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LOCAL WELL-POSEDNESS FOR THE QUANTUM ZAKHAROV SYSTEM*

YUNG-FU FANG† AND KUAN-HSIANG WANG‡

Abstract. We consider the quantum Zakharov system in spatial dimensions greater than 1. The local well-posedness is obtained for initial data of the electric field and of the ion density lying in some Sobolev spaces with certain regularities. For higher dimensions, the results cover the subcritical region. We get major part of the subcritical region for lower dimensions. For the quantum Zakharov system with initial data possessing the critical regularities, the local well-posedness is also proved for spatial dimensions greater than 7. As the quantum parameter approaches zero, we prove the local well-posedness for Zakharov system which improves the known result.

Keywords. quantum Zakharov system; Zakharov system; local well-posedness; quantum parameter; Strichartz estimates; Fourier restriction norm method.

AMS subject classifications. Primary 35L30; Secondary 35L05; 35Q55.

1. Introduction

In this paper, we consider the quantum Zakharov system which reads as

$$\begin{cases} iE_t + \Delta E - \varepsilon^2 \Delta^2 E = nE, \quad x \in \mathbb{R}^d; \\ n_{tt} - \Delta n + \varepsilon^2 \Delta^2 n = \Delta |E|^2; \\ E(0) = E_0,\, n(0) = n_0,\, n_t(0) = n_1, \end{cases} \tag{1.1}$$

where E is the slowly varying envelope of the rapidly oscillating electric field and n is the deviation of the ion density from its mean value. E is complex valued and n is real valued, see [20]. Equation (1.1) describes the propagation of Langmuir waves in an ionized plasma with quantum effect. The readers are referred to [20] for a more physical background. The system (1.1) has the conservation of mass

$$\int |E(t)|^2 dx = \int |E(0)|^2 dx \tag{1.2}$$

and the conservation of the Hamiltonian

$$\frac{1}{2}\|\nabla E\|_{L^2}^2 + \frac{1}{2}\varepsilon^2 \|\Delta E\|_{L^2}^2 + \frac{1}{4}\Big(\|n_t\|_{\dot{H}^{-1}}^2 + \|n\|_{L^2}^2 + \varepsilon^2 \|\nabla n\|_{L^2}^2\Big) + \frac{1}{2}\int_{\mathbb{R}^d} n|E|^2 dx. \tag{1.3}$$

In the absence of quantum effect, *i.e.* $\varepsilon = 0$, we have the classical Zakharov system,

$$\begin{cases} iE_t + \Delta E = nE, \quad x \in \mathbb{R}^d; \\ n_{tt} - \Delta n = \Delta |E|^2; \\ E(0) = E_0,\, n(0) = n_0,\, n_t(0) = n_1, \end{cases} \tag{1.4}$$

which also possesses the conservation of mass and the conservation of the Hamiltonian, see [31]. The Zakharov system (1.4) has been extensively studied for the local well-posedness (LWP) and global well-posedness (GWP) [1–4, 8, 11, 25, 27], for ill-posedness

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†Department of Mathematics, National Cheng Kung University, No. 1, Dasyue Rd., Tainan City 70101, Taiwan (yffang@mail.ncku.edu.tw). http://www.math.ncku.edu.tw/∼fang

‡Department of Applied Mathematics, National University of Kaohsiung, 700 Kaohsiung University Rd., Nanzih District, Kaohsiung 811, Taiwan (khwang0511@gmail.com).

[21], for blow-up [12,13], for scattering results [14,16,29], and for adiabatic limit of the solution [26,28].

The works on (1.1) are less than those on (1.4). We list some known results for (1.1). For dimension one, the system is studied for the LWP and GWP, see [7,10], and [22]. For dimensions $d=1,2,3$, the GWP, the stability of solution, and the classical limit is proved for (1.1), see [19]. In 2019, Fang-Nakanishi showed LWP, GWP, and scattering for (1.1) with L^2 data for $1\leq d\leq 8$, see [9].

For the sake of simplicity, we transform (1.1) into the first-order equations in time t and denote

$$\langle\varepsilon\mathcal{D}\rangle:=\sqrt{1-\varepsilon^2\Delta},\quad \mathcal{D}:=\sqrt{-\Delta},\quad \mathcal{N}:=n+i(\mathcal{D}\langle\varepsilon\mathcal{D}\rangle)^{-1}\partial_t n,\quad n=\mathcal{R}e(\mathcal{N}), \tag{1.5}$$

where $\mathcal{R}e(\mathcal{N})$ is the real part of $\mathcal{N}$. Thus the quantum Zakharov system (1.1) becomes

$$\begin{cases} iE_t-(\mathcal{D}\langle\varepsilon\mathcal{D}\rangle)^2E=\mathcal{R}e(\mathcal{N})E, & (t,x)\in\mathbb{R}\times\mathbb{R}^d;\\ i\mathcal{N}_t-\mathcal{D}\langle\varepsilon\mathcal{D}\rangle\mathcal{N}=\mathcal{D}\langle\varepsilon\mathcal{D}\rangle^{-1}|E|^2; & \\ E(0)=E_0,\quad \mathcal{N}(0)=\mathcal{N}_0, & x\in\mathbb{R}^d; \end{cases} \tag{1.6}$$

where $\mathcal{N}_0:=n_0+i(\mathcal{D}\langle\varepsilon\mathcal{D}\rangle)^{-1}n_1$. The corresponding Hamiltonian becomes

$$\frac{1}{2}\|\mathcal{D}\langle\varepsilon\mathcal{D}\rangle E\|_{L^2}^2+\frac{1}{4}\|\langle\varepsilon\mathcal{D}\rangle\mathcal{N}\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}^d}\mathcal{R}e(\mathcal{N})|E|^2\,dx=\text{constant}. \tag{1.7}$$

Ignoring the terms ΔE and $\mathcal{D}\langle\varepsilon\mathcal{D}\rangle\mathcal{N}$, and replacing $\mathcal{D}\langle\varepsilon\mathcal{D}\rangle^{-1}$ by I in system (1.6) gives the corresponding system which is invariant under the dilation

$$E(t,x)\to E_\lambda(t,x)=\lambda^4E(\lambda^4t,\lambda x),\quad \mathcal{N}(t,x)\to\mathcal{N}_\lambda(t,x)=\lambda^4\mathcal{N}(\lambda^4t,\lambda x), \tag{1.8}$$

see [11]. Since

$$\|E_\lambda(0,\cdot)\|_{\dot H^k}=\lambda^{4-\frac{d}{2}+k}\|E_0\|_{\dot H^k}\quad\text{and}\quad \|\mathcal{N}_\lambda(0,\cdot)\|_{\dot H^\ell}=\lambda^{4-\frac{d}{2}+\ell}\|\mathcal{N}_0\|_{\dot H^\ell},$$

the system (1.6) is critical for

$$(k,\ell)=\big((d-8)/2,(d-8)/2\big). \tag{1.9}$$

Throughout the paper, we set $\varepsilon=1$, unless it is specified. We denote the region $R_{QZ,d}$ by

$$\{(k,\ell):\max\{-\ell,\ell-2k,-2k\}\leq d/2,\text{and}-5/2<\ell-k<4\}\text{ for }d=2,3; \tag{1.10}$$
$$\{(k,\ell):\max\{-\ell,\ell-2k,|k|-2\ell-2\}<4-d/2,k>-1,\text{and}-5/2<\ell-k<4\}\text{ for }d\geq 4. \tag{1.11}$$

Our main results are as follows.

THEOREM 1.1 (LWP for QZ). *Let $d\geq 2$. If $(k,\ell)\in R_{QZ,d}$, then the quantum Zakharov system (1.6) is locally well-posed for initial data $(E_0,\mathcal{N}_0)\in H^k(\mathbb{R}^d)\times H^\ell(\mathbb{R}^d)$. The solution $(E,\mathcal{N})\in C([0,T];H^k(\mathbb{R}^d))\times C([0,T];H^\ell(\mathbb{R}^d))$.*

Notice that $(k,\ell)=(\frac{d-8}{2}+\sigma,\frac{d-8}{2})$ lies on the boundary of the region $R_{QZ,d}$ for the LWP stated in Theorem 1.1, but not included.

THEOREM 1.2 (LWP for the critical case). *Let $d\geq 8$. If $(k,\ell)=(\frac{d-8}{2}+\sigma,\frac{d-8}{2})$ with $0\leq\sigma\leq 1$, then the quantum Zakharov system (1.6) is locally well-posed for initial data*

$(E_0,\mathcal{N}_0)\in H^{\frac{d-8}{2}+\sigma}(\mathbb{R}^d)\times H^{\frac{d-8}{2}}(\mathbb{R}^d)$. *The solution* $(E,\mathcal{N})\in C([0,T];H^{\frac{d-8}{2}+\sigma}(\mathbb{R}^d))\times C([0,T];H^{\frac{d-8}{2}}(\mathbb{R}^d))$.

We are also interested in the LWP of (1.4) and we improve the result in [11] for $d\geq 2$.

THEOREM 1.3 (LWP for Z). *Let* $d\geq 2$. *If* (k,ℓ) *satisfies*

$$\begin{array}{lll} \ell-1<k\leq \ell+1 & & \text{for all } d, \\ \ell>d/2-2, & 2k-(\ell+1)>d/2-2 & \text{for } d\geq 4, \\ \ell\geq 0, & 2k-(\ell+1)\geq 0 & \text{for } d=2,3, \end{array} \tag{1.12}$$

then the system (1.6) *with* $\varepsilon=0$ *is locally well-posed for initial data* $(E_0,\mathcal{N}_0)\in H^k(\mathbb{R}^d)\times H^\ell(\mathbb{R}^d)$. *The solution* $(E,\mathcal{N})\in C([0,T];H^k(\mathbb{R}^d))\times C([0,T];H^\ell(\mathbb{R}^d))$.

The outline of the paper is as follows. In Section 1, we introduced the QZ system and the main results. In Section 2, we define some notations and discuss some basic estimates. In Sections 3–5, we discuss the multilinear estimates which is the key ingredient. In Section 6, we prove the Theorem 1.1. In Section 7, we prove Theorem 1.2. Finally, in Section 8, we show Theorem 1.3.

2. Notations and basic estimates

We denote the Fourier transform and its inverse transform of $u(t,x)$ over the space variable by $\mathcal{F}u(t,\xi)$ and $\mathcal{F}^{-1}u(t,\xi)$, the Fourier transform and its inverse transform of $u(t,x)$ over time and space variables by $\widehat{u}(\tau,\xi)$ and $u^\vee(\tau,\xi)$. Consider the $4th$ order Schrödinger equation,

$$iE_t-(\mathcal{D}\langle\mathcal{D}\rangle)^2E=F. \tag{2.1}$$

We can obtain the solution formula

$$E(t,x)=U(t)E_0(x)+U*_RF(t,x), \tag{2.2}$$

where $U(t):=e^{it(\mathcal{D}\langle\mathcal{D}\rangle)^2}$ is the $4th$ order Schrödinger propagator and the Duhamel operator is

$$U*_RF(t,x):=-i\int_0^t U(t-s)F(s,x)ds. \tag{2.3}$$

Analogously for the $4th$ order wave equation,

$$i\mathcal{N}_t-\mathcal{D}\langle\mathcal{D}\rangle\mathcal{N}=G, \tag{2.4}$$

we can obtain the solution formula

$$\mathcal{N}(t,x)=W(t)\mathcal{N}_0(x)+W*_RG(t,x), \tag{2.5}$$

where $W(t):=e^{it\mathcal{D}\langle\mathcal{D}\rangle}$ is the $4th$ order wave propagator and the Duhamel operator is

$$W*_RG(t,x):=-i\int_0^t W(t-s)G(s,x)ds. \tag{2.6}$$

For $s\in\mathbb{R}$, we denote by $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$ the usual inhomogeneous and homogeneous Sobolev spaces equipped with the norms, respectively,

$$\|u\|_{H^s}:=\|\langle\mathcal{D}\rangle^s u\|_{L^2}=\|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u\|_{L^2} \quad \text{and} \quad \|u\|_{\dot{H}^s}:=\|\mathcal{D}^s u\|_{L^2}=\||\xi|^s\mathcal{F}u\|_{L^2}. \tag{2.7}$$

Let us define some Bourgain spaces $X^{\phi}_{s,\alpha}$ and Y^{ϕ}_{s} , $s,\alpha\in\mathbb{R}$, with the norms

$$\|v\|_{X^{\phi}_{s,\alpha}}:=\Big(\iint\langle\xi\rangle^{2s}\langle\tau+\phi(\xi)\rangle^{2\alpha}|\widehat{v}(\tau,\xi)|^2d\tau d\xi\Big)^{\frac{1}{2}} \tag{2.8}$$

and

$$\|v\|_{Y^{\phi}_{s}}:=\Big(\int\Big(\int\langle\xi\rangle^{s}\langle\tau+\phi(\xi)\rangle^{-1}|\widehat{v}(\tau,\xi)|d\tau\Big)^2d\xi\Big)^{\frac{1}{2}}. \tag{2.9}$$

It is known that $X^{\phi}_{s,\alpha}\subset C([0,T];H^s)$ for $\alpha>\frac{1}{2}$, see [11]. $X^{\phi}_{s,\alpha}$ and Y^{ϕ}_{s} are called *4th order Schrödinger spaces* if $\phi(\xi):=|\xi|^2\langle\xi\rangle^2$, and denoted by $X^{S}_{s,\alpha}$ and Y^{S}_{s}, while $X^{\phi}_{s,\alpha}$ and Y^{ϕ}_{s} are called reduced wave spaces if $\phi(\xi):=|\xi|\langle\xi\rangle$, and denoted by $X^{W}_{s,\alpha}$ and Y^{W}_{s}. We denote by I the arbitary time interval and $L^q_IL^r_x=L^q_t(I,L^r_x(\mathbb{R}^d))$. Especially, we denote $L^q_tL^r_x=L^q_t(\mathbb{R},L^r_x(\mathbb{R}^d))$ and $L^q_{t,x}=L^q_tL^q_x$.

Let ψ be a cut-off function such that $\psi(t)$ is 1 for $|t|\le 1$, 0 for $|t|>2$, and $\psi_T(t)=\psi(\frac{t}{T})$. Also let $\chi_S(\tau)$ be the indicator function on the set S, that is 1 if $\tau\in S$, 0 if $\tau\notin S$.

Lemma 2.1 (Homogeneous estimates). *Suppose $T\le 1$. We have the following:*

(S1) $\|U(t)E_0\|_{C([0,T];H^k)}=\|E_0\|_{H^k}$.

(S2) If $0\le b_1$, *then* $\|\psi_TU(t)E_0\|_{X^S_{k,b_1}}\le T^{\frac{1}{2}-b_1}\|\psi\|_{H^{b_1}}\|E_0\|_{H^k}$.

(W1) $\|W(t)\mathcal{N}_0\|_{C([0,T];H^\ell)}=\|\mathcal{N}_0\|_{H^\ell}$.

(W2) If $0\le b$, *then* $\|\psi_TW(t)\mathcal{N}_0\|_{X^W_{\ell,b}}\le T^{\frac{1}{2}-b}\|\psi\|_{H^b}\|\mathcal{N}_0\|_{H^\ell}$.

Lemma 2.2 (Duhamel estimates). *Suppose $T\le 1$.*

(S1) If $0\le c_1,b_1$, *and* $b_1+c_1\le 1$, *then*

$$\|\psi_TU*_RF\|_{X^S_{k,b_1}}\lesssim T^{1-b_1-c_1}\|F\|_{X^S_{k,-c_1}}+T^{\frac{1}{2}-b_1}\|\big(\chi_{\{|\tau+|\xi|^2\langle\xi\rangle^2|\ge T^{-1}\}}\widehat{F}\big)^{\vee}\|_{Y^S_k}.$$

(S2) If $0\le c_1<\frac{1}{2}$, $0\le b_1,b_1+c_1\le 1$, *then* $\|\psi_TU*_RF\|_{X^S_{k,b_1}}\lesssim T^{1-b_1-c_1}\|F\|_{X^S_{k,-c_1}}$.

(S3) If $F\in Y^S_k$, *then* $\|U*_RF\|_{C([0,T];H^k)}\lesssim\|F\|_{Y^S_k}$.

(W1) If $0\le c,b$, *and* $b+c\le 1$ *then*

$$\|\psi_TW*_RG\|_{X^W_{\ell,b}}\lesssim T^{1-b-c}\|G\|_{X^W_{\ell,-c}}+T^{\frac{1}{2}-b}\|\big(\chi_{\{|\tau+|\xi|\langle\xi\rangle|\ge T^{-1}\}}\widehat{G}\big)^{\vee}\|_{Y^W_\ell}.$$

(W2) If $0\le c<\frac{1}{2}$, $0\le b$, *and* $b+c\le 1$, *then* $\|\psi_TW*_RG\|_{X^W_{\ell,b}}\lesssim T^{1-b-c}\|G\|_{X^W_{\ell,-c}}$.

(W3) If $G\in Y^W_\ell$, *then* $\|W*_RG\|_{C([0,T];H^\ell)}\lesssim\|G\|_{Y^W_\ell}$.

We skip the proofs of Lemmas 2.1-2.2 and the readers are referred to [8, 11, 17] and [18]. We recall the Strichartz estimates for the operators $\mathcal{D}\langle\mathcal{D}\rangle$ and $(\mathcal{D}\langle\mathcal{D}\rangle)^2$. A pair (q,r) is called *Schrödinger admissible*, for short *S-admissible*, if

$$2\le q,r\le\infty,\quad (q,r,d)\ne(2,\infty,2),\quad \frac{2}{q}+\frac{d}{r}=\frac{d}{2}. \tag{2.10}$$

A pair (q,r) is called *biharmonic admissible*, for short *B-admissible*, if

$$2\le q,r\le\infty,\quad (q,r,d)\ne(2,\infty,4),\quad \frac{4}{q}+\frac{d}{r}=\frac{d}{2}, \tag{2.11}$$

see [30]. We also denote the notation $\rho(r):=\frac{d}{2}-\frac{d}{r}$.

PROPOSITION 2.1 (Pausader [30]). *Let $E\in C([0,T],H^{-4}(\mathbb{R}^d))$ be a solution of (2.1). For any B-admissible pairs (q_1,r_1) and (q_2,r_2), it satisfies*

$$\|E\|_{L^{q_1}_{[0,T]}L^{r_1}_x}\lesssim\|E_0\|_{L^2_x}+\|F\|_{L^{q_2'}_{[0,T]}L^{r_2'}_x}, \tag{2.12}$$

where the constant depends only on q_2 and r_2. Besides, for any S-admissible pairs (q_1,r_1) and (q_2,r_2), we have

$$\|\langle\mathcal{D}\rangle^{\frac{2}{q_1}}E\|_{L^{q_1}_{[0,T]}L^{r_1}_x}\lesssim\|E_0\|_{L^2_x}+\|\langle\mathcal{D}\rangle^{-\frac{2}{q_2}}F\|_{L^{q_2'}_{[0,T]}L^{r_2'}_x}, \tag{2.13}$$

where the constant depends only on q_2 and r_2.

The proof is based on the work of Kenig-Ponce-Vega [24], or the works of Ben-Artzi-Koch-Saut [6], Pausader [30], and Keel-Tao [23], together with some modifications. The readers are also referred to [9] for the supplement. Now we can use the interpolation between (2.12) and (2.13) to get the following.

COROLLARY 2.1. *Assume that $0\le\theta\le1, 4\le\widetilde{q}\le\infty$ for $d=1$, $2<\widetilde{q}\le\infty$ for $d=2,4$, and $2\le\widetilde{q}\le\infty$ for $d=3,d\ge5$. Then the following inequalities hold. For $d=1,2,3$,*

$$\|\langle\mathcal{D}\rangle^{\frac{2}{\widetilde{q}}\theta}U(t)E_0\|_{L^q_IL^r_x}\lesssim\|E_0\|_{L^2_x}, \tag{2.14}$$

where $\frac{1}{q}=\frac{\theta}{\widetilde{q}}+\frac{d}{8}(1-\theta)$ and $\rho(r)=\frac{d}{2}(1-\theta)+\frac{2}{\widetilde{q}}\theta$. For $d\ge4$,

$$\|\langle\mathcal{D}\rangle^{\theta}U(t)E_0\|_{L^q_IL^r_x}\lesssim\|E_0\|_{L^2_x}, \tag{2.15}$$

where $\frac{1}{q}=\frac{1-\theta}{\widetilde{q}}+\frac{\theta}{2}$ and $\rho(r)=\frac{4}{\widetilde{q}}(1-\theta)+\theta$.

Proof. We first consider $1\le d\le3$. Let the pair $(\widetilde{q},\widetilde{r})$ be S-admissible. The interpolation between (2.13) with $(\widetilde{q},\widetilde{r})$ and (2.12) with $(\frac{8}{d},\infty)$ gives (2.14).

We next consider $d\ge4$. Let the pairs $(2,r_1)$ be S-admissible and (q_2,r_2) be B-admissible. The interpolation between (2.13) with $(2,r_1)$ and (2.12) with (q_2,r_2) gives (2.15). □

Invoking the above corollary, we can obtain a variant version of Strichartz estimate for (2.1).

LEMMA 2.3. *Under the assumptions of Corollary 2.1, assume that $\beta>\frac{1}{2}$ and $0\le\tilde{a}\le\beta$. Then the following inequalities hold. For $d=1,2,3$,*

$$\|(\langle\xi\rangle^{\frac{2}{\widetilde{q}}\theta\frac{\widetilde{a}}{\beta}}\langle\tau+|\xi|^2\langle\xi^2\rangle^{-\widetilde{a}}|\widehat{u}|)^\vee\|_{L^q_IL^r_x}\lesssim\|u\|_{L^2_{t,x}}, \tag{2.16}$$

where

$$\frac{1}{q}=\left(\frac{d}{8}(1-\theta)+\frac{\theta}{\widetilde{q}}\right)\frac{\widetilde{a}}{\beta}+\frac{1}{2}\left(1-\frac{\widetilde{a}}{\beta}\right)\quad \textit{and}\quad \rho(r)=\left(\frac{d}{2}(1-\theta)+\frac{2}{\widetilde{q}}\theta\right)\frac{\widetilde{a}}{\beta}. \tag{2.17}$$

For $d\ge4$,

$$\|(\langle\xi\rangle^{\theta\frac{\widetilde{a}}{\beta}}\langle\tau+|\xi|^2\langle\xi^2\rangle^{-\widetilde{a}}|\widehat{u}|)^\vee\|_{L^q_IL^r_x}\lesssim\|u\|_{L^2_{t,x}}, \tag{2.18}$$

where

$$\frac{1}{q}=\left(\frac{1-\theta}{\widetilde{q}}+\frac{\theta}{2}\right)\frac{\widetilde{a}}{\beta}+\frac{1}{2}\left(1-\frac{\widetilde{a}}{\beta}\right) \quad \text{and} \quad \rho(r)=\left(\frac{4}{\widetilde{q}}(1-\theta)+\theta\right)\frac{\widetilde{a}}{\beta}. \tag{2.19}$$

Proof. For $1\leq d\leq 3$, we first get the following estimate

$$\left\|\langle\mathcal{D}\rangle^{\frac{2}{q}\theta}f\right\|_{L_I^{q_1}L_x^{r_1}}\lesssim\|f\|_{X_{0,\beta}^S}, \tag{2.20}$$

where $\frac{1}{q_1}=\frac{d}{8}(1-\theta)+\frac{\theta}{\widetilde{q}}$ and $\rho(r_1)=\frac{d}{2}(1-\theta)+\frac{2}{\widetilde{q}}\theta$. Using Fourier transform and its inverse, we can rewrite $f=\mathcal{F}_\tau^{-1}U(t)\mathcal{F}_t(U(-t)f)$. Invoking (2.14), we first compute

$$\begin{aligned}\|\langle\mathcal{D}\rangle^{\frac{2}{q}\theta}f\|_{L_I^{q_1}L_x^{r_1}}&\lesssim\int\left\|\mathcal{F}_t(U(-t)f)(\tau)\right\|_{L_x^2}d\tau\\&\lesssim\left(\int\langle\tau+|\xi|^2\langle\xi\rangle^2\rangle^{2\beta}\left|\widehat{f}(\tau,\xi)\right|^2d\xi d\tau\right)^{\frac{1}{2}}=\|f\|_{X_{0,\beta}^S},\end{aligned} \tag{2.21}$$

where we use (2.14) and $\beta>\frac{1}{2}$. Invoking the interpolation between (2.20) and $\|f\|_{L_I^2L_x^2}\leq\|f\|_{X_{0,0}^S}$, we then obtain

$$\left\|\langle\mathcal{D}\rangle^{\frac{2}{q}\theta\frac{\widetilde{a}}{\beta}}f\right\|_{L_I^qL_x^r}\lesssim\|f\|_{X_{0,\widetilde{a}}^S},$$

where (2.17) holds.

For $d\geq 4$, we analogously get the following estimate

$$\left\|\langle\mathcal{D}\rangle^{\theta}f\right\|_{L_I^{q_1}L_x^{r_1}}\lesssim\|f\|_{X_{0,\beta}^S}, \tag{2.22}$$

where $\frac{1}{q_1}=\frac{1-\theta}{\widetilde{q}}+\frac{\theta}{2}$ and $\rho(r_1)=\frac{4}{\widetilde{q}}(1-\theta)+\theta$. Invoking the interpolation between (2.22) and $\|f\|_{L_I^2L_x^2}\leq\|f\|_{X_{0,0}^S}$, we then obtain

$$\left\|\langle\mathcal{D}\rangle^{\theta\frac{\widetilde{a}}{\beta}}f\right\|_{L_t^qL_x^r}\lesssim\|f\|_{X_{0,\widetilde{a}}^S},$$

where (2.19) holds. Finally, we let $\widehat{f}=\langle\tau+|\xi|^2\langle\xi\rangle^2\rangle^{-\widetilde{a}}|\widehat{u}|$ and complete the proof. ∎

PROPOSITION 2.2 (Gustafson-Nakanishi-Tsai [15], Theorem 2.1, and [9]). *Let $\mathcal{N}$ be a solution of (2.4). If (q_i,r_i) are S-admissible for $i=1,2$, we have, with $\gamma_i:=(1-2/d)/q_i$,*

$$\left\|\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}\right)^{-\gamma_1}\mathcal{N}\right\|_{L_{[0,T]}^{q_1}L_x^{r_1}}\lesssim\|\mathcal{N}_0\|_{L_x^2}+\left\|\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}\right)^{\gamma_2}G\right\|_{L_{[0,T]}^{q_2'}L_x^{r_2'}}, \tag{2.23}$$

where the implicit constant depends only on d, q_1, and q_2.

REMARK 2.1. For $d\geq 2$, invoking (2.23) and Plancherel Theorem, we have

$$\|W(t)\mathcal{N}_0\|_{L_I^{q_1}L_x^{r_1}}=\left\|\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}\right)^{-\gamma_1}W(t)\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}\right)^{\gamma_1}\mathcal{N}_0\right\|_{L_I^{q_1}L_x^{r_1}}\lesssim\left\|\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}\right)^{\gamma_1}\mathcal{N}_0\right\|_{L_x^2}\lesssim\|\mathcal{N}_0\|_{L_x^2}. \tag{2.24}$$

We can also obtain a variant version of Strichartz estimate for (2.4).

Lemma 2.4. *For $d\geq 2$, let $\beta_0>\frac{1}{2}$, $0\leq a\leq\beta_0$, $0<\eta\leq 1$ for $d=2$, and $0\leq\eta\leq 1$ for $d\geq 3$. Then*

$$\|(\langle\tau\pm|\xi|\langle\xi\rangle\rangle^{-a}|\widehat{u}|)^{\vee}\|_{L_I^qL_x^r}\lesssim\|u\|_{L^2_{t,x}}, \tag{2.25}$$

where

$$\frac{2}{q}=1-\eta\frac{a}{\beta_0}\quad\text{and}\quad\rho(r)=(1-\eta)\frac{a}{\beta_0}. \tag{2.26}$$

For the multilinear estimates, we state the following calculus lemma. Define $[a]_+=a$ if $a>0$, δ if $a=0$, 0 if $a<0$, where δ is an arbitrary small number.

Lemma 2.5 ([11] Lemma 4.2). *Let $0\leq a_-\leq a_+$, $a_++a_->\frac{1}{2}$, and $\alpha=2a_--[1-2a_+]_+$. Then the following estimate holds for all $s\in\mathbb{R}$*

$$\int_{\mathbb{R}}\langle y-s\rangle^{-2a_+}\langle y+s\rangle^{-2a_-}dy\leq c\langle s\rangle^{-\alpha}.$$

3. Multilinear estimates

For the local well-posedness of (1.6), we need to verify the following multilinear estimates,

$$\|\mathcal{R}e(\mathcal{N})E\|_{X^S_{k,-c_1}}\lesssim\|\mathcal{N}\|_{X^W_{\ell,b}}\|E\|_{X^S_{k,b_1}}, \tag{3.1}$$

$$\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}(E_1\bar{E}_2)\|_{X^W_{\ell,-c}}\lesssim\|E_1\|_{X^S_{k,b_1}}\|E_2\|_{X^S_{k,b_1}}, \tag{3.2}$$

$$\|\mathcal{R}e(\mathcal{N})E\|_{Y^S_k}\lesssim\|\mathcal{N}\|_{X^W_{\ell,b}}\|E\|_{X^S_{k,b_1}}, \tag{3.3}$$

$$\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}\big(E_1\bar{E}_2\big)\|_{Y^W_\ell}\lesssim\|E_1\|_{X^S_{k,b_1}}\|E_2\|_{X^S_{k,b_1}}, \tag{3.4}$$

whose proof will be given in the next section. For estimate (3.1), it is sufficient to obtain

$$\|\mathcal{N}E\|_{X^S_{k,-c_1}}+\|\bar{\mathcal{N}}E\|_{X^S_{k,-c_1}}\lesssim\|\mathcal{N}\|_{X^W_{\ell,b}}\|E\|_{X^S_{k,b_1}}. \tag{3.5}$$

For (3.3), it is sufficient to estimate the following inequality.

$$\|\mathcal{N}E\|_{Y^S_k}+\|\bar{\mathcal{N}}E\|_{Y^S_k}\lesssim\|\mathcal{N}\|_{X^W_{\ell,b}}\|E\|_{X^S_{k,b_1}}. \tag{3.6}$$

The proofs for $\mathcal{N}E$ and $\bar{\mathcal{N}}E$ are similar, thus we only discuss the case of $\mathcal{N}E$. By the duality argument, (3.5) and (3.2) are equivalent to

$$|\langle\mathcal{N}E,g\rangle|\lesssim\|\mathcal{N}\|_{X^W_{\ell,b}}\|E\|_{X^S_{k,b_1}}\|g\|_{X^S_{-k,c_1}}$$

and

$$|\langle\mathcal{D}\langle\mathcal{D}\rangle^{-1}(E_1\bar{E}_2),g\rangle|\lesssim\|E_1\|_{X^S_{k,b_1}}\|E_2\|_{X^S_{k,b_1}}\|g\|_{X^W_{-\ell,c}}$$

respectively. Thus we know that (3.5) holds if and only if $S\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}$ and (3.2) holds if and only if $W\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}$ for all $u,u_1,u_2\in L^2_{t,x}$, where

$$S:=\int\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{c_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}\frac{\langle\xi_1\rangle^k}{\langle\xi_2\rangle^k}\langle\xi\rangle^{-\ell}d\mu, \tag{3.7}$$

$$W := \int \frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^c}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|}{\langle\xi\rangle}\frac{\langle\xi\rangle^\ell}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k}\,d\mu, \tag{3.8}$$

$$\xi=\xi_1-\xi_2,\quad \tau=\tau_1-\tau_2,\quad \sigma=\tau+|\xi|\langle\xi\rangle,\quad \sigma_j=\tau_j+|\xi_j|^2\langle\xi_j\rangle^2,\text{ for } j=1,2, \tag{3.9}$$

and $d\mu=d\tau_2 d\xi_2 d\tau_1 d\xi_1$, we denote

$$A_S := \langle\xi_1\rangle^k\langle\xi_2\rangle^{-k}\langle\xi\rangle^{-\ell} \quad\text{and}\quad A_W := \langle\xi\rangle^\ell\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}. \tag{3.10}$$

For (3.3) and (3.4), it is equivalent to prove

$$\Big|\int\Big(\int\langle\xi_1\rangle^k\langle\tau_1+|\xi_1|^2\langle\xi_1\rangle^2\rangle^{-1}|\widehat{\mathcal{N}E}(\tau_1,\xi_1)|d\tau_1\Big)\overline{\widehat{v_1}(\xi_1)}d\xi_1\Big| \lesssim \|\mathcal{N}\|_{X^W_{\ell,b}}\|E\|_{X^S_{k,b_1}}\|v_1\|_{L^2_x}$$

and

$$\Big|\int\Big(\int\langle\xi\rangle^\ell\langle\tau+|\xi|\langle\xi\rangle\rangle^{-1}\frac{|\xi|}{\langle\xi\rangle}|\widehat{E_1\bar{E}_2}(\tau,\xi)|d\tau\Big)\overline{\widehat{v}(\xi)}d\xi\Big| \lesssim \|E_1\|_{X^S_{k,b_1}}\|E_2\|_{X^S_{k,b_1}}\|v\|_{L^2_x}$$

respectively. Analogously we need to obtain the estimates $\widetilde{S}\lesssim\|u\|_{L^2_{t,x}}\|v_1\|_{L^2_x}\|u_2\|_{L^2_{t,x}}$ and $\widetilde{W}\lesssim\|v\|_{L^2_x}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}$, where

$$\widetilde{S} := \int \frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}A_S\,d\mu, \tag{3.11}$$

$$\widetilde{W} := \int \frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^a}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|}{\langle\xi\rangle}A_W\,d\mu, \tag{3.12}$$

$d\mu=d\tau_2 d\xi_2 d\tau_1 d\xi_1$, $|\widehat{u}_1|=\langle\sigma_1\rangle^{a_1-1}|\mathcal{F}v_1(\xi_1)|$ in $\widetilde{S}$, and $|\widehat{u}|=\langle\sigma\rangle^{a-1}|\mathcal{F}v(\xi)|$ in $\widetilde{W}$. We split the space of $(\tau_2,\xi_2,\tau_1,\xi_1)$ into three parts. Let

$$\Omega_1 := \{2|\xi_2|<|\xi_1|\},\quad \Omega_2 := \{2|\xi_1|<|\xi_2|\},\quad \Omega_3 := \{1/2|\xi_1|\le|\xi_2|\le 2|\xi_1|\} \tag{3.13}$$

On each of Ω_j, $j=1,2,3$, we can simplify the quantities A_S and A_W as follows:

$$\begin{array}{llll} k=0, & & A_S\sim\langle\xi\rangle^{-\ell}, & A_W\sim\langle\xi\rangle^{\ell};\\ k>0, & \text{on }\Omega_1, & A_S\sim\langle\xi\rangle^{k-\ell}\langle\xi_2\rangle^{-k}, & A_W\sim\langle\xi\rangle^{\ell-k}\langle\xi_2\rangle^{-k};\\ k>0, & \text{on }\Omega_2, & A_S\lesssim\langle\xi\rangle^{-\ell}, & A_W\sim\langle\xi\rangle^{\ell-k}\langle\xi_1\rangle^{-k};\\ k>0, & \text{on }\Omega_3, & A_S\sim\langle\xi\rangle^{-\ell}, & A_W\lesssim\langle\xi\rangle^{\ell-2k};\\ k<0, & \text{on }\Omega_1, & A_S\lesssim\langle\xi\rangle^{-\ell}, & A_W\lesssim\langle\xi\rangle^{\ell-2k};\\ k<0, & \text{on }\Omega_2, & A_S\sim\langle\xi\rangle^{-k-\ell}\langle\xi_1\rangle^{k}, & A_W\lesssim\langle\xi\rangle^{\ell-2k};\\ k<0, & \text{on }\Omega_3, & A_S\sim\langle\xi\rangle^{-\ell}, & A_W\sim\langle\xi\rangle^{\ell}\langle\xi_1\rangle^{-2k}. \end{array} \tag{3.14}$$

Correspondingly we split the integral as follows:

$$S_j := \int_{\Omega_j}\frac{|\widehat{u}|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{c_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}A_S\,d\mu,\quad\text{and}\quad W_j := \int_{\Omega_j}\frac{|\widehat{u}|}{\langle\sigma\rangle^c}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|}{\langle\xi\rangle}A_W\,d\mu, \tag{3.15}$$

for $j=1,2,3$. Also we define $\widetilde{S}_j$ and $\widetilde{W}_j$ accordingly, for $j=1,2,3$,

$$\widetilde{S}_j := \int_{\Omega_j}\frac{|\widehat{u}|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}A_S\,d\mu,\quad\text{and}\quad \widetilde{W}_j := \int_{\Omega_j}\frac{|\widehat{u}|}{\langle\sigma\rangle^a}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|}{\langle\xi\rangle}A_W\,d\mu. \tag{3.16}$$

We decomposed the region Ω_j into three parts for $j=1,2$, as follows:

$$\Omega_{j,\sigma}:=\Omega_j\big|_{|\sigma|\geq\max\{|\sigma_1|,|\sigma_2|\}},\ \Omega_{j,\sigma_1}:=\Omega_j\big|_{|\sigma_1|\geq\max\{|\sigma|,|\sigma_2|\}},\ \Omega_{j,\sigma_2}:=\Omega_j\big|_{|\sigma_2|\geq\max\{|\sigma|,|\sigma_1|\}}. \tag{3.17}$$

We also split the integral S_j into three parts $S_{j,\sigma}$, S_{j,σ_1}, and S_{j,σ_2} with respect to the regions $\Omega_{j,\sigma}$, Ω_{j,σ_1}, Ω_{j,σ_2} for $j=1,2$. Thus W_j is split into three parts $W_{j,\sigma}$, W_{j,σ_1}, and W_{j,σ_2} in the same fashion for $j=1,2$. So are the integrals $\widetilde{S}_j$ and $\widetilde{W}_j$. Since

$$\max\{\langle\sigma\rangle,\langle\sigma_1\rangle,\langle\sigma_2\rangle\}\gtrsim|\sigma_1-\sigma_2-\sigma|+1=\big||\xi_1|^2\langle\xi_1\rangle^2-|\xi_2|^2\langle\xi_2\rangle^2-|\xi|\langle\xi\rangle\big|+1\gtrsim\langle\xi\rangle^4 \tag{3.18}$$

on $\Omega_1\cup\Omega_2$, thus we have the following inequality

$$\max\{\langle\xi\rangle^4,\langle\xi_1\rangle^4,\langle\xi_2\rangle^4\}\lesssim\max\{\langle\sigma\rangle,\langle\sigma_1\rangle,\langle\sigma_2\rangle\}, \tag{3.19}$$

which will be used frequently in the proof of multilinear estimates. The proofs of Theorems 1.1 - 1.2 will be given in the later sections.

4. Multilinear estimates for the X-norm

We first pay attention to the integration region Ω_3 since we shall not have any inequality like (3.19) that is applicable. In fact, the multilinear estimates on Ω_3 gives the limit condition for (k,ℓ). We denote $n(r,r'):=\rho(r)-\rho(r')$ if $2\leq r'\leq r<\infty$, $>\frac{d}{2}-\rho(r')$ if $r=\infty$.

Lemma 4.1. *Let $a,a_1,a_2\geq 0$ and $m\in\mathbb{R}$. For $d=2,3$, let $m\leq\frac{d}{2}$ and one of the following conditions is satisfied. (a) $a,a_2>\frac12,a_1>\frac14$; (b) $a,a_1>\frac12,a_2>\frac14$; (c) $a_1,a_2>\frac12,a>\frac14$. Then we have*

$$\int_{\Omega_3}\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^a}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{a_2}}\langle\xi\rangle^m\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}. \tag{4.1}$$

For $d\geq 4$, let $m\leq 4-\frac{d}{2}$ and one of the following conditions is satisfied. (i) $a,a_2>\frac12,a_1>\frac{d}{8}-\frac12+\frac{m}{4}$; (ii) $a,a_1>\frac12,a_2>\frac{d}{8}-\frac12+\frac{m}{4}$; (iii) $a_1,a_2>\frac12,a>\frac{d}{8}-\frac12+\frac{m}{4}$. Then we have (4.1).

Proof. Without loss of generality, we can assume that $m\geq 0$ for $2\leq d\leq 7$. Set $m=\alpha_1+\alpha_2$, by the Hölder inequality, we have

$$\begin{aligned}&\text{LHS of (4.1)}\\&\lesssim\int_{\Omega_3}\frac{|\widehat{u}|}{\langle\sigma\rangle^a}\frac{\langle\xi_1\rangle^{\alpha_1}|\widehat{u}_1|}{\langle\sigma_1\rangle^{a_1}}\frac{\langle\xi_2\rangle^{\alpha_2}|\widehat{u}_2|}{\langle\sigma_2\rangle^{a_2}}d\mu\\&\lesssim\|(\langle\sigma\rangle^{-a}|\widehat{u}|)^\vee\|_{L^q_tL^r_x}\|(\langle\xi_1\rangle^{\alpha_1}\langle\sigma_1\rangle^{-a_1}|\widehat{u}_1|)^\vee\|_{L^{q_1}_tL^{r_1}_x}\|(\langle\xi_2\rangle^{\alpha_2}\langle\sigma_2\rangle^{-a_2}|\widehat{u}_2|)^\vee\|_{L^{q_2}_tL^{r_2}_x},\end{aligned} \tag{4.2}$$

where

$$\frac1q+\frac1{q_1}+\frac1{q_2}=1\quad\text{and}\quad\rho(r)+\rho(r_1)+\rho(r_2)=\frac{d}{2}. \tag{4.3}$$

Invoking Lemma 2.4, if (2.26) holds, we can obtain $\|(\langle\sigma\rangle^{-a}|\widehat{u}|)^\vee\|_{L^q_tL^r_x}\lesssim\|u\|_{L^2_{t,x}}$ provided that β_0,a, and η satisfy the assumption in Lemma 2.4.

For last two norms in (4.2), invoking Lemma 2.3, if

$$\frac{2}{q_j}=1-(1-\gamma_j(1-\theta_j))\frac{a_j}{\beta_j}-\alpha_j,\quad\rho(r_j)=(2\gamma_j(1-\theta_j))\frac{a_j}{\beta_j}+\alpha_j,\quad\alpha_j=\zeta_j\theta_j\frac{a_j}{\beta_j}, \tag{4.4}$$

$$\gamma_1=\gamma_2=\frac{d}{4}\ \text{ for }\ d=2,3;\gamma_j=\frac{2}{\widetilde{q}_j}\ \text{ for }\ d\geq 4,\zeta_j=\frac{2}{\widetilde{q}_j}\ \text{ for }\ d=2,3;\zeta_1=\zeta_2=1\ \text{ for }\ d\geq 4, \tag{4.5}$$

we can get

$$\|(\langle\xi_j\rangle^{\alpha_j}\langle\sigma_j\rangle^{-a_j}|\widehat{u}_j|)^\vee\|_{L_t^{q_j}L_x^{r_j}}\lesssim\|u_j\|_{L^2_{t,x}},\quad \text{for}\quad j=1,2,$$

provided that β_j,a_j,θ_j, and $\widetilde{q}_j$ satisfy the assumption in Lemma 2.3. Now we have

$$m=\alpha_1+\alpha_2=\zeta_1\theta_1\frac{a_1}{\beta_1}+\zeta_2\theta_2\frac{a_2}{\beta_2}\begin{cases}<2 & \text{if}\quad d=2,\\ \leq 2 & \text{if}\quad d\geq 3.\end{cases} \tag{4.6}$$

Combining (4.3), (2.26), and (4.4) gives

$$1=\eta\frac{a}{\beta_0}+(1-\gamma_1(1-\theta_1))\frac{a_1}{\beta_1}+(1-\gamma_2(1-\theta_2))\frac{a_2}{\beta_2}-m \tag{4.7}$$

and

$$\frac{d}{2}=(1-\eta)\frac{a}{\beta_0}+2\gamma_1(1-\theta_1)\frac{a_1}{\beta_1}+2\gamma_2(1-\theta_2)\frac{a_2}{\beta_2}+m. \tag{4.8}$$

Then we add up (4.7) with (4.8) and add up $2\times$(4.7) with (4.8) to obtain

$$1+\frac{d}{2}=\frac{a}{\beta_0}+(1+\gamma_1(1-\theta_1))\frac{a_1}{\beta_1}+(1+\gamma_2(1-\theta_2))\frac{a_2}{\beta_2} \tag{4.9}$$

and

$$2+\frac{d}{2}=(1+\eta)\frac{a}{\beta_0}+2\frac{a_1}{\beta_1}+2\frac{a_2}{\beta_2}-m. \tag{4.10}$$

From (4.8) and (4.10), we have the conditions $m\leq\frac{d}{2}$ and $m\leq 4-\frac{d}{2}$ respectively. Therefore, we obtain the limit conditions $m\leq\frac{d}{2}$ for $d=2,3$, and $m\leq 4-\frac{d}{2}$ for $d\geq 4$.

From the equalities (4.9) and (4.10), we may get these conditions $(a)-(c)$ and $(i)-(iii)$. For example, if $a,a_2>\frac{1}{2}$, we may choose $\eta=1$, and thus we get $a_1\geq(\frac{d}{4}-1+\frac{m}{2})\beta_1>\frac{d}{8}-\frac{1}{2}+\frac{m}{4}$ by (4.10). However, such a condition is not enough when the equality (4.9) holds for $d=3$. More precisely, the lower bound is too small for $d=3$. Since $\frac{d}{8}-\frac{1}{2}+\frac{m}{4}\leq\frac{1}{4}$ when $d=3$, we assume that $a_1>\frac{1}{4}$.

For $d\geq 8$, we have

$$\text{LHS of (4.1)}\lesssim\|(\langle\xi\rangle^m\langle\sigma\rangle^{-a}|\widehat{u}|)^\vee\|_{L_t^qL_x^r}\|(\langle\sigma_1\rangle^{-a_1}|\widehat{u}_1|)^\vee\|_{L_t^{q_1}L_x^{r_1}}\|(\langle\sigma_2\rangle^{-a_2}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2}} \tag{4.11}$$

with (4.3). Since $m\leq 0$, we apply Sobolev inequality and Lemma 2.4 to obtain

$$\|(\langle\xi\rangle^m\langle\sigma\rangle^{-a}|\widehat{u}|)^\vee\|_{L_t^qL_x^r}\lesssim\|(\langle\xi\rangle^{m+n(r,r')}\langle\sigma\rangle^{-a}|\widehat{u}|)^\vee\|_{L_t^qL_x^{r'}}\lesssim\|u\|_{L^2_{t,x}}, \tag{4.12}$$

where $n(r,r')\leq -m$, and (q,r') satisfies (2.26) provided that $\beta_0>\frac{1}{2},0\leq a\leq\beta_0$, and $0\leq\eta\leq 1$. Invoking Lemma 2.3, if (4.4) with $\theta_1=\theta_2=0$ holds, we can get

$$\|(\langle\sigma_j\rangle^{-a_j}|\widehat{u}_j|)^\vee\|_{L_t^{q_j}L_x^{r_j}}\lesssim\|u_j\|_{L^2_{t,x}},\quad j=1,2,$$

provided that $\beta_j > \frac{1}{2}$, $0 \le a_j \le \beta_j$, and $2 \le \widetilde{q}_j \le \infty$. Combining all needed conditions, we need to verify that the following system holds.

$$\begin{cases} 1 = \eta \frac{a}{\beta_0} + \left(1 - \frac{2}{\widetilde{q}_1}\right) \frac{a_1}{\beta_1} + \left(1 - \frac{2}{\widetilde{q}_2}\right) \frac{a_2}{\beta_2}, \\ \rho(r) = \frac{d}{2} - \frac{4}{\widetilde{q}_1} \frac{a_1}{\beta_1} - \frac{4}{\widetilde{q}_2} \frac{a_2}{\beta_2} < \frac{d}{2}, \\ -m \ge \rho(r) - (1-\eta) \frac{a}{\beta_0} \ge 0. \end{cases} \tag{4.13}$$

From the system (4.13), we have

$$-m \ge \frac{d}{2} + 2 - (1+\eta) \frac{a}{\beta_0} - 2\frac{a_1}{\beta_1} - 2\frac{a_2}{\beta_2} \ge 0 \tag{4.14}$$

which implies the conditions $(i)-(iii)$. Under the assumptions in Lemma 4.1, we can choose the appropriate parameters to verify the Equations (4.9), (4.10), and (4.13). □

REMARK 4.1. For $d \ge 4$, from the Equations (4.10) and (4.14), we have to choose all parameters $a, a_1, a_2 > \frac{1}{2}$ when $m = 4 - \frac{d}{2}$. That is, we have to choose $b_1, b, c_1 > \frac{1}{2}$ when the integral S_3 in (3.15) with $\ell = \frac{d}{2} - 4$ is derived from Lemma 4.1. However, the condition $b_1 + c_1 > 1$ so that the corresponding Duhamel estimate does not hold. In order to obtain LWP on boundary, we apply the method used in [9]. However, we only obtain the result on some parts of the boundary.

Next we deal with the multilinear estimates on Ω_3 for the wave case with $k < 0$. Note when $d \ge 4$, we obtain the conditions $-\ell, \ell - 2k \le 4 - \frac{d}{2}$ by Lemma 4.1, so $k \ge 0$ for $d \ge 8$. We split Ω_3 into two parts: (i) $\Omega_3 \cup \{|\xi_2| \le 2|\xi|\}$ and (ii) $\Omega_3 \cap \{|\xi_2| > 2|\xi|\}$. For the case (i), we have $\langle \xi \rangle^{\ell} \langle \xi_1 \rangle^{-k} \langle \xi_2 \rangle^{-k} \sim \langle \xi \rangle^{\ell - 2k}$, and thus we may estimate it by Lemma 4.1. Hence, we only need to consider the case of (ii).

LEMMA 4.2. *For $4 \le d \le 7$, let $\ell - 2k \le 4 - \frac{d}{2}$ and $-1 \le k < 0$. If $b_1 > \frac{1}{2}$, $c > \frac{d}{8} - \frac{1}{2} + \frac{\ell - 2k}{4}$ and $c > \frac{-k}{2}$, we have*

$$\int_{\Omega_3 \cap \{|\xi_2| > 2|\xi|\}} \frac{|\widehat{u}(\tau,\xi)|}{\langle \sigma \rangle^c} \frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle \sigma_1 \rangle^{b_1}} \frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle \sigma_2 \rangle^{b_1}} \langle \xi \rangle^{\ell} \langle \xi_1 \rangle^{-2k} \, d\mu \lesssim \|u\|_{L^2_{t,x}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}. \tag{4.15}$$

Proof. For $\ell \ge 0$, we have $\langle \xi \rangle^{\ell} \langle \xi_1 \rangle^{-2k} \lesssim \langle \xi_1 \rangle^{\ell - 2k}$. Moreover, the proof of this case is the same as Lemma 4.1. Thus, we may obtain that (4.15) holds if $\ell - 2k \le 4 - \frac{d}{2}, b_1 > \frac{1}{2}, c > \frac{d}{8} - \frac{1}{2} + \frac{\ell - 2k}{4}$.

For $\ell < 0$, set $\alpha_1 + \alpha_2 = -2k$. Thus, by the Hölder inequality, we have

$$\begin{aligned} \text{LHS of (4.23)} &\sim \int_{\Omega_3 \cap \{|\xi_2| > 2|\xi|\}} \frac{|\widehat{u}|}{\langle \sigma \rangle^c} \frac{\langle \xi_1 \rangle^{\alpha_1} |\widehat{u}_1|}{\langle \sigma_1 \rangle^{b_1}} \frac{\langle \xi_2 \rangle^{\alpha_2} |\widehat{u}_2|}{\langle \sigma_2 \rangle^{b_1}} \langle \xi \rangle^{\ell} d\mu \\ &\lesssim \| \left(\langle \xi \rangle^{\ell} \langle \sigma \rangle^{-c} |\widehat{u}| \right)^{\vee} \|_{L^q_t L^r_x} \| \left(\langle \xi_1 \rangle^{\alpha_1} \langle \sigma_1 \rangle^{-b_1} |\widehat{u}_1| \right)^{\vee} \|_{L^{q_1}_t L^{r_1}_x} \| \left(\langle \xi_2 \rangle^{\alpha_2} \langle \sigma_2 \rangle^{-b_1} |\widehat{u}_2| \right)^{\vee} \|_{L^{q_2}_t L^{r_2}_x} \end{aligned} \tag{4.16}$$

with (4.3). Since $\ell < 0$, we apply Sobolev inequality and Lemma 2.4 to obtain

$$\| (\langle \xi \rangle^{\ell} \langle \sigma \rangle^{-c} |\widehat{u}|)^{\vee} \|_{L^q_t L^r_x} \lesssim \| (\langle \xi \rangle^{\ell + n(r,r')} \langle \sigma \rangle^{-c} |\widehat{u}|)^{\vee} \|_{L^q_t L^{r'}_x} \lesssim \|u\|_{L^2_{t,x}}, \tag{4.17}$$

where $n(r,r') \le -\ell$ and (q,r') satisfies (2.26) provided $\beta_0 > \frac{1}{2}, 0 \le c \le \beta_0$, and $0 \le \eta \le 1$. Invoking Lemma 2.3, if (4.4) with $\beta_1 = \beta_2 = b_1$ holds, we can get

$$\| (\langle \xi_j \rangle^{\alpha_j} \langle \sigma_j \rangle^{-b_1} |\widehat{u}_j|)^{\vee} \|_{L^{q_j}_t L^{r_j}_x} \lesssim \|u_j\|_{L^2_{t,x}}, \quad j = 1, 2,$$

provided θ_j and $\widetilde{q}_j$ satisfty the assumptions in Lemma 2.3. Combining all needed conditions, we have that the following system holds.

$$\begin{cases} 1=\eta\frac{c}{\beta_0}+\frac{2}{\widetilde{q}_1}(1-\theta_1)+\frac{2}{\widetilde{q}_2}(1-\theta_2)-2k, \\ -2k=\theta_1+\theta_2, \\ -\ell\geq\frac{d}{2}-2k-2-(1+\eta)\frac{c}{\beta_0}\geq 0. \end{cases} \tag{4.18}$$

From the last inequality in (4.18), we get the condition $c>\frac{-k}{2}$ when $-\ell\geq\frac{d}{2}-2$. Under the assumptions, we can choose the appropriate parameters to verify the system (4.18). ∎

REMARK 4.2. For $d=2,3$, we may omit the term $\langle\xi\rangle^{\ell}$ in (4.15) as $\ell<0$. Then we obtain the corresponding estimate by the argument in the proof of Lemma 4.1, see Lemma 4.6.

LEMMA 4.3. *Let $a,a_1,a_2\geq 0$ and $m\in\mathbb{R}$. For $d=2,3$, let $a,a_1,a_2\geq\frac{m}{4}$ and one of the following conditions is satisfied. (a) $a,a_2>\frac{1}{2},a_1>\frac{d}{8}$; (b) $a,a_1>\frac{1}{2},a_2>\frac{d}{8}$; (c) $a,a_2>\frac{1}{2},a>\frac{d}{8}$. Then we have*

$$\int_{\Omega_j}\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^{a}}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{a_2}}\langle\xi\rangle^m\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}\quad \text{for}\quad j=1,2, \tag{4.19}$$

where Ω_1, and Ω_2 are given in (3.13).

For $d\geq 4$, let $m\leq 4-\frac{d}{2}$ and one of the following conditions is satisfied. (i) $a,a_2>\frac{1}{2},a_1>\frac{d}{8}-\frac{1}{2}+\frac{m}{4}$; (ii) $a,a_1>\frac{1}{2},a_2>\frac{d}{8}-\frac{1}{2}+\frac{m}{4}$; (iii) $a_1,a_2>\frac{1}{2},a>\frac{d}{8}-\frac{1}{2}+\frac{m}{4}$. Then we have (4.19).

Proof. We only prove the case of Ω_1, while the case of Ω_2 is analogous. The case of Ω_{1,σ_2} is symmetric to Ω_{1,σ_1} for all d, so we skip the proof. Without loss of generality, we assume that $m\geq 0$ for $2\leq d\leq 7$.

For $d=2,3$, on $\Omega_{1,\sigma}$, invoking (3.18) and the condition $a\geq\frac{m}{4}$, we obtain

$$\int_{\Omega_{1,\sigma}}\frac{|\widehat{u}|}{\langle\sigma\rangle^{a}}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{a_2}}\langle\xi\rangle^m\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|(\langle\sigma_1\rangle^{-a_1}|\widehat{u}_1|)^{\vee}\|_{L^{q_1}_tL^{r_1}_x}\|(\langle\sigma_2\rangle^{-a_2}|\widehat{u}_2|)^{\vee}\|_{L^{q_2}_tL^{r_2}_x}, \tag{4.20}$$

where $q=2$, q_1, q_2, $r=2$, r_1, and r_2 satisfy (4.3). Invoking Lemma 2.3, if (4.4) with $\widetilde{q}_1=\widetilde{q}_2=\infty$ holds, we have

$$\|(\langle\sigma_j\rangle^{-a_j}|\widehat{u}_j|)^{\vee}\|_{L^{q_j}_tL^{r_j}_x}\lesssim\|u_j\|_{L^2_tL^2_x},\quad j=1,2,$$

provided that $\beta_j>\frac{1}{2}$, $0\leq a_j\leq\beta_j$, and $0\leq\theta_j\leq 1$. For the conditions (a) and (c), we choose $\beta_1=\frac{4a_1}{d},\beta_2=a_2,\theta_1=1$, and $\theta_2=0$ which satisfy (4.3). For the condition (b), we choose $\beta_1=a_1,\beta_2=\frac{4a_2}{d},\theta_1=0$, and $\theta_2=1$ which satisfy (4.3).

On Ω_{1,σ_1}, invoking (3.18) and $a_1\geq\frac{m}{4}$, for the conditions (a) and (c), we have

$$\begin{aligned}&\int_{\Omega_{1,\sigma_1}}\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^{a}}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{a_2}}\langle\xi\rangle^m\,d\mu\\ &\lesssim\Big\|\Big(\frac{|\widehat{u}|}{\langle\sigma\rangle^{a}}\Big)^{\vee}\Big\|_{L^q_tL^r_x}\|u_1\|_{L^2_{t,x}}\Big\|\Big(\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{a_2}}\Big)^{\vee}\Big\|_{L^{q_2}_tL^{r_2}_x},\end{aligned} \tag{4.21}$$

where $q=\frac{8}{4-d}, q_2=\frac{8}{d}, \rho(r)=0$, and $\rho(r_2)=\frac{d}{2}$. Invoking Lemma 2.4 with $\beta_0=\frac{4a}{d}$ and $\eta=1$ and Lemma 2.3 with $\beta=a_2$ and $\theta=0$, we obtain

$$\text{RHS of (4.21)} \lesssim \|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}.$$

For the condition (b), we first consider the case $0<m\le 2$. By the Hölder inequality, we have

$$\int_{\Omega_{1,\sigma_1}} \frac{|\widehat{u}||\widehat{u}_1||\widehat{u}_2|}{\langle\sigma\rangle^{a}\langle\sigma_1\rangle^{a_1}\langle\sigma_2\rangle^{a_2}}\langle\xi\rangle^m \lesssim \Big\|\Big[\frac{|\widehat{u}|}{\langle\sigma\rangle^a}\Big]^\vee\Big\|_{L^q_tL^r_x}\Big\|\Big[\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{a_1-\frac{m}{4}}}\Big]^\vee\Big\|_{L^{q_1}_tL^{r_1}_x}\Big\|\Big[\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{a_2}}\Big]^\vee\Big\|_{L^{q_2}_tL^{r_2}_x} \tag{4.22}$$

with (4.3). Invoking Lemma 2.4 with $\beta_0=a$ and $\eta=\frac{d}{2}-1+2\delta$, if $\frac{2}{q}=\rho(r)=1-\eta$, we have

$$\|(\langle\sigma\rangle^{-a}|\widehat{u}|)^\vee\|_{L^q_tL^r_x}\lesssim\|u\|_{L^2_{t,x}}.$$

For the remaining two terms, we can set two quantities $A_1=1-\frac{m}{2}+\delta$ and $A_2=\frac{m}{2}-2\delta$ so that $0<A_1,A_2\le 1$. Invoking Lemma 2.3, if $\theta=\frac{4}{d}-\frac{1+\delta}{1-\delta}$, $\widetilde{q}=\infty$,

$$\frac{2}{q_j}=1-\left(1-\frac{d(1-\theta)}{4}\right)A_j,\quad\text{and}\quad \rho(r_j)=\frac{d(1-\theta)}{2}A_j,$$

for $j=1,2$, we obtain

$$\|(\langle\sigma_1\rangle^{-(a_1-\frac{m}{4})}|\widehat{u}_1|)^\vee\|_{L^{q_1}_tL^{r_1}_x}\lesssim\|u_1\|_{L^2_{t,x}}\quad\text{and}\quad\|(\langle\sigma_2\rangle^{-a_2}|\widehat{u}_2|)^\vee\|_{L^{q_2}_tL^{r_2}_x}\lesssim\|u_2\|_{L^2_{t,x}}.$$

It is easy to verify the condition (4.3). For $m>2$, since $a>\frac{1}{2}$, $a_1\ge\frac{m}{4}$, and $a_2>\frac{1}{2}$, we have

$$\int_{\Omega_{1,\sigma_1}}\frac{|\widehat{u}|}{\langle\sigma\rangle^a}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{a_2}}\langle\xi\rangle^m\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}},$$

where the argument is analogous to that of the part (a).

For $4\le d\le 7$. On $\Omega_{1,\sigma}$ and Ω_{1,σ_1}, by the Hölder inequality, we always obtain

$$\text{LHS of (4.19)}\lesssim\|(\langle\sigma\rangle^{-\tilde{a}_0}|\widehat{u}|)^\vee\|_{L^q_tL^r_x}\|(\langle\sigma_1\rangle^{-\tilde{a}_1}|\widehat{u}_1|)^\vee\|_{L^{q_1}_tL^{r_1}_x}\|(\langle\sigma_2\rangle^{-\tilde{a}_2}|\widehat{u}_2|)^\vee\|_{L^{q_2}_tL^{r_2}_x} \tag{4.23}$$

with (4.3), where $\tilde{a}_2=a_2$ on $\Omega_{1,\sigma}\cup\Omega_{1,\sigma_1}$,

$$\tilde{a}_0=\begin{cases} a-\dfrac{m}{4} & \text{on } \Omega_{1,\sigma},\\ a & \text{on } \Omega_{1,\sigma_1},\end{cases}\qquad \tilde{a}_1=\begin{cases} a_1 & \text{on } \Omega_{1,\sigma},\\ a_1-\dfrac{m}{4} & \text{on } \Omega_{1,\sigma_1}.\end{cases} \tag{4.24}$$

To estimate the RHS of (4.23) is similar to the RHS of (4.2). Thus, we obtain that the following system holds when we assume that (2.26) and (4.4) hold and satisfy the condition (4.3).

$$\begin{cases} \dfrac{d}{2}+1=\dfrac{\tilde{a}_0}{\beta_0}+\left(1+\dfrac{2}{\widetilde{q}_1}\right)\dfrac{\tilde{a}_1}{\beta_1}+\left(1+\dfrac{2}{\widetilde{q}_2}\right)\dfrac{\tilde{a}_2}{\beta_2},\\ \dfrac{d}{2}+2=\left(1+\dfrac{2}{\widetilde{q}_0}\right)\dfrac{\tilde{a}_0}{\beta_0}+2\dfrac{\tilde{a}_1}{\beta_1}+2\dfrac{\tilde{a}_2}{\beta_2},\end{cases} \tag{4.25}$$

where $\beta_j>\frac{1}{2}, 0\le\tilde{a}_j\le\beta_j$, and $2\le\widetilde{q}_j\le\infty$ ($\widetilde{q}_j>2$ for $d=4, j=1,2$) for $j=0,1,2$.

The conditions (i) and (ii) are symmetric on the case $\Omega_{1,\sigma}$ and the conditions (ii) and (iii) are similar on the case Ω_{1,σ_1}, so we consider only the first one respectively. We may take $\frac{a-\frac{m}{4}}{\beta_0}+\frac{a_1}{\beta_1}=\frac{d}{4}$, $\frac{a_2}{\beta_2}=1$ for the conditions (i), (iii) on $\Omega_{1,\sigma}$ and $\frac{a_1-\frac{m}{4}}{\beta_1}+\frac{a_2}{\beta_2}=\frac{d}{4}+\delta$, $\frac{a}{\beta_0}=1$ for the conditions (i), (ii) on Ω_{1,σ_1}. Then we may choose the appropriate parameters so that (4.25) holds.

Finally, the proof of the case $d\geq 8$ is the same as Lemma 4.1, so we complete the proof. ∎

Lemma 4.4. *Let $a,a_1,a_2\geq 0, m\in\mathbb{R}$ and $m_1\geq 0$. For $d\geq 4$, let $m_1\geq m+\frac{d}{2}-4$, $a,a_1,a_2\geq\frac{m}{4}$, and one of the following conditions holds. (i) $a,a_2>\frac{1}{2}, a_1>\frac{d}{8}-\frac{1}{2}+\frac{m-m_1}{4}$; (ii) $a_1,a_2>\frac{1}{2}, a>\frac{d}{8}-\frac{1}{2}+\frac{m-m_1}{4}$. Then, we can derive*

$$\int_{\Omega_j}\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^{a}}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{a_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{a_2}}\frac{\langle\xi\rangle^{m}}{\langle\xi_j+(-1)^j\xi\rangle^{m_1}}\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}, \quad (4.26)$$

for $j=1,2$. Note $\xi_j+(-1)^j\xi=\xi_2$ if $j=1$ and $\xi_j+(-1)^j\xi=\xi_1$ if $j=2$.

Proof. We only prove the case for $j=1$, while the other case $j=2$ is analogous. Without loss of generality, we assume that $m\geq 0$. For (4.26) with $j=1$, we split the integral into three parts depending on the dominants of $\{\sigma,\sigma_1,\sigma_2\}$. For each of the dominanted regions, by the Hölder inequality, we have

$$\text{LHS of (4.26)}\lesssim\|(\frac{|\widehat{u}|}{\langle\sigma\rangle^{\tilde{a}}})^{\vee}\|_{L^q_tL^r_x}\|(\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{\tilde{a}_1}})^{\vee}\|_{L^{q_1}_tL^{r_1}_x}\|(\frac{|\widehat{u}_2|}{\langle\xi_2\rangle^{m_1}\langle\sigma_2\rangle^{\tilde{a}_2}})^{\vee}\|_{L^{q_2}_tL^{r_2}_x} \quad (4.27)$$

with (4.3), where

$$\tilde{a}=\begin{cases} a-\dfrac{m}{4} & \text{on } \Omega_{1,\sigma},\\ a & \text{otherwise,}\end{cases}\quad \tilde{a}_1=\begin{cases} a_1-\dfrac{m}{4} & \text{on } \Omega_{1,\sigma_1},\\ a_1 & \text{otherwise,}\end{cases}\quad \tilde{a}_2=\begin{cases} a_2-\dfrac{m}{4} & \text{on } \Omega_{1,\sigma_2},\\ a_2 & \text{otherwise.}\end{cases} \quad (4.28)$$

Invoking Lemma 2.4, if (q,r) satisfies (2.26), we get $\|(\langle\sigma\rangle^{-\tilde{a}}|\widehat{u}|)^{\vee}\|_{L^q_tL^r_x}\lesssim\|u\|_{L^2_{t,x}}$ provided that $\beta_0>\frac{1}{2}$, $0\leq\tilde{a}\leq\beta_0$, and $0\leq\eta\leq 1$. Invoking Lemma 2.3, if (4.4) with $\theta_1=0$ holds for $j=1$, we can get $\|(\langle\sigma_1\rangle^{-\tilde{a}_1}|\widehat{u}_1|)^{\vee}\|_{L^{q_1}_tL^{r_1}_x}\lesssim\|u_1\|_{L^2_{t,x}}$ provided that $\beta_1,\tilde{a}_1,\widetilde{q}_1$ satisfy the assumptions in Lemma 2.3. For the remaining norms, invoke Sobolev inequality and Lemma 2.3, then we can derive

$$\|(\langle\xi_2\rangle^{-m_1}\langle\sigma_2\rangle^{-\tilde{a}_2}|\widehat{u}_2|)^{\vee}\|_{L^{q_2}_tL^{r_2}_x}\lesssim\|(\langle\xi_2\rangle^{-m_1+n(r_2,r_2')}\langle\sigma_2\rangle^{-\tilde{a}_2}|\widehat{u}_2|)^{\vee}\|_{L^{q_2}_tL^{r_2'}_x}\lesssim\|u_2\|_{L^2_{t,x}}$$

provided that $\beta_2,\tilde{a}_2,\theta_2,\widetilde{q}_2$ satisfy the assumption in Lemma 2.3, (q_2,r_2') satisfies (4.4) and $n(r_2,r_2')-m_1\leq\theta_2\frac{\tilde{a}_2}{\beta_2}$.

Combining all needed conditions, we have that the following system holds.

$$\begin{cases} 1=\eta\frac{\tilde{a}}{\beta_0}+\left(1-\frac{2}{\widetilde{q}_1}\right)\frac{\tilde{a}_1}{\beta_1}+\left(1-\frac{2(1-\theta_2)}{\widetilde{q}_2}-\theta_2\right)\frac{\tilde{a}_2}{\beta_2},\\ \rho(r_2)=\frac{d}{2}-(1-\eta)\frac{\tilde{a}}{\beta_0}-\frac{4}{\widetilde{q}_1}\frac{\tilde{a}_1}{\beta_1}<\frac{d}{2},\\ m_1+\theta_2\frac{\tilde{a}_2}{\beta_2}\geq\rho(r_2)-\left(\frac{4}{\widetilde{q}_2}(1-\theta_2)+\theta_2\right)\frac{\tilde{a}_2}{\beta_2}\geq 0.\end{cases} \quad (4.29)$$

From the above system, we have

$$m_1\geq\frac{d}{2}+2-(1+\eta)\frac{\tilde{a}}{\beta_0}-2\frac{\tilde{a}_1}{\beta_1}-2\frac{\tilde{a}_2}{\beta_2}\geq-\theta_2\frac{\tilde{a}_2}{\beta_2}$$

which implies the conditions described in Lemma 4.4. ∎

REMARK 4.3. For $d=2,3$, we may omit the term $\langle \xi_j+(-1)^j\xi\rangle^{m_1}$ in (4.26), and then treat the corresponding estimate by Lemma 4.3.

We have to choose both b_1 and c_1 bigger than $\frac{1}{2}$ when we derive the multilinear estimate on Ω_1 for the Schrödinger case with $|k|-\ell>2$ by Lemma 4.4. However, the condition $b_1+c_1>1$ such that the Duhamel estimate will not hold. Thus, we try to relax the conditions of b_1 so that we may obtain a larger LWP region.

LEMMA 4.5. *Let $|k|-\ell\geq 2$ and $-\ell\leq\frac{d}{2}$ for $d=2,3$, $-\ell\leq 4-\frac{d}{2}$ for $d\geq 4$.*

(a) For $k\geq 0$, if $b,c_1>\frac{1}{2}$, $c_1\geq\frac{k-\ell}{4}>b_1>\frac{k-\ell-1}{4}$, $b_1>\frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}$, and $b>\frac{k-\ell-1}{4}$, then

$$\int_{\Omega_1}\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{c_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}\frac{\langle\xi_1\rangle^{k-\ell}}{\langle\xi_2\rangle^k}\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}. \tag{4.30}$$

(b) For $k<0$, if $b,b_1>\frac{1}{2}$, $b_1\geq\frac{-k-\ell}{4}>c_1>\frac{-k-\ell-1}{4}$, $c_1>\frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}$, and $b>\frac{-k-\ell-1}{4}$, then

$$\int_{\Omega_2}\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{c_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}\frac{\langle\xi_1\rangle^{k}}{\langle\xi_2\rangle^{k+\ell}}\,d\mu\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}. \tag{4.31}$$

Proof. First we consider the case of $d=2,3$. For the part (a), on Ω_{1,σ_2}, we have

$$\begin{aligned}&\int_{\Omega_{1,\sigma_2}}\frac{|\widehat{u}|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{c_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\frac{\langle\xi_1\rangle^{k-\ell}}{\langle\xi_2\rangle^k}\,d\mu\\&\lesssim\left\|\left(\frac{|\widehat{u}|}{\langle\sigma\rangle^b}\right)^\vee\right\|_{L^q_tL^r_x}\left\|\left(\frac{\langle\xi_1\rangle^{k-\ell-4b_1}|\widehat{u}_1|}{\langle\sigma_1\rangle^{c_1}}\right)^\vee\right\|_{L^{q_1}_tL^{r_1}_x}\left\|\left(\frac{|\widehat{u}_2|}{\langle\xi_2\rangle^k}\right)^\vee\right\|_{L^2_tL^{r_2}_x},\end{aligned}$$

where q, q_1, $q_2=2$, r, r_1, and r_2 satisfy (4.3) and $0<k-\ell-4b_1<1$. Invoke Lemma 2.4 with $\beta_0=b$ and $\eta=k-\ell-4b_1$, if $\frac{2}{q}=\rho(r)=1-(k-\ell-4b_1)$, then

$$\|(\langle\sigma\rangle^{-b}|\widehat{u}|)^\vee\|_{L^q_tL^r_x}\lesssim\|u\|_{L^2_{t,x}}.$$

Invoke Lemma 2.3, if (4.4) with $\beta_1=c_1,\theta_1=1$, and $\widetilde{q}_1=\frac{2}{k-\ell-4b_1}$ holds for $j=1$, then we obtain

$$\|(\langle\xi_1\rangle^{k-\ell-4b_1}\langle\sigma_1\rangle^{-c_1}|\widehat{u}_1|)^\vee\|_{L^{q_1}_tL^{r_1}_x}\lesssim\|u_1\|_{L^2_{t,x}}.$$

One may check that $\frac{1}{q}+\frac{1}{q_1}=\frac{1}{2}$ and we now have $\rho(r_2)=\frac{d}{2}-1$. By the conditions $k-\ell\geq 2$ and $-\ell\leq\frac{d}{2}$, we have $k\geq 2-\frac{d}{2}\geq\rho(r_2)$. Invoke Sobolev inequality, then we derive

$$\|(\langle\xi_2\rangle^{-k}|\widehat{u}_2|)^\vee\|_{L^2_tL^{r_2}_x}\lesssim\|u_2\|_{L^2_{t,x}}.$$

On $\Omega_{1,\sigma}$, by the conditions $b>\frac{1}{2}$ and $b>\frac{k-\ell-1}{4}$, we have

$$\int_{\Omega_{1,\sigma}}\frac{|\widehat{u}||\widehat{u}_1||\widehat{u}_2|\langle\xi_1\rangle^{k-\ell}}{\langle\langle\sigma\rangle^b\sigma_1\rangle^{c_1}\langle\sigma_2\rangle^{b_1}\langle\xi_2\rangle^k}\lesssim\|u\|_{L^2_{t,x}}\left\|\left(\frac{\langle\xi_1\rangle^{m_1}|\widehat{u}_1|}{\langle\sigma_1\rangle^{c_1}}\right)^\vee\right\|_{L^{q_1}_tL^{r_1}_x}\left\|\left(\frac{|\widehat{u}_2|}{\langle\xi_2\rangle^k\langle\sigma_2\rangle^{b_1}}\right)^\vee\right\|_{L^{q_2}_tL^{r_2}_x},$$

where $q=2$, q_1, q_2, $r=2$, r_1, and r_2 satisfy (4.3) and $m_1=0$ for $k-\ell=2$, $m_1=k-\ell-2-4\delta$ for $2<k-\ell<3$, $m_1=1-4\delta$ for $k-\ell\geq 3$. Invoke Lemma 2.3, if (4.4) with

$\beta_1 = c_1, \theta_1 = \frac{3}{2} - \frac{2}{d}$ for $k-\ell=2$, $\theta_1 = 1$ for $k-\ell>2$, and $\widetilde{q}_1 = \frac{2}{m_1}$ holds for $j=1$, then we obtain

$$\|(\langle\xi_1\rangle^{m_1}\langle\sigma_1\rangle^{-c_1}|\widehat{u}_1|)^\vee\|_{L_t^{q_1}L_x^{r_1}} \lesssim \|u_1\|_{L^2_{t,x}}.$$

For the remaining terms, we can choose the suitable parameters β_2, θ_2, and $\widetilde{q}_2$ so that (q_2, r_2') satisfy (4.4). Then invoking Sobolev inequality and Lemma 2.3, we have

$$\|(\langle\xi_2\rangle^{-k}\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2}} \lesssim \|(\langle\xi_2\rangle^{-k+n(r_2,r_2')}\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2'}} \lesssim \|u_2\|_{L^2_{t,x}}.$$

For the region Ω_{1,σ_1}, it is similar to the proof in Lemma 4.4, so we omit the proof. The argument for the case $k<0$ and (4.31) is analogous to that of $k\geq 0$ and (4.30), thus we omit the proof. Also the proof for the case $d\geq 4$ is similar to that of the case $d=2,3$. Again we skip the proof. ∎

We now apply Lemmata 4.1-4.4 to obtain the multilinear estimates (3.1) and (3.2). By the limited conditions in Lemmata 4.1-4.4, we divide the region of LWP in Theorem 1.1 into several subregions. For $d=2,3$, we set

$$\begin{aligned}
D_1 &:= \{(k,\ell) : \max\{-\ell, \ell-2k, -2k\} \leq d/2,\ \max\{|k|-\ell, \ell-k\} < 2\},\\
D_2 &:= \{(k,\ell) : \ell \geq -d/2,\ 2 \leq k-\ell < 5/2\},\\
D_3 &:= \{(k,\ell) : \ell-2k \leq d/2,\ -4 < k-\ell \leq -2\},\\
D_4 &:= \{(k,\ell) : \max\{\ell, 2k\} \geq -3/2,\ k+\ell \leq -2\}.
\end{aligned} \tag{4.32}$$

Note $R_{QZ,2} = D_1 \cup D_2 \cup D_3$ and $R_{QZ,3} = D_1 \cup D_2 \cup D_3 \cup D_4$. For $d\geq 4$, we set

$$\begin{aligned}
D_5 &:= \{(k,\ell) : \max\{-\ell, \ell-2k\} < 4-d/2,\ k>-1,\ \max\{|k|-\ell, \ell-k\} < 2\},\\
D_6 &:= \{(k,\ell) : k-2\ell < 6-d/2,\ 2 \leq k-\ell < 5/2\},\\
D_7 &:= \{(k,\ell) : \ell-2k < 4-d/2,\ -4 < k-\ell \leq -2\}.\\
D_8 &:= \{(k,\ell) : k>-1,\ 2 \leq -k-\ell < \ell+6-d/2\},
\end{aligned} \tag{4.33}$$

Note $R_{QZ,d} = D_5 \cup D_6 \cup D_7 \cup D_8$ for $d\geq 4$, which is given in (1.11). Also note that the subregion $D_8 = \emptyset$ for $d \geq 6$.

LEMMA 4.6. *For $d=2,3$, we have that the multilinear estimates (3.1) and (3.2) hold provided*

$b_1, b > 1/2, c_1 \geq (\lvert k\rvert-\ell)/4, c \geq (\ell-k)/4, c_1, c > d/8$	*for* $(k,\ell) \in D_1$,
$b, c_1 > 1/2, c_1 \geq (k-\ell)/4 > b_1 > (k-\ell-1)/4, b_1 > d/8, c > 1/4$	*for* $(k,\ell) \in D_2$,
$b_1 > 1/2, b_1, c \geq (\ell-k)/4, b, c_1 \geq 0$	*for* $(k,\ell) \in D_3$,
$b_1, b > 1/2, c > 3/8, b_1 \geq (-k-\ell)/4 > c_1 > 3/8$	*for* $(k,\ell) \in D_4$.

Proof. By (3.15), we have $S = S_1 + S_2 + S_3$ and $W = W_1 + W_2 + W_3$. For $(k,\ell) \in D_1$, we treat S_1 by Lemma 4.3(*a*) with $m = k-\ell$, S_2 by Lemma 4.3(*a*) with $m=-\ell$, and S_3 by Lemma 4.1(*a*) with $m=-\ell$. For $W_1 - W_3$, we separate into two cases: $k\geq 0$ and $k<0$. For $k\geq 0$, we treat W_1 and W_2 by Lemma 4.3(c) with $m=\ell-k$, and W_3 by Lemma 4.1(c) with $m = \ell-2k$. Thus, for $k\geq 0$, we obtain the estimates (3.1) and (3.2) provided $-\ell, \ell-2k \leq \frac{d}{2}, b_1, b > \frac{1}{2}, c_1 \geq \frac{k-\ell}{4}, c_1, c > \frac{d}{8}, c \geq \frac{\ell-k}{4}$.

For $k<0$, we treat W_1 and W_2 by Lemma 4.3(c) with $m=\ell-2k$. For W_3, we have $A_W \lesssim \langle\xi_1\rangle^{\ell-2k}$ for $\ell \geq 0$ and $A_W \lesssim \langle\xi_1\rangle^{-2k}$ for $\ell<0$. We may obtain the corresponding inequality for W_3 by the same argument in the proof of Lemma 4.1. Thus, for $k<0$, we assume that $-\ell, \ell-2k, -2k \le \frac{d}{2}, b_1, b > \frac{1}{2}, c_1 \ge \frac{-k-\ell}{4}, c_1, c > \frac{d}{8}$.

For D_2, we invoke Lemmata 4.5(a), 4.3(b) and 4.1(b) to treat S_1, S_2 and S_3, respectively. Therefore, we have that the estimate (3.1) holds provided $b, c_1 > \frac{1}{2}, c_1 \ge \frac{k-\ell}{4} > b_1 > \frac{k-\ell-1}{4}, b_1 > \frac{d}{8}$. For $W_1 - W_3$, we have $A_W \lesssim 1$. Thus, by $c > \frac{1}{4}, b_1 > \frac{d}{8}$, and the Hölder inequality, we obtain

$$W \lesssim \|(\langle\sigma\rangle^{-(\frac{1}{4}+\delta)}|\widehat{u}|)^\vee\|_{L_t^q L_x^r} \|(\langle\sigma_1\rangle^{-(\frac{d}{8}+\delta)}|\widehat{u}_1|)^\vee\|_{L_t^{q_1} L_x^{r_1}} \|(\langle\sigma_2\rangle^{-(\frac{d}{8}+\delta)}|\widehat{u}_2|)^\vee\|_{L_t^{q_1} L_x^{r_1}},$$

where $q = \frac{4}{d-1}, q_1 = \frac{8}{5-d}, \rho(r) = \frac{d-2}{2}$ and $\rho(r_1) = \frac{1}{2}$. Then invoke Lemma 2.4 with $\beta_0 = \frac{1}{2} + 2\delta, \eta = 3-d$ to process the first norm and Lemma 2.3 with $\beta = \frac{d+8\delta}{2d}, \theta = 1 - \frac{4}{d^2}, \tilde{q} = \infty$ to process the last two norms, so we have $W \lesssim \|u\|_{L_{t,x}^2} \|u_1\|_{L_{t,x}^2} \|u_2\|_{L_{t,x}^2}$.

For D_3, we have the fact that $\ell \ge 2 + k \ge 4 - \frac{d}{2} > \frac{d}{2}$, and thus $A_S \lesssim \langle\xi\rangle^{-(\frac{d}{2}+)}$. Therefore, by $b_1 > \frac{1}{2}, b, c_1 \ge 0$, the Hölder inequality and Sobolev inequality, we obtain

$$S \lesssim \|(\langle\xi\rangle^{-(\frac{d}{2}+)}|\widehat{u}|)^\vee\|_{L_t^2 L_x^\infty} \|u_1\|_{L_t^2 L_x^2} \|(\langle\sigma_2\rangle^{-(\frac{1}{2}+)}|\widehat{u}_2|)^\vee\|_{L_t^\infty L_x^2} \lesssim \|u\|_{L_{t,x}^2} \|u_1\|_{L_{t,x}^2} \|u_2\|_{L_{t,x}^2}.$$

Invoke Lemma 4.3(c) to process W_1 and W_2 and Lemma 4.1(c) to process W_3. Thus, we obtain $W \lesssim \|u\|_{L_{t,x}^2} \|u_1\|_{L_{t,x}^2} \|u_2\|_{L_{t,x}^2}$ provided $b_1 > \frac{1}{2}$ and $b_1, c \ge \frac{\ell-k}{4}$.

For $(k,\ell) \in D_4$, we apply Lemmata 4.3(a), 4.1(a) and 4.5(b) to treat S_1, S_3 and S_2 respectively. Thus, $S \lesssim \|u\|_{L_{t,x}^2} \|u_1\|_{L_{t,x}^2} \|u_2\|_{L_{t,x}^2}$ provided $b_1, b > \frac{1}{2}, b_1 \ge \frac{-k-\ell}{4} > c_1 > \frac{3}{8}$. Then apply Lemmata 4.3(c) and 4.1(c) to obtain $W_1 + W_2 + W_3 \lesssim \|u\|_{L_{t,x}^2} \|u_1\|_{L_{t,x}^2} \|u_2\|_{L_{t,x}^2}$ provided $-2k \le \frac{3}{2}$, $b_1 > \frac{1}{2}$, and $c > \frac{3}{8}$. □

LEMMA 4.7. *For $d \ge 4$, we have that the multilinear estimates (3.1) and (3.2) hold provided*

$$\begin{aligned} &b_1, b > 1/2, c_1 \ge (|k|-\ell)/4, c_1 > d/8 - 1/2 - \ell/4, c \ge (\ell-k)/4, \\ &c > d/8 - 1/2 + (\ell-2k)/4, c > -k/2, c, c_1 \ge 0 && \text{for} \quad (k,\ell) \in D_5, \\ &b, c_1 > 1/2, c_1 \ge (k-\ell)/4 > b_1 > (k-\ell-1)/4, b_1 > d/8 - 1/2 - \ell/4, c \ge 0 && \text{for} \quad (k,\ell) \in D_6, \\ &b_1 > 1/2, b_1, c \ge (\ell-k)/4, c > d/8 - 1/2 + (\ell-2k)/4, b, c_1 \ge 0 && \text{for} \quad (k,\ell) \in D_7, \\ &b_1, b > 1/2, b_1 \ge (-k-\ell)/4 > c_1 > d/8 - 1/2 - \ell/4, c > -k/2 && \text{for} \quad (k,\ell) \in D_8. \end{aligned}$$

Proof. For $(k,\ell) \in D_5$, we separate into two cases: $k \ge 0$ and $k < 0$. For $k \ge 0$, we treat S_1 by Lemma 4.4(i) with $m = k - \ell$ and $m_1 = k$, S_2 by Lemma 4.3(i) with $m = -\ell$, and S_3 by Lemma 4.1(i) with $m = -\ell$. For $W_1 - W_3$, we treat W_1 and W_2 by Lemma 4.4(ii) with $m = \ell - k$ and $m_1 = k$, and W_3 by Lemma 4.1(iii) with $m = \ell - 2k$. Thus, we assume that $-\ell, \ell - 2k \le 4 - \frac{d}{2}, b, b_1 > \frac{1}{2}, c_1 \ge \frac{k-\ell}{4}, c_1 > \frac{d}{8} - \frac{1}{2} - \frac{\ell}{4}, c \ge \frac{\ell-k}{4}, c > \frac{d}{8} - \frac{1}{2} + \frac{\ell-2k}{4}, c_1, c \ge 0$.

For $k < 0$, we treat S_1 by Lemma 4.3(i) with $m = -\ell$, S_2 by Lemma 4.4(i) with $m = -k-\ell$ and $m_1 = -k$, and S_3 by Lemma 4.1(i) with $m = -\ell$. For $W_1 - W_3$, we treat W_1 and W_2 by Lemma 4.3(iii) with $m = \ell - 2k$. For W_3, we separate into two subregions: $U_1 = \Omega_3 \cap \{|\xi_2| \le 2|\xi|\}$ and $U_2 = \Omega_3 \cap \{|\xi_2| > 2|\xi|\}$. For U_1, we have $A_W \lesssim \langle\xi\rangle^{\ell-2k}$, and then the corresponding integral be treated by Lemma 4.1(iii) with $m = \ell - 2k$. For U_2, we have $A_W \sim \langle\xi\rangle^{\ell}\langle\xi_1\rangle^{-2k}$, and then the corresponding integral be treated by Lemma 4.2. Thus, we assume that $-\ell, \ell - 2k \le 4 - \frac{d}{2}, k > -1; b, b_1 > \frac{1}{2}, c_1 \ge \frac{-k-\ell}{4}, c_1 > \frac{d}{8} - \frac{1}{2} - \frac{\ell}{4}, c > \frac{-k}{2}, c > \frac{d}{8} - \frac{1}{2} + \frac{\ell-2k}{4}, c_1, c \ge 0$.

For D_6, we apply Lemmata 4.5(i), 4.3(ii) and 4.1(ii) to treat S_1, S_2 and S_3 respectively. Thus, we get $-\ell \le 4-\frac{d}{2}, b, c_1 > \frac{1}{2}, c_1 \ge \frac{k-\ell}{4} > b_1 > \frac{k-\ell-1}{4}, b > \frac{k-\ell-1}{4}, b_1 > \frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}$. For W_1, we have $A_W \lesssim \langle \xi_2\rangle^{-\frac{d}{2}}$. Thus, by $b_1 > \frac{1}{4}, c \ge 0$ and the Hölder inequality, we obtain

$$W_1 \lesssim \|u\|_{L_t^2 L_x^2} \|(\langle \sigma_1\rangle^{-(\frac{1}{4}+\delta)}|\widehat{u}_1|)^\vee\|_{L_t^4 L_x^{r_1}} \|(\langle \xi\rangle_2^{-\frac{d}{2}} \langle \sigma_2\rangle^{-(\frac{1}{4}+\delta)}|\widehat{u}_2|)^\vee\|_{L_t^4 L_x^{r_2}},$$

where $\rho(r_1)=2\delta$ and $\rho(r_2)=\frac{d}{2}-2\delta$. Then invoke Lemma 2.3 with $\beta=\frac{1+4\delta}{2+4\delta}, \theta=0, \tilde{q}=\frac{1+2\delta}{\delta}$ to process the second norm, and treat the last norm by the Sobolev inequality, and thus we have $W_1 \lesssim \|u\|_{L^2_{t,x}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}$. The estimates of W_2 and W_3 are similar as W_1, so we obtain $W \lesssim \|u\|_{L^2_{t,x}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}$.

For D_7, we have the fact that $\ell > \frac{d}{2}$, so $A_S \lesssim \langle \xi_2\rangle^{-(\frac{d}{2}+\delta)}$ on $\Omega_1 \cup \Omega_2$, $\lesssim \langle \xi\rangle^{-(\frac{d}{2}+\delta)}$ on Ω_3. For S_1 and S_2, by $b_1 > \frac{1}{2}, b, c_1 \ge 0$, the Hölder inequality and Sobolev inequality, we obtain

$$S_1+S_2 \lesssim \|u\|_{L_t^2 L_x^2} \|u_1\|_{L_t^2 L_x^2} \|(\frac{|\widehat{u}_2|}{\langle \xi_2\rangle^{\frac{d}{2}+\delta}\langle \sigma_2\rangle^{\frac{1}{2}+\delta}})^\vee\|_{L_t^\infty L_x^\infty} \lesssim \|u\|_{L^2_{t,x}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}.$$

The estimate of S_3 is similar as S_1, so we have $S \lesssim \|u\|_{L^2_{t,x}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}$. For $W_1 - W_3$, invoke Lemma 4.4(ii) and 4.1(iii) respectively, and thus we obtain $W \lesssim \|u\|_{L^2_{t,x}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}$ provided $\ell - 2k \le 4-\frac{d}{2}, b_1 > \frac{1}{2}, b_1, c \ge \frac{\ell-k}{4}, c > \frac{d}{8}-\frac{1}{2}-\frac{\ell-2k}{4}$.

For $(k,\ell) \in D_8$, the estimates for $S_1, S_3, W_1 - W_3$ are the same as the estimates in D_5. Moreover, the estimate of S_2 is dealt with Lemma 4.5(b), so we assume that $b_1, b > \frac{1}{2}, b_1 \ge \frac{-k-\ell}{4} > c_1 > \frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}$ and $c > \frac{-k}{2}$. □

5. Multilinear estimates for Y-norm

We have to choose $c_1 \ge \frac{1}{2}$ when we derive the multilinear estimates on Ω_1 for the Schrödinger case with $k-\ell \ge 2$. The situation for c in the multilinear estimates for the wave case with $\ell - k \ge 2$ is analogous. Thus, we need to verify the estimates (3.3) and (3.4).

LEMMA 5.1. *Let* $2 \le k-\ell \le 4$ *and* $-\ell \le \frac{d}{2}$ *for* $d=2,3$, $-\ell < 4-\frac{d}{2}$ *for* $d \ge 4$. *If* $b > \frac{1}{2}$, $b \ge \frac{k-\ell}{4} > b_1 > \frac{k-\ell-1}{4}$, *and* $b_1 > \frac{d}{8}$ *for* $d=2,3$, $b_1 > \frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}$ *for* $d \ge 4$, *then the estimate* (3.3) *holds.*

Proof. We first consider the integral (3.11) on Ω_1. By $b > \frac{1}{2}$, (3.18), and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\widetilde{S}_{1,\sigma_1} &\lesssim \int_{\Omega_{1,\sigma_1}} \frac{|\widehat{u}(\tau,\xi)|}{\langle \sigma\rangle^b} \frac{|\mathcal{F}v_1(\xi_1)|}{\langle \sigma_1\rangle^{a_1}} \frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle \sigma_2\rangle^{b_1}} \frac{\langle \xi\rangle^m}{\langle \xi_2\rangle^k} d\mu \\ &\lesssim \|u\|_{L^2_{t,x}} M^{\frac{1}{2}} \Big(\int \langle \sigma\rangle^{-2b} |\mathcal{F}v_1(\xi+\xi_2)\widehat{u}_2(\sigma_2 - |\xi_2|^2\langle \xi_2\rangle^2)|^2 d\nu\Big)^{\frac{1}{2}} \\ &\lesssim \|u\|_{L^2_{t,x}} M^{\frac{1}{2}} \|v_1\|_{L^2_x} \|u_2\|_{L^2_{t,x}}, \end{aligned} \tag{5.1}$$

where $m = k-\ell-4(1-a_1)$, $d\nu = d\sigma_2 d\xi_2 d\sigma d\xi$, and

$$M := \sup_{\sigma,\xi \in \Omega_{1,\sigma_1}} \int_{\Omega_{1,\sigma_1}} \langle \xi\rangle^{2m} \langle \xi_2\rangle^{-2k} \langle \sigma_1\rangle^{-2a_1} \langle \sigma_2\rangle^{-2b_1} d\sigma_2 d\xi_2. \tag{5.2}$$

To estimate the above supremum, we apply Lemma 2.5, if $a_1+b_1>\frac{1}{2}$, then

$$\begin{aligned}M &\lesssim \sup_{\sigma,\xi\in\Omega_{1,\sigma_1}}\langle\xi\rangle^{2m}\int_{|\xi_2|\le|\xi|}\langle\xi_2\rangle^{-2k}\langle\sigma+|\xi|\langle\xi\rangle-|\xi|^2\langle\xi\rangle^2+|\xi+\xi_2|^2\langle\xi+\xi_2\rangle^2\rangle^{-\alpha(a_1,b_1)}d\xi_2\\ &\lesssim \sup_{\xi}\langle\xi\rangle^{2m}\int_{|\xi_2|\le|\xi|}(1+|\xi_2|^2)^{-k}d\xi_2\\ &\lesssim \sup_{\xi}\langle\xi\rangle^{2m+[d-2k]_+}<\infty, \end{aligned} \tag{5.3}$$

where $\alpha(a_1,b_1)=2\min\{a_1,b_1\}-[1-2\max\{a_1,b_1\}]_+\ge 0$ and $2m+[d-2k]_+\le 0$. One may verify the conditions $a_1+b_1>\frac{1}{2}$ and $2m+[d-2k]_+\le 0$ under the assumptions in Lemma 5.1.

For the subregion $\Omega_{1,\sigma}$, we may treat the corresponding integral $\widetilde{S}_{1,\sigma}$ as in the proof of Lemma 4.4 with $a_1<\frac{1}{2}$ since we assume that $-\ell<4-\frac{d}{2}$. So in the subregion Ω_{1,σ_2}, we may treat the corresponding integral $\widetilde{S}_{1,\sigma_2}$ with $a_1<\frac{1}{2}$ by the proof of Lemma 4.5. Therefore, we have

$$\widetilde{S}_{1,\sigma}+\widetilde{S}_{1,\sigma_2}\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}\lesssim\|u\|_{L^2_{t,x}}\|v_1\|_{L^2_x}\|u_2\|_{L^2_{t,x}}.$$

For $\widetilde{S}_2$ and $\widetilde{S}_3$, we have the same argument as $\widetilde{S}_{1,\sigma}$. Thus, by Lemmata 4.1 and 4.3 respectively, we can take $a_1<\frac{1}{2}$ so that the following inequality

$$\widetilde{S}_j\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}\lesssim\|u\|_{L^2_{t,x}}\|v_1\|_{L^2_x}\|u_2\|_{L^2_{t,x}}$$

still holds for $j=2,3$. We complete the proof. ∎

LEMMA 5.2. *Let $2\le\ell-k\le 4$ and $\ell-2k\le\frac{d}{2}$ for $d=2,3$, $\ell-2k<4-\frac{d}{2}$ for $d\ge 4$. If $b_1>\frac{1}{2}$, $b_1\ge\frac{\ell-2k}{4}+\frac{d}{8}$ for $k<\frac{d}{2}$, $b_1>\frac{\ell-k}{4}$ for $k=\frac{d}{2}$, and $b_1\ge\frac{\ell-k}{4}+\frac{d}{8}$ for $k>\frac{d}{2}$, then the estimate (3.4) holds.*

Proof. For $\widetilde{W}_3$, invoke Lemma 4.1(c) with $m=\ell-2k$, $a=\frac{d}{8}$, and $a_1=a_2=b_1$ for $d=2,3$, and Lemma 4.1(iii) with $m=\ell-2k$, $a=\frac{d}{8}-\frac{1}{2}+\frac{\ell-2k}{4}+\delta$, and $a_1=a_2=b_1$ for $d\ge 4$, then we have

$$\widetilde{W}_3\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}\lesssim\|v\|_{L^2_x}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}.$$

Note that we can take $a<\frac{1}{2}$ for $d\ge 4$ since $\ell-2k<4-\frac{d}{2}$.

For the integral $\widetilde{W}_1$, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\widetilde{W}_1 &\lesssim\int_{\Omega_1}\frac{|\widehat{v}(\xi)|}{\langle\sigma\rangle}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}\frac{\langle\xi\rangle^{\ell-k}}{\langle\xi_2\rangle^{k}}d\mu\\ &\lesssim\|v\|_{L^2}M^{\frac{1}{2}}\left(\int_{\Omega_1}|\widehat{u}_1(\sigma-|\xi|\langle\xi\rangle+\sigma_2-|\xi_2|^2\langle\xi_2\rangle^2,\xi+\xi_2)\widehat{u}_2(\sigma_2-|\xi_2|^2\langle\xi_2\rangle^2,\xi_2)|^2d\nu\right)^{\frac{1}{2}}\\ &\lesssim\|v\|_{L^2_x}M^{\frac{1}{2}}\|u_1\|_{L^2}\|u_2\|_{L^2}\end{aligned}$$

where $d\nu=d\sigma_2d\xi_2d\sigma d\xi$ and

$$M:=\sup_{\xi\in\Omega_1}\int_{\Omega_1}\frac{\langle\xi\rangle^{2(\ell-k)}\langle\xi_2\rangle^{-2k}}{\langle\sigma\rangle^2\langle\sigma-|\xi|\langle\xi\rangle+\sigma_2-|\xi_2|^2\langle\xi_2\rangle^2+|\xi+\xi_2|^2\langle\xi+\xi_2\rangle^2\rangle^{2b_1}\langle\sigma_2\rangle^{2b_1}}d\sigma_2d\xi_2d\sigma.$$

To estimate the above supremum, we apply Lemma 2.5 twice to obtain

$$\begin{aligned} M &\lesssim \sup_{\xi\in\Omega_1} \langle\xi\rangle^{2(\ell-k)} \int \frac{1}{\langle\xi_2\rangle^{2k}} \int \frac{1}{\langle\sigma\rangle^2 \langle \sigma - |\xi|\langle\xi\rangle - |\xi_2|^2\langle\xi_2\rangle^2 + |\xi+\xi_2|^2\langle\xi+\xi_2\rangle^2\rangle^{2b_1}} d\sigma d\xi_2 \\ &\lesssim \sup_{\xi} \langle\xi\rangle^{2(\ell-k)} \int \langle\xi_2\rangle^{-2k} \langle |\xi+\xi_2|^2\langle\xi+\xi_2\rangle^2 - |\xi|\langle\xi\rangle - |\xi_2|^2\langle\xi_2\rangle^2\rangle^{-2\mu} d\xi_2 \\ &\lesssim \sup_{\xi} \langle\xi\rangle^{2(\ell-k)-8\mu+[d-2k]_+} < \infty, \end{aligned} \tag{5.4}$$

where $\mu := \min\{b_1, 1\}$ and $2(\ell-k)-8\mu+[d-2k]_+ \le 0$. One may check the condition $2(\ell-k)-8\mu+[d-2k]_+ \le 0$ easily, so we skip this step.

For the integral $\widetilde{W}_2$, through the same argument, we can obtain

$$\widetilde{W}_2 \lesssim \int_{\Omega_2} \frac{|\widehat{v}(\xi)|}{\langle\sigma\rangle} \frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{b_1}} \frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}} \frac{\langle\xi\rangle^{\ell-k}}{\langle\xi_1\rangle^k} d\mu \lesssim \|v\|_{L^2_x} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}.$$

□

6. Proof of the main Theorem

Following the solution formulae (2.2) and (2.5), we can construct the iteration map

$$\Lambda(E,\mathcal{N}) := \big(\Phi(E,\mathcal{N}), \Psi(E)\big), \tag{6.1}$$

where

$$\begin{aligned} \Phi(E,\mathcal{N}) &:= \psi_1(t)U(t)E_0 + \psi_T(t)U *_R \big(\mathcal{R}e(\mathcal{N})E\big), \\ \Psi(E) &:= \psi_1(t)W(t)\mathcal{N}_0 + \psi_T(t)W *_R \big(\mathcal{D}\langle\mathcal{D}\rangle^{-1}|E|^2\big). \end{aligned} \tag{6.2}$$

We want to find a fixed point in a set B of an appropriate Banach space. We choose

$$B := \{(E,\mathcal{N}) : \|E\|_{X^S_{k,b_1}} \le M_1, \ \|\mathcal{N}\|_{X^W_{\ell,b}} \le M_2\},$$

where $M_1 = 2\|\psi\|_{H^{b_1}}\|E_0\|_{H^k} + 1$ and $M_2 = 2\|\psi\|_{H^b}\|\mathcal{N}_0\|_{H^\ell} + 1$.

Proof. **(Proof of Theorem 1.1 for d=2, 3.)** For $(k,\ell) \in D_1$, we can choose $c_1 = \max\{\frac{d}{8}+\delta, \frac{|k|-\ell}{4}\}$, $c = \max\{\frac{d}{8}+\delta, \frac{\ell-k}{4}\}$, $b_1 > \frac{1}{2}$, and $b > \frac{1}{2}$ such that $b_1 + c_1 < 1$ and $b + c < 1$. Invoking Lemmata 2.1(S2), 2.2(S2), and 4.6, we can derive

$$\begin{aligned} \|\Phi(E,\mathcal{N})\|_{X^S_{k,b_1}} &\lesssim \|\psi\|_{H^{b_1}}\|E_0\|_{H^k} + T^{1-b_1-c_1}\|\mathcal{R}e(\mathcal{N})E\|_{X^S_{k,-c_1}} \\ &\lesssim M_1/2 + T^{1-b_1-c_1} M_2 M_1. \end{aligned} \tag{6.3}$$

Invoking Lemmata 2.1(W2), 2.2(W2), and 4.6, we can obtain

$$\begin{aligned} \|\Psi(E)\|_{X^W_{\ell,b}} &\lesssim \|\psi\|_{H^{b_1}}\|\mathcal{N}_0\|_{H^\ell} + T^{1-b-c}\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}|E|^2\|_{X^S_{k,-c_1}} \\ &\lesssim M_2/2 + T^{1-b-c} M_1^2. \end{aligned} \tag{6.4}$$

Also with the aid of Lemmata 2.2(S2), 2.2(W2), and 4.6, we can estimate the differences

$$\|\Phi(E_1,\mathcal{N}_1) - \Phi(E_2,\mathcal{N}_2)\|_{X^S_{k,b_1}} \lesssim T^{1-b_1-c_1}\big(M_1+M_2\big)\big(\|\mathcal{N}_1-\mathcal{N}_2\|_{X^W_{\ell,b}} + \|E_1-E_2\|_{X^S_{k,b_1}}\big) \tag{6.5}$$

and

$$\|\Psi(E_1)-\Psi(E_2)\|_{X^W_{\ell,b}} \lesssim T^{1-b-c}M_1\|E_1-E_2\|_{X^S_{k,b_1}}. \tag{6.6}$$

Hence Λ is a contraction map on B if we choose T sufficiently small such that

$$T^{1-b_1-c_1}\big(M_1+M_2\big)\ll 1, \quad T^{1-b-c}M_1^2/M_2\ll 1, \quad \text{and} \quad T^{1-b-c}M_1\ll 1. \tag{6.7}$$

For $(k,\ell)\in D_2$, we can choose $b=c_1=\max\{\frac{1}{2}+\delta, \frac{k-\ell}{4}\}$, $c=\frac{1}{4}+\delta$, and $b_1=\frac{3}{8}+\delta$ such that $b_1+c_1<1$ and $b+c<1$. Invoking Lemmata 2.1(S2), 2.2(S1), 4.6, and 5.1, we can derive

$$\begin{aligned}\|\Phi(E,\mathcal{N})\|_{X^S_{k,b_1}} &\lesssim \|\psi\|_{H^{b_1}}\|E_0\|_{H^k}+T^{1-b_1-c_1}\|\mathcal{R}e(\mathcal{N})E\|_{X^S_{k,-c_1}}+T^{\frac{1}{2}-b_1}\|\mathcal{R}e(\mathcal{N})E\|_{Y^S_k}\\ &\lesssim M_1/2+T^{1-b_1-c_1}M_2M_1.\end{aligned}$$

Invoking Lemmata 2.1(W2), 2.2(W2), and 4.6, we can obtain (6.4). Also with the aid of Lemmata 2.2(S1), 2.2(W2), 4.6, and 5.1, we can estimate the differences

$$\begin{aligned}&\|\Phi(E_1,\mathcal{N}_1)-\Phi(E_2,\mathcal{N}_2)\|_{X^S_{k,b_1}}\\ &\lesssim T^{1-b_1-c_1}\big(\|\mathcal{N}_1-\mathcal{N}_2\|_{X^W_{\ell,b}}\|E_1\|_{X^S_{k,b_1}}+\|\mathcal{N}_2\|_{X^W_{\ell,b}}\|E_1-E_2\|_{X^S_{k,b_1}}\big)\\ &\quad +T^{\frac{1}{2}-b_1}\big(\|\mathcal{R}e(\mathcal{N}_1)(E_1-E_2)\|_{Y^S_k}+\|\mathcal{R}e(\mathcal{N}_1-\mathcal{N}_2)E_2\|_{Y^S_k}\big)\\ &\lesssim T^{1-b_1-c_1}\big(M_1+M_2\big)\big(\|\mathcal{N}_1-\mathcal{N}_2\|_{X^W_{\ell,b}}+\|E_1-E_2\|_{X^S_{k,b_1}}\big)\end{aligned}$$

and (6.6) holds. Hence Λ is a contraction map on B if we choose T such that (6.7) holds.

For $(k,\ell)\in D_3$, we choose $b=c_1=0$, $c=\frac{\ell-k}{4}$, and $b_1=\max\{\frac{\ell-2k}{4}+\frac{d}{8},\frac{1}{2}+\delta\}$ for $k<\frac{d}{2}$, $\frac{\ell-k}{4}+\delta$ for $k=\frac{d}{2}$, $\frac{\ell-k}{4}$ for $k>\frac{d}{2}$. Thus we have $b_1+c_1<1$ and $b+c<1$. Invoking Lemmata 2.1(S2), 2.2(S2), and 4.6, we derive (6.3). Invoking Lemmata 2.1(W2), 2.2(W1), 4.6, and 5.2, we obtain

$$\begin{aligned}\|\Psi(E)\|_{X^W_{\ell,b}} &\lesssim \|\psi\|_{H^{b_1}}\|\mathcal{N}_0\|_{H^\ell}+T^{1-b-c}\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}|E|^2\|_{X^S_{k,-c_1}}+T^{\frac{1}{2}-b}\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}|E|^2\|_{Y^W_\ell}\\ &\lesssim M_2/2+T^{1-b-c}M_1^2.\end{aligned}$$

With the aid of Lemmata 2.2(S2), 2.2(W1), 4.6, and 5.2, we estimate the differences to get (6.5) and

$$\begin{aligned}&\|\Psi(E_1)-\Psi(E_2)\|_{X^W_{\ell,b}}\\ &\lesssim T^{1-b-c}\Big(\|E_1-E_2\|_{X^S_{k,b_1}}\|E_1\|_{X^S_{k,b_1}}+\|E_2\|_{X^S_{k,b_1}}\|E_1-E_2\|_{X^S_{k,b_1}}\big)+\\ &\quad T^{\frac{1}{2}-b}\big(\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}E_1(\bar{E}_1-\bar{E}_2)\|_{Y^W_\ell}+\|\mathcal{D}\langle\mathcal{D}\rangle^{-1}(E_1-E_2)\bar{E}_2\|_{Y^W_\ell}\big)\\ &\lesssim T^{1-b-c}M_1\|E_1-E_2\|_{X^S_{k,b_1}}.\end{aligned}$$

Hence Λ is a contraction map on B if we choose T sufficiently small such that (6.7) holds.

For $(k,\ell)\in D_4$, we can choose $b_1=\max\{\frac{1}{2}+\delta, \frac{-\ell-k}{4}\}$, $c=c_1=\frac{3}{8}+\delta$, and $b=\frac{1}{2}+\delta$ such that $b_1+c_1<1$ and $b+c<1$. The remaining argument here is the same with that of the case for D_1.

Since $X^\phi_{p,q}\subset C([0,T];H^p)$ for $q>\frac{1}{2}$ and Lemma 2.2 (S3), (W3) for $q\le\frac{1}{2}$, then we have $(E,\mathcal{N})\in C([0,T];H^k)\times C([0,T];H^\ell)$. The continuity of the solution map can be shown in the standard way by using the same estimates on difference. □

Proof. **(Proof of Theorem 1.1 for $d \geq 4$.)** For $(k,\ell)\in D_5$, we can choose $c_1 = \max\{\frac{|k|-\ell}{4}, \frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}+\delta, \delta\} < \frac{1}{2}$, $c = \max\{\frac{\ell-k}{4}, \frac{d}{8}-\frac{1}{2}+\frac{\ell-2k}{4}+\delta, -\frac{k}{2}+\delta, \delta\} < \frac{1}{2}$, $b_1 > \frac{1}{2}$, and $b > \frac{1}{2}$. For $(k,\ell)\in D_6$, we can choose $b_1 = \max\{\frac{k-\ell-1}{4}+\delta, \frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}+\delta\}$, $c_1 = \max\{\frac{k-\ell}{4}, \frac{1}{2}+\delta\}$, $b = \max\{\frac{k-\ell}{4}, \frac{1}{2}+\delta\}$, and $c = \frac{1}{8}$. map on B if we choose T sufficiently small such that (6.7) holds. For $(k,\ell)\in D_7$, we can choose $b_1 = \max\{\frac{1}{2}+\delta, \frac{\ell-2k}{4}+\frac{d}{8}\}$ for $k<\frac{d}{2}$, $\max\{\frac{1}{2}+\delta, \frac{\ell-k}{4}+\delta\}$ for $k=\frac{d}{2}$, $\max\{\frac{1}{2}+\delta, \frac{\ell-k}{4}\}$ for $k>\frac{d}{2}$, $c = \max\{\frac{\ell-k}{4}, \frac{d}{8}-\frac{1}{2}+\frac{\ell-2k}{4}+\delta\}$, $c_1>0$, and $b=0$. For $(k,\ell)\in D_8$, we can choose $c_1 = \frac{d}{8}-\frac{1}{2}-\frac{\ell}{4}+\delta<\frac{1}{2}$, $b_1 = \max\{\frac{-\ell-k}{4}, \frac{1}{2}+\delta\}$, $c = \frac{-k}{2}+\delta < \frac{1}{2} < b$.
Here we obtain $b_1+c_1<1$ and $b+c<1$ for all $(k,\ell)\in R_{QZ,d}$. Thus, we can derive that Λ is a contraction map on B if we choose T sufficiently small such that (6.7) holds. ∎

7. Critical cases for quantum Zakharov system

In this section, we fill up some boundary of the region of LWP for (1.6) which is not included in the statement of Theorem 1.1. The method for Theorem 1.1 fails, due to the fact that the related estimates can no longer provide any power of T for the boundary case. Two cases are particularly of interest. At the mass level, the system (1.6) in $\mathbb{R}^8$ is critical, see (1.9). At the energy level, (1.6) in $\mathbb{R}^{10}$ is critical in the energy space. It worth mentioning that the corresponding homogenized system which can be regarded as a high frequency limit of (1.6), does not have ground states for $d=10$. The readers are also referred to [5] and [9] for more discussion.

Proof. **(Proof of Theorem 1.2.)** First we list some B-admissible pairs like $(2, \frac{2d}{d-4})$, $(4, \frac{2d}{d-2})$, $(8, \frac{2d}{d-1})$, and $(\infty, 2)$; S-admissible pairs like $(2, \frac{2d}{d-2})$, $(4, \frac{2d}{d-1})$, and $(\infty, 2)$, see (2.10) and (2.11). We denote some function spaces as follows.

$$S := L_I^\infty L^2 \cap L_I^2 B^1_{\frac{2d}{d-2},2}, \quad W := L_I^\infty L^2 \cap L_I^2 L^{\frac{2d}{d-2}},$$

$$\widetilde{S} := L_I^2 L^{\frac{2d}{d-4}} \cap L_I^4 L^{\frac{2d}{d-2}}, \quad \widetilde{W} := L_I^\infty L^2 + L_I^4 L^{\frac{2d}{d-2}},$$

where the time interval $I=[0,T]$ and $B^s_{p,q}(\mathbb{R}^d)$ is the usual Besov space. Analogous with (6.1), we set the iteration map as follows.

$$\Lambda(E,\mathcal{N}) = \big(\langle\mathcal{D}\rangle^{\zeta+\sigma}\Phi(E,\mathcal{N}), \langle\mathcal{D}\rangle^{\zeta}\Psi(E)\big),$$

where $\zeta = \frac{d-8}{2}$, $0\leq\sigma\leq 1$, $\Phi(E,\mathcal{N})$, and $\Psi(E)$ are defined in (6.2). We also denote $\Psi(E)=\Psi(E,E)$ and skip the index I in the later discussion. Invoking Sobolev inequality and interpolation, we have

$$\|E\|_{L^2 L^{\frac{2d}{d-4}}} \lesssim \|\langle\mathcal{D}\rangle E\|_{L^2 L^{\frac{2d}{d-2}}} \quad \text{and} \quad \|E\|_{L^4 L^{\frac{2d}{d-2}}} \lesssim \|E\|^{1/2}_{L^\infty L^2} \|\langle\mathcal{D}\rangle E\|^{1/2}_{L^2 L^{\frac{2d}{d-2}}},$$

which implies that

$$\|E\|_{\widetilde{S}} \lesssim \|E\|_S \quad \text{and} \quad \|\mathcal{N}\|_{\widetilde{W}} \lesssim \|\mathcal{N}\|_{L^\infty L^2} \lesssim \|\mathcal{N}\|_W. \tag{7.1}$$

Now we want to derive the Duhamel estimate for (2.3). Invoking Strichartz estimate (2.13) and Sobolev inequality, we obtain

$$\begin{aligned}\|\langle\mathcal{D}\rangle^{\zeta+\sigma} U *_R (\mathcal{N}E)\|_S &\lesssim \|\langle\mathcal{D}\rangle^{\zeta+\sigma} U *_R (\mathcal{N}E)\|_{L^\infty L^2} + \|\langle\mathcal{D}\rangle^{\zeta+\sigma+\frac{2}{2}} U *_R (\mathcal{N}E)\|_{L^2 L^{\frac{2d}{d-2}}} \\ &\lesssim \|\langle\mathcal{D}\rangle^{\zeta+\sigma-\frac{2}{2}} (\mathcal{N}E)\|_{L^{2'} L^{(\frac{2d}{d-2})'}}\end{aligned}$$

For $\zeta+\sigma-1\geq 0$, we split the wave $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$ to proceed with the estimate

$$\begin{aligned}
&\|\langle\mathcal{D}\rangle^{\zeta+\sigma-1}\mathcal{N}E\|_{L^2L^{\frac{2d}{d+2}}}+\|\mathcal{N}\langle\mathcal{D}\rangle^{\zeta+\sigma-1}E\|_{L^2L^{\frac{2d}{d+2}}}\\
\lesssim&\|\langle\mathcal{D}\rangle^{\zeta+\sigma-1}\mathcal{N}_1\|_{L^\infty L^{\frac{2d}{d-2+2\sigma}}}\|E\|_{L^2L^{\frac{2d}{4-2\sigma}}}+\|\mathcal{N}_1\|_{L^\infty L^{\frac{2d}{d-2\zeta}}}\|\langle\mathcal{D}\rangle^{\zeta+\sigma-1}E\|_{L^2L^{\frac{2d}{d-6}}}\\
&+\|\langle\mathcal{D}\rangle^{\zeta+\sigma-1}\mathcal{N}_2\|_{L^4L^{\frac{2d}{d-4+2\sigma}}}\|E\|_{L^4L^{\frac{2d}{6-2\sigma}}}+\|\mathcal{N}_2\|_{L^4L^{\frac{2d}{6}}}\|\langle\mathcal{D}\rangle^{\zeta+\sigma-1}E\|_{L^4L^{\frac{2d}{d-4}}}\\
\lesssim&\left(\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}_1\|_{L^\infty L^2}+\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}_2\|_{L^4L^{\frac{2d}{d-2}}}\right)\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}},
\end{aligned}$$

which holds for all decompositions, $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$. Thus we have

$$\begin{aligned}
\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U*_R\left(\mathcal{R}e(\mathcal{N})E\right)\|_S\lesssim&\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}\|_{L^\infty L^2+L^4L^{\frac{2d}{d-2}}}\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}}\\
\lesssim&\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}\|_{\widetilde{W}}\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}}.
\end{aligned}\tag{7.2}$$

For $\zeta+\sigma-1<0$, we again split the wave $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$ to proceed with the estimate

$$\begin{aligned}
\|\langle\mathcal{D}\rangle^{\zeta+\sigma-1}\left(\mathcal{R}e(\mathcal{N})E\right)\|_{L^2L^{\frac{2d}{d+2}}}\lesssim&\|\mathcal{N}E\|_{L^2L^{\frac{2d}{12-2\sigma}}}\\
\lesssim&\left(\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}_1\|_{L^\infty L^2}+\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}_2\|_{L^4L^{\frac{2d}{d-2}}}\right)\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}},
\end{aligned}$$

which holds for all decompositions, $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$. Thus we have

$$\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U*_R\left(\mathcal{R}e(\mathcal{N})E\right)\|_S\lesssim\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}\|_{\widetilde{W}}\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}}.\tag{7.3}$$

Next we want to derive the Duhamel estimate for (2.6). Invoking Strichartz estimate (2.23) and Sobolev inequality, we obtain

$$\begin{aligned}
\|\langle\mathcal{D}\rangle^{\zeta}W*_R\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}|E|^2\right)\|_W\lesssim&\|\left(\mathcal{D}\langle\mathcal{D}\rangle^{-1}\right)^{\gamma_2}\langle\mathcal{D}\rangle^{\zeta}\mathcal{D}\langle\mathcal{D}\rangle^{-1}|E|^2\|_{L^{\infty'}L^{2'}}\\
\lesssim&\|\langle\mathcal{D}\rangle^{\zeta}E\|_{L^2L^{\frac{2d}{d-4}}}\|E\|_{L^2L^{\frac{2d}{4}}}\\
\lesssim&\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}}^2.
\end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned}
&\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U*_R\left(\mathcal{R}e(\mathcal{N})E\right)\|_{\widetilde{S}}+\|\langle\mathcal{D}\rangle^{\zeta}W*_R\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}|E|^2\right)\|_{\widetilde{W}}\\
\lesssim&\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U*_R\left(\mathcal{R}e(\mathcal{N})E\right)\|_S+\|\langle\mathcal{D}\rangle^{\zeta}W*_R\left(\frac{\mathcal{D}}{\langle\mathcal{D}\rangle}|E|^2\right)\|_W\\
\lesssim&\Big(\|\langle\mathcal{D}\rangle^{\zeta+\sigma}E\|_{\widetilde{S}}+\|\langle\mathcal{D}\rangle^{\zeta}\mathcal{N}\|_{\widetilde{W}}\Big)^2.
\end{aligned}\tag{7.4}$$

For the homogeneous estimates, we can apply (7.1), (2.12) and (2.23) to obtain

$$\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U(t)E_0\|_{\widetilde{S}}\lesssim\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U(t)E_0\|_S\lesssim\|E_0\|_{H^{\zeta+\sigma}}$$

and

$$\|\langle\mathcal{D}\rangle^{\zeta}W(t)\mathcal{N}_0\|_{\widetilde{W}}\lesssim\|\langle\mathcal{D}\rangle^{\zeta}W(t)\mathcal{N}_0\|_W\lesssim\|\mathcal{N}_0\|_{H^{\zeta}}.$$

Invoking the domination convergence theorem, we have

$$\lim_{T\to+0}\|\langle\mathcal{D}\rangle^{\zeta+\sigma}U(t)E_0\|_{\widetilde{S}}=0.$$

As for the free solution of wave, the situation is more subtle. For any $\varepsilon > 0$, there is a function $g \in H^{\zeta+\frac{1}{2}}$ such that $\|\mathcal{N}_0 - g\|_{H^\zeta} < \varepsilon$. Invoking Sobolev inequality and Strichartz estimate (2.23), we have

$$\|\langle \mathcal{D}\rangle^\zeta W(t)g\|_{L^4 L^{\frac{2d}{d-2}}} \lesssim \|\langle \mathcal{D}\rangle^{\zeta+\frac{1}{2}} W(t)g\|_{L^4 L^{\frac{2d}{d-1}}} \lesssim \|g\|_{H^{\zeta+\frac{1}{2}}} < \infty$$

which gives

$$\lim_{T\to+0} \|\langle \mathcal{D}\rangle^\zeta W(t)g\|_{L^4 L^{\frac{2d}{d-2}}} = 0.$$

Hence we can estimate

$$\|\langle \mathcal{D}\rangle^\zeta W(t)\mathcal{N}_0\|_{\widetilde{W}} \lesssim \|\mathcal{N}_0 - g\|_{H^\zeta} + \|\langle \mathcal{D}\rangle^\zeta W(t)g\|_{L^4 L^{\frac{2d}{d-2}}}. \tag{7.5}$$

Taking the limsup of the above estimate, we get

$$\limsup_{T\to+0} \|\langle \mathcal{D}\rangle^\zeta W(t)\mathcal{N}_0\|_{\widetilde{W}} \le \varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Thus we can conclude that

$$\lim_{T\to+0} \|\langle \mathcal{D}\rangle^\zeta W(t)\mathcal{N}_0\|_{\widetilde{W}} = 0.$$

Finally we can proceed with the iteration argument. Given $\delta > 0$, we can choose a sufficiently small $T^* > 0$ such that

$$\|\langle \mathcal{D}\rangle^{\zeta+\sigma} U(t)E_0\|_{\widetilde{S}} + \|\langle \mathcal{D}\rangle^\zeta W(t)\mathcal{N}_0\|_{\widetilde{W}} \le \delta/2 \quad \text{for all} \quad T < T^*. \tag{7.6}$$

We choose $\|(E_0, \mathcal{N}_0)\|_{H^{\zeta+\sigma}\times H^\zeta} \le M/2$ for some $M > 0$ and define

$$X_M = \{(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N}) \in S \times W : \|(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N})\|_{S\times W} \le M\}. \tag{7.7}$$

$$\widetilde{X}_\delta = \{(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N}) \in \widetilde{S} \times \widetilde{W} : (7.6) \text{ holds and } \|(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N})\|_{\widetilde{S}\times\widetilde{W}} \le \delta\}. \tag{7.8}$$

Note $X_\delta \subset \widetilde{X}_\delta$. Since

$$\|\Lambda\big(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N}\big)\|_{\widetilde{S}\times\widetilde{W}} \lesssim \|\langle \mathcal{D}\rangle^{\zeta+\sigma} \Phi(E,\mathcal{N})\|_{\widetilde{S}} + \|\langle \mathcal{D}\rangle^\zeta \Psi(E)\|_{\widetilde{W}} \lesssim \delta,$$

provided that $\frac{1}{2} + C\delta \le 1$. We can compute

$$\|\Lambda\big(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N}\big)\|_{\widetilde{S}\times\widetilde{W}} \lesssim \|\langle \mathcal{D}\rangle^{\zeta+\sigma} \Phi(E,\mathcal{N})\|_{\widetilde{S}} + \|\langle \mathcal{D}\rangle^\zeta \Psi(E)\|_{\widetilde{W}} \lesssim \delta,$$

provided that $C\delta \le 1/2$. Then we can compute

$$\|\Lambda\big(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle \mathcal{N}\big)\|_{S\times W} \lesssim \|\langle \mathcal{D}\rangle^{\zeta+\sigma} \Phi(E,\mathcal{N})\|_S + \|\langle \mathcal{D}\rangle^\zeta \Psi(E)\|_W \lesssim M,$$

provided that $C\delta^2 \le M/2$. We can also compute the difference

$$\begin{aligned} &\|\Lambda\big(\langle \mathcal{D}\rangle^{\zeta+\sigma} E, \langle \mathcal{D}\rangle^\zeta \mathcal{N}\big) - \Lambda\big(\langle \mathcal{D}\rangle^{\zeta+\sigma} E', \langle \mathcal{D}\rangle^\zeta \mathcal{N}'\big)\|_{S\times W} \\ &\lesssim \big(\|\langle \mathcal{D}\rangle^{\zeta+\sigma}(E-E')\|_{\widetilde{S}} + \|\langle \mathcal{D}\rangle^\zeta(\mathcal{N}-\mathcal{N}')\|_{\widetilde{W}}\big)\cdot \\ &\quad \big(\|\langle \mathcal{D}\rangle^{\zeta+\sigma} E\|_{\widetilde{S}} + \|\langle \mathcal{D}\rangle^{\zeta+\sigma} E'\|_{\widetilde{S}} + \|\langle \mathcal{D}\rangle^\zeta \mathcal{N}\|_{\widetilde{W}}\big) \\ &\lesssim \frac{1}{2}\big(\|\langle \mathcal{D}\rangle^{\zeta+\sigma}(E-E')\|_S + \|\langle \mathcal{D}\rangle^\zeta(\mathcal{N}-\mathcal{N}')\|_W\big), \end{aligned}$$

provided that $C\delta \le \frac{1}{2}$. Therefore Λ is a contraction on X_M. □

8. The improved result for Zakharov system

In the section, we only consider the improved region of local well-posedness for Zakharov system. Let $s,\alpha\in\mathbb{R}$, we define the Bourgain spaces $X^{S_0}_{s,\alpha}$, $X^{W_0}_{s,\alpha}$ and $Y^{W_0}_s$ with the norms

$$\|v\|_{X^{S_0}_{s,\alpha}} := \|\langle\xi\rangle^s\langle\tau+|\xi|^2\rangle^\alpha\widehat{v}(\tau,\xi)\|_{L^2_{\tau,\xi}}, \quad \|v\|_{X^{W_0}_{s,\alpha}} := \|\langle\xi\rangle^s\langle\tau+|\xi|\rangle^\alpha\widehat{v}(\tau,\xi)\|_{L^2_{\tau,\xi}}, \tag{8.1}$$

and

$$\|v\|_{Y^{W_0}_s} := \|\langle\xi\rangle^s\langle\tau+|\xi|\rangle^{-1}\widehat{v}(\tau,\xi)\|_{L^2_\xi L^1_\tau}. \tag{8.2}$$

The homogeneous estimates are given in [11]. For the multilinear estimates, we state the following lemma.

LEMMA 8.1 ([11], Lemma 3.1). *For $d\ge 2$, let $\beta>\frac{1}{2}$, $0\le\widetilde{a}\le\beta_0$, and $0<\eta\le 1$. Define q and r by $\frac{2}{q}=1-\eta\frac{\widetilde{a}}{\beta}$ and $\rho(r)=(1-\eta)\frac{\widetilde{a}}{\beta}$. Then*

$$\|(\langle\tau+|\xi|^2\rangle^{-\widetilde{a}}|\widehat{u}|)^\vee\|_{L^q_tL^r_x}\lesssim\|u\|_{L^2_{t,x}}. \tag{8.3}$$

The corresponding multilinear estimates are as follows.

$$\|\mathcal{N}E\|_{X^{S_0}_{k,-c_1}}+\|\bar{\mathcal{N}}E\|_{X^{S_0}_{k,-c_1}}\lesssim\|\mathcal{N}\|_{X^{W_0}_{\ell,b}}\|E\|_{X^{S_0}_{k,b_1}}. \tag{8.4}$$

$$\|\mathcal{D}(E_1\bar{E}_2)\|_{X^{W_0}_{\ell,-c}}\lesssim\|E_1\|_{X^{S_0}_{k,b_1}}\|E_2\|_{X^{S_0}_{k,b_1}}. \tag{8.5}$$

$$\|\mathcal{D}\big(E_1\bar{E}_2\big)\|_{Y^{W_0}_\ell}\lesssim\|E_1\|_{X^{S_0}_{k,b_1}}\|E_2\|_{X^{S_0}_{k,b_1}}. \tag{8.6}$$

The proofs for $\mathcal{N}E$ and $\bar{\mathcal{N}}E$ are similar, thus we only discuss the case of $\mathcal{N}E$. By the duality argument, we need to obtain the estimates $S_0+W_0\lesssim\|u\|_{L^2_{t,x}}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}$ and $\widetilde{W}_0\lesssim\|v\|_{L^2_x}\|u_1\|_{L^2_{t,x}}\|u_2\|_{L^2_{t,x}}$, where

$$S_0 := \int\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^b}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{c_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}A_S\,d\mu, \quad W_0 := \int\frac{|\widehat{u}(\tau,\xi)|}{\langle\sigma\rangle^c}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}|\xi|A_W\,d\mu,$$

and

$$\widetilde{W}_0 := \int\frac{|\mathcal{F}v(\xi)|}{\langle\sigma\rangle}\frac{|\widehat{u}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}}|\xi|A_W\,d\mu,$$

$d\mu=d\tau_2d\xi_2d\tau_1d\xi_1,\xi=\xi_1-\xi_2,\tau=\tau_1-\tau_2,\sigma=\tau+|\xi|$, and $\sigma_i=\tau_i+|\xi_i|^2$, $i=1,2$.

LEMMA 8.2. *Let $0<\ell-k<1$ and let $2k\ge\ell+1$ for $d=2,3$ and $2k>\ell+\frac{d}{2}-1$ for $d\ge 4$. If $b_1>\frac{1}{2},b,c_1\ge 0$, and $b>\frac{d}{4}-\frac{\ell}{2}$, then the estimate (8.4) holds.*

Proof. For the subregion $2|\xi_2|\le|\xi_1|$, by $k-\ell<0,c_1\ge 0$ and the Hölder inequality, we have

$$S_0\lesssim\|(\langle\sigma\rangle^{-b}|\widehat{u}|)^\vee\|_{L^q_tL^2_x}\|u_1\|_{L^2_{t,x}}\|(\langle\xi_2\rangle^{-\ell}\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L^{q_2}_tL^\infty_x}, \tag{8.7}$$

where $\frac{1}{q}+\frac{1}{q_2}=\frac{1}{2}$. Note that $\ell>1$ for $d=2,3$ and $\ell>\frac{d}{2}-1$ for $d\ge 4$, thus we can choose $\eta\in(0,1]$ such that $\ell>\frac{d}{2}-1+\eta$. Then invoking Sobolev inequality and Lemma 8.1 with $\beta=b_1$, if $q_2=\frac{2}{1-\eta}$, we obtain

$$\|(\langle\xi_2\rangle^{-\ell}\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L^{q_2}_tL^\infty_x}\lesssim\|(\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L^{q_2}_tL^{r_2}_x}\lesssim\|u_2\|_{L^2_{t,x}},$$

where $\rho(r_2)=1-\eta$. Now we have $q=\frac{2}{\eta}<\infty$. By Sobolev inequality, if $b\ge\frac{1-\eta}{2}$, we have

$$\|(\langle\sigma\rangle^{-b}|\widehat{u}|)^\vee\|_{L_t^qL_x^2}\lesssim\|u\|_{L^2_{t,x}}.$$

For any $b>\frac{d}{4}-\frac{\ell}{2}$, we can choose suitable η so that $b\ge\frac{1-\eta}{2}$.

For the subregion $2|\xi_2|>|\xi_1|$, by $c_1\ge0$ and the Hölder inequality, we have

$$S_0\lesssim\|(\langle\xi\rangle^{-\ell}\langle\sigma\rangle^{-b}|\widehat{u}|)^\vee\|_{L_t^qL_x^r}\|u_1\|_{L^2_{t,x}}\|(\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2}},\tag{8.8}$$

where $\frac{1}{q}+\frac{1}{q_2}=\frac{1}{2}$ and $\rho(r)+\rho(r_2)=\frac{d}{2}$. Invoke Lemma 8.1 with $\beta=b_1$, if $\frac{2}{q_2}=\rho(r_2)=1-\eta_1$ for $0<\eta_1\le1$, then $\|(\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2}}\lesssim\|u_2\|_{L^2_{t,x}}$. We now have $\frac{1}{q}=\frac{\eta_1}{2}$ and $\rho(r)=\frac{d}{2}-1+\eta_1$. By Sobolev inequality, if $b=\frac{1-\eta_1}{2}$ and $\ell\ge\rho(r)$ or $\ell>\frac{d}{2}$ for $\eta_1=1$, then $\|(\langle\xi\rangle^{-\ell}\langle\sigma\rangle^{-b}|\widehat{u}|)^\vee\|_{L_t^qL_x^r}\lesssim\|u\|_{L^2_{t,x}}$. ∎

Lemma 8.3. *Let $0<\ell-k<1$ and let $2k\ge\ell+1$ for $d=2,3$ and $2k>\ell+\frac{d}{2}-1$ for $d\ge4$. If $b_1,c\ge\frac{\ell-k+1}{2}$, then the estimate (8.5) holds.*

Proof. We split the integration region into three subregions Ω_1,Ω_2 and Ω_3 which are defined in (3.13). For Ω_3, we have

$$W_0\lesssim\|(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-c}|\widehat{u}|)^\vee\|_{L_t^qL_x^r}\|(\langle\sigma_1\rangle^{-b_1}|\widehat{u}_1|)^\vee\|_{L_t^{q_1}L_x^{r_1}}\|(\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_1}L_x^{r_1}},$$

where $m=2k-\ell+1,\frac{1}{q}+\frac{2}{q_1}=1$ and $\rho(r)+2\rho(r_1)=\frac{d}{2}$. Invoking Lemma 8.1 with $\beta=b_1$, if $\frac{2}{q_1}=\rho(r_1)=1-\eta$ for $0<\eta\le1$, we obtain $\|(\langle\sigma_j\rangle^{-b_1}|\widehat{u}_j|)^\vee\|_{L_t^{q_1}L_x^{r_1}}\lesssim\|u_j\|_{L^2_{t,x}}$ for $j=1,2$. Now $q=\eta^{-1}$ and $\rho(r)=\frac{d}{2}-2+2\eta$. Let $\eta\le\frac{1}{2}$ so that $q\ge2$ and $r<\infty$. Invoking Sobolev inequality, if $m\ge\rho(r)$ and $c\ge\frac{1}{2}-\frac{1}{q}$, then $\|(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-c}|\widehat{u}|)^\vee\|_{L_t^qL_x^r}\lesssim\|u\|_{L^2_{t,x}}$. Since $2k\ge\ell+1$ for $d=2,3$ and $2k>\ell+\frac{d}{2}-1$ for $d\ge4$, we can choose η so that $m\ge\rho(r)$.

The two remaining subregions Ω_1 and Ω_2 are symmetric and we consider only the first one. For Ω_1, we again split the integral into three parts depending on the dominants of $\{\sigma,\sigma_1,\sigma_2\}$. On $\Omega_{1,\sigma}$, we have the inequality $\langle\xi\rangle^2\lesssim\langle\sigma\rangle$. By $2c\ge\ell-k+1$, we have

$$\int_{\Omega_{1,\sigma}}\frac{|\widehat{u}|}{\langle\sigma\rangle^c}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|\langle\xi\rangle^\ell}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k}\lesssim\|u\|_{L^2_{t,x}}\Big\|\Big(\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\Big)^\vee\Big\|_{L_t^{q_1}L_x^{r_1}}\Big\|\Big(\frac{\langle\xi_2\rangle^{-k}|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\Big)^\vee\Big\|_{L_t^{q_2}L_x^{r_2}},$$

where $\frac{1}{q_1}+\frac{1}{q_2}=1$ and $\rho(r_1)+\rho(r_2)=\frac{d}{2}$. Invoke Lemma 8.1, if $\frac{2}{q_1}=\rho(r_1)=1-\eta$ for $0<\eta<1$, then $\|(\langle\sigma_1\rangle^{-b_1}|\widehat{u}_1|)^\vee\|_{L_t^{q_1}L_x^{r_1}}\lesssim\|u_1\|_{L^2_{t,x}}$. Now $\frac{1}{q_2}=\frac{\eta}{2}$ and $\rho(r_2)=\frac{d}{2}-1+\eta<\frac{d}{2}$. Invoke Sobolev inequality and Lemma 8.1, if $k\ge\rho(r_2)-\eta$, then

$$\|(\langle\xi_2\rangle^{-k}\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2}}\lesssim\|(\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2'}}\lesssim\|u_2\|_{L^2_{t,x}},$$

where $\rho(r_2')=\eta$.

On Ω_{1,σ_1}, we have $\langle\xi\rangle^2\lesssim\langle\sigma_1\rangle$ and $2b_1\ge\ell-k+1$, so

$$\int_{\Omega_{1,\sigma_1}}\frac{|\widehat{u}|}{\langle\sigma\rangle^c}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|\langle\xi\rangle^\ell}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k}\lesssim\Big\|\Big(\frac{|\widehat{u}|}{\langle\sigma\rangle^c}\Big)^\vee\Big\|_{L_t^qL_x^2}\|u_1\|_{L^2_{t,x}}\Big\|\Big(\frac{\langle\xi_2\rangle^{-k}|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\Big)^\vee\Big\|_{L_t^{q_2}L_x^\infty},$$

where $\frac{1}{q}+\frac{1}{q_2}=\frac{1}{2}$. Invoke Sobolev inequality and Lemma 8.1, if $k>\frac{d}{2}-1+\eta$ and $\frac{2}{q_2}=1-\eta$ for some $0<\eta\le1$, then we get

$$\|(\langle\xi_2\rangle^{-k}\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^\infty}\lesssim\|(\langle\sigma_2\rangle^{-b_1}|\widehat{u}_2|)^\vee\|_{L_t^{q_2}L_x^{r_2}}\lesssim\|u_2\|_{L^2_{t,x}},$$

where $\rho(r_2)=1-\eta$. Now $\frac{2}{q}=\eta$. By Sobolev inequality, we have $\|(\langle\sigma\rangle^{-c}|\widehat{u}|)^\vee\|_{L_t^q L_x^2}\lesssim \|u\|_{L_t^2 L_x^2}$. Since $k>\frac{d}{2}-1$, we can choose η small enough such that $k>\frac{d}{2}-1+\eta$.

Finally we estimate the integral over Ω_{1,σ_2}.

$$\begin{aligned}&\int_{\Omega_{1,\sigma_2}} \frac{|\widehat{u}|}{\langle\sigma\rangle^c}\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\frac{|\widehat{u}_2|}{\langle\sigma_2\rangle^{b_1}}\frac{|\xi|\langle\xi\rangle^\ell}{\langle\xi_1\rangle^k\langle\xi_2\rangle^k}\,d\mu\\ &\lesssim \Big\|\Big(\frac{|\widehat{u}|}{\langle\sigma\rangle^c}\Big)^\vee\Big\|_{L_t^q L_x^2}\Big\|\Big(\frac{|\widehat{u}_1|}{\langle\sigma_1\rangle^{b_1}}\Big)^\vee\Big\|_{L_t^{q_1}L_x^{r_1}}\|(\langle\xi_2\rangle^{-k}|\widehat{u}_2|)^\vee\|_{L_t^2 L_x^{r_2}},\end{aligned}\tag{8.9}$$

where $\frac{1}{q}+\frac{1}{q_1}=\frac{1}{2}$ and $\rho(r_1)+\rho(r_2)=\frac{d}{2}$. Invoke Lemma 8.1, if $\frac{2}{q_1}=\rho(r_1)=1-\eta$ for some $0<\eta<1$, then $\|(\langle\sigma_1\rangle^{-b_1}|\widehat{u}_1|)^\vee\|_{L_t^{q_1}L_x^{r_1}}\lesssim\|u_1\|_{L_{t,x}^2}$. Now $\frac{2}{q}=\eta$ and $\rho(r_2)=\frac{d}{2}-1+\eta<\frac{d}{2}$. By Sobolev inequality, if $k\ge\rho(r_2)$, then we have $\|(\langle\sigma\rangle^{-c}|\widehat{u}|)^\vee\|_{L_t^qL_x^2}\lesssim\|u\|_{L_{t,x}^2}$ and $\|(\langle\xi_2\rangle^{-k}|\widehat{u}_2|)^\vee\|_{L_t^2L_x^{r_2}}\lesssim\|u_2\|_{L_{t,x}^2}$. Since $k>\frac{d}{2}-1$, we can choose η small enough such that $k>\frac{d}{2}-1+\eta$. ∎

Next we apply the argument in Lemma 5.2 to obtain the estimate (8.6).

Lemma 8.4. *Let $0<\ell-k<1$ and let $2k\ge\ell+1$ for $d=2,3$ and $2k>\ell+\frac{d}{2}-1$ for $d\ge4$. If $b_1\ge\frac{\ell-2k+1}{2}+\frac{d}{4}$ for $k<\frac{d}{2}$, $b_1>\frac{\ell-k+1}{2}$ for $k=\frac{d}{2}$, and $b_1\ge\frac{\ell-k+1}{2}$ for $k>\frac{d}{2}$, then the estimate (8.6) holds.*

Proof. We split the region into three subregions Ω_1,Ω_2 and Ω_3 which are defined in (3.13).

The proof of the case Ω_3 is the same as Lemma 8.3, so we have, if $a\ge\frac{1}{2}-\eta$ and $2k-\ell-1\ge\frac{d}{2}-2+2\eta$ for some $0<\eta\le\frac{1}{2}$,

$$\widetilde{W_0}\lesssim\|\langle\sigma\rangle^{a-1}|\mathcal{F}v|\|_{L_{\sigma,\xi}^2}\|u_1\|_{L_{t,x}^2}\|u_2\|_{L_{t,x}^2}.$$

We can choose $a=\frac{1}{2}-\eta<\frac{1}{2}$, and then $\widetilde{W_0}\lesssim\|v\|_{L_x^2}\|u_1\|_{L_{t,x}^2}\|u_2\|_{L_{t,x}^2}$.

The subregions Ω_1 and Ω_2 are symmetric and we consider only the first one. For Ω_1, the proof is similar as Lemma 5.2. Therefore, we have $\widetilde{W_1}\lesssim\|v\|_{L_x^2}M^{\frac{1}{2}}\|u_1\|_{L^2}\|u_2\|_{L^2}$, where

$$M:=\sup_{\xi\in\Omega_1}\int_{\Omega_1}\frac{|\xi|^2\langle\xi\rangle^{2(\ell-k)}\langle\xi_2\rangle^{-2k}}{\langle\sigma\rangle^2\langle\sigma-|\xi|+\sigma_2-|\xi_2|^2+|\xi+\xi_2|^2\rangle^{2b_1}\langle\sigma_2\rangle^{2b_1}}d\sigma_2d\xi_2d\sigma.$$

To estimate the above supremum, we apply Lemma 2.5 twice to obtain

$$M\lesssim\sup_\xi\langle\xi\rangle^{2(\ell-k+1)-4\mu+[d-2k]_+}<\infty,\tag{8.10}$$

where $\mu:=\min\{b_1,1\}$ and $2(\ell-k+1)-4\mu+[d-2k]_+\le0$. ∎

Proof. **(Proof of Theorem 1.3.)** The proof is the same as the proof of Theorem 1.1, so we only show that these parameters we choose satisfy all needed lemmata and conditions. Let $b_1=\frac{\ell-2k+1}{2}+\frac{d}{4}$ for $k<\frac{d}{2}$, $\frac{\ell-k+1}{2}+\delta$ for $k=\frac{d}{2}$, $\frac{\ell-k+1}{2}$ for $k>\frac{d}{2}$. Let $c_1=0, b=\max\{\frac{d}{4}-\frac{\ell}{2}+\delta,0\}$ and $c=\frac{\ell-k+1}{2}$. Clearly, (b_1,c_1,b,c) satisfies Lemmata 8.2-8.4. Following we check that $b_1+c_1<1$ and $b+c<1$.

By $2k\ge\ell+1$ for $d=3$ and $2k>\ell+\frac{d}{2}-1$ for $d\ge4$, we have $b_1\le\frac{d}{4}<1$ for $d=3$ and $b_1<\frac{4-d}{4}+\frac{d}{4}=1$ for $d\ge4$. Thus, $b_1<1$ as $k<\frac{d}{2}$. For $k\ge\frac{d}{2}$, we have $b_1<1$ since $\ell-1<k$. This shows that $b_1+c_1<1$.

It is easy to obtain that $k > \frac{d}{2} - 1$ for $d \geq 2$, so $\frac{d}{4} - \frac{\ell}{2} + \frac{\ell - k + 1}{2} < 1$. This implies that $b + c < 1$. □

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