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GLOBAL WEAK SOLUTIONS TO INVISCID BURGERS-VLASOV EQUATIONS*

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Abstract. In this paper, we consider the existence of global weak solutions to a one dimensional fluid-particles interaction model: inviscid Burgers-Vlasov equations with fluid velocity in L^∞ and particles' probability density in L^1. Our weak solution is also an entropy solution to inviscid Burgers' equation. The approach is to ingeniously add artificial viscosity to construct approximate solutions satisfying L^∞ compensated compactness framework and weak L^1 compactness framework. It is worthy to be pointed out that the bounds of fluid velocity and the kinetic energy of particles' probability density are both independent of time.

Keywords. weak solution; fluid-particles interaction; L^∞ velocity; L^1 density; compensated compactness; Dunford-Pettis theorem.

AMS subject classifications. 76T10; 35F20; 35Q35; 35Q72; 45K05; 82D05.

1. Introduction

We consider the following inviscid Burgers-Vlasov equations:

$$\begin{cases} u_t+uu_x=\int_{\mathbb{R}} fv dv - u\int_{\mathbb{R}} f dv, \\ f_t+vf_x+(f(u-v))_v=0, \end{cases} \tag{1.1}$$

with the initial data

$$u(x,0)=u_0(x), \quad f(x,v,0)=f_0(x,v)\geq 0. \tag{1.2}$$

The system (1.1) is one kind of simple model about inviscid fluid-particles interaction. The motion of the fluid with bulk velocity $u(x,t)$ is modeled by the inviscid Burgers equation, while the dispersed particles with probability density function $f(x,v,t)$ is described by a Vlasov-like equation. The interaction between the fluid and the dispersed particles is achieved by a friction term between the bulk velocity of fluid and velocity of the dispersed particles, namely the drag force term $\int_{\mathbb{R}} f(v-u)dv$.

For related fluid-structure models, we shall first mention the following diffusive system, Burgers-Vlasov equations:

$$\begin{cases} \rho_g(u_t+uu_x-\nu_g u_{xx})=E_d, \\ f_t+vf_x+(F_d f)_v=0, \end{cases} \tag{1.3}$$

in which a dispersed phase interacts with a viscous gas. Here ρ_g is the density of gas. The force term E_d describes the exchange of impulse between the gas and particles, and the drag force F_d is used to describe the friction of viscous gas on the droplets. The force terms are related by the following formulas:

$$E_d=C(r)\rho_p(u_p-u), \quad F_d=C(r)(u(x,t)-v),$$

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$$\rho_p = \frac{4\pi}{3}\rho_l r^3 \int_{\mathbb{R}} f(x,v,t)dv, \quad \rho_p u_p = \frac{4\pi}{3}\rho_l r^3 \int_{\mathbb{R}} f(x,v,t)vdv. \tag{1.4}$$

In (1.4), ρ_l is the density of the liquid and $C(r)$ is a constant depending on the radius r of droplets in the spray consisting of dispersed particles. After assuming that the gas is of constant mass density ρ_g and simplifying the momentum equation of the gas, the Burgers' equation, i.e. the first equation in (1.3), is utilized to model the evolution of the viscous gas. Further assuming the spray is of enough dilution and neglecting gravity effect, a Vlasov-like equation, i.e. the second equation in (1.3), is then applied to govern the evolution of the particles. Other detailed information about the derivations and assumptions on (1.3) can also be found in [10,12,28]. For the mathematical analysis of (1.3), as far as we know, the first global existence and uniqueness of classical solutions to the Cauchy problem has been considered in [10], in which the Burgers-Vlasov equations are equipped with regular and compactly supported initial data. Meanwhile, the stability of travelling waves are also considered. When the initial data is less regular than that in [10], the global existence and uniqueness of finite energy solutions are proved in [12]. The second related simpler 1-D model on fluid-structure interaction

$$\begin{cases} u_t + uu_x = \lambda(h'(t) - u(t,h(t)))\delta_{h(t)}, \\ mh''(t) = -\lambda(h'(t) - u(t,h(t))), \end{cases} \tag{1.5}$$

is also considered in [19], where u is the velocity of the inviscid fluid and $h(t)$ is the location of the particles. λ is the positive friction constant and $\delta_{h(t)}$ is the Dirac measure at $h(t)$. Global entropy weak solutions involving shock waves to the system (1.5) are obtained in [19]. There are also some other fluid-kinetic models: compressible/incompressible Euler/Navier-Stokes equations coupled with Vlasov/Vlasov-Fokker-Planck equations. Weak solutions or classical solution close to the equilibrium are studied in [20, 22, 25, 26, 29]. Some asymptotic problems such as hydrodynamic limit/stratified limit of viscous Burgers-Vlasov equation and Euler/Navier-Stokes equations coupling with Vlasov equation are also considered in [3, 12–16, 21, 24].

In this paper, we investigate the existence of global weak solutions to the Cauchy problem (1.1)-(1.2). Since the derivation of the model in [9] is in 1D, and as far as the authors know, references on 2D or 3D cases are not found for such model, we consider the 1D invisid Burgers-Vlasov equations. Comparing with the diffusive system (1.3), without viscosity term, shock wave may exist when the initial data are given arbitrarily large. Hence we consider the entropy weak solutions to inviscid Burgers' equation. Because of the nonlocal source term in the Burgers' equation, we require the solution of the Vlasov equation be of finite kinetic energy. Consequently, we define $L^\infty - L^1$ weak solution to (1.1).

Definition 1.1. *For any fixed $T \in (0,\infty)$, a pair of functions $u : \mathbb{R}\times[0,T] \to \mathbb{R}$, $f : \mathbb{R}^2 \times [0,T] \to [0,\infty)$ is called a global $L^\infty - L^1$ weak solution of Cauchy problem (1.1)-(1.2) if the following statements hold:*

(1) $u(x,t) \in L^\infty(\mathbb{R}\times[0,T])$ and $f(x,v,t) \in L^\infty([0,T],(1+v^2)L^1(\mathbb{R}^2))$.

(2) $u(x,t)$ is an entropy solution to Burgers' equation, i.e. for any $\phi \in C_c^1(\mathbb{R}\times [0,T))$,

$$\int_{\mathbb{R}} \phi(x,0)u_0(x)dx + \int_0^T \int_{\mathbb{R}} \left(u\phi_t + \frac{1}{2}u^2\phi_x + \phi \int_{\mathbb{R}} f(v-u)dv \right) dxdt = 0, \tag{1.6}$$

and for any convex entropy pair (η,q) *the following entropy inequality*

$$\eta(u)_t+q(u)_x+\eta'(u)\int_{\mathbb{R}} f(u-v)dv\le 0 \tag{1.7}$$

holds in the sense of distributions, where (η,q) *satisfies* $\eta'(u)u=q'(u)$ *and* $\eta(u)$ *is convex with respect to* u.

(3) $f(x,v,t)$ *is a weak solution to Vlasov equation, i.e. for any* $\psi\in C_c^1(\mathbb{R}^2\times[0,T))$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(x,v,0)f_0(x,v)dxdv+\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}(f\psi_t+vf\psi_x+\psi_v f(u-v))dvdxdt=0. \tag{1.8}$$

Now we are ready to state our main result.

Theorem 1.1 (Main Theorem). *Let initial data* (u_0,f_0) *satisfy*

$$\|u_0(x)\|_{L^\infty(\mathbb{R})}+\|(1+v^2)f_0(x,v)\|_{L^1(\mathbb{R}^2)}\le M_0 \tag{1.9}$$

for some positive constant M_0. *Then there exists a global* $L^\infty-L^1$ *weak solution to* (1.1)-(1.2) *in the sense of Definition 1.1 and there is a constant* M *depending solely on* M_0 *such that*

$$\|u(x,t)\|_{L^\infty(\mathbb{R}\times[0,T])}+\|(1+v^2)f(x,v,t)\|_{L^\infty([0,T],L^1(\mathbb{R}^2))}\le M. \tag{1.10}$$

Hereafter, M denotes the constant depending only on M_0 and it may vary from line to line.

Remark 1.1. Our uniform bounds of velocity $\|u\|_{L^\infty}$ and kinetic energy of Vlasov equation $\int_{\mathbb{R}}\int_{\mathbb{R}} f(1+v^2)dvdx$ are both independent of time T.

Our strategy of proving Theorem 1.1 is to construct approximate solutions by adding artificial viscosity to Burgers equation technically and regard the nonlocal term $\int_{\mathbb{R}} f(v-u)dv$ as a dissipative source term in some sense. Maximum principles of parabolic equation and transport equation are applied to establish the uniform bound. Besides, in order to get the uniform estimates (also be independent of time T) of approximate viscosity solution u^ε , we add some novel viscosity term and choose a special control function. In proving almost everywhere convergence of u^ε, we employ L^∞ compensated compactness framework. More information about compensated compactness framework of L^p or L^∞ space can be found in [6–8,11,23] and the references therein. On the other hand, to show the weak L^1 convergence of f^ε, we apply Dunford-Pettis theorem and analyze kinetic energy of f^ε and evolution of sets. In the proof, we also came across the difficulty on the weak convergence of $\int vf^\varepsilon dv$, which is overcome again by applying the uniform bound of kinetic energy.

To be concise, in the present paper, we use $\int$ instead of $\int_{\mathbb{R}}$. $C(\cdot)$ denotes constant depending on the parameters in the bracket. The rest of the paper is organised as follows. Section 2 is devoted to construct approximate solutions and prove their global existence. The proof of Theorem 1.1 is given in Section 3.

2. Approximate solutions

In this section, we construct the globally existing approximate solutions to problem (1.1)-(1.2) by adding artificial viscosity and choosing the initial data technically. That is, we consider the approximate problem

$$\begin{cases} u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon + \varepsilon \left(\int f^\varepsilon dv\right)_x + \int f^\varepsilon v dv - u^\varepsilon \int f^\varepsilon dv, \\ f_t^\varepsilon + v f_x^\varepsilon + (f^\varepsilon (u^\varepsilon - v))_v = 0, \end{cases} \tag{2.1}$$

and the carefully selected initial data

$$\begin{aligned} &u^\varepsilon(x,0) = u_0^\varepsilon(x) = u_0(x) * j_\varepsilon(x), \\ &f^\varepsilon(x,v,0) = f_0^\varepsilon(x,v) = \left[\min\{\varepsilon^{-1/6}, f_0(x,v)\mathbf{1}_{\{|x|+|v|\le \varepsilon^{-1/6}\}}\}\right] * j_\varepsilon(x) * j_\varepsilon(v) \ge 0, \end{aligned} \tag{2.2}$$

where j_ε is the standard one-dimensional mollifier with parameter ε. The initial data here is chosen to make f^ε be of explicit ε-depending compact support and L^∞ bound for later use. Our idea of adding the above viscosity is based on the following reasons. As is known, linear transport equation preserves the regularity of initial data and inviscid Burgers' equation may formulate shock wave. Thus we add parabolic viscosity term to Burgers' equation only. Moreover, to gain the uniform bound of u^ε, we not only make full use of Vlasov equation and flux term in Burgers equation but also carefully choose control function. The viscosity term $\varepsilon(\int_{\mathbb{R}} f^\varepsilon dv)_x$ is introduced due to derivatives of our control function.

For any $\sigma \in (0,1)$, we use $C^{2+\sigma}(\mathbb{R}^2 \times [0,T])$ and $C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,T])$ to denote usual and parabolic Hölder spaces respectively. We now would like to consider the global existence and uniqueness of smooth solutions $(u^\varepsilon, f^\varepsilon)$ to the Cauchy problem (2.1)-(2.2) with $u^\varepsilon(x,t) \in C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,T])$, $0 \le f^\varepsilon(x,v,t) \in C^{2+\sigma}(\mathbb{R}^2\times[0,T])$. For simplicity, in this section, the superscript ε in u^ε and f^ε will be dropped.

2.1. Local existence. We first consider the local existence of smooth solution of the Cauchy problem (2.1)-(2.2). Let

$$G(x,t) = \begin{cases} \delta(x), & t = 0, \\ \frac{1}{\sqrt{4\pi\varepsilon t}} e^{\frac{-x^2}{4\varepsilon t}}, & t > 0, \end{cases}$$

denote the kernel of the homogeneous heat equation $u_t = \varepsilon u_{xx}$. Then from the Burgers' equation, i.e. the first equation in (2.1), Duhamel principle tells us

$$\begin{aligned} u(x,t) = &\int G(x-y,t)u_0(y)dy + \int_0^t \int G(x-y,t-s)\left(\int f v dv - u\int f dv\right)(y,s)dyds \\ &+ \int_0^t \int G(x-y,t-s)\left(\varepsilon\int f dv - \frac{1}{2}u^2\right)_y (y,s)dyds. \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} u(x,t) = &\int G(x-y,t)u_0(y)dy + \int_0^t \int G(x-y,t-s)\left(\int f v dv - u\int f dv\right)(y,s)dyds \\ &+ \int_0^t \int G_y(x-y,t-s)\left(\varepsilon\int f dv - \frac{1}{2}u^2\right)(y,s)dyds, \end{aligned} \tag{2.3}$$

which implicitly gives the solution to the Burgers' equation. For the Vlasov equation, i.e. the second equation in (2.1), we rewrite it as

$$f_t + v f_x + (u-v) f_v = f,$$

which is a transport equation. One can integrate along the backward characteristic curves

$$\frac{d}{ds}X(s;x,v,t)=V(s;x,v,t), \quad X(t;x,v,t)=x, \tag{2.4}$$

$$\frac{d}{ds}V(s;x,v,t)=u(X(s;x,v,t),s)-V(s;x,v,t), \quad V(t;x,v,t)=v, \tag{2.5}$$

to get

$$f(x,v,t)=f_0(X(0;x,v,t),V(0;x,v,t))e^t. \tag{2.6}$$

Note that for smooth u, the system of (2.4) and (2.5) has a unique smooth solution $(X(s;x,v,t),V(s;x,v,t))$:

$$X(s;x,v,t)=x+\int_t^s V(\tau;x,v,t)d\tau, \tag{2.7}$$

$$V(s;x,v,t)=ve^{t-s}+\int_t^s e^{\tau-s}u(X(\tau;x,v,t),\tau)d\tau. \tag{2.8}$$

Hence, from (2.3) and (2.6) we construct the approximate solutions of (2.1) in the following way. Set $u^{(0)}=u_0^\varepsilon, f^{(0)}=f_0^\varepsilon$, then there exists a $K=C(\varepsilon,M_0)$ such that for any $t>0$

$$\|u^{(0)}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t])}\le K, \quad \|f^{(0)}\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,t])}\le K, \quad |\text{supp} f^{(0)}|\le K,$$

where we used the definition

$$|\text{supp} f|=\max\{|x|,|v|: f(x,v,\cdot)>0\}.$$

For $k\ge 1$, we define

$$\begin{aligned} u^{(k)}(x,t)= &\int G(x-y,t)u_0(y)dy \\ &+\int_0^t\int G(x-y,t-s)\left(\int f^{(k-1)}vdv-u^{(k-1)}\int f^{(k-1)}dv\right)(y,s)dyds \\ &+\int_0^t\int G_y(x-y,t-s)\left(\varepsilon\int f^{(k-1)}dv-\frac{1}{2}(u^{(k-1)})^2\right)(y,s)dyds, \end{aligned}$$

and

$$f^{(k)}(x,v,t)=f_0(X^{(k)}(0;x,v,t),V^{(k)}(0;x,v,t))e^t,$$

where $(X^{(k)}(s;x,v,t),V^{(k)}(s;x,v,t))$ is defined using (2.4) and (2.5) as

$$\begin{aligned} &\frac{d}{ds}X^{(k)}(s;x,v,t)=V^{(k)}(s;x,v,t), \\ &\quad X^{(k)}(t;x,v,t)=x, \\ &\frac{d}{ds}V^{(k)}(s;x,v,t)=u^{(k-1)}(X^{(k)}(s;x,v,t),s)-V^{(k)}(s;x,v,t), \\ &\quad V^{(k)}(t;x,v,t)=v, \end{aligned}$$

and $(x,v)\in \mathrm{supp} f^{(k)}$. It is easy to see that $(X^{(k)},V^{(k)})$ is well-defined. Thus $f^{(k)}$ and $u^{(k)}$ make sense. Besides, one can also see that $(u^{(k)},f^{(k)})$ solves the following approximate equations

$$\begin{cases} u_t^{(k)}+u^{(k-1)}u_x^{(k-1)}=\varepsilon u_{xx}^{(k)}+\varepsilon\left(\int f^{(k-1)}dv\right)_x+\int f^{(k-1)}vdv-u^{(k-1)}\int f^{(k-1)}dv, \\ f_t^{(k)}+vf_x^{(k)}+(u^{(k-1)}-v)f_v^{(k)}=f^{(k)}, \end{cases} \tag{2.9}$$

Obviously, from (2.2), for any $(x,v)\in \mathrm{supp} f^{(1)}$,

$$|X^{(1)}(0;x,v,t)|+|V^{(1)}(0;x,v,t)|\le \varepsilon^{-1/6}.$$

Using (2.7), (2.8) and the bound of $u^{(0)}$, we have

$$\begin{aligned} |v|&\le |V^{(1)}(0;x,v,t)e^{-t}|+\left|\int_0^t e^{\tau-t}u^{(0)}(X^{(1)}(\tau;x,v,t),\tau)d\tau\right|\le \varepsilon^{-1/6}+K\le 2K,\\ |x|&\le |X^{(1)}(0;x,v,t)|+\left|\int_0^t V^{(1)}(\tau;x,v,t)d\tau\right|\le \varepsilon^{-1/6}+3Kte^t\le 2K, \end{aligned}$$

provided t is sufficiently small, where we have used the fact that

$$\begin{aligned} |V^{(1)}(s;x,v,t)|&\le |ve^{t-s}|+\left|\int_t^s e^{\tau-s}u^{(0)}(X^{(1)}(\tau;x,v,t),\tau)d\tau\right| \\ &\le e^t(2K+\|u^{(0)}\|_{L^\infty})\le 3Ke^t, \quad \textit{for } 0\le s\le t. \end{aligned}$$

Thus $|\mathrm{supp} f^{(1)}|\le 2K$. Moreover, we have the following conclusion.

LEMMA 2.1. *There exists a small $t_0>0$ such that the sequence $\{(u^{(k)},f^{(k)})\}_{k\ge 0}$ constructed above is a contraction sequence in*

$$\mathcal{S}(t)=\{(u,f)\,|\,\|u\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t])}\le 2K,\quad \|f\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,t])}\le 2K,\quad |\mathrm{supp} f|\le 2K\}$$

for all $t\in(0,t_0)$.

Proof. It is easy to see that $(u^{(0)},f^{(0)})\in S(t)$. Suppose that for $k\ge 1$, $(u^{(k-1)},f^{(k-1)})$ has been shown in $\mathcal{S}(t)$. We then estimate $(u^{(k)},f^{(k)})$.

Taking derivatives with respect to x, one has for $\ell=0,1,2$,

$$\begin{aligned} \partial_x^\ell u^{(k)}(x,t)=&\int \partial_x^\ell G(x-y,t)u_0(y)dy \\ &+\int_0^t\int \partial_x^\ell G(x-y,t-s)\left(\int f^{(k-1)}vdv-u^{(k-1)}\int f^{(k-1)}dv\right)(y,s)dyds \\ &+\int_0^t\int \partial_x^\ell G_y(x-y,t-s)\left(\varepsilon\int f^{(k-1)}dv-\frac{1}{2}(u^{(k-1)})^2\right)(y,s)dyds. \end{aligned}$$

Using the symmetry of $G(x-y)$, one further has

$$\begin{aligned} \partial_x^\ell u^{(k)}(x,t)=&\int \partial_x^\ell G(x-y,t)u_0(y)dy \\ &+\int_0^t\int (-1)^\ell\partial_y^\ell G(x-y,t-s)\left(\int f^{(k-1)}vdv-u^{(k-1)}\int f^{(k-1)}dv\right)(y,s)dyds \end{aligned}$$

$$+\int_0^t\int(-1)^\ell\partial_y^\ell G_y(x-y,t-s)\left(\varepsilon\int f^{(k-1)}dv-\frac{1}{2}(u^{(k-1)})^2\right)(y,s)dyds.$$

Integration by parts gives

$$\begin{aligned}\partial_x^\ell u^{(k)}(x,t)=&\int G(x-y,t)\partial_y^\ell u_0(y)dy\\&+\int_0^t\int G(x-y,t-s)\left(\partial_y^\ell\int f^{(k-1)}vdv\right)(y,s)dyds\\&-\int_0^t\int G(x-y,t-s)\partial_y^\ell\left(u^{(k-1)}\int f^{(k-1)}dv\right)(y,s)dyds\\&+\int_0^t\int G_y(x-y,t-s)\partial_y^\ell\left(\varepsilon\int f^{(k-1)}dv\right)(y,s)dyds,\\&-\int_0^t\int G_y(x-y,t-s)\partial_y^\ell\left(\frac{1}{2}(u^{(k-1)})^2\right)(y,s)dyds.\end{aligned}$$

For any $(x,v)\in\mathrm{supp}f^{(k)}$, $t<t_1\ll 1$, one has

$$|X^{(k)}(0;x,v,t)|+|V^{(k)}(0;x,v,t)|\le\varepsilon^{-1/6},$$

and

$$\begin{aligned}&|v|\le|V^{(k)}(0;x,v,t)e^{-t}|+\left|\int_0^t e^{\tau-t}u^{(k-1)}(X^{(k)}(\tau;x,v,t),\tau)d\tau\right|\le K+2Kt\le 2K,\\&|V^{(k)}(s;x,v,t)|\le|ve^{t-s}|+|\int_t^s e^{\tau-s}u^{(k-1)}(X^{(k)}(\tau;x,v,t),\tau)d\tau|\le 4e^tK\le 6K,\\&|x|\le|X^{(k)}(0;x,v,t)|+\left|\int_0^t V^{(k)}(\tau;x,v,t)d\tau\right|\le K+4Kte^t\le 2K\\&|X^{(k)}(s;x,v,t)|\le|x|+t|V^{(k)(s;x,v,t)}|\le 2K+4te^tK\le 6K\end{aligned}$$

Thus

$$|\mathrm{supp}f^{(k)}|\le 2K. \tag{2.10}$$

Noticing the estimates of heat kernel

$$\int G(x,t)dx=1,\quad \int|G_x(x,t)|dx\le\frac{C}{\sqrt{\varepsilon t}},$$

using the bound of $\mathrm{supp}f^{(k-1)}$, one has

$$\|u^{(k)}(\cdot,t)\|_{C^2(\mathbb{R})}\le K+C(\varepsilon,K)(t+\sqrt{t}),$$

then further gets

$$\|u^{(k)}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t])}\le K+C(\varepsilon,K)(t+\sqrt{t})\le 2K, \tag{2.11}$$

provided $t<t_2\ll 1$.

For the estimates of $f^{(k)}$ and for any $0\le|\alpha|\le 2$, taking derivatives one has

$$(\partial^\alpha f^{(k)})_t+v(\partial^\alpha f^{(k)})_x+(u^{(k-1)}-v)(\partial^\alpha f^{(k)})_v=B(\alpha)\partial^\alpha f^{(k)}+B(\alpha'), \tag{2.12}$$

where ∂^α denotes the mixed derivatives of x,v and t with order $|\alpha|$, $B(\alpha)$ is a linear function of v and $\partial_{x,t}^\beta u^{(k-1)}$, with $|\beta|\le|\alpha|$ and $B(\alpha')$ is a linear combination of $\partial^{\alpha'}f^{(k)}$ with $|\alpha'|<|\alpha|$, whose coefficients are linear functions of v and $\partial_{x,t}^\beta u^{(k-1)}$, with $|\beta|\le|\alpha|$. Integrating (2.12) along the characteristic curves $(X^{(k)},V^{(k)})$, one has

$$\begin{aligned}\partial^\alpha f^k=&(\partial^\alpha f_0)(X^{(k)},V^{(k)})e^{\int_0^t B(\alpha)(X^{(k)},V^{(k)})d\tau}\\&+\int_0^t e^{\int_\tau^t B(\alpha)(X^{(k)},V^{(k)})ds}B(\alpha')(X^{(k)},V^{(k)})d\tau,\end{aligned}$$

which then implies

$$\|f^{(k)}\|_{C^2(\mathbb{R}^2\times[0,t])}\le Ke^{tC(\varepsilon,K)}+C(\varepsilon,K)te^{C(\varepsilon,K)t}\le 2K.$$

We further gain the following Hölder estimate

$$\|f^{(k)}\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,t])}\le 2K, \tag{2.13}$$

provided $t<t_3\ll 1$. Hence from (2.10), (2.11) and (2.13), one has

$$(u^{(k)},f^{(k)})\in\mathcal{S}(t).$$

Moreover, consider the equation for $u^{(k)}-u^{(k-1)}$ and $f^{(k)}-f^{(k-1)}$, similar to the above calculation, we get

$$\begin{aligned}\|u^{(k+1)}-u^{(k)}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t])}\le&C(\varepsilon,K)t\|u^{(k)}-u^{(k-1)}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t])}\\\le&\frac{1}{2}\|u^{(k)}-u^{(k-1)}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t])}\\\|f^{(k+1)}-f^{(k)}\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,t])}\le&C(\varepsilon,K)(e^t-1)\|f^{(k)}-f^{(k-1)}\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,t])}\\\le&\frac{1}{2}\|f^{(k)}-f^{(k-1)}\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,t])},\end{aligned}$$

provided $t<t_4\ll 1$. Let $t_0=\min\{t_i,i=1,\cdots,4\}$. Then we gain $(u^{(k)},f^{(k)})$ is a contraction sequence in $\mathcal{S}(t)$ with $t\in(0,t_0)$ and end the proof. □

Applying fixed-point theorem to $(u^{(k)},f^{(k)})$, combining with Lemma 2.1, one gains that there exists a pair of functions (u,f) such that

$$\begin{aligned}&u^{(k)}\to u \text{ in } C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,t_0]),\\&f^{(k)}\to f \text{ in } C^{2+\sigma}(\mathbb{R}^2\times[0,t_0]),\end{aligned}$$

and (u,f) is the unique smooth solution of Cauchy problem (2.1)-(2.2) by taking limit in (2.9).

2.2. Uniform estimates. We will apply maximum principles of parabolic equation and transport equation to bound $\|u\|_{L^\infty(\mathbb{R}\times[0,T])}$ and $\|f\|_{L^\infty([0,T],(1+v^2)L^1(\mathbb{R}^2))}$.

Lemma 2.2. *For the approximate solutions constructed above, there exists a constant M depending only on M_0 such that (1.10) holds for $(u^\varepsilon,f^\varepsilon)$.*

Proof. Obviously, from (2.2) and (2.6) one has $f(x,v,t)\ge 0$. For any $(x,v)\in\operatorname{supp} f$, by (2.2) and (2.8), one has

$$|v|\le|V(0;x,v,t)e^{-t}|+\left|\int_0^t e^{\tau-t}u(X(\tau;x,v,t),\tau)d\tau\right|$$

$$\leq \varepsilon^{-1/6} + \hat{C}(\varepsilon, M_0, T), \tag{2.14}$$

after a priori assuming

$$\|u\|_{L^\infty} \leq \hat{C}(\varepsilon, M_0, T) \text{ for some large enough } \hat{C}(\varepsilon, M_0, T).$$

Besides,

$$\begin{aligned} |x| &\leq |X(0;x,v,t)| + \left| \int_0^t V(\tau;x,v,t) d\tau \right| \\ &\leq \varepsilon^{-1/6} + (2\hat{C}(\varepsilon, M_0, T) + \varepsilon^{-1/6}) T e^T. \end{aligned} \tag{2.15}$$

Hence by (2.14)-(2.15), f enjoys compact support (depending on ε, T) and

$$\lim_{|x|\to\infty \text{ or } |v|\to\infty} f(x,v,t) = 0.$$

Thus integrating the Vlasov equation over $\mathbb{R}^2 \times [0,t]$ with respect to (x,v,t) gives

$$\int\int f(x,v,t) dx dv = \int\int f_0(x,v) dx dv \leq M_0, \tag{2.16}$$

where we have used (1.9). Moreover, integration over $\mathbb{R}$ with respect to v also gives

$$\left(\int f dv \right)_t + \left(\int v f dv \right)_x = 0. \tag{2.17}$$

Define control function

$$\psi(x,t) = \int_x^\infty \int f(y,v,t) dv dy,$$

then using (2.17) one has

$$\begin{aligned} &\psi_t = \int_x^\infty \left(\int f(y,v,t) dv \right)_t dy = -\int_x^\infty \left(\int v f(y,v,t) dv \right)_y dy = \int f v dv, \\ &\psi_x = -\int f dv, \quad \psi_{xx} = -\left(\int f dv \right)_x. \end{aligned}$$

Thus one is also able to derive the equation for $u - \psi$

$$\begin{aligned} (u-\psi)_t + u(u-\psi)_x &= \varepsilon u_{xx} + \varepsilon \left(\int f dv \right)_x + \int f v dv - u \int f dv - \psi_t - u \psi_x \\ &= \varepsilon (u-\psi)_{xx}. \end{aligned}$$

Applying maximum principle for the above parabolic equation with respect to $u - \psi$, we obtain

$$\|u - \psi\|_{L^\infty} \leq \|u_0 - \psi(x,0)\|_{L^\infty} \leq M_0 + \int\int f_0 dv dx \leq 2M_0.$$

Thus, we have

$$\|u(x,t)\|_{L^\infty} \leq \|\psi\|_{L^\infty} + \|u - \psi\|_{L^\infty} \leq 3M_0 \tag{2.18}$$

by using (2.16), which also closes our a priori assumption

$$\|u\|_{L^\infty}\le 3M_0<\hat{C}(\varepsilon,M_0,T).$$

Besides, similar to the calculations of (2.14) and (2.15), one has

$$|v|\le \varepsilon^{-1/6}+3M_0,\quad |x|\le \varepsilon^{-1/6}+(\varepsilon^{-1/6}+2M_0)Te^T. \tag{2.19}$$

Furthermore, multiplying the Vlasov equation by v^2 and integrating over $\mathbb{R}^2$ with respect to (v,x) we have

$$\begin{aligned}\frac{d}{dt}\int\int fv^2dvdx&=\int\int(2fvu-2fv^2)dvdx\le\int\int f(u^2-v^2)dvdx\\&\le\|u\|_{L^\infty}^2\int\int fdvdx-\int\int fv^2dvdx\\&\le 9M_0^3-\int\int fv^2dvdx,\end{aligned}$$

where we have used (2.16). Grönwall's inequality yields

$$\int\int fv^2dvdx\le e^{-t}\int\int f_0v^2dvdx+9M_0^3(1-e^{-t})\le M_0+9M_0^3. \tag{2.20}$$

Thus (2.16), (2.18) and (2.20) conclude the present lemma. ∎

2.3. Conclusion. Standard theory of quasilinear parabolic equations (see [18]) can be applied to the equation for $u-\psi$

$$(u-\psi)_t+u(u-\psi)_x=\varepsilon(u-\psi)_{xx}$$

to get

$$\|(u-\psi)_x\|_{C^0(\mathbb{R})}\le C(\varepsilon,T),$$

where we have used the uniform bound on $\|u\|_{C^0(\mathbb{R})}\le C$ and $C(\varepsilon,T)$ is an increasing function of T. So we have

$$\|u_x\|_{C^0(\mathbb{R})}\le C(\varepsilon,T).$$

Then using the Vlasov equation, we can also get

$$\|f_x\|_{C^0(\mathbb{R}^2)}+\|f_v\|_{C^0(\mathbb{R}^2)}\le C(\varepsilon,T).$$

By standard bootstrap argument we have the following estimate

$$\|u(x,t)\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,T])}+\|f(x,v,t)\|_{C^{2+\sigma}(\mathbb{R}^2\times[0,T])}\le C(\varepsilon,T),$$

With the local existence in Subsection 2.1, the global existence of solution $u(x,t)\in C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,T])$ and $0\le f(x,v,t)\in C^{2+\sigma}(\mathbb{R}^2\times[0,T])$ to the Cauchy problem (2.1)-(2.2) is obtained. Thus we get the following conclusion.

THEOREM 2.1. *For any $T>0$, any fixed ε, there exists a unique global solution $u^\varepsilon(x,t)\in C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R}\times[0,T])$, $0\le f^\varepsilon(x,v,t)\in C^{2+\sigma}(\mathbb{R}^2\times[0,T])$ to Cauchy problem (2.1)-(2.2).*

3. Proof of main theorem

In this section, we will establish the convergence of $(u^\varepsilon, f^\varepsilon)$, whose limit is just a $L^\infty - L^1$ weak solution to Cauchy problem (1.1)-(1.2).

3.1. Limit of functions.

We first show some convergence results related to f^ε. Recall the well-known weak L^1 compactness framework, i.e. Dunford-Pettis Theorem (see [17] Theorem 8 or [2], page 167).

Proposition 3.1 (Dunford-Pettis). *A sequence $\{f^\varepsilon\}$ is weakly compact in $L^1(\mathbb{R}^2)$ if and only if $\{f^\varepsilon\}$ satisfies the following conditions:*

(1) The sequence f^ε is equibounded in $L^1(\mathbb{R}^2)$, i.e.

$$\sup_\varepsilon \|f^\varepsilon\|_{L^1(\mathbb{R}^2)} < \infty.$$

(2) The sequence f^ε is equiintegrable, i.e.

(2a) For any $\delta > 0$, there exists measurable set $A \subset \mathbb{R}^2$ with $|A| < \infty$ such that

$$\int_{\mathbb{R}^2 \setminus A} f^\varepsilon dx dv < \delta.$$

(2b) For any $\delta > 0$, there exists $\kappa > 0$ such that for any measurable set $E \subset \mathbb{R}^2$, with $|E| \le \kappa$, there holds

$$\int_E f^\varepsilon dx dv < \delta.$$

We shall verify (1) and (2) for $f^\varepsilon(x,v,t)$ with any fixed $t > 0$.

Verification of (1): Obviously, from Lemma 2.2, one finds that

$$\|f^\varepsilon\|_{L^\infty([0,T],(1+v^2)L^1(\mathbb{R}^2))} \le M, \tag{3.1}$$

which means $f^\varepsilon(\cdot,\cdot,t)$ is uniformly equibounded with respect to ε and t in $L^1(\mathbb{R}^2)$.

Verification of (2a): For any $\delta > 0$, we can choose $A = \{(x,v) \,|\, |x| \le \Lambda, |v| \le \Lambda\}$ with $\Lambda \ge \frac{M}{\delta}$ where M comes from (3.1), so we have for any $t > 0$,

$$\int_{\mathbb{R}^2 \setminus A} f^\varepsilon(x,v,t) dx dv \le \frac{1}{\Lambda^2 + 1} \int_{\mathbb{R}^2 \setminus A} f^\varepsilon(x,v,t)(1+v^2) dx dv \le \delta,$$

which implies that (2a) is satisfied by $f^\varepsilon(x,v,t)$.

Verification of (2b): By the fact that $f_0^\varepsilon \rightharpoonup f_0$ in $L^1(\mathbb{R}^2)$, Dunford-Pettis theorem tells us that for any δ, there exists κ_0 such that for any $E_0 \subset \mathbb{R}^2$ with $|E_0| \le \kappa_0$ it holds that

$$\int_{E_0} f_0^\varepsilon(x,v) dx dv \le \delta.$$

On the other hand, considering the variable transformation

$$\mathcal{J} : (x,v) \mapsto (X^\varepsilon, V^\varepsilon),$$

from (2.4) and (2.5), one is able to show that the Jacobian

$$J(\tau) = \det \nabla_{x,v}(X^\varepsilon, V^\varepsilon)$$

of map $\mathcal{J}$ is positive and satisfies the following ODE

$$\begin{cases} \frac{dJ(\tau)}{d\tau} = -J(\tau), \\ J(t) = 1. \end{cases}$$

Then we have $J(\tau) = e^{t-\tau}$. For any $T \in (0,\infty)$, one can take $\kappa = e^{-T}\kappa_0$. Then for any measurable set $E \subset \mathbb{R}^2$ with $|E| \leq \kappa$, one has $|\mathcal{J}(0)(E)| \leq e^t e^{-T}\kappa_0 \leq \kappa_0$ for any $t \in [0,T]$. Hence one further has

$$\int_{\mathcal{J}(0)(E)} f_0^\varepsilon(x,v)dxdv \leq \delta.$$

Finally, with (2.6), one gains

$$\begin{aligned} \int_E f^\varepsilon(x,v,t)dxdv &= \int_{\mathcal{J}(0)(E)} f_0^\varepsilon(X^\varepsilon(0;x,v,t),V^\varepsilon(0;x,v,t))e^t J(0)^{-1}dX^\varepsilon dV^\varepsilon \\ &= \int_{\mathcal{J}(0)(E)} f_0^\varepsilon(X^\varepsilon(0;x,v,t),V^\varepsilon(0;x,v,t))dX^\varepsilon dV^\varepsilon \leq \delta, \end{aligned}$$

which then gives (2b).

Therefore, applying Proposition 3.1, we get some subsequence (for simplicity we still denote) f^ε and a nonnegative function $f \in L^\infty([0,T],L^1(\mathbb{R}^2))$ such that

$$f^\varepsilon(x,v,t) \rightharpoonup f(x,v,t) \text{ weakly in } L^1(\mathbb{R}^2), \text{ for any } t > 0 \tag{3.2}$$

and

$$\int\int f dxdv \leq M. \tag{3.3}$$

Next we shall show the convergence of $\int v f^\varepsilon dv$. In fact, using the convexity of kinetic energy and weak convergence of f^ε, one is able to get from (2.20) that

$$\int\int f v^2 dvdx \leq \liminf_{\varepsilon\to 0} \int\int f^\varepsilon v^2 dvdx \leq M. \tag{3.4}$$

For the weak convergence of $\int f^\varepsilon v dv$, we utilize the approach in [1]. Denote $\mathbf{1}_{[-1,1]}(s)$ as $\omega(s)$, for any $\varphi \in C_c^\infty(\mathbb{R}\times[0,T])$, for arbitrary $L > 0$, one can derive

$$\begin{aligned} \int_0^T \int \left(\int v f^\varepsilon dv - \int v f dv \right) \varphi dxdt &= \int_0^T \int \left(\int \omega(\frac{v}{L}) v f^\varepsilon dv - \int \omega(\frac{v}{L}) v f dv \right) \varphi dxdt \\ &\quad + \int_0^T \int\int v f^\varepsilon (1-\omega(\frac{v}{L})) dv \varphi dxdt \\ &\quad - \int_0^T \int\int v f (1-\omega(\frac{v}{L})) dv \varphi dxdt. \end{aligned}$$

By (3.2), i.e. the weak convergence of f^ε in L^1 , one can get that the first term on the right-hand side converges to 0 as $\varepsilon \to 0$. For the last two terms, using (2.20) and (3.4) we also have

$$\left| \int_0^T \int\int v f^\varepsilon (1-\omega(\frac{v}{L})) dv \varphi dxdt \right| \leq \frac{\|\varphi\|_{L^\infty}}{L} \int_0^T \int\int f^\varepsilon v^2 dvdxdt \leq \frac{MT\|\varphi\|_{L^\infty}}{L},$$

$$\left|\int_0^T\int\int vf(1-\omega(\frac{v}{L}))dv\varphi dxdt\right|\le\frac{\|\varphi\|_{L^\infty}}{L}\int_0^T\int\int fv^2dvdxdt\le\frac{MT\|\varphi\|_{L^\infty}}{L},$$

both of which go to 0 when $L\to\infty$. Therefore, we have

$$\int f^\varepsilon vdv\to\int vfdv \text{ in the sense of distributions.} \tag{3.5}$$

Similarly, we also have

$$\int f^\varepsilon dv\to\int fdv \text{ in the sense of distributions.} \tag{3.6}$$

Now, we consider the convergence of u^ε. We also recall the L^∞ compensated compactness framework and Murat's Lemma:

PROPOSITION 3.2 ([5]). *Assume that a sequence $u^\varepsilon(x,t)$ satisfies*

$$\|u^\varepsilon\|_{L^\infty}\le C,$$

and

$$\eta(u^\varepsilon)_t+q(u^\varepsilon)_x \text{ is compact in } H^{-1}_{loc}(\mathbb{R}\times[0,T])$$

for any entropy pair (η,q) with $\eta'(u)u=q'(u)$ (or two special entropy pairs in Theorem 2.7 of [5]). Then there exists a subsequence $\{u^{\varepsilon_k}\}_{k=1}^\infty\subset\{u^\varepsilon\}_{\varepsilon>0}$ and function $u(x,t)$ such that

$$u^{\varepsilon_k}\to u,\quad (u^{\varepsilon_k})^2\to u^2, a.e. \text{ as } k\to\infty.$$

LEMMA 3.1 ([4,27]). *Let $\Omega\in\mathbb{R}^n$ be an open bounded subset, then*

$$(\text{compact set of } W^{-1,a}_{loc}(\Omega))\cap(\text{bounded set of } W^{-1,b}_{loc}(\Omega))\subset(\text{compact set of } H^{-1}_{loc}(\Omega)),$$

where a and b are constants satisfying $1<a\le 2<b$.

From uniform estimate (2.18), there exists a $u(x,t)\in L^\infty(\mathbb{R}\times[0,T])$ such that

$$\|u\|_{L^\infty(\mathbb{R}\times[0,T])}\le M \text{ and } u^\varepsilon(x,t)\rightharpoonup u(x,t), \text{ weak * in } L^\infty(\mathbb{R}\times[0,T]). \tag{3.7}$$

To prove the strong convergence of u^ε, we need to get entropy dissipation estimate. For any compact set $\Omega\subset\mathbb{R}\times[0,T]$, for any $\varphi\in C_c^\infty(\mathbb{R}\times[0,T])$ with $\varphi|_\Omega=1$, multiplying the first equation in (2.1) by $u^\varepsilon\varphi^2$, integrating over $\mathbb{R}\times[0,T]$, one has

$$\int_0^T\int\left[\frac12(u^\varepsilon)^2_t\varphi^2+\frac13(u^\varepsilon)^3_x\varphi^2\right]dxdt=\int_0^T\int\left[\varepsilon u^\varepsilon_{xx}u^\varepsilon\varphi^2+\varepsilon\left(\int f^\varepsilon dv\right)_x u^\varepsilon\varphi^2\right]dxdt$$
$$+\int_0^T\int\left[u^\varepsilon\varphi^2\int f^\varepsilon vdv-\varphi^2(u^\varepsilon)^2\int f^\varepsilon dv\right]dxdt.$$

Integrating by parts gives

$$\int_0^T\int\varepsilon\varphi^2(u^\varepsilon_x)^2dxdt=\int_0^T\int\left[(u^\varepsilon)^2\varphi\varphi_t+\frac23(u^\varepsilon)^3\varphi\varphi_x-2\varepsilon\varphi\varphi_xu^\varepsilon\int f^\varepsilon dv\right]dxdt$$

$$+\int_0^T\int\left[u^\varepsilon\varphi^2\int f^\varepsilon v dv-\varphi^2(u^\varepsilon)^2\int f^\varepsilon dv\right]dxdt$$
$$-\int_0^T\int\left[2\varepsilon\varphi\varphi_x u^\varepsilon u_x^\varepsilon+\varepsilon\varphi^2 u_x^\varepsilon\int f^\varepsilon dv\right]dxdt.$$

From Lemma 2.2, one can easily see that the first two terms are bounded. Applying Hölder's inequality and (3.1), the last term is bounded by

$$\frac{1}{2}\int_0^T\int_\Omega \varepsilon(u_x^\varepsilon)^2 dxdt+M\varepsilon\int_0^T\int\varphi^2\left(\int f^\varepsilon dv\right)^2 dxdt+C(M_0,T).$$

From (2.2) and (2.6), one has

$$\|f\|_{L^\infty}\le e^t\|f_0^\varepsilon\|_{L^\infty}\le\varepsilon^{-1/6}e^T.$$

By (2.19), we further have

$$\varepsilon\int_0^T\int\varphi^2\left(\int f^\varepsilon dv\right)^2 dxdt\le C(M_0,T)\varepsilon^{1/3}. \tag{3.8}$$

So we gain

$$\iint_\Omega \varepsilon(u_x^\varepsilon)^2 dxdt\le C(M_0,T). \tag{3.9}$$

Now we are ready to show the entropy dissipation of (2.1). For any weak entropy-entropy flux (η,q) with $\eta\in C^2$, one has

$$\eta(u^\varepsilon)_t+q(u^\varepsilon)_x=\varepsilon\eta(u^\varepsilon)_{xx}-\varepsilon\eta''(u^\varepsilon)(u_x^\varepsilon)^2+\eta'(u^\varepsilon)\int f^\varepsilon(v-u^\varepsilon)dv$$
$$+\varepsilon\eta'(u^\varepsilon)\left(\int f^\varepsilon dv\right)_x=:\sum_{k=1}^4 I_k,$$

which is obtained by multiplying (2.1) by $\eta'(u^\varepsilon)$. Obviously, from the uniform boundedness of u^ε (2.18), using the estimates (3.1) and (3.9), we obtain that I_2+I_3 is bounded in $L^1_{loc}(\mathbb{R}\times[0,T])$. Thus by embedding theorem and Schauder's theorem I_2+I_3 is compact in $W^{-1,\alpha}_{loc}(\mathbb{R}\times[0,T])$ with some $1<\alpha<2$. For I_1, we also have

$$\left|\int_0^T\int\varepsilon\eta(u^\varepsilon)_{xx}\varphi dxdt\right|=\left|\int_0^T\int\varepsilon\eta'(u^\varepsilon)u_x^\varepsilon\varphi_x dxdt\right|$$
$$\le M\sqrt{\varepsilon}\left(\int_0^T\int_\Omega\varepsilon(u_x^\varepsilon)^2 dxdt\right)^{1/2},$$

for any compact set $\Omega\subset\mathbb{R}\times[0,T]$, and any $\varphi\in C_c^\infty(\Omega)$. So we have that I_1 is compact in $H^{-1}_{loc}(\mathbb{R}\times[0,T])$. For I_4, using (3.8) and (3.9), one also has

$$\left|\int_0^T\int\varepsilon\eta'(u^\varepsilon)\left(\int f^\varepsilon dv\right)_x\varphi dxdt\right|$$

$$
\begin{aligned}
&\leq \left|\int_0^T \int \varepsilon\eta'(u^\varepsilon)\int f^\varepsilon dv \varphi_x dxdt\right| + \left|\int_0^T \int \varepsilon\varphi\eta''(u^\varepsilon)u_x^\varepsilon \int f^\varepsilon dvdxdt\right| \\
&\leq M\varepsilon + M\left(\int_0^T \int \varepsilon(u_x^\varepsilon)^2 dxdt\right)^{1/2}\left(\varepsilon\int_0^T \int \varphi^2\left(\int f^\varepsilon dv\right)^2 dxdt\right)^{1/2} \\
&\leq M\varepsilon^{1/6},
\end{aligned}
$$

which also implies that I_4 is compact in $H_{loc}^{-1}(\mathbb{R}\times[0,T])$. Putting things together, one has

$$
\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } W_{loc}^{-1,\alpha}(\mathbb{R}\times[0,T]) \text{ for some } 1<\alpha<2. \tag{3.10}
$$

From the L^∞ bound of u^ε, one also has

$$
\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is bounded in } W_{loc}^{-1,\infty}(\mathbb{R}\times[0,T]). \tag{3.11}
$$

Applying Murat's lemma (see Lemma 3.1) to (3.10) and (3.11), we finally have

$$
\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } H_{loc}^{-1}(\mathbb{R}\times[0,T]). \tag{3.12}
$$

Therefore, using L^∞ compensated compactness framework (see Proposition 3.2 or Theorem 2.7 in [5]), (3.7) and (3.12), we have

$$
\begin{aligned}
&u^\varepsilon(x,t) \to u(x,t), \quad \text{a.e. in } \mathbb{R}\times[0,T], \\
&u^\varepsilon(x,t) \to u(x,t), \text{ in } L_{loc}^p(\mathbb{R}\times[0,T]), \quad \forall p\in[1,\infty).
\end{aligned} \tag{3.13}
$$

3.2. Limit of equations. To show that (u,f) is a weak solution to (1.1), we need to verify (1.6) and (1.8). For simplicity, here we only show (1.6), since (1.8) can be treated similarly. Multiplying the first equation in (2.1) by $\phi\in C_c^\infty(\mathbb{R}\times[0,T))$, integrating over $\mathbb{R}\times[0,T]$ and using integration by parts, we obtain

$$
\begin{aligned}
&\int \phi(x,0)u_0^\varepsilon(x)dx + \int_0^T\int\left(u^\varepsilon\phi_t + \frac{1}{2}(u^\varepsilon)^2\phi_x + \phi\int f^\varepsilon(v-u^\varepsilon)dv\right)dxdt \\
&\quad + \varepsilon\int_0^T\int\left(u_x^\varepsilon\phi_x + \int f^\varepsilon dv\phi_x\right)dxdt = 0.
\end{aligned}
$$

For the last two terms in the left-hand side, using (3.9), one can derive

$$
\begin{aligned}
\left|\varepsilon\int_0^T\int u_x^\varepsilon\phi_x dxdt\right| &\leq \sqrt{\varepsilon}\left(\int_0^T\int \varepsilon(u_x^\varepsilon)^2 dxdt\right)^{\frac{1}{2}}\left(\int_0^T\int \phi_x^2 dxdt\right)^{\frac{1}{2}} \\
&\leq C(M_0, \|\phi\|_{H^1(\mathbb{R})}, T)\sqrt{\varepsilon},
\end{aligned}
$$

and

$$
\left|\varepsilon\int_0^T\int \phi_x\int f dvdxdt\right| \leq C(M_0, T, \|\phi\|_{C^1(\mathbb{R})})\varepsilon,
$$

both of which go to 0 when $\varepsilon\to 0$. It only remains to show the convergence of $u^\varepsilon\int f^\varepsilon dv$. In fact, observing that

$$
u^\varepsilon\int f^\varepsilon dv - u\int f dv = (u^\varepsilon - u)\int f^\varepsilon dv + \left(\int f^\varepsilon dv - \int f dv\right)u,
$$

using (3.6) and (3.13) we have

$$u^\varepsilon \int f^\varepsilon dv \to u \int f dv \text{ in the sense of distributions.}$$

Thus we have (1.6).

3.3. Entropy inequality. We shall also show entropy inequality for Burgers equation, i.e. (1.7). Multiply the first equation in (2.1) by $\eta'(u^\varepsilon)\varphi$, where η is convex and $\varphi \in C_c^\infty(\mathbb{R}\times(0,T))$ is nonnegative, and integrate the result over $\mathbb{R}\times(0,T)$ to get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\left(\eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x + \varphi\eta'(u^\varepsilon)\int_{\mathbb{R}} f^\varepsilon(v-u^\varepsilon)dv\right)dxdt\\
&= -\int_0^T\int\left(\varepsilon u^\varepsilon_{xx} + \varepsilon\left(\int f^\varepsilon dv\right)_x\right)\eta'(u^\varepsilon)\varphi dxdt\\
&= \varepsilon\int_0^T\int u^\varepsilon_x\eta'(u^\varepsilon)\varphi_x dxdt + \varepsilon\int_0^T\int \eta''(u^\varepsilon)(u^\varepsilon_x)^2\varphi dxdt\\
&\qquad + \int_0^T\int \varepsilon\varphi_x\eta'(u^\varepsilon)\int f^\varepsilon dvdxdt + \int_0^T\int \varepsilon\eta''(u^\varepsilon)u^\varepsilon_x\varphi\int f^\varepsilon dvdxdt\\
&\geq -C(\varphi)(\varepsilon + \varepsilon^{1/2} + \varepsilon^{1/6}),
\end{aligned}$$

where we have used (3.8) and (3.9). We also have

$$\eta'(u^\varepsilon)\int vf^\varepsilon dv - \eta'(u)\int vf dv = (\eta'(u^\varepsilon)-\eta'(u))\int vf^\varepsilon dv + \left(\int vf^\varepsilon dv - \int vf dv\right)\eta'(u),$$

and

$$u^\varepsilon\eta'(u^\varepsilon)\int f^\varepsilon dv - u\eta'(u)\int f dv = (u^\varepsilon\eta'(u^\varepsilon) - u\eta'(u))\int f^\varepsilon dv + \left(\int f^\varepsilon dv - \int f dv\right)u\eta'(u).$$

Using (3.5), (3.6) and (3.13) we get the entropy inequality (1.7) by letting $\varepsilon\to 0$.

Finally, combining with (3.3), (3.4), (3.7), we complete the proof of Theorem 1.1.

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