\partial_z^2[(\xi)^2]-\frac{\nu}{2\rho_0}\partial_z^3\xi+\frac{c}{2}\Delta_y\xi=0. \tag{5.11}$$

Looking at Figures 4.1 and 5.1 together with Equations (4.11) and (5.10), we see that there is a bijection between the variables of the KZK and NPE equations defined by the relations

$$z_{NPE}=-c\tau_{KZK} \text{ and } \tau_{NPE}=\varepsilon\tau_{KZK}+\frac{z_{KZK}}{c}, \tag{5.12}$$

which implies for the derivatives

$$\partial_{\tau_{NPE}}=c\partial_{z_{KZK}} \text{ and } \partial_{z_{NPE}}=-\frac{1}{c}\partial_{\tau_{KZK}}.$$

Thus, as it was mentioned in the introduction, the known mathematical results for the KZK equation can be directly applied for the NPE equation.

5.2. Well posedness of the NPE equation. We consider the Cauchy problem:

$$\begin{cases}\partial_{\tau z}^2\xi+\frac{(\gamma+1)c}{4\rho_0}\partial_z^2[(\xi)^2]-\frac{\nu}{2\rho_0}\partial_z^3\xi+\frac{c}{2}\Delta_y\xi=0 \text{ on } \mathbb{R}_+\times\mathbb{T}_z\times\mathbb{R}^{n-1},\\ \xi(0,z,y)=\xi_0(z,y) \text{ on } \mathbb{T}_z\times\mathbb{R}^{n-1},\end{cases} \tag{5.13}$$

in the class of $L-$periodic functions with respect to the variable z and with mean value zero along z. The use of the operator ∂_z^{-1}, identically defined as ∂_τ^{-1} in Equation (4.19), allows us to consider instead of Equation (5.11) the following equivalent equation

$$\partial_\tau\xi+\frac{(\gamma+1)c}{4\rho_0}\partial_z[(\xi)^2]-\frac{\nu}{2\rho_0}\partial_z^2\xi+\frac{c}{2}\partial_z^{-1}\Delta_y\xi=0 \text{ on } \mathbb{R}_+\times\mathbb{T}_z\times\mathbb{R}^{n-1}.$$

As a consequence we can use the results of Subsection 4.2 if we replace τ by z. In the same time for the viscous case the following theorem holds:

THEOREM 5.1. *Let $n \geq 2$, $\nu > 0$, $s > \max\left(4, \left[\frac{n}{2}\right]+1\right)$ and $\xi_0 \in H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})$ with zero mean value along z. Then there exists a constant $k_2 > 0$ such that if*

$$\|\xi_0\|_{H^s(\mathbb{T}_z\times\mathbb{R}^2)} < k_2, \tag{5.14}$$

then the Cauchy problem for the NPE Equation (5.13) has a unique global-in-time solution

$$\xi \in \bigcap_{\ell=0}^{2} C^\ell([0,+\infty[, H^{s-2\ell}(\mathbb{T}_z \times \mathbb{R}^2)), \tag{5.15}$$

satisfying the zero mean value condition along z. Moreover, for Ψ according with Equation (5.5) we have

$$\Psi := -\frac{c}{\rho_0}\partial_z^{-1}\xi \in \bigcap_{\ell=0}^{2} C^\ell([0,+\infty[, H^{s-2\ell}(\mathbb{T}_z \times \mathbb{R}^2)),$$

also satisfying the zero mean value condition along z, i.e. $\int_0^L \Psi(\tau,z,y)\mathrm{d}z = 0$.

Proof. For $\xi_0 \in H^s(\mathbb{T}_z \times \mathbb{R}^{n-1})$ small enough the existence of a global-in-time solution

$$\xi \in \bigcap_{\ell=0}^{1} C^\ell([0,+\infty[, H^{s-2\ell}(\mathbb{T}_z \times \mathbb{R}^{n-1}))$$

of the Cauchy problem for the NPE Equation (5.13) comes from Theorem 4.1. We also have the desired regularity by a simple bootstrap argument. Moreover, the formula for ∂_z^{-1} (see the equivalent definition of ∂_τ^{-1} in Equation (4.19)) implies for $s \geq 1$ the Poincaré inequality

$$\|\partial_z^{-1}\xi\|_{H^s(\mathbb{T}_z\times\mathbb{R}^{n-1})} \leq C\|\partial_z\partial_z^{-1}\xi\|_{H^s(\mathbb{T}_z\times\mathbb{R}^{n-1})} \leq C\|\xi\|_{H^s(\mathbb{T}_z\times\mathbb{R}^{n-1})},$$

which gives us the same regularity for Ψ. ∎

5.3. Approximation of the solutions of the isentropic Navier-Stokes system by the solutions of the NPE equation. By Subsections 4.2 and 5.2, this time the approximation domain is $\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$. Let ξ be a sufficiently regular solution of the Cauchy problem (5.13) for the NPE equation in $\mathbb{T}_z \times \mathbb{R}^{n-1}$. Then, taking ξ and χ according to formulas (5.5)-(5.6), with Ψ defined using the operator ∂_z^{-1} equivalent to ∂_τ^{-1} (see Equation (4.19)), we define $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ by formulas (5.3)–(5.4). For $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ we obtain a solution of the approximate system (3.17)–(3.18) defined on $\mathbb{R}_+ \times \mathbb{T}_{x_1} \times \mathbb{R}^{n-1}$ with $p(\overline{\rho}_\varepsilon)$ from the state law (2.8), but with the remainder terms R_1^{NS-NPE} and $\mathbf{R}_2^{NS-NPE}$ defined respectively in Equations (5.7)–(5.9) instead of R_1^{NS-Kuz} and $\mathbf{R}_2^{NS-Kuz}$.

In what follows we consider the three dimensional case, knowing, thanks to the energy method used in Ref. [29] on $\mathbb{R}^3$, that the Cauchy problem for the Navier-Stokes system is globally well-posed in $\mathbb{T}_{x_1} \times \mathbb{R}^2$ for sufficiently small initial data (see Ref. [29] Theorem 7.1, p. 100 or Ref. [7]):

THEOREM 5.2. *There exists a constant $k_1 > 0$ such that if the initial data*

$$\rho_\varepsilon(0) - \rho_0 \in H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2),\ \ \mathbf{v}_\varepsilon(0) \in H^3(\mathbb{T}_{x_1} \times \mathbb{R}^2) \tag{5.16}$$

satisfy

$$\|\rho_\varepsilon(0)-\rho_0\|_{H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2)}+\|\mathbf{v}_\varepsilon(0)\|_{H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2)}<k_1,$$

and $\rho_\varepsilon(0)-\rho_0$ *and* $\mathbf{v}_\varepsilon(0)$ *have the zero mean value among* x_1 *then the Cauchy problem (2.6)-(2.8) on* $\mathbb{T}_{x_1}\times\mathbb{R}^2$ *with the initial data (5.16) has a unique global-in-time solution* $(\rho_\varepsilon,\,\mathbf{v}_\varepsilon)$ *such that*

$$\rho_\varepsilon-\rho_0\in C([0,+\infty[;H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2))\cap C^1([0,+\infty[;H^2(\mathbb{T}_{x_1}\times\mathbb{R}^2)), \tag{5.17}$$

which implies

$$\rho_\varepsilon-\rho_0 \quad \text{and} \quad \mathbf{v}_\varepsilon\in C([0,+\infty[;H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2))\cap C^1([0,+\infty[;H^1(\mathbb{T}_{x_1}\times\mathbb{R}^2)). \tag{5.18}$$

Moreover for all times for $\rho_\varepsilon-\rho_0$ *and* $\mathbf{v}_\varepsilon$ *have the mean value zero along* x_1.

The existence results for the Cauchy problems of the Navier-Stokes system (2.6)-(2.8) and the NPE Equation (5.13) allow us to establish the global existence of $\mathbf{U}_\varepsilon$ and $\overline{\mathbf{U}}_\varepsilon$, considered in the NPE approximation framework:

THEOREM 5.3. *Let* $n=3$. *There exists a constant* $k>0$ *such that if the initial datum* $\xi_0\in H^5(\mathbb{T}_z\times\mathbb{R}^2)$ *for the Cauchy problem for the NPE Equation (5.13) (necessarily* $k\le k_2$*, see Theorem 5.1) is sufficiently small*

$$\|\xi_0\|_{H^5(\mathbb{T}_z\times\mathbb{R}^{n-1})}<k,$$

has the mean value zero, then there exist global-in-time solutions $\overline{\mathbf{U}}_\varepsilon=(\overline{\rho}_\varepsilon,\overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)^t$ *of the approximate Navier-Stokes system (3.20) and* $\mathbf{U}_\varepsilon=(\rho_\varepsilon,\rho_\varepsilon\mathbf{v}_\varepsilon)^t$ *of the exact Navier-Stokes system (3.19) respectively, with the same regularity corresponding to (5.18) and with the mean value zero in the* x_1*-direction, both considered with the state law (2.8) and with the same initial data*

$$(\bar\rho_\varepsilon-\rho_\varepsilon)|_{t=0}=0, \quad (\bar{\mathbf{v}}_\varepsilon-\mathbf{v}_\varepsilon)|_{t=0}=0. \tag{5.19}$$

Here $\bar\rho_\varepsilon|_{t=0}$ *and* $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ *are constructed as the functions of the initial datum for NPE equation* ξ_0 *according to formulas (5.3)–(5.6).*

Proof. The proof is essentially the same as for Theorem 3.1. According to Theorem 5.1 with $s=5$, the datum ξ_0 is regular enough so that

$$\rho_\varepsilon-\rho_0|_{t=0}\in H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2) \text{ and } \mathbf{v}_\varepsilon|_{t=0}\in[H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2)]^3$$

constructed with the help of formulas (5.3)–(5.6) in order to apply Theorem 5.2. These formulas together with Theorem 5.1 imply that $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ have the desired regularity. ∎

Thanks to Theorem 5.3 we validate the approximation of the solution of the Navier-Stokes system $\mathbf{U}_\varepsilon$ by the solution of the NPE equation $\overline{\mathbf{U}}_\varepsilon$ following Ref. [35]:

THEOREM 5.4. *Let* $\nu>0$ *and* $\varepsilon>0$ *be fixed and all assumptions of Theorem 5.3 hold. Then estimates of Theorem 3.3 hold in* $L^2(\mathbb{T}_{x_1}\times\mathbb{R}^2)$. *Moreover, if* $\xi_0\in H^s(\mathbb{T}_{x_1}\times\mathbb{R}^2)$ *with* $s\ge 4$*, then we have the stability estimate (1.2) with*

$$\overline{\rho}_\varepsilon(t,x_1,x')-\rho_0\in C([0,+\infty[;H^2(\mathbb{T}_{x_1}\times\mathbb{R}^2))\cap C^1([0,+\infty[;L^2(\mathbb{T}_{x_1}\times\mathbb{R}^2)), \tag{5.20}$$

$$\overline{\boldsymbol{v}}_\varepsilon(t,x_1,x')\in C([0,+\infty[;H^3(\mathbb{T}_{x_1}\times\mathbb{R}^2))\cap C^1([0,+\infty[;H^1(\mathbb{T}_{x_1}\times\mathbb{R}^2)) \tag{5.21}$$

and

$$R_1^{NS-NPE} \text{ and } \boldsymbol{R}_2^{NS-NPE} \in C([0,+\infty[;L^2(\mathbb{T}_{x_1}\times\mathbb{R}^2)). \tag{5.22}$$

Proof. The proof, being the same as in Theorem 3.3, is omitted. In fact it is due to the same Equations (3.19) and (3.20) with just different remainder terms of the same order on ε.

It is also easy to see using the previous arguments that the minimum regularity of the initial data (see Table 7.1) to have the remainder terms in $C([0,+\infty[;L^2(\mathbb{T}_{x_1}\times\mathbb{R}^2))$ (see (5.22)) corresponds to $\xi_0\in H^s(\mathbb{T}_{x_1}\times\mathbb{R}^2)$ for $s\geq 3$. Indeed, if $s\geq 3$, then

$$\text{for } 0\leq \ell\leq 1 \quad \xi(\tau,z,y)\in C^\ell([0,+\infty[;H^{s-2\ell}(\mathbb{T}_z\times\mathbb{R}^2)).$$

As in addition the least regular term in the remainders is $\partial_\tau^2\Psi$ coming from $\partial_\tau\chi$, this finally implies with formulas (5.3)–(5.6) the desired regularities of $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ given in Equations (5.20) and (5.21) respectively. ∎

6. Approximations of the Euler system

Let us consider the following isentropic Euler system:

$$\partial_t\rho_\varepsilon+\operatorname{div}(\rho_\varepsilon\mathbf{v}_\varepsilon)=0, \tag{6.1}$$
$$\rho_\varepsilon[\partial_t\mathbf{v}_\varepsilon+(\mathbf{v}_\varepsilon.\nabla)\mathbf{v}_\varepsilon]+\nabla p(\rho_\varepsilon)=0 \tag{6.2}$$

with $p(\rho_\varepsilon)$ given in Equation (2.8). We use all notations of previous sections just taking $\nu=0$.

Let us consider two and three dimensional cases. The entropy η of the isentropic Euler system, defined in Equation (3.31), is of class C^3 and in addition $\eta''(U_\varepsilon)$ is positive definite for $\rho_\varepsilon>0$. Moreover, from (3.19) we see that $G_i\in C^\infty$ with respect to U_ε for $\rho_\varepsilon>0$. Then we can apply Theorem 5.1.1 p. 98 in Ref. [8], which gives us the local well-posedness of the Euler system:

THEOREM 6.1 ([8]). *In $\mathbb{R}^n$ for $n=2$ or 3, suppose the initial data $\mathbf{U}_\varepsilon(0)$ be continuously differentiable on $\mathbb{R}^n$, take value in some compact set with $\rho_\varepsilon(0)>0$, and*

$$\text{for } i=1,\ldots,n,\ \partial_{x_i}\mathbf{U}_\varepsilon(0)\in[H^s(\mathbb{R}^n)]^{n+1} \text{ with } s>n/2.$$

Then there exists $0<T_\infty\leq+\infty$, and a unique continuously differentiable function $\mathbf{U}_\varepsilon$ on $\mathbb{R}^3\times[0,T_\infty[$ taking value with $\rho_\varepsilon>0$, which is a classical solution of the Cauchy problem associated to (3.19) with $\nu=0$. Furthermore for $i=1,\ldots,n$

$$\partial_{x_i}\mathbf{U}_\varepsilon(t)\in\bigcap_{k=0}^{s}C^k([0,T_\infty[;[H^{s-k}(\mathbb{R}^n)]^{n+1}).$$

The interval $[0,T_\infty[$ is maximal in that if $T_\infty<+\infty$ then

$$\int_0^{T_\infty}\sup_{i=1,\ldots,n}\|\partial_{x_i}\mathbf{U}_\varepsilon\|_{[L^\infty(\mathbb{R}^n)]^{n+1}}dt=+\infty,$$

and/or the range of $\mathbf{U}_\varepsilon(t)$ escapes from every compact subset of $\mathbb{R}_+^*\times\mathbb{R}^n$ as $t\to T_\infty$.

REMARK 6.1. A sufficient condition for the initial data to apply Theorem 6.1 is to have $\rho_\varepsilon(0)-\rho_0\in H^3(\mathbb{R}^n)$ and $\mathbf{v}_\varepsilon(0)\in(H^3(\mathbb{R}^n))^n$ with $\rho_\varepsilon(0)>0$.

To approximate the solutions of the Euler system and the Kuznetsov, the NPE or the KZK equations, we need to know for which time (how long) they exist. As opposed to the viscous case, the inviscid models can provide blow-up phenomena as indicated in Theorem 6.1 for the Euler system, in Theorem 6.5 for the Kuznetsov equation and for the KZK and the NPE equations see Theorem 1.3 in Ref. [34]. Let us start by summarizing what is known on the blow-up time for the Euler system [2, 36–40].

Due to our framework of the non-linear acoustic, it is important for us to have a potential motion (the irrotational case) and to consider the compressible isentropic Euler system (6.1)–(6.2) with initial data defining a perturbation of order ε around the constant state $(\rho_0, 0)$. The following theorem estimates the existence time of its solutions:

THEOREM 6.2.

(1) [8] In $\mathbb{R}^n$ for $n=2$ or 3, suppose the initial data

$$\mathbf{U}_\varepsilon(0) = (\rho_{\varepsilon,0}, \rho_{\varepsilon,0}\mathbf{v}_{\varepsilon,0})^t$$

be a perturbation of order ε around the constant state $(\rho_0, 0)$ (see Equation (6.3)) and take value such that for $i=1,\dots,n$

$$\partial_{x_i}\mathbf{U}_\varepsilon(0) \in [H^s(\mathbb{R}^n)]^{n+1}$$

with $s > n/2$. Then according to Theorem 6.1 there exists a unique classical solution of the Cauchy problem associated to (3.19) with $\nu = 0$ with a regularity given in Theorem 6.1. Moreover considering a generic constant $C > 0$ independent of ε, the existence time T_ε is estimated by $T_\varepsilon \geq \frac{C}{\varepsilon}$.

(2) [36–39] If $\nabla \times \mathbf{v}_{\varepsilon,0} = 0$ and if

$$\left(\frac{\rho_{\varepsilon,0}}{\rho_0}\right)^{\frac{\gamma-1}{2}} - 1 \text{ and } \boldsymbol{v}_{\varepsilon,0} \text{ belong to the energy space } X^m$$

a dense subspace of $H^m(\mathbb{R}^n)$ with $m \geq 4$ (for instance $X^m \subset \mathcal{D}(\mathbb{R}^n)$, see p.7-8 in Ref. [38] for the exact definition of X^m), then

$$T_\varepsilon \geq \frac{C}{\varepsilon^2} \text{ for } n=2, \text{ and } T_\varepsilon \geq \exp\left(\frac{C}{\varepsilon}\right) - 1 \text{ for } n=3.$$

The regularity is given by energy estimates on X^m which implies at least the same regularity as in Theorem 6.1 if for $i=1,\dots,n$

$$\partial_{x_i}\mathbf{U}_\varepsilon(0) \in [H^{m-1}(\mathbb{R}^n)]^{n+1}.$$

Proof. The first point is a direct consequence of the proof of Theorem 5.1.1 p. 98 in Ref. [8]. For the second point we refer to Refs. [36–39] in order to have estimations of T_ε with the help of energy estimates in the considered energy spaces which are dense subspaces of the usual Sobolev spaces. ∎

Let us pay attention to the optimality of the lifespan in the previous results for two [2] and three dimensional cases [40]. The following theorem tells us that the lower bound for the lifespan of the compressible Euler system in the irrotational case found in Theorem 6.2 is optimal thanks to the estimation of the blow-up time:

THEOREM 6.3.
(1) [2] In $\mathbb{R}^2$, we consider the initial data given by

$$\rho_\varepsilon(0) = \rho_0 + \varepsilon\rho_{\varepsilon,0} \text{ and } \mathbf{v}_\varepsilon(0) = \varepsilon\mathbf{v}_{\varepsilon,0}, \tag{6.3}$$

with $\rho_{\varepsilon,0}$ and $\mathbf{v}_{\varepsilon,0}$ of regularity C^∞ with a compact support. Moreover

$$\mathbf{v}_{\varepsilon,0}(x) = v_r|x|_2\overrightarrow{e}_r + v_\theta|x|_2\overrightarrow{e}_\theta,$$

with $\rho_{\varepsilon,0}, v_r, v_\theta \in \mathcal{D}(\mathbb{R}^2)$ depending only on $r = |x|_2 = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2)^t$. Then the Euler system (6.1)–(6.2) with initial data (6.3) admits a C^∞ solution for $t \in [0, T_\varepsilon[$ with

$$\lim_{\varepsilon \to 0} \varepsilon^2 T_\varepsilon = C > 0.$$

(2) [40] In $\mathbb{R}^3$, we consider the initial data given by (6.3) with $\rho_{\varepsilon,0}$ and $\mathbf{v}_{\varepsilon,0}$ of regularity C^∞ with a compact support. Moreover

$$\mathbf{v}_{\varepsilon,0}(x) = v_r|x|_3\overrightarrow{e}_r,$$

with $\rho_{\varepsilon,0}$ and $v_r \in \mathcal{D}(\mathbb{R}^3)$ depending only on $r = |x|_3 = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3)^t$. Then the Euler system (6.1)–(6.2) with initial data (6.3) admits a C^∞ solution for $t \in [0, T_\varepsilon[$ with

$$\lim_{\varepsilon \to 0} \varepsilon \ln(T_\varepsilon) = C > 0.$$

Now let us consider the derivation of the Kuznetsov equation of Subsection 3.1 in the assumption $\nu = 0$. Taking *ansatz* (3.2)–(3.3) for ρ_ε and $\mathbf{v}_\varepsilon$ and imposing (3.5)–(3.6) for ρ_1 and ρ_2 with $\nu = 0$, we derive as in Subsection 3.1 the inviscid Kuznetsov equation with the notation $\alpha = \frac{\gamma-1}{c^2}$

$$\begin{cases} \partial_t^2 u - c^2\Delta u = \varepsilon\partial_t\left((\nabla u)^2 + \frac{\alpha}{2}(\partial_t u)^2\right), \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases} \tag{6.4}$$

Thanks to Theorem 1.1 in Ref. [9], we have the following local well posedness result for the inviscid Kuznetsov equation:

THEOREM 6.4 ([9]). *Let $\nu = 0$, $n \in \mathbb{N}^*$ and $s > \frac{n}{2} + 1$. For all $u_0 \in H^{s+1}(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n)$ such that*

$$\|u_1\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon},\ \|u_0\|_{L^\infty(\mathbb{R}^n)} < M_1 \text{ and } \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} < M_2,$$

with M_1 and M_2 in $\mathbb{R}_+^*$, the following results hold:

(1) There exists $T^* > 0$, finite or not, such that there exists a unique solution u of the inviscid Kuznetsov system (6.4) with the following regularity

$$u \in C^r([0, T^*[; H^{s+1-r}(\mathbb{R}^n)) \text{ for } 0 \le r \le s, \tag{6.5}$$

$$\forall t \in [0, T^*[,\ \|u_t(t)\|_{L^\infty(\mathbb{R}^n)} < \frac{1}{2\alpha\varepsilon},\ \|u\|_{L^\infty(\mathbb{R}^n)} < M_1,\ \|\nabla u\|_{L^\infty(\mathbb{R}^n)} < M_2. \tag{6.6}$$

(2) The map $(u_0, u_1) \mapsto (u(t,.), \partial_t u(t,.))$ is continuous in the topology of $H^{s+1} \times H^s$ uniformly in $t \in [0, T^*[$.

Ref. [9] allows us to give a result on the lower bound of the lifespan T_ε of the Kuznetsov equation. The method is similar to the case of the Euler system (6.1)–(6.2). It is based on the use of a group of linear transformations preserving the wave equation $u_{tt}-\Delta u=0$, initially proposed by John [16]. Let us briefly summarize the lifespan and blow-up time results for the inviscid Kuznetsov equation in the following theorem:

THEOREM 6.5.

(1) [9] *Let $m\in\mathbb{N}$, $m\geq\left[\frac{n}{2}+2\right]$. For $u_0\in H^{m+1}(\mathbb{R}^n)$ and $u_1\in H^m(\mathbb{R}^n)$ such that the results of Theorem 6.4 hold for $s=m$, let u_0 and u_1 be also small enough in the sense of an energy defined in point 3 of Theorem 1.1 in Ref.* [9]*. Then there exists a generic constant $C>0$ independent of ε such that $T_\varepsilon\geq\frac{C}{\varepsilon}$.*

(2) [9] *Let $m\in\mathbb{N}$, $m\geq n+2$ if n is even and $m\geq n+1$ if n is odd. For $u_0\in H^{m+1}(\mathbb{R}^n)$ and $u_1\in H^m(\mathbb{R}^n)$ such that the results of Theorem 6.4 hold for $s=m$, let u_0 and u_1 be also small enough in the sense of a generalized energy defined in Theorem 3.3 in Ref.* [9]*. Then there exists a generic constant $C>0$ independent of ε such that*

$$T_\varepsilon\geq\frac{C}{\varepsilon^2}\ \text{for}\ n=2, T_\varepsilon\geq\exp\left(\frac{C}{\varepsilon}\right)-1\ \text{for}\ n=3\ \text{and}\ T_\varepsilon=+\infty\ \text{for}\ n\geq 4.$$

(3) [3] *In dimension $n=2$ and 3, there exist functions $u_0\in\mathcal{D}(\mathbb{R}^n)$ and $u_1\in\mathcal{D}(\mathbb{R}^n)$ such that the solution u of the Cauchy problem for the inviscid Kuznetsov Equation (6.4) has a geometric blow-up for the time of order $T_\varepsilon=O\left(\frac{1}{\varepsilon^2}\right)$ and $T_\varepsilon=O\left(\exp\left(\frac{1}{\varepsilon}\right)\right)$ respectively.*

REMARK 6.2. In $\mathbb{R}^2$ and $\mathbb{R}^3$ we see that the lifespan of the inviscid Kuznetsov equation corresponds to the blow-up time estimation for the compressible isentropic Euler system in Theorems 6.2 and 6.3, a result in accordance with the fact that the inviscid Kuznetsov equation is an approximation of the Euler system. We also notice that in the two cases (for the Euler system and the Kuznetsov equation) having longer existence time requires more regularity on the initial data.

Relying now the existence results for the Euler system and the Kuznetsov equation, we formulate our approximation result:

THEOREM 6.6. *Let $n=2$ or 3. If the initial data $u_0\in H^4(\mathbb{R}^n)$ and $u_1\in H^3(\mathbb{R}^n)$ for the Cauchy problem for the inviscid Kuznetsov Equation (6.4) satisfy*

$$\|u_0\|_{H^4(\mathbb{R}^n)}+\|u_1\|_{H^3(\mathbb{R}^n)}\leq k \tag{6.7}$$

with a constant $k>0$ small enough, there exists $T^*_\varepsilon>0$ and $C>0$, independent of ε, satisfying

$$T^*_\varepsilon\geq\frac{C}{\varepsilon}$$

such that there exist local in time solutions

$$\overline{\mathbf{U}}_\varepsilon=(\overline{\rho}_\varepsilon,\overline{\rho}_\varepsilon\overline{\mathbf{v}}_\varepsilon)^t\ \text{and}\ \mathbf{U}_\varepsilon=(\rho_\varepsilon,\rho_\varepsilon\mathbf{v}_\varepsilon)^t\ \text{on}\ [0,T^*_\varepsilon[$$

of the approximate Euler system given by (3.20) and of the exact Euler system given by (3.19) with $\nu=0$, both considered with the state law (2.8) and with the same initial data (3.24). In addition, the solutions have the same regularity corresponding to

$$\mathbf{U}_\varepsilon-(\rho_0,0)^t\in\bigcap_{\ell=0}^{3}C^\ell([0,T^*_\varepsilon[;[H^{3-\ell}(\mathbb{R}^n)]^{n+1}). \tag{6.8}$$

Here $\bar\rho_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial data for the Kuznetsov equation u_0 and u_1 by formulas (3.25)–(3.26) according to (3.2)–(3.3) and (3.5)–(3.6) taken with $\nu=0$.

Moreover, there exist constants $C>0$ and $K>0$ independent of ε and on time t, such that

$$\forall t \le \frac{C}{\varepsilon} \quad \|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(\mathbb{R}^3)} \le Kt\varepsilon^3 e^{K\varepsilon t} \le 4\varepsilon^2. \tag{6.9}$$

If $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ with $s>\frac{n}{2}$ and there exists a classical solution of the Euler system found for the initial data satisfying (1.1), then estimate (1.2) holds with

$$\overline{\rho}_\varepsilon - \rho_0 \in C([0,T^*_\varepsilon[;H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T^*_\varepsilon[;H^s(\mathbb{R}^n)), \tag{6.10}$$

$$\overline{\boldsymbol{v}}_\varepsilon \in C([0,T^*_\varepsilon[;H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T^*_\varepsilon[;H^s(\mathbb{R}^n)) \tag{6.11}$$

and with the remainder terms $R_1^{Euler-Kuz}$ and $\mathbf{R}_2^{Euler-Kuz}$ (see (3.14)–(3.15) with $\nu=0$) belonging to $C([0,T^*_\varepsilon[;H^s(\mathbb{R}^n))$.

Proof. Taking $u_0 \in H^4(\mathbb{R}^n)$ and $u_1 \in H^3(\mathbb{R}^n)$ satisfying Equation (6.7) with a $k>0$ small enough, the Cauchy problem for the inviscid Kuznetsov Equation (6.4) is locally well-posed according to Theorem 6.5. Moreover the solution u belongs to

$$\bigcap_{\ell=0}^{4} C^\ell([0,T_{\varepsilon,1}[;H^{4-\ell}(\mathbb{R}^n))$$

with $T_{\varepsilon,1} \ge \frac{C_1}{\varepsilon}$ and $C_1>0$ independent of ε.

As $u_0 \in H^4(\mathbb{R}^n)$ and $u_1 \in H^3(\mathbb{R}^n)$, it ensures that

$$\rho_\varepsilon - \rho_0|_{t=0} \in H^3(\mathbb{R}^n) \text{ and } \mathbf{v}_\varepsilon|_{t=0} \in [H^3(\mathbb{R}^n)]^3.$$

Therefore $\rho_\varepsilon|_{t=0}>0$ if u_0 and u_1 are small enough. By Theorem 6.2 it is sufficient to have a local solution $\mathbf{U}_\varepsilon$ on $[0,T_{\varepsilon,2}[$ of the exact Euler system (see (3.19) with $\nu=0$) verifying (6.8) with T^*_ε corresponding to $T_{\varepsilon,2}$, $T_{\varepsilon,2} \ge \frac{C_2}{\varepsilon}$ with $C_2>0$ independent of ε.

Now we consider $T^*_\varepsilon = \min(T_{\varepsilon,1},T_{\varepsilon,2})$, and we have $T^*_\varepsilon \ge \frac{C}{\varepsilon}$ with $C>0$ independent of ε. As $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ are defined by *ansatz* (3.2)-(3.3) with ρ_1 and ρ_2 given in Equations (3.5)–(3.6), the regularity of u implies for $\overline{\mathbf{U}}_\varepsilon$ at least the same regularity as given in (6.8). To find it we use the Sobolev embedding (3.30) for the multiplication.

Knowing the existence results for the two problems, we validate the approximation of $\mathbf{U}_\varepsilon$ by the solution of the Kuznetsov equation, *i.e.* by $\overline{\mathbf{U}}_\varepsilon$, following Ref. [35]: we make use of the convex entropy as in Ref. [8] for the isentropic Euler equation and the rest follows exactly as in the proof of Theorem 3.3 except that $\nu=0$.

Let us finish the proof with the remark on the minimal regularity of the initial data for the Kuznetsov equation such that the approximation of the Euler system is possible, *i.e.* the remainder terms $R_1^{Euler-Kuz}$ and $\mathbf{R}_2^{Euler-Kuz}$ must be kept bounded for a finite time interval. Indeed, if we take $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$ with $s>\frac{n}{2}$, then $u \in C([0,T^*_\varepsilon[;H^{s+2}(\mathbb{R}^n))$ and

$$u_t \in C([0,T^*_\varepsilon[;H^{s+1}(\mathbb{R}^n)), \quad u_{tt} \in C([0,T^*_\varepsilon[;H^s(\mathbb{R}^n)).$$

Since $\overline{\rho}_\varepsilon$ is defined by (3.2) with (3.5)–(3.6) and $\overline{\mathbf{v}}_\varepsilon$ by (3.3) with $\nu=0$, respectively, we deduce regularity (6.10)–(6.11). As this time for $\nu=0$, we don't have the term $\Delta\partial_t u$ as in the viscous case in Equation (3.14), the remainder terms belong to $C([0,T^*_\varepsilon[;H^s(\mathbb{R}^n))$. ∎

REMARK 6.3. If we allow the Euler system to have, not the classical, but an admissible weak solution with the bounded energy (see Definition 3.1 and take $\nu=0$) taking the initial data in a small (on ε) L^2-neighborhood of $\overline{\mathbf{U}}_\varepsilon(0)$, then we also formally have estimate (1.2). But, thanks to Ref. [26] it is known that the Euler system can provide infinitely many admissible weak solutions, and thus there is no sense to approximate them.

For the approximation of solutions of the Euler system by the solutions of the NPE equation we obtain the following theorem:

THEOREM 6.7. *Let $n=2$ or 3. There exists a constant $k>0$ such that if the initial datum $\xi_0\in H^5(\mathbb{T}_z\times\mathbb{R}^{n-1})$ for the Cauchy problem for the NPE Equation (5.13) with $\nu=0$ is sufficiently small*

$$\|\xi_0\|_{H^5(\mathbb{T}_z\times\mathbb{R}^{n-1})}<k\varepsilon,$$

has the mean value zero, then

(1) There exist unique local in time solutions $\overline{\mathbf{U}}_\varepsilon$ of the approximate Euler system (3.20) and $\mathbf{U}_\varepsilon$ of the exact Euler system (3.19) with $\nu=0$ respectively. The solutions $\overline{\mathbf{U}}_\varepsilon$ and $\mathbf{U}_\varepsilon$ are of the same regularity corresponding to (5.18) on $[0,T^*_\varepsilon[$ instead of $[0,+\infty[$ and of mean value zero in the x_1-direction, both considered with the state law (2.8) and with the same initial data (5.19). Here $\bar{\rho}_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ are constructed as the functions of the initial datum for NPE equation ξ_0 according to formulas (5.3)–(5.6) with $\nu=0$.

(2) Moreover, there exists $C>0$ independent of ε such that $T^*_\varepsilon>\frac{C}{\varepsilon}$ and for $t\leq\frac{C}{\varepsilon}$ inequality (6.9) holds on $\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}$.

If $\xi_0\in H^s(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1})$ with $s>3$ and there exists a classical solution of the Euler system found for the initial data satisfying (1.1), then estimate (1.2) holds with

$$\overline{\rho}_\varepsilon-\rho_0\in C([0,T^*_\varepsilon[;H^2(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}))\cap C^1([0,T^*_\varepsilon[;L^2(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1})), \tag{6.12}$$

$$\overline{\boldsymbol{v}}_\varepsilon\in C([0,T^*_\varepsilon[;H^3(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}))\cap C^1([0,T^*_\varepsilon[;H^1(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1})) \tag{6.13}$$

and with the remainder terms $R_1^{Euler-NPE}$ and $\mathbf{R}_2^{Euler-NPE}$ (see (5.7)–(5.9) with $\nu=0$) belonging to $C([0,T^*_\varepsilon[;L^2(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}))$.

Proof. The work of Dafermos in Ref. [8] can always be applied on $\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}$ for $n=2$ or 3 instead of $\mathbb{R}^n$ so we have an equivalent of Theorem 6.1 and we also have the same equivalent of Theorem 6.2. This is due to the fact that the energy estimates in the articles of Sideris [36–39] are always true on $\mathbb{T}_{x_1}\times\mathbb{R}$ and $\mathbb{T}_{x_1}\times\mathbb{R}^2$. In all these cases we must also suppose that we have the mean value equal to zero in the direction x_1. As by Theorem 4.1 the NPE equation is locally well posed on $[0,T_\varepsilon[$ with $T_\varepsilon\geq\frac{C}{\varepsilon}$ if $\|\xi_0\|_{H^5(\mathbb{T}_z\times\mathbb{R}^{n-1})}<k\varepsilon$, we have an equivalent of Theorem 6.6 for the exact compressible isentropic Euler system and its approximation by the NPE equation on $\mathbb{T}_{x_1}\times\mathbb{R}^{n-1}$ for $n=2$ or 3, as $\xi_0\in H^5(\mathbb{T}_z\times\mathbb{R}^{n-1})$ also implies $\bar{\rho}_\varepsilon|_{t=0}$ and $\bar{\mathbf{v}}_\varepsilon|_{t=0}$ in $H^3(\mathbb{T}_{x_1}\times\mathbb{R}^{n-1})$.

The minimum regularity of the initial data (see Table 7.1) to have the remainder terms well defined is found exactly in the same way as in Theorem 5.4 for the viscous case, as soon as the least regular term does not disappear taking $\nu=0$. ∎

For the approximation by the KZK equation the inviscid case has already been studied in Ref. [35]. The key point is that we must restrict our spatial domain to a cone in order to take into account the fact that the KZK equation is only locally well posed. For the completeness of the article and for the reader's convenience, we give, updating for our new *ansatz*, the Euler-KZK approximation result, proved in detail in Ref. [35].

THEOREM 6.8. *Suppose that there exists the solution I of the KZK Cauchy problem (4.22) with $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ for $s > \max\{10, \left[\frac{n}{2}\right]+1\}$ and $\nu = 0$ such that $I(\tau,z,y)$ is $L-$periodic with respect to τ and defined for $|z| \le R$ and $y \in \mathbb{R}^{n-1}_y$.*

Let $\overline{\mathbf{U}}_\varepsilon = (\overline{\rho}_\varepsilon, \overline{\rho}_\varepsilon \overline{\boldsymbol{v}}_\varepsilon)^t$ be the approximate solution of the isentropic Euler system (4.35)–(4.36) with $\nu = 0$ deduced from a solution of the KZK equation. Then the function $\overline{\mathbf{U}}_\varepsilon(t,x_1,x')$ is defined in $\mathbb{T}_t \times \Omega_\varepsilon$ with

$$\Omega_\varepsilon = \{x_1 \in \mathbb{R} | \quad |x_1| < \frac{R}{\varepsilon} - ct\} \times \mathbb{R}^{n-1}_{x'}$$

and is smooth enough according to the regularity of I:

$$\overline{\rho}_\varepsilon \in C(\mathbb{T}_t; H^4(\Omega_\varepsilon)) \cap C^1(\mathbb{T}_t; H^3(\Omega_\varepsilon)) \quad \text{and} \quad \overline{\boldsymbol{v}}_\varepsilon \in C(\mathbb{T}_t; H^3(\Omega_\varepsilon)) \cap C^1(\mathbb{T}_t; H^2(\Omega_\varepsilon)).$$

Let us now consider the Euler system (3.19) with $\nu = 0$ in a cone

$$C(t) = \{0 < s < t\} \times Q_\varepsilon(s) = \{x = (x_1,x') : |x_1| \le \frac{R}{\varepsilon} - Ms, M \ge c, x' \in \mathbb{R}^{n-1}\}$$

with the same initial data

$$(\rho_\varepsilon - \overline{\rho}_\varepsilon)|_{t=0} = 0 \quad \text{and} \quad (\mathbf{v}_\varepsilon - \overline{\mathbf{v}}_\varepsilon)|_{t=0} = 0.$$

Consequently, (see Ref. [8] p. 62) there exists $T_0 > 0$, such that for the time interval $0 \le t \le \frac{T_0}{\varepsilon}$ there exists the classical solution $\mathbf{U}_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon \mathbf{v}_\varepsilon)$ of the Euler system (3.19) with $\nu = 0$ in a cone

$$C(T) = \{0 < t < T | T < \frac{T_0}{\varepsilon}\} \times Q_\varepsilon(t)$$

with

$$\|\nabla \mathbf{U}_\varepsilon\|_{L^\infty([0,\frac{T_0}{\varepsilon}[;H^{s-1}(Q_\varepsilon))} < \varepsilon C \text{ for } s > \left[\frac{n}{2}\right] + 1.$$

Moreover, there exists $K > 0$, such that for any ε small enough the solutions $\mathbf{U}_\varepsilon$ and $\overline{\mathbf{U}}_\varepsilon$, which were determined as above in cone $C(T)$ with the same initial data, satisfy the estimate for $0 < t < \frac{T_0}{\varepsilon}$

$$\|(\mathbf{U}_\varepsilon - \overline{\mathbf{U}}_\varepsilon)(t)\|^2_{L^2(Q_\varepsilon(t))} \le c_0^2 \varepsilon^3 t e^{2K\varepsilon t} \le 4\varepsilon^2$$

with $c_0^2 > 0$.

If $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with $s \ge 6$ then

$$\overline{\rho}_\varepsilon(t,x_1,x') - \rho_0 \quad \text{and} \quad \overline{\boldsymbol{v}}_\varepsilon(t,x_1,x') \in C^1([0,\frac{T_0}{\varepsilon}[;H^1(Q_\varepsilon)) \tag{6.14}$$

and R_1^{NS-KZK} and $\boldsymbol{R}_2^{NS-KZK}$ (for the definitions see the appendix) are in $C([0,\frac{T_0}{\varepsilon}[;L^2(Q_\varepsilon))$ and hence estimate (1.2) holds as soon as the initial data of the classical solution of the Euler system $\mathbf{U}_\varepsilon$ are taken in their small L^2-neighborhood defined by (1.1).

Proof. Considering expressions for $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ in (4.32) with $\nu=0$, the term asking the most regularity of I_0 is the same as for the viscous case and given by $\partial_\tau^{-1}\partial_z I$. Thus, we need to impose the same regularity of $I_0\in H^s$ with $s>10$ as for $\nu>0$ to ensure $\overline{\mathbf{v}}_\varepsilon\in C(\mathbb{T}_t,H^3(\Omega_\varepsilon))$. This regularity, if we take the same initial data, implies the existence of the classical solution $\mathbf{U}_\varepsilon$ of the Euler system.

Now, if the initial data are taken in a small L^2 neighborhood, according to (1.1), we can find the minimal regularity on I_0 ensuring that the remainder terms are bounded and well defined.

If $I_0\in H^s$ with $s\geq 6$ we have for $0\leq \ell\leq 1$ that the initial data found from I_0 for the Navier-Stokes system satisfies Theorem 4.2. Indeed, if $I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1})$ with $s>\max\{8,\frac{n}{2}\}$, then for $0\leq k\leq 4$

$$I(\tau,z,y)\in C^\ell(]-R,R[;H^{s-2\ell}(\mathbb{T}_\tau\times\mathbb{R}^{n-1})).$$

Let us denote $\Omega=\mathbb{T}_\tau\times\mathbb{R}^{n-1}$. Defining $\overline{\rho}_\varepsilon$ by Equation (4.8) with Equations (4.5) and (4.6) and $\overline{\mathbf{v}}_\varepsilon$ by (4.16) with $\nu=0$ respectively, we have as in the proof of Theorem 4.3 for $0\leq\ell\leq 1$ considering $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ as functions of (τ,z,y):

$$\partial_z^\ell\partial_\tau\overline{\rho}_\varepsilon\in C(]-R,R[;H^{s-1-2\ell}(\Omega)),\ \partial_z^\ell\partial_\tau\overline{\mathbf{v}}_\varepsilon\in C(]-R,R[;H^{s-3-2\ell}(\Omega)),$$

from where we deduce (6.14). These regularities of $\overline{\rho}_\varepsilon$ and $\overline{\mathbf{v}}_\varepsilon$ viewed as functions of (t,x_1,x') allow to have all left-hand terms in the approximate Euler system (4.35)–(4.36) with $\nu=0$ of the regularity $C([0,\frac{T_0}{\varepsilon}[;L^2(Q_\varepsilon))$ and the remainder terms in the right-hand side inherit it. Since the least regular term in the remainder terms is $\frac{J}{2}\partial_z[(\partial_z\Phi)^2]$, the regularity of $I_0\in H^s(\mathbb{T}_t\times\mathbb{R}^{n-1})$ with $s\geq 6$ (see also Table 7.1) is minimal to ensure that R_1^{NS-KZK} and $\mathbf{R}_2^{NS-KZK}$ are in $C([0,\frac{T_0}{\varepsilon}[;L^2(Q_\varepsilon))$. ∎

7. Conclusion

We summarize all obtained approximation results in the comparative Table 7.1.

Appendix. Expressions of the remainder terms. The expression of H, the profile of ρ_2, in the paraxial variables of the KZK *ansatz* is:

$$\begin{aligned}H(\tau,z,y)=&-\frac{\rho_0(\gamma-1)}{2c^4}(\partial_\tau\Phi)^2-\frac{\nu}{c^4}\partial_\tau^2\Phi\\&+\varepsilon\left[-\frac{\rho_0}{2c^2}[(\nabla_y\Phi)^2-\frac{2}{c}\partial_z\Phi\,\partial_\tau\Phi]-\frac{\nu}{c^2}[\Delta_y\Phi-\frac{2}{c}\partial_{z\tau}^2\Phi]\right]\\&+\varepsilon^2[-\frac{\rho_0}{2c^2}(\partial_z\Phi)^2-\frac{\nu}{c^2}\partial_z^2\Phi].\end{aligned}\tag{A.1}$$

If we consider (4.33)-(4.34) the expressions of R_1^{NS-KZK} and $\mathbf{R}_2^{NS-KZK}$ are written with the terms I and J defined by (4.5) and (4.6) respectively:

$$\begin{aligned}&\varepsilon^3R_1^{NS-KZK}\\=&\ \varepsilon^3\left[-\rho_0\partial_z^2\Phi+\frac{1}{c}\partial_zI\partial_\tau\Phi+\frac{1}{c}\partial_\tau I\partial_z\Phi-\nabla_yI.\nabla_y\Phi\right.\\&\left.+\frac{2}{c}I\partial_{\tau z}^2\Phi-I\Delta_y\Phi-\frac{1}{c^2}\partial_\tau J\partial_\tau\Phi-\frac{1}{c^2}J\partial_\tau^2\Phi\right]\\&+\varepsilon^4\left[-\partial_zI\partial_z\Phi-I\partial_z^2\Phi+\frac{1}{c}\partial_zJ\partial_\tau J+\frac{1}{c}\partial_\tau J\partial_z\Phi\right.\\&\left.-\nabla_yJ.\nabla_y\Phi+\frac{2}{c}J\partial_{\tau z}^2\Phi-J\Delta_y\Phi\right]+\varepsilon^5[-\partial_zJ\partial_z\Phi-J\partial_z^2\Phi];\end{aligned}$$

Table 7.1. *Approximation results for models derived from Navier-Stokes and Euler systems*

	Kuznetsov		KZK		NPE	
	Navier-Stokes	Euler	Navier-Stokes	Euler	Navier-Stokes	Euler
Theorem	Theorem 3.3	Theorem 6.6	Theorem 4.3	Theorem 6.8	Theorem 5.4	Theorem 6.7
Ansatz	$\rho_\varepsilon = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2$, $\mathbf{v}_\varepsilon = -\varepsilon\nabla u$, $\rho_1 = \frac{\rho_0}{c^2}\partial_t u$, ρ_2 from (3.6)		paraxial approximation $u = \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon}\mathbf{x}')$ $\rho_\varepsilon = \rho_0 + \varepsilon I + \varepsilon^2 J$, $\mathbf{v}_\varepsilon$ from (4.16), $I = \frac{\rho_0}{c^2}\partial_\tau\Phi$, J from (4.6)		paraxial approximation $u = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon}\mathbf{x}')$ $\rho_\varepsilon = \rho_0 + \varepsilon\xi + \varepsilon^2\chi$, $\mathbf{v}_\varepsilon$ from (5.3), $\xi = -\frac{\rho_0}{c}\partial_z\Psi$, χ from (5.6)	
Models	$\partial_t^2 u - c^2\Delta u = \varepsilon\partial_t\left((\nabla u)^2 + \frac{\gamma-1}{2c^2}(\partial_t u)^2 + \frac{\nu}{\rho_0}\Delta u\right)$		$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\Delta_y I = 0$		$\partial^2_{\tau z}\xi + \frac{(\gamma+1)c}{4\rho_0}\partial^2_z(\xi^2) - \frac{\nu}{2\rho_0}\partial^3_z\xi + \frac{c}{2}\Delta_y\xi = 0$	
Approximation Order	$O(\varepsilon^3)$					
Domain Ω	$\mathbb{R}^3$		the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$	the cone $\{\lvert x_1\rvert < \frac{R}{\varepsilon} - ct\} \times \mathbb{R}^{n-1}_{x'}$	$\mathbb{T}_{x_1} \times \mathbb{R}^2$	
Approximation	$\lVert U_\varepsilon - \overline{U}_\varepsilon\rVert_{L^2} \le \varepsilon$ for $t \le \frac{T}{\varepsilon}$					
Initial data regularity	$u_0 \in H^5(\Omega)$ $u_1 \in H^4(\Omega)$	$u_0 \in H^4(\Omega)$ $u_1 \in H^3(\Omega)$	$I_0 \in H^{10}(\Omega)$	$I_0 \in H^{10}(\Omega)$	$\xi_0 \in H^5(\Omega)$	$\xi_0 \in H^5(\Omega)$
Data regularity for remainder boundedness	$u_0 \in H^{s+2}(\Omega)$ $u_1 \in H^{s+1}(\Omega)$ $s > \frac{n}{2}$	$u_0 \in H^{s+2}(\Omega)$ $u_1 \in H^{s+1}(\Omega)$ $s > \frac{n}{2}$	$I_0 \in H^8(\Omega)$	$I_0 \in H^6(\Omega)$	$\xi_0 \in H^4(\Omega)$	$\xi_0 \in H^4(\Omega)$

among the x_1 axis

$$\begin{aligned}
&\varepsilon^3\mathbf{R}_2^{NS-KZK}.\overrightarrow{e}_1\\
=\ &\varepsilon^3\left[-\frac{\rho_0}{2c}\partial_\tau[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]-\frac{\nu}{c}\partial_\tau[-\frac{2}{c}\partial^2_{\tau z}\Phi+\Delta_y\Phi]\right.\\
&\left.-\frac{I}{2c}\partial_\tau[\frac{1}{c^2}(\partial_\tau\Phi)^2]+\frac{J}{c}\partial^2_\tau\Phi\right]\\
&+\varepsilon^4\left[\frac{\rho_0}{2}\partial_z[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]+\nu\partial_z[-\frac{2}{c}\partial^2_{\tau z}\Phi+\Delta_y\Phi]\right.\\
&-\frac{I}{2c}\partial_\tau[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]+\frac{I}{2}\partial_z[\frac{1}{c^2}(\partial_\tau\Phi)^2]-J\partial^2_{\tau z}\Phi\\
&\left.-\frac{J}{2c}\partial_\tau[\frac{1}{c^2}(\partial_\tau\Phi)^2]-\frac{\rho_0}{2c}\partial_\tau[(\partial_z\Phi)^2]-\frac{\nu}{c}\partial_\tau\partial^2_z\Phi\right]\\
&+\varepsilon^5\left[-\frac{I}{2c}\partial_\tau[(\partial_z\Phi)^2]+\frac{I}{2}\partial_z[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]\right.\\
&+\frac{J}{2}\partial_z[\frac{1}{c^2}(\partial_\tau\Phi)^2]-\frac{J}{2c}\partial_\tau[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]\\
&\left.+\frac{\rho_0}{2}\partial_z[(\partial_z\Phi)^2]+\nu\partial^3_z\Phi\right]\\
&+\varepsilon^6\left[\frac{I}{2}\partial_z[(\partial_z\Phi)^2]-\frac{J}{2c}\partial_\tau[(\partial_z\Phi)^2]+\frac{J}{2}[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]\right]+\varepsilon^7\left[\frac{J}{2}\partial_z[(\partial_z\Phi)^2]\right]
\end{aligned}$$

and in the hyperplane orthogonal to the x_1 axis

$$\begin{aligned}
&\sum_{i=2}^n(\mathbf{R}_2^{NS-KZK}.\overrightarrow{e}_i)\overrightarrow{e}_i\\
=\ &\varepsilon^{\frac{7}{2}}\left[\frac{\rho_0}{2}\nabla_y[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]+\nu\nabla_y[-\frac{2}{c}\partial^2_{\tau z}\Phi+\Delta_y\Phi]\right.\\
&\left.+\frac{I}{2}\nabla_y[\frac{1}{c^2}(\partial_\tau\Phi)^2]-J\nabla_y[\partial_\tau\Phi]\right]\\
&+\varepsilon^{\frac{9}{2}}\left[\frac{I}{2}\nabla_y[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]+\frac{J}{2}\nabla_y[\frac{1}{c^2}(\partial_\tau\Phi)^2]\right.\\
&\left.+\frac{\rho_0}{2}\nabla_y[(\partial_z\Phi)^2]+\nu\nabla_y[\partial^2_z\Phi]\right]\\
&+\varepsilon^{\frac{11}{2}}\left[\frac{I}{2}\nabla_y[(\partial_z\Phi)^2]+\frac{J}{2}\nabla_y[-\frac{2}{c}\partial_z\Phi\partial_\tau\Phi+(\nabla_y\Phi)^2]\right]+\varepsilon^{\frac{13}{2}}\left[\frac{J}{2}\nabla_y[(\partial_z\Phi)^2]\right].
\end{aligned}$$

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