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CRITERIA FOR THE A-CONTRACTION AND STABILITY FOR THE PIECEWISE-SMOOTH SOLUTIONS TO HYPERBOLIC BALANCE LAWS*

SAM G. KRUPA†

Abstract. We show uniqueness and stability in L^2 and for all time for piecewise-smooth solutions to hyperbolic balance laws. We have in mind applications to gas dynamics, the isentropic Euler system and the full Euler system for a polytropic gas in particular. We assume the discontinuity in the piecewise-smooth solution is an extremal shock. We use only mild hypotheses on the system. Our techniques and result hold without smallness assumptions on the solutions. We can handle shocks of any size. We work in the class of bounded, measurable solutions satisfying a single entropy condition. We also assume a strong trace condition on the solutions, but this is weaker than BV_{loc}. We use the theory of a-contraction (see Kang and Vasseur [Arch. Ration. Mech. Anal., 222(1):343–391, 2016]) developed for the stability of pure shocks in the case without source.

Keywords. System of conservation laws; compressible Euler equation; Euler system; isentropic solutions; generalized Riemann problem; piecewise-smooth solutions; Rankine–Hugoniot discontinuity; shock; stability; uniqueness.

AMS subject classifications. Primary 35L65; Secondary 76N15; 35L45; 35A02; 35B35; 35D30; 35L67; 35Q31; 76L05; 35Q35; 76N10.

1. Introduction

We consider an $n\times n$ system of balance laws,

$$\begin{cases}\partial_t u+\partial_x f(u)=G(u(\cdot,t))(x), \text{ for } x\in\mathbb{R},\ t>0,\\ u(x,0)=u^0(x) \text{ for } x\in\mathbb{R}.\end{cases} \tag{1.1}$$

For a fixed $T>0$ (including possibly $T=\infty$), the *unknown* is $u\colon \mathbb{R}\times[0,T)\to\mathbb{M}^{n\times 1}$. The function $u^0\colon \mathbb{R}\to\mathbb{M}^{n\times 1}$ is in $L^\infty(\mathbb{R})$ and is the *initial data*. The function $f\colon \mathbb{M}^{n\times 1}\to\mathbb{M}^{n\times 1}$ is the *flux function* for the system. The *source term* $G\colon (L^2(\mathbb{R}))^n\to(L^2(\mathbb{R}))^n$ is translation invariant, i.e.

$$G(g(\cdot))(x+y)=G(g(x+\cdot))(y) \tag{1.2}$$

for every $g\in(L^\infty(\mathbb{R}))^n$ and for all $x,y\in\mathbb{R}$. We also ask that G be Lipschitz continuous from $(L^2(I))^n\to(L^2(I))^n$ for every interval $I\subseteq\mathbb{R}$, with a Lipschitz constant uniform in I. In other words, there exists $C_G>0$ such that

$$\left\|G(g_1)-G(g_2)\right\|_{L^2(I)}\le C_G\|g_1-g_2\|_{L^2(I)}, \tag{1.3}$$

for every $g_1,g_2\in(L^2(\mathbb{R}))^n$ and for every interval $I\subseteq\mathbb{R}$. Furthermore, we require that G is bounded on $(L^\infty(\mathbb{R}))^n$:

$$\left\|G(g)\right\|_{L^\infty(\mathbb{R})}\le C_G\|g\|_{L^\infty(\mathbb{R})}, \tag{1.4}$$

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†Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA (skrupa@math.utexas.edu). https://www.samkrupa.com

for every $g\in(L^\infty(\mathbb{R}))^n$. Examples for possible such source terms include the identity: $G(g)=g$ for every $g\in(L^\infty(\mathbb{R}))^n$. The equation $\partial_t u+\partial_x f(u)=u$ is a balance law with linear excitation. Dafermos considers this equation in the scalar case in one space dimension (see [17, p. 399]). Another important example is also zero ($G\equiv 0$) – in this case (1.1) becomes a hyperbolic system of conservation laws without source.

We assume the system (1.1) is endowed with a strictly convex entropy η and associated entropy flux q. Note the system will be hyperbolic on the state space where η exists. We assume the functions f,η, and q are defined on an open convex state space $\mathcal{V}\subset\mathbb{R}^n$. We assume $f,q\in C^2(\mathcal{V})$ and $\eta\in C^3(\mathcal{V})$. By assumption, the entropy η and its associated entropy flux q verify the following compatibility relation:

$$\partial_j q=\sum_{i=1}^{n}\partial_i\eta\partial_j f_i,\qquad 1\leq j\leq n. \tag{1.5}$$

By convention, the relation (1.5) is rewritten as

$$\nabla q=\nabla\eta\nabla f, \tag{1.6}$$

where ∇f denotes the matrix $(\partial_j f_i)_{i,j}$.

For $u\in\mathcal{V}$ where η exists, the system (1.1) is hyperbolic, and the matrix $\nabla f(u)$ is diagonalizable, with eigenvalues

$$\lambda_1(u)\leq\cdots\leq\lambda_n(u), \tag{1.7}$$

called *characteristic speeds*.

We consider both bounded *classical* and bounded *weak* solutions to (1.1). A weak solution u is bounded and measurable and satisfies (1.1) in the sense of distributions. I.e., for every Lipschitz continuous test function $\Phi:\mathbb{R}\times[0,T)\to\mathbb{M}^{1\times n}$ with compact support,

$$\int_0^T\int_{-\infty}^{\infty}\Big[\partial_t\Phi u+\partial_x\Phi f(u)\Big]\,dxdt+\int_{-\infty}^{\infty}\Phi(x,0)u^0(x)\,dx=-\int_0^T\int_{-\infty}^{\infty}\Phi G(u(\cdot,t))(x)\,dxdt. \tag{1.8}$$

We only consider solutions u which are entropic for the entropy η. That is, they satisfy the following entropy condition:

$$\partial_t\eta(u)+\partial_x q(u)\leq\nabla\eta(u)G(u(\cdot,t))(x), \tag{1.9}$$

in the sense of distributions. I.e., for all positive, Lipschitz continuous test functions $\phi:\mathbb{R}\times[0,T)\to\mathbb{R}$ with compact support:

$$\begin{aligned}&\int_0^T\int_{-\infty}^{\infty}\Big[\partial_t\phi\big(\eta(u(x,t))\big)+\partial_x\phi\big(q(u(x,t))\big)\Big]\,dxdt+\int_{-\infty}^{\infty}\phi(x,0)\eta(u^0(x))\,dx\\&\geq-\int_0^T\int_{-\infty}^{\infty}\phi\nabla\eta(u(x,t))G(u(\cdot,t))(x)\,dxdt.\end{aligned} \tag{1.10}$$

In the case when $G\equiv 0$: For $u_L,u_R\in\mathbb{R}^n$, the function $u:\mathbb{R}\times[0,\infty)\to\mathbb{R}^n$ defined by

$$u(x,t):=\begin{cases}u_L & \text{if } x<\sigma t,\\ u_R & \text{if } x>\sigma t\end{cases} \tag{1.11}$$

is a weak solution to (1.1) if and only if u_L, u_R, and σ satisfy the Rankine-Hugoniot jump compatibility relation:

$$f(u_R) - f(u_L) = \sigma(u_R - u_L), \tag{1.12}$$

in which case (1.11) is called a *shock* solution.

Moreover, when $G \equiv 0$, the solution (1.11) will be entropic for η (according to (1.10)) if and only if,

$$q(u_R) - q(u_L) \leq \sigma(\eta(u_R) - \eta(u_L)). \tag{1.13}$$

In this case, (u_L, u_R, σ) is an *entropic Rankine–Hugoniot discontinuity*.

Further, remark that at a discontinuity in a solution to (1.1), the presence of the source term G does not modify the Rankine-Hugoniot jump compatibility relation (1.12) or the entropy condition (1.13) because due to (1.4), the map $(x,t) \mapsto G(u(\cdot,t))(x)$ will be in L^1_{loc} and so in the standard proof of Rankine-Hugoniot (see for example [22, p. 612-4]), the source term does not play a role.

For a fixed u_L, we consider the set of u_R which satisfy (1.12) and (1.13) for some σ. For a general $n \times n$ strictly hyperbolic system of conservation laws endowed with a strictly convex entropy, we know that locally this set of u_R values is made up of n curves (see for example [33, p. 140-6]).

The present paper concerns the finite-time stability of piecewise-smooth solutions to (1.1), working in the L^2 setting. We work in a very general setting. Our techniques are based on the theory of shifts as developed by Vasseur within the context of the relative entropy method (see [46]). We consider systems of the form (1.1), with minimal assumptions on the shock families. We ask that the extremal shock speeds (1-shock and n-shock speeds) are separated from the intermediate shock families. If we want to consider 1-shocks, we ask that the 1-shock family satisfy the Liu entropy condition (shock speed decreases as the right-hand state travels down the 1-shock curve), and we ask that the shock strength increase in the sense of relative entropy (an L^2 requirement) as the right-hand state travels down the 1-shock curve. If we want to consider n-shocks, we ask for similar requirements on the n-shock family.

The intermediate wave families have far fewer requirements. The intermediate shock curves might not even be well-defined and characteristic speeds might cross.

In particular, the results in this article apply to both the isentropic Euler system and the full Euler system for a polytropic gas, viewing both systems in Eulerian coordinates.

We study solutions $\bar{u}$ which are piecewise-Lipschitz continuous in the space variable x. We study the stability and uniqueness of these solutions among a large class of weak solutions u which are bounded, measurable, entropic for at least one strictly convex entropy, and verify a strong trace condition (weaker than BV_{loc}). We do not make small data assumptions. We require the piecewise-smooth $\bar{u}$ contain a single shock of extremal family. However, the rougher solutions u, which we compare to this solution $\bar{u}$, may have shocks of any type or family.

Previous results in the theory of stability and a-contraction have only been able to consider initial data which is pure shock (piecewise constant). This present paper extends the ideas in the theory of a-contraction (in particular as developed in [27]).

The point of the present article is this: As discussed for the case of nonlocal scalar balance laws in [30], when studying the stability up to a translation in space of solutions piecewise-constant in space, we can view the shift function which is doing the translation as simply determining at which points do we want to see the left-hand state of our solution, and at which points do we want to see the right-hand state of our solution.

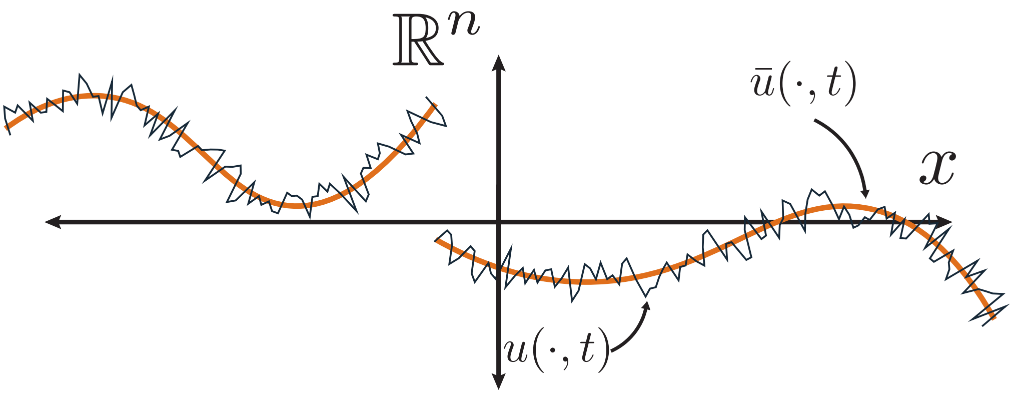

FIG. 1.1. *In this paper, we study the stability of solutions u (to (1.1)) which are L^2 perturbations of a piecewise-smooth solution $\bar{u}$, as shown in this schematic. The nonlinearity in the solution $\bar{u}$ causes significant technical challenges not present in the piecewise-constant case (for the piecewise-constant case, see [27, 35]).*

However, for piecewise-*smooth* data, the shift function cannot be viewed like this. Instead, the shift function is viewed as artificially translating in space our solution. If the solution is non-constant away from the discontinuity, this artificial translation creates a linear term in the entropy dissipation (see Lemma 3.3), which we cannot Gronwall in comparison with the quadratic terms. The answer is to create a shift function which not only neutralizes entropy production at the discontinuity of the solution, but also creates additional negative entropy (see Proposition 4.1) we can use to cancel out the linear term in the Gronwall argument (see Figure 1.1). Regarding the idea of additional negative entropy caused by a shift, see [26].

This work is related to the *generalized Riemann problem*, which concerns solutions with initial data which is piecewise-smooth instead of simply piecewise-constant across a single jump discontinuity. For existence and uniqueness results for the generalized Riemann problem, see [36, 37]. However, these results have small data limitations.

Previous results in this direction include Chen, Frid, and Li [9] where for the full Euler system, they show uniqueness and long-time stability for perturbations of Riemann initial data among a large class of entropy solutions (locally BV and without smallness conditions) for the 3×3 Euler system in Lagrangian coordinates. They also show uniqueness for solutions piecewise-Lipschitz in x. For an extension to the relativistic Euler equations, see Chen and Li [10]. However, these papers do not give L^2 stability results for all time.

We study the stability in L^2 of piecewise-smooth solutions to the system of balance laws (1.1). The study of piecewise-smooth solutions takes us a step beyond the classical Riemann problem, which considers piecewise-constant initial data. Furthermore, when the system (1.1) has the source term G, it is important to study piecewise-smooth solutions and not just piecewise-constant, for the source term may mean that even pure shock wave initial data evolves into something more complicated. For a nonlocal example of this phenomenon, consider the solution to the Riemann problems for the Burgers–Hilbert equation, which is Burgers equation with a nonlocal source term [7, 8, 24, 25].

Our method is the relative entropy method, a technique created by Dafermos [14, 15] and DiPerna [21] to give L^2-type stability estimates between a Lipschitz continuous solution and a rougher solution, which is only weak and entropic for a strictly convex entropy (the so-called *weak-strong* stability theory). For a system (1.1) endowed with an entropy η, the technique of relative entropy considers the quantity called the *relative entropy*, defined as

$$\eta(u|v) := \eta(u) - \eta(v) - \nabla\eta(v)(u - v). \tag{1.14}$$

Similarly, we define relative entropy-flux,

$$q(u;v) := q(u) - q(v) - \nabla\eta(v)(f(u) - f(v)). \tag{1.15}$$

Remark that for any constant $v \in \mathbb{R}^n$, the map $u \mapsto \eta(u|v)$ is an entropy for the system (1.1), with associated entropy flux $u \mapsto q(u;v)$. Furthermore, if u is a weak solution to (1.1) and entropic for η, then u will also be entropic for $\eta(\cdot|v)$. This can be calculated directly from (1.1) and (1.9) – note that the map $u \mapsto \eta(u|v)$ is basically η plus a linear term.

Moreover, by virtue of η being *strictly* convex, the relative entropy is comparable to the L^2 distance, in the following sense:

LEMMA 1.1. *For any fixed compact set $V \subset \mathcal{V}$, there exists $c^*, c^{**} > 0$ such that for all $u, v \in V$,*

$$c^*|u - v|^2 \leq \eta(u|v) \leq c^{**}|u - v|^2. \tag{1.16}$$

The constants c^*, c^{**} depend on V and bounds on the second derivative of η.

This lemma follows from Taylor's theorem; for a proof see [35, 46].

Given a Lipschitz solution $\bar{u}$ to (1.1), and a weak, entropic solution u, the method of relative entropy gives estimates on the growth in time of the quantity

$$\left\|\bar{u}(\cdot,t) - u(\cdot,t)\right\|_{L^2(\mathbb{R})}$$

by studying the time derivative $\partial_t \int \eta(u|\bar{u})\, dx$ and using the entropy inequality (1.9). By Lemma 1.1, we get L^2-type stability estimates.

Introducing a discontinuity into $\bar{u}$ causes difficulties in the method of relative entropy. In particular, simple examples for the scalar conservation laws show that a discontinuity in $\bar{u}$ prevents stability between $\bar{u}$ and u in the form of the classical weak-strong estimates.

However, by allowing the discontinuity in $\bar{u}$ to move with an artificial speed which depends on u, we can recover weak-strong type estimates. Within the context of the relative entropy method, this theory of stability up to a shift was initiated in [46] by Vasseur. Over the last decade, this theory of stability up to a shift has been matured and developed by Vasseur and his team. The first result was for pure shock wave initial data for the scalar conservation laws [34]. Further results include work on the scalar viscous conservation laws in both one space dimension [28] and multiple [29]. Recently, work on the scalar conservation laws has allowed for many discontinuities to exist in the otherwise smooth $\bar{u}$ – with each discontinuity shifted in such a way as to maintain L^2 stability between $\bar{u}$ and an arbitrary weak solution u entropic for at least one entropy. With this, it is possible to make comparisons between two solutions which satisfy only one entropy condition, and thus show that one entropy condition is enough for uniqueness. See [31] (and the references therein) for more details. To study the L^2 stability of pure shock wave initial data in the systems case, the technique of a-contraction was introduced [27, 35, 43, 45, 47]. For a general overview of theory of shifts and the relative entropy method, see [44, Section 3-5]. By considering stability up to a shift, the method of relative entropy can also be used to study the asymptotic limit when the limit is discontinuous (see [13] for the scalar case, [48] for systems). There is a long history of using the relative entropy method to study the asymptotic limit. However, without the theory of shifts, it appears that only limits which are Lipschitz continuous can be studied (see [1, 2, 4, 5, 23, 38, 41, 49] and [46] for a survey).

The present article is a further step in the program of stability up to a shift.

In this paper, we continue the ideas introduced in [30]. In [30], it is shown that the generalized characteristics of u can be used as shift functions to kill growth in L^2 between a piecewise-smooth solution $\bar{u}$ and weak solution to (1.1) entropic for the entropy η. Further, using the generalized characteristic as a shift function provides various benefits over using the previous shift function constructions, as discussed in [30].

In this paper, we bring novel ideas from the scalar case in [30] to the systems case. In the systems case, we need to use the theory of a-contraction.

For the scalar case, the generalized characteristics for u are the natural shift functions to be used. In the systems case, we use a shift function which again is based on the generalized characteristics, but with a correction where the shift travels at greater-than-characteristic-speed due to a-contraction and the existence of multiple shock families in the systems case.

On top of the benefits for generalized-characteristic-based shifts mentioned in [30] (such as simplicity of analysis, ease of construction, enhanced control on the shifts, and strictly negative entropy creation) the use of generalized-characteristic-based shifts for the *systems case* allows for simplified proofs compared to the previous state-of-the-art a-contraction result, [27]. By having very obvious control on the speed of generalized-characteristic-based shifts, we are able to obviate the need for many of the computations in the foregoing analysis [27].

For systems of conservation laws in one space dimension such as (1.1) (including the scalar conservation laws), we have non-uniqueness for solutions. We impose entropy conditions such as (1.9), motivated by physics, to try to weed out "nonphysical" solutions which have physical entropy decreasing (or according to (1.9), mathematical entropy increasing). Remark that requiring more than one entropy condition (for more than one entropy) is impractical – many systems only admit a single nontrivial entropy. In the scalar case, this approach has had tremendous success. In fact, requiring solutions satisfy the entropy condition (1.9) for at least one strictly convex entropy in C^1 is enough to get uniqueness for solutions (see [18, 31, 42]). However, even for the scalar case proving uniqueness with a single entropy condition has proved difficult. The first result [42] was not until 1994. Furthermore, the first two results [18, 42] use techniques limited to the scalar case. They use the special connection between scalar conservation laws in one space dimension and Hamilton–Jacobi equations: the space derivative of the solution to a Hamilton–Jacobi equation is formally the solution to the associated scalar conservation law. Notably, [31] gives a proof of the single entropy condition for scalar conservation laws which works directly on the conservation law and utilizes the theory of shifts. Moreover, progress for uniqueness of entropic solutions to *systems* of conservation laws has been slow. The best theory so far is the Bressan, Crasta, and Piccoli L^1 theory [6] for uniqueness in the class of solutions with small total variation. It would be interesting however to study the uniqueness of these solutions amongst a larger class. For example, existence of solutions with large data is known for the 2×2 Euler system – but the uniqueness theory for such solutions with large data lags behind.

The situation for the hyperbolic conservation laws in multiple space dimensions is even more dire – there is non-uniqueness for entropic solutions to incompressible and compressible Euler by virtue of the many highly oscillatory solutions created via convex integration or related techniques. For incompressible Euler, see two papers by De Lellis and Székelyhidi [19,20]. For compressible Euler, see [11,12,40].

However, there is still the possibility of pushing forward the theory of *uniqueness* for hyperbolic systems of conservation laws in one space dimension. The current paper is a step in that direction – utilizing the L^2-type relative entropy method and the constantly evolving theory of shifts.

In this article, we use the method of relative entropy, the theory of shifts and a-contraction. These theories are not perturbative. They enable us to get results without small data limitations. Further, by the nature of these theories, we only use a single entropy condition.

We present our main and most important theorem regarding L^2-type stability and uniqueness results. The hypotheses $(\mathcal{H})$ and $(\mathcal{H})^*$ in the theorem depend only on the hyperbolic part of the system (1.1) and the fixed piecewise-smooth solution $\bar{u}$. The hypotheses are related to conditions on 1-shocks and n-shocks and in particular are satisfied by the isentropic Euler and full Euler systems. These hypotheses are explained in detail in Section 2.

THEOREM 1.1 (Main theorem – L^2 stability for entropic piecewise-Lipschitz solutions to hyperbolic systems of balance laws). *Fix $R,T>0$. Fix $i\in\{1,n\}$. Assume that $u,\bar{u}\in L^\infty(\mathbb{R}\times[0,T))$ are weak solutions to (1.1) with initial data u^0 and $\bar{u}^0$, respectively. If $\bar{u}$ contains a 1-shock, assume the hypotheses $(\mathcal{H})$ hold. Likewise, if $\bar{u}$ contains an n-shock, assume the hypotheses $(\mathcal{H})^*$ hold. Assume that u and $\bar{u}$ are entropic for the entropy $\eta\in C^3(\mathbb{R}^n)$. Assume that $\bar{u}$ is Lipschitz continuous on $\{(x,t)\in\mathbb{R}\times[0,T)|x<s(t)\}$ and on $\{(x,t)\in\mathbb{R}\times[0,T)|x>s(t)\}$, where $s:[0,T)\to\mathbb{R}$ is a Lipschitz function. Assume also that u verifies the strong trace property (Definition 2.1). Assume also that there exists $\rho>0$ such that for all $t\in[0,T)$*

$$\left|\bar{u}(s(t)+,t)-\bar{u}(s(t)-,t)\right|>\rho. \tag{1.17}$$

Then there exists a Lipschitz continuous function $X\colon[0,T)\to\mathbb{R}$ with $X(0)=0$ and constants $\mu_1,\mu_2,r>0$ such that,

$$\int\limits_{-R+s(0)}^{R+s(0)} \left|u(x,t_0)-\bar{u}(x+X(t_0),t_0)\right|^2 dx \le \mu_2 e^{\mu_1 t_0} \int\limits_{-R-rt_0+s(0)}^{R+rt_0+s(0)} \left|u^0(x)-\bar{u}^0(x)\right|^2 dx, \tag{1.18}$$

for all $t_0\in[0,T)$.

Moreover, we have control on X:

$$\int\limits_0^{t_0} (\dot{X}(t))^2\,dt \le \mu_2(1+e^{\mu_1 t_0}) \int\limits_{-R-rt_0+s(0)}^{R+rt_0+s(0)} \left|u^0(x)-\bar{u}^0(x)\right|^2 dx. \tag{1.19}$$

REMARK 1.1.

- The theorem gives the uniqueness of piecewise-smooth solutions to (1.1) amongst the large class of bounded weak solutions, entropic for a convex entropy and verifying a strong trace condition.

- The constants $\mu_1,\mu_2>0$ depend on a, ρ, $\|u\|_{L^\infty}$, $\|\bar{u}\|_{L^\infty}$, and bounds on the derivatives of η on the range of u and $\bar{u}$. In addition, μ_1 depends on C_G (see (1.3)), Lip$[\bar{u}]$, R, T, and bounds on the derivatives of f on the range of u and $\bar{u}$. Note that r only depends on bounds on the derivatives of f and η (on the range of u and $\bar{u}$).
- As opposed to (1.4), the proof of Theorem 1.1 will in fact go through whenever we have an estimate of the form

$$\left|\int_{x_1}^{x_2} \nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)\,dx\right|$$
$$\leq C\int_{x_1}^{x_2} \left|\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))\right|dx, \tag{1.20}$$

for $x_1,x_2\in\mathbb{R}$ and some constant $C>0$. Note that $u\in L^\infty$ and (1.4) implies (1.20).
- Note that Hölder's inequality and (1.19) give control on the shift in the form of

$$\frac{1}{t_0}\int_0^{t_0}\left|\dot{X}(t)\right|dt\leq\frac{\sqrt{\mu_2(1+e^{\mu_1 t_0})}}{\sqrt{t_0}}\left\|u^0(\cdot)-\bar{u}^0(\cdot)\right\|_{L^2(-R-rt_0+s(0),R+rt_0+s(0))}. \tag{1.21}$$

- Note that by (2.11), condition (1.17) is equivalent to the existence of a $\tilde{\rho}>0$ such that for all $t\in[0,T)$

$$r(t)>\tilde{\rho}, \tag{1.22}$$

where $r(t)$ satisfies $S^i_{\bar{u}(s(t)-,t)}(r(t))=\bar{u}(s(t)+,t)$.

The outline of the paper is as follows: in Section 2, we give our hypotheses on the system. In Section 3, we present technical lemmas. In Section 4, we construct the shift with the additional entropy dissipation. Finally, in Section 5 we prove the main theorem by using the additional entropy dissipation from the shift to translate in x the piecewise-smooth solution artificially.

2. Hypotheses on the system

We will consider the following structural hypotheses $(\mathcal{H})$, $(\mathcal{H})^*$ on the system (1.1), (1.9) regarding the 1-shock and n-shock curves (they are closely related to hypotheses in [35] and [27]). For a fixed piecewise-smooth solution $\bar{u}$ (as in the context of the main theorem Theorem 1.1):

- $(\mathcal{H}1)$: (Family of 1-shocks verifying the Liu condition) There exists $r_0>0$ such that for all $u_L\in\{\bar{u}(s(t)-,t)|t\in[0,T)\}:=I_-$, and for all $u\in B_{r_0}(u_L)$, there is a 1-shock curve (issuing from u) $S^1_u\colon[0,s_u)\to\mathcal{V}$ (possibly $s_u=\infty$) parameterized by arc length. Moreover, $S^1_u(0)=u$ and the Rankine-Hugoniot jump condition holds:

$$f(S^1_u(s))-f(u)=\sigma^1_u(s)(S^1_u(s)-u), \tag{2.1}$$

where $\sigma_u^1(s)$ is the velocity function. The map $u \mapsto s_u$ is Lipschitz on $\mathcal{V}$. Further, the maps $(s,u) \mapsto S_u^1(s)$ and $(s,u) \mapsto \sigma_u^1(s)$ are both C^1 on $\{(s,u) | s \in [0,s_u), u \in \mathcal{V}\}$, and the following conditions are satisfied:

(a) (Liu entropy condition) $\dfrac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s) < 0, \quad \sigma_u^1(0) = \lambda_1(u)$,

(b) (shock "strengthens" with s) $\dfrac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s)) > 0, \quad$ for all $s > 0$,

(c) (the shock curve cannot wrap tightly around itself)

For all $R > 0$, there exists $\tilde{S} > 0$ such that

$$\left\{S_u^1(s)\middle| s \in [0,s_u), |u| \leq R \text{ and } \left|S_u^1(s)\right| \leq R\right\} \subseteq \left\{S_u^1(s)\middle| |u| \leq R \text{ and } s \leq \tilde{S}\right\}.$$

- $(\mathcal{H}2)$: If (u_L, u_R) is an entropic Rankine-Hugoniot discontinuity with shock speed σ, then $\sigma > \lambda_1(u_R)$.
- $(\mathcal{H}3)$: If (u_L, u_R) (with $u_L \in B_{r_0}(\tilde{u}_L)$, for $\tilde{u}_L \in I_-$) is an entropic Rankine-Hugoniot discontinuity with shock speed σ verifying

$$\sigma \leq \lambda_1(u_L), \tag{2.2}$$

then u_R is in the image of $S_{u_L}^1$. In other words, there exists $s_{u_R} \in [0, s_{u_L})$ such that $S_{u_L}^1(s_{u_R}) = u_R$ (and by implication, $\sigma = \sigma_{u_L}^1(s_{u_R})$).

Similarly, we will consider the following structural hypotheses $(\mathcal{H})^*$ on the system (1.1), (1.9) regarding the n-shock curves:

- $(\mathcal{H}1)^*$: (Family of n-shocks verifying the Liu condition) There exists $r_0 > 0$ such that for all $u_R \in \{\bar{u}(s(t)+,t) | t \in [0,T)\} := I_+$, and for all $u \in B_{r_0}(u_R)$, there is an n-shock curve (issuing from u) $S_u^n : [0,s_u) \to \mathcal{V}$ (possibly $s_u = \infty$) parameterized by arc length. Moreover, $S_u^n(0) = u$ and the Rankine-Hugoniot jump condition holds:

$$f(S_u^n(s)) - f(u) = \sigma_u^n(s)(S_u^n(s) - u), \tag{2.3}$$

where $\sigma_u^n(s)$ is the velocity function. The map $u \mapsto s_u$ is Lipschitz on $\mathcal{V}$. Further, the maps $(s,u) \mapsto S_u^n(s)$ and $(s,u) \mapsto \sigma_u^n(s)$ are both C^1 on $\{(s,u) | s \in [0,s_u), u \in \mathcal{V}\}$, and the following conditions are satisfied:

(a) (Liu entropy condition) $\dfrac{\mathrm{d}}{\mathrm{d}s}\sigma_u^n(s) > 0, \quad \sigma_u^n(0) = \lambda_n(u)$,

(b) (shock "strengthens" with s) $\dfrac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^n(s)) > 0, \quad$ for all $s > 0$,

(c) (the shock curve cannot wrap tightly around itself)

For all $R > 0$, there exists $\tilde{S} > 0$ such that

$$\left\{S_u^n(s)\middle| s \in [0,s_u), |u| \leq R \text{ and } \left|S_u^n(s)\right| \leq R\right\} \subseteq \left\{S_u^n(s)\middle| |u| \leq R \text{ and } s \leq \tilde{S}\right\}.$$

- $(\mathcal{H}2)^*$: If (u_R, u_L) is an entropic Rankine-Hugoniot discontinuity with shock speed σ, then $\sigma < \lambda_n(u_L)$.
- $(\mathcal{H}3)^*$: If (u_R, u_L) (with $u_R \in B_{r_0}(\tilde{u}_R)$, for $\tilde{u}_R \in I_+$) is an entropic Rankine-Hugoniot discontinuity with shock speed σ verifying

$$\sigma \geq \lambda_n(u_R), \tag{2.4}$$

then u_L is in the image of $S^n_{u_R}$. In other words, there exists $s_{u_L} \in [0, s_{u_R})$ such that $S^n_{u_R}(s_{u_L}) = u_L$ (and by implication, $\sigma = \sigma^n_{u_R}(s_{u_L})$).

Remark 2.1. See [27, 35] for remarks on these hypotheses. We include them here for completeness. In particular,

- Note that the system (1.1) verifies the hypotheses ($\mathcal{H}1$)-($\mathcal{H}3$) on the 1-shock family if and only if the system

$$\begin{cases} \partial_t u - \partial_x f(u) = \tilde{G}(u(\cdot,t))(x), \text{ for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = u^0(x) \text{ for } x \in \mathbb{R}, \end{cases} \tag{2.5}$$

  verifies the properties ($\mathcal{H}1$)*-($\mathcal{H}3$)* for the n-shock family, and where

$$\tilde{G}(g(\cdot))(x) := G(g(-\cdot))(-x) \tag{2.6}$$

  for every $g \in (L^\infty(\mathbb{R}))^n$ and for all $x \in \mathbb{R}$. Notice that if G verifies properties (1.2), (1.3), and (1.4), then $\tilde{G}$ does as well. It is in this way that ($\mathcal{H}1$)-($\mathcal{H}3$) are dual to ($\mathcal{H}1$)*-($\mathcal{H}3$)*.
- On top of the Liu entropy condition (Property (a) in ($\mathcal{H}1$)), we also assume Property (b), which says that the 1-shock strength grows along the 1-shock curve $S^1_{u_L}$ when measured via the pseudo-distance of the relative entropy (recall that the map $(u,v) \mapsto \eta(u|v)$ measures L^2-distance somehow – see Lemma 1.1). This growth condition arises naturally in the study of admissibility criteria for systems of conservation laws. In particular, Property (b) ensures that Liu admissible shocks are entropic for the entropy η even for moderate-to-strong shocks (see [16, 32, 39]).
  In [3], Barker, Freistühler, and Zumbrun show that stability and in particular contraction fails to hold for the full Euler system if we replace Property (b) with

$$\frac{\mathrm{d}}{\mathrm{d}s}\eta(S^1_u(s)) > 0, \qquad s > 0. \tag{2.7}$$

  This shows that it is better to measure shock strength using the relative entropy rather than the entropy itself.
- Recall the famous Lax E-condition for an i-shock (u_L, u_R, σ),

$$\lambda_i(u_R) \le \sigma \le \lambda_i(u_L). \tag{2.8}$$

  The hypothesis ($\mathcal{H}2$) is implied by the first half of the Lax E-condition along with the hyperbolicity of the system (1.1). In addition, we do not allow for right 1-contact discontinuities.
- The hypothesis ($\mathcal{H}3$) is a statement about the well-separation of the 1-shocks from all other Rankine-Hugoniot discontinuities entropic for η; the 1-shocks do not interfere with any other shocks. In particular, ($\mathcal{H}3$) will hold for any strictly hyperbolic system in the form (1.1) if all Rankine-Hugoniot discontinuities (u_L, u_R, σ) entropic for η lie on an i-shock curve for some i and the extended Lax admissibility condition holds:

$$\lambda_{i-1}(u_L) \le \sigma \le \lambda_{i+1}(u_R), \tag{2.9}$$

  where $\lambda_0 := -\infty$ and $\lambda_{n+1} := \infty$. Moreover, we only use the first inequality in (2.9) and the fact that $\lambda_1(u) \le \lambda_{i-1}(u)$ for all $u \in \mathcal{V}$ and for all $i > 1$.

Furthermore, note that for *any* strictly hyperbolic system in the form (1.1), if u_R and u_L live in a fixed compact set, then there exists $\delta>0$ such that (2.9) will hold if $|u_R-u_L|\leq\delta$. Similarly, for any strictly hyperbolic system endowed with a strictly convex entropy, all Rankine-Hugoniot discontinuities (u_L,u_R,σ) entropic for η will locally be in the form $S^i_{u_L}(s)=u_R$ for some $s>0$, and where $S^i_{u_L}$ is the i-shock curve issuing from u_L. See [33, Theorem 1.1, p. 140] and more generally [33, p. 140-6]. For the full Euler system, $(\mathcal{H}3)$ will hold regardless of the size of the shock (u_L,u_R).

- Note that due to the map $(s,u)\mapsto S^1_u(s)$ being Lipschitz, we have

$$\left|S^1_u(s)-u\right|=\left|S^1_u(s)-S^1_u(0)\right|\leq \mathrm{Lip}\left[(s,u)\mapsto S^1_u(s)\right]s, \tag{2.10}$$

for all $u\in B_{r_0}(I_-)$ and all $s\in[0,s_u)$. Equivalently,

$$\frac{1}{\mathrm{Lip}\left[(s,u)\mapsto S^1_u(s)\right]}\left|S^1_u(s)-u\right|\leq s. \tag{2.11}$$

- On the state space $\mathcal{V}$ where the strictly convex entropy η is defined, the system (1.1) is hyperbolic. Further, by virtue of $f\in C^2(\mathcal{V})$, the eigenvalues of $\nabla f(u)$ vary continuously on the state space $\mathcal{V}$. Further, if the eigenvalue $\lambda_1(u)$ $(\lambda_n(u))$ is simple for $u\in\mathcal{V}$ (such as when the system (1.1) is strictly hyperbolic), the map $u\mapsto\lambda_1(u)$ $(u\mapsto\lambda_n(u))$ will be in $C^1(\mathcal{V})$ due to the implicit function theorem.

We study solutions u to (1.1) among the class of functions verifying a strong trace property (first introduced in [35]):

DEFINITION 2.1. *Fix $T>0$. Let $u\colon\mathbb{R}\times[0,T)\to\mathbb{R}^n$ verify $u\in L^\infty(\mathbb{R}\times[0,T))$. We say u has the* strong trace property *if for every fixed Lipschitz continuous map $h\colon[0,T)\to\mathbb{R}$, there exists $u_+,u_-\colon[0,T)\to\mathbb{R}^n$ such that*

$$\lim_{n\to\infty}\int_0^{t_0}\operatorname*{ess\,sup}_{y\in(0,\frac{1}{n})}\left|u(h(t)+y,t)-u_+(t)\right|dt=\lim_{n\to\infty}\int_0^{t_0}\operatorname*{ess\,sup}_{y\in(-\frac{1}{n},0)}\left|u(h(t)+y,t)-u_-(t)\right|dt=0 \tag{2.12}$$

for all $t_0\in(0,T)$.

Note that for example a function $u\in L^\infty(\mathbb{R}\times[0,T))$ will satisfy the strong trace property if for each fixed h, the right and left limits

$$\lim_{y\to0^+}u(h(t)+y,t)\qquad\text{and}\qquad\lim_{y\to0^-}u(h(t)+y,t) \tag{2.13}$$

exist for almost every t. In particular, a function $u\in L^\infty(\mathbb{R}\times[0,T))$ will have strong traces according to Definition 2.1 if u has a representative which is in BV_{loc}. However, the strong trace property is weaker than BV_{loc}.

3. Technical lemmas

Throughout this paper, we use the following definition for the relative flux

$$f(a|b):=f(a)-f(b)-\nabla f(b)(a-b), \tag{3.1}$$

and the relative $\nabla\eta$: for $a,b\in\mathbb{M}^{n\times1}$,

$$\nabla\eta(a|b):=\nabla\eta(a)-\nabla\eta(b)-[a-b]^T\nabla^2\eta(b). \tag{3.2}$$

The following lemma from [47] describes how the relative entropy obeys a sort of triangle inequality:

Lemma 3.1 (Structural lemma from [47]-triangle inequality for the relative entropy). *For any $u,v,w\in\mathcal{V}$, we have*

$$\eta(u|w)+\eta(w|v)=\eta(u|v)+(\nabla\eta(w)-\nabla\eta(v))(w-u), \tag{3.3}$$

and

$$q(u;w)+q(w;v)=q(u;v)+(\nabla\eta(w)-\nabla\eta(v))(f(w)-f(u)). \tag{3.4}$$

Thus, for any $\sigma\in\mathbb{R}$,

$$\begin{aligned} q(u;v)-\sigma\eta(u|v)=&(q(u;w)-\sigma\eta(u;w))+(q(w;v)-\sigma\eta(w|v))\\ &-(\nabla\eta(w)-\nabla\eta(v))(f(w)-f(u)-\sigma(w-u)). \end{aligned}\tag{3.5}$$

The proof of Lemma 3.1 follows immediately from the definition of $q(\cdot;\cdot)$ and $\eta(\cdot|\cdot)$. In particular, see [27, p. 360-1] for a simple proof.

Lemma 3.2. *Fix $B>0$. Then there exists a constant $C>0$ depending on B such that the following holds:*

If $u_L,u_R\in\mathcal{V}$ with $|u_L|,|u_R|\leq B$, then whenever $\alpha,\theta\in(0,1)$ verify

$$\alpha<\frac{\theta^2}{C}, \tag{3.6}$$

then $R_a:=\{u|\eta(u|u_L)\leq a\eta(u|u_R)\}\subset B_\theta(u_L)$ for all $0<a<\alpha$.

Remark 3.1. The set R_a is compact.

The proof of Lemma 3.2 is found in the proof of Lemma 4.3 in [27]. We repeat the proof in the Appendix for the reader's convenience.

The following lemma gives the entropy dissipation caused by changing the domain of integration, translating the solution $\bar{u}$ in x (by a function $X(t)$), and from the source term G.

Lemma 3.3 (Local entropy dissipation rate). *Fix $T>0$. Let $u,\bar{u}\in L^\infty(\mathbb{R}\times[0,T))$ be weak solutions to (1.1). Assume that u and $\bar{u}$ are entropic for the entropy η. Assume that $\bar{u}$ is Lipschitz continuous on $\{(x,t)\in\mathbb{R}\times[0,T)|x<s(t)\}$ and on $\{(x,t)\in\mathbb{R}\times[0,T)|x>s(t)\}$, where $s:[0,T)\to\mathbb{R}$ is a Lipschitz function . Assume also that u verifies the strong trace property (Definition 2.1). Let $h_1,h_2,X:[0,T)\to\mathbb{R}$ be Lipschitz continuous functions with the property that there exists $\delta>0$ such that $h_2(t)-h_1(t)\geq\delta$ for all $t\in[0,T)$. Assume also that for all $t\in[0,T)$, $s(t)-X(t)$ is not in the open set $(h_1(t),h_2(t))$.*

Then,

$$\begin{aligned} \int_0^{t_0}\Big[&q(u(h_1(t)+,t);\bar{u}((h_1(t)+X(t))+,t))-q(u(h_2(t)-,t);\bar{u}((h_2(t)+X(t))-,t))\\ &+\dot{h}_2(t)\eta(u(h_2(t)-,t)|\bar{u}((h_2(t)+X(t))-,t))\\ &-\dot{h}_1(t)\eta(u(h_1(t)+,t)|\bar{u}((h_1(t)+X(t))+,t))\Big]\,dt \end{aligned}$$

$$\geq \int_{h_1(t_0)}^{h_2(t_0)} \eta(u(x,t_0)|\bar{u}(x+X(t_0),t_0))\,dx - \int_{h_1(0)}^{h_2(0)} \eta(u^0(x)|\bar{u}^0(x))\,dx$$

$$+\int_0^{t_0}\int_{h_1(t)}^{h_2(t)} \Bigg(\partial_x\Big|_{(x+X(t),t)} \nabla\eta(\bar{u}(x,t))\Bigg) f(u(x,t)|\bar{u}(x+X(t),t))$$

$$+\Bigg(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\dot{X}(t)\Bigg)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]$$

$$-\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)$$

$$+\Bigg(G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\Bigg)^T \nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)$$

$$-\bar{u}(x+X(t),t)]\,dxdt. \tag{3.7}$$

Proof. This proof is based on a similar argument in [30].

Step 1. We show that for all positive, Lipschitz continuous test functions $\phi:\mathbb{R}\times[0,T)\to\mathbb{R}$ with compact support and that vanish on the set $\{(x,t)\in\mathbb{R}\times[0,T) | x=s(t)-X(t)\}$, we have

$$\int_0^T\int_{-\infty}^{\infty} [\partial_t\phi\eta(u(x,t)|\bar{u}(x+X(t),t))+\partial_x\phi q(u(x,t);\bar{u}(x+X(t),t))]\,dxdt$$

$$+\int_{-\infty}^{\infty}\phi(x,0)\eta(u^0(x)|\bar{u}^0(x))\,dx$$

$$\geq \int_0^T\int_{-\infty}^{\infty}\phi\Bigg[\Bigg(\partial_x\Big|_{(x+X(t),t)} \nabla\eta(\bar{u}(x,t))\Bigg) f(u(x,t)|\bar{u}(x+X(t),t))$$

$$+\Bigg(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\dot{X}(t)\Bigg)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]$$

$$-\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)$$

$$+\Bigg(G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\Bigg)^T \nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)$$

$$-\bar{u}(x+X(t),t)]\Bigg]\,dxdt. \tag{3.8}$$

Note that (3.8) is the analogue in our case of the key estimate used in Dafermos's proof of weak-strong stability, which gives a relative version of the entropy inequality (see equation (5.2.10) in [17, p. 122-5]). The proof of (3.8) is based on the famous weak-strong stability proof of Dafermos and DiPerna [17, p. 122-5]. To take into account the entropy production due to translating the solution $\bar{u}$ by the function X, we use the argument introduced in [30].

Note that on the complement of the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)\}$, $\bar{u}$ is smooth and so we have the exact equalities,

$$\partial_t\Big|_{(x,t)}\big(\bar{u}(x,t)\big)+\partial_x\Big|_{(x,t)}\big(f(\bar{u}(x,t))\big)=G(\bar{u}(\cdot,t))(x), \tag{3.9}$$

$$\partial_t\Big|_{(x,t)}\big(\eta(\bar{u}(x,t))\big)+\partial_x\Big|_{(x,t)}\big(q(\bar{u}(x,t))\big)=\nabla\eta(\bar{u}(x,t))G(\bar{u}(\cdot,t))(x). \tag{3.10}$$

Thus for any Lipschitz continuous function $X:[0,T)\to\mathbb{R}$ with $X(0)=0$ we have on the complement of the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$,

$$\begin{aligned}&\partial_t\Big|_{(x,t)}\big(\bar{u}(x+X(t),t)\big)+\partial_x\Big|_{(x,t)}\big(f(\bar{u}(x+X(t),t))\big)\\&=\left(\partial_x\Big|_{(x+X(t),t)}\big(\bar{u}(x,t)\big)\right)\dot{X}(t)+G(\bar{u}(\cdot,t))(x+X(t)),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\partial_t\Big|_{(x,t)}\big(\eta(\bar{u}(x+X(t),t))\big)+\partial_x\Big|_{(x,t)}\big(q(\bar{u}(x+X(t),t))\big)\\&=\nabla\eta(\bar{u}(x+X(t),t))\left(\partial_x\Big|_{(x+X(t),t)}\big(\bar{u}(x,t)\big)\right)\dot{X}(t)+\nabla\eta(\bar{u}(x+X(t),t))G(\bar{u}(\cdot,t))(x+X(t)).\end{aligned} \tag{3.12}$$

We can now imitate the weak-strong stability proof in [17, p. 122-5], using (3.11) and (3.12) instead of (3.9) and (3.10).

Recall (3.1), which says

$$f(u|\bar{u}):=f(u)-f(\bar{u})-\nabla f(\bar{u})(u-\bar{u}). \tag{3.13}$$

Remark that $f(u|\bar{u})$ is locally quadratic in $u-\bar{u}$.

Fix any positive, Lipschitz continuous test function $\phi:\mathbb{R}\times[0,T)\to\mathbb{R}$ with compact support. Assume also that ϕ vanishes on the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$. Then, we use that u satisfies the entropy inequality in a distributional sense:

$$\begin{aligned}&\int_0^T\int_{-\infty}^{\infty}\Big[\partial_t\phi\big(\eta(u(x,t))\big)+\partial_x\phi\big(q(u(x,t))\big)\Big]\,dxdt+\int_{-\infty}^{\infty}\phi(x,0)\eta(u^0(x))\,dx\\&\geq-\int_0^T\int_{-\infty}^{\infty}\phi\nabla\eta(u(x,t))G(u(\cdot,t))(x)\,dxdt.\end{aligned} \tag{3.14}$$

We also view (3.12) as a distributional equality:

$$\begin{aligned}&\int_0^T\int_{-\infty}^{\infty}\Big[\partial_t\phi\big(\eta(\bar{u}(x+X(t),t))\big)+\partial_x\phi\big(q(\bar{u}(x+X(t),t))\big)\Big]\,dxdt+\int_{-\infty}^{\infty}\phi(x,0)\eta(\bar{u}^0(x))\,dx\\&=-\int_0^T\int_{-\infty}^{\infty}\phi\Bigg[\nabla\eta(\bar{u}(x+X(t),t))\left(\partial_x\Big|_{(x+X(t),t)}\big(\bar{u}(x,t)\big)\right)\dot{X}(t)\end{aligned}$$

$$+\nabla\eta(\bar{u}(x+X(t),t))G(\bar{u}(\cdot,t))(x+X(t))\Bigg]\,dxdt. \tag{3.15}$$

To get (3.15), we do integration by parts twice on the left-hand side of (3.12). Once on the domain $\{(x,t)\in\mathbb{R}\times[0,T)|x<s(t)-X(t)\}$ and once on the domain $\{(x,t)\in\mathbb{R}\times[0,T)|x>s(t)-X(t)\}$. We don't have a boundary term along the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$ because ϕ vanishes on this set.

We subtract (3.15) from (3.14), to get

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty}[\partial_t\phi\eta(u(x,t)|\bar{u}(x+X(t),t))+\partial_x\phi q(u(x,t);\bar{u}(x+X(t),t))]\,dxdt\\
&\quad+\int_{-\infty}^{\infty}\phi(x,0)\eta(u^0(x)|\bar{u}^0(x))\,dx\\
&\geq-\int_0^T\int_{-\infty}^{\infty}\Big(\partial_t\phi\nabla\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&\quad+\partial_x\phi\nabla\eta(\bar{u}(x+X(t),t))[f(u(x,t))-f(\bar{u}(x+X(t),t))]\Big)\,dxdt\\
&\quad-\int_{-\infty}^{\infty}\phi(x,0)\nabla\eta(\bar{u}^0(x))[u^0(x)-\bar{u}^0(x)]\,dx\\
&\quad+\int_0^T\int_{-\infty}^{\infty}\phi\Bigg[\nabla\eta(\bar{u}(x+X(t),t))\left(\partial_x\Big|_{(x+X(t),t)}(\bar{u}(x,t))\right)\dot{X}(t)\\
&+\nabla\eta(\bar{u}(x+X(t),t))G(\bar{u}(\cdot,t))(x+X(t))-\nabla\eta(u(x,t))G(u(\cdot,t))(x)\Bigg]\,dxdt.
\end{aligned}\tag{3.16}$$

The function u is a distributional solution to the system of conservation laws. Thus, for every Lipschitz continuous test function $\Phi:\mathbb{R}\times[0,T)\to\mathbb{M}^{1\times n}$ with compact support,

$$\int_0^T\int_{-\infty}^{\infty}\Big[\partial_t\Phi u+\partial_x\Phi f(u)\Big]\,dxdt+\int_{-\infty}^{\infty}\Phi(x,0)u^0(x)\,dx=-\int_0^T\int_{-\infty}^{\infty}\Phi G(u(\cdot,t))(x)\,dxdt. \tag{3.17}$$

We also can rewrite (3.11) in a distributional way, for Φ which have the additional property of vanishing on $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$:

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty}\Big[\partial_t\Phi\bar{u}(x+X(t),t)+\partial_x\Phi f(\bar{u}(x+X(t),t))\Big]\,dxdt+\int_{-\infty}^{\infty}\Phi(x,0)\bar{u}^0(x)\,dx\\
&=-\int_0^T\int_{-\infty}^{\infty}\Phi\left[\left(\partial_x\Big|_{(x+X(t),t)}(\bar{u}(x,t))\right)\dot{X}(t)+G(\bar{u}(\cdot,t))(x+X(t))\right]\,dxdt.
\end{aligned}\tag{3.18}$$

To prove (3.18), on the left-hand side of (3.11) we again do integration by parts twice. Once on the domain $\{(x,t)\in\mathbb{R}\times[0,T)|x<s(t)-X(t)\}$ and once on the domain $\{(x,t)\in\mathbb{R}\times[0,T)|x>s(t)-X(t)\}$. We lose the boundary terms along $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$ because Φ vanishes there.

Then, we can choose

$$\phi\nabla\eta(\bar{u}(x+X(t),t)) \tag{3.19}$$

as the test function Φ, and subtract (3.18) from (3.17). We can extend the function (3.19) to the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$ by defining it to be zero. This extension is still Lipschitz continuous.

This yields,

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty}\Bigg[\partial_t[\phi\nabla\eta(\bar{u}(x+X(t),t))][u(x,t)-\bar{u}(x+X(t),t)]\\
&\qquad+\partial_x[\phi\nabla\eta(\bar{u}(x+X(t),t))][f(u(x,t))-f(\bar{u}(x+X(t),t))]\Bigg]\,dxdt\\
&\qquad+\int_{-\infty}^{\infty}\phi(x,0)\nabla\eta(\bar{u}^0(x))[u^0(x)-\bar{u}^0(x)]\,dx\\
&=\int_0^T\int_{-\infty}^{\infty}\phi\nabla\eta(\bar{u}(x+X(t),t))\Bigg[\left(\partial_x\Big|_{(x+X(t),t)}(\bar{u}(x,t))\right)\dot{X}(t)\\
&\qquad+G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\Bigg]\,dxdt.
\end{aligned} \tag{3.20}$$

Recall $\bar{u}$ is a classical solution on the complement of the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)\}$ and verifies (3.11). Thus, on the complement of the set $\{(x,t)\in\mathbb{R}\times[0,T)|x=s(t)-X(t)\}$,

$$\begin{aligned}
&\partial_t\Big|_{(x,t)}\big(\nabla\eta(\bar{u}(x+X(t),t))\big)\\
&=\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)+\partial_t\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))\\
&=\Bigg(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)-\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\big[\nabla f(\bar{u}(x+X(t),t))\big]^T\\
&\qquad+G^T(\bar{u}(\cdot,t))(x+X(t))\Bigg)\nabla^2\eta(\bar{u}(x+X(t),t))\\
&=\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)+G^T(\bar{u}(\cdot,t))(x+X(t))\right)\nabla^2\eta(\bar{u}(x+X(t),t))\\
&\qquad-\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\nabla^2\eta(\bar{u}(x+X(t),t))\nabla f(\bar{u}(x+X(t),t)),
\end{aligned} \tag{3.21}$$

because $\big[\nabla f(\bar{u})\big]^T\nabla^2\eta(\bar{u})=\nabla^2\eta(\bar{u})\nabla f(\bar{u})$.

Thus, by (3.21) and the definition of the relative flux in (3.1),

$$
\begin{aligned}
&\partial_t\Big|_{(x,t)} \big(\nabla\eta(\bar{u}(x+X(t),t))\big)[u(x,t)-\bar{u}(x+X(t),t)]\\
&\quad+\partial_x\Big|_{(x,t)} \big(\nabla\eta(\bar{u}(x+X(t),t))\big)[f(u(x,t))-f(\bar{u}(x+X(t),t))]\\
&=\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\nabla^2\eta(\bar{u}(x+X(t),t))f(u(x,t)|\bar{u}(x+X(t),t))\\
&\quad+\left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\dot{X}(t)+G^T(\bar{u}(\cdot,t))(x+X(t))\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)\\
&\quad-\bar{u}(x+X(t),t)].
\end{aligned}
\tag{3.22}
$$

We combine (3.16), (3.20), and (3.22) to get

$$
\begin{aligned}
&\int_0^T\int_{-\infty}^{\infty} [\partial_t\phi\eta(u(x,t)|\bar{u}(x+X(t),t))+\partial_x\phi q(u(x,t);\bar{u}(x+X(t),t))]\,dxdt\\
&\quad+\int_{-\infty}^{\infty}\phi(x,0)\eta(u^0(x)|\bar{u}^0(x))\,dx\\
&\geq\int_0^T\int_{-\infty}^{\infty}\phi\Bigg[\nabla\eta(\bar{u}(x+X(t),t))\left(\partial_x\Big|_{(x+X(t),t)}(\bar{u}(x,t))\right)\dot{X}(t)\\
&\quad+\nabla\eta(\bar{u}(x+X(t),t))G(\bar{u}(\cdot,t))(x+X(t))-\nabla\eta(u(x,t))G(u(\cdot,t))(x)\\
&\quad+\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))f(u(x,t)|\bar{u}(x+X(t),t))\\
&\quad+\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)\right.\\
&\quad\left.+G^T(\bar{u}(\cdot,t))(x+X(t))\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&-\nabla\eta(\bar{u}(x+X(t),t))\left[\left(\partial_x\Big|_{(x+X(t),t)}(\bar{u}(x,t))\right)\dot{X}(t)+G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\right]\Bigg]\,dxdt\\
&=\int_0^T\int_{-\infty}^{\infty}\phi\Bigg[-\nabla\eta(u(x,t))G(u(\cdot,t))(x)\\
&\quad+\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))f(u(x,t)|\bar{u}(x+X(t),t))\\
&\quad+\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)\right.\\
&\quad\left.+G^T(\bar{u}(\cdot,t))(x+X(t))\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&\quad-\nabla\eta(\bar{u}(x+X(t),t))\Big[-G(u(\cdot,t))(x)\Big]\Bigg]\,dxdt.
\end{aligned}
\tag{3.23}
$$

Note that we can add zero, to get

$$
\begin{aligned}
&-\nabla\eta(u(x,t))G(u(\cdot,t))(x)+G^T(\bar{u}(\cdot,t))(x+X(t))\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&\quad-\nabla\eta(\bar{u}(x+X(t),t))\Big[-G(u(\cdot,t))(x)\Big]\\
&=-G^T(u(\cdot,t))(x)\Bigg(\big(\nabla\eta(u(x,t))\big)^T-\big(\nabla\eta(\bar{u}(x+X(t),t))\big)^T\\
&\quad-\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\Bigg)\\
&\quad+\Bigg(G^T(\bar{u}(\cdot,t))(x+X(t))-G^T(u(\cdot,t))(x)\Bigg)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&=-G^T(u(\cdot,t))(x)(\nabla\eta(u(x,t)|\bar{u}(x+X(t),t)))^T\\
&\quad+\Bigg(G^T(\bar{u}(\cdot,t))(x+X(t))-G^T(u(\cdot,t))(x)\Bigg)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&=-\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)\\
&\quad+\Bigg(G^T(\bar{u}(\cdot,t))(x+X(t))-G^T(u(\cdot,t))(x)\Bigg)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)].
\end{aligned}
\tag{3.24}
$$

This calculation is from [46].

Then, from (3.23) and (3.24), we get (3.8).

Step 2. Choose $0<\epsilon<\min\{T-t_0,\frac{1}{2}\delta\}$.

We apply the test function $\omega(t)\chi(x,t)$ to (3.8), where

$$
\omega(t):=\begin{cases}1 & \text{if } 0\le t<t_0\\ \frac{1}{\epsilon}(t_0-t)+1 & \text{if } t_0\le t<t_0+\epsilon\\ 0 & \text{if } t_0+\epsilon\le t,\end{cases}
\tag{3.25}
$$

and

$$
\chi(x,t):=\begin{cases}0 & \text{if } x<h_1(t)\\ \frac{1}{\epsilon}(x-h_1(t)) & \text{if } h_1(t)\le x<h_1(t)+\epsilon\\ 1 & \text{if } h_1(t)+\epsilon\le x\le h_2(t)-\epsilon\\ -\frac{1}{\epsilon}(x-h_2(t)) & \text{if } h_2(t)-\epsilon<x\le h_2(t)\\ 0 & \text{if } h_2(t)<x.\end{cases}
\tag{3.26}
$$

The function ω is modeled from [17, p. 124]. The function χ is from [34, p. 765]. We get,

$$
\begin{aligned}
&\int\limits_0^{t_0}\Bigg[-\int\limits_{h_1(t)}^{h_1(t)+\epsilon}\frac{1}{\epsilon}\dot{h}_1(t)\eta(u(x,t)|\bar{u}(x+X(t),t))\,dx+\int\limits_{h_1(t)}^{h_1(t)+\epsilon}\frac{1}{\epsilon}q(u(x,t);\bar{u}(x+X(t),t))\,dx\\
&\quad+\int\limits_{h_2(t)-\epsilon}^{h_2(t)}\frac{1}{\epsilon}\dot{h}_2(t)\eta(u(x,t)|\bar{u}(x+X(t),t))\,dx-\int\limits_{h_2(t)-\epsilon}^{h_2(t)}\frac{1}{\epsilon}q(u(x,t);\bar{u}(x+X(t),t))\,dx\Bigg]\,dt
\end{aligned}
$$

$$+\int_{h_1(0)}^{h_2(0)} \eta(u^0(x)|\bar{u}^0(x))\,dx - \int_{t_0}^{t_0+\epsilon} \frac{1}{\epsilon} \int_{h_1(t)}^{h_2(t)} \eta(u(x,t)|\bar{u}(x+X(t),t))\,dxdt + \mathcal{O}(\epsilon)$$

$$\geq \int_0^{t_0} \int_{h_1(t)}^{h_2(t)} \text{RHS}\,dxdt, \tag{3.27}$$

where RHS represents everything being multiplied by ϕ in the integral on the right-hand side of (3.8).

We let $\epsilon \to 0$ in (3.27). We use dominated convergence, the Lebegue differentiation theorem, and recall that u satisfies the strong trace property (Definition 2.1). This yields,

$$\begin{aligned}
&\int_0^{t_0} \Big[q(u(h_1(t)+,t);\bar{u}((h_1(t)+X(t))+,t)) - q(u(h_2(t)-,t);\bar{u}((h_2(t)+X(t))-,t)) \\
&\qquad + \dot{h}_2(t)\eta(u(h_2(t)-,t)|\bar{u}((h_2(t)+X(t))-,t)) \\
&\qquad - \dot{h}_1(t)\eta(u(h_1(t)+,t)|\bar{u}((h_1(t)+X(t))+,t)) \Big]\,dt \\
&\geq \int_{h_1(t_0)}^{h_2(t_0)} \eta(u(x,t_0)|\bar{u}(x+X(t_0),t_0))\,dx - \int_{h_1(0)}^{h_2(0)} \eta(u^0(x)|\bar{u}^0(x))\,dx + \int_0^{t_0}\int_{h_1(t)}^{h_2(t)} \text{RHS}\,dxdt,
\end{aligned} \tag{3.28}$$

where we also used the convexity of η to take the limit of the term

$$\int_{t_0}^{t_0+\epsilon} \frac{1}{\epsilon} \int_{h_1(t)}^{h_2(t)} \eta(u(x,t)|\bar{u}(x+X(t),t))\,dxdt \tag{3.29}$$

for every t_0 and not just almost every t_0.

We receive (3.7). □

4. Construction of the shift

In this section, we prove

PROPOSITION 4.1 (Existence of the shift function). *Fix $T>0$. Assume u is a bounded weak solution to (1.1). Assume u is entropic for the entropy η, and u has strong traces (Definition 2.1). Fix $i \in \{1,n\}$. Then let $(\bar{u}_+(t),\bar{u}_-(t),\dot{s}(t))$ be an i-shock for all $t \in [0,T)$, where $s:[0,T)\to\mathbb{R}$ is a Lipschitz continuous function. Assume also that the map $t \mapsto (\bar{u}_+(t),\bar{u}_-(t))$ is bounded. For $i=1$, assume the hypotheses $(\mathcal{H})$ hold. Likewise, if $i=n$, assume the hypotheses $(\mathcal{H})^*$ hold.*

Assume also that there exists $\rho>0$ such that for all $t\in[0,T)$

$$r(t) > \rho, \tag{4.1}$$

where $r(t)$ satisfies $S^1_{\bar{u}_-(t)}(r(t)) = \bar{u}_+(t)$.

Then, there exists a constant $a>0$ and a Lipschitz continuous map $h:[0,T)\to\mathbb{R}$ with $h(0)=s(0)$ and such that for almost every t,

$$a\big(q(u_+;\bar{u}_+(t)) - \dot{h}(t)\eta(u_+|\bar{u}_+(t))\big) - q(u_-;\bar{u}_-(t)) + \dot{h}(t)\eta(u_-|\bar{u}_-(t)) \leq -c\Big|\dot{s}(t)-\dot{h}(t)\Big|^2, \tag{4.2}$$

where $u_\pm := u(h(t)\pm,t)$. *The constants* $c,a>0$ *depend on* $\|u\|_{L^\infty}$, $\left\|\bar{u}_+(\cdot)\right\|_{L^\infty([0,T))}$, $\left\|\bar{u}_-(\cdot)\right\|_{L^\infty([0,T))}$, *and* ρ.

The proof of Proposition 4.1 uses

PROPOSITION 4.2. *Assume the hypotheses* $(\mathcal{H})$ *hold.*

Let $B,\rho>0$. *Then there exists a constant* $a_*\in(0,1)$ *depending on* B *and* ρ *such that the following is true:*

For any $a\in(0,a_*)$, *there exists a constant* c_1 *depending on* B, ρ, *and* a *such that*

$$\begin{aligned}&a\big(q(S^1_u(s);S^1_{u_L}(s_R))-\sigma^1_u(s)\eta(S^1_u(s)|S^1_{u_L}(s_R))\big)-q(u;u_L)+\sigma^1_u(s)\eta(u|u_L)\\&\leq -c_1\Big|\sigma^1_{u_L}(s_R)-\sigma^1_u(s)\Big|^2,\end{aligned}\tag{4.3}$$

for all $u_L\in\mathcal{V}$ *with* $|u_L|\leq B$, *all* $u\in\{u|\eta(u|u_L)\leq a\eta(u|S^1_{u_L}(s_R))\}$, *any* $s\in[0,B]$, *and any* $s_R\in[\rho,B]$.

Moreover,

$$a\big(q(u;S^1_{u_L}(s_R))-\lambda_1(u)\eta(u|S^1_{u_L}(s_R))\big)-q(u;u_L)+\lambda_1(u)\eta(u|u_L)\leq -c_1,\tag{4.4}$$

for all $u\in\{u|\eta(u|u_L)\leq a\eta(u|S^1_{u_L}(s_R))\}$ *and for the same constant* c_1.

REMARK 4.1. The proof of Proposition 4.2 holds when we only have $\eta\in C^2$.

Proposition 4.2 uses ideas from the proof of Lemma 4.3 in [27], but to prove Proposition 4.2 we keep careful track of the dependencies on the constants and make sure in our calculations to leave some extra negativity in the entropy dissipation lost at the shock $(u_L,u_R,\sigma_{L,R})$ (thus we have a negative right-hand side in our (4.3) and (4.4)). The idea of the extra negativity in the entropy dissipation is similar to the work [26,30].

To prove Proposition 4.2, we will need

COROLLARY 4.1. *Assume the system* (1.1) *satisfies the hypothesis* $(\mathcal{H}1)$. *Fix* $B,\rho>0$. *Then there exists* $k,\delta_0>0$ *depending on* B *and* ρ *such that for any* $\delta\in(0,\delta_0]$, $u\in\mathcal{V}\cap B_{r_0}(I_-)$ *with* $|u|\leq B$ *and for any* $s_0\in(\rho,B)$ *and* $s\geq 0$,

$$\begin{aligned}&q(S^1_u(s);S^1_u(s_0))-\sigma^1_u(s)\eta(S^1_u(s)|S^1_u(s_0))\leq -k\Big|\sigma^1_u(s)-\sigma^1_u(s_0)\Big|^2, &&\textit{for } |s-s_0|<\delta,\\&q(S^1_u(s);S^1_u(s_0))-\sigma^1_u(s)\eta(S^1_u(s)|S^1_u(s_0))\leq -k\delta\Big|\sigma^1_u(s)-\sigma^1_u(s_0)\Big|, &&\textit{for } |s-s_0|\geq\delta.\end{aligned}\tag{4.5}$$

Our proof of Corollary 4.1 is based on the proof of a very similar result in [27, p. 387-9]. We modify the proof in [27, p. 387-9] – being careful to keep the constants k and δ_0 uniform in s_0 and u.

The proof of Proposition 4.2 is based on the formulas (4.5), and this is where the negative right-hand sides in (4.3) and (4.4) come from.

Corollary 4.1 itself follows from Lemma 4.1 giving us an explicit formula for the entropy lost at an entropic i-shock $(u,S^i_u(s))$, for any i-family:

LEMMA 4.1. *For any i-shock* ($i\in\{1,\ldots,n\}$) $(u,S^i_u(s),\sigma^i_u(s))$ *and any* $v\in\mathbb{R}^n$,

$$q(S^i_u(s);v)-\sigma^i_u(s)\eta(S^i_u(s)|v)=q(u;v)-\sigma^i_u(s)\eta(u|v)+\int_0^s\frac{d}{dt}\sigma^i_u(t)\eta(u|S^i_u(t))\,dt.\tag{4.6}$$

Therefore, for any $s \geq 0, s_0 > 0$,

$$q(S_u^i(s);S_u^i(s_0)) - \sigma_u^i(s)\eta(S_u^i(s)|S_u^i(s_0)) = \int_{s_0}^{s} \frac{d}{dt}\sigma_u^i(t)\Big(\eta(u|S_u^i(t)) - \eta(u|S_u^i(s_0))\Big)\,dt. \tag{4.7}$$

The formulas (4.7) and (4.5) are modifications on a key lemma due to DiPerna [21]. See Lax [32] for the formula (4.6). For a proof of (4.6), see [47]. Note that (4.6) and (4.7) hold for a shock $(u, S_u^i(s), \sigma_u^i(s))$ from any i-family, $i = 1, 2, \ldots, n$, and not just extremal families (1-family or n-family) – the relation (4.6) is a direct consequence of the Rankine-Hugoniot condition. Further, (4.7) comes from applying (4.6) twice.

4.1. Proof of Corollary 4.1. This is based on the proof of a similar result in [27, p. 387-9].

Define

$$M \coloneqq \sup_{s\in(0,B),\ |u|\leq B} \frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s), \tag{4.8}$$

$$P \coloneqq \inf_{s\in(\rho,B),\ |u|\leq B} \frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s)). \tag{4.9}$$

Note that by Property (a) of $(\mathcal{H}1)$ $M < 0$ and by Property (b) of $(\mathcal{H}1)$ $P > 0$. Furthermore, note that M and P depend only on the system (1.1), (1.9), B and ρ.

Then by uniform continuity on the compact set $\{(s,u) | s \in [0,B] \text{ and } |u| \leq B\}$, there exists $\delta_0 > 0$ such that for all $s_0 \in (\rho, B)$ and for all $s \geq 0$ with $|s_0 - s| \leq \delta_0$,

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s) - \frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s_0)\right| \leq \frac{1}{2}|M|,\\
&\left|\frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s)) - \frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s_0))\right| \leq \frac{1}{2}P,
\end{aligned} \tag{4.10}$$

Note that δ_0 only depends on the system (1.1), (1.9), B and ρ.

In particular,

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s) - \frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s_0)\right| \leq \frac{1}{2}|M| \leq \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s_0)\right|,\\
&\left|\frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s)) - \frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s_0))\right| \leq \frac{1}{2}P \leq \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s_0))\right|.
\end{aligned} \tag{4.11}$$

From (4.11), we get the estimates

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s) = -\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right| \leq -\frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s_0)\right|,\\
&\frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s)) = \left|\frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s))\right| \geq \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s_0))\right|.
\end{aligned} \tag{4.12}$$

We use (4.7) and (4.12) to get for all s with $|s - s_0| < \delta_0$,

$$q(S_u^1(s);S_u^1(s_0)) - \sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_0)) = \int_{s_0}^{s} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_u^1(t)\Big(\eta(u|S_u^1(t)) - \eta(u|S_u^1(s_0))\Big)\,dt$$

$$\le -\frac{1}{4}\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_u^1(s_0)\right|\frac{\mathrm{d}}{\mathrm{d}t}\eta(u|S_u^1(s_0))\int_{s_0}^{s}(t-s_0)\,dt$$
$$= -\frac{1}{8}\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_u^1(s_0)\right|\frac{\mathrm{d}}{\mathrm{d}t}\eta(u|S_u^1(s_0))|s-s_0|^2. \tag{4.13}$$

Note that due to (4.11),

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right| \le \frac{3}{2}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s_0)\right|. \tag{4.14}$$

Thus,

$$\left|\sigma_u^1(s)-\sigma_u^1(s_0)\right| \le \frac{3}{2}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s_0)\right||s-s_0|, \tag{4.15}$$

which gives us that for all s verifying $|s-s_0|<\delta_0$,

$$q(S_u^1(s);S_u^1(s_0))-\sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_0)) \le -k_1\left|\sigma_u^1(s)-\sigma_u^1(s_0)\right|^2, \tag{4.16}$$

where we define

$$k_1 := \frac{1}{18}P\inf_{s\in(0,B),\ |u|\le B}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right|^{-1}. \tag{4.17}$$

Note that k_1 only depends on B and ρ.

On the other hand, we now show (4.5) for $|s-s_0|\ge\delta_0$. For all s verifying $s\le s_0-\delta_0$, we get from (4.7)

$$\begin{aligned}
&q(S_u^1(s);S_u^1(s_0))-\sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_0))\\
&=\int_s^{s_0-\delta_0}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_u^1(t)\Big(\eta(u|S_u^1(s_0))-\eta(u|S_u^1(t))\Big)\,\mathrm{d}t+\int_{s_0-\delta_0}^{s_0}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_u^1(t)\Big(\eta(u|S_u^1(s_0))-\eta(u|S_u^1(t))\Big)\,\mathrm{d}t\\
&:=I_1+I_2.
\end{aligned} \tag{4.18}$$

Note that for a positive constant c_1 satisfying

$$c_1 \le \inf_{s_0\in[\delta_0,B]\text{ and }|u|\le B}\Big(\eta(u|S_u^1(s_0))-\eta(u|S_u^1(s_0-\delta_0))\Big), \tag{4.19}$$

then we have (recalling Property (a) of hypothesis $(\mathcal{H}1)$)

$$\begin{aligned}
I_1 &\le \int_s^{s_0-\delta_0}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_u^1(t)\Big(\eta(u|S_u^1(s_0))-\eta(u|S_u^1(s_0-\delta_0))\Big)\,\mathrm{d}t\\
&\le -c_1\left|\sigma_u^1(s_0-\delta_0)-\sigma_u^1(s)\right|\\
&\le -c_1\left|\sigma_u^1(s_0)-\sigma_u^1(s)\right|+c_1\left|\sigma_u^1(s_0)-\sigma_u^1(s_0-\delta_0)\right|\\
&\le -c_1\left|\sigma_u^1(s_0)-\sigma_u^1(s)\right|+c_1\delta_0\sup_{s\in(0,B),\ |u|\le B}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right|.
\end{aligned} \tag{4.20}$$

Recall that δ_0 depends only on B and ρ. Thus, we can find a c_1 which satisfies (4.19) and depends only on B and ρ. In particular, note that

$$\delta_0 P \leq \inf_{s_0\in[\delta_0,B] \text{ and } |u|\leq B} \Big(\eta(u|S_u^1(s_0)) - \eta(u|S_u^1(s_0-\delta_0))\Big). \tag{4.21}$$

Note that for $t\in(s_0-\delta_0,s_0)$,

$$\eta(u|S_u^1(s_0)) - \eta(u|S_u^1(t)) = \int_t^{s_0} \frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s))\,ds \geq P(s_0-t). \tag{4.22}$$

Thus,

$$I_2 \leq PM \int_{s_0-\delta_0}^{s_0} (s_0-t)\,dt = \frac{\delta_0^2 PM}{2}. \tag{4.23}$$

Recall $M<0$.

Pick

$$c_1 := -\delta_0 k_2, \tag{4.24}$$

where

$$k_2 := \min\left\{\frac{PM}{2\sup_{s\in(0,B),\ |u|\leq B}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right|}, P\right\}. \tag{4.25}$$

Note that k_2 depends only on B and ρ.

Then from (4.18), (4.21), (4.20), and (4.23), we get

$$q(S_u^1(s);S_u^1(s_0)) - \sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_0)) \leq -\delta_0 k_2\left|\sigma_u^1(s_0)-\sigma_u^1(s)\right|. \tag{4.26}$$

The case for $s>s_0+\delta_0$ is analogous to the case for $s\leq s_0-\delta_0$: For $s>s_0+\delta_0$, consider a constant $c_2>0$ such that

$$c_2 \leq \inf_{s_0\in[\rho,B] \text{ and } |u|\leq B} \Big(\eta(u|S_u^1(s_0+\delta_0)) - \eta(u|S_u^1(s_0))\Big), \tag{4.27}$$

Note that δ_0 only depends on B and ρ. Thus, we can find a constant c_2 verifying (4.27) and depending only on B and ρ. In particular, note that

$$\delta_0 P \leq \inf_{s_0\in[\rho,B] \text{ and } |u|\leq B} \Big(\eta(u|S_u^1(s_0+\delta_0)) - \eta(u|S_u^1(s_0))\Big). \tag{4.28}$$

Then write (recalling (4.7)),

$$\begin{aligned}
&q(S_u^1(s);S_u^1(s_0)) - \sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_0)) \\
&= \int_{s_0}^{s_0+\delta_0} \frac{\mathrm{d}}{\mathrm{dt}}\sigma_u^1(t)\Big(\eta(u|S_u^1(t)) - \eta(u|S_u^1(s_0))\Big)\,\mathrm{dt}
\end{aligned}$$

$$+\int\limits_{s_0+\delta_0}^{s} \frac{\mathrm{d}}{\mathrm{dt}}\sigma_u^1(t)\Big(\eta(u|S_u^1(t))-\eta(u|S_u^1(s_0))\Big)\,\mathrm{d}t$$
$$:=J_1+J_2. \tag{4.29}$$

Then,

$$J_2 \le \int\limits_{s_0+\delta_0}^{s} \frac{\mathrm{d}}{\mathrm{dt}}\sigma_u^1(t)\Big(\eta(u|S_u^1(s_0+\delta_0))-\eta(u|S_u^1(s_0))\Big)\,\mathrm{d}t$$
$$\le c_2 \int\limits_{s_0+\delta_0}^{s} \frac{\mathrm{d}}{\mathrm{dt}}\sigma_u^1(t)\,\mathrm{d}t \tag{4.30}$$

Then, by Property (a) of hypothesis ($\mathcal{H}1$),

$$= -c_2\Big|\sigma_u^1(s)-\sigma_u^1(s_0+\delta_0)\Big|$$
$$\le -c_2\Big|\sigma_u^1(s)-\sigma_u^1(s_0)\Big| + c_2\Big|\sigma_u^1(s_0+\delta_0)-\sigma_u^1(s_0)\Big|$$
$$\le -c_2\Big|\sigma_u^1(s)-\sigma_u^1(s_0)\Big| + c_2\delta_0 \sup_{s\in(0,B),\ |u|\le B}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right|.$$

Note that for $t\in(s_0,s_0+\delta_0)$,

$$\eta(u|S_u^1(t))-\eta(u|S_u^1(s_0)) = \int\limits_{s_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\eta(u|S_u^1(s))\,ds$$
$$\ge P(t-s_0). \tag{4.31}$$

Thus,

$$J_1 \le PM \int\limits_{s_0}^{s_0+\delta_0} (t-s_0)\,dt$$
$$= \frac{\delta_0^2 PM}{2}. \tag{4.32}$$

Recall $M<0$.

Pick

$$c_2 := -\delta_0 k_3, \tag{4.33}$$

where

$$k_3 := \min\left\{\frac{PM}{2\sup_{s\in(0,B),\ |u|\le B}\left|\frac{\mathrm{d}}{\mathrm{d}s}\sigma_u^1(s)\right|}, P\right\}. \tag{4.34}$$

Note that k_3 depends only on B and ρ.

Then from (4.29), (4.28), (4.30), and (4.32), we get

$$q(S_u^1(s);S_u^1(s_0)) - \sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_0)) \le -\delta_0 k_3\Big|\sigma_u^1(s_0)-\sigma_u^1(s)\Big|. \tag{4.35}$$

REMARK 4.2. Note that in hypothesis ($\mathcal{H}1$), we assume the 1-shock curve S_u^1 is parameterized by arc length. Thus, if $s<B$ then $\left|S_u^1(s)\right|<B$.

4.2. Proof of Proposition 4.2. This proof is based on the proof of Lemma 4.3 in [27].

In what follows, we use C to denote a generic constant which only depends on B and ρ.

Also, for convenience we define

$$u_R \coloneqq S^1_{u_L}(s_R) \tag{4.36}$$
$$R_a \coloneqq \{u | \eta(u|u_L) \leq a\eta(u|u_R)\}. \tag{4.37}$$

Step 1. We first need to show that for any fixed $\sigma_0 \in \mathbb{R}$ such that $\lambda_1(u_L) > \sigma_0$, there exists $\beta, \epsilon_0 > 0$ such that

$$-q(u;u_L) + \sigma_0 \eta(u|u_L) \leq -\beta \eta(u|u_L), \tag{4.38}$$

for all $u \in B_{\epsilon_0}(u_L)$.

The difference between $\lambda_1(u_L)$ and σ_0 will power the proof of (4.3). We will choose a σ_0 later.

We use Taylor expansion to prove (4.38):

$$-q(u;u_L) + \sigma_0 \eta(u|u_L) = (u-u_L)^T \nabla^2 \eta(u_L)(\sigma_0 I - \nabla f(u_L))(u-u_L) + \mathcal{O}(|u-u_L|^3) \tag{4.39}$$

Due to the strict convexity of η, $\nabla^2\eta(u_L)$ is symmetric and strictly positive definite. Also, by assumption $\nabla^2\eta(u_L)\nabla f(u_L)$ is symmetric. Thus these two matrices are diagonalizable in the same basis. We receive,

$$\nabla^2 \eta(u_L) \nabla f(u_L) \geq \lambda_1(u_L) \nabla^2 \eta(u_L). \tag{4.40}$$

Let $C_1 > 0$ be a constant such that the term $\mathcal{O}(|u-u_L|^3)$ in (4.39) satisfies $\mathcal{O}(|u-u_L|^3) \leq C_1 |u-u_L|^3$ for all $|u_L| \leq B$ and all $u \in B_1(u_L)$. Note C_1 depends only on B. Let

$$C_2 \coloneqq \inf_{|x|=1\,,\,|u_L|\leq B} x^T \nabla^2 \eta(u_L) x. \tag{4.41}$$

Note that because η is strictly convex, $C_2 > 0$. Note C_2 depends only on B.

Then, for all

$$\epsilon_0 < \min\{\frac{C_2}{2C_1}(\lambda_1(u_L) - \sigma_0), 1\} \tag{4.42}$$

and for all $u \in B_{\epsilon_0}(u_L)$, we have from (4.40) and because $\lambda_1(u_L) > \sigma_0$,

$$\begin{aligned} -q(u;u_L) + \sigma_0 \eta(u|u_L) &\leq -(\lambda_1(u_L) - \sigma_0)(u-u_L)^T \nabla^2 \eta(u_L)(u-u_L) + \mathcal{O}(|u-u_L|^3) \\ &\leq -\frac{(\lambda_1(u_L) - \sigma_0)}{2}(u-u_L)^T \nabla^2 \eta(u_L)(u-u_L) \\ &\leq -C\frac{(\lambda_1(u_L) - \sigma_0)}{2} \eta(u|u_L) \end{aligned} \tag{4.43}$$

by Lemma 1.1. This proves (4.38), with

$$\beta = C\frac{(\lambda_1(u_L) - \sigma_0)}{2}. \tag{4.44}$$

Step 2. We can now compute to show (4.3).

In the context of Corollary 4.1, we can use the same value of B in Corollary 4.1 as in Proposition 4.2. In Corollary 4.1, we have constants k and δ_0. Note that these constants depend on B and ρ. In the context of Corollary 4.1, we are allowed to choose δ as long as it is sufficiently small. Choose

$$\delta := \min\{\delta_0, \frac{s_R}{2}\} \tag{4.45}$$

for the δ in Corollary 4.1. Note that δ depends on B and ρ. Then, define

$$k^* := \min\{\delta k, k\}. \tag{4.46}$$

Note that k^* depends on B and ρ.

Define the following quantities,

$$M := \sup_{0\le s\le B\ ,\ |u|\le B+1} \frac{d}{ds}\sigma_u^1(s), \tag{4.47}$$

where the constant M exists and satisfies $M<0$ because by the hypotheses $(\mathcal{H}1)$, $(s,u)\mapsto \sigma_u^1(s)$ is C^1 and $\frac{d}{ds}\sigma_u^1(s)<0$. We further define,

$$L := \sup_{|u|\le B+1} \|\nabla\lambda_1\|, \tag{4.48}$$

$$\sigma_0 := \lambda_1(u_L) + \frac{k^* M}{16C_3}\frac{s_R}{2}, \tag{4.49}$$

where C_3 will appear later, in (4.73) – and C_3 will depend on B. The constant L exists because by assumption the flux $f\in C^2(\mathcal{V})$ (see the remarks after the hypotheses $(\mathcal{H})$ and $(\mathcal{H})^*$). Note M and L depend only on B.

We choose ϵ_0 such that

$$\epsilon_0 < \min\left\{ -\frac{k^* M}{16C_3}\frac{s_R}{2}\frac{1}{L}, -\frac{C_2}{C_1}\frac{k^* M}{16C_3}\frac{s_R}{2}\frac{1}{L}, -\frac{C_2}{C_1}\frac{k^* M}{16C_3}\frac{s_R}{2}, 1\right\}. \tag{4.50}$$

Note the right-hand side of (4.50) depends on B and ρ. We also need to make sure that a_* is small enough such that $R_a\subset B_{\epsilon_0}(u_L)$ for all $0<a<a_*$. Recall (3.6).

We claim that for all $u\in B_{\epsilon_0}(u_L)$,

$$\sigma_u^1(s)\le\sigma_0, \quad \text{for } s\ge\frac{s_R}{2}, \tag{4.51}$$

and

$$\lambda_1(u)-\sigma_0 \le \frac{k^*}{8C_3}\left|\sigma_u^1(\frac{s_R}{2})-\sigma_u^1(s_R)\right|. \tag{4.52}$$

We show (4.51): for $s\ge\frac{s_R}{2}$,

$$\begin{aligned}\sigma_u^1(s) &\le \sigma_u^1(0)+sM\\ &=\lambda_1(u)+sM\\ &\le \lambda_1(u_L)-\frac{k^* M}{16C_3}\frac{s_R}{2}+sM\end{aligned}$$

$$
\begin{aligned}
&= \lambda_1(u_L) + M\Big(s - \frac{k^*}{16C_3}\frac{s_R}{2}\Big)\\
&\leq \lambda_1(u_L) + M\Big(\frac{s_R}{2} - \frac{k^*}{16C_3}\frac{s_R}{2}\Big)\\
&= \lambda_1(u_L) + M\frac{s_R}{2}\Big(1 - \frac{k^*}{16C_3}\Big)\\
&< \sigma_0, \qquad (4.53)
\end{aligned}
$$

where to get the last inequality we can make C_3 larger if necessary such that $\frac{k^*}{16C_3} < \frac{1}{2}$, noting C_3 will then depend on ρ and B.

We now show (4.52):

$$
\begin{aligned}
\lambda_1(u) - \sigma_0 &\leq \lambda_1(u_L) - \frac{k^* M}{16C_3}\frac{s_R}{2} - \sigma_0\\
&= \lambda_1(u_L) - \frac{k^* M}{16C_3}\frac{s_R}{2} - \lambda_1(u_L) - \frac{k^* M}{16C_3}\frac{s_R}{2}\\
&= -\frac{k^* M}{8C_3}\frac{s_R}{2} \leq \frac{k^*}{8C_3}\Big|\sigma_u^1\Big(\frac{s_R}{2}\Big) - \sigma_u^1(s_R)\Big|, \qquad (4.54)
\end{aligned}
$$

by definition of M.

To prove (4.3), we consider two cases: $s \geq \frac{s_R}{2}$ and $s < \frac{s_R}{2}$.

We first consider $s \geq \frac{s_R}{2}$. From (3.5), we get

$$
\begin{aligned}
&q(S_u^1(s); u_R) - \sigma_u^1(s)\eta(S_u^1(s)|u_R)\\
&= -\big(q(u_R; S_u^1(s_R)) - \sigma_u^1(s)\eta(u_R|S_u^1(s_R))\big) + \big(q(S_u^1(s); S_u^1(s_R)) - \sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_R))\big)\\
&\quad + \big(\nabla\eta(u_R) - \nabla\eta(S_u^1(s_R))\big)\big(f(u_R) - f(S_u^1(s)) - \sigma_u^1(s)(u_R - S_u^1(s))\big). \qquad (4.55)
\end{aligned}
$$

By using the Rankine-Hugoniot jump compatibility conditions

$$
f(u_R) - f(u_L) = \sigma_{u_L}^1(s_R)(u_R - u_L), \qquad (4.56)
$$

$$
f(S_u^1(s)) - f(u) = \sigma_u^1(s)(S_u^1(s) - u), \qquad (4.57)
$$

we can rewrite

$$
\begin{aligned}
&q(S_u^1(s); u_R) - \sigma_u^1(s)\eta(S_u^1(s)|u_R)\\
&= -\big(q(u_R; S_u^1(s_R)) - \sigma_u^1(s)\eta(u_R|S_u^1(s_R))\big)\\
&\quad + \big(q(S_u^1(s); S_u^1(s_R)) - \sigma_u^1(s)\eta(S_u^1(s)|S_u^1(s_R))\big)\\
&\quad + \big(\nabla\eta(u_R) - \nabla\eta(S_u^1(s_R))\big)\big(f(u_L) - f(u) - \sigma_u^1(s)(u_L - u)\\
&\quad + (\sigma_{u_L}^1(s_R) - \sigma_u^1(s))(u_R - u_L)\big)\\
&:= I_1 + I_2 + I_3. \qquad (4.58)
\end{aligned}
$$

To estimate I_2 and I_3, we use the following rough estimates. In these estimates, the constants are uniform in u_L (with $|u_L| \leq B$) and $s_R \in [\rho, B]$. The estimates hold for any $u \in B_{\epsilon_0}(u_L)$ (recall by (4.50), $\epsilon_0 < 1$). Recall that by the hypothesis ($\mathcal{H}1$), $(s,u) \mapsto S_u^1(s)$ is C^1. Then,

$$
\Big|\eta(S_{u_L}^1(s_R)|S_u^1(s_R))\Big| \leq C\Big|S_{u_L}^1(s_R) - S_u^1(s_R)\Big|^2 \leq C|u_L - u|^2,
$$

because $\eta\in C^2$ and by Lemma 1.1, $\eta(a|b)$ is locally quadratic in $a-b$. Continuing,

$$\left|(q(S^1_{u_L}(s_R);S^1_u(s_R))\right|\leq C\left|S^1_{u_L}(s_R)-S^1_u(s_R)\right|^2\leq C|u_L-u|^2,$$

because $q\in C^2$ and from Taylor's theorem $q(a;b)$ is locally quadratic in $a-b$ (expand $q(a)$ and $f(a)$ around the point b).

Further,

$$\left|\nabla\eta(S^1_{u_L}(s_R))-\nabla\eta(S^1_u(s_R))\right|\leq C\left|S^1_{u_L}(s_R)-S^1_u(s_R)\right|\leq C|u_L-u|,$$

because $\eta\in C^2(\mathcal{V})$. Lastly,

$$\left|\sigma^1_{u_L}(s_R)-\sigma^1_u(s_R)\right|\leq C|u_L-u|,$$

because by the hypothesis $(\mathcal{H}1)$, $(s,u)\mapsto\sigma^1_u(s)$ is C^1. (4.59)

Then, from the estimates (4.59), we get

$$\begin{aligned}I_1&=-q(u_R;S^1_u(s_R))+\sigma^1_u(s)\eta(S^1_{u_L}(s_R)|S^1_u(s_R))\\&=-q(u_R;S^1_u(s_R))+\sigma^1_u(s_R)\eta(S^1_{u_L}(s_R)|S^1_u(s_R))+(\sigma^1_u(s)-\sigma^1_u(s_R))\eta(S^1_{u_L}(s_R)|S^1_u(s_R))\\&\leq C|u_L-u|^2(1+\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|),\end{aligned}\tag{4.60}$$

and

$$\begin{aligned}I_3=&\left(\nabla\eta(u_R)-\nabla\eta(S^1_u(s_R))\right)\left(f(u_L)-f(u)-\sigma^1_u(s)(u_L-u)\right.\\&\qquad\left.+(\sigma^1_{u_L}(s_R)-\sigma^1_u(s))(u_R-u_L)\right)\\&\leq C|u_L-u|\left(|u_L-u|+\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right||u_L-u|+\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|\right).\end{aligned}\tag{4.61}$$

To control I_2, we use Corollary 4.1. Note first that

$$\begin{aligned}\left|\sigma^1_u(s)-\sigma^1_{u_L}(s_R)\right|^2&\leq\left(\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|+\left|\sigma^1_u(s_R)-\sigma^1_{u_L}(s_R)\right|\right)^2\\&\leq\left(\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|+C|u-u_L|\right)^2\\&=\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|^2+2C|u-u_L|\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|+C^2|u-u_L|^2.\end{aligned}\tag{4.62}$$

Then, for $|s-s_R|<\delta$ we use Corollary 4.1 and (4.62) above:

$$\begin{aligned}I_2&=q(S^1_u(s);S^1_u(s_R))-\sigma^1_u(s)\eta(S^1_u(s)|S^1_u(s_R))\\&\leq-k^*\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|^2\\&=-\frac{k^*}{2}\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|^2-\frac{k^*}{2}\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|^2\\&\leq-\frac{k^*}{2}\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|^2-\frac{k^*}{2}\left|\sigma^1_u(s)-\sigma^1_{u_L}(s_R)\right|^2\\&\qquad+Ck^*|u-u_L|\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|+\frac{k^*}{2}C^2|u-u_L|^2\\&=-\frac{k^*}{2}\left|\sigma^1_u(s)-\sigma^1_u(s_R)\right|^2-\frac{k^*}{2}\left|\sigma^1_u(s)-\sigma^1_{u_L}(s_R)\right|^2\end{aligned}$$

$$+C|u-u_L|\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2, \tag{4.63}$$

where in the last equality we just absorb some constants into the C.

Then, if $|s-s_R|<\delta$, we use our estimates on I_1, I_2, and I_3 to get

$$q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\leq -\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|^2-\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_{u_L}^1(s_R)\right|^2$$
$$+C|u-u_L|\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2$$
$$\leq -\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_{u_L}^1(s_R)\right|^2+C(|u-u_L|^2+|u-u_L|^4),$$

where we have used the version of Young's inequality with ϵ. Continuing,

$$\leq -\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_{u_L}^1(s_R)\right|^2+C|u-u_L|^2, \tag{4.64}$$

because $u\in B_{\epsilon_0}(u_L)$ and by (4.50), $\epsilon_0<1$.

Thus, putting everything together, we have for $s\geq \frac{s_R}{2}$ and $|s-s_R|<\delta$,

$$a\big(q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\big)-q(u;u_L)+\sigma_u^1(s)\eta(u|u_L)$$
$$\leq aC|u-u_L|^2-\frac{ak^*}{2}\left|\sigma_u^1(s)-\sigma_{u_L}^1(s_R)\right|^2-q(u;u_L)+\sigma_0\eta(u|u_L), \tag{4.65}$$

by (4.64) and (4.51). Continuing,

$$\leq aC|u-u_L|^2-\frac{ak^*}{2}\left|\sigma_u^1(s)-\sigma_{u_L}^1(s_R)\right|^2-\beta\eta(u|u_L), \tag{4.66}$$

by (4.38). We recall Lemma 1.1, and choose a_* small enough such that $aC|u-u_L|^2-\beta\eta(u|u_L)\leq 0$ for all u. As always, we also require that a_* is small enough such that $R_a\subset B_{\epsilon_0}(u_L)$ for all $0<a<a_*$ (recall the condition (3.6)). This proves (4.3).

When $s\geq\frac{s_R}{2}$ and $|s-s_R|>\delta$, using Corollary 4.1 and our estimates on I_1 and I_3 (4.60) and (4.61),

$$q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\leq -k^*\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|$$
$$+C|u-u_L|^2\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2$$
$$=-\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|-\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|$$
$$+C|u-u_L|\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2$$
$$\leq -\frac{k^*}{2}\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|+C|u-u_L|^2, \tag{4.67}$$

because $u\in B_{\epsilon_0}(u_L)$ and we pick ϵ_0 even smaller such that $\epsilon_0<\min\{\frac{k^*}{4C},1\}$. Recall we require that a_* is small enough such that $R_a\subset B_{\epsilon_0}(u_L)$ for all $0<a<a_*$ (see (3.6)).

Putting everything together, for $s\geq\frac{s_R}{2}$ and $|s-s_R|>\delta$,

$$a\big(q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\big)-q(u;u_L)+\sigma_u^1(s)\eta(u|u_L)$$
$$\leq aC|u-u_L|^2-\frac{ak^*}{2}\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|-q(u;u_L)+\sigma_0\eta(u|u_L) \tag{4.68}$$

by (4.67) and (4.51). Continuing,

$$\leq aC|u-u_L|^2 - \frac{ak^*}{2}\Big|\sigma_u^1(s)-\sigma_u^1(s_R)\Big| - \beta\eta(u|u_L) \tag{4.69}$$

by (4.38). We again recall Lemma 1.1, and choose a_* small enough such that $aC|u-u_L|^2-\beta\eta(u|u_L)\leq 0$ for all u. Recall, we always require that a_* is small enough such that $R_a\subset B_{\epsilon_0}(u_L)$ for all $0<a<a_*$ (use condition (3.6)). Again note that with σ_0 defined in (4.49) and β defined in (4.44), $\beta = Cs_R$. Finally, we get the right-hand side of (4.3) by noting that $\left|\sigma_u^1(s)-\sigma_u^1(s_R)\right|$ will be uniformly bounded from below for all $|s-s_R|>\delta$ (with $s\in[0,B]$ and $s_R\in[\rho,B]$), because by Property (a) of $(\mathcal{H}1)$, $\frac{d}{ds}\sigma_u^1(s)<0$. Furthermore, the term $\left|\sigma_{u_L}^1(s_R)-\sigma_u^1(s)\right|^2$ on the right-hand side of (4.3) is bounded (with the bound depending on B). Thus, by making c_1 sufficiently small, this proves (4.3). Recall also that δ depends on B and ρ. Thus, c_1 depends on B and ρ.

On the other hand, we now consider $s<\frac{s_R}{2}$. From (4.45), we have $\delta<\frac{s_R}{2}$. Thus when $s<\frac{s_R}{2}$, $|s-s_R|>\delta$.

The computations in (4.67) apply exactly. We get again,

$$q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\leq -\frac{k^*}{2}\Big|\sigma_u^1(s)-\sigma_u^1(s_R)\Big| + C|u-u_L|^2, \tag{4.70}$$

again because $u\in B_{\epsilon_0}(u_L)$ and ϵ_0 verifies $\epsilon_0<\frac{k^*}{4C}$.

Then, because by the assumptions $(\mathcal{H})$ $\frac{d}{ds}\sigma_u^1(s)<0$, we have for all $s<\frac{s_R}{2}$,

$$q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\leq -\frac{k^*}{2}\left|\sigma_u^1\Big(\frac{s_R}{2}\Big)-\sigma_u^1(s_R)\right| + C|u-u_L|^2 \tag{4.71}$$

Then, for ϵ_0 small enough such that $C\epsilon_0^2\leq\frac{k^*Ms_R}{8}$ (where M is from (4.47)),

$$\leq -\frac{k^*}{4}\left|\sigma_u^1\Big(\frac{s_R}{2}\Big)-\sigma_u^1(s_R)\right|. \tag{4.72}$$

Recall we also need a_* sufficiently small so that $R_a\subset B_{\epsilon_0}(u_L)$ for all $0<a<a_*$. See (3.6).

To control the left-hand side of the entropy dissipation in (4.3), we estimate

$$-q(u;u_L)+\sigma_u^1(s)\eta(u|u_L)\leq -q(u;u_L)+\lambda_1(u)\eta(u|u_L),$$

because by the assumptions $(\mathcal{H})$ $\frac{d}{ds}\sigma_u^1(s)<0$ and $\sigma_u^1(0)=\lambda_1(u)$. Continuing,

$$\begin{aligned}&= -q(u;u_L)+\sigma_0\eta(u|u_L)+(\lambda_1(u)-\sigma_0)\eta(u|u_L)\\ &\leq (\lambda_1(u)-\sigma_0)\eta(u|u_L),\end{aligned}$$

by (4.38). Continuing,

$$\leq a(\lambda_1(u)-\sigma_0)\eta(u|u_R),$$

because $u\in R_a\subset B_{\epsilon_0}(u_L)$. Continuing, recall $\epsilon_0<1$ by (4.50). Furthermore, recall Lemma 1.1, $|u_L|\leq B$, $s_R\leq B$, and $S_{u_L}^1$ is parameterized by arc length. Then, we get

$$\leq aC_3(\lambda_1(u)-\sigma_0). \tag{4.73}$$

Note C_3 is a constant which depends on B.

Putting everything together, for all $s < \frac{s_R}{2}$,

$$\begin{aligned}&a\Big(q(S_u^1(s);u_R)-\sigma_u^1(s)\eta(S_u^1(s)|u_R)\Big)-q(u;u_L)+\sigma_u^1(s)\eta(u|u_L)\\&\le -a\Big(\frac{k^*}{4}\Big|\sigma_u^1(\frac{s_R}{2})-\sigma_u^1(s_R)\Big|-C_3(\lambda_1(u)-\sigma_0)\Big)\\&\le -a\Big(\frac{k^*}{8}\Big|\sigma_u^1(\frac{s_R}{2})-\sigma_u^1(s_R)\Big|\Big),\end{aligned}$$

by (4.52). Continuing,

$$\le \frac{aMk^*s_R}{16}, \tag{4.74}$$

where M is from (4.47). Recall $M<0$.

Note that the term $\left|\sigma_{u_L}^1(s_R)-\sigma_u^1(s)\right|^2$ on the right-hand side of (4.3) is bounded (with the bound depending on B), so we get the right-hand side of (4.3) by making c_1 smaller if necessary. Note that in making this adjustment to c_1, c_1 will depend on B and ρ. This proves (4.3).

Lastly, we get (4.4) by the same computation (4.74) and taking $s=0$. Recall that by the hypothesis $(\mathcal{H}1)$, $\sigma_u^1(0)=\lambda_1(u)$.

4.3. Proof of Proposition 4.1. By the remark about taking the negative of the flux $(-f)$ if necessary, we can assume that $(\bar{u}_+(t),\bar{u}_-(t),\dot{s}(t))$ is a 1-shock.

We will use Proposition 4.2. The 1-shock $(\bar{u}_+(t),\bar{u}_-(t),\dot{s}(t))$ in Proposition 4.1 will play the role of $(u_L,S_{u_L}^1(s_R))$ in Proposition 4.2. Take $R:=\max\{\|u\|_{L^\infty},\left\|\bar{u}_-(\cdot)\right\|_{L^\infty([0,T))}\}$ and then take the $\tilde{S}$ corresponding to this R as in Property (c) of $(\mathcal{H}1)$. Define the B in Proposition 4.2 to be $B:=\max\{R,\tilde{S},\left\|\bar{u}_+(\cdot)\right\|_{L^\infty([0,T))}\}$. Then, we have that for all (u_-,u_+,σ) 1-shock with $u_+,u_-<R$, there exists $s\in(0,B)$ such that $u_+=S_{u_-}^1(s)$. Further, note that B depends on $\|u\|_{L^\infty}$ and $\left\|\bar{u}_-(\cdot)\right\|_{L^\infty([0,T))}$.

Then, pick $0<a<1$ as in Proposition 4.2. Here, a is playing the same role as the a in Proposition 4.2.

Throughout this proof, c denotes a generic constant that depends on $\|u\|_{L^\infty}$, ρ, $\left\|\bar{u}_+(\cdot)\right\|_{L^\infty([0,T))}$, $\left\|\bar{u}_-(\cdot)\right\|_{L^\infty([0,T))}$, and a.

Note by Proposition 4.2, the constant a depends on $\|u\|_{L^\infty}$, $\left\|\bar{u}_-(\cdot)\right\|_{L^\infty([0,T))}$, $\left\|\bar{u}_+(\cdot)\right\|_{L^\infty([0,T))}$, and ρ.

Step 1. We now show that for any $\gamma_0>0$,

$$\inf \eta(u|u_L)-a\eta(u|u_R)\ge c_4\gamma_0^2 \tag{4.75}$$

for a constant $c_4>0$, where the infimum runs over all (u,u_L,u_R) such that $\operatorname{dist}(u,\{w|\eta(w|u_L)\le a\eta(w|u_R)\})\ge\gamma_0$ and $|u_L|,|u_R|\le B$. Here, B is from Proposition 4.2 and the distance $\operatorname{dist}(x,A)$ between a point x and a set A is defined in the usual way,

$$\operatorname{dist}(x,A):=\inf_{y\in A}|x-y|. \tag{4.76}$$

Consider any triple (u,u_L,u_R) such that $\operatorname{dist}(u,\{w|\eta(w|u_L)\le a\eta(w|u_R)\})\ge\gamma_0$ and $|u_L|,|u_R|\le B$.

By Proposition 4.2, the set $\{w|\eta(w|u_L)\le a\eta(w|u_R)\}$ is compact. Thus, there exists $w_0\in\{w|\eta(w|u_L)\le a\eta(w|u_R)\}$ such that

$$|u-w_0|=\operatorname{dist}(u,\{w|\eta(w|u_L)\le a\eta(w|u_R)\}). \tag{4.77}$$

We Taylor expand the function

$$\Gamma(u) := \eta(u|u_L) - a\eta(u|u_R) \tag{4.78}$$

around the point w_0:

$$\Gamma(u) = \Gamma(w_0) + \nabla\Gamma(w_0)(u-w_0) + \int_0^1 (1-t)(u-w_0)^T \nabla^2\Gamma(w_0 + t(u-w_0))(u-w_0)\,dt. \tag{4.79}$$

By definition of w_0, we must have $\Gamma(w_0) = 0$ and $\nabla\Gamma(w_0)(u-w_0) \geq 0$.

Note that $\nabla^2\Gamma = (1-a)\nabla^2\eta$. Thus, by strict convexity of η and because $0 < a < 1$, we have $\nabla^2\Gamma \geq cI$ for some constant $c > 0$.

We then calculate,

$$\begin{aligned} &\int_0^1 (1-t)(u-w_0)^T \nabla^2\Gamma(w_0 + t(u-w_0))(u-w_0)\,dt \\ &\geq \int_0^{\frac{1}{2}} (1-t)(u-w_0)^T \nabla^2\Gamma(w_0 + t(u-w_0))(u-w_0)\,dt, \end{aligned} \tag{4.80}$$

where we have changed the limits of integration. Continuing,

$$\geq \frac{c}{2}|u-w_0|^2 \geq \frac{c}{2}\gamma_0^2, \tag{4.81}$$

where the last inequality comes from $\operatorname{dist}(u, \{w|\eta(w|u_L) \leq a\eta(w|u_R)\}) \geq \gamma_0$. This proves (4.75).

We choose

$$\gamma_0 := \frac{c_1}{2L_*}, \tag{4.82}$$

where c_1 is from Proposition 4.2 and L_* is the Lipschitz constant of the map

$$(u, u_L, u_R) \mapsto a\big(q(u;u_R) - \lambda_1(u)\eta(u|u_R)\big) - q(u;u_L) + \lambda_1(u)\eta(u|u_L). \tag{4.83}$$

Step 2. Define

$$V(u,t) := \lambda_1(u) - C_* \mathbb{1}_{\{u|a\eta(u|\bar{u}_+(t)) < \eta(u|\bar{u}_-(t))\}}(u), \tag{4.84}$$

where $C_* > 0$ is a large constant, which we can pick to be

$$C_* := \frac{1}{c_4\gamma_0^2}\left(\sup_{u,u_L,u_R \in B_B(0)} \big|aq(u;u_R) - q(u;u_L)\big| + 1\right) + 2\sup_{u \in B_B(0)} \big|\lambda_1(u)\big|, \tag{4.85}$$

where c_4 is from (4.75).

We solve the following ODE in the sense of Filippov flows,

$$\begin{cases} \dot{h}(t) = V(u(h(t),t),t) \\ h(0) = s(0), \end{cases} \tag{4.86}$$

The existence of such an h comes from the following lemma,

LEMMA 4.2 (Existence of Filippov flows). *Let $V(u,t):\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be bounded on $\mathbb{R}^n\times[0,\infty)$, upper semi-continuous in u, and measurable in t. Let u be a bounded, weak solution to (1.1), entropic for the entropy η. Assume also that u verifies the strong trace property (Definition 2.1). Let $x_0\in\mathbb{R}$. Then we can solve*

$$\begin{cases}\dot{h}(t)=V(u(h(t),t),t)\\ h(0)=x_0,\end{cases} \tag{4.87}$$

in the Filippov sense. That is, there exists a Lipschitz function $h:[0,\infty)\to\mathbb{R}$ such that

$$Lip[h]\le\|V\|_{L^\infty}, \tag{4.88}$$

$$h(0)=x_0, \tag{4.89}$$

and

$$\dot{h}(t)\in I[V(u_+,t),V(u_-,t)], \tag{4.90}$$

for almost every t, where $u_\pm:=u(h(t)\pm,t)$ and $I[a,b]$ denotes the closed interval with endpoints a and b.

Moreover, for almost every t,

$$f(u_+)-f(u_-)=\dot{h}(u_+-u_-), \tag{4.91}$$

$$q(u_+)-q(u_-)\le\dot{h}(\eta(u_+)-\eta(u_-)), \tag{4.92}$$

which means that for almost every t, either $(u_+,u_-,\dot{h})$ is an entropic shock (for η) or $u_+=u_-$.

The proof of (4.88), (4.89), and (4.90) is very similar to the proof of Proposition 1 in [35]. A proof of (4.88), (4.89), and (4.90) is included in the Appendix for the reader's convenience.

It is well known that (4.91) and (4.92) are true for any Lipschitz continuous function $h:[0,\infty)\to\mathbb{R}$ when u is BV. When instead u is only known to have strong traces (Definition 2.1), then (4.91) and (4.92) are given in Lemma 6 in [35]. We do not prove (4.91) and (4.92) here; their proof is in the appendix in [35].

Note that V (see (4.84)) is upper semi-continuous in u because indicator functions of open sets are lower semi-continuous and the negative of a lower semi-continuous function is upper semi-continuous.

Step 3. Let $u_\pm:=u(h(t)\pm,t)$.
Note that by Lemma 4.2,

$$\begin{aligned}\dot{h}(t)\in I\Bigg[&\lambda_1(u_+)-C_*\mathbb{1}_{\{u|a\eta(u|\bar{u}_+(t))<\eta(u|\bar{u}_-(t))\}}(u_+),\\ &\lambda_1(u_-)-C_*\mathbb{1}_{\{u|a\eta(u|\bar{u}_+(t))<\eta(u|\bar{u}_-(t))\}}(u_-)\Bigg].\end{aligned} \tag{4.93}$$

We are now ready to show (4.2).

For each fixed time t, we have 4 cases to consider to prove (4.2):
Case 1.

$$a\eta(u_-|\bar{u}_+(t)) < \eta(u_-|\bar{u}_-(t)), \tag{4.94}$$
$$a\eta(u_+|\bar{u}_+(t)) < \eta(u_+|\bar{u}_-(t)). \tag{4.95}$$

Case 2.

$$a\eta(u_-|\bar{u}_+(t)) < \eta(u_-|\bar{u}_-(t)), \tag{4.96}$$
$$a\eta(u_+|\bar{u}_+(t)) \geq \eta(u_+|\bar{u}_-(t)). \tag{4.97}$$

Case 3.

$$a\eta(u_-|\bar{u}_+(t)) \geq \eta(u_-|\bar{u}_-(t)), \tag{4.98}$$
$$a\eta(u_+|\bar{u}_+(t)) < \eta(u_+|\bar{u}_-(t)). \tag{4.99}$$

Case 4.

$$a\eta(u_-|\bar{u}_+(t)) \geq \eta(u_-|\bar{u}_-(t)), \tag{4.100}$$
$$a\eta(u_+|\bar{u}_+(t)) \geq \eta(u_+|\bar{u}_-(t)). \tag{4.101}$$

Note that we allow for $u_+ = u_-$.
We start with

Case 1. In this case, by (4.90), (4.85), and (4.93) we know that

$$\begin{aligned}\dot{h}(t) &\leq -\frac{1}{c_4\gamma_0^2}\left(\sup_{u,u_L,u_R\in B_B(0)}\left|aq(u;u_R)-q(u;u_L)\right|+1\right)-\sup_{u\in B_B(0)}\left|\lambda_1(u)\right|\\ &< \inf_{u\in B_B(0)}\lambda_1(u).\end{aligned} \tag{4.102}$$

If $u_+ \neq u_-$, then we have (4.91) and (4.92). But then, (4.102) contradicts ($\mathcal{H}2$). Thus, $u_+ = u_-$.

Let $v := u_+ = u_-$.

If $\operatorname{dist}(v, \{w|\eta(w|\bar{u}_-(t)) \leq a\eta(w|\bar{u}_+(t))\}) \geq \gamma_0$, then

$$\begin{aligned}&a\Big(q(u_+;\bar{u}_+(t))-\dot{h}(t)\eta(u_+|\bar{u}_+(t))\Big)-q(u_-;\bar{u}_-(t))+\dot{h}(t)\eta(u_-|\bar{u}_-(t))\\ &=a\Big(q(v;\bar{u}_+(t))-\dot{h}(t)\eta(v|\bar{u}_+(t))\Big)-q(v;\bar{u}_-(t))+\dot{h}(t)\eta(v|\bar{u}_-(t))\\ &=aq(v;\bar{u}_+(t))-q(v;\bar{u}_-(t))-\dot{h}(t)\big(a\eta(v|\bar{u}_+(t))-\eta(v|\bar{u}_-(t))\big)\\ &\leq -1,\end{aligned} \tag{4.103}$$

because of (4.102) and (4.75). Because the term $\left|\dot{s}(t)-\dot{h}(t)\right|^2$ on the right-hand side of (4.2) is bounded due to (4.88) and s being Lipschitz, we have proven (4.2) by choosing c sufficiently small.

If on the other hand, $\operatorname{dist}(v, \{w|\eta(w|\bar{u}_-(t)) \leq a\eta(w|\bar{u}_+(t))\}) < \gamma_0$, then

$$a\Big(q(u_+;\bar{u}_+(t))-\dot{h}(t)\eta(u_+|\bar{u}_+(t))\Big)-q(u_-;\bar{u}_-(t))+\dot{h}(t)\eta(u_-|\bar{u}_-(t))$$

$$
\begin{aligned}
&= a\Big(q(v;\bar{u}_+(t)) - \dot{h}(t)\eta(v|\bar{u}_+(t))\Big) - q(v;\bar{u}_-(t)) + \dot{h}(t)\eta(v|\bar{u}_-(t))\\
&= aq(v;\bar{u}_+(t)) - q(v;\bar{u}_-(t)) - \dot{h}(t)\big(a\eta(v|\bar{u}_+(t)) - \eta(v|\bar{u}_-(t))\big)\\
&\leq a\Big(q(v;\bar{u}_+(t)) - \lambda_1(v)\eta(v|\bar{u}_+(t))\Big) - q(v;\bar{u}_-(t)) + \lambda_1(v)\eta(v|\bar{u}_-(t)),
\end{aligned}
$$

because $\eta(v|\bar{u}_-(t)) - a\eta(v|\bar{u}_+(t)) \geq 0$ and $\dot{h} \leq -\sup_{u\in B_B(0)}|\lambda_1(u)|$. Continuing, we get

$$
\leq -\frac{1}{2}c_1, \tag{4.104}
$$

from (4.4), the definition of γ_0 (4.82), the assumption that

$$
\operatorname{dist}(v,\{w|\eta(w|\bar{u}_-(t)) \leq a\eta(w|\bar{u}_+(t))\}) < \gamma_0 \tag{4.105}
$$

and the assumption that $r(t) > \rho$ for all t, where $r(t)$ satisfies $S^1_{\bar{u}_-(t)}(r(t)) = \bar{u}_+(t)$. Again because the term $\left|\dot{s}(t) - \dot{h}(t)\right|^2$ on the right-hand side of (4.2) is bounded due to (4.88) and s being Lipschitz, we have proven (4.2) by choosing c sufficiently small. Note c will depend on ρ.

Case 2. In this case, we must have $u_- \neq u_+$. Recall also that (1.1) is hyperbolic. Furthermore, we have from (4.90) that

$$
\dot{h} \in \left[-\tfrac{1}{c_4\gamma_0^2}\left(\sup_{u,u_L,u_R\in B_B(0)}\left|aq(u;u_R) - q(u;u_L)\right| + 1\right) - \sup_{u\in B_B(0)}\left|\lambda_1(u)\right|, \lambda_1(u_+)\right].
$$

However, this implies that $(u_+,u_-,\dot{h})$ is a right 1-contact discontinuity (see [17, p. 274]). This contradicts the hypothesis ($\mathcal{H}2$) on the shock $(u_+,u_-,\dot{h})$, which is entropic for η because of (4.91) and (4.92). The hypothesis ($\mathcal{H}2$) forbids right 1-contact discontinuities. Thus, we conclude that this case (*Case 2*) cannot actually occur.

Case 3. In this case, we have from (4.90) that

$$
\dot{h} \in \left[-\frac{1}{c_4\gamma_0^2}\left(\sup_{u,u_L,u_R\in B_B(0)}\left|aq(u;u_R) - q(u;u_L)\right| + 1\right) - \sup_{u\in B_B(0)}\left|\lambda_1(u)\right|, \lambda_1(u_-)\right]. \tag{4.106}
$$

By the hypothesis ($\mathcal{H}3$), along with (4.91), (4.92), we have that $(u_+,u_-,\dot{h})$ must be a 1-shock. Also, u_- verifies $a\eta(u_-|\bar{u}_+(t)) \geq \eta(u_-|\bar{u}_-(t))$. Thus, we can apply Proposition 4.2. Recall that $r(t) > \rho$ for all t, where $r(t)$ satisfies $S^1_{\bar{u}_-(t)}(r(t)) = \bar{u}_+(t)$. We receive (4.2).

Case 4. In this case, we have from (4.90) that $\dot{h} \in I[\lambda_1(u_+),\lambda_1(u_-)]$. Then, by the hypothesis ($\mathcal{H}2$), along with (4.91), (4.92), we know that we cannot have

$$
I[\lambda_1(u_+),\lambda_1(u_-)] = (\lambda_1(u_-),\lambda_1(u_+)) \tag{4.107}
$$

because then (4.107) would imply that $(u_+,u_-,\dot{h})$ is a right 1-contact discontinuity. However, ($\mathcal{H}2$) prevents right 1-contact discontinuities. Recall ($\mathcal{H}3$). We conclude that $(u_+,u_-,\dot{h})$ is a 1-shock. Moreover, u_- verifies $a\eta(u_-|\bar{u}_+(t)) \geq \eta(u_-|\bar{u}_-(t))$. We can now apply Proposition 4.2. Recall that $r(t) > \rho$ for all t, where $r(t)$ satisfies $S^1_{\bar{u}_-(t)}(r(t)) = \bar{u}_+(t)$. This gives (4.2).

5. Proof of main theorem Theorem 1.1

Note that if $\bar{u}$ contains an n-shock, then the solution $(x,t)\mapsto\bar{u}(-x,t)$ to the system $\partial_t u-\partial_x f(u)=\tilde{G}(u)$ will have a 1-shock for this system (with $\tilde{G}$ defined as in (2.6)). Thus, we can always assume $\bar{u}$ has a 1-shock.

Let h be as in Proposition 4.1.

Define

$$\begin{aligned} h_1(t) &:= -R+s(0)+r(t-t_0),\\ h_2(t) &:= R+s(0)-r(t-t_0), \end{aligned} \tag{5.1}$$

where $r>0$ verifies

$$\left|q(u;\bar{u})\right|\leq r\eta(u|\bar{u}). \tag{5.2}$$

Such an $r>0$ exists because u and $\bar{u}$ are bounded, $q(a;b)$ and $\eta(a|b)$ are both locally quadratic in $a-b$, and η is strictly convex.

Then we apply Lemma 3.3 to h_1 and h. This yields,

$$\begin{aligned}
&\int_0^{t_0}\Big[q(u(h_1(t)+,t);\bar{u}((h_1(t)+X(t))+,t))-q(u(h(t)-,t);\bar{u}((h(t)+X(t))-,t))\\
&\quad+\dot{h}(t)\eta(u(h(t)-,t)|\bar{u}((h(t)+X(t))-,t))\\
&\quad-\dot{h}_1(t)\eta(u(h_1(t)+,t)|\bar{u}((h_1(t)+X(t))+,t))\Big]\,dt\\
&\geq\int_{h_1(t_0)}^{h(t_0)}\eta(u(x,t_0)|\bar{u}(x+X(t_0),t_0))\,dx-\int_{h_1(0)}^{h(0)}\eta(u^0(x)|\bar{u}^0(x))\,dx\\
&\quad+\int_0^{t_0}\int_{h_1(t)}^{h(t)}\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))f(u(x,t)|\bar{u}(x+X(t),t))\\
&\quad+\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\\
&\quad-\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)\\
&\quad+\Big(G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\Big)^T\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)\\
&\quad-\bar{u}(x+X(t),t)]\,dxdt,
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned} f(u|\bar{u}) &:= f(u)-f(\bar{u})-\nabla f(\bar{u})(u-\bar{u}),\\ X(t) &:= s(t)-h(t). \end{aligned} \tag{5.4}$$

Similarly, we apply Lemma 3.3 to h and h_2. This yields,

$$\int_0^{t_0}\Big[q(u(h(t)+,t);\bar{u}((h(t)+X(t))+,t))-q(u(h_2(t)-,t);\bar{u}((h_2(t)+X(t))-,t))$$

$$+\dot{h}_2(t)\eta(u(h_2(t)-,t)|\bar{u}((h_2(t)+X(t))-,t))$$
$$-\dot{h}(t)\eta(u(h(t)+,t)|\bar{u}((h(t)+X(t))+,t))\Big]\,dt$$
$$\geq \int\limits_{h(t_0)}^{h_2(t_0)} \eta(u(x,t_0)|\bar{u}(x+X(t_0),t_0))\,dx - \int\limits_{h(0)}^{h_2(0)} \eta(u^0(x)|\bar{u}^0(x))\,dx$$
$$+\int\limits_0^{t_0}\int\limits_{h(t)}^{h_2(t)} \left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))f(u(x,t)|\bar{u}(x+X(t),t))$$
$$+\left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\dot{X}(t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]$$
$$-\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)$$
$$+\left(G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\right)^T\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)$$
$$-\bar{u}(x+X(t),t)]\,dxdt. \tag{5.5}$$

We combine (5.3) and a multiples of (5.5). This gives,

$$\int\limits_0^{t_0}\Big[a\Big(q(u(h(t)+,t);\bar{u}((h(t)+X(t))+,t))-\dot{h}(t)\eta(u(h(t)+,t)|\bar{u}((h(t)+X(t))+,t))\Big)$$
$$+\dot{h}(t)\eta(u(h(t)-,t)|\bar{u}((h(t)+X(t))-,t))-q(u(h(t)-,t);\bar{u}((h(t)+X(t))-,t))$$
$$+aq(u(h_1(t)+,t);\bar{u}((h_1(t)+X(t))+,t))-a\dot{h}_1(t)\eta(u(h_1(t)+,t)|\bar{u}((h_1(t)+X(t))+,t))$$
$$-q(u(h_2(t)-,t);\bar{u}((h_2(t)+X(t))-,t))$$
$$+\dot{h}_2(t)\eta(u(h_2(t)-,t)|\bar{u}((h_2(t)+X(t))-,t))\Big]\,dt$$
$$\geq\Bigg[a\int\limits_{h_1(t_0)}^{h(t_0)} \eta(u(x,t_0)|\bar{u}(x+X(t_0),t_0))\,dx+\int\limits_{h(t_0)}^{h_2(t_0)} \eta(u(x,t_0)|\bar{u}(x+X(t_0),t_0))\,dx\Bigg]$$
$$-\Bigg[a\int\limits_{h_1(0)}^{h(0)} \eta(u^0(x)|\bar{u}^0(x))\,dx+\int\limits_{h(0)}^{h_2(0)} \eta(u^0(x)|\bar{u}^0(x))\,dx\Bigg]$$
$$+\int\limits_0^{t_0}\int\limits_{\mathbb{R}} \mathbb{1}_a(x)\Bigg[\left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))f(u(x,t)|\bar{u}(x+X(t),t))$$
$$+\left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\dot{X}(t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]$$
$$-\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))G(u(\cdot,t))(x)$$
$$+\left(G(\bar{u}(\cdot,t))(x+X(t))-G(u(\cdot,t))(x)\right)^T\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)$$

$$\left. -\bar{u}(x+X(t),t)]\right]\, dxdt, \tag{5.6}$$

where

$$\mathbb{1}_a(x) := a\mathbb{1}_{\{x|h_1(t)<x<h(t)\}}(x) + \mathbb{1}_{\{x|h(t)<x<h_2(t)\}}(x). \tag{5.7}$$

We estimate the last term on the right-hand side of (5.6), which is of the form

$$\int_0^{t_0}\int_{\mathbb{R}} \overbrace{\mathbb{1}_a(x)}^{L^\infty(\mathbb{R})} \quad \overbrace{\Big[\cdots\Big]}^{L^1([h_1(t),h_2(t)])} \quad dxdt, \tag{5.8}$$

using the indicated Hölder dualities.

We then want to estimate from above the term

$$\begin{aligned}
&\int_{h_1(t)}^{h_2(t)} \Bigg| \overbrace{\left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))}^{L^\infty([h_1(t),h_2(t)])} \overbrace{f(u(x,t)|\bar{u}(x+X(t),t))}^{L^1([h_1(t),h_2(t)])} \\
&\quad + \dot{X}(t)\overbrace{\left(\partial_x\Big|_{(x+X(t),t)} \bar{u}^T(x,t)\right)}^{L^2([h_1(t),h_2(t)])} \overbrace{\nabla^2\eta(\bar{u}(x+X(t),t))}^{L^\infty([h_1(t),h_2(t)])} \overbrace{[u(x,t)-\bar{u}(x+X(t),t)]}^{L^2([h_1(t),h_2(t)])} \\
&\quad - \overbrace{\nabla\eta(u(x,t)|\bar{u}(x+X(t),t))}^{L^1([h_1(t),h_2(t)])}\ \overbrace{G(u(\cdot,t))(x)}^{L^\infty([h_1(t),h_2(t)])} \\
&\quad + \overbrace{\Big(G(\bar{u}(\cdot,t))(x+X(t)) - G(u(\cdot,t))(x)\Big)}^{L^2([h_1(t),h_2(t)])}{}^T\ \overbrace{\nabla^2\eta(\bar{u}(x+X(t),t))}^{L^\infty([h_1(t),h_2(t)])} \times \\
&\quad \overbrace{[u(x,t)-\bar{u}(x+X(t),t)]}^{L^2([h_1(t),h_2(t)])}\Bigg|\, dx,
\end{aligned} \tag{5.9}$$

where $\times$ is matrix multiplication (a matrix times a vector).

We use the Hölder dualities indicated above. In particular, recall that $f(a|b)$ is locally quadratic in $a-b$ and that $\partial_x\bar{u}\in L^\infty(\mathbb{R}\times[0,T))$ due to $\bar{u}$ being Lipschitz continuous.

Note that from $G:(L^2(\mathbb{R}))^n\to(L^2(\mathbb{R}))^n$ being translation invariant (see (1.2)) and from (1.3), we have

$$\begin{aligned}
&\big\|G(\bar{u}(\cdot,t))(\cdot+X(t)) - G(u(\cdot,t))(\cdot)\big\|_{L^2([h_1(t),h_2(t)]} \\
=&\big\|G(\bar{u}(\cdot+X(t),t))(\cdot) - G(u(\cdot,t))(\cdot)\big\|_{L^2([h_1(t),h_2(t)])} \\
\le& C_G\big\|\bar{u}(\cdot+X(t),t) - u(\cdot,t)\big\|_{L^2([h_1(t),h_2(t)])},
\end{aligned} \tag{5.10}$$

where C_G is from (1.3).

Recall also (1.4).

Note also that we can estimate,

$$\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])} \leq \sqrt{2(R+rT)}\|\partial_x \bar{u}\|_{L^\infty(\mathbb{R}\times[0,T))} = \sqrt{2(R+rT)}\mathrm{Lip}[\bar{u}]. \tag{5.11}$$

For $\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|_{L^\infty}\neq 0$ we have, from using the 'Young's inequality with ϵ,'

$$\begin{aligned}
&\left|\dot{X}(t)\right|\left\|u(\cdot,t)-\bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\\
&\leq \frac{c}{2\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|_{L^\infty}}(\dot{X}(t))^2\\
&\quad+\frac{\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|_{L^\infty}}{2c}\left\|u(\cdot,t)-\bar{u}(\cdot+X(t),t)\right\|^2_{L^2([h_1(t),h_2(t)])},
\end{aligned} \tag{5.12}$$

where c is from the right-hand side of (4.2). Note that c depends on ρ, $\|u\|_{L^\infty}$, $\left\|\bar{u}(s(t)+,t)\right\|_{L^\infty([0,T))}$, $\left\|\bar{u}(s(t)-,t)\right\|_{L^\infty([0,T))}$, and a. From (5.12), we get

$$\begin{aligned}
&\left|\dot{X}(t)\right|\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|_{L^\infty}\left\|u(\cdot,t)-\bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\\
&\leq \frac{c}{2}(\dot{X}(t))^2\\
&\quad+\frac{\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|^2_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|^2_{L^\infty}}{2c}\left\|u(\cdot,t)-\bar{u}(\cdot+X(t),t)\right\|^2_{L^2([h_1(t),h_2(t)])}.
\end{aligned} \tag{5.13}$$

If for some t, $\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|_{L^\infty}=0$, then we don't have to estimate the term

$$\dot{X}(t)\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]. \tag{5.14}$$

Recall (5.1) and (5.2). Note in particular we have $\dot{h}_1=r$ and $\dot{h}_2=-r$. Then from (4.2) (in Proposition 4.1) and (5.13), we get

$$\begin{aligned}
&-\int_0^{t_0}\int_{\mathbb{R}}\left[\left(\partial_x\Big|_{(x+X(t),t)}\bar{u}^T(x,t)\dot{X}(t)\right)\nabla^2\eta(\bar{u}(x+X(t),t))[u(x,t)-\bar{u}(x+X(t),t)]\right]dxdt\\
&+\int_0^{t_0}\Bigg[a\bigg(q(u(h(t)+,t);\bar{u}((h(t)+X(t))+,t))-\dot{h}(t)\eta(u(h(t)+,t)|\bar{u}((h(t)+X(t))+,t))\bigg)\\
&+\dot{h}(t)\eta(u(h(t)-,t)|\bar{u}((h(t)+X(t))-,t))-q(u(h(t)-,t);\bar{u}((h(t)+X(t))-,t))\\
&+aq(u(h_1(t)+,t);\bar{u}((h_1(t)+X(t))+,t))-a\dot{h}_1(t)\eta(u(h_1(t)+,t)|\bar{u}((h_1(t)+X(t))+,t))\\
&-q(u(h_2(t)-,t);\bar{u}((h_2(t)+X(t))-,t))+\dot{h}_2(t)\eta(u(h_2(t)-,t)|\bar{u}((h_2(t)+X(t))-,t))\Bigg]dt\\
&\leq\int_0^{t_0}-\frac{c}{2}(\dot{X}(t))^2
\end{aligned}$$

$$+\frac{\left\|\partial_x \bar{u}(\cdot+X(t),t)\right\|^2_{L^2([h_1(t),h_2(t)])}\left\|\nabla^2\eta(\bar{u})\right\|^2_{L^\infty}}{2c}\left\|u(\cdot,t)-\bar{u}(\cdot+X(t),t)\right\|^2_{L^2([h_1(t),h_2(t)])}\,dt. \tag{5.15}$$

Recall (5.10), (5.11), and (5.15). Recall also (5.1) and (5.4). Further, recall from Proposition 4.1 that $h(0)=s(0)$. Recall also that from Proposition 4.1, we know the constant c depends on ρ, $\|u\|_{L^\infty}$, and $\|\bar{u}\|_{L^\infty}$. Lastly, recall that $f(a|b)$, $\eta(a|b)$, and $\nabla\eta(a|b)$ are locally quadratic in $a-b$ (recall $\eta\in C^3(\mathbb{R}^n)$), and from the strict convexity of η we in fact have Lemma 1.1. Then, from (5.6), we receive

$$\begin{aligned}
&\mu_1\int_0^{t_0}\int_{h_1(t)}^{h_2(t)}\left|u(x,t)-\bar{u}(x+X(t),t)\right|^2dxdt+\mu_2\int_{-R-rt_0+s(0)}^{R+rt_0+s(0)}\left|u^0(x)-\bar{u}^0(x)\right|^2dx\\
&\quad-\frac{1}{\mu_2}\int_0^{t_0}(\dot{X}(t))^2dt\geq\int_{-R+s(0)}^{R+s(0)}\left|u(x,t_0)-\bar{u}(x+X(t_0),t_0)\right|^2dx
\end{aligned}\tag{5.16}$$

for all $t_0\in[0,T)$, where $\mu_1,\mu_2>0$ are constants depending on a, ρ, $\|u\|_{L^\infty}$, $\|\bar{u}\|_{L^\infty}$, and bounds on the derivatives of η on the range of u and $\bar{u}$. Furthermore, μ_1 also depends on C_G (see (1.3) and (1.4)), Lip$[\bar{u}]$, ρ, R, T, and bounds on the derivatives of f on the range of u and $\bar{u}$. Note that r (see (5.2)) only depends on bounds on the derivatives of f and η on the (range of u and $\bar{u}$). The constant a then itself depends on ρ, $\|u\|_{L^\infty}$, and $\|\bar{u}\|_{L^\infty}$ (see Proposition 4.2).

We can drop the last term on the left-hand side of (5.16), to get

$$\begin{aligned}
&\mu_1\int_0^{t_0}\int_{h_1(t)}^{h_2(t)}\left|u(x,t)-\bar{u}(x+X(t),t)\right|^2dxdt+\mu_2\int_{-R-rt_0+s(0)}^{R+rt_0+s(0)}\left|u^0(x)-\bar{u}^0(x)\right|^2dx\\
&\geq\int_{-R+s(0)}^{R+s(0)}\left|u(x,t_0)-\bar{u}(x+X(t_0),t_0)\right|^2dx.
\end{aligned}\tag{5.17}$$

We then apply the Gronwall inequality to (5.17). This yields,

$$\begin{aligned}
&\int_{-R+s(0)}^{R+s(0)}\left|u(x,t_0)-\bar{u}(x+X(t_0),t_0)\right|^2dx\\
&\leq\mu_2e^{\mu_1t_0}\left(\int_{-R-rt_0+s(0)}^{R+rt_0+s(0)}\left|u^0(x)-\bar{u}^0(x)\right|^2dx\right).
\end{aligned}\tag{5.18}$$

From (5.18), we get (1.18).

We now show (1.19). From (5.16), we get

$$\mu_1\int_0^{t_0}\int_{h_1(t)}^{h_2(t)}\left|u(x,t)-\bar{u}(x+X(t),t)\right|^2dxdt+\mu_2\int_{-R-rt_0+s(0)}^{R+rt_0+s(0)}\left|u^0(x)-\bar{u}^0(x)\right|^2dx$$

$$\geq \frac{1}{\mu_2} \int_0^{t_0} (\dot{X}(t))^2 \, dt. \tag{5.19}$$

Then we bootstrap, and use (1.18) to estimate the term

$$\int_{h_1(t)}^{h_2(t)} \left| u(x,t) - \bar{u}(x+X(t),t) \right|^2 dx$$

in (5.19). This gives (1.19).

This proves Theorem 1.1.

Appendix.

A.1. Proof of Lemma 3.2. Throughout this proof, C will denote a generic constant depending only on B.

We will first show that for $0 < a < 1$, the set R_a is convex.

For $a < 1$, we can rewrite

$$\eta(u|u_L) \leq a\eta(u|u_R) \tag{A.1}$$

as

$$\eta(u) \leq \frac{1}{1-a}(\eta(u_L) - a\eta(u_R) - \nabla\eta(u_L)\cdot u_L + a\nabla\eta(u_R)\cdot u_R + (\nabla\eta(u_L) - a\nabla\eta(u_R))\cdot u). \tag{A.2}$$

The right-hand side of (A.2) is (affine) linear in u. Thus the convexity of η implies that $R_a = \{u | \eta(u|u_L) \leq a\eta(u|u_R)\}$ is convex.

For $a < \frac{1}{2}$, we can rewrite (A.2) to get

$$\begin{aligned} \eta(u|u_L) &\leq \frac{a}{1-a}(\eta(u_L) - \eta(u_R) - \nabla\eta(u_L)\cdot u_L + \nabla\eta(u_R)\cdot u_R + (\nabla\eta(u_L) - \nabla\eta(u_R))\cdot u) \\ &\leq Ca(1+|u|). \end{aligned} \tag{A.3}$$

We combine this with Lemma 1.1 to get that for all $u \in R_a \cap B_\theta(u_L)$ (recalling $\theta < 1$),

$$|u - u_L|^2 \leq Ca(1+|u|) \leq Ca. \tag{A.4}$$

Thus, when α satisfies (3.6) with C as in (A.4), and $0 < a < \alpha$, we have

$$|u - u_L|^2 \leq Ca < \frac{\theta^2}{2}. \tag{A.5}$$

Thus $R_a \cap B_\theta(u_L)$ is strictly contained in $B_\theta(u_L)$. As we have shown, the set R_a is convex. Thus R_a is also connected, which implies that

$$R_a = R_a \cap B_\theta(u_L). \tag{A.6}$$

We conclude that $R_a \subset B_\theta(u_L)$ for all $0 < a < \alpha$. This completes the proof.

A.2. Proof of Lemma 4.2. The following proof of (4.88), (4.89), and (4.90) is based on the proof of Proposition 1 in [35], the proof of Lemma 2.2 in [43], and the proof of Lemma 3.5 in [31]. We do not prove (4.91) or (4.92) here; these properties are in Lemma 6 in [35], and their proofs are in the appendix in [35].

Define

$$v_n(x,t) := \int_0^1 V\left(u(x+\frac{y}{n},t),t\right) dy. \tag{A.7}$$

Let h_n be the solution to the ODE:

$$\begin{cases} \dot{h}_n(t) = v_n(h_n(t),t), \text{ for } t>0 \\ h_n(0) = x_0. \end{cases} \tag{A.8}$$

The v_n are uniformly bounded in n because by assumption V is bounded ($\|v_n\|_{L^\infty} \leq \|V\|_{L^\infty}$). The v_n are measurable in t, and due to the mollification by $\frac{1}{n}$ are also Lipschitz continuous in x. Thus (A.8) has a unique solution in the sense of Carathéodory.

The h_n are Lipschitz continuous with Lipschitz constants uniform in n, due to the v_n being uniformly bounded in n. Thus, by Arzelà–Ascoli the h_n converge in $C^0(0,T)$ for any fixed $T>0$ to a Lipschitz continuous function h (passing to a subsequence if necessary). Note that $\dot{h}_n$ converges in L^∞ weak* to $\dot{h}$.

We define

$$V_{\max}(t) := \max\{V(u_-,t), V(u_+,t)\}, \tag{A.9}$$

$$V_{\min}(t) := \min\{V(u_-,t), V(u_+,t)\}, \tag{A.10}$$

where $u_\pm := u(h(t)\pm,t)$.

To show (4.90), we will first prove that for almost every $t>0$

$$\lim_{n\to\infty} [\dot{h}_n(t) - V_{\max}(t)]_+ = 0, \tag{A.11}$$

$$\lim_{n\to\infty} [V_{\min}(t) - \dot{h}_n(t)]_+ = 0, \tag{A.12}$$

where $[\,\cdot\,]_+ := \max(0,\cdot)$.

The proofs of (A.11) and (A.12) are similar; we only show the first one.

$$\begin{aligned}
[\dot{h}_n(t) - V_{\max}(t)]_+ &= \left[\int_0^1 V\left(u(h_n(t)+\frac{y}{n},t),t\right) dy - V_{\max}(t)\right]_+ \\
&= \left[\int_0^1 V\left(u(h_n(t)+\frac{y}{n},t),t\right) - V_{\max}(t)\, dy\right]_+ \\
&\leq \int_0^1 \left[V\left(u(h_n(t)+\frac{y}{n},t),t\right) - V_{\max}(t)\right]_+ dy \\
&\leq \operatorname*{ess\,sup}_{y\in(0,\frac{1}{n})} \left[V\left(u(h_n(t)+y,t),t\right) - V_{\max}(t)\right]_+ \\
&\leq \operatorname*{ess\,sup}_{y\in(-\epsilon_n,\epsilon_n)} \left[V\left(u(h(t)+y,t),t\right) - V_{\max}(t)\right]_+,
\end{aligned} \tag{A.13}$$

where $\epsilon_n := \big|h_n(t)-h(t)\big| + \frac{1}{n}$. Note $\epsilon_n \to 0^+$.

Fix a $t \geq 0$ such that u has a strong trace in the sense of Definition 2.1. Then because the map $u \mapsto V(u,t)$ is upper semi-continuous,

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{y\in(0,\frac{1}{n})} \left[V\Big(u(h(t)\pm y,t),t\Big) - V\big(u_\pm,t\big)\right]_+ = 0, \tag{A.14}$$

where $u_\pm := u(h(t)\pm,t)$. Recall that the map $u \mapsto V(u,t)$ being upper semi-continuous at the point u_0 means that

$$\limsup_{u\to u_0} V(u,t) \leq V(u_0,t). \tag{A.15}$$

From (A.14), we get

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{y\in(0,\frac{1}{n})} \left[V\Big(u(h(t)\pm y,t),t\Big) - V_{\max}(t)\right]_+ = 0. \tag{A.16}$$

We can control (A.13) from above by the quantity

$$\operatorname*{ess\,sup}_{y\in(-\epsilon_n,0)} \left[V\Big(u(h(t)+y,t),t\Big) - V_{\max}(t)\right]_+ + \operatorname*{ess\,sup}_{y\in(0,\epsilon_n)} \left[V\Big(u(h(t)+y,t),t\Big) - V_{\max}(t)\right]_+. \tag{A.17}$$

By (A.16), we have that (A.17) goes to 0 as $n\to\infty$. This proves (A.11).

Recall that $\dot{h}_n$ converges in L^∞ weak* to $\dot{h}$. Thus, due to the convexity of the function $[\,\cdot\,]_+$,

$$\int_0^T [\dot{h}(t) - V_{\max}(t)]_+ \, dt \leq \liminf_{n\to\infty} \int_0^T [\dot{h}_n(t) - V_{\max}(t)]_+ \, dt. \tag{A.18}$$

By the dominated convergence theorem and (A.11),

$$\liminf_{n\to\infty} \int_0^T [\dot{h}_n(t) - V_{\max}(t)]_+ \, dt = 0. \tag{A.19}$$

We conclude,

$$\int_0^T [\dot{h}(t) - V_{\max}(t)]_+ \, dt = 0. \tag{A.20}$$

From a similar argument,

$$\int_0^T [V_{\min}(t) - \dot{h}(t)]_+ \, dt = 0. \tag{A.21}$$

This proves (4.90).

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