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ON A NONLOCAL DIFFERENTIAL EQUATION DESCRIBING ROOTS OF POLYNOMIALS UNDER DIFFERENTIATION*

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Abstract. In this work we study the nonlocal transport equation derived recently by Steinerberger [Proc. Amer. Math. Soc., 147(11):4733–4744, 2019]. When this equation is considered on the real line, it describes how the distribution of roots of a polynomial behaves under iterated differentiation of the function. This equation can also be seen as a nonlocal fast diffusion equation. In particular, we study the well-posedness of the equation, establish some qualitative properties of the solution and give conditions ensuring the global existence of both weak and strong solutions. Finally, we present a link between the equation obtained by Steinerberger and a one-dimensional model of the surface quasi-geostrophic equation used by Chae et al. [Adv. Math., 194(1):203–223, 2005].

Keywords. Nonlocal fast diffusion; Global existence; Maximum principle.

AMS subject classifications. 35K65; 35K55; 35B65; 35S30.

1. Introduction and main results

In this paper, we consider the following one-dimensional nonlinear transport equation

$$\partial_t u+\partial_x \arctan\left(\frac{Hu}{u}\right)=0 \qquad (x,t) \text{ on } \mathbb{S}\times[0,T], \tag{1.1}$$

where

$$Hu(x,t)=\frac{1}{2\pi}\text{p.v.}\int_{\mathbb{S}}\frac{u(y,t)}{\tan\left(\frac{x-y}{2}\right)}dy,$$

is the periodic Hilbert transform and $\mathcal{S}$ is the one-dimensional circle (or, equivalently, the interval $[-\pi,\pi]$ with periodic boundary conditions). The previous equation needs to be supplemented with the initial data

$$u(x,0)=u_0(x). \tag{1.2}$$

This equation (when posed on the real line $\mathbb{R}$) has been derived by S. Steinerberger [33] when studying how the distribution of roots behaves under iterated differentiation. Besides the derivation of Equation (1.1), Steinerberger also found certain explicit solutions. Furthermore, Steinerberger obtained that the arcsine distribution

$$U(x)=\frac{C}{\sqrt{1-x^2}}\mathbf{1}_{|x|<1}, \tag{1.3}$$

is a steady state when considering the equation only in the interval $(-1,1)$. We observe that this steady state is not smooth because the derivative blows up at the boundary of its support. Whether such singularities occur in the evolution problem (1.1) is an interesting open question.

The purpose of this work is to study the properties of the transport Equation (1.1) and, in particular, to obtain global-in-time results that exclude the formation in

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finite time of such singularities. The study of nonlocal and nonlinear one-dimensional equations is a wide research area with a large literature. For other similar equations and related results we refer to [5, 6, 9, 13, 14, 24, 15, 16, 25, 27, 30, 26].

In this paper we prove the following results: first we establish the local existence of smooth solution emanating from smooth initial data together with some qualitative properties of such solutions. Even if this is a rather basic result, the nonlinearity of the equation makes it non-trivial.

THEOREM 1.1. *Let $0<u_0\in H^2(\mathbb{S})$ be the initial data. Then there exists a time $0<T\leq\infty$, $T=T(\|u_0\|_{H^2},\min_x u_0(x))$ and a unique positive classical solution to* (1.1)

$$0<u\in C([0,T],H^2(\mathbb{S})).$$

Furthermore, this solution verifies the following properties:

- $$\|u(t)\|_{L^2}^2+\int_0^t \mathcal{D}[u(s)]ds=\|u_0\|_{L^2}^2. \tag{1.4}$$

  where

  $$\mathcal{D}[u(t)]=\frac{1}{4\pi}\int_{\mathbb{S}} p.v.\int_{\mathbb{S}} \frac{u(x,t)-u(y,t)}{\sin\left(\frac{x-y}{2}\right)^2}\log\left(\frac{u(x,t)^2+(Hu(x,t))^2}{u(y,t)^2+(Hu(y,t))^2}\right)dxdy,$$

- $\|u(t)\|_{L^1}=\|u_0\|_{L^1}$,
- *if* $u_0(x)$ *is even,* $u(x,t)$ *is even for all* $t\geq 0$,
- *Maximum principle:* $\max_x u(x,t)\leq \max_x u_0(x)$,
- *Minimum principle:* $\min_x u_0(x)\leq \min_x u(x,t)$.

REMARK 1.1. We remark that, for an arbitrary $u(x,t)$, we are not able to give a sign to $\mathcal{D}[u(t)]$ (compare with [1, 19]). In other words, we are not able to show whether the L^2 norm decays.

Before stating the rest of our results, we observe that, in the case where the domain is a re-scaled torus or the whole real line, there is a one-parameter family of scale invariant transformations for this equation. Indeed, if $u(x,t)$ is a solution to (1.1), then

$$u_\lambda(x,t)=\lambda^\alpha u(\lambda x,\lambda^{1-\alpha}t)\,\forall\alpha\in\mathbb{R}, \tag{1.5}$$

is also a solution. In other words, the previous scaling is invariant for the equation.

In addition, under certain restrictions in the Wiener spaces $\dot{A}^s$ (see (1.7) for the definition), we can ensure that the solution is global. Although this size constraint may be seen as very restrictive, we think it is neeeded in order that the solution remains smooth. In particular, we observe the fact that the solution approaching the homogeneous steady state in the space

$$L^\infty(0,T;A^1)$$

excludes two main scenarios. On the one hand, due to the size restriction and the decay, the solution cannot vanish. On the other hand the derivative of the solution cannot blow up. In other words, even if we cannot prove it (and so far this remains an open question), we somehow expect finite time singularity for certain vanishing initial data with large initial slope. The precise statement of the theorem is

Theorem 1.2. *Let* $0<u_0\in H^2(\mathbb{S})$ *be the initial data and denote*

$$\langle u_0\rangle=\frac{1}{2\pi}\int_{\mathbb{S}}u_0(x)dx.$$

There exists $0<\mathcal{C}$ *such that if*

$$\frac{\|u_0-\langle u_0\rangle\|_{\dot{A}^1}}{\langle u_0\rangle}\leq\mathcal{C},$$

where the Wiener space $\dot{A}^1$ *is defined in* (1.7), *then the solution (from Theorem 1.1) is global and satisfies*

$$\|u(t)-\langle u_0\rangle\|_{\dot{A}^1}\leq\|u_0-\langle u_0\rangle\|_{\dot{A}^1}e^{-\delta t}$$

for certain $0<\delta(\langle u_0\rangle)$ *small enough.*

Remark 1.2. The explicit lower bound $0.13<\mathcal{C}$ is obtained as a byproduct.

We observe that the

$$L^\infty(0,\infty;\dot{A}^1)$$

norm is also invariant by the scaling

$$u_\lambda(x,t)=\frac{1}{\lambda}u(\lambda x,\lambda^2 t).$$

This scaling corresponds to the scaling of the Equation (1.5) when $\alpha=-1$. As a consequence, $\dot{A}^1$ is a (scaling) critical space which makes Theorem 1.2 a global existence result in a critical space.

Finally, we study the existence of weak solutions *i.e.* solutions that satisfy the equation in the following sense:

$$-\int_0^T\int_{\mathbb{S}}u(x,s)\partial_t\phi(x,s)+\arctan\left(\frac{Hu(x,s)}{u(x,s)}\right)\partial_x\phi(x,s)dxds=\int_{\mathbb{S}}u_0(x)\phi(x,0)dx,$$

for all test functions $\phi\in C^\infty(\mathbb{S}\times[0,T))$.

In that regards, we prove the global existence of weak solution for initial data satisfying certain size conditions in a critical space:

Theorem 1.3. *Let* $0<u_0\in A^0(\mathbb{S})$ *be the initial data and denote*

$$\langle u_0\rangle=\frac{1}{2\pi}\int_{\mathbb{S}}u_0(x)dx.$$

There exists $0<\tilde{\mathcal{C}}$ *such that if*

$$\frac{\|u_0-\langle u_0\rangle\|_{A^0}}{\langle u_0\rangle}\leq\tilde{\mathcal{C}},$$

where the Wiener space A^0 *is defined in* (1.7), *then there exists a unique global weak solution*

$$u\in C([0,T],A^0)\cap L^1(0,T;A^1),\forall 0<T<\infty$$

and this solution satisfies

$$\|u(t)-\langle u_0\rangle\|_{C^0}\leq \|u_0-\langle u_0\rangle\|_{A^0}e^{-\delta t}$$

for certain $0<\delta(\langle u_0\rangle)$ *small enough.*

REMARK 1.3. The explicit lower bound $0.24<\tilde{\mathcal{C}}$ is obtained as a byproduct.

We observe that the

$$L^\infty(0,\infty;A^0)$$

norm is also invariant by the scaling

$$u_\lambda(x,t)=u(\lambda x,\lambda t).$$

This scaling corresponds to the scaling of the Equation (1.5) when $\alpha=0$. As a consequence, A^0 is a (scaling) critical space. Thus, Theorem 1.3 is a global existence result in a critical space.

REMARK 1.4. Similar results can be proved for the nonlocal fast diffusion equation (see also [32, 31, 3])

$$\partial_t u+\partial_x\left(\frac{Hu}{u^m}\right)=0,\, m\in\mathbb{N}.$$

The rest of the paper is devoted to the proofs of the results (Sections 2-4) and the link between (1.1) and the equation

$$\partial_t g+\Lambda g=\partial_x(gHg), \tag{1.6}$$

(see Section 5). We would like to remark that (1.6) was proposed as a one-dimensional model of the 2D surface quasi-geostrophic equation by Chae, Córdoba, Córdoba and Fontelos [7] (see also the papers by Matsuno [29] and Baker, Li and Morlet [2]). It is well-known that the solutions of (1.6) blow up in finite time for certain vanishing initial data [5]. The (so far formal) link between (1.1) and (1.6) suggests the possible occurence of finite time blow up for (1.1) for solutions corresponding to vanishing initial data. To the best of the author's knowledge, this question is still open and should be the object of a future study elsewhere.

Notation. We denote

$$\Lambda u=H\partial_x u(x)=\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{S}}\frac{u(x)-u(x-y)}{\sin^2(y/2)}dy.$$

We define the (homogeneous) L^2-based Sobolev spaces

$$\dot{H}^s=\left\{u(x)=\sum_{n\in\mathbb{Z}}\hat{u}(n)e^{inx}\text{ with }\sum_{n\in\mathbb{Z}}|n|^{2s}|\hat{u}(n)|^2<\infty\right\},$$

with norm $\|u\|_{\dot{H}^s}=\|\Lambda^s u\|_{L^2}$. The standard non-homogeneous Sobolev spaces are then

$$H^s=\left\{u(x)=\sum_{n\in\mathbb{Z}}\hat{u}(n)e^{inx}\text{ with }\sum_{n\in\mathbb{Z}}(1+|n|^{2s})|\hat{u}(n)|^2<\infty\right\}.$$

Similarly, we recall the definition of the (homogeneous) Wiener spaces

$$\dot{A}^s=\left\{u(x)=\sum_{n\in\mathbb{Z}}\hat{u}(n)e^{inx}\text{ with }\sum_{n\in\mathbb{Z}}|n|^s|\hat{u}(n)|<\infty\right\}. \tag{1.7}$$

with norm $\|u\|_{\dot{A}^s}=\|\widehat{\Lambda^s u}\|_{\ell^1}$. The standard non-homogeneous Wiener spaces are then

$$A^s=\left\{u(x)=\sum_{n\in\mathbb{Z}}\hat{u}(n)e^{inx}\text{ with }\sum_{n\in\mathbb{Z}}(1+|n|^s)|\hat{u}(n)|<\infty\right\}.$$

2. Proof of Theorem 1.1

Well-posedness. The existence will follow using the energy method [28, Chapter 3] once the appropriate *a priori* estimates are obtained. We define the energy

$$\mathcal{E}(t)=\frac{1}{\min_x u(x,t)}+\|u(t)\|_{H^2}. \tag{2.1}$$

We have to prove an inequality of the type

$$\frac{d}{dt}\mathcal{E}(t)\leq C(1+\mathcal{E}(t))^p,$$

for certain C and p.

To estimate the first term in the energy we use a pointwise argument (see [12, 10, 21, 4] for more details). The solution has at least a minimum:

$$m(t)=\min_x u(x,t)=u(\underline{x}_t,t).$$

Because of the positivity of the initial data, we have that $m(0)>0$. Following the argument in [12, 21, 4], we have that

$$\frac{d}{dt}m(t)=\partial_t u(\underline{x}_t,t)=-\frac{m(t)\Lambda u(\underline{x}_t,t)}{m(t)^2+(Hu(\underline{x}_t,t))^2}\text{ a.e..}$$

Then,

$$\frac{d}{dt}\frac{1}{\min_x u(x,t)}=-\frac{\partial_t u(\underline{x}_t,t)}{m(t)^2}\leq C\frac{\|u\|_{H^2}}{m(t)^3}\leq C(\mathcal{E}(t))^4 \tag{2.2}$$

For the sake of brevity we only provide with the estimates for the higher order terms (the lower order terms being easier). We take two derivatives of the equation and test against $\partial_x^2 u$. We find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\dot{H}^2}^2=&-\int_{\mathbb{S}}\frac{(\partial_x^2u\Lambda u+u\Lambda\partial_x^2u)\partial_x^2u}{u^2+(Hu)^2}dx+\int_{\mathbb{S}}\frac{u\Lambda u(2u\partial_x^2u+2Hu\partial_x\Lambda u)\partial_x^2u}{(u^2+(Hu)^2)^2}dx\\&+\int_{\mathbb{S}}\frac{(\partial_x\Lambda u\partial_x u+Hu\partial_x^3u)\partial_x^2u}{u^2+(Hu)^2}dx-\int_{\mathbb{S}}\frac{Hu\partial_x u(2u\partial_x^2u+2Hu\Lambda\partial_x u)\partial_x^2u}{(u^2+(Hu)^2)^2}dx+\mathrm{R}\\=&I_1+I_2+I_3+I_4+\mathrm{R},\end{aligned}$$

with

$$I_1=-\int_{\mathbb{S}}\frac{(\partial_x^2u\Lambda u+u\Lambda\partial_x^2u)\partial_x^2u}{u^2+(Hu)^2}dx,$$

$$I_2 = \int_{\mathbb{S}} \frac{u\Lambda u(2u\partial_x^2 u + 2Hu\partial_x\Lambda u)\partial_x^2 u}{(u^2+(Hu)^2)^2} dx,$$

$$I_3 = \int_{\mathbb{S}} \frac{(\partial_x\Lambda u\partial_x u + Hu\partial_x^3 u)\partial_x^2 u}{u^2+(Hu)^2} dx,$$

$$I_4 = -\int_{\mathbb{S}} \frac{Hu\partial_x u(2u\partial_x^2 u + 2Hu\Lambda\partial_x u)\partial_x^2 u}{(u^2+(Hu)^2)^2} dx,$$

and R being either lower order terms or terms akin to I_2 and I_4. Using that

$$\left\|\frac{1}{u^2+(Hu)^2}\right\|_{L^\infty} \leq \left\|\frac{1}{u^2}\right\|_{L^\infty} \leq \frac{1}{m(t)^2} \leq \mathcal{E}(t)^2,$$

together with Hölder and Sobolev inequalities and interpolation between Sobolev spaces, we obtain that

$$\text{l.o.t.} \leq C(\mathcal{E}(t))^8 \|u(t)\|_{H^2}. \tag{2.3}$$

We recall the Córdoba-Córdoba inequality [11]

$$\theta\Lambda\theta \geq \frac{1}{2}\Lambda(\theta^2),$$

to find that

$$\int_{\mathbb{S}} \frac{u\Lambda\partial_x^2 u\partial_x^2 u}{u^2+(Hu)^2} dx \geq \frac{1}{2}\int_{\mathbb{S}} H\partial_x\left(\frac{u}{u^2+(Hu)^2}\right)(\partial_x^2 u)^2 dx.$$

Thus,

$$-\int_{\mathbb{S}} \frac{u\Lambda\partial_x^2 u\partial_x^2 u}{u^2+(Hu)^2} dx \leq -\frac{1}{2}\int_{\mathbb{S}} H\partial_x\left(\frac{u}{u^2+(Hu)^2}\right)(\partial_x^2 u)^2 dx$$

and we have that

$$\begin{aligned} I_1 &\leq \frac{\|u(t)\|_{H^2}^2\|u\|_{\dot{A}^1}}{m(t)^2} - \frac{1}{2}\int_{\mathbb{S}} H\partial_x\left(\frac{u}{u^2+(Hu)^2}\right)(\partial_x^2 u)^2 dx \\ &\leq \frac{\|u(t)\|_{H^2}^2\|u\|_{\dot{A}^1}}{m(t)^2} + \frac{1}{2}\frac{\|u(t)\|_{H^2}^2\|u\|_{\dot{A}^1}}{m(t)^2} + \frac{\|u(t)\|_{H^2}^2\|u\|_{A^0}^2\|u\|_{\dot{A}^1}}{m(t)^4} \\ &\leq C(1+\mathcal{E}(t))^8\|u(t)\|_{H^2}. \end{aligned}$$

Similarly, using

$$\|Hf\|_{L^\infty} + \|f\|_{L^\infty} \leq 2\|f\|_{A^0} \leq C\|f\|_{H^1},$$

together with Hölder inequality and the Sobolev embedding we have that

$$I_2 \leq C\|u(t)\|_{H^2}\mathcal{E}(t)^8.$$

We observe that I_4 can be estimated as before and we find that

$$I_2 + I_4 \leq C(1+\mathcal{E}(t))^8\|u(t)\|_{H^2}.$$

Finally, we observe that an integration by parts allows us to write

$$I_3 = \int_{\mathbb{S}} \frac{\partial_x \Lambda u \partial_x u \partial_x^2 u}{u^2+(Hu)^2} dx - \int_{\mathbb{S}} \partial_x \left(\frac{Hu}{u^2+(Hu)^2} \right) \frac{(\partial_x^2 u)^2}{2} dx.$$

From here it is possible to conclude that

$$I_3 \leq C(1+\mathcal{E}(t))^8 \|u(t)\|_{H^2}.$$

Then, we obtain that

$$\frac{d}{dt}\|u(t)\|_{H^2}^2 \leq C(1+\mathcal{E}(t))^8 \|u(t)\|_{H^2}. \tag{2.4}$$

Thus, collecting (2.2), (2.3) and (2.4), we conclude the desired inequality

$$\frac{d}{dt}\mathcal{E}(t) \leq C(1+\mathcal{E}(t))^8.$$

Once the energy $\mathcal{E}$ is bounded, we would like to remark that, besides remaining in H^2, the solution is positive.

From here we only have to appropriately mollify the equation as in [28, Chapter 3] to obtain approximate problems having local-in-time existence. In particular, being

$$\mathcal{J}_\epsilon(x)$$

the periodic heat kernel at time $t=\epsilon$, we define

$$\partial_t u^\epsilon + \mathcal{J}_\epsilon * \partial_x \arctan\left(\frac{H\mathcal{J}_\epsilon * u^\epsilon}{\epsilon + \mathcal{J}_\epsilon * u^\epsilon} \right) = 0 \qquad (x,t) \text{ on } \mathbb{S}\times[0,T]. \tag{2.5}$$

To ensure a common lifespan for these approximate problems we just invoke the previous energy estimates. The final step is to obtain that the sequence of approximating problems u^ϵ is Cauchy in the low order norm

$$L^\infty(0,T;L^2)$$

(see [28, Chapter 3] for further details). This concludes with the existence part. The uniqueness follows from a standard contradiction argument together with the smoothness and positivity of the approximate solutions.

An identity for the evolution of the L^2 norm. We test (1.1) against u. We find that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = \int_{\mathbb{S}} -\frac{u^2 \Lambda u}{u^2+(Hu)^2} + \frac{uHu\partial_x u}{u^2+(Hu)^2} dx.$$

We compute

$$\begin{aligned}
\mathcal{D}[u(t)] &= -\frac{1}{2}\int_{\mathbb{S}} Hu \partial_x \log\left(u^2+(Hu)^2\right) dx \\
&= -\int_{\mathbb{S}} Hu \frac{u\partial_x u + Hu\Lambda u}{u^2+(Hu)^2} dx \\
&= -\int_{\mathbb{S}} \frac{u\partial_x u Hu}{u^2+(Hu)^2} - \frac{u^2 \Lambda u}{u^2+(Hu)^2} + \Lambda u dx
\end{aligned}$$

$$=-\int_{\mathbb{S}}\frac{u\partial_x uHu}{u^2+(Hu)^2}-\frac{u^2\Lambda u}{u^2+(Hu)^2}dx.$$

As a consequence,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2=-\mathcal{D}[u(t)].$$

Furthermore,

$$\begin{aligned}\mathcal{D}[u(t)]&=\frac{1}{2}\int_{\mathbb{S}}\Lambda u\log\left(u^2+(Hu)^2\right)dx\\&=\frac{1}{8\pi}\int_{\mathbb{S}}\text{p.v.}\int_{\mathbb{S}}\frac{u(x)-u(x-y)}{\sin(y/2)^2}\log\left(u(x)^2+(Hu(x))^2\right)dxdy\\&=\frac{1}{8\pi}\int_{\mathbb{S}}\text{p.v.}\int_{\mathbb{S}}\frac{u(x)-u(y)}{\sin((x-y)/2)^2}\log\left(u(x)^2+(Hu(x))^2\right)dxdy\\&=\frac{1}{8\pi}\int_{\mathbb{S}}\text{p.v.}\int_{\mathbb{S}}\frac{u(y)-u(x)}{\sin((x-y)/2)^2}\log\left(u(y)^2+(Hu(y))^2\right)dxdy.\end{aligned}$$

Then, we have identity (1.4).

Propagation of the L^1 norm. Once the solution remains positive, the L^1 norm is preserved due to the divergence form of the equation.

Propagation of the even symmetry. This is a straightforward consequence of the fact that the Hilbert transform H maps even functions into odd functions.

Maximum principle. We define

$$M(t)=\max_x u(x,t)=u(\overline{x}_t,t)$$

Then (see [18, 21, 4] for more details) we have that

$$\frac{d}{dt}M(t)=\partial_t u(\overline{x}_t,t)\ \text{ a.e.},$$

Then

$$\frac{d}{dt}M(t)+\frac{M(t)\Lambda u(\overline{x}_t)}{M(t)^2+(Hu(\overline{x}_t))^2}=0,$$

We observe that

$$\Lambda u(\overline{x}_t)\geq 0.$$

Thus, using $0\leq M(t)$, we obtain that

$$M(t)\leq M(0).$$

Minimum principle. With the previous definition for $m(t)$, we have that

$$\frac{d}{dt}m(t)+\frac{m(t)\Lambda u(\underline{x}_t)}{m(t)^2+(Hu(\underline{x}_t))^2}=0.$$

Thus, using $\Lambda u(\underline{x}_t,t)\leq 0$, we find that

$$0<m(0)\leq m(t).$$

3. Proof of Theorem 1.2

The proof of this theorem follows the approach in [20]. We define the new variable

$$v(x,t)=u(x,t)-\langle u_0\rangle.$$

This variable quantifies the difference between the steady state $u_\infty=\langle u_0\rangle$ and u. The idea of the theorem is first to linearize around the stady state $\langle u_0\rangle$. Secondly, we obtain an inequality of the form

$$\frac{d}{dt}\|v(t)\|_{\dot{A}^1}+\frac{\|v(t)\|_{\dot{A}^2}}{\langle u_0\rangle}\leq\mathcal{F}\left(\frac{\|v(t)\|_{\dot{A}^1}}{\langle u_0\rangle}\right)\frac{\|v(t)\|_{\dot{A}^2}}{\langle u_0\rangle},$$

with $\mathcal{F}(0)=0$ and $\mathcal{F}$ continuous. We observe that this inequality guarantees $v(t)\to 0$ in $\dot{A}^1$ for small enough $\|v_0\|_{\dot{A}^1}/\langle u_0\rangle$.

In what follows we assume that

$$r=\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle}<\frac{1}{2},$$

so that

$$\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle-\|v\|_{\dot{A}^1}}=\frac{r}{1-r}<1.$$

Since we have the following Poincaré-type inequality

$$\|v\|_{\dot{A}^s}\leq\|v\|_{\dot{A}^r},\ \forall 0\leq s<r,$$

we observe that

$$\left|\frac{Hu}{u}\right|=\left|\frac{Hv}{u}\right|\leq\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\leq\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle-\|v\|_{\dot{A}^1}}<1.$$

As a consequence, we can expand the nonlinearity as a power series

$$\arctan\left(\frac{Hu}{u}\right)=\sum_{n\in\mathbb{Z}^+\cup\{0\}}\frac{(-1)^n}{1+2n}\left(\frac{Hu}{u}\right)^{1+2n},$$

so

$$\partial_t u=-\sum_{n\in\mathbb{Z}^+\cup\{0\}}(-1)^n\left(\frac{Hu}{u}\right)^{2n}\left(\frac{\Lambda u}{u}-\frac{Hu\partial_x u}{u^2}\right).$$

In the new variable, this latter equation reads

$$\begin{aligned}\partial_t v=&-\sum_{n\in\mathbb{Z}^+}(-1)^n\left(\frac{Hv}{v+\langle u_0\rangle}\right)^{2n}\left(\frac{\Lambda v}{v+\langle u_0\rangle}-\frac{Hv\partial_x v}{(v+\langle u_0\rangle)^2}\right)\\&-\left(\frac{\Lambda v}{v+\langle u_0\rangle}-\frac{Hv\partial_x v}{(v+\langle u_0\rangle)^2}\right).\end{aligned}$$

We recall the following Taylor series

$$\frac{1}{\langle u_0\rangle+v}=\frac{1}{\langle u_0\rangle}+\frac{1}{\langle u_0\rangle}\sum_{n\in\mathbb{Z}^+}(-1)^n\left(\frac{v}{\langle u_0\rangle}\right)^n,$$

$$\frac{1}{(\langle u_0\rangle+v)^2}=\frac{1}{\langle u_0\rangle^2}+\frac{1}{\langle u_0\rangle^2}\sum_{n\in\mathbb{Z}^+}(-1)^n(1+n)\left(\frac{v}{\langle u_0\rangle}\right)^n.$$

We define

$$\mathfrak{S}_1=\frac{Hv}{\langle u_0\rangle}+\frac{Hv}{\langle u_0\rangle}\sum_{m\in\mathbb{Z}^+}(-1)^m\left(\frac{v}{\langle u_0\rangle}\right)^m$$
$$\mathfrak{S}_2=\frac{Hv\partial_x v}{\langle u_0\rangle^2}+\frac{Hv\partial_x v}{\langle u_0\rangle^2}\sum_{m\in\mathbb{Z}^+}(-1)^m(1+m)\left(\frac{v}{\langle u_0\rangle}\right)^m$$
$$\mathfrak{S}_3=\frac{\Lambda v}{\langle u_0\rangle}\sum_{m\in\mathbb{Z}^+}(-1)^m\left(\frac{v}{\langle u_0\rangle}\right)^m.$$

Using the previous Taylor series together with the previous definitions, we find that

$$\begin{aligned}\partial_t v+\frac{\Lambda v}{\langle u_0\rangle}=&-\sum_{n\in\mathbb{Z}^+}(-1)^n(\mathfrak{S}_1)^{2n}\left(\frac{\Lambda v}{\langle u_0\rangle}+\mathfrak{S}_3\right)\\&+\sum_{n\in\mathbb{Z}^+}(-1)^n(\mathfrak{S}_1)^{2n}\mathfrak{S}_2-\mathfrak{S}_3+\mathfrak{S}_2.\end{aligned}\tag{3.1}$$

We take a derivative of (3.1) to obtain that

$$\begin{aligned}\partial_t\partial_x v+\frac{\Lambda\partial_x v}{\langle u_0\rangle}=&\sum_{n\in\mathbb{Z}^+}(-1)^{n+1}2n(\mathfrak{S}_1)^{2n-1}\partial_x\mathfrak{S}_1\left(\frac{\Lambda v}{\langle u_0\rangle}+\mathfrak{S}_3\right)\\&-\sum_{n\in\mathbb{Z}^+}(-1)^n(\mathfrak{S}_1)^{2n}\left(\frac{\Lambda\partial_x v}{\langle u_0\rangle}+\partial_x\mathfrak{S}_3\right)\\&+\sum_{n\in\mathbb{Z}^+}(-1)^n 2n(\mathfrak{S}_1)^{2n-1}\partial_x\mathfrak{S}_1\mathfrak{S}_2+\sum_{n\in\mathbb{Z}^+}(-1)^n(\mathfrak{S}_1)^{2n}\partial_x\mathfrak{S}_2\\&-\partial_x\mathfrak{S}_3+\partial_x\mathfrak{S}_2\end{aligned}\tag{3.2}$$

We want to estimate

$$\|v\|_{\dot{A}^1}=\|\partial_x v\|_{A^0}.$$

To do that we first observe that A^0 is an algebra, thus,

$$\|\mathfrak{S}_1^{2n}\|_{A^0}\leq\|\mathfrak{S}_1\|_{A^0}^{2n}\leq\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle}+\frac{\|v\|_{A^0}}{\langle u_0\rangle}\sum_{m\in\mathbb{Z}^+}\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle}\right)^m\right)^{2n}.$$

Summing up the series, we find the estimate

$$\|\mathfrak{S}_1^{2n}\|_{A^0}\leq\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^{2n}.$$

Similarly,

$$\|\mathfrak{S}_1^{2n-1}\|_{A^0}\leq\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^{2n-1}$$

$$\|\mathfrak{S}_2\|_{A^0} \leq \frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{\langle u_0\rangle^2} + \frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{\langle u_0\rangle^2}\sum_{m\in\mathbb{Z}^+}(1+m)\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle}\right)^m \leq \frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{(\langle u_0\rangle - \|v\|_{A^0})^2}$$

$$\|\mathfrak{S}_3\|_{A^0} \leq \frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle}\sum_{m\in\mathbb{Z}^+}\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle}\right)^m \leq \frac{\|v\|_{\dot{A}^1}\|v\|_{A^0}}{\langle u_0\rangle(\langle u_0\rangle - \|v\|_{A^0})}$$

$$\|\partial_x\mathfrak{S}_1\|_{A^0} \leq \frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle - \|v\|_{A^0}} + \frac{\|v\|_{\dot{A}^1}\|v\|_{A^0}}{(\langle u_0\rangle - \|v\|_{A^0})^2}$$

$$\|\partial_x\mathfrak{S}_2\|_{A^0} \leq \frac{\|v\|_{\dot{A}^1}^2 + \|v\|_{A^0}\|v\|_{\dot{A}^2}}{(\langle u_0\rangle - \|v\|_{A^0})^2} + \frac{2\|v\|_{A^0}\|v\|_{\dot{A}^1}^2}{(\langle u_0\rangle - \|v\|_{A^0})^3}$$

$$\|\partial_x\mathfrak{S}_3\|_{A^0} \leq \frac{\|v\|_{\dot{A}^2}\|v\|_{A^0}}{\langle u_0\rangle(\langle u_0\rangle - \|v\|_{A^0})} + \frac{\|v\|_{\dot{A}^1}^2}{(\langle u_0\rangle - \|v\|_{A^0})^2}.$$

We obtain that

$$\begin{aligned}\frac{d}{dt}\|v\|_{\dot{A}^1} + \frac{\|v\|_{\dot{A}^2}}{\langle u_0\rangle} \leq \frac{\|v\|_{\dot{A}^2}}{\langle u_0\rangle}\Bigg\{&\sum_{n\in\mathbb{Z}^+}2n\left(\frac{r}{1-r}\right)^{2n-1}\left[\frac{r}{1-r}+\frac{r^2}{(1-r)^2}\right]\frac{1}{1-r}\\ &+\sum_{n\in\mathbb{Z}^+}\left(\frac{r}{1-r}\right)^{2n}\left(1+\frac{r}{1-r}+\frac{r}{(1-r)^2}\right)\\ &+\sum_{n\in\mathbb{Z}^+}2n\left(\frac{r}{1-r}\right)^{2n-1}\left[\frac{r}{1-r}+\frac{r^2}{(1-r)^2}\right]\frac{r}{(1-r)^2}\\ &+2\sum_{n\in\mathbb{Z}^+}\left(\frac{r}{1-r}\right)^{2n}\left(\frac{r}{(1-r)^2}+\frac{r^2}{(1-r)^3}\right)\\ &+\frac{r}{1-r}+\frac{r}{(1-r)^2}+2\left(\frac{r}{(1-r)^2}+\frac{r^2}{(1-r)^3}\right)\Bigg\}.\end{aligned} \tag{3.3}$$

Using

$$\frac{z^2}{1-z^2} = \sum_{n\in\mathbb{Z}^+} z^{2n},$$

we find

$$\begin{aligned}\frac{d}{dt}\|v\|_{\dot{A}^1} + \frac{\|v\|_{\dot{A}^2}}{\langle u_0\rangle} \leq \frac{\|v\|_{\dot{A}^2}}{\langle u_0\rangle}\Bigg\{&\frac{2\frac{r}{1-r}}{\left(1-\left(\frac{r}{1-r}\right)^2\right)^2}\left[\frac{r}{1-r}+\frac{r^2}{(1-r)^2}\right]\frac{1}{1-r}\\ &+\frac{\left(\frac{r}{1-r}\right)^2}{1-\left(\frac{r}{1-r}\right)^2}\left(1+\frac{r}{1-r}+\frac{r}{(1-r)^2}\right)\\ &+\frac{2\frac{r}{1-r}}{\left(1-\left(\frac{r}{1-r}\right)^2\right)^2}\left[\frac{r}{1-r}+\frac{r^2}{(1-r)^2}\right]\frac{r}{(1-r)^2}\\ &+2\frac{\left(\frac{r}{1-r}\right)^2}{1-\left(\frac{r}{1-r}\right)^2}\left(\frac{r}{(1-r)^2}+\frac{r^2}{(1-r)^3}\right)\\ &+\frac{r}{1-r}+\frac{3r}{(1-r)^2}+\frac{r^2}{(1-r)^3}\Bigg\}.\end{aligned} \tag{3.4}$$

Finally, we can simplify the previous expression and find that

$$\mathcal{F}(r)=\frac{2\frac{r}{1-r}}{\left(1-\left(\frac{r}{1-r}\right)^2\right)^2}\left(\frac{r}{(1-r)^3}+\frac{r}{(1-r)^4}\right)+\frac{\left(\frac{r}{1-r}\right)^2}{1-2r}$$
$$+2\frac{\left(\frac{r}{1-r}\right)^2}{1-\left(\frac{r}{1-r}\right)^2}\frac{1}{(1-r)^3}+\frac{r}{1-r}+\frac{3r}{(1-r)^2}+\frac{r^2}{(1-r)^3}.$$

We observe that $\mathcal{F}$ is a continuous function in a neighborhood of $r=0$ and satisfies $\mathcal{F}(0)=0$. Thus, there exists $0<\mathcal{C}$ such that $\mathcal{F}(\mathcal{C})<1$. We finally observe that if $\|v_0\|_{\dot{A}^1}/\langle u_0\rangle<\mathcal{C}$ this condition propagates in time and ensures the following bound

$$\|v(t)\|_{\dot{A}^1}\leq\|v_0\|_{\dot{A}^1}e^{-\delta t},$$

for small enough $0<\delta\ll 1$. This last inequality together with a close inspection of the energy estimates in Theorem 1.1 lead to the following inequality

$$\frac{d}{dt}\|u\|^2_{H^2}\leq C(u_0)\|u\|^2_{H^2},$$

and then we conclude the global bound for the H^2 norm using Grönwall's inequality.

4. Proof of Theorem 1.3

In this section we prove the existence of global weak solutions for certain initial data satisfying appropriate size restriction in the space A^0. We emphasize that this space is scale invariant with respect to the scaling of the equation. First we obtain a priori estimates, then we consider a vanishing viscosity approximation and prove the convergence of the approximate solutions.

***A priori* estimates.** Following the previous ideas, the first nonlinear term in (3.1) contributes with

$$\left\|\sum_{n\in\mathbb{Z}^+}(-1)^n(\mathfrak{S}_1)^{2n}\left(\frac{\Lambda v}{\langle u_0\rangle}+\frac{\Lambda v}{\langle u_0\rangle}\sum_{m\in\mathbb{Z}^+}(-1)^m\left(\frac{v}{\langle u_0\rangle}\right)^m\right)\right\|_{A^0}$$
$$\leq\sum_{n\in\mathbb{Z}^+}\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^{2n}\left(\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle-\|v\|_{A^0}}\right)$$
$$\leq\left(\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle-\|v\|_{A^0}}\right)\left(\frac{1}{1-\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^2}-1\right)$$
$$\leq\left(\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle-\|v\|_{A^0}}\right)\left(\frac{\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^2}{1-\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^2}\right).$$

The second nonlinear term in (3.1) can be estimated as

$$\left\|\sum_{n\in\mathbb{Z}^+}(-1)^n(\mathfrak{S}_1)^{2n}\left(\frac{Hv\partial_x v}{\langle u_0\rangle^2}+\frac{Hv\partial_x v}{\langle u_0\rangle^2}\sum_{m\in\mathbb{Z}^+}(-1)^m(1+m)\left(\frac{v}{\langle u_0\rangle}\right)^m\right)\right\|_{A^0}$$

$$\le \sum_{n\in\mathbb{Z}^+}\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^{2n}\left(\frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{(\langle u_0\rangle-\|v\|_{A^0})^2}\right)$$

$$\le\left(\frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{(\langle u_0\rangle-\|v\|_{A^0})^2}\right)\left(\frac{\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^2}{1-\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle-\|v\|_{A^0}}\right)^2}\right).$$

Finally, we find that

$$\begin{aligned}\|\mathfrak{S}_3\|_{A^0}&=\left\|\Lambda v\frac{1}{\langle u_0\rangle}\sum_{n\in\mathbb{Z}^+}(-1)^n\left(\frac{v}{\langle u_0\rangle}\right)^n\right\|_{A^0}\\&\le\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle}\sum_{n\in\mathbb{Z}^+}\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle}\right)^n\\&\le\|v\|_{\dot{A}^1}\left(\frac{1}{\langle u_0\rangle-\|v\|_{A^0}}-\frac{1}{\langle u_0\rangle}\right),\end{aligned}$$

$$\begin{aligned}\|\mathfrak{S}_2\|_{A^0}&=\left\|\left(\frac{Hv\partial_x v}{\langle u_0\rangle^2}+\frac{Hv\partial_x v}{\langle u_0\rangle^2}\sum_{n\in\mathbb{Z}^+}(-1)^n(1+n)\left(\frac{v}{\langle u_0\rangle}\right)^n\right)\right\|_{A^0}\\&\le\frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{\langle u_0\rangle^2}+\frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{\langle u_0\rangle^2}\sum_{n\in\mathbb{Z}^+}(1+n)\left(\frac{\|v\|_{A^0}}{\langle u_0\rangle}\right)^n\\&\le\frac{\|v\|_{A^0}\|v\|_{\dot{A}^1}}{(\langle u_0\rangle-\|v\|_{A^0})^2}.\end{aligned}$$

We define

$$s=\frac{\|v\|_{A^0}}{\langle u_0\rangle}.$$

Collecting the previous estimates, we find that

$$\begin{aligned}\frac{d}{dt}\|v\|_{A^0}+\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle}\le\frac{\|v\|_{\dot{A}^1}}{\langle u_0\rangle}\Bigg[&\frac{s}{(1-s)^2}+\frac{s}{1-s}+\left(\frac{s}{(1-s)^2}\right)\left(\frac{\left(\frac{s}{1-s}\right)^2}{1-\left(\frac{s}{1-s}\right)^2}\right)\\&+\left(\frac{1}{1-s}\right)\left(\frac{\left(\frac{s}{1-s}\right)^2}{1-\left(\frac{s}{1-s}\right)^2}\right)\Bigg]\end{aligned}$$

Using the hypotheses on $\tilde{\mathcal{C}}$, we conclude that

$$s\le\tilde{\mathcal{C}}$$

implies

$$\frac{d}{dt}\|v\|_{A^0}+\delta\|v\|_{\dot{A}^1}\le 0,$$

and that, thanks to a Poincaré-type inequality, leads to

$$\|v(t)\|_{A^0}\le\|v_0\|_{A^0}e^{-\delta t}.$$

Furthermore, the solution also enjoys the following parabolic gain of regularity

$$\int_0^t \|v(s)\|_{H^{0.5}}^2 ds \leq \int_0^t \|v(s)\|_{\dot{A}^1} ds \sup_s \|v(s)\|_{L^1} \leq \int_0^t \|v(s)\|_{\dot{A}^1} ds \|v_0\|_{A^0} 2\pi.$$

Approximated solutions. To construct the approximate solutions, we consider the following vanishing viscosity approximated problem

$$\partial_t u^\varepsilon + \partial_x \arctan\left(\frac{Hu^\varepsilon}{u^\varepsilon}\right) = \varepsilon \partial_x^2 u^\varepsilon \qquad (x,t) \text{ on } \mathbb{S}\times[0,T], \tag{4.1}$$

with a mollified initial data

$$u^\varepsilon(x,0) = \mathcal{M}_\varepsilon * u_0(x).$$

The corresponding approximate solution exists globally and remains smooth.

Compactness. Fix $0<T<\infty$. We have that u^ε is uniformly bounded in

$$L^\infty(0,T;A^0)\cap L^2(0,T;H^{0.5}).$$

This implies weak-* convergence

$$u^\varepsilon \overset{*}{\rightharpoonup} u,$$

in

$$L^\infty([0,T]\times\mathbb{S}),$$

and weak convergence

$$u^\varepsilon \rightharpoonup u,$$

in

$$L^2(0,T;H^{0.5}(\mathbb{S})).$$

Furthermore, $\partial_t u^\epsilon$ is uniformly bounded in

$$L^2(0,T;H^{-1.5}).$$

A standard application of Aubin-Lions Theorem [34] ensures the strong convergence (after maybe taking a subsequence)

$$u^\varepsilon \to u,\, Hu^\varepsilon \to Hu$$

in

$$L^2(0,T;L^2).$$

Taking another subsequence if necessary, we obtain that

$$u^\varepsilon(x,t)\to u(x,t) \text{ a.e in } \mathbb{S}\times[0,T]$$

In particular, we conclude the lower bound

$$\min_x u_0(x) \leq u(x,t) \text{ a.e in } \mathbb{S}\times[0,T].$$

Using that $\partial_t u^\epsilon$ is uniformly bounded in

$$L^1(0,T;A^0),$$

another application of Aubin-Lions compactness lemma (see [17] for further details), allows us to also ensure that

$$u^\varepsilon \to u,\, Hu^\varepsilon \to Hu$$

in

$$L^r(0,T;A^0)\cap L^q(0,T;A^{0.5}),\, 1\le r<\infty,\, 1\le q<2.$$

To conclude that the solution u is in fact a $C([0,T],A^0)$ function, we invoke Fatou lemma as in [17] to obtain that, for a small enough δ,

$$\|u(t)\|_{A^0}+\delta\int_0^t \|u(s)\|_{A^1}ds \le \liminf_{\varepsilon\to 0}\left(\|u^\varepsilon(t)\|_{A^0}+\delta\int_0^t \|u^\varepsilon(s)\|_{A^1}ds\right)\le C(u_0).$$

This in particular implies that

$$u\in L^\infty(0,T;A^0).$$

Now we use the fundamental theorem of calculus to obtain

$$\|u(t_2)-u(t_1)\|_{A^0}\le \int_{t_1}^{t_2}\|\partial_t u(s)\|_{A^0}ds,$$

from where the continuity in time follows.

Passing to the limit. The other terms being linear, we only have to take into consideration the convergence of

$$J=\int_0^T\int_{\mathbb{S}}\left(\arctan\left(\frac{Hu^\varepsilon}{u^\varepsilon}\right)-\arctan\left(\frac{Hu}{u}\right)\right)\partial_x\phi dxds.$$

We have that

$$J\le \int_0^T\int_{\mathbb{S}}\left|\frac{Hu^\varepsilon}{u^\varepsilon}-\frac{Hu}{u}\right||\partial_x\phi|dxds.$$

Using the lower bounds for u and u^ϵ together with Hölder inequality, we conclude that

$$J\to 0.$$

This concludes the proof of the existence of a global weak solution u.

Uniqueness. The uniqueness of the solution follows from a standard contradiction argument once the control of

$$\int_0^t \|u(s)\|_{A^1}ds$$

is ensured.

5. Link between (1.1) and (1.6)

We now look for a solution of (1.1) having the following form

$$u(x,t)=\langle u_0\rangle+\varepsilon\sum_{j=0}^{\infty}\varepsilon^j f^{(j)}(x,t),$$

(here ε can be thought as the displacement from the homogeneous state $\langle u_0\rangle$). The idea is to truncate the series up to certain order, say, two,

$$f(x,t)=\varepsilon f^{(0)}(x,t)+\varepsilon^2 f^{(1)}(x,t),$$

and see what f solves. In this way we will obtain that (up to $O(\varepsilon^3)$), f solves (1.6). A similar approach has been used in the study of free boundary problems for incompressible fluids (see [8, 22, 23] and the references therein). First, we observe that (1.1) can be equivalently written as

$$\partial_t u+\frac{u\Lambda u-Hu\partial_x u}{u^2+(Hu)^2}=0 \qquad (x,t) \text{ on } \mathbb{S}\times[0,T]. \tag{5.1}$$

Thus,

$$\partial_t u\left(\langle u_0\rangle^2+2(u-\langle u_0\rangle)\langle u_0\rangle+(u-\langle u_0\rangle)^2+(Hu)^2\right)+u\Lambda u-Hu\partial_x u=0.$$

Forcing the previous ansatz and matching the powers of ε, we find that $f^{(0)}$ solves

$$\partial_t f^{(0)}+\frac{\Lambda f^{(0)}}{\langle u_0\rangle}=0.$$

Similarly, $f^{(1)}$ solves

$$\partial_t f^{(1)}\langle u_0\rangle^2+2\partial_t f^{(0)}f^{(0)}\langle u_0\rangle+\langle u_0\rangle\Lambda f^{(1)}+f^{(0)}\Lambda f^{(0)}-Hf^{(0)}\partial_x f^{(0)}=0.$$

Thus, substituting $\langle u_0\rangle\partial_t f^{(0)}$ by $-\Lambda f^{(0)}$, we find that f solves

$$\partial_t f+\frac{1}{\langle u_0\rangle}\Lambda f-\frac{1}{\langle u_0\rangle^2}\partial_x(Hff)=O(\varepsilon^3).$$

Thus, neglecting the $O(\varepsilon^3)$ terms we find that

$$g(x,t)=\frac{f(x,t\langle u_0\rangle)}{\langle u_0\rangle}$$

solves (1.6).

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