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SUPPRESSION OF BLOW UP BY MIXING IN GENERALIZED KELLER-SEGEL SYSTEM WITH FRACTIONAL DISSIPATION*

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Abstract. In this paper, we consider the Cauchy problem for a generalized parabolic-elliptic Keller-Segel equation with a fractional dissipation and an additional mixing effect of advection by an incompressible flow. Under a suitable mixing condition on the advection, we study well-posedness of solution with large initial data. We establish the global L^∞ estimate of the solution through nonlinear maximum principle, and obtain the global existence of classical solution.

Keywords. Generalized Keller-Segel system; Mixing; Fractional dissipation; Suppression of blow up.

AMS subject classifications. 35A01; 35B45; 35R11; 35Q92.

1. Introduction

We consider the following generalized parabolic-elliptic Keller-Segel system on torus $\mathbb{T}^d$ with fractional dissipation and the additional mixing effect of advection by an incompressible flow

$$\begin{cases} \partial_t \rho + u\cdot\nabla\rho + (-\Delta)^{\frac{\alpha}{2}}\rho + \nabla\cdot(\rho B(\rho)) = 0, & t>0, x\in\mathbb{T}^d, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1.1}$$

Here $\rho(t,x)$ is the unknown function of t and x, $0<\alpha<2$, $\mathbb{T}^d$ is the periodic box with dimension $d\geq 2$. The quantity ρ denotes the density of microorganisms, u is a given divergence-free vector field which is an ambient flow. The nonlocal operator $(-\Delta)^{\frac{\alpha}{2}}$ is known as the Laplacian of the order $\frac{\alpha}{2}$, which is given by

$$(-\Delta)^{\frac{\alpha}{2}}\phi(x) = \mathcal{F}^{-1}(|\xi|^\alpha\hat{\phi}(\xi))(x), \quad x\in\mathbb{R}^d, \tag{1.2}$$

where

$$\hat{\phi}(\xi) = \mathcal{F}(\phi(x)) = \int_{\mathbb{R}^d}\phi(x)e^{-ix\cdot\xi}dx,$$

and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transformation and its inverse transformation. The linear vector operator B is called attractive kernel, which could be formally represented as

$$B(\rho) = \nabla((-\Delta)^{-\frac{d+2-\beta}{2}}\rho) = \nabla K * \rho, \tag{1.3}$$

where

$$\nabla K \sim -\frac{x}{|x|^\beta}, \quad \beta\in[2,d+1), \quad x\in\mathbb{R}^d. \tag{1.4}$$

In this paper, we consider the tours $\mathbb{T}^d$ and $\beta\in[2,d]$, the definitions of $(-\Delta)^{\frac{\alpha}{2}}$ and B are different from those in (1.2) and (1.3). The fractional Laplacian operator needs a kernel representation, the details can be found in Section 2. We pose the following assumptions on the attractive kernel B (see [25,28])

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(1) when $\beta=d$, the $B(\rho)$ is written as

$$B(\rho)=\nabla(-\Delta)^{-1}(\rho-\overline{\rho}), \tag{1.5}$$

where $\overline{\rho}$ is the mean value of ρ_0 over $\mathbb{T}^d$, with the following definition

$$\overline{\rho}=\frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d}\rho_0(x)dx.$$

(2) When $2\le\beta<d$, K is a periodic convolution kernel, which is smooth away from the origin, and $\nabla K\sim-\frac{x}{|x|^\beta}$ near $x=0$, and we denote

$$B(\rho)=\nabla K*\rho. \tag{1.6}$$

Without advection, the Equation (1.1) is the generalized Keller-Segel system with fractional dissipation

$$\partial_t\rho+(-\Delta)^{\frac{\alpha}{2}}\rho+\nabla\cdot(\rho B(\rho))=0,\quad \rho(0,x)=\rho_0(x),\quad x\in\Omega, \tag{1.7}$$

where Ω is $\mathbb{R}^d$ or $\mathbb{T}^d$, and the Equation (1.7) describes many physical processes involving diffusion and interaction of particles (see [5,8]). In one space dimension, the equation admits global-in-time smooth solutions for large initial data, while the solutions may blow up in finite time for large initial data. Specifically, when $\alpha=2,\beta=d$, the Equation (1.7) is called classical attractive-type Keller-Segel system. In one space dimension, the equation admits large data global-in-time smooth solution (see [24,34]). In high dimensions, there are global-in-time smooth solutions when the initial data is small, while the solutions may exhibit finite-time blowup for large data (see [6,14,28,33,35]). When $0<\alpha<2,\beta=d$, the Equation (1.7) is a classical Keller-Segel system with fractional dissipation, which has been studied by many people. For $d=1$ and $0<\alpha\le1$, the solution of Equation (1.7) is global if $\|\rho_0\|_{L^{\frac{1}{\alpha}}}\le C(\alpha)$, and the solution of Equation (1.7) is global if $1<\alpha<2$ (see [7]). While $d\ge2$, the solution of Equation (1.7) would blow up in finite time with large data (see [4,25,29,30]). In the case of $0<\alpha<2,\beta\in[2,d+1),d\ge2$, the Equation (1.7) is called a generalized Keller-Segel system with fractional dissipation, the solution of Equation (1.7) always blows up in finite time when the initial data is large (see [4,25,30]).

Recently, the chemotactic models with other mechanisms have been extensively studied, and some interesting results have been obtained. For example, Burczak, Belinchón (see [9]) and Tello, Winkler (see [36]) proved that a logistic source could prevent the singularity of the solution. A more interesting problem is the chemotactic process taking place in fluid, the agent involved in chemotaxis is also advected by the ambient flow. The problem of chemotaxis in fluid flow has been studied (see [17,18,31,32]). For the possible effects resulting from the interaction of chemotaxis and fluid transport process, many people get interested in the suppression of blow up in the chemotactic model by fluid effect. Kiselev, Xu (see [28]) and Hopf, Rodrigo (see [25]) obtained the global solution of the Equation (1.1) by the mixing effect of fluid. Bedrossian and He (see [2]) showed that the shear flow was dissipation enhancing for the Keller-Segel system. In this paper, we continue to study the mixing effect of fluid to chemotactic model.

Mixing was studied by Constantin, Kiselev, Ryzhik, and Zlatoš (see [11]) as the fluid effect. In order to describe the mixing effect, Constantin et al. considered the following heat equation with advection

$$\phi_t^A(t,x)+Au\cdot\nabla\phi^A(t,x)-\Delta\phi^A(t,x)=0,\quad \phi^A(0,x)=\phi_0(x), \tag{1.8}$$

and they defined the relaxation enhancing flow. Namely, for every $\tau>0,\delta>0$, there exists a positive constant $A_0=A(\tau,\delta)$, such that for any $A\geq A_0$ and any $\phi_0(x)\in L^2$

$$\|\phi^A(\tau,\cdot)\|_{L^2}\leq\delta\|\phi_0\|_{L^2},$$

then incompressible flow u is called relaxation enhancing flow. Here $\phi^A(t,x)$ is the solution of (1.8), $\overline{\phi}$ is the average of ϕ_0 and $\overline{\phi}=0$. The authors provided a necessary and sufficient condition for the relaxation enhancing flow. Notice that if there is no dissipation term in (1.8), the L^2 norm is conserved, namely $\|\phi^A\|_{L^2}=\|\phi_0\|_{L^2}$. The result in [11] means that combination of mixing and dissipation produces a significantly stronger dissipative effect than dissipation alone. Specifically, for a fixed time τ, $\|\phi^A(\tau,\cdot)\|_{\dot{H}^1}$ is large enough in some sense when A is large enough. So the mixing term is enhancing for the dissipation, it can be useful in the model describing a physical situation which involves fast unitary dynamics with dissipation (see [27,28]). We will briefly introduce relaxation enhancing flow and weakly mixing (see, Definition 2.1) in Section 2.3, the reader can refer to [11] for more details.

For the Equation (1.1), mixing effect is included in chemotactic model, and our main concern is whether mixing can suppress the blowup phenomenon in finite time. When $\alpha=2,\beta=d,d=2,3$, Kiselev and Xu (see [28]) established the L^2 estimate of the solution in the case of weakly mixing, and obtained the global smooth solution by L^2-criterion. Namely, the blowup solution of Keller-Segel system was prevented. For $0<\alpha<2,\beta\in[2,d+1),d\geq2$, Hopf and Rodrigo proved that there exists L^2 estimate of the solution by relaxation enhancing flow, and also got the global smooth solution if $\alpha>\max\{\beta-\frac{d}{2},1\}$ (see [25], Theorem 4.5). In particular, for classical Keller-Segel system with fractional dissipation, when $\alpha>\frac{d}{2},d=2,3$, the solution of (1.1) was globally smooth. For the smaller lower bounds on α and higher dimension d, we require the $L^p(p>2)$ estimate of the solution instead of the L^2 estimate. Hopf and Rodrigo only considered the case $\alpha=2,\beta=d$, with $d\geq4$ (see [25], Theorem 4.6), they got the $L^p\,(2<p<\infty)$ estimate of the solution by relaxation enhancing flow, and obtained the global smooth solution by L^p-criterion.

At the same time, Hopf and Rodrigo thought that the $L^p(p>2)$ estimate of the solution for Equation (1.1) is hard to achieve in the case of $0<\alpha<2,\beta\in[2,d+1),d\geq2$. So it is not obvious to extend the result to the generalized Keller-Segel system with fractional dissipation of any strength α and in any dimension $d\geq2$ by energy method.

In this paper, we consider the generalized Keller-Segel system with fractional dissipation and weakly mixing in the case of any $0<\alpha<2,\beta\in[2,d],d\geq2$ and for convenience, we consider $\mathbb{T}^d=[-\frac{1}{2},\frac{1}{2})^d$. In order to get L^p estimate of the solution to Equation (1.1), we introduce a nonlinear maximum principle on $\mathbb{T}^d$ (see Appendix). Due to mixing effect, we obtain the $L^p(p=\infty)$ estimate of the solution through nonlinear maximum principle, then we get the global classical solution by L^∞-criterion. We believe that the range of α and d are more general in our results, as compared to other results in [25,28]. Due to technical difficulties, we don't consider the case of $d<\beta<d+1$.

Let us now state our main result.

THEOREM 1.1. *Let $0<\alpha<2,\beta\in[2,d],d\geq2$, for any initial data $\rho_0\geq0,\rho_0\in H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$, there exists a smooth incompressible flow u, such that the unique solution $\rho(t,x)$ of Equation (1.1) is global in time, and we have*

$$\rho(t,x)\in C(\mathbb{R}^+;H^3(\mathbb{T}^d)).$$

REMARK 1.1. The smooth incompressible flow u is weakly mixing (see Definition 2.1), and the result is still open for the general relaxation enhancing flow (see [11,25]).

REMARK 1.2. The result can seen as an extension of Kiselev et al. (see [28]) and Hopf et al. (see [25]). For the case of $d<\beta<d+1$, we need some new ideas.

REMARK 1.3. If $\rho_0(x)$ is a constant, then the solution of Equation (1.1) obviously has global existence, so we discuss only the case where $\rho_0(x)$ is not constant in this paper.

In the following, we briefly state our main ideas of the proof. Firstly, we establish the L^∞-criterion of solution to Equation (1.1). Namely, we can get the higher order energy estimate of the solution if the L^∞ norm of solution is uniformly bounded, thus there is a global classical solution for the Equation (1.1). Next, we obtain the L^∞ estimate of the solution to Equation (1.1). According to the L^1 norm conservation of the solution and nonlinear maximum principle, we can get the local L^∞ estimate of the solution, it follows that the local L^2 estimate of the solution is small by mixing effect. Combining with the local L^2 and L^∞ estimate of the solution, we deduce that the local $L^p\,(p>\frac{d}{a})$ estimate of the solution is controlled by its initial data. Using the nonlinear maximum principle again, the local L^∞ norm is estimated by the initial data. Repeating the above process, we extend the local L^p and L^∞ estimate of the solution to all time. Thus, we get the uniform L^∞ estimate. In the details of the proof, we will discuss $\beta=d$ and $\beta\in[2,d)$ respectively, due to the different properties of $B(\rho)$. When $\beta=d$, the attractive kernel is written as $B(\rho)=\nabla((-\Delta)^{-1}(\rho-\overline{\rho}))$. While $2\leq\beta<d$, the attractive kernel can be expressed by $B(\rho)=\nabla K*\rho$. So, some different techniques are required to deal with the two cases.

This paper is organized as follows. In Section 2, we introduce the properties of the nonlocal operator and the functional space. We give the local well-posedness and basic properties for the generalized Keller-Segel system with fractional dissipation and weakly mixing. The mixing effect of the solution is also introduced in this section. In Section 3, we establish the L^∞ estimate of the solution to Equation (1.1) when $\beta=d$ with $d\geq 2$, and we give the proof of Theorem 1.1 by L^∞-criterion. In Section 4, we finish the proof of Theorem 1.1 through a similar method in the case of $\beta\in[2,d), d>2$. Because of different properties of $B(\rho)$, we introduce some different techniques to complete the proof. In the Appendix, we prove a nonlinear maximum principle on the periodic box.

Throughout the paper, C stands for universal constants that may change from line to line.

2. Preliminaries

In what follows, we provide some auxiliary results and notations.

2.1. Nonlocal operator. The fractional Laplacian is a nonlocal operator and it has the following kernel representation on $\mathbb{T}^d$ (see [9, 10])

$$(-\Delta)^{\frac{\alpha}{2}}f(x)=C_{\alpha,d}\sum_{k\in\mathbb{Z}^d}P.V.\int_{\mathbb{T}^d}\frac{f(x)-f(y)}{|x-y+k|^{d+\alpha}}dy, \tag{2.1}$$

where

$$C_{\alpha,d}=\frac{2^\alpha\Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}}|\Gamma(-\frac{\alpha}{2})|},$$

and

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt.$$

Recall that we denote by

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

the eigenvalue of the operator $-\Delta$ on $\mathbb{T}^d$, then the eigenvalue of operator $(-\Delta)^{\frac{\alpha}{2}}$ is as follows (see [15])

$$0 \le \lambda_1^{\frac{\alpha}{2}} \le \lambda_2^{\frac{\alpha}{2}} \le \cdots \le \lambda_n^{\frac{\alpha}{2}} \le \cdots. \tag{2.2}$$

The following results are two important lemmas, see [1, 13, 26] for the details.

LEMMA 2.1 (Positivity Lemma). *Suppose* $0 \le \alpha \le 2, \Omega = \mathbb{R}^d, \mathbb{T}^d$ *and* $f, (-\Delta)^{\frac{\alpha}{2}} f \in L^p$, *where* $p \ge 2$. *Then*

$$\frac{2}{p} \int_\Omega ((-\Delta)^{\frac{\alpha}{4}} |f|^{\frac{p}{2}})^2 dx \le \int_\Omega |f|^{p-2} f (-\Delta)^{\frac{\alpha}{2}} f dx.$$

LEMMA 2.2. *Suppose* $0 < \alpha < 2, \Omega = \mathbb{R}^d, \mathbb{T}^d$ *and* $f \in \mathcal{S}(\Omega)$. *Then*

$$\int_\Omega (-\Delta)^{\frac{\alpha}{2}} f(x) dx = 0.$$

2.2. Functional spaces and inequalities. We write $L^p(\mathbb{T}^d)$ for the usual Lebesgue space

$$L^p(\mathbb{T}^d) = \left\{ f \text{ measurable s.t.} \int_{\mathbb{T}^d} |f(x)|^p dx < \infty \right\},$$

the norm for the L^p space is denoted as $\|\cdot\|_{L^p}$, it means

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}^d} |f|^p dx \right)^{\frac{1}{p}},$$

with natural adjustment when $p = \infty$. The homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^s}$,

$$\|f\|_{\dot{H}^s}^2 = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{2s} |\hat{f}(k)|^2,$$

and the non-homogeneous Sobolev norm $\|\cdot\|_{H^s}$,

$$\|f\|_{H^s}^2 = \|f\|_{L^2}^2 + \|f\|_{\dot{H}^s}^2.$$

For some standard inequalities, we can refer to [19, 22]. The following inequality is a Sobolev embedding for the fractional derivative (see [3]).

LEMMA 2.3 (Homogeneous Sobolev embedding). *Suppose* $0 < \frac{\sigma}{d} < \frac{1}{p} < 1$ *and define* $q \in (p, \infty)$ *via*

$$\frac{\sigma}{d} = \frac{1}{p} - \frac{1}{q}.$$

Then for all $f \in C^\infty(\mathbb{T}^d)$ *with zero mean*

$$\|f\|_{L^q} \le C \|(-\Delta)^{\frac{\sigma}{2}} f\|_{L^p}.$$

2.3. Mixing effect. Given an incompressible vector field u which is Lipschitz in spatial variables, if we defined the trajectories map by (see [11, 28])

$$\frac{d}{dt}\Phi_t(x) = u(t,\Phi_t(x)), \quad \Phi_0(x) = x.$$

Then define a unitary operator U^t acting on $L^2(\mathbb{T}^d)$ as follows

$$U^t f(x) = f(\Phi_t^{-1}(x)),\ f \in L^2(\mathbb{T}^d),$$

for simplicity, we denote

$$U = U^t, \tag{2.3}$$

and

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{T}^d) \Big| \int_{\mathbb{T}^d} f(x)dx = 0, f \neq 0 \right\}.$$

Next, we give the definition of weakly mixing (see [28]).

DEFINITION 2.1. *The incompressible flow u is called weakly mixing, if $u = u(t,x)$ is smooth and the spectrum of the operator U is purely continuous on $\mathcal{G}$, where U be defined in (2.3) with u.*

REMARK 2.1. If u is weakly mixing, for any $f \in L^2(\mathbb{T}^d)$ and f is not constant, we can get $f - \overline{f} \in \mathcal{G}$, then $f - \overline{f}$ is not an eigenfunction of U. So U has no nontrivial eigenfunction on $\mathcal{G}$, where $\overline{f}$ is mean value of f.

REMARK 2.2. The incompressible flow u is called relaxation enhancing (see [11]) if the operator U has no eigenfunctions in $\dot{H}^1$ other than a constant function. Obviously, the weakly mixing is relaxation enhancing flow.

Let us denote $\omega(t,x)$ is the unique solution of the equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \omega(0,x) = \rho_0(x), \tag{2.4}$$

there is the following lemma,

LEMMA 2.4. *Suppose that $0 < \alpha < 2$, $u(t,x)$ is a smooth divergence-free vector field for each $t \geq 0$. Let $\omega(t,x)$ be the solution of (2.4). Then for every $t \geq 0$, and for every $\rho_0 \in \dot{H}^{\frac{\alpha}{2}}$, we have*

$$\|\omega(t,\cdot)\|_{\dot{H}^{\frac{\alpha}{2}}} \leq F(t)\|\rho_0\|_{\dot{H}^{\frac{\alpha}{2}}},$$

where

$$F(t) = \exp\left(\int_0^t D(s)ds \right),$$

and

$$D(t) \leq C\|(-\Delta)^{\frac{2\alpha+d+2}{4}} u(t,\cdot)\|_{L^2}.$$

Proof. We can refer to [25, 28]. ∎

REMARK 2.3. For the examples of relaxation enhancing flow and weakly mixing, we can refer to [11, 20, 21, 25, 28].

2.4. Local well-posedness of (1.1). We provide a local existence of the solution to (1.1) and some basic properties.

THEOREM 2.1. *Let $0<\alpha\leq 2,\beta\in[2,d+1),d\geq 2$, $\rho_0\in H^3(\mathbb{T}^d)$ be a non-negative initial data, then there exist lifespan time $T=T(\rho_0,\alpha)>0$ and unique non-negative solution $\rho(t,x)$ of (1.1), such that*

$$\rho(t,x)\in C([0,T];H^3(\mathbb{T}^d)),$$

and

$$\|\rho(t,\cdot)\|_{L^1}=\|\rho_0\|_{L^1}.$$

Furthermore, under the restriction $\alpha>1$, the solution is smooth.

Proof. The proof of Theorem 2.1 is standard and it is similar to the one in [1,29]. □

3. Proof of Theorem 1.1 ($\beta=d,d\geq 2$)

In this section, we consider the classical Keller-Segel system with fractional dissipation and weakly mixing. We establish the L^∞-criterion, and get the L^∞ estimate of the solution.

3.1. L^∞-criterion. We show that to get the global classical solution of (1.1), we only need to have certain control of spatial L^∞ norm of the solution.

PROPOSITION 3.1. *Suppose that $0<\alpha<2,\beta=d,d\geq 2$, for any initial data $\rho_0\geq 0,\rho_0\in H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$. Then the following criterion holds: either the local solution to (1.1) extends to a global classical solution or there exists $T^*\in(0,\infty)$, such that*

$$\lim_{t\to T^*}\|\rho(t,\cdot)\|_{L^\infty}=\infty.$$

Proof. We only need to derive a priori bounds on higher order derivatives in terms of L^∞ norm of the solution. Assume $\rho(t,x)$ is the solution of Equation (1.1), and $\|\rho(t,\cdot)\|_{L^\infty}$ is bound. Let us multiply both sides of (1.1) by $(-\Delta)^3\rho$ and integrate over $\mathbb{T}^d$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho\|_{\dot H^3}^2&+\int_{\mathbb{T}^d}u\cdot\nabla\rho(-\Delta)^3\rho dx\\&+\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}\rho(-\Delta)^3\rho dx+\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(-\Delta)^3\rho dx=0.\end{aligned}\tag{3.1}$$

We use step-by-step integration and the incompressibility of u to obtain

$$\left|\int_{\mathbb{T}^d}u\cdot\nabla\rho(-\Delta)^3\rho dx\right|\leq C\|u\|_{C^3}\|\rho\|_{\dot H^3}^2.\tag{3.2}$$

And the third term of the left-hand side of (3.1) is equal to

$$\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}\rho(-\Delta)^3\rho dx=\|\rho\|_{\dot H^{3+\frac{\alpha}{2}}}^2.\tag{3.3}$$

For the fourth term of the left-hand side of (3.1), we split it into two pieces

$$\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(-\Delta)^3\rho dx=\int_{\mathbb{T}^d}\nabla\rho\cdot(\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))(-\Delta)^3\rho dx-\int_{\mathbb{T}^d}\rho(\rho-\overline{\rho})(-\Delta)^3\rho dx.\tag{3.4}$$

Integrating by parts, the second term of the right-hand side of (3.4) is expressed as follows

$$\int_{\mathbb{T}^d} \rho(\rho-\overline{\rho})(-\Delta)^3\rho dx \sim \sum_{l=0}^{3}\int_{\mathbb{T}^d} D^l\rho D^{3-l}(\rho-\overline{\rho})D^3\rho dx,$$

where $l=0,1,2.3$ and D denotes any partial derivative. By Hölder's inequality, one has

$$\sum_{l=0,3}\int_{\mathbb{T}^d} D^l\rho D^{3-l}(\rho-\overline{\rho})D^3\rho dx \le (2\|\rho\|_{L^\infty}+\overline{\rho})\|\rho\|_{\dot{H}^3}^2,$$

and

$$\sum_{l=1}^{2}\int_{\mathbb{T}^d} D^l\rho D^{3-l}(\rho-\overline{\rho})D^3\rho dx \le 2\|D\rho\|_{L^6}\|D^2\rho\|_{L^3}\|\rho\|_{\dot{H}^3}.$$

For any $1\le q_0'\le\infty$, we deduce by interpolation inequality that

$$\|\rho\|_{L^{q_0'}} \le \|\rho\|_{L^1}^{\frac{1}{q_0'}}\|\rho\|_{L^\infty}^{1-\frac{1}{q_0'}}, \tag{3.5}$$

combining with the Gagliardo-Nirenberg inequality, then there exist $1\le q_1',q_2'\le\infty$, such that

$$\|D\rho\|_{L^6} \le C\|\rho\|_{L^{q_1'}}^{1-\theta_1}\|\rho\|_{\dot{H}^3}^{\theta_1}, \tag{3.6}$$

and

$$\|D^2\rho\|_{L^3} \le C\|\rho\|_{L^{q_2'}}^{1-\theta_2}\|\rho\|_{\dot{H}^3}^{\theta_2}, \tag{3.7}$$

where

$$\theta_1 = \frac{6-d(1-\frac{6}{q_1'})}{18-d(3-\frac{6}{q_1'})}, \quad \theta_2 = \frac{12-d(2-\frac{6}{q_2'})}{18-d(3-\frac{6}{q_2'})}.$$

Due to $\|\rho\|_{L^1}$ conservation and the fact tha $\|\rho\|_{L^\infty}$ is bounded, according to (3.5), (3.6) and (3.7), we obtain

$$\|D\rho\|_{L^6}\|D^2\rho\|_{L^3}\|\rho\|_{\dot{H}^3} \le C\|\rho\|_{L^{q_1'}}^{1-\theta_1}\|\rho\|_{L^{q_2'}}^{1-\theta_2}\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2} \le C\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}.$$

Therefore, we have

$$\int_{\mathbb{T}^d} \rho(\rho-\overline{\rho})(-\Delta)^3\rho dx \le C(\|\rho\|_{L^\infty}+\overline{\rho})\|\rho\|_{\dot{H}^3}^2 + C\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}. \tag{3.8}$$

Integrating by parts the first term of the right-hand side of (3.4), we get terms that can be estimated similarly to the second term of the right-hand side of (3.4). The only exceptional terms that appear which have different structure (see [28]) are

$$\int_{\mathbb{T}^d} (\partial_{i_1}\partial_{i_2}\partial_{i_3}\nabla\rho)\cdot(\nabla(-\Delta)^{-1}\rho)\partial_{i_1}\partial_{i_2}\partial_{i_3}\rho dx,$$

while these can be reduced to

$$\int_{\mathbb{T}^d}(\rho-\overline{\rho})(\partial_{i_1}\partial_{i_2}\partial_{i_3}\rho)^2dx,$$

and according to the estimation as before, we get

$$\int_{\mathbb{T}^d}\nabla\rho\cdot(\nabla(-\Delta)^{-1}\rho)(-\Delta)^3\rho dx\leq C(\|\rho\|_{L^\infty}+\overline{\rho})\|\rho\|^2_{\dot{H}^3}+C\|\rho\|^{1+\theta_1+\theta_2}_{\dot{H}^3}. \tag{3.9}$$

Thus, we deduce by (3.8) and (3.9) that

$$\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(-\Delta)^3\rho dx\leq C(\|\rho\|_{L^\infty}+\overline{\rho})\|\rho\|^2_{\dot{H}^3}+C\|\rho\|^{1+\theta_1+\theta_2}_{\dot{H}^3}. \tag{3.10}$$

Combining (3.1), (3.2), (3.3) and (3.10), we have

$$\frac{d}{dt}\|\rho\|^2_{\dot{H}^3}\leq -2\|\rho\|^2_{\dot{H}^{3+\frac{\alpha}{2}}}+C(\|u\|_{C^3}+\|\rho\|_{L^\infty}+\overline{\rho})\|\rho\|^2_{\dot{H}^3}+C\|\rho\|^{1+\theta_1+\theta_2}_{\dot{H}^3}. \tag{3.11}$$

By Gagliardo-Nirenberg inequality, for any $1\leq q_3'\leq\infty$, we obtain

$$\|\rho\|_{\dot{H}^3}\leq C\|\rho\|^{1-\theta}_{L^{q_3'}}\|\rho\|^{\theta}_{\dot{H}^{3+\frac{\alpha}{2}}}, \tag{3.12}$$

where

$$\theta=\frac{\frac{2d}{q_3'}+6-d}{\frac{2d}{q_3'}+6-d+\alpha}.$$

We denote

$$\gamma=\frac{2}{\theta}=\frac{\frac{4d}{q_3'}+12-2d+2\alpha}{\frac{2d}{q_3'}+6-d},$$

according to (3.5) and (3.12), we get

$$\|\rho\|^{\gamma}_{\dot{H}^3}\leq C\|\rho\|^{(1-\theta)\gamma}_{L^{q_3'}}\|\rho\|^2_{\dot{H}^{3+\frac{\alpha}{2}}}\leq C_4\|\rho\|^2_{\dot{H}^{3+\frac{\alpha}{2}}},$$

thus, we have

$$-\|\rho\|^2_{\dot{H}^{3+\frac{\alpha}{2}}}\leq -C_4^{-1}\|\rho\|^{\gamma}_{\dot{H}^3}\leq -C\|\rho\|^{\gamma}_{\dot{H}^3}. \tag{3.13}$$

According to (3.11) and (3.13), one has

$$\frac{d}{dt}\|\rho\|^2_{\dot{H}^3}\leq -C\|\rho\|^{\gamma}_{\dot{H}^3}+C(\|u\|_{C^3}+\|\rho\|_{L^\infty}+\overline{\rho})\|\rho\|^2_{\dot{H}^3}+C\|\rho\|^{1+\theta_1+\theta_2}_{\dot{H}^3}. \tag{3.14}$$

As $\|u\|_{C^3}$, $\|\rho\|_{L^\infty}$ are bounded, and we choose q_3', such that

$$2<1+\theta_1+\theta_2<\gamma,$$

then we know that $\|\rho\|_{\dot{H}^3}$ is bounded. Because $\|\rho\|_{L^2}$ bound is obvious, by the definition of $\|\rho\|_{H^3}$, we imply that $\|\rho(t,\cdot)\|_{H^3}$ is bounded. Namely, there exists a constant $C_{H^3}=C(\|\rho\|_{L^\infty},\|\rho_0\|_{H^3})$, such that

$$\|\rho(t,\cdot)\|_{H^3}\leq C_{H^3}.$$

This completes the proof of Proposition 3.1. ∎

Remark 3.1. Particularly, if $\|\rho\|_{L^\infty}$ is bounded only in $[0,T]$, then $\|\rho(t,\cdot)\|_{H^3}$ is bounded in $[0,T]$.

3.2. L^∞ estimate of ρ. We establish the L^∞ estimate of the solution to Equation (1.1). The important technique we use is the nonlinear maximum principle on periodic box, the details can be found in the Appendix. A useful lemma is as follows:

LEMMA 3.1. *Let $0<\alpha<2, \beta=d, d\geq 2$, $\rho(t,x)$ is the local solution of Equation (1.1) with initial data $\rho_0(x)\geq 0$. If $\rho\in L^p(\mathbb{T}^d)$, $1\leq p<\infty$, and we denote*

$$\widetilde{\rho}(t)=\rho(t,\overline{x}_t)=\max_{x\in\mathbb{T}^d}\rho(t,x).$$

Then we have

$$\widetilde{\rho}(t)\leq C(d,p)\|\rho\|_{L^p}, \tag{3.15}$$

or

$$\frac{d}{dt}\widetilde{\rho}\leq \widetilde{\rho}^2-\widetilde{\rho}-C(\alpha,d,p)\frac{\widetilde{\rho}^{1+\frac{p\alpha}{d}}}{\|\rho\|_{L^p}^{\frac{p\alpha}{d}}}. \tag{3.16}$$

Proof. For any fixed $t\geq 0$, using the vanishing of a derivative at the point of maximum, we see that

$$\partial_t\rho(t,\overline{x}_t)=\frac{d}{dt}\widetilde{\rho}(t), \quad (u\cdot\nabla\rho)(t,\overline{x}_t)=u\cdot\nabla\widetilde{\rho}(t)=0,$$

and

$$\left(\nabla\cdot(\rho B(\rho))\right)(t,\overline{x}_t)=-(\widetilde{\rho}(t))^2+\widetilde{\rho}(t),$$

if we denote

$$(-\Delta)^{\frac{\alpha}{2}}\rho(t,x)\big|_{x=\overline{x}_t}=(-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho}(t),$$

we deduce that by (1.1) the evolution of $\widetilde{\rho}$ follows

$$\frac{d}{dt}\widetilde{\rho}+(-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho}-\widetilde{\rho}^2+\widetilde{\rho}=0. \tag{3.17}$$

According to the nonlinear maximum principle (see Lemma), if $\widetilde{\rho}$ satisfies

$$\widetilde{\rho}(t)\leq C(d,p)\|\rho\|_{L^p},$$

we finish the proof of (3.15). If not, $\widetilde{\rho}$ must satisfy

$$(-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho}(t)\geq C(\alpha,d,p)\frac{(\widetilde{\rho}(t))^{1+\frac{p\alpha}{d}}}{\|\rho\|_{L^p}^{\frac{p\alpha}{d}}}. \tag{3.18}$$

Thus, we deduce by (3.17) and (3.18) that

$$\frac{d}{dt}\widetilde{\rho}=\widetilde{\rho}^2-\widetilde{\rho}-(-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho}\leq\widetilde{\rho}^2-\widetilde{\rho}-C(\alpha,d,p)\frac{\widetilde{\rho}^{1+\frac{p\alpha}{d}}}{\|\rho\|_{L^p}^{\frac{p\alpha}{d}}},$$

so we have

$$\frac{d}{dt}\widetilde{\rho}\leq\widetilde{\rho}^2-\widetilde{\rho}-C(\alpha,d,p)\frac{\widetilde{\rho}^{1+\frac{p\alpha}{d}}}{\|\rho\|_{L^p}^{\frac{p\alpha}{d}}},$$

we finish the proof of (3.16). This completes the proof of Lemma 3.1. □

First, we need the local L^2, $L^p(p>\frac{d}{\alpha})$ and L^∞ estimates of the solution.

Lemma 3.2. *Let $0<\alpha<2, \beta=d, d\geq 2$, $\rho(t,x)$ is the local solution of Equation (1.1) with initial data $\rho_0(x)\geq 0$. Suppose that $\|\rho_0\|_{L^2}=B_0$, $\|\rho_0\|_{L^p}\leq C_p$, $\|\rho_0\|_{L^\infty}\leq C_\infty$. Then there exists a time $\tau_1>0$, for any $0\leq t\leq \tau_1$, such that*

$$\|\rho(t,\cdot)-\overline{\rho}\|_{L^2}\leq 2(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}, \quad \|\rho(t,\cdot)\|_{L^p}\leq 2C_p, \qquad \|\rho(t,\cdot)\|_{L^\infty}\leq 2C_\infty,$$

where $\overline{\rho}$ is the mean-value of ρ_0 and $p>\frac{d}{\alpha}$.

Proof. According to the proof of Lemma 3.1, one has

$$\frac{d}{dt}\widetilde{\rho}=\widetilde{\rho}^2-\widetilde{\rho}-(-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho},$$

due to $\widetilde{\rho}\geq 0, (-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho}\geq 0$, we get

$$\frac{d}{dt}\widetilde{\rho}\leq \widetilde{\rho}^2. \tag{3.19}$$

If we define

$$\tau_0=\min\left\{\frac{1}{2C_\infty}, T\right\},$$

where T is defined in Theorem 2.1. Because $\|\rho_0\|_{L^\infty}\leq C_\infty$, by solving the differential inequality in (3.19), for any $0\leq t\leq \tau_0$, we have

$$\|\rho(t,\cdot)\|_{L^\infty}\leq 2C_\infty. \tag{3.20}$$

Let us multiply both sides of (1.1) by $|\rho|^{p-2}\rho$ and integrate over $\mathbb{T}^d$, to obtain that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\rho\|_{L^p}^p&+\int_{\mathbb{T}^d}u\cdot\nabla\rho|\rho|^{p-2}\rho dx\\&+\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}\rho|\rho|^{p-2}\rho dx+\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))|\rho|^{p-2}\rho dx=0,\end{aligned} \tag{3.21}$$

for any $0\leq t\leq \tau_0$, we deduce by the standard energy estimate, (3.20) and (3.21) that

$$\frac{d}{dt}\|\rho\|_{L^p}^p\leq -2\||\rho|^{\frac{p}{2}}\|_{\dot{H}^{\frac{\alpha}{2}}}^2+2(p-1)(C_\infty+\overline{\rho})\|\rho\|_{L^p}^p. \tag{3.22}$$

And let us multiply both sides of (1.1) by $\rho-\overline{\rho}$ and integrate over $\mathbb{T}^d$, to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho-\overline{\rho}\|_{L^2}^2&+\int_{\mathbb{T}^d}u\cdot\nabla\rho(\rho-\overline{\rho})dx\\&+\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}\rho(\rho-\overline{\rho})dx+\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(\rho-\overline{\rho})dx=0.\end{aligned} \tag{3.23}$$

and for any $0\leq t\leq \tau_0$, we obtain

$$\frac{d}{dt}\|\rho-\overline{\rho}\|_{L^2}^2\leq -2\|\rho\|_{\dot{H}^{\frac{\alpha}{2}}}^2+2(C_\infty+\overline{\rho})\|\rho-\overline{\rho}\|_{L^2}^2. \tag{3.24}$$

We denote

$$\tau_1 = \min\left\{\tau_0, \frac{\ln 2}{(C_\infty + \overline{\rho})}\right\}, \tag{3.25}$$

due to $\|\rho_0 - \overline{\rho}\|_{L^2}^2 = B_0^2 - \overline{\rho}^2, \|\rho_0\|_{L^p} \leq C_p$, by solving the differential inequality in (3.22) and (3.24), for any $0 \leq t \leq \tau_1$, we get

$$\|\rho(t,\cdot)\|_{L^p} \leq 2C_p,$$

and

$$\|\rho(t,\cdot) - \overline{\rho}\|_{L^2} \leq 2(B_0^2 - \overline{\rho}^2)^{\frac{1}{2}}.$$

According to (3.20) and the definition of τ_1, for any $0 \leq t \leq \tau_1$, we obtain

$$\|\rho(t,\cdot)\|_{L^\infty} \leq 2C_\infty.$$

This completes the proof of Lemma 3.2. □

REMARK 3.2. If there exists $0 < \tau' \leq \tau_1$, such that $\int_0^{\tau'} \|\rho(t,\cdot)\|_{\dot{H}^{\frac{\alpha}{2}}}^2 dt$ is large enough, then $\|\rho(\tau',\cdot) - \overline{\rho}\|_{L^2}$ is obviously small. If not, we need an approximation to $\rho(t,x)$. We also get the local L^2 estimate of the solution only dependent on L^∞ estimate of the solution.

Next, we give an approximation lemma.

LEMMA 3.3. *Let $0 < \alpha < 2, \beta = d, d \geq 2$, suppose that the vector field $u(t,x)$ is smooth incompressible flow. Let $\rho(t,x), \omega(t,x)$ be the local solution of Equation (1.1) and (2.4) respectively with $\rho_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d), \rho_0 \geq 0$. Then for every $t \in [0,T]$, we have*

$$\begin{aligned}\frac{d}{dt}\|\rho - \omega\|_{L^2}^2 \leq &-\|\rho\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + F^2(t)\|\rho_0\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + \|\rho\|_{L^4}^2\|\rho - \overline{\rho}\|_{L^2} \\ &+ C(\|\nabla\rho\|_{L^2} + \|\rho\|_{L^\infty})\|\rho - \overline{\rho}\|_{L^2}\|\rho_0\|_{L^\infty},\end{aligned}$$

where $F(t)$ be defined in Lemma 2.4, and $F(t) \in L^\infty_{loc}[0,\infty)$.

Proof. By (1.1) and (2.4), we obtain the equation

$$\partial_t(\rho - \omega) + u \cdot \nabla(\rho - \omega) + (-\Delta)^{\frac{\alpha}{2}}\rho + \nabla \cdot (\rho B(\rho)) = 0. \tag{3.26}$$

Let us multiply both sides of (3.26) by $\rho - \omega$ and integrate over $\mathbb{T}^d$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho - \omega\|_{L^2}^2 &+ \int_{\mathbb{T}^d} u \cdot \nabla(\rho - \omega)(\rho - \omega)dx \\ &+ \int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}}\rho(\rho - \omega)dx + \int_{\mathbb{T}^d} \nabla \cdot (\rho B(\rho))(\rho - \omega)dx = 0.\end{aligned} \tag{3.27}$$

For the second term of the left-hand side of (3.27), according to the incompressibility of u, we easily get

$$\int_{\mathbb{T}^d} u \cdot \nabla(\rho - \omega)(\rho - \omega)dx = 0. \tag{3.28}$$

The third term of the left-hand side of (3.27) can be estimated as

$$\int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}}\rho(\rho - \omega)dx = \int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}}\rho\rho dx - \int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}}\rho\omega dx,$$

then we deduce by Lemma 2.1 and Hölder's inequality that

$$\int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}} \rho\rho = \|\rho\|^2_{\dot{H}^{\frac{\alpha}{2}}},$$

and

$$\int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}} \rho\omega dx = \int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{4}} \rho(-\Delta)^{\frac{\alpha}{4}} \omega\, dx \le \|\rho\|_{\dot{H}^{\frac{\alpha}{2}}} \|\omega\|_{\dot{H}^{\frac{\alpha}{2}}},$$

so we get

$$\int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}} \rho(\rho-\omega) dx \ge \|\rho\|^2_{\dot{H}^{\frac{\alpha}{2}}} - \|\rho\|_{\dot{H}^{\frac{\alpha}{2}}} \|\omega\|_{\dot{H}^{\frac{\alpha}{2}}}. \tag{3.29}$$

The fourth term of the left-hand side of (3.27) can be estimated as

$$\int_{\mathbb{T}^d} \nabla\cdot(\rho B(\rho))(\rho-\omega)dx = \int_{\mathbb{T}^d} \nabla\cdot(\rho B(\rho))\rho dx - \int_{\mathbb{T}^d} \nabla\cdot(\rho B(\rho))\omega dx. \tag{3.30}$$

For the first term of the right-hand side of (3.30), we get

$$\int_{\mathbb{T}^d} \nabla\cdot(\rho B(\rho))\rho dx = -\frac{1}{2}\int_{\mathbb{T}^d} \rho^2(\rho-\overline{\rho})dx \le \frac{1}{2}\|\rho\|^2_{L^4}\|\rho-\overline{\rho}\|_{L^2}, \tag{3.31}$$

and for the second term of the right-hand side of (3.30), by Hölder's inequality and Poincaré's inequality, we obtain that

$$\begin{aligned}
\int_{\mathbb{T}^d} \nabla\cdot(\rho B(\rho))\omega dx &= \int_{\mathbb{T}^d} \nabla\cdot(\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))\omega dx \\
&\le \|\nabla\cdot(\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))\|_{L^1}\|\omega\|_{L^\infty} \\
&\le C(\|\nabla\rho\cdot(\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))\|_{L^1} + \|\rho(\rho-\overline{\rho})\|_{L^1})\|\omega\|_{L^\infty} \\
&\le C(\|\nabla\rho\|_{L^2}\|\rho-\overline{\rho}\|_{L^2} + \|\rho\|_{L^\infty}\|\rho-\overline{\rho}\|_{L^2})\|\omega\|_{L^\infty} \\
&\le C(\|\nabla\rho\|_{L^2} + \|\rho\|_{L^\infty})\|\rho-\overline{\rho}\|_{L^2}\|\omega\|_{L^\infty}.
\end{aligned} \tag{3.32}$$

Therefore, we deduce by (3.30), (3.31) and (3.32) that

$$\int_{\mathbb{T}^d} \nabla\cdot(\rho B(\rho))(\rho-\omega)dx \le \frac{1}{2}\|\rho\|^2_{L^4}\|\rho-\overline{\rho}\|_{L^2} + C(\|\nabla\rho\|_{L^2} + \|\rho\|_{L^\infty})\|\rho-\overline{\rho}\|_{L^2}\|\omega\|_{L^\infty}. \tag{3.33}$$

Combining (3.27), (3.28), (3.29) and (3.33), one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\rho-\omega\|^2_{L^2} \le -\|\rho\|^2_{\dot{H}^{\frac{\alpha}{2}}} + \|\rho\|_{\dot{H}^{\frac{\alpha}{2}}}\|\omega\|_{\dot{H}^{\frac{\alpha}{2}}} + \frac{1}{2}\|\rho\|^2_{L^4}\|\rho-\overline{\rho}\|_{L^2} \\
+ C(\|\nabla\rho\|_{L^2} + \|\rho\|_{L^\infty})\|\rho-\overline{\rho}\|_{L^2}\|\omega\|_{L^\infty}.
\end{aligned} \tag{3.34}$$

For the the second term of the right-hand side of (3.34), Young's inequality yields that

$$\|\rho\|_{\dot{H}^{\frac{\alpha}{2}}}\|\omega\|_{\dot{H}^{\frac{\alpha}{2}}} \le \frac{1}{2}\|\rho\|^2_{\dot{H}^{\frac{\alpha}{2}}} + \frac{1}{2}\|\omega\|^2_{\dot{H}^{\frac{\alpha}{2}}}, \tag{3.35}$$

thus, we deduce by (3.34) and (3.35) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\rho-\omega\|^2_{L^2} \le -\frac{1}{2}\|\rho\|^2_{\dot{H}^{\frac{\alpha}{2}}} + \frac{1}{2}\|\omega\|^2_{\dot{H}^{\frac{\alpha}{2}}} + \frac{1}{2}\|\rho\|^2_{L^4}\|\rho-\overline{\rho}\|_{L^2} \\
+ C(\|\nabla\rho\|_{L^2} + \|\rho\|_{L^\infty})\|\rho-\overline{\rho}\|_{L^2}\|\omega\|_{L^\infty}.
\end{aligned}$$

By Lemma 2.4, we have

$$\frac{d}{dt}\|\rho-\omega\|_{L^2}^2 \le -\|\rho\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + F^2(t)\|\rho_0\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + \|\rho\|_{L^4}^2\|\rho-\overline{\rho}\|_{L^2} + C(\|\nabla\rho\|_{L^2}+\|\rho\|_{L^\infty})\|\rho-\overline{\rho}\|_{L^2}\|\rho_0\|_{L^\infty}.$$

This completes the proof of Lemma 3.3. □

Now, we establish global L^∞ estimate of the solution to Equation (1.1) in the case of weakly mixing.

PROPOSITION 3.2 (Global L^∞ estimate). *Let $0<\alpha<2, \beta=d, d\ge 2$, suppose $\rho(t,x)$ is the solution of Equation (1.1) with initial data $\rho_0\ge 0, \rho_0\in H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$. Then there exist weakly mixing u and a positive constant C_{L^∞}, such that*

$$\|\rho(t,\cdot)\|_{L^\infty}\le C_{L^\infty}, \quad t\in[0,+\infty].$$

Before starting the proof of Proposition 3.2, we need one auxiliary result (see [11, 15, 28]). On $\mathbb{T}^d$, we denote by $0\le\lambda_1^{\frac{\alpha}{2}}\le\lambda_2^{\frac{\alpha}{2}}\le\cdots\le\lambda_n^{\frac{\alpha}{2}}\le\cdots$ the eigenvalues of $(-\Delta)^{\frac{\alpha}{2}}$ and by $e_1,e_2,\cdots,e_n,\cdots$ the corresponding orthogonal eigenvectors. Let us denote by P_N the orthogonal projection on the subspace spanned by the first N eigenvectors $e_1,e_2,\cdots,e_N$ and

$$S=\{\phi\in L^2\big|\|\phi\|_{L^2}=1\}.$$

The following lemma is an extension of the well-known RAGE theorem (see [11, 16, 28]).

LEMMA 3.4. *Let U be a unitary operator with purely continuous spectrum defined on $L^2(\mathbb{T}^d)$. Let $K\subset S$ be a compact set. Then for every N and $\sigma>0$, there exists $T_c=T(N,\sigma,K,U)$ such that for all $T\ge T_c$ and every $\phi\in K$, we have*

$$\frac{1}{T}\int_0^T\|P_N U^t\phi\|_{L^2}^2 dt\le\sigma.$$

REMARK 3.3. We denote $\chi=\chi(|x|\le R)$ as a cutoff function, if we have χ instead of P_N, then the RAGE theorem tells us that any state in continuous spectrum space will "infinitely often leave" the ball of radius R. This is indeed what we expect physically.

Let us consider the equation

$$\begin{cases}\partial_t\rho^A + Au\cdot\nabla\rho^A + (-\Delta)^{\frac{\alpha}{2}}\rho^A + \nabla\cdot(\rho^A B(\rho^A))=0, & t>0, x\in\mathbb{T}^d\\ \rho^A(0,x)=\rho_0(x).\end{cases} \tag{3.36}$$

Here A is the coupling constant regulating strength of the fluid flow that we assume to be large and Au is weakly mixing.

We are ready to give the proof of the Proposition 3.2.

Proof. **(Proof of Proposition 3.2.)** As $\rho_0\in H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$, without loss of generality, we assume that there exist positive constants B_0, E, C_∞ and $C_\infty\ge 2C(d,p)(E+\overline{\rho})$ such that

$$\|\rho_0\|_{L^2}=B_0, \quad \|\rho_0-\overline{\rho}\|_{L^p}\le E, \quad \|\rho_0\|_{L^\infty}\le C_\infty, \tag{3.37}$$

where $p>\frac{d}{\alpha}$, and $C(d,p)$ be as defined in Lemma 5.1. We denote

$$B_1=\min\left\{(B_0^2-\overline{\rho}^2)^{\frac{1}{2}},\left(\frac{E}{(2C_\infty+\overline{\rho})^{1-\frac{2}{p}}}\right)^{\frac{p}{2}}\right\}.$$

As $\lambda_n^{\frac{\alpha}{2}}$ are the eigenvalues of $(-\Delta)^{\frac{\alpha}{2}}$ on $\mathbb{T}^d$, and

$$\lambda_n^{\frac{\alpha}{2}}\to\infty,\quad n\to\infty,$$

so we choose N, such that

$$\lambda_N^{\frac{\alpha}{2}}\geq\max\left\{\frac{800}{23}(C_\infty+\overline{\rho}),\left(1-\frac{B_1^2}{B_0^2-\overline{\rho}^2}\right)\frac{1}{\tau_1}+(C_\infty+\overline{\rho}),\frac{2}{\tau_1}\ln\frac{B_0^2-\overline{\rho}^2}{B_1^2}\right\},\tag{3.38}$$

where τ_1 is defined in (3.25). Define the compact set $K\subset S$ by

$$K=\{\phi\in S\big|\|\phi\|_{\dot{H}^{\frac{\alpha}{2}}}^2\leq\lambda_N^{\frac{\alpha}{2}}\}.\tag{3.39}$$

Let U^t is the unitary operator associated with weakly mixing flow u in the Definition 2.1. Fix $\sigma=\frac{1}{100}$, then we get $T_c=T_c(N,\sigma,K,U)$, which is the time provided by Lemma 3.4. We proceed to impose the first condition on $A_0=A_0(T_c,\rho_0,\tau_1)$. For any $A\geq A_0$, we define τ as follows

$$\tau=\frac{T_c}{A}\leq\tau_1,$$

where τ_1 be defined in (3.25). As $\|\rho_0\|_{L^2}=B_0$, then for the solution $\rho^A(t,x)$ of Equation (3.36), we deduce by Lemma 3.2 and (3.37) that

$$\|\rho^A(t,\cdot)-\overline{\rho}\|_{L^2}\leq 2(B_0^2-\overline{\rho}^2)^{\frac{1}{2}},\quad\|\rho^A(t,\cdot)\|_{L^\infty}\leq 2C_\infty,\quad 0\leq t\leq\tau_1.\tag{3.40}$$

Next, we consider the equation

$$\partial_t\omega^A+Au\cdot\nabla\omega^A=0,\quad\omega^A(0,x)=\rho_0(x),$$

according to the definition of U^t, one has

$$\omega^A(t,x)-\overline{\rho}=U^{At}(\rho_0(x)-\overline{\rho}).$$

Let $(\rho_0-\overline{\rho})/\|\rho_0-\overline{\rho}\|_{L^2}\in K$, we obtain by the Lemma 3.4 and the definition of τ that

$$\begin{aligned}\frac{1}{\tau}\int_0^\tau\|P_N(\omega^A-\overline{\rho})\|_{L^2}^2dt&=\frac{1}{\tau}\int_0^\tau\|P_NU^{At}(\rho_0-\overline{\rho})\|_{L^2}^2dt\\&=\frac{\|\rho_0-\overline{\rho}\|_{L^2}^2}{\tau}\int_0^\tau\|P_NU^{At}\frac{(\rho_0-\overline{\rho})}{\|\rho_0-\overline{\rho}\|_{L^2}}\|_{L^2}^2dt\\&=\frac{\|\rho_0-\overline{\rho}\|_{L^2}^2}{A\tau}\int_0^\tau\|P_NU^{At}\frac{(\rho_0-\overline{\rho})}{\|\rho_0-\overline{\rho}\|_{L^2}}\|_{L^2}^2dAt\\&=\frac{\|\rho_0-\overline{\rho}\|_{L^2}^2}{T_c}\int_0^{T_c}\|P_NU^{s}\frac{(\rho_0-\overline{\rho})}{\|\rho_0-\overline{\rho}\|_{L^2}}\|_{L^2}^2dsv\\&\leq\sigma\|\rho_0-\overline{\rho}\|_{L^2}^2\leq\frac{1}{100}(B_0^2-\overline{\rho}^2).\end{aligned}\tag{3.41}$$

Since $(\rho_0-\overline{\rho})/\|\rho_0-\overline{\rho}\|_{L^2}\in K$, by the definition of K in (3.39), one has

$$\|\rho_0\|^2_{\dot{H}^{\frac{\alpha}{2}}}\leq \lambda_N^{\frac{\alpha}{2}}\|\rho_0-\overline{\rho}\|^2_{L^2}\leq \lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{\rho}^2). \tag{3.42}$$

For any fixed $p^*\in[1,\infty)$, according to (3.5) and (3.40), there exists a positive constant $C=C(p^*)$, such that

$$\|\rho^A(t,\cdot)\|_{L^{p^*}}\leq C,\quad t\in[0,\tau_1],$$

and we deduce by (3.14) and (3.40) that $\|\rho^A(t,\cdot)\|_{H^3}$ is bounded for any $t\in[0,\tau_1]$. Namely, there is a positive constant $C^*_{H^3}=C(A)$, such that

$$\|\rho^A(t,\cdot)\|_{H^3}\leq C^*_{H^3},\quad t\in[0,\tau_1],$$

by Gagliardo-Nirenberg inequality, we obtain

$$\|\nabla\rho^A\|_{L^2}\leq C\|\rho^A\|^{1-\theta_0}_{L^{p^*}}\|\rho^A\|^{\theta_0}_{\dot{H}^3}\leq C_4,\quad t\in[0,\tau_1], \tag{3.43}$$

where

$$\theta_0=\frac{\frac{1}{2}-\frac{1}{d}-\frac{1}{p^*}}{\frac{1}{2}-\frac{3}{d}-\frac{1}{p^*}}.$$

Combining (3.37), (3.40), (3.42), (3.43) and Lemma 3.3, for any $0<t\leq\tau_1$, we get

$$\begin{aligned}\frac{d}{dt}\|\rho^A-\omega^A\|^2_{L^2}\leq&\lambda_N^{\frac{\alpha}{2}}F(At)^2(B_0^2-\overline{\rho}^2)+C(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}+C(C_4+2C_\infty)C_\infty(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}\\ \leq&\lambda_N^{\frac{\alpha}{2}}F(At)^2(B_0^2-\overline{\rho}^2)+C(1+C_4C_\infty+2C_\infty^2)(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}.\end{aligned} \tag{3.44}$$

As $F(t)$ is a locally bounded function, then there is a $A_1\geq A_0$, when $A\geq A_1$, we have

$$\begin{aligned}&\int_0^\tau \lambda_N^{\frac{\alpha}{2}}F(At)^2(B_0^2-\overline{\rho}^2)+C(1+C_4C_\infty+2C_\infty^2)(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}\,dt\\ \leq&\frac{\lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{\rho}^2)}{A}\int_0^{T_c}F(t)^2dt+C(1+C_4C_\infty+2C_\infty^2)(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}\tau\\ \leq&\frac{B_0^2-\overline{\rho}^2}{100}.\end{aligned}$$

Therefore, we integrate from 0 to t on both sides of (3.44), where $0\leq t\leq\tau$, we obtain

$$\|\rho^A(t,\cdot)-\omega^A(t,\cdot)\|^2_{L^2}\leq\frac{B_0^2-\overline{\rho}^2}{100}, \tag{3.45}$$

so we deduce by $\|\omega^A(t,\cdot)-\overline{\rho}\|^2_{L^2}=\|\rho_0-\overline{\rho}\|^2_{L^2}=B_0^2-\overline{\rho}^2$ that

$$\frac{81}{100}(B_0^2-\overline{\rho}^2)\leq\|\rho^A(t,\cdot)-\overline{\rho}\|^2_{L^2}\leq\frac{121}{100}(B_0^2-\overline{\rho}^2),\quad 0\leq t\leq\tau. \tag{3.46}$$

Furthermore, by the estimates (3.41) and (3.45), we get

$$\frac{1}{\tau}\int_0^\tau\|P_N(\rho^A(t,\cdot)-\overline{\rho})\|^2_{L^2}dt$$

$$\leq \frac{2}{\tau}\int_0^\tau \|P_N(\omega^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 dt + \frac{2}{\tau}\int_0^\tau \|P_N(\rho^A(t,\cdot)-\omega^A(t,\cdot))\|_{L^2}^2 dt$$
$$\leq \frac{B_0^2-\overline{\rho}^2}{25}. \tag{3.47}$$

For $\|\rho^A(t,\cdot)\|_{\dot H^{\frac{\alpha}{2}}}^2$, we have

$$\begin{aligned}
\|\rho^A(t,\cdot)\|_{\dot H^{\frac{\alpha}{2}}}^2 &= \|\rho^A(t,\cdot)-\overline{\rho}\|_{\dot H^{\frac{\alpha}{2}}}^2 \\
&\geq \|(I-P_N)(\rho^A(t,\cdot)-\overline{\rho})\|_{\dot H^{\frac{\alpha}{2}}}^2 \\
&= \|(-\Delta)^{\frac{\alpha}{4}}(I-P_N)(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 \\
&\geq \lambda_N^{\frac{\alpha}{2}}\|(I-P_N)(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2,
\end{aligned}$$

and

$$\|(I-P_N)(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 \geq \frac{1}{2}\|(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 - \|P_N(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2.$$

Thus, we deduce by (3.46) and (3.47) that

$$\begin{aligned}
\frac{1}{\tau}\int_0^\tau \|\rho^A(t,\cdot)\|_{\dot H^{\frac{\alpha}{2}}}^2 dt &\geq \frac{1}{\tau}\int_0^\tau \lambda_N^{\frac{\alpha}{2}}\|(I-P_N)(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 dt \\
&\geq \frac{\lambda_N^{\frac{\alpha}{2}}}{2\tau}\int_0^\tau \|(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 dt - \frac{\lambda_N^{\frac{\alpha}{2}}}{\tau}\int_0^\tau \|P_N(\rho^A(t,\cdot)-\overline{\rho})\|_{L^2}^2 dt \\
&\geq \frac{81}{200}\lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{\rho}^2) - \frac{1}{25}\lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{\rho}^2) \\
&\geq \frac{73}{200}\lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{\rho}^2).
\end{aligned} \tag{3.48}$$

According to (3.24), we obtain

$$\frac{d}{dt}\|\rho^A-\overline{\rho}\|_{L^2}^2 \leq -2\|\rho^A\|_{\dot H^{\frac{\alpha}{2}}}^2 + 2(C_\infty+\overline{\rho})\|\rho^A-\overline{\rho}\|_{L^2}^2, \tag{3.49}$$

we integrate from 0 to τ on both sides of (3.49), to get

$$\|\rho^A(\tau,\cdot)-\overline{\rho}\|_{L^2}^2 \leq -2\int_0^\tau \|\rho^A\|_{\dot H^{\frac{\alpha}{2}}}^2 dt + \int_0^\tau 2(C_\infty+\overline{\rho})\|\rho^A-\overline{\rho}\|_{L^2}^2 dt + \|\rho_0-\overline{\rho}\|_{L^2}^2.$$

Combining (3.37), (3.38), (3.40) and (3.48), we have

$$\begin{aligned}
\|\rho^A(\tau,\cdot)-\overline{\rho}\|_{L^2}^2 &\leq (B_0^2-\overline{\rho}^2) - 2\tau\left(\frac{1}{\tau}\int_0^\tau \|\rho^A\|_{\dot H^{\frac{\alpha}{2}}}^2 dt\right) + \int_0^\tau 8(C_\infty+\overline{\rho})(B_0^2-\overline{\rho}^2)dt \\
&\leq -\frac{73}{100}\lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{\rho}^2)\tau + 8(C_\infty+\overline{\rho})(B_0^2-\overline{\rho}^2)\tau + (B_0^2-\overline{\rho}^2) \\
&\leq \left(-\frac{73}{100}\lambda_N^{\frac{\alpha}{2}} + 8(C_\infty+\overline{\rho})\right)(B_0^2-\overline{\rho}^2)\tau + (B_0^2-\overline{\rho}^2) \\
&\leq (1-\frac{1}{2}\lambda_N^{\frac{\alpha}{2}}\tau)(B_0^2-\overline{\rho}^2).
\end{aligned}$$

We define

$$k = \left\lfloor \frac{A\tau_1}{2T_c} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ is downward rectification. Then there exists a $A_2>A_1$, when $A\geq A_2$, using the same argument k times, we get

$$\|\rho^A(k\tau,\cdot)-\overline{\rho}\|_{L^2}\leq(1-\frac{1}{2}\lambda_N^{\frac{\alpha}{2}}\tau)^{\frac{k}{2}}(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}\leq B_1. \tag{3.50}$$

We deduce by (3.20), (3.50) and interpolation inequality that

$$\|\rho^A(k\tau,\cdot)-\overline{\rho}\|_{L^p}\leq\|\rho^A-\overline{\rho}\|_{L^2}^{\frac{2}{p}}\|\rho^A-\overline{\rho}\|_{L^\infty}^{1-\frac{2}{p}}\leq E,$$

so we have

$$\|\rho^A(k\tau,\cdot)\|_{L^p}\leq E+\overline{\rho}, \tag{3.51}$$

and according to $\|\rho_0-\overline{\rho}\|_{L^p}\leq E$ and Lemma 3.2, for any $0\leq t\leq k\tau$, one has

$$\|\rho^A(t,\cdot)\|_{L^p}\leq 2(E+\overline{\rho}).$$

We denote

$$\widetilde{\rho}(t)=\rho^A(t,\overline{x}_t)=\max_{x\in\mathbb{T}^d}\rho^A(t,x),$$

then by nonlinear maximum principle, if $t\in[0,k\tau]$ and $\widetilde{\rho}(t)$ satisfies (3.15), then we have

$$\widetilde{\rho}(t)\leq C(d,p)\|\rho^A(t,\cdot)\|_{L^p}\leq 2C(d,p)(E+\overline{\rho})\leq C_\infty. \tag{3.52}$$

If not, then $\widetilde{\rho}(t)\geq 2C(d,p)(E+\overline{\rho})$, and $\widetilde{\rho}(t)$ satisfies (3.16), so we have

$$\frac{d}{dt}\widetilde{\rho}\leq\widetilde{\rho}^2-\widetilde{\rho}-C_3\widetilde{\rho}^{1+\frac{p\alpha}{d}},\quad t\in[0,k\tau], \tag{3.53}$$

where $C_3=C(\alpha,d,p)/(2(E+\overline{\rho}))^{\frac{p\alpha}{d}}$. We set

$$M_0=\max\{x|x^2-x-C_3x^{1+\frac{p\alpha}{d}}=0\},$$

and we denote

$$C_{L^\infty}=\max\{2C(d,p)(E+\overline{\rho}),M_0,\|\rho_0\|_{L^\infty}\},$$

as $\alpha>\frac{d}{p}$, then

$$1+\frac{p\alpha}{d}>2.$$

Solving the differential inequality of (3.53), we deduce that

$$\widetilde{\rho}(t)\leq C_{L^\infty}, \tag{3.54}$$

for any $t\in[0,k\tau]$, combining with (3.52) and (3.54), we have

$$\|\rho^A(t,\cdot)\|_{L^\infty}\leq C_{L^\infty}.$$

For the solution $\rho(t,x)$ of Equation (1.1), by the same argument as above, we deduce that for any $n\in\mathbb{Z}^+$, one has

$$\|\rho(nk\tau,\cdot)\|_{L^p}\leq E+\overline{\rho},$$

then by the similarity with (3.51) and (3.54), for any $t \geq 0$, we have

$$\|\rho(t,\cdot)\|_{L^\infty} \leq C_{L^\infty}.$$

This completes the proof of Proposition 3.2. ∎

REMARK 3.4. Without loss of generality, we can assume $C_\infty = C_{L^\infty}$ for the completeness of proof.

Let us prove the Theorem 1.1 briefly.

Proof. **(Proof of Theorem 1.1.)** According to Proposition 3.2, for the solution ρ of Equation (1.1), we have

$$\|\rho(t,\cdot)\|_{L^\infty} \leq C_{L^\infty} \quad 0 \leq t < \infty,$$

then $\|\rho\|_{H^3}$ is uniformly bounded by the L^∞-criterion. Namely, we deduce by $\|\rho\|_{L^2}$ estimate of the solution and solving different inequality (3.14) that

$$\|\rho\|_{H^3} \leq C_{H^3}.$$

By using standard continuation argument, we have

$$\rho(t,x) \in C(\mathbb{R}^+; H^3(\mathbb{T}^d)).$$

This completes the proof of Theorem 1.1. ∎

REMARK 3.5. In fact, for any $k \geq 2$, $\rho_0 \in H^k(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, we can get

$$\rho(t,x) \in C(\mathbb{R}^+; H^k(\mathbb{T}^d)).$$

4. Proof of Theorem 1.1 ($\beta \in [2,d), d > 2$)

In this section, we consider the generalized Keller-Segel system with fractional diffusion and weakly mixing in the case of $\beta \in [2,d), d > 2$. As the proof is similar to Theorem 1.1, so we only deal with the details that are different.

4.1. L^∞-criterion.

We get the global classical solution of Equation (1.1) if L^∞ estimate of the solution is a global bound.

PROPOSITION 4.1. *Suppose that $0 < \alpha < 2, \beta \in [2,d), d > 2$, for any initial data $\rho_0 \geq 0, \rho_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$. Then the following criterion holds: either the local solution to (1.1) extends to a global classical solution or there exists $T^* \in (0,\infty)$, such that*

$$\lim_{t \to T^*} \|\rho(t,\cdot)\|_{L^\infty} = \infty.$$

Proof. For the fourth term of (3.1), we deduce that

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla \cdot (\rho B(\rho))(-\Delta)^3 \rho dx &= \int_{\mathbb{T}^d} \nabla \cdot (\rho \nabla K * \rho)(-\Delta)^3 \rho dx \\ &= \int_{\mathbb{T}^d} \nabla \rho \cdot \nabla K * \rho (-\Delta)^3 \rho dx + \int_{\mathbb{T}^d} \rho \Delta K * \rho (-\Delta)^3 \rho dx. \qquad (4.1)\end{aligned}$$

Integrating by parts the first term of the right-hand side of (4.1), we obtain

$$\int_{\mathbb{T}^d} \nabla \rho \cdot \nabla K * \rho (-\Delta)^3 \rho dx \sim \sum_{l=0}^{3} \int_{\mathbb{T}^d} D^l(\nabla \rho) \cdot D^{3-l}(\nabla K * \rho) D^3 \rho dx. \qquad (4.2)$$

According to the definition of periodic convolution kernel K in Section 1, we know that $\nabla K\in L^1(\mathbb{T}^d),\Delta K\in L^1(\mathbb{T}^d)$, then there exists a constant $C_0>0$, one has

$$\|\nabla K\|_{L^1}\leq C_0,\qquad \|\Delta K\|_{L^1}\leq C_0.$$

When $l=0,1$, according to the definition of K in (1.6), we deduce by Young's and Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{T}^d}\nabla\rho\cdot D^3(\nabla K*\rho)D^3\rho dx&\leq\|D\rho\|_{L^{p_1}}\|D^3(\nabla K*\rho)\|_{L^q}\|\rho\|_{\dot{H}^3}\\&\leq\|D\rho\|_{L^{p_1}}\|\Delta K*D^2\rho\|_{L^q}\|\rho\|_{\dot{H}^3}\\&\leq C\|D\rho\|_{L^{p_1}}\|D^2\rho\|_{L^q}\|\rho\|_{\dot{H}^3},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{T}^d}D(\nabla\rho)\cdot D^2(\nabla K*\rho)D^3\rho dx&\leq\|D^2\rho\|_{L^q}\|D^2(\nabla K*\rho)\|_{L^{p_1}}\|\rho\|_{\dot{H}^3}\\&\leq\|D^2\rho\|_{L^q}\|\Delta K*D\rho\|_{L^{p_1}}\|\rho\|_{\dot{H}^3}\\&\leq C\|D\rho\|_{L^{p_1}}\|D^2\rho\|_{L^q}\|\rho\|_{\dot{H}^3},\end{aligned}$$

where

$$\frac{1}{p_1}+\frac{1}{q}=\frac{1}{2}.$$

By Gagliardo-Nirenberg inequality, for $1\leq q_1,q_2<\infty$, one has

$$\|D\rho\|_{L^{p_1}}\|D^2\rho\|_{L^q}\|\rho\|_{\dot{H}^3}\leq C\|\rho\|_{L^{q_1}}^{1-\theta_1}\|\rho\|_{L^{q_2}}^{1-\theta_2}\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}\leq C\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2},$$

where

$$\theta_1=\frac{\frac{1}{p_1}-\frac{1}{d}-\frac{1}{q_1}}{\frac{1}{2}-\frac{3}{d}-\frac{1}{q_1}},\quad \theta_2=\frac{\frac{1}{q}-\frac{2}{d}-\frac{1}{q_2}}{\frac{1}{2}-\frac{3}{d}-\frac{1}{q_2}}.$$

Then for $l=2$, we get

$$\int_{\mathbb{T}^d}D^2(\nabla\rho)\cdot D(\nabla K*\rho)D^3\rho dx\leq\|\Delta K*\rho\|_{L^\infty}\|\rho\|_{\dot{H}^3}^2\leq C_0\|\rho\|_{L^\infty}\|\rho\|_{\dot{H}^3}^2,$$

and when $l=3$, we obtain

$$\int_{\mathbb{T}^d}D^3(\nabla\rho)\cdot(\nabla K*\rho)D^3\rho dx=-\frac{1}{2}\int_{\mathbb{T}^d}(D^3\rho)^2\Delta K*\rho dx.$$

Therefore, we have

$$\int_{\mathbb{T}^d}\nabla\rho\cdot\nabla K*\rho(-\Delta)^3\rho dx\leq C(\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}+\|\rho\|_{L^\infty}\|\rho\|_{\dot{H}^3}^2).\tag{4.3}$$

For the second term of the right-hand side of (4.1), we get

$$\int_{\mathbb{T}^d}\rho\Delta K*\rho(-\Delta)^3\rho dx\sim\sum_{l=0}^{3}\int_{\mathbb{T}^d}D^l\rho D^{3-l}(\Delta K*\rho)D^3\rho dx.\tag{4.4}$$

when $l=1,2$, similar to above, we obtain

$$\sum_{l=1}^{2}\int_{\mathbb{T}^d} D^l\rho D^{3-l}(\Delta K*\rho)D^3\rho dx \le C\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2},$$

if $l=0$, for $2<p_2<\infty$, we deduce that

$$\begin{aligned}\int_{\mathbb{T}^d}\rho D^3(\Delta K*\rho)D^3\rho dx &\le \|\rho\|_{L^\infty}\|\Delta K*D^3\rho\|_{L^2}\|\rho\|_{\dot{H}^3}\\ &\le C_0\|\rho\|_{L^\infty}\|\rho\|_{\dot{H}^3}^2\\ &\le C\|\rho\|_{\dot{H}^3}^2,\end{aligned}$$

and when $l=3$, we get

$$\int_{\mathbb{T}^d} D^3\rho(\Delta K*\rho)D^3\rho dx \le \|\Delta K*\rho\|_{L^\infty}\|\rho\|_{\dot{H}^3}^2 \le C\|\rho\|_{\dot{H}^3}^2,$$

so we have

$$\int_{\mathbb{T}^d}\rho\Delta K*\rho(-\Delta)^3\rho dx \le C(\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}+\|\rho\|_{\dot{H}^3}^2). \tag{4.5}$$

Thus, we deduce by (4.3) and (4.5) that

$$\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(-\Delta)^3\rho dx \le C(\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}+\|\rho\|_{\dot{H}^3}^2). \tag{4.6}$$

Combining (3.1), (3.2), (3.3) and (4.6), we have

$$\frac{d}{dt}\|\rho\|_{\dot{H}^3}^2 \le -2\|\rho\|_{\dot{H}^{3+\frac{\alpha}{2}}}^2 + C(\|u\|_{C^3}+1)\|\rho\|_{\dot{H}^3}^2 + C\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}. \tag{4.7}$$

According to (3.13) and for $1\le p_0<\infty$, one has

$$-\|\rho\|_{\dot{H}^{3+\frac{\alpha}{2}}}^2 \le -C_4^{-1}\|\rho\|_{\dot{H}^3}^{\gamma} \le -C\|\rho\|_{\dot{H}^3}^{\gamma}. \tag{4.8}$$

where

$$\gamma=\frac{\frac{4d}{p_0}+12-2d+2\alpha}{\frac{2d}{p_0}+6-d}, \quad 1\le p_0<\infty.$$

By (4.7) and (4.8), we have

$$\frac{d}{dt}\|\rho\|_{\dot{H}^3}^2 \le -C\|\rho\|_{\dot{H}^3}^{\gamma} + C(\|u\|_{C^3}+1)\|\rho\|_{\dot{H}^3}^2 + C\|\rho\|_{\dot{H}^3}^{1+\theta_1+\theta_2}, \tag{4.9}$$

we can choose p_0, such that

$$\gamma>\max\{2,1+\theta_1+\theta_2\}.$$

By the differential inequality (4.9), the conclusion can easily be deduced. This completes the proof of Proposition 4.1. ∎

4.2. L^∞ estimate of ρ. We obtain the L^∞ estimate of the solution by weakly mixing. The same idea from Section 3 is employed.

LEMMA 4.1. *Let $0<\alpha<2, \beta\in[2,d), d>2$, $\rho(t,x)$ is the local solution of Equation (1.1) with initial data $\rho_0(x)\geq 0$. If $\rho\in L^p(\mathbb{T}^d)$, $1\leq p<\infty$, and we denote*

$$\widetilde{\rho}(t)=\rho(t,\overline{x}_t)=\max_{x\in\mathbb{T}^d}\rho(t,x).$$

Then we have

$$\widetilde{\rho}(t)\leq C(d,p)\|\rho\|_{L^p}, \tag{4.10}$$

or

$$\frac{d}{dt}\widetilde{\rho}\leq C_0\widetilde{\rho}^2-C(\alpha,d,p)\frac{\widetilde{\rho}^{1+\frac{p\alpha}{d}}}{\|\rho\|_{L^p}^{\frac{p\alpha}{d}}}. \tag{4.11}$$

Proof. Let us denote by $\overline{x}_t$ the point such that

$$\widetilde{\rho}(t)=\rho(t,\overline{x}_t)=\max_{x\in\mathbb{T}^d}\rho(t,x),$$

then for a fixed $t\geq 0$, for a derivative at the point of maximum, we see that

$$(\nabla\cdot(\rho B(\rho)))(t,\overline{x}_t)=\nabla\rho\cdot\nabla K*\rho(t,\overline{x}_t)+\rho\Delta K*\rho(t,\overline{x}_t). \tag{4.12}$$

For the first term of the right-hand side in (4.12), one has

$$\nabla\rho\cdot\nabla K*\rho(t,\overline{x}_t)=0, \tag{4.13}$$

and for the second term of the right-hand side of (4.12), we deduce by Young's inequality that

$$\begin{aligned}\rho\Delta K*\rho(t,\overline{x}_t)&\leq\|\rho\Delta K*\rho\|_{L^\infty}\\&\leq\|\rho\|_{L^\infty}\|\Delta K*\rho\|_{L^\infty}\\&\leq\|\rho\|_{L^\infty}^2\|\Delta K\|_{L^1}=\widetilde{\rho}^2\|\Delta K\|_{L^1}.\end{aligned} \tag{4.14}$$

Thus, combining (1.1), (4.12), (4.13) and (4.14), we imply that the evolution of $\widetilde{\rho}$ follows

$$\frac{d}{dt}\widetilde{\rho}+(-\Delta)^{\frac{\alpha}{2}}\widetilde{\rho}-C_0\widetilde{\rho}^2\leq 0. \tag{4.15}$$

According to the nonlinear maximum principle, one has

$$\widetilde{\rho}(t)\leq C(d,p)\|\rho\|_{L^p},$$

if not, we have

$$\frac{d}{dt}\widetilde{\rho}\leq C_0\widetilde{\rho}^2-C(\alpha,d,p)\frac{\widetilde{\rho}^{1+\frac{p\alpha}{d}}}{\|\rho\|_{L^p}^{\frac{p\alpha}{d}}}. \tag{4.16}$$

This completes the proof of Lemma 4.1. ∎

We give the local L^2, $L^p(p>\frac{d}{\alpha})$ and L^∞ estimates of the solution.

LEMMA 4.2. *Let $0<\alpha<2, \beta\in[2,d), d>2$, $\rho(t,x)$ is the local solution of Equation (1.1) with initial data $\rho_0(x)\geq 0$. Suppose that $\|\rho_0\|_{L^2}=B_0$, $\|\rho_0\|_{L^p}\leq C_p$, $\|\rho_0\|_{L^\infty}\leq C_\infty$. Then there exists a time $\tau_1>0$, for any $0\leq t\leq\tau_1$, such that*

$$\|\rho(t,\cdot)-\overline{\rho}\|_{L^2}\leq 2(B_0^2-\overline{\rho}^2)^{\frac{1}{2}}, \quad \|\rho(t,\cdot)\|_{L^p}\leq 2C_p, \qquad \|\rho(t,\cdot)\|_{L^\infty}\leq 2C_\infty,$$

where $\overline{\rho}$ is the mean-value of ρ_0 and $p>\frac{d}{\alpha}$.

Proof. As $\|\rho_0\|_{L^\infty}\leq C_\infty$, we define

$$\tau_0=\min\left\{\frac{1}{2C_0C_\infty},T\right\},$$

then for any $0\leq t\leq\tau_0$, we get

$$\|\rho(t,\cdot)\|_{L^\infty}\leq 2C_\infty, \quad 0\leq t\leq\tau_0. \tag{4.17}$$

When $0\leq t\leq\tau_0$, the fourth term of the left-hand side of (3.21) can be estimated as

$$\begin{aligned}\int_{\mathbb{T}^d}\nabla(\rho B(\rho))|\rho|^{p-2}\rho dx&=\int_{\mathbb{T}^d}\nabla(\rho\nabla K*\rho)|\rho|^{p-2}\rho dx\\&=\int_{\mathbb{T}^d}\nabla\rho\cdot\nabla K*\rho|\rho|^{p-2}\rho dx+\int_{\mathbb{T}^d}|\rho|^p\Delta K*\rho dx\\&=-\frac{1}{p}\int_{\mathbb{T}^d}|\rho|^p\Delta K*\rho dx+\int_{\mathbb{T}^d}|\rho|^p\Delta K*\rho dx\\&\leq\frac{p-1}{p}\|\Delta K*\rho\|_{L^\infty}\|\rho\|_{L^p}^p\\&\leq\frac{2C_0C_\infty(p-1)}{p}\|\rho\|_{L^p}^p.\end{aligned}\tag{4.18}$$

So we have

$$\frac{d}{dt}\|\rho\|_{L^p}^p\leq -2\||\rho|^{\frac{p}{2}}\|_{\dot{H}^{\frac{\alpha}{2}}}^2+2C_0C_\infty(p-1)\|\rho\|_{L^p}^p. \tag{4.19}$$

For any $0\leq t\leq\tau_0$, the fourth term of the left-hand side of (3.23) can be estimated as

$$\begin{aligned}\int_{\mathbb{T}^d}\nabla(\rho B(\rho))(\rho-\overline{\rho})dx&=\int_{\mathbb{T}^d}\nabla(\rho\nabla K*\rho)(\rho-\overline{\rho})dx\\&=-\frac{1}{2}\int_{\mathbb{T}^d}(\rho-\overline{\rho})^2\Delta K*\rho dx+\int_{\mathbb{T}^d}\rho\Delta K*(\rho-\overline{\rho})(\rho-\overline{\rho})dx\\&\leq\frac{1}{2}\|\Delta K\|_{L^1}\|\rho\|_{L^\infty}\|\rho-\overline{\rho}\|_{L^2}^2+\|\Delta K\|_{L^1}\|\rho\|_{L^\infty}\|\rho-\overline{\rho}\|_{L^2}^2\\&\leq 3C_0C_\infty\|\rho-\overline{\rho}\|_{L^2}^2.\end{aligned}\tag{4.20}$$

We deduce by (3.23) and (4.20) that

$$\frac{d}{dt}\|\rho-\overline{\rho}\|_{L^2}^2\leq -2\|\rho\|_{\dot{H}^{\frac{\alpha}{2}}}^2+6C_0C_\infty\|\rho-\overline{\rho}\|_{L^2}^2. \tag{4.21}$$

We denote

$$\tau_1=\min\left\{\tau_0,\frac{p\ln 2}{2C_0C_\infty(p-1)},\frac{\ln 2}{3C_0C_\infty}\right\}, \tag{4.22}$$

for any $0<t\le\tau_1$, we deduce by (4.17) and (4.21) that

$$\|\rho(t,\cdot)-\overline{\rho}\|_{L^2}\le 2(B_0^2-\overline{\rho}^2)^{\frac{1}{2}},\quad \|\rho(t,\cdot)\|_{L^p}\le 2C_p.$$

According to (4.17) and the definition of τ_1, for any $0\le t\le\tau_1$, we have

$$\|\rho(t,\cdot)\|_{L^\infty}\le 2C_\infty,\quad 0\le t\le\tau_1.$$

This completes the proof of Lemma 4.2. □

Next, we give an approximation lemma.

LEMMA 4.3. *Let $0<\alpha<2,\beta\in[2,d),d>2$, suppose that the vector field $u(t,x)$ is smooth incompressible flow. Let $\rho(t,x),\omega(t,x)$ be the local solution of Equations (1.1) and (2.4) respectively with $\rho_0\in H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d),\rho_0\ge 0$. Then for every $t\in[0,T]$, we have*

$$\frac{d}{dt}\|\rho-\omega\|_{L^2}^2\le -\|\rho\|_{\dot H^{\frac{\alpha}{2}}}^2+F(t)^2\|\rho_0\|_{\dot H^{\frac{\alpha}{2}}}^2+C_0\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2 +2C_0(\|\nabla\rho\|_{L^2}\|\rho\|_{L^2}+\|\rho\|_{L^2}^2)\|\rho_0\|_{L^\infty},$$

where $F(t)$ be as defined in Lemma 2.4, and $F(t)\in L^\infty_{loc}[0,\infty)$.

Proof. The fourth term of the left-hand side of (3.27) can be estimated as

$$\begin{aligned}
&\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(\rho-\omega)dx=\int_{\mathbb{T}^d}\nabla\cdot(\rho\nabla K*\rho)(\rho-\omega)dx\\
&=\int_{\mathbb{T}^d}\nabla\cdot(\rho\nabla K*\rho)\rho dx-\int_{\mathbb{T}^d}\nabla\cdot(\rho\nabla K*\rho)\omega dx\\
&=\frac{1}{2}\int_{\mathbb{T}^d}\rho^2\Delta K*\rho dx-\int_{\mathbb{T}^d}\nabla\cdot(\rho\nabla K*\rho)\omega dx\\
&\le\frac{1}{2}\|\Delta K*\rho\|_{L^\infty}\|\rho\|_{L^2}^2+\|\nabla\cdot(\rho\nabla K*\rho)\|_{L^1}\|\omega\|_{L^\infty}\\
&\le\frac{1}{2}\|\Delta K\|_{L^1}\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2+(\|\nabla\rho\|_{L^2}\|\nabla K*\rho\|_{L^2}+\|\rho\|_{L^2}\|\Delta K*\rho\|_{L^2})\|\omega\|_{L^\infty}\\
&\le\frac{C_0}{2}\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2+C_0(\|\nabla\rho\|_{L^2}\|\rho\|_{L^2}+\|\rho\|_{L^2}^2)\|\omega\|_{L^\infty},
\end{aligned}\tag{4.23}$$

where $\|\nabla K\|_{L^1},\|\Delta K\|_{L^1}$ is bounded since $\beta\in[2,d)$. By Young's inequality and (4.23), one has

$$\int_{\mathbb{T}^d}\nabla\cdot(\rho B(\rho))(\rho-\omega)dx\le\frac{C_0}{2}\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2+C_0(\|\nabla\rho\|_{L^2}\|\rho\|_{L^2}+\|\rho\|_{L^2}^2)\|\omega\|_{L^\infty}.\tag{4.24}$$

Combining (3.27), (3.28), (3.29), (3.35), (4.24) and Lemma 2.4, we have

$$\frac{d}{dt}\|\rho-\omega\|_{L^2}^2\le -\|\rho\|_{\dot H^{\frac{\alpha}{2}}}^2+F(t)^2\|\rho_0\|_{\dot H^{\frac{\alpha}{2}}}^2+C_0\|\rho\|_{L^\infty}\|\rho\|_{L^2}^2 +2C_0(\|\nabla\rho\|_{L^2}\|\rho\|_{L^2}+\|\rho\|_{L^2}^2)\|\rho_0\|_{L^\infty}.$$

This completes the proof of Lemma 4.3. □

Next, we establish the global L^∞ estimate of the solution.

PROPOSITION 4.2 (Global L^∞ estimate). *Let $0<\alpha<2,\beta\in[2,d),d>2$, suppose $\rho(t,x)$ is the solution of Equation (1.1) with initial data $\rho_0\ge 0,\rho_0\in H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$. Then there exist weakly mixing u and a positive constant C_{L^∞}, such that*

$$\|\rho(t,\cdot)\|_{L^\infty}\le C_{L^\infty},\quad t\in[0,+\infty].$$

Proof. According to the proof of Proposition 3.2. □

REMARK 4.1. According to Proposition 4.1 and 4.2, we can finish the proof of Theorem 1.1 when $\beta \in [2,d), d>2$.

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Appendix. Nonlinear maximum principle. In this section, we recall nonlinear maximum principle on $\mathbb{T}^d$, the main idea of its proof comes from [9, 12, 23].

LEMMA 5.1. *Let $f \in \mathcal{S}(\mathbb{T}^d)$ and denote by $\overline{x}$ the point such that*

$$f(\overline{x}) = \max_{x\in\mathbb{T}^d} f(x),$$

and $f(\overline{x}) > 0$, then we have the following

$$(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) \geq C(\alpha,d,p) \frac{f(\overline{x})^{1+\frac{p\alpha}{d}}}{\|f\|_{L^p}^{\frac{p\alpha}{d}}},$$

or

$$f(\overline{x}) \leq C(d,p)\|f\|_{L^p}.$$

Proof. We take $R>0$ a positive number and define

$$N_1(R) = \{\lambda \in B(0,R) \big| f(\overline{x}) - f(\overline{x}-\lambda) > \frac{f(\overline{x})}{2}\}.$$

and

$$M = \min_{y\in\partial\mathbb{T}^d} |\overline{x}-y|,$$

where $\partial\mathbb{T}^d$ represents the boundary of the periodic box $\mathbb{T}^d$. Without loss of generality, we assume that $M \geq \frac{1}{4}$. If

$$R \leq M, \tag{5.1}$$

then, we have

$$B(0,R) \subset \mathbb{T}^d.$$

If we denote

$$N_2(R) = B(0,R) - N_1(R),$$

then

$$N_2(R) = \{\lambda \in B(0,R) \big| f(\overline{x}) - f(\overline{x}-\lambda) \leq \frac{f(\overline{x})}{2}\},$$

and

$$\|f\|_{L^p}^p \geq \int_{\mathbb{T}^d} |f(\overline{x}-\lambda)|^p d\lambda \geq \int_{N_2(R)} |f(\overline{x}-\lambda)|^p d\lambda \geq \left(\frac{|f(\overline{x})|}{2}\right)^p |N_2(R)|,$$

thus, we obtain

$$|N_2(R)| \leq \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p. \tag{5.2}$$

According to the definition of (2.1), we have

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) &\geq C_{\alpha,d} P.V. \int_{\mathbb{T}^d} \frac{f(\overline{x}) - f(\overline{x}-\lambda)}{|\lambda|^{d+\alpha}} d\lambda \\ &\geq C_{\alpha,d} P.V. \int_{N_1(R)} \frac{f(\overline{x}) - f(\overline{x}-\lambda)}{|\lambda|^{d+\alpha}} d\lambda \\ &\geq C_{\alpha,d} \frac{f(\overline{x})}{2} \frac{1}{R^{d+\alpha}} |N_1(R)|.\end{aligned}$$

We deduce by (5.2), the definition of $N_1(R)$ and $N_2(R)$ that

$$|N_1(R)| = |B(0,R)| - |N_2(R)| \geq \omega_d R^d - \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p,$$

where ω_d is the volume per sphere, then we get

$$(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) \geq C_{\alpha,d} \frac{f(\overline{x})}{2R^{d+\alpha}} \left(\omega_d R^d - \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p\right). \tag{5.3}$$

We take R such that

$$\omega_d R^d = 2\left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p,$$

thus

$$R = \left(\frac{2}{\omega_d}\left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p\right)^{\frac{1}{d}} = \left(\frac{2}{\omega_d}\right)^{\frac{1}{d}} \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^{\frac{p}{d}}. \tag{5.4}$$

By (5.3) and (5.4), we have

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) &\geq C_{\alpha,d} \frac{f(\overline{x})}{2R^{d+\alpha}} \left(\omega_d R^d - \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p\right) = C_{\alpha,d} \frac{f(\overline{x})}{2R^{d+\alpha}} \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p \\ &= \frac{C_{\alpha,d} 2^p}{2\left(\frac{2}{\omega_d}\right)^{\frac{d+\alpha}{d}} 2^{\frac{p(d+\alpha)}{d}}} \frac{\|f\|_{L^p}^p f(\overline{x})^{\frac{p(d+\alpha)}{d}} f(\overline{x})}{(\|f\|_{L^p})^{\frac{p(d+\alpha)}{d}} f(\overline{x})^p} \\ &= C(\alpha,d,p) \frac{f(\overline{x})^{1+\frac{p\alpha}{d}}}{\|f\|_{L^p}^{\frac{p\alpha}{d}}}.\end{aligned}$$

If R does not fulfill (5.1), then

$$\left(\frac{2}{\omega_d}\right)^{\frac{1}{d}} \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^{\frac{p}{d}} > M,$$

so we conclude that

$$f(\overline{x}) \leq \frac{2}{M^{\frac{d}{p}}\left(\frac{\omega_d}{2}\right)^{\frac{1}{p}}}\|f\|_{L^p} \leq C(d,p)\|f\|_{L^p}.$$

This completes the proof of Lemma 5.1. □

REMARK 5.1. For the case of $\mathbb{R}^d$, we can refer to [23].

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