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GLOBAL SOLUTIONS OF A DIFFUSE INTERFACE MODEL FOR THE TWO-PHASE FLOW OF COMPRESSIBLE VISCOUS FLUIDS IN 1D*

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Abstract. This paper is concerned with a coupled Navier-Stokes/Cahn-Hilliard system describing a diffuse interface model for the two-phase flow of compressible viscous fluids in a bounded domain in one dimension. We prove the existence and uniqueness of global classical solutions for $\rho_0 \in C^{3,\alpha}(I)$. Moreover, we also obtain the global existence of weak solutions and unique strong solutions for $\rho_0 \in H^1(I)$ and $\rho_0 \in H^2(I)$, respectively. In these cases, the initial density function ρ_0 has a positive lower bound.

Keywords. Compressible; Navier-Stokes; Cahn-Hilliard; Global solutions.

AMS subject classifications. 35A01; 35A02; 35Q35.

1. Introduction

In this paper, we investigate a diffusive interface model, which describes the motion of a mixture of two compressible viscous fluids with different densities. Classically, the fluids, which are macroscopically immiscible, are assumed to be separated by a sharp interface. But, in order to describe topological transitions, such as droplet formation, coalescence of several droplet or droplet breakup, we need to take into account a partial mixing on a small length scale in the model. As a result, the sharp interface of the two fluids is replaced by a narrow transition layer, and an order parameter related to the concentration difference of both fluids is introduced. This model can be described by coupled Navier-Stokes/Cahn-Hilliard equations. Navier-Stokes equations govern the dynamic character of the fluids, such as velocity. The interaction of the fluids on the interface, such as the change of the concentration caused by diffusion, is described by Cahn-Hilliard equations. It is evident that, the change of the concentration is effected by the velocity of the fluids. And the velocity of the fluids is also related with the concentration, because of the surface tension. Therefore, one obtains coupled Navier-Stokes/Cahn-Hilliard equations both governing the fluid velocity and describing the concentration difference of the two fluids. In fact, the concentration difference can also be assumed to satisfy different variants of Allen-Cahn or other types of dynamics [6, 15]. However, numerical simulations show that the Cahn-Hilliard model is much more effective for predicting droplet breakup phenomenon (see [24]). In this work, we are interested in the Navier-Stokes/Cahn-Hilliard system.

The model considered here was first deduced by Lowengrub and Truskinovsky [29].

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It has been modified and studied by Abels and Feireisl [4] in the following form

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0, \\ \rho\partial_t\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}-\operatorname{div}\mathbb{S}+\nabla p=-\operatorname{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}\right), \\ \rho\partial_t\chi+\rho\mathbf{u}\cdot\nabla\chi=\Delta\mu, \\ \rho\mu=\rho\frac{\partial f}{\partial\chi}-\Delta\chi \end{cases} \tag{$\star$}$$

with $\mathbb{S}=2\lambda(\chi)\mathbb{D}(\mathbf{u})+\eta(\chi)\operatorname{div}\mathbf{u}\mathbb{I}$, $\mathbb{D}(\mathbf{u})=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)-\frac{1}{3}\operatorname{div}\mathbf{u}\mathbb{I}$ and the pressure $p=\rho^2\frac{\partial f}{\partial\rho}(\rho,\chi)$, where $\rho\geq 0$, $\mathbf{u}$, χ, μ denote the total density, the mean velocity of the fluid mixture, the (mass) concentration difference of the two components and the chemical potential, respectively. The functions $\lambda(\chi)>0$, $\eta(\chi)\geq 0$ and the free energy density $f(\rho,\chi)$ are to be specified later. The first and the second equations of ($\star$) are compressible Navier-Stokes equations, which has an extra term $\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}$ describing capillary effect related to the free energy

$$E_{free}(\rho,\chi)=\int_\Omega\left(\rho f(\rho,\chi)+\frac{1}{2}|\nabla\chi|^2\right)\mathrm{d}x.$$

The third and the last equations in ($\star$) are Cahn-Hilliard equations.

When the difference of the densities of two components is negligible, or the densities of both components as well as the density of the mixture are constant, the system reduces to an incompressible one. In this case, Gurtin et al. [16] derived an incompressible model, which has been paid much attention. Boyer [8] studied this flow under shear in detail, where the diffusion coefficient is allowed to be degenerate, the viscosity depends on the concentration, and logarithmic-type potentials are included. Under these assumptions, Boyer proved the existence and uniqueness of global weak and strong solutions in 2D, the existence of global weak and local strong solutions in 3D, as well as local asymptotic stability of suitable stationary solutions. Abels [1] investigated this model in the case of constant mobility, nonconstant viscosity and singular potentials. In [1], Abels proved the existence and uniqueness results, the regularity of solutions and the convergence to a single equilibrium. Moreover, there are also other results about this model, concerning the well-posedness, asymptotic behavior of solutions, global attractor, numerical simulations, etc. We refer the readers to [9, 17, 18, 21, 24] and references therein.

For incompressible fluids with general densities, Abels et al. [5] established a model by defining the mean velocity of the mixture as volume-averaged velocity. Such a mean velocity field is divergence free. By sending the interfacial thickness to zero, they obtained various sharp interface models. The authors proved that all sharp interface models fulfill natural energy inequalities. In another paper [2], Abels considered a different model, which assumes that the velocity field is no longer divergence free, and the pressure enters the equation as the chemical potential. With the aid of a two-level approximation, the author proved the existence of weak solutions for the non-stationary system in 2D and 3D. Recently, Abels et al. [3] showed the existence of weak solutions for a new model. Boyer [7] supposed that the velocity field is divergence free, and he showed the local existence of unique strong solutions. The author also proved that if the densities tend to 1, i.e. in the slightly nonhomogeneous case, there exist global weak solutions and unique local strong solutions, which are in fact global in 2D. An asymptotic stability result for the metastable states was also given. In 2015 Liu et al. [28] deduced another kind of Navier-Stokes/Cahn-Hilliard system by the energetic variational approaches, and gave some numerical experiments. Later Jiang et al. [20] derived a similar Navier-Stokes/Allen-Cahn system, and proved the existence of weak solutions in 3D, the well-posedness of strong solutions in 2D, and the longtime behavior of the 2D strong solutions. All these results are obtained under the assumption that the density is a function of the concentration. For incompressible Navier-Stokes/Allen-Cahn system with free density function, Li et al. studied the existence of unique local strong solutions [25], and the main mechanism for possible breakdown of such a local strong solution [26].

For compressible fluids with general densities, a case more closer to the physical reality, Lowengrub and Truskinovsky [29] derived a thermodynamically consistent model. The authors defined the mean velocity as mass-averaged velocity, which yields the conservation of mass. They showed that, when the densities of the components are not perfectly matched, the evolution of the concentration field always leads to the fluid motion, even if the fluids are inviscid. This model can also be found in [6]. As far as we know, there are only a few theoretical results about compressible models. Kotschote and Zacher [23] proved the existence and uniqueness of local strong solutions of the model derived in [29]. By neglecting the effect of the density with respect to the gradient of the concentration in the free energy, Abels and Feireisl [4] deduced a variant model. The authors showed the existence of weak solutions in 3D, by adding artificial pressure and implicit time discretization, where the density is a renormalized solution. For the compressible Navier-Stokes/Allen-Cahn system proposed by Blesgen [10], Kotschote [22] got the local existence of unique strong solutions, Feireisl et al. [14] proved the existence of weak solutions in 3D, where the density ρ is a measurable function. In [13], Ding et al. studied 1D case and obtained the well-posedness of the solutions. A different compressible Navier-Stokes/Allen-Cahn system, arising from the biological material change in the process of stem cell differentiation, has been studied in [33]. The existence of spherically symmetric weak solutions was obtained.

In this paper, we deal with the solvability of the one dimensional compressible Navier-Stokes/Cahn-Hilliard system. We prove the existence of unique classical solutions, unique strong solutions and weak solutions, when the initial density ρ_0 is away from vacuum states and belongs to $C^{3,\alpha}(I)$, $H^2(I)$ and $H^1(I)$, respectively.

It is well known that, for ideal polytropic fluids, the pressure $p=R\rho^\gamma$ with constants $R>0$ and $\gamma>1$, see [12,19] for example. On the other hand, in the theory of the Cahn-Hilliard equation, double-well structural potential is often considered. A typical example of such potential is the logarithmic type, which is suggested by Cahn and Hilliard [11]. However, this potential is usually replaced by a polynomial approximation of the type $\gamma_1\chi^4-\gamma_2\chi^2$, where γ_1 and γ_2 are positive constants, see [35,36] and references therein. Therefore, it is reasonable to take a specific free energy f as follows

$$f(\rho,\chi)=\frac{R\rho^{\gamma-1}}{\gamma-1}+\frac{\chi^4}{4}-\frac{\chi^2}{2}.$$

Moreover, we assume that the functions $\lambda(\chi)=\nu$ and $\eta(\chi)=-\frac{1}{3}\nu$ are constants. Then the system $(\star)$ in one dimension is simplified into the following form

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ \rho u_t+\rho u u_x+R(\rho^\gamma)_x=\nu u_{xx}-\frac{1}{2}\left(\chi_x^2\right)_x,\\ \rho\chi_t+\rho u\chi_x=\mu_{xx},\\ \rho\mu=\rho(\chi^3-\chi)-\chi_{xx},\end{cases} \tag{1.1}$$

where $(\rho,u,\chi):(0,1)\times\mathbb{R}_+\to\mathbb{R}_+\times\mathbb{R}^2$, $\rho\ge 0$ is the total density, u denotes the mean velocity of the fluid mixture, χ represents the concentration difference of the two fluids, μ is the chemical potential and $\nu>0$ is the viscous coefficient. Moreover, we supplement the system (1.1) with the following initial value condition

$$(\rho,u,\chi)\Big|_{t=0}=(\rho_0,u_0,\chi_0),\quad x\in[0,1] \tag{1.2}$$

and the boundary value condition

$$(u,\chi_x,\mu_x)\Big|_{x=0,1}=(0,0,0),\qquad t\ge 0. \tag{1.3}$$

Remark 1.1. For the free energy specified above, one gets a strongly coupled system, including a fourth order diffusion Equation $(1.1)_{3,4}$. Therefore, we are unable to ensure the concentration difference staying in the physical reasonable interval $[-1,1]$, since we do not have the comparison principle for such a fourth order diffusion equation.

On the other hand, the density ρ itself and its derivatives up to second order, with respect to x-variable, enter the coefficients of the Cahn-Hilliard Equation $(1.1)_{3,4}$. In fact, this is the main difference from the models for incompressible fluids. Hence, to prove the existence of classical solutions, we have to estimate ρ_{xxx} first. Meanwhile, one observes that the system (1.1) is strongly coupled and the equations therein are strongly nonlinear. All of these suggest the main difficulties in the *a priori* estimates.

Notation 1.1.

(1) $I=(0,1)$, $\partial I=\{0,1\}$, $Q_T=I\times(0,T)$ for $T>0$.

(2) For $p\ge 1$, denote $L^p=L^p(I)$ as the L^p space with the norm $\|\cdot\|_{L^p}$. For $k\ge 1$ and $p\ge 1$, denote $W^{k,p}=W^{k,p}(I)$ for the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k=W^{k,2}(I)$.

(3) For any nonnegative integer k and $0<\alpha<1$, denote the Hölder spaces

$$C^{2k+\alpha,k+\frac{\alpha}{2}}(\overline{Q}_T)=\{u;\ \partial_x^\beta\partial_t^r\in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T),\ \text{for any } \beta,r \text{ such that } \beta+2r\le 2k\},$$
$$C^{4k+\alpha,k+\frac{\alpha}{4}}(\overline{Q}_T)=\{u;\ \partial_x^\beta\partial_t^r\in C^{\alpha,\frac{\alpha}{4}}(\overline{Q}_T),\ \text{for any } \beta,r \text{ such that } \beta+4r\le 4k\}.$$

The main purpose of this paper is to investigate the solvability of the problem (1.1)–(1.3) with $\rho_0\ge c_0>0$. Our first result is about global classical solutions.

Theorem 1.1. *Assume that $\rho_0\in C^{3,\alpha}(I)$ satisfies $0<c_0^{-1}\le\rho_0\le c_0$ for some constants $\alpha\in(0,1)$ and c_0, $u_0\in C^{3,\alpha}(I)$ with $u_0(0)=u_0(1)=0$, $\chi_0\in C^{4,\alpha}(I)$. Then there exists a unique classical solution $(\rho,u,\chi):I\times\mathbb{R}_+\to\mathbb{R}_+\times\mathbb{R}^2$ of the initial boundary value problem (1.1)–(1.3) satisfying that, for any $T>0$, there exists a constant $c=c(c_0,T)>0$ such that*

$$(\rho_{xxx},\rho_{xxt})\in C^{\frac{\alpha}{2},\frac{\alpha}{4}}(\overline{Q}_T),\quad 0<c^{-1}\le\rho\le c \text{ on } Q_T,$$
$$u_x\in C^{2+\frac{\alpha}{2},1+\frac{\alpha}{4}}(\overline{Q}_T),\qquad \chi\in C^{4+\alpha,1+\frac{\alpha}{4}}(\overline{Q}_T).$$

We also obtain the existence of unique strong solutions and weak solutions which are defined as follows.

Definition 1.1. *Let $\rho_0\in H^2(I)$ satisfies $0<c_0^{-1}\le\rho_0\le c_0$ for some constant c_0, $u_0\in H_0^1(I)$ and $\chi_0\in H^4(I)$. A triplet (ρ,u,χ) is called a strong solution to the problem (1.1)–(1.3), if*

$$\rho\in L^\infty(0,T;H^1),\quad \rho_t\in L^\infty(0,T;L^2),\quad 0<c^{-1}\le\rho\le c,$$

$$
\begin{aligned}
&u\in L^\infty(0,T;H_0^1)\cap L^2(0,T;H^2), \qquad u_t\in L^2(0,T;L^2),\\
&\chi\in L^\infty(0,T;H^3), \qquad \chi_t\in L^\infty(0,T;L^2)\cap L^2(0,T;H^2),\\
&\mu\in L^\infty(0,T;H^2)\cap L^2(0,T;H^3), \qquad \mu_t\in L^2(0,T;L^2),
\end{aligned}
$$

where (ρ,u,χ) *satisfies (1.1)* a.e. *in* Q_T*, and*

$$
\begin{aligned}
&(\rho,u,\chi)\big|_{t=0}=(\rho_0,u_0,\chi_0) \quad \text{a.e. } in\ I,\\
&(u,\chi_x,\mu_x)\big|_{x=0,1}=(0,0,0), \quad t\geq 0
\end{aligned}
$$

in the sense of trace.

Definition 1.2. *Let* $\rho_0\in H^1(I)$ *satisfies* $0<c_0^{-1}\leq\rho_0\leq c_0$ *for some constant* c_0*,* $u_0\in L^2(I)$ *and* $\chi_0\in H^1(I)$*. A triplet* (ρ,u,χ) *is called a weak solution to the problem (1.1)–(1.3), if*

$$
\begin{aligned}
&\rho\in L^\infty(0,T;H^1), \quad \rho_t\in L^2(0,T;L^2), \quad 0<c^{-1}\leq\rho\leq c,\\
&u\in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1),\\
&\chi\in L^\infty(0,T;H^1)\cap L^2(0,T;H^3), \qquad \mu\in L^2(0,T;H^1)
\end{aligned}
$$

such that

$$
\iint_{Q_T}\Big(\rho_t\zeta(x)-\rho u\zeta'(x)\Big)\mathrm{d}x\mathrm{d}t=0, \qquad \text{for any } \zeta(x)\in C^1(I),
$$

and

$$
\begin{aligned}
&-\iint_{Q_T}\Big(\rho u\xi(x)\eta'(t)+\rho u^2\xi'(x)\eta(t)+\rho^\gamma\xi'(x)\eta(t)\Big)\mathrm{d}x\mathrm{d}t\\
&=\int_I\rho_0u_0\xi(x)\eta(0)\mathrm{d}x-\iint_{Q_T}\left(u_x\xi'(x)\eta(t)-\frac{1}{2}\chi_x^2\xi'(x)\eta(t)\right)\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

for any $\xi(x)\in C_0^1(I)$*,* $\eta(t)\in C^1[0,T]$ *with* $\eta(T)=0$*. Moreover,*

$$
-\iint_{Q_T}(\rho\chi\phi_t+\rho u\chi\phi_x)\mathrm{d}x\mathrm{d}t=\int_I\rho_0\chi_0\phi(0)\mathrm{d}x-\iint_{Q_T}\mu_x\phi_x\mathrm{d}x\mathrm{d}t
$$

and

$$
\iint_{Q_T}\rho\mu\phi\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\big(\rho(\chi^3-\chi)\phi+\chi_x\phi_x\big)\mathrm{d}x\mathrm{d}t
$$

hold for any $\phi\in C^1(Q_T)$ *with* $\phi(\cdot,T)=0$*.*

Theorem 1.2. *Let* $\rho_0\in H^2(I)$ *satisfies* $0<c_0^{-1}\leq\rho_0\leq c_0$ *for some constant* c_0*,* $u_0\in H_0^1(I)$ *and* $\chi_0\in H^4(I)$*. Then the problem* (1.1)–(1.3) *admits a unique strong solution.*

Theorem 1.3. *Let* $\rho_0\in H^1(I)$ *satisfies* $0<c_0^{-1}\leq\rho_0\leq c_0$ *for some constant* c_0*,* $u_0\in L^2(I)$ *and* $\chi_0\in H^1(I)$*. Then the problem* (1.1)–(1.3) *admits at least one weak solution.*

Remark 1.2. To our knowledge, there are few theoretical results about compressible Navier-Stokes/Cahn-Hilliard system. Abels and Feireisl [4] obtained the existence of weak solutions, where the density ρ is a renormalized solution. Kotschote and Zacher [23] established the local existence of unique strong solutions. Even for compressible

Navier-Stokes/Allen-Cahn system, only the existence of weak solutions and spherically symmetric weak solutions have been obtained, see Feireisl et al. [14] and Witterstein [33]. In present paper, we only consider the 1D problem with the specified free energy, but we hope that our study can be a good beginning for further investigations.

Since the constants R and ν play no role in the analysis, we assume henceforth that $R=\nu=1$.

This paper is organized as follows. In Section 2, we discuss the local existence of a unique strong solution to the problem (1.1)–(1.3) by the Schauder fixed-point theorem. Then we show that, if the initial data is smooth enough, the local strong solution is classical. In Section 3, we obtain *a priori* estimates for the classical solution of the problem (1.1)–(1.3). In Section 4, we prove our main results by weakly convergent method and energy argument.

2. Local classical solutions

In this section, we investigate the existence and uniqueness of local classical solutions to the problem (1.1)–(1.3). Our main result is as follows.

THEOREM 2.1. *Assume that $\rho_0\in C^{3,\alpha}(I)$ satisfies $0<c_0^{-1}\le\rho_0\le c_0$ for some constants c_0 and $\alpha\in(0,1)$, $u_0\in C^{3,\alpha}(I)$ with $u_0(0)=u_0(1)=0$, $\chi_0\in C^{4,\alpha}(I)$. Then there exist a small time $T_*>0$, a constant $c=c(c_0,T_*)$ and a unique classical solution (ρ,u,χ): $I\times[0,T_*)\to\mathbb{R}_+\times\mathbb{R}^2$ to the initial boundary value problem (1.1)–(1.3) such that*

$$
\begin{aligned}
&(\rho_{xxx},\rho_{xxt})\in C^{\frac{\alpha}{2},\frac{\alpha}{4}}(Q_{T_*}), \quad 0<c^{-1}\le\rho\le c \quad \text{on } Q_{T_*},\\
&u_x\in C^{2+\frac{\alpha}{2},1+\frac{\alpha}{4}}(Q_{T_*}), \qquad \chi\in C^{4+\alpha,1+\frac{\alpha}{4}}(Q_{T_*}).
\end{aligned}
$$

Before proving this theorem, we show the local existence of unique strong solutions under the assumptions $\rho_0\in H^3(I)$ with $0<c_0^{-1}\le\rho_0\le c_0$, $u_0\in H^3(I)$ and $\chi_0\in H^4(I)$, which is much stronger than the assumptions in Theorem 1.2 for the global existence of unique strong solutions. After that, we will prove that, if the initial data is smooth enough satisfying the assumptions in Theorem 2.1, the unique local strong solution is classical.

PROPOSITION 2.1. *Let $\rho_0\in H^3(I)$, $0<c_0^{-1}\le\rho_0\le c_0$ for some constant c_0, and $u_0\in H_0^1(I)\cap H^3(I)$, $\chi_0\in H^4(I)$. Then there exist a small time $T_*>0$, a constant $c=c(c_0,T_*)$ and a unique strong solution (ρ,u,χ) to the problem (1.1)–(1.3) such that*

$$
\begin{aligned}
&\rho\in L^\infty(0,T_*;H^3), \qquad \rho_t\in L^\infty(0,T_*;H^2), \qquad 0<c^{-1}\le\rho\le c,\\
&u\in L^\infty(0,T_*;H_0^1\cap H^3)\cap L^2(0,T_*;H^4), \quad u_t\in L^\infty(0,T_*;H_0^1)\cap L^2(0,T_*;H^2),\\
&\chi\in L^\infty(0,T_*;H^3)\cap L^2(0,T_*;H^4), \qquad \chi_t\in L^\infty(0,T_*;L^2)\cap L^2(0,T_*;H^2),\\
&\mu\in L^\infty(0,T_*;H^2)\cap L^2(0,T_*;H^4), \qquad \mu_t\in L^2(0,T_*;L^2).
\end{aligned}
$$

In order to prove this proposition, we consider the following auxiliary system

$$
\begin{cases}
\rho_t+(\rho v)_x=0,\\
\rho u_t+\rho v u_x+(\rho^\gamma)_x=u_{xx}-\dfrac{1}{2}\left(\chi_x^2\right)_x,\\
\rho\chi_t+\rho v\chi_x=\mu_{xx},\\
\rho\mu=\rho(\varphi^3-\varphi)-\chi_{xx}
\end{cases}
\tag{2.1}
$$

subject to the initial boundary value conditions (1.2) and (1.3), where v and φ are known functions which satisfy the boundary value conditions $v\big|_{\partial I}=0$ and $\varphi_x\big|_{\partial I}=0$ for $t\ge 0$.

The following result for the auxiliary system is sufficient to prove the local existence of strong solutions to the problem (1.1)–(1.3).

LEMMA 2.1. *Let T be a fixed time with $0<T<1$. Assume that $v(x,0)=u_0(x)$, $\varphi(x,0)=\chi_0(x)$ for $x\in I$ and*

$$\sup_{0\le t\le T}(\|v\|^2_{H_0^1\cap H^3}+\|v_t\|^2_{H_0^1})+\int_0^T\left(\|v\|^2_{H_0^1\cap H^4}+\|v_t\|^2_{H^2}\right)\le K_1, \tag{2.2}$$

$$\sup_{0\le t\le T}(\|\varphi\|^2_{H^3}+\|\varphi_t\|^2_{L^2})+\int_0^T(\|\varphi\|^2_{H^4}+\|\varphi_t\|^2_{H^2})\le K_2 \tag{2.3}$$

holds for some constants $K_1,K_2>1$. Then there exists a unique strong solution (ρ,u,χ) to the problem (2.1), (1.2) and (1.3) such that

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho\|^2_{H^3}+K_1^{-1}\|\rho_t\|^2_{H^2}+\|\rho^{-1}\|^2_{L^\infty}+\|u\|^2_{H_0^1\cap H^3}+\|u_t\|^2_{H_0^1}+\|\chi\|^2_{H^3}+\|\chi_t\|^2_{L^2}+\|\mu\|^2_{H^2}\right)\\&\quad+\int_0^T\left(\|u\|^2_{H_0^1\cap H^4}+\|u_t\|^2_{H^2}+\|\chi\|^2_{H^4}+\|\chi_t\|^2_{H^2}\right)+\|\mu\|^2_{H^4}+\|\mu_t\|^2_{L^2}\le\overline{C},\end{aligned}$$

where $\overline{C}:=C(K_1T^{1/2},K_1^2T,K_2^3T,T)>0$ is a constant depending only on $K_1T^{1/2}$, K_1^2T, K_2^3T and T.

The existence and uniqueness of strong solutions to the hyperbolic Equation $(2.1)_1$ is well known. Moreover, the solution ρ satisfies the following estimates

$$\sup_{0\le t\le T}\left(\|\rho\|_{H^3}+K_1^{-1/2}\|\rho_t\|_{H^2}+\|\rho^{-1}\|_{L^\infty}\right)\le c\exp\{cK_1T^{1/2}\}. \tag{2.4}$$

For the proof of this result, we refer to [32] and remind that v satisfies (2.2). From $(2.1)_2$ and $(2.1)_{3,4}$, we have

$$u_t=\frac{1}{\rho}u_{xx}-vu_x-\gamma\rho^{\gamma-2}\rho_x-\frac{1}{\rho}\chi_x\chi_{xx}, \tag{2.5}$$

$$\chi_t=-\frac{1}{\rho^2}\chi_{xxxx}+2\frac{\rho_x}{\rho^3}\chi_{xxx}+\left(\frac{\rho_{xx}}{\rho^3}-2\frac{\rho_x^2}{\rho^4}\right)\chi_{xx}-v\chi_x+\frac{1}{\rho}(3\varphi^2-1)\varphi_{xx}+\frac{6}{\rho}\varphi\varphi_x^2. \tag{2.6}$$

It follows from classical arguments (see [27,30]) that the above linear parabolic equations subject to (1.2) and (1.3) have a unique strong solution (u,χ). It remains for us to do some necessary a priori estimates for u, χ and μ. We begin with χ and μ.

LEMMA 2.2. *It holds that*

$$\sup_{0\le t\le T}\int_I\chi^2(t)+\iint_{Q_T}(\chi_{xx}^2+\mu^2)\le C(K_1T^{1/2},K_2^3T). \tag{2.7}$$

Proof. Multiplying $(2.1)_3$ by χ, then integrating the result with respect to x over I, using integration by parts, $(2.1)_{1,4}$ and (2.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi^2&=\int_I\mu\chi_{xx}=\int_I(\varphi^3-\varphi)\chi_{xx}-\int_I\frac{1}{\rho}\chi_{xx}^2\\&\le-\frac{1}{2}\int_I\frac{1}{\rho}\chi_{xx}^2+\frac{1}{2}\int_I\rho(\varphi^3-\varphi)^2,\end{aligned}$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho\chi^2+\int_I \frac{1}{\rho}\chi_{xx}^2\le 2\|\rho\|_{L^\infty}(\|\varphi\|_{L^\infty}^6+\|\varphi\|_{L^\infty}^2).$$

Integrating the above inequality over $(0,t)$, by (2.3) and (2.4), we obtain

$$\int_I \rho\chi^2(t)+\int_0^t\int_I \frac{1}{\rho}\chi_{xx}^2\le c_0\|\chi_0\|_{L^2}^2+C(K_1T^{1/2})t\|\varphi\|_{L^\infty}^6\le C+C(K_1T^{1/2})K_2^3t,$$

from which and the equation $(2.1)_4$, we see that (2.7) holds. The proof is complete. □

If we choose $0<T<T_*:=\min\{\frac{1}{K_1^2},\frac{1}{K_2^3}\}$, then $C(K_1T^{1/2},\, K_1^2T,\, K_2^3T,\, T)\le C$, where C is a constant independent of K_1 and K_2. For convenience, here and below, we denote by C a constant, whose value may be different from line to line but is independent of K_1 and K_2.

Lemma 2.3. *We have the inequality*

$$\sup_{0\le t\le T}\int_I \chi_x^2(t)+\iint_{Q_T}(\chi_{xxx}^2+\mu_x^2)\le C,\qquad 0<T<T_*. \tag{2.8}$$

Proof. Multiplying $(2.1)_3$ by μ, then integration the result over I, using integration by parts, $(2.1)_{1,3,4}$ and (2.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I \chi_x^2+\int_I \mu_x^2&=\int_I v\chi_x\chi_{xx}-\int_I \rho(\varphi^3-\varphi)(\chi_t+v\chi_x)\\&=\frac{1}{2}\int_I v(\chi_x^2)_x-\int_I(\varphi^3-\varphi)\mu_{xx}\\&=-\frac{1}{2}\int_I v_x\chi_x^2+\int_I(3\varphi^2-1)\varphi_x\mu_x\\&\le\|v_x\|_{L^\infty}\int_I\chi_x^2+\frac{1}{2}\int_I\mu_x^2+\frac{1}{2}\int_I(3\varphi^2-1)^2\varphi_x^2,\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I \chi_x^2+\int_I \mu_x^2\le\|v_x\|_{L^\infty}\int_I\chi_x^2+C\|\varphi\|_{L^\infty}^4\|\varphi\|_{H^1}^2\le K_1^{1/2}\int_I\chi_x^2+CK_2^3. \tag{2.9}$$

By Grönwall's inequality, we get

$$\int_I\chi_x^2\le\exp\left\{K_1^{1/2}t\right\}\left(\|\chi_0\|_{H^1}^2+CK_2^3t\right)\le C,$$

provided that $0<t<T_*$. Integrating (2.9) from 0 to $T(<T_*)$, we have

$$\int_I\chi_x^2+\iint_{Q_T}\mu_x^2\le C. \tag{2.10}$$

Differentiating $(2.1)_4$ with respect to x and using (2.3), (2.4), (2.7), (2.10), we obtain (2.8). This completes the proof. □

Lemma 2.4. *There holds*

$$\sup_{0\le t\le T}\int_I \chi_t^2(t)+\iint_{Q_T}(\chi_{xxt}^2+\mu_t^2+\mu_{xxxx}^2)\le C,\qquad 0<T<T_*. \tag{2.11}$$

Proof. Differentiating $(2.1)_3$ with respect to t, then multiplying the result by χ_t, integrating over I and using $(2.1)_{1,4}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi_t^2+\int_I\rho_t\chi_t^2+\int_I\rho_t v\chi_x\chi_t+\int_I\rho v_t\chi_x\chi_t\\=&\int_I\mu_{xxt}\chi_t=-\int_I\mu_{xt}\chi_{xt}=\int_I\mu_t\chi_{xxt}\\=&-\int_I\frac{1}{\rho}\chi_{xxt}^2-\int_I\left(\frac{1}{\rho}\right)_t\chi_{xx}\chi_{xxt}+\int_I(3\varphi^2-1)\varphi_t\chi_{xxt},\end{aligned}$$

from which we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi_t^2+\int_I\frac{1}{\rho}\chi_{xxt}^2\\=&-\int_I\rho_t\chi_t^2-\int_I\rho_t v\chi_x\chi_t-\int_I\rho v_t\chi_x\chi_t-\int_I\left(\frac{1}{\rho}\right)_t\chi_{xx}\chi_{xxt}+\int_I(3\varphi^2-1)\varphi_t\chi_{xxt}\\\le&\frac{1}{2}\int_I\frac{1}{\rho}\chi_{xxt}^2+C(\|\rho_t\|_{L^\infty}+\|\rho_t\|_{L^\infty}^2+\|\rho\|_{L^\infty})\int_I\rho\chi_t^2+C\|v\|_{L^\infty}^2\int_I\chi_x^2\\&+C\|\chi_x\|_{L^\infty}^2\int_I v_t^2+C\|\rho_t\|_{L^\infty}^2\int_I\chi_{xx}^2+\int_I\rho(3\varphi^2-1)^2\varphi_t^2\\\le&\frac{1}{2}\int_I\frac{1}{\rho}\chi_{xxt}^2+CK_1\int_I\rho\chi_t^2+C(\|v\|_{L^\infty}^2+\|v_t\|_{L^2}^2+\|\rho_t\|_{L^\infty}^4)\\&+C\int_I\chi_{xxx}^2+C\|3\varphi^2-1\|_{L^\infty}^2\|\varphi_t\|_{L^2}^2,\end{aligned}$$

where we have used the inequality

$$\|\rho_t\|_{L^\infty}^2\int_I\chi_{xx}^2=-\|\rho_t\|_{L^\infty}^2\int_I\chi_{xxx}\chi_x\le\int_I\chi_{xxx}^2+\|\rho_t\|_{L^\infty}^4\int_I\chi_x^2.$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi_t^2+\int_I\frac{1}{\rho}\chi_{xxt}^2\le CK_1\int_I\rho\chi_t^2+CK_1^2+C\int_I\chi_{xxx}^2+CK_2^3.\tag{2.12}$$

Recalling (2.6), we see that

$$\|\sqrt{\rho}\chi_t(0)\|_{L^2}\le C(\|\rho_0\|_{H^2},\|u_0\|_{L^2},\|\chi_0\|_{H^4}).\tag{2.13}$$

Then Grönwall's inequality implies

$$\int_I\rho\chi_t^2\le\exp\{CK_1t\}\left(\int_I\rho\chi_t^2(0)+CK_1^2t+C\iint_{Q_t}\chi_{xxx}^2+CK_2^3t\right)\le C,$$

provided that $0<t<T_*$. Integrating (2.12) over $(0,T)$, we have

$$\int_I\rho\chi_t^2+\iint_{Q_T}\frac{1}{\rho}\chi_{xxt}^2\le C.\tag{2.14}$$

Differentiating $(2.1)_4$ with respect to t and $(2.1)_3$ with respect to x twice, we have

$$\mu_t=(3\varphi^2-1)\varphi_t+\frac{\rho_t}{\rho^2}\chi_{xx}-\frac{1}{\rho}\chi_{xxt},$$

$$\mu_{xxxx} = \rho_{xx}\chi_t + 2\rho_x\chi_{xt} + \rho\chi_{xxt} + \rho_{xx}v\chi_x + \rho v_{xx}\chi_x + \rho v\chi_{xxx} \\ + 2\rho_x v_x\chi_x + 2\rho_x v\chi_{xx} + 2\rho v_x\chi_{xx}.$$

By (2.3), (2.2), (2.4), (2.8) and (2.14), Lemma 2.4 follows. □

LEMMA 2.5. *We have the inequality*

$$\sup_{0\le t\le T}\int_I \chi_{xx}^2(t) + \iint_{Q_T}\chi_{xxxx}^2 \le C, \qquad 0<T<T_*. \tag{2.15}$$

Proof. Multiplying $(2.1)_3$ by χ_{xxxx} and integrating the result over I, we have

$$\int_I \rho\chi_t\chi_{xxxx} + \int_I \rho v\chi_x\chi_{xxxx} = \int_I \mu_{xx}\chi_{xxxx}. \tag{2.16}$$

From $(2.1)_4$ and the boundary conditions $\mu_x\big|_{\partial I} = \varphi_x\big|_{\partial I} = 0$, we get $\left(\frac{1}{\rho}\chi_{xx}\right)_x\Big|_{\partial I} = 0$. Moreover,

$$\frac{1}{\rho}\chi_{xxxx} = \left(\frac{1}{\rho}\chi_{xx}\right)_{xx} - 2\left(\frac{1}{\rho}\right)_x\chi_{xxx} - \left(\frac{1}{\rho}\right)_{xx}\chi_{xx}.$$

Using the above equality and integrating by parts, we have

$$\begin{aligned}
&\int_I \rho\chi_t\chi_{xxxx} = \int_I \rho^2\chi_t\frac{1}{\rho}\chi_{xxxx}\\
&= \int_I \rho^2\chi_t\left(\frac{1}{\rho}\chi_{xx}\right)_{xx} - 2\int_I \rho^2\chi_t\left(\frac{1}{\rho}\right)_x\chi_{xxx} - \int_I \rho^2\chi_t\left(\frac{1}{\rho}\right)_{xx}\chi_{xx}\\
&= -\int_I \rho^2\chi_{xt}\left(\frac{1}{\rho}\chi_{xx}\right)_x - 2\int_I \rho\rho_x\chi_t\left(\frac{1}{\rho}\chi_{xx}\right)_x + 2\int_I \chi_t\rho_x\chi_{xxx} + \int_I \rho^2\chi_t\left(\frac{\rho_x}{\rho^2}\right)_x\chi_{xx}\\
&= \int_I \rho\chi_{xxt}\chi_{xx} + 2\int_I \rho_x\chi_{xt}\chi_{xx} + 2\int_I \chi_t\frac{\rho_x^2}{\rho}\chi_{xx} - 2\int_I \rho_x\chi_t\chi_{xxx}\\
&\quad + 2\int_I \chi_t\rho_x\chi_{xxx} + \int_I \chi_t\rho_{xx}\chi_{xx} - 2\int_I \chi_t\frac{\rho_x^2}{\rho}\chi_{xx}\\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho\chi_{xx}^2 - \frac{1}{2}\int_I \rho_t\chi_{xx}^2 + 2\int_I \rho_x\chi_{xt}\chi_{xx} + \int_I \chi_t\rho_{xx}\chi_{xx}.
\end{aligned} \tag{2.17}$$

On the other hand, $(2.1)_4$ implies

$$\begin{aligned}
\mu_{xx} &= -\frac{1}{\rho}\chi_{xxxx} + 2\frac{\rho_x}{\rho^2}\chi_{xxx} + \left(\frac{\rho_{xx}}{\rho^2} - 2\frac{\rho_x^2}{\rho^3}\right)\chi_{xx} + (3\varphi^2-1)\varphi_{xx} + 6\varphi\varphi_x^2\\
&:= -\frac{1}{\rho}\chi_{xxxx} + A,
\end{aligned} \tag{2.18}$$

where

$$\int_I A^2 \le C\int_I \chi_{xxx}^2 + C\|\chi_{xx}\|_{L^\infty}^2\int_I \rho_{xx}^2 + C\int_I \chi_{xx}^2 + C\|\varphi\|_{L^\infty}^4\|\varphi_{xx}\|_{L^2}^2.$$

Substitute (2.17) and (2.18) into (2.16) to give

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho\chi_{xx}^2 + \int_I \frac{1}{\rho}\chi_{xxxx}^2$$

$$
\begin{aligned}
&=\frac{1}{2}\int_I \rho_t\chi_{xx}^2-2\int_I \rho_x\chi_{xt}\chi_{xx}-\int_I \chi_t\rho_{xx}\chi_{xx}-\int_I \rho v\chi_x\chi_{xxxx}+\int_I A\chi_{xxxx}\\
&\le\frac{1}{2}\int_I \frac{1}{\rho}\chi_{xxxx}^2+C\left(\|\rho_t\|_{L^\infty}+\|\rho_x\|_{L^\infty}^2\right)\int_I \rho\chi_{xx}^2+C\int_I \chi_{xt}^2+C\|\chi_{xx}\|_{L^\infty}^2\int_I \rho_{xx}^2\\
&\quad+C\int_I \chi_t^2+C\|v\|_{L^\infty}^2\int_I \chi_x^2+C\int_I \chi_{xxx}^2+C\int_I \chi_{xx}^2+C\|\varphi\|_{H^2}^6\\
&\le\frac{1}{2}\int_I \frac{1}{\rho}\chi_{xxxx}^2+CK_1\int_I \rho\chi_{xx}^2+C\int_I (\chi_{xxt}^2+\chi_{xxx}^2)+CK_1+C\|\varphi\|_{H^2}^6,
\end{aligned}
$$

where we have used the inequalities

$$
\begin{aligned}
&\int_I \chi_{xt}^2=\int_I \chi_{xt}\chi_{xt}=-\int_I \chi_t\chi_{xxt}\le\int_I \chi_t^2+\int_I \chi_{xxt}^2\le C+\int_I \chi_{xxt}^2,\\
&\int_I \chi_{xx}^2=\int_I \chi_{xx}\chi_{xx}=-\int_I \chi_x\chi_{xxx}\le\int_I \chi_x^2+\int_I \chi_{xxx}^2\le C+\int_I \chi_{xxx}^2.
\end{aligned}
$$

Thus, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho\chi_{xx}^2+\int_I \frac{1}{\rho}\chi_{xxxx}^2\le CK_1\int_I \rho\chi_{xx}^2+C\int_I (\chi_{xxt}^2+\chi_{xxx}^2)+CK_1+CK_2^3. \tag{2.19}
$$

Grönwall's inequality implies

$$
\int_I \rho\chi_{xx}^2\le\exp\{CK_1t\}\left(\int_I \rho\chi_{xx}^2(0)+C\iint_{Q_t}(\chi_{xxt}^2+\chi_{xxx}^2)+CK_1t+CK_2^3t\right)\le C,
$$

provided that $0<t<T_*$. For any $0<T<T_*$, integrating (2.19) over $(0,T)$, (2.15) holds. The proof of Lemma 2.5 is complete. □

LEMMA 2.6. *It holds that*

$$
\sup_{0\le t\le T}\int_I \left(\chi_{xxx}^2+\mu^2+\mu_{xx}^2\right)(t)\le C,\quad 0<T<T_*. \tag{2.20}
$$

Proof. From $(2.1)_{3,4}$ and (2.4), (2.11), (2.15), we have

$$
\begin{aligned}
&\int_I \mu^2\le\int_I \varphi^6+\int_I \varphi^2+\int_I \frac{1}{\rho^2}\chi_{xx}^2\le C\|\varphi\|_{L^\infty}^6+C,\\
&\int_I \mu_{xx}^2\le\int_I \rho^2\chi_t^2+\int_I \rho^2v^2\chi_x^2\le C\|v\|_{L^\infty}^2+C.
\end{aligned}
$$

Noticing

$$
\begin{aligned}
|\varphi(x,t)|&\le|\varphi(x,t)-\varphi(x,0)|+|\varphi(x,0)|\le\int_0^t\|\varphi_\tau(\tau)\|_{L^\infty}d\tau+|\chi_0(x)|\\
&\le C\int_0^t\|\varphi_\tau(\tau)\|_{W^{1,1}}d\tau+|\chi_0(x)|=C\int_0^t\int_I(|\varphi_t|+|\varphi_{xt}|)+|\chi_0(x)|\\
&\le C|Q_t|^{1/2}\left(\iint_{Q_t}\varphi_t^2\right)^{1/2}+C|Q_t|^{1/2}\left(\iint_{Q_t}\varphi_{xt}^2\right)^{1/2}+\|\chi_0\|_{H^1}
\end{aligned} \tag{2.21}
$$

and

$$
|v(x,t)|\le|v(x,t)-v(x,0)|+|v(x,0)|=\left|\int_0^t(v_\tau(x,\tau)-v_\tau(0,\tau))d\tau\right|+|u_0(x)|
$$

$$=\left|\int_0^t\int_0^x v_{y\tau}(y,\tau)dyd\tau\right|+|u_0(x)|\le|Q_t|^{1/2}\left(\iint_{Q_t}v_{xt}^2\right)^{1/2}+\|u_0\|_{H^1},\qquad(2.22)$$

we have

$$\int_I\mu^2\le C(K_2t)^3+C\le C,$$
$$\int_I\mu_{xx}^2\le CK_1t+C\le C,$$

for any $0<t<T_*$. Differentiating $(2.1)_4$ with respect to x, we have

$$\chi_{xxx}=-\rho\mu_x-\rho_x\mu+\rho_x(\varphi^3-\varphi)+\rho(3\varphi^2-1)\varphi_x.$$

Similar to (2.22), we can deduce that

$$|\varphi_x(x,t)|\le|Q_t|^{1/2}\left(\iint_{Q_t}\varphi_{xxt}^2\right)^{1/2}+\|\chi_0\|_{H^2}.$$

Hence, we obtain

$$\int_I\chi_{xxx}^2\le C\int_I\mu^2+C\int_I\mu_x^2+C\|\varphi\|_{L^\infty}^6+C\|\varphi\|_{L^\infty}^2+C(\|\varphi\|_{L^\infty}^4+1)\|\varphi_x\|_{L^\infty}^2\le C,$$

for any $0\le t\le T_*$. Lemma 2.6 follows. □

In what follows, we turn to do some a priori estimates for u.

LEMMA 2.7. *For any $0<T<T_*$, we have*

$$\sup_{0\le t\le T}\int_I u_x^2(t)+\iint_{Q_T}u_t^2\le C.\qquad(2.23)$$

Proof. Multiplying $(2.1)_2$ by u_t, then integrating the result over I, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I u_x^2+\int_I\rho u_t^2&=-\int_I\rho vu_xu_t-\gamma\int_I\rho^{\gamma-1}\rho_xu_t-\int_I\chi_x\chi_{xx}u_t\\&\le\frac{1}{2}\int_I\rho u_t^2+C\|v\|_{L^\infty}^2\int_I u_x^2+C+C\|\chi_x\|_{L^\infty}^2\int_I\chi_{xx}^2\\&\le\frac{1}{2}\int_I\rho u_t^2+CK_1\int_I u_x^2+C,\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I u_x^2+\int_I\rho u_t^2\le CK_1\int_I u_x^2+C.$$

By Grönwall's inequality and (2.4), we can deduce that (2.23) holds. This completes the proof. □

LEMMA 2.8. *There holds*

$$\sup_{0\le t\le T}\int_I(u_t^2+u_{xx}^2)(t)+\iint_{Q_T}u_{xt}^2\le C,\qquad 0<T<T_*.\qquad(2.24)$$

Proof. Differentiating $(2.1)_2$ with respect to t and multiplying the result by u_t, then integrating with respect to x over I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho u_t^2+\int_I u_{xt}^2\\
=&-\frac{1}{2}\int_I \rho_t u_t^2-\int_I \rho_t v u_x u_t-\int_I \rho v_t u_x u_t-\int_I \rho v u_{xt}u_t+\gamma\int_I \rho^{\gamma-1}\rho_t u_{xt}+\int_I \chi_x\chi_{xt}u_{xt}\\
\le&\frac{1}{2}\int_I u_{xt}^2+C(\|\rho_t\|_{L^\infty}+\|\rho_t\|_{L^\infty}^2+1+\|v\|_{L^\infty}^2)\int_I \rho u_t^2\\
&+C(\|v\|_{L^\infty}^2+\|v_t\|_{L^\infty}^2)\int_I u_x^2+C\|\rho_t\|_{L^\infty}^2+C\|\chi_x\|_{L^\infty}^2\int_I \chi_{xt}^2,
\end{aligned}$$

from which we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho u_t^2+\int_I u_{xt}^2\le CK_1\int_I \rho u_t^2+CK_1+C\int_I (v_{xt}^2+\chi_{xxt}^2).$$

Recalling (2.5), we get

$$\|\sqrt{\rho}u_t(0)\|_{L^2}\le C(\|\rho_0\|_{H^1},\|u_0\|_{H^2},\|\chi_0\|_{H^2}). \tag{2.25}$$

Grönwall's inequality implies

$$\int_I u_t^2(t)+\iint_{Q_t} u_{xt}^2\le C, \qquad 0<t<T_*.$$

From $(2.1)_2$, (2.4), (2.15), (2.23), (2.24) and (2.22), we have

$$\begin{aligned}
\int_I u_{xx}^2&\le C\int_I u_t^2+C\|v\|_{L^\infty}^2\int_I u_x^2+C+\|\chi_x\|_{L^\infty}^2\int_I \chi_{xx}^2\le C+C\|v\|_{L^\infty}^2\\
&\le C+C\left(|Q_t|\iint_{Q_t} v_{xt}^2+u_0^2(x)\right)\le C+CK_1 t\le C, \quad 0<t<T_*.
\end{aligned}$$

The proof is complete. ∎

LEMMA 2.9. *For any $0<T<T_*$, we have*

$$\sup_{0\le t\le T}\int_I u_{xt}^2(t)+\iint_{Q_T} u_{xxt}^2\le C. \tag{2.26}$$

Proof. Differentiating $(2.1)_2$ with respect to t, then multiplying the result by u_{xxt} and integrating with respect to x over I, we have

$$\int_I (\rho u_t)_t u_{xxt}+\int_I (\rho v u_x)_t u_{xxt}+\int_I (\rho^\gamma)_{xt}u_{xxt}=\int_I u_{xxt}^2-\int_I (\chi_x\chi_{xx})_t u_{xxt}, \tag{2.27}$$

where

$$\begin{aligned}
\int_I (\rho u_t)_t u_{xxt}&=\int_I \rho u_{tt}u_{xxt}+\int_I \rho_t u_t u_{xxt}=-\int_I \rho u_{xtt}u_{xt}-\int_I \rho_x u_{tt}u_{xt}+\int_I \rho_t u_t u_{xxt}\\
&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho u_{xt}^2+\frac{1}{2}\int_I \rho_t u_{xt}^2-\int_I \rho_x u_{tt}u_{xt}+\int_I \rho_t u_t u_{xxt}.
\end{aligned} \tag{2.28}$$

Differentiating (2.1)$_2$ with respect to t, we see that

$$u_{tt}=-\frac{\rho_t}{\rho}u_t-\frac{\rho_t}{\rho}vu_x-v_tu_x-vu_{xt}-\frac{(\rho^\gamma)_{xt}}{\rho}+\frac{1}{\rho}u_{xxt}-\frac{1}{\rho}\chi_{xt}\chi_{xx}-\frac{1}{\rho}\chi_x\chi_{xxt}:=\frac{1}{\rho}u_{xxt}+B, \tag{2.29}$$

where

$$\int_I B^2\le CK_1\int_I \rho u_{xt}^2+C\int_I \chi_{xxt}^2+CK_1^2.$$

Substitute (2.28) and (2.29) into (2.27) to give

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho u_{xt}^2+\int_I u_{xxt}^2=&\frac{1}{2}\int_I \rho_t u_{xt}^2-\int_I\rho_x\left(\frac{1}{\rho}u_{xxt}+B\right)u_{xt}+\int_I \rho_t u_t u_{xxt}\\&+\int_I(\rho_t v u_x+\rho v_t u_x+\rho v u_{xt})u_{xxt}+\int_I \chi_{xt}\chi_{xx}u_{xxt}\\&+\int_I(\gamma(\gamma-1)\rho^{\gamma-2}\rho_x\rho_t+\gamma\rho^{\gamma-1}\rho_{xt})u_{xxt}+\int_I\chi_x\chi_{xxt}u_{xxt}\\\le&\frac{1}{2}\int_I u_{xxt}^2+CK_1\int_I\rho u_{xt}^2+C\int_I\chi_{xxt}^2+CK_1^2,\end{aligned}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho u_{xt}^2+\int_I u_{xxt}^2\le CK_1\int_I\rho u_{xt}^2+C\int_I\chi_{xxt}^2+CK_1^2.$$

Noticing

$$\|\sqrt{\rho}u_{xt}(0)\|_{L^2}\le C(\|\rho_0\|_{H^2},\|u_0\|_{H^3},\|\chi_0\|_{H^3}),$$

and applying Grönwall's inequality, Lemma 2.9 follows. ∎

LEMMA 2.10. *For any $0<T<T_*$, there holds*

$$\sup_{0\le t\le T}\int_I u_{xxx}^2(t)+\iint_{Q_T}u_{xxxx}^2\le C. \tag{2.30}$$

Proof. Differentiating (2.1)$_2$ with respect to x, we see that

$$\begin{aligned}u_{xxx}=&\rho u_{xt}+\rho_x u_t+\rho_x v u_x+\rho v_x u_x+\rho v u_{xx}\\&+\gamma(\gamma-1)\rho^{\gamma-2}\rho_x^2+\gamma\rho^{\gamma-1}\rho_{xx}+\chi_{xx}^2+\chi_x\chi_{xxx}.\end{aligned} \tag{2.31}$$

Similar to (2.21), we have

$$|v_x(x,t)|\le C|Q_t|^{1/2}\left(\iint_{Q_t}v_{xt}^2\right)^{1/2}+C|Q_t|^{1/2}\left(\iint_{Q_t}v_{xxt}^2\right)^{1/2}+\|u_0\|_{H^2}.$$

Hence, we get

$$\begin{aligned}\int_I u_{xxx}^2\le& C\int_I u_{xt}^2+C\int_I u_t^2+C(\|v\|_{L^\infty}^2+\|v_x\|_{L^\infty}^2)\int_I u_x^2+C\|v\|_{L^\infty}^2\int_I u_{xx}^2\\&+C+\|\chi_{xx}\|_{L^\infty}^2\int_I\chi_{xx}^2+\|\chi_x\|_{L^\infty}^2\int_I\chi_{xxx}^2\le C,\qquad 0\le t\le T_*.\end{aligned}$$

Differentiating (2.31) with respect to x again and applying the estimates obtained before, we can easily arrive at (2.30). The proof is complete. □

Lemma 2.1 is an immediate consequence of Lemmas 2.2–2.10.

Proof. (**Proof of Proposition 2.1.**) Let X be the set of all functions (v,φ) satisfying conditions (2.2) and (2.3). For each $(v,\varphi)\in X$, set $(u,\chi)=\Phi(v,\varphi)$ be the solution of the problem (2.1), (1.2) and (1.3). By Lemma 2.1, we can choose large positive constants K_1 and K_2 satisfying $\min\{K_1,K_2\}>\overline{C}$ such that Φ maps X into X for $0<T<T_*:=\min\{\frac{1}{K_1^2},\frac{1}{K_2^3}\}$. It is clear that the set X is a convex and compact subset of $C([0,T];H^1)$. The continuity of Φ can be easily proved by an energy method. Therefore, it follows from the Schauder fixed-point theorem that Φ has a fixed point. This proves the existence of strong solutions to the problem (1.1)–(1.3). Furthermore, a simple energy argument shows the uniqueness. The proof of this Proposition is complete. □

Before proving Theorem 2.1, we recall the following well-known lemma.

LEMMA 2.11 ([31]). *Assume $X\subset E\subset Y$ are Banach spaces and $X\hookrightarrow\hookrightarrow E$. Then the following embeddings are compact:*

$$\text{(i)}\quad \left\{\varphi:\varphi\in L^q(0,T;X),\frac{\partial\varphi}{\partial t}\in L^1(0,T;Y)\right\}\hookrightarrow\hookrightarrow L^q(0,T;E),\quad \text{if}\quad 1\le q\le\infty;$$

$$\text{(ii)}\quad \left\{\varphi:\varphi\in L^\infty(0,T;X),\frac{\partial\varphi}{\partial t}\in L^r(0,T;Y)\right\}\hookrightarrow\hookrightarrow C([0,T];E),\quad \text{if}\quad 1<r\le\infty.$$

Proof. (**Proof of Theorem 2.1.**) Obviously, the assumptions on initial data in Theorem 2.1 satisfy the assumptions in Proposition 2.1. So the problem (1.1)–(1.3) admits a unique local strong solution (ρ,u,χ) and a small time T_*, such that

$$\begin{aligned}\sup_{0\le t<T_*}&\left(\|\rho\|_{H^3}^2+\|\rho_t\|_{H^2}^2+\|\rho^{-1}\|_{L^\infty}^2+\|u\|_{H_0^1\cap H^3}^2+\|u_t\|_{H_0^1}^2+\|\chi\|_{H^3}^2+\|\chi_t\|_{L^2}^2+\|\mu\|_{H^2}^2\right)\\&+\int_0^{T_*}\left(\|u\|_{H_0^1\cap H^4}^2+\|u_t\|_{H^2}^2+\|\chi\|_{H^4}^2+\|\chi_t\|_{H^2}^2+\|\mu\|_{H^4}^2+\|\mu_t\|_{L^2}^2\right)\le C<+\infty.\end{aligned} \tag{2.32}$$

In what follows, starting from (2.32), the assumptions on initial data and the equations in (1.1), we discuss the regularities of the local strong solution. We claim that

$$\max\left\{\|\rho_{xx}\|_{C^{\frac12,\frac12}(\overline{Q}_T)},\|\chi_{xx}\|_{C^{\frac12,\frac14}(\overline{Q}_T)},\|u_x\|_{C^{1,\frac12}(\overline{Q}_T)}\right\}\le C,\quad 0<T<T_*. \tag{2.33}$$

We prove the estimate of ρ for example. The other two are similar. For any $(x_1,t),(x_2,t)\in\overline{Q}_T$, we have

$$|\rho_{xx}(x_1,t)-\rho_{xx}(x_2,t)|=\left|\int_{x_2}^{x_1}\rho_{xxx}(x,t)dx\right|\le\left(\int_I\rho_{xxx}^2\right)^{1/2}\left(\int_{x_2}^{x_1}1^2\right)^{1/2}\le C|x_1-x_2|^{1/2}. \tag{2.34}$$

Differentiating $(1.1)_1$ with respect to x twice, we get

$$\rho_{xxt}+\rho_{xxx}u+3\rho_{xx}u_x+3\rho_xu_{xx}+\rho u_{xxx}=0.$$

For any $(x,t_1),(x,t_2)\in\overline{Q}_T$, we consider the case of $x\in[0,1/2]$. Suppose that $\Delta t=t_2-t_1>0$ satisfying $\Delta t\le 1/2$. Integrating the above equation over $(x,x+\Delta t)\times(t_1,t_2)$, we have

$$\begin{aligned}
&\int_x^{x+\Delta t}(\rho_{yy}(y,t_2)-\rho_{yy}(y,t_1))dy\\
=&-\int_{t_1}^{t_2}\int_x^{x+\Delta t}(\rho_{yyy}u+3\rho_{yy}u_y+3\rho_y u_{yy}+\rho u_{yyy})(y,t)dydt\\
\le&\int_{t_1}^{t_2}\left[\|u\|_{L^\infty}\|\rho_{xxx}\|_{L^2}(\Delta t)^{1/2}+\|\rho\|_{L^\infty}\|u_{xxx}\|_{L^2}(\Delta t)^{1/2}\right]dt\\
&+3(\|\rho_{xx}\|_{L^\infty}\|u_x\|_{L^\infty}+\|\rho_y\|_{L^\infty}\|u_{xx}\|_{L^\infty})(\Delta t)^2\\
\le& C(\Delta t)^{3/2}.
\end{aligned}$$

Noticing $\rho_{xx}\in L^\infty(0,T;H^1)$ and $\rho_{xxt}\in L^\infty(0,T;L^2)$, by Lemma 2.11 we have $\rho_{xx}\in C([0,T];L^2)$. For the left-hand side of the above inequality, by the integral mean value theorem, there exists a point $x^*\in[x,x+\Delta t]\in[0,1]$ such that

$$|\rho_{xx}(x^*,t_2)-\rho_{xx}(x^*,t_1)|\le C|t_2-t_1|^{1/2}.$$

Combine the above inequality with (2.34) to give

$$\begin{aligned}
&|\rho_{xx}(x,t_1)-\rho_{xx}(x,t_2)|\\
\le& |\rho_{xx}(x,t_1)-\rho_{xx}(x^*,t_1)|+|\rho_{xx}(x^*,t_1)-\rho_{xx}(x^*,t_2)|+|\rho(x,t_2)_{xx}-\rho_{xx}(x^*,t_2)|\\
\le& C|x-x^*|^{1/2}+C|t_1-t_2|^{1/2}\le C|t_1-t_2|^{1/2}.
\end{aligned}\tag{2.35}$$

For the case of $x\in[1/2,1]$, integrating the equation over $(x-\Delta t,x)\times(t_1,t_2)$, then we can also get the above inequality. From (2.34) and (2.35), for any $(x_1,t_1),(x_2,t_2)\in\overline{Q}_T$, there holds

$$\begin{aligned}
|\rho_{xx}(x_1,t_1)-\rho_{xx}(x_2,t_2)|\le&|\rho_{xx}(x_1,t_1)-\rho_{xx}(x_2,t_1)|+|\rho_{xx}(x_2,t_1)-\rho_{xx}(x_2,t_2)|\\
\le& C(|x_1-x_2|^{1/2}+|t_1-t_2|^{1/2}).
\end{aligned}$$

From $(1.1)_2$, we have

$$u_t=\frac{1}{\rho}u_{xx}+f_1(x,t),\tag{2.36}$$

where $f_1(x,t)=-uu_x-\gamma\rho^{\gamma-2}\rho_x-\frac{1}{\rho}\chi_x\chi_{xx}$. Noticing that $\|\frac{1}{\rho}\|_{C^{1,1}(\overline{Q}_T)}$, $\|f_1\|_{C^{\frac12,\frac14}(\overline{Q}_T)}\le C$ and $u_0\in C^{3,\alpha}(I)$, applying the Schauder theory, we have

$$\|u\|_{C^{2+\frac12,1+\frac14}(\overline{Q}_T)}\le C.$$

From $(1.1)_{3,4}$, we get

$$\chi_t=-\frac{1}{\rho^2}\chi_{xxxx}+2\frac{\rho_x}{\rho^3}\chi_{xxx}+\left(\frac{\rho_{xx}}{\rho^3}-2\frac{\rho_x^2}{\rho^4}+\frac{3\chi^2-1}{\rho}\right)\chi_{xx}+f_2(x,t),$$

where $f_2(x,t)=-u\chi_x+\frac{6}{\rho}\chi\chi_x^2$. Observing that the coefficients and f_2 are bounded in $C^{\frac12,\frac14}(\overline{Q}_T)$ and $\chi_0\in C^{4,\alpha}(I)$, by the Schauder theory (see [34]), we have

$$\|\chi\|_{C^{4+\frac12,1+\frac18}(\overline{Q}_T)}\le C.$$

In particular,

$$\|\chi_{xxx}\|_{C^{\alpha,\frac{\alpha}{4}}(\overline{Q}_T)}\le C.$$

Differentiating (2.36) with respect to x, using the above results and the Schauder theory again, we obtain

$$\|u_x\|_{C^{2+\frac{\alpha}{2},1+\frac{\alpha}{4}}(\overline{Q}_T)}\le C.$$

In order to study the regularity of ρ_{xxx}, we introduce the Lagrangian mass coordinates defined by

$$y(x,t)=\int_0^x \rho(\xi,t)d\xi, \qquad \tau(x,t)=t.$$

It is easy to see that $(x,t)\to(y,\tau)$ is a C^1-bijective map from $I\times[0,T)\to I\times[0,T)$, provided that $\rho(x,t)\in C^1(I\times[0,T))$ is positive and $\int_0^1\rho(\xi,t)d\xi=\int_0^1\rho_0(\xi)d\xi=1$ for all $t\in[0,T)$. By direct calculations, we see that

$$\frac{\partial}{\partial t}=-\rho u\frac{\partial}{\partial y}+\frac{\partial}{\partial\tau}, \qquad \frac{\partial}{\partial x}=\rho\frac{\partial}{\partial y}. \tag{2.37}$$

If we write $F(x,t)=-\frac{1}{2}(\chi_x^2)_x$, we know that $\|F_{xx}\|_{C^{\frac{1}{2},\frac{1}{8}}(\overline{Q}_T)}\le C$. Hence, $(1.1)_{1,2}$ can be rewritten in the Lagrangian coordinates as

$$\begin{cases}\rho_\tau+\rho^2u_y=0,\\ u_\tau+(\rho^\gamma)_y=(\rho u_y)_y+F,\end{cases} \tag{2.38}$$

which satisfies the initial boundary value conditions

$$(\rho,u)\Big|_{\tau=0}=(\rho_0,u_0)\ \text{ for }\ y\in I \qquad \text{and} \qquad u\Big|_{\partial I}=0\ \text{ for }\ \tau\ge 0.$$

By (2.38), direction calculations imply

$$\left(u+(\ln\rho)_y\right)_\tau=F-\gamma\rho^\gamma\left(u+(\ln\rho)_y\right)+\gamma\rho^\gamma u,$$

from which we have

$$\rho_y=\rho\left(u_0+(\ln\rho_0)_y\right)\exp\left\{-\gamma\int_0^\tau\rho^\gamma\right\}+\rho\int_0^\tau\left(F+\gamma\rho^\gamma u\right)\exp\left\{-\gamma\int_s^\tau\rho^\gamma\right\}\mathrm{d}s-\rho u.$$

Recalling $\rho_0\in C^{3,\alpha}(I)$, $u_0\in C^{3,\alpha}(I)$ and $\|\rho_{yy}\|_{C^{\frac{1}{2},\frac{1}{2}}(\overline{Q}_T)}$, $\|u_{yy}\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}$, $\|F_{yy}\|_{C^{\frac{1}{2},\frac{1}{8}}(\overline{Q}_T)}\le C$, we have

$$\|\rho_{yyy}\|_{C^{\frac{1}{2},\frac{1}{8}}(\overline{Q}_T)}\le C,$$

from which and $(1.1)_1$, we conclude that

$$\|\rho_{yy}\|_{C^{1,1}(\overline{Q}_T)}\le C.$$

By the Schauder theory and the equations in (1.1), repeating the above arguments once again, we obtain

$$\max\left\{\|\rho_{xxx}\|_{C^{\frac{\alpha}{2},\frac{\alpha}{4}}(\overline{Q}_T)},\ \|\rho_{xxt}\|_{C^{\frac{\alpha}{2},\frac{\alpha}{4}}(\overline{Q}_T)},\ \|u_x\|_{C^{2+\frac{\alpha}{2},1+\frac{\alpha}{4}}(\overline{Q}_T)},\ \|\chi\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\overline{Q}_T)}\right\}\le C,$$

for any $0<T<T_*$. ∎

3. A priori estimates

In this section, we are going to do some a priori estimates about the classical solution which has been obtained in Theorem 2.1. The first estimate is a basic energy equality.

LEMMA 3.1. *For any $0\le t<T$, it holds that*

$$\int_I\Big(\frac{\rho u^2}{2}+\frac{\rho^\gamma}{\gamma-1}+\frac{\rho(\chi^2-1)^2}{4}+\frac{\chi_x^2}{2}\Big)(t)+\int_0^t\int_I(u_x^2+\mu_x^2)=E_0, \tag{3.1}$$

where

$$\int_I\rho(t)=\int_I\rho_0\equiv 1\quad\text{and}\quad E_0:=\int_I\Big(\frac{\rho_0u_0^2}{2}+\frac{\rho_0^\gamma}{\gamma-1}+\frac{\rho_0(\chi_0^2-1)^2}{4}+\frac{\chi_{0x}^2}{2}\Big)$$

denotes the total energy of the initial data.

Proof. Multiplying $(1.1)_2$ by u and integrating the result over I, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I\frac{\rho u^2}{2}+\int_I(\rho^\gamma)_xu+\int_Iu_x^2=-\frac12\int_I(\chi_x^2)_xu. \tag{3.2}$$

Now we claim that

$$\int_I(\rho^\gamma)_xu=\frac{\mathrm{d}}{\mathrm{dt}}\int_I\frac{\rho^\gamma}{\gamma-1}. \tag{3.3}$$

In fact, by $(1.1)_1$ we have

$$\int_I(\rho^\gamma)_xu=\int_I\rho^{\gamma-1}(\rho_t+\rho_xu)=\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho^\gamma+\frac{1}{\gamma}\int_I(\rho^\gamma)_xu,$$

from which we obtain (3.3). Multiplying $(1.1)_3$ by μ, then integrating the result over I with respect to x, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I\left(\rho\left(\frac{\chi^4}{4}-\frac{\chi^2}{2}\right)+\frac{\chi_x^2}{2}\right)+\int_I\mu_x^2=\frac12\int_I(\chi_x^2)_xu. \tag{3.4}$$

Putting (3.2), (3.3) and (3.4) together, we see that (3.1) holds. The proof is complete. □

To obtain the upper and lower bounds of the density ρ, we need the following estimate.

LEMMA 3.2. *For any $0\le t<T$, there holds*

$$\int_0^t\int_I\frac{1}{\rho}\chi_{xx}^2\le C(E_0,T). \tag{3.5}$$

Proof. Multiplying $(1.1)_3$ by χ, using $(1.1)_1$ and $(1.1)_{3,4}$, we have

$$\frac12\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi^2=-\int_I\mu_x\chi_x=-3\int_I\chi^2\chi_x^2+\int_I\chi_x^2-\int_I\frac{1}{\rho}\chi_{xx}^2,$$

which implies that

$$\frac12\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi^2+3\int_I\chi^2\chi_x^2+\int_I\frac{1}{\rho}\chi_{xx}^2=\int_I\chi_x^2\le C(E_0),$$

where we have used (3.1). Integrating the above inequality over $(0,t)$, Lemma 3.2 follows. ∎

Then we can prove the upper and lower bounds of ρ.

LEMMA 3.3. *We have the inequalities*

$$\int_I \rho_x^2(t)\le C_1, \qquad t\in[0,T), \tag{3.6}$$

$$C_1^{-1}\le \rho(x,t)\le C_1, \quad (x,t)\in I\times[0,T), \tag{3.7}$$

where $C_1>0$ is a constant depending only on $c_0, E_0, \|\rho_0\|_{H^1}$, γ and T.

Proof. Using $(1.1)_1$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2 &= \int_I \rho_t\left|\left(\frac{1}{\rho}\right)_x\right|^2 + 2\int_I \rho\left(\frac{1}{\rho}\right)_x\left(\frac{1}{\rho}\right)_{xt}\\
&=-\int_I (\rho u)_x\left|(\frac{1}{\rho})_x\right|^2 + 2\int_I \rho\left(\frac{1}{\rho}\right)_x\left(-\frac{\rho_t}{\rho^2}\right)_x\\
&=-\int_I (\rho u)_x\left|\left(\frac{1}{\rho}\right)_x\right|^2 - 2\int_I \rho u\left(\frac{1}{\rho}\right)_x\left(\frac{1}{\rho}\right)_{xx} + 2\int_I\left(\frac{1}{\rho}\right)_x u_{xx}\\
&=-\int_I (\rho u)_x\left|\left(\frac{1}{\rho}\right)_x\right|^2 + \int_I (\rho u)_x\left|\left(\frac{1}{\rho}\right)_x\right|^2 + 2\int_I\left(\frac{1}{\rho}\right)_x u_{xx}\\
&=2\int_I\left(\frac{1}{\rho}\right)_x u_{xx},
\end{aligned}$$

i.e.

$$\int_I\left(\frac{1}{\rho}\right)_x u_{xx} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2.$$

Multiplying $(1.1)_2$ by $\left(\frac{1}{\rho}\right)_x$, integrating the result over I, using the above equality and $(1.1)_1$ again, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2 + \gamma\int_I \rho^{\gamma-3}\rho_x^2 - \frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho u\left(\frac{1}{\rho}\right)_x\\
&=-\int_I \rho u\left(\frac{1}{\rho}\right)_{xt} + \int_I (\rho u^2)_x\left(\frac{1}{\rho}\right)_x + \int_I \chi_x\chi_{xx}\left(\frac{1}{\rho}\right)_x\\
&=\int_I (\rho u)_x\left(-\frac{\rho_t}{\rho^2}\right) + \int_I (\rho u^2)_x\left(-\frac{\rho_x}{\rho^2}\right) + \int_I \chi_x\chi_{xx}\left(\frac{1}{\rho}\right)_x\\
&\le \int_I \frac{|(\rho u)_x|^2 - (\rho u^2)_x\rho_x}{\rho^2} + \|\chi_x\|_{L^\infty}\int_I |\chi_{xx}|\left|\left(\frac{1}{\rho}\right)_x\right|\\
&\le \int_I u_x^2 + \|\chi_x\|_{W^{1,1}}\left(\int_I \frac{1}{\rho}\chi_{xx}^2\right)^{1/2}\left(\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2\right)^{1/2}\\
&\le \int_I u_x^2 + \|\chi_x\|_{W^{1,1}}^2 + \int_I \frac{1}{\rho}\chi_{xx}^2\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2
\end{aligned}$$

$$\leq \int_I u_x^2 + 2\int_I \chi_x^2 + 2\int_I \rho \int_I \frac{1}{\rho}\chi_{xx}^2 + \int_I \frac{1}{\rho}\chi_{xx}^2 \int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2$$

$$\leq \int_I u_x^2 + C(E_0) + 2\int_I \frac{1}{\rho}\chi_{xx}^2 + \int_I \frac{1}{\rho}\chi_{xx}^2 \int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2,$$

where we have used the embedding equality for one dimension $\|\chi_x\|_{L^\infty(I)} \leq \|\chi_x\|_{W^{1,1}(I)}$, Cauchy's inequality and Hölder's inequality. Integrating the above inequality over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2 + \gamma \int_0^t \int_I \rho^{\gamma-3}\rho_x^2 \\
&\leq \int_I \rho u \left(\frac{1}{\rho}\right)_x - \int_I \rho_0 u_0 \left(\frac{1}{\rho_0}\right)_x + \frac{1}{2}\int_I \rho_0 \left|\left(\frac{1}{\rho_0}\right)_x\right|^2 \\
&\quad + \int_0^t \int_I u_x^2 + C(E_0)t + 2\int_0^t \int_I \frac{1}{\rho}\chi_{xx}^2 + \int_0^t \left(\int_I \frac{1}{\rho}\chi_{xx}^2 \int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2\right) \\
&\leq \frac{1}{4}\int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2 + C(c_0, E_0, \|\rho_0\|_{H^1}, t) + \int_0^t \left(\int_I \frac{1}{\rho}\chi_{xx}^2 \int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2\right),
\end{aligned}$$

which implies that

$$\frac{1}{4}\int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2 + \gamma \int_0^t \int_I \rho^{\gamma-3}\rho_x^2 \leq C(c_0, E_0, \|\rho_0\|_{H^1}, t) + \int_0^t \left(\int_I \frac{1}{\rho}\chi_{xx}^2 \int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2\right).$$

Applying Grönwall's inequality, we have

$$\int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2 + \gamma \int_0^t \int_I \rho^{\gamma-3}\rho_x^2 \leq C(c_0, E_0, \|\rho_0\|_{H^1}, t). \tag{3.8}$$

Since $\int_I \rho(t) = \int_I \rho_0 = 1$, using the mean value theorem, there exists $a(t) \in I$ such that $\rho(a(t),t) = \int_I \rho(t) = 1$. Hence, we have

$$\begin{aligned}
\frac{1}{\rho(x,t)} &= \frac{1}{\rho(x,t)} - \frac{1}{\rho(a(t),t)} + \frac{1}{\rho(a(t),t)} = \int_{a(t)}^x \left(\frac{1}{\rho(\xi,t)}\right)_\xi d\xi + 1 \\
&= \int_{a(t)}^x \frac{-\rho_\xi}{\rho^2} + 1 \leq \left\|\frac{1}{\rho}\right\|_{L^\infty}^{\frac{1}{2}} \left(\int_I \frac{\rho_x^2}{\rho^3}\right)^{\frac{1}{2}} + 1 \leq \frac{1}{2}\left\|\frac{1}{\rho}\right\|_{L^\infty} + \frac{1}{2}\int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2 + 1.
\end{aligned}$$

Taking the supremum over $x \in I$ yields

$$\left\|\frac{1}{\rho}\right\|_{L^\infty} \leq 2 + \int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2 \leq C(c_0, E_0, \|\rho_0\|_{H^1}, t). \tag{3.9}$$

On the other hand, since $\gamma > 1$, we write $\gamma = 1 + 2\delta$ for some $\delta > 0$. Then, we get

$$\begin{aligned}
\|\rho^\delta\|_{L^\infty} &\leq c\int_I \rho^\delta + c\int_I (\rho^\delta)_x \leq c\left(\int_I \rho^\gamma\right)^{\delta/\gamma} + c\int_I \rho^{\delta-1}|\rho_x| \\
&\leq C(E_0, \gamma) + c\left(\int_I \rho^\gamma\right)^{1/2} \left(\int_I \rho \left|\left(\frac{1}{\rho}\right)_x\right|^2\right)^{1/2}
\end{aligned}$$

$$\leq C(c_0, E_0, \|\rho_0\|_{H^1}, \gamma, t). \tag{3.10}$$

From (3.9) and (3.10), we see that (3.7) holds. Combining (3.8) with (3.10), we obtain (3.6). The proof of Lemma 3.3 is complete. □

Observing that χ satisfies Neumann boundary value condition, we should estimate the upper bound of the concentration χ.

LEMMA 3.4. *For any $T > 0$, there holds*

$$\|\chi\|_{L^\infty(I\times(0,T))} \leq C(E_0). \tag{3.11}$$

Proof. From $(1.1)_3$ and the boundary value condition (1.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho\chi = -\rho u\chi\Big|_{\partial I} + \mu_x\Big|_{\partial I} = 0,$$

which implies that

$$\int_I \rho\chi = \int_I \rho_0\chi_0 \leq \int_I \rho_0 + \int_I \rho_0\chi_0^2 \leq C(E_0).$$

Thus, noticing that $\int_I \rho(x,t)dx = 1$ and $\rho \geq 0$ for any $(x,t) \in I \times (0,T)$, we get

$$\begin{aligned}
|\chi(x,t)| &= \left|\chi(x,t)\int_I \rho(y,t)dy\right| \\
&\leq \left|\int_I \rho(y,t)\left(\chi(x,t) - \chi(y,t)\right)dy\right| + \left|\int_I \rho(y,t)\chi(y,t)dy\right| \\
&\leq \left|\int_I \rho(y,t)\left(\int_y^x \chi_\xi(\xi,t)d\xi\right)dy\right| + C(E_0) \\
&\leq \int_I |\chi_x| \int_I \rho(y,t)dy + C(E_0) \\
&\leq \left(\int_I \chi_x^2\right)^{\frac{1}{2}} + C(E_0) \leq C(E_0).
\end{aligned}$$

This completes the proof. □

From $(1.1)_4$, using the boundary value condition $\chi_x\big|_{\partial I} = 0$ and (3.1), we have

$$\int_I \rho\mu = \int_I \rho(\chi^3 - \chi) - \chi_x\Big|_{\partial I} \leq C(E_0)\int_I \rho \leq C(E_0).$$

Similar to the proof of Lemma 3.4, we can deduce that

LEMMA 3.5. *For any $0 \leq t < T$, we have*

$$\int_0^t \int_I \mu^2 \leq C(E_0, T). \tag{3.12}$$

The lemma below is useful in the proof of the forthcoming lemma.

LEMMA 3.6. *For any $0 \leq t < T$, it holds that*

$$\int_0^t \int_I \chi_{xxx}^2 \leq C_1. \tag{3.13}$$

Proof. Differentiating (1.1)$_4$ with respect to x, we have

$$\chi_{xxx}=\rho_x\left(\chi^3-\chi\right)+\rho(3\chi^2-1)\chi_x-\rho_x\mu-\rho\mu_x.$$

From (3.1), (3.6), (3.7), (3.11) and (3.12), we see that (3.13) holds. ∎

In terms of above lemmas, we obtain the following important lemma. This turns out to be the most difficult step.

Lemma 3.7. *For any $0\le t<T$, there exists a positive constant C_2 depending only on c_0, E_0, $\|\rho_0\|_{H^2}$, $\|u_0\|_{H^1}$, $\|\chi_0\|_{H^4}$, γ and T, such that*

$$\int_I\left(\frac{1}{\rho}\chi_{xx}^2+\rho\chi_t^2+u_x^2\right)(t)+\int_0^t\int_I\left(\frac{1}{\rho}\chi_{xxt}^2+\rho u_t^2+u_{xx}^2\right)\le C_2. \tag{3.14}$$

Proof. Firstly, differentiating (1.1)$_3$ with respect to t, then multiplying the result by χ_t, integrating over I, using (1.1)$_1$ and (1.1)$_4$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi_t^2+\int_I\frac{1}{\rho}\chi_{xxt}^2\\
&=-2\int_I\rho u\chi_t\chi_{xt}+\int_I\rho u_x u\chi_x\chi_t+\int_I\rho_x u^2\chi_x\chi_t-\int_I\rho u_t\chi_x\chi_t\\
&\quad-\int_I\frac{u_x}{\rho}\chi_{xx}\chi_{xxt}-\int_I\frac{\rho_x u}{\rho^2}\chi_{xx}\chi_{xxt}+\int_I(3\chi^2-1)\chi_t\chi_{xxt}\\
&\le\frac{1}{4}\int_I\frac{1}{\rho}\chi_{xxt}^2+c\|\chi_{xx}\|_{L^\infty}^2\left(\left\|\frac{1}{\rho}\right\|_{L^\infty}\int_I u_x^2+\left\|\frac{1}{\rho}\right\|_{L^\infty}^3\|u\|_{L^\infty}^2\int_I\rho_x^2\right)\\
&\quad+c\left(\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2+\|\rho\|_{L^\infty}\|\chi_x\|_{L^\infty}^2+\left\|\frac{1}{\rho}\right\|_{L^\infty}\|\chi_x\|_{L^\infty}^2+\|\chi_x\|_{L^\infty}^2+\|3\chi^2-1\|_{L^\infty}\right)\int_I\rho\chi_t^2\\
&\quad+c\|u\|_{L^\infty}^2\int_I u_x^2+c\|u\|_{L^\infty}^4\int_I\rho_x^2+\frac{1}{4}\int_I\rho u_t^2+\int_I\chi_{xt}^2\\
&\le\frac{1}{2}\int_I\frac{1}{\rho}\chi_{xxt}^2+C_1\left(1+\|u\|_{L^\infty}^2+\|\chi_x\|_{L^\infty}^2\right)\int_I\rho\chi_t^2\\
&\quad+C_1\int_I(\chi_{xx}^2+\chi_{xxx}^2)\int_I u_x^2+C_1\left(\int_I u_x^2\right)^2+\frac{1}{4}\int_I\rho u_t^2\\
&\le\frac{1}{2}\int_I\frac{1}{\rho}\chi_{xxt}^2+C_1\left(1+\int_I(u_x^2+\chi_{xx}^2)\right)\int_I\rho\chi_t^2+C_1\int_I\left(\chi_{xx}^2+\chi_{xxx}^2+u_x^2\right)\int_I u_x^2+\frac{1}{4}\int_I\rho u_t^2,
\end{aligned}$$

where we have used the following Sobolev embedding inequalities

$$\|\chi_{xx}\|_{L^\infty(I)}\le c\|\chi_{xx}\|_{W^{1,2}(I)},\quad \|\chi_x\|_{L^\infty(I)}\le c\|\chi_{xx}\|_{L^2(I)},\quad \|u\|_{L^\infty(I)}\le c\|u_x\|_{L^2(I)}.$$

Thus we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho\chi_t^2+\int_I\frac{1}{\rho}\chi_{xxt}^2\\
&\le\frac{1}{2}\int_I\rho u_t^2+C_1\left(1+\int_I(u_x^2+\chi_{xx}^2)\right)\int_I\rho\chi_t^2+C_1\int_I\left(\chi_{xx}^2+\chi_{xxx}^2+u_x^2\right)\int_I u_x^2.
\end{aligned}\tag{3.15}$$

Next, multiplying (1.1)$_2$ by u_t and integrating the result over I, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I u_x^2+\int_I\rho u_t^2$$

$$
\begin{aligned}
&= -\int_I \rho u u_x u_t - \gamma \int_I \rho^{\gamma-1}\rho_x u_t - \int_I \chi_x \chi_{xx} u_t \\
&\le \frac{1}{4}\int_I \rho u_t^2 + c\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2 \int_I u_x^2 + c\gamma^2 \|\rho\|_{L^\infty}^{2\gamma-3}\int_I \rho_x^2 + c\|\chi_x\|_{L^\infty}^2 \int_I \frac{1}{\rho}\chi_{xx}^2 \\
&\le \frac{1}{4}\int_I \rho u_t^2 + C_1\left(\int_I u_x^2\right)^2 + c\int_I \chi_{xx}^2 \int_I \frac{1}{\rho}\chi_{xx}^2 + C_1,
\end{aligned}
$$

which implies that

$$
\frac{\mathrm{d}}{\mathrm{dt}}\int_I u_x^2 + \frac{3}{2}\int_I \rho u_t^2 \le C_1\left(\int_I u_x^2\right)^2 + c\int_I \chi_{xx}^2 \int_I \frac{1}{\rho}\chi_{xx}^2 + C_1. \tag{3.16}
$$

In the following, we deal with $\int_I \frac{1}{\rho}\chi_{xx}^2$. Using $(1.1)_3$, $(1.1)_4$ and integrating by parts, we get

$$
\begin{aligned}
\int_I \frac{1}{\rho}\chi_{xx}^2 &= \int_I (\chi^3-\chi)\chi_{xx} - \int_I \mu\chi_{xx} = \int_I (\chi^3-\chi)\chi_{xx} - \int_I \mu_{xx}\chi \\
&= \int_I (\chi^3-\chi)\chi_{xx} - \int_I \rho\chi_t\chi - \int_I \rho u \chi_x \chi \\
&\le \frac{1}{2}\int_I \frac{1}{\rho}\chi_{xx}^2 + c\int_I \rho\chi_t^2 + C_1,
\end{aligned}
$$

from which we have

$$
\int_I \frac{1}{\rho}\chi_{xx}^2 \le c\int_I \rho\chi_t^2 + C_1. \tag{3.17}
$$

Put (3.15), (3.16) and (3.17) together, to get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_I (\rho\chi_t^2 + u_x^2) + \int_I \left(\frac{1}{\rho}\chi_{xxt}^2 + \rho u_t^2\right) \\
\le & C_1\left(1 + \int_I (u_x^2 + \chi_{xx}^2)\right)\int_I \rho\chi_t^2 + C_1\int_I (\chi_{xx}^2 + \chi_{xxx}^2 + u_x^2)\int_I u_x^2 + C_1\int_I \chi_{xx}^2 + C_1 \\
\le & C_1\left(1 + \int_I (\chi_{xxx}^2 + u_x^2)\right)\int_I (\rho\chi_t^2 + u_x^2) + C_1\left(1 + \int_I \chi_{xxx}^2\right).
\end{aligned} \tag{3.18}
$$

Recalling (2.13) and applying Grönwall's inequality to (3.18), we have

$$
\int_I (\rho\chi_t^2 + u_x^2) + \int_0^t \int_I \left(\frac{1}{\rho}\chi_{xxt}^2 + \rho u_t^2\right) \le C(C_1, \|\rho_0\|_{H^2}, \|u_0\|_{H^1}, \|\chi_0\|_{H^4}).
$$

By (3.17) and $(1.1)_2$, we obtain (3.14). The proof of Lemma 3.7 is complete. □

From the above estimates, we can directly calculate that

LEMMA 3.8. *For any* $0 \le t < T$, *there holds*

$$
\int_I (\rho_t^2 + \chi_{xxx}^2 + \mu^2 + \mu_{xx}^2)(t) + \int_0^t \int_I (\mu_t^2 + \mu_{xxx}^2) \le C_2. \tag{3.19}
$$

Next, we continue to do some estimates for u.

LEMMA 3.9. *For any $0\le t<T$, we have*

$$\int_I(\rho u_t^2+u_{xx}^2)(t)+\int_0^t\int_I u_{xt}^2\le C_3, \tag{3.20}$$

where C_3 is a positive constant depending only on c_0, E_0, $\|\rho_0\|_{H^2}$, $\|u_0\|_{H^2}$, $\|\chi_0\|_{H^4}$, γ and T.

Proof. Differentiating $(1.1)_2$ with respect to t, multiplying the result by u_t and integrating over I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I\rho u_t^2+\int_I u_{xt}^2\\
&=-2\int_I\rho u u_t u_{xt}-\int_I\rho u u_x^2 u_t-\int_I\rho u^2 u_{xx}u_t-\int_I\rho u^2 u_x u_{xt}-\int_I\rho u_x u_t^2\\
&\quad-\gamma\int_I\rho^\gamma u_x u_{xt}-\gamma\int_I\rho^{\gamma-1}\rho_x u u_{xt}+\int_I\chi_x\chi_{xt}u_{xt}\\
&\le\frac{1}{2}\int_I u_{xt}^2+c(\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2+\|u_x\|_{L^\infty})\int_I\rho u_t^2+c\|u_x\|_{L^\infty}^2\int_I u_x^2+c\|u\|_{L^\infty}^2\int_I u_{xx}^2\\
&\quad+c\|\rho\|_{L^\infty}^2\|u\|_{L^\infty}^4\int_I u_x^2+c\gamma^2\|\rho\|_{L^\infty}^{2\gamma}\int_I u_x^2+c\gamma^2\|\rho\|_{L^\infty}^{2\gamma-2}\|u\|_{L^\infty}^2\int_I\rho_x^2+c\|\chi_x\|_{L^\infty}^2\int_I\chi_{xt}^2\\
&\le\frac{1}{2}\int_I u_{xt}^2+C_1(\|u_x\|_{L^2}^2+\|u_x\|_{W^{1,2}})\int_I\rho u_t^2+C_2\|u_x\|_{W^{1,2}}^2+\|u_x\|_{L^2}^2\int_I u_{xx}^2\\
&\quad+C_2\|u_x\|_{L^2}^4+C_2+C_1\|u_x\|_{L^2}^2+c\|\chi_{xx}\|_{L^2}^2\left(\int_I\chi_t^2+\int_I\chi_{xxt}^2\right)\\
&\le\frac{1}{2}\int_I u_{xt}^2+C_2\left(1+\int_I u_{xx}^2\right)\int_I\rho u_t^2+C_2\left(1+\int_I(u_{xx}^2+\chi_{xxt}^2)\right),
\end{aligned}$$

from which we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I\rho u_t^2+\int_I u_{xt}^2\le C_2\left(1+\int_I u_{xx}^2\right)\int_I\rho u_t^2+C_2\left(1+\int_I(u_{xx}^2+\chi_{xxt}^2)\right). \tag{3.21}$$

Applying Grönwall's inequality to (3.21), using (3.14) and recalling (2.25), we get

$$\int_I\rho u_t^2+\int_0^t\int_I u_{xt}^2\le C(C_2,\|u_0\|_{H^2}).$$

Combining the above inequality with $(1.1)_2$, Lemma 3.9 follows. ∎

LEMMA 3.10. *For any $0\le t<T$, we have*

$$\int_I(\rho_{xx}^2+\rho_{xt}^2+\chi_{xxxx}^2)(t)+\int_0^t\int_I\mu_{xxxx}^2\le C_3. \tag{3.22}$$

Proof. Just as the proof of Theorem 2.1, we introduce the Lagrangian coordinates defined by (2.37). Write $F(x,t)=-\frac{1}{2}(\chi_x^2)_x$, from (3.19) we know that $\|F\|_{L^\infty(0,T;H^1)}\le C_2$. Hence, $(1.1)_{1,2}$ can be rewritten in the Lagrangian coordinates as

$$\begin{cases}\rho_\tau+\rho^2u_y=0,\\ u_\tau+(\rho^\gamma)_y=(\rho u_y)_y+F,\end{cases} \tag{3.23}$$

which satisfies the initial boundary value conditions

$$(\rho,u)\Big|_{\tau=0}=(\rho_0,u_0)\ \text{ for }\ y\in I \qquad \text{and} \qquad u\Big|_{\partial I}=0\ \text{ for }\ \tau>0.$$

Similar to the proof of Theorem 2.1, after direct calculations, we have

$$\begin{aligned}\rho_{yy}=&\Big(\rho_y\,(u_0+(\ln\rho_0)_y)+\rho\,(u_{0y}+(\ln\rho_0)_{yy})-\rho\,(u_0+(\ln\rho_0)_y)\gamma^2\int_0^\tau\rho^{\gamma-1}\rho_y\Big)\\ &\cdot\exp\Big\{-\gamma\int_0^\tau\rho^\gamma\Big\}+\rho_y\int_0^\tau(F+\gamma\rho^\gamma u)\exp\Big\{-\gamma\int_s^\tau\rho^\gamma\Big\}\mathrm{d}s\\ &+\rho\int_0^\tau\Big((F_y+\gamma^2\rho^{\gamma-1}\rho_y u+\gamma\rho^\gamma u_y)-(F+\gamma\rho^\gamma u)\gamma^2\int_s^\tau\rho^{\gamma-1}\rho_y\Big)\\ &\cdot\exp\Big\{-\gamma\int_s^\tau\rho^\gamma\Big\}\mathrm{d}s-\rho_y u-\rho u_y.\end{aligned}\tag{3.24}$$

Moreover, the estimates (3.6), (3.7), (3.14) and (3.20) in the Lagrangian coordinate become

$$\int_I\rho_y^2\le C_1,\qquad 0<C_1^{-1}\le\rho\le C_1,\qquad \int_I(u_y^2+u_{yy}^2)\le C_3,$$

from which and (3.24), we can deduce that

$$\int_I\rho_{yy}^2\le C_3\int_I(\rho_{0yy}^2+u_{0y}^2+\rho_y^2+u_y^2)+C_3\int_0^\tau\int_I(\rho_y^2+F_y^2+u_y^2)\le C_3,\tag{3.25}$$

where we have used the inequalities

$$\|\rho_y\|_{L^\infty}^2=\|\rho_y^2\|_{L^\infty}\le\int_I(|\rho_y^2|+|(\rho_y^2)_y|)=\int_I\rho_y^2+2\int_I|\rho_y\rho_{yy}|\le\varepsilon\int_I\rho_{yy}^2+c(\varepsilon)\int_I\rho_y^2$$

with ε small enough. From the definition of Lagrangian coordinates (2.37), we see that

$$\rho_{xx}=\rho(\rho_x)_y=\rho(\rho\rho_y)_y=\rho^2\rho_{yy}+\rho\rho_y^2,$$

which implies

$$\int_I\rho_{xx}^2\le C_3.$$

Moreover, from $(1.1)_1$ we get

$$\rho_{xt}=-\rho_{xx}u-2\rho_x u_x-\rho u_{xx},$$

from which we have

$$\int_I\rho_{xt}^2\le C_3.$$

Differentiating $(1.1)_{3,4}$ with respect to x twice, using (3.7), (3.19) and (3.22), we have

$$\int_I\chi_{xxxx}+\int_0^t\int_I\mu_{xxxx}^2\le C_3.$$

Therefore, (3.22) holds. The proof of Lemma 3.10 is complete. □

Furthermore, we also have

LEMMA 3.11. *For any $0\le t<T$, there holds*

$$\int_I (\rho_{xxx}^2+\rho_{xxt}^2)(t)\le C_4, \tag{3.26}$$

where C_4 is a positive constant depending only on c_0, E_0, $\|\rho_0\|_{H^3}$, $\|u_0\|_{H^2}$, $\|\chi_0\|_{H^4}$, γ and T.

Proof. Similar to the proof of Lemma 3.10, from (3.22) we know that $\|F\|_{L^\infty(0,T;H^2)}\le C_3$, where $F(x,t)=-\frac{1}{2}(\chi_x^2)_x$. In the Lagrangian coordinate, we can calculate ρ_{yyy} and obtain

$$\int_I \rho_{yyy}^2\le C_3\int_I(\rho_{0yyy}^2+u_{0yy}^2+\rho_{yy}^2+u_{yy}^2)+C_3\int_0^\tau\int_I(\rho_{yy}^2+F_{yy}^2+u_{yy}^2)\le C(C_3,\|\rho_0\|_{H^3}).$$

From (2.37), we see that

$$\rho_{xxx}=\rho(\rho^2\rho_{yy}+\rho\rho_y^2)_y=\rho^3\rho_{yyy}+4\rho^2\rho_y\rho_{yy}+\rho\rho_y^3.$$

Therefore, Lemma 3.11 follows. □

Similar to the proof of Lemma 2.9 and 2.10 in Section 3, we can also derive that

LEMMA 3.12. *For any $0\le t<T$, there exists a positive constant C_5 depending only on c_0, E_0, $\|\rho_0\|_{H^2}$, $\|u_0\|_{H^3}$, $\|\chi_0\|_{H^4}$, γ and T, such that*

$$\int_I (u_{xt}^2+u_{xxx}^2)(t)+\int_0^t\int_I(u_{xxt}^2+u_{xxxx}^2)\le C_5. \tag{3.27}$$

4. Proof of the main results

This section is devoted to the proof of our main results, Theorems 1.1–1.3, which have been stated in Section 1.

Firstly, in view of the local existence of the classical solutions and the a priori estimates obtained in Section 3, one may finish the proof of the existence and uniqueness of global classical solutions by standard arguments. Theorem 1.1 follows.

Next, by virtue of Lemmas 3.1–3.6, we show that there exist global weak solutions to the problem (1.1)–(1.3) under the assumptions $\rho_0\in H^1(I)$ with $0<c_0^{-1}\le\rho_0\le c_0$ and $u_0\in L^2(I)$, $\chi_0\in H^1(I)$.

Proof. (**Proof of Theorem 1.3.**) By the standard mollification , we may assume that for any $\alpha\in(0,1)$, there exists a sequence of initial data $(\rho_0^\varepsilon,u_0^\varepsilon,\chi_0^\varepsilon)\in C^{3,\alpha}(I)\times C^{3,\alpha}(I)\times C^{4,\alpha}(I)$ such that

$$0<c_0^{-1}\le\rho_0^\varepsilon\le c_0<+\infty \quad \text{on } I,$$
$$\lim_{t\to 0}(\|\rho_0^\varepsilon-\rho_0\|_{H^1}+\|u_0^\varepsilon-u_0\|_{L^2}+\|\chi_0^\varepsilon-\chi_0\|_{H^1})=0.$$

Let $(\rho^\varepsilon,u^\varepsilon,\chi^\varepsilon)$ be the unique global classical solution of (1.1) with the initial conditions $(\rho_0^\varepsilon,u_0^\varepsilon,\chi_0^\varepsilon)$ and the boundary value conditions $(u^\varepsilon,\chi_x^\varepsilon,\mu_x^\varepsilon)\big|_{\partial I}=0$ for $t>0$. It follows from Lemmas 3.1–3.6 and the equations (1.1) that, for any $0<T<+\infty$, the following properties hold

$$\begin{aligned}&\frac{1}{C(T)}\le\rho^\varepsilon\le C(T),\quad \text{in } I\times[0,T],\\&\|\rho^\varepsilon\|_{L^\infty(0,T;H^1)}+\|\rho_t^\varepsilon\|_{L^2(0,T;L^2)}\le C(T),\\&\|u^\varepsilon\|_{L^\infty(0,T;L^2)}+\|u^\varepsilon\|_{L^2(0,T;H_0^1)}\le C(T),\\&\|\chi^\varepsilon\|_{L^\infty(0,T;H^1)}+\|\chi^\varepsilon\|_{L^2(0,T;H^3)}\le C(T),\\&\|\mu^\varepsilon\|_{L^2(0,T;H^1)}\le C(T).\end{aligned}$$

After taking possible subsequences (denoted by itself for convenience), taking $\varepsilon\to 0$, by (1.1) and Lemma 2.11, we have

$$(\rho^\varepsilon,\rho_x^\varepsilon)\rightharpoonup(\rho,\rho_x)\quad\text{weak}^*\text{ in } L^\infty(0,T;L^2),\tag{4.1}$$
$$\rho_t^\varepsilon\rightharpoonup\rho_t\quad\text{weakly in } L^2(0,T;L^2),\tag{4.2}$$
$$\rho^\varepsilon\to\rho\quad\text{strongly in } C(Q_T),\tag{4.3}$$
$$u^\varepsilon\rightharpoonup u\quad\text{weak}^*\text{ in } L^\infty\left(0,T;L^2\right),\tag{4.4}$$
$$(u^\varepsilon,u_x^\varepsilon)\rightharpoonup(u,u_x)\qquad\text{weakly in } L^2\left(0,T;L^2\right),\tag{4.5}$$
$$(\chi^\varepsilon,\chi_x^\varepsilon,\chi_{xx}^\varepsilon,\chi_{xxx}^\varepsilon)\rightharpoonup(\chi,\chi_x,\chi_{xx},\chi_{xxx})\quad\text{weakly in } L^2(0,T;L^2),\tag{4.6}$$
$$(\chi^\varepsilon,\chi_x^\varepsilon)\rightharpoonup(\chi,\chi_x)\quad\text{weak}^*\text{ in } L^\infty\left(0,T;L^2\right),\tag{4.7}$$
$$(\mu^\varepsilon,\mu_x^\varepsilon)\rightharpoonup(\mu,\mu_x)\quad\text{weakly in } L^2(0,T;L^2).\tag{4.8}$$

It's easy to see that $(\rho^\varepsilon,u^\varepsilon,\chi^\varepsilon)$ satisfy

$$\begin{aligned}&\iint_{Q_T}\left(\rho_t^\varepsilon\zeta(x)-\rho^\varepsilon u^\varepsilon\zeta'(x)\right)=0,\\&-\iint_{Q_T}\left(\rho^\varepsilon u^\varepsilon\xi(x)\eta'(t)+\rho^\varepsilon(u^\varepsilon)^2\xi'(x)\eta(t)+(\rho^\varepsilon)^\gamma\xi'(x)\eta(t)\right)\\&=\int_I\rho_0^\varepsilon u_0^\varepsilon\xi(x)\eta(0)-\iint_{Q_T}\left(u_x^\varepsilon\xi'(x)\eta(t)-\frac{1}{2}(\chi_x^\varepsilon)^2\xi'(x)\eta(t)\right),\\&-\iint_{Q_T}(\rho^\varepsilon\chi^\varepsilon\phi_t+\rho^\varepsilon u^\varepsilon\chi^\varepsilon\phi_x)=\int_I\rho_0^\varepsilon\chi_0^\varepsilon\phi(0)-\iint_{Q_T}\mu_x^\varepsilon\phi_x,\\&\iint_{Q_T}\rho^\varepsilon\mu^\varepsilon\phi=\iint_{Q_T}\left(\rho^\varepsilon((\chi^\varepsilon)^3-\chi^\varepsilon)\phi+\chi_x^\varepsilon\phi_x\right)\end{aligned}$$

for any $\zeta(x)\in C^1(I)$, $\xi(x)\in C_0^1(I)$, $\eta(t)\in C^1[0,T]$ with $\eta(T)=0$ and $\phi\in C^1(Q_T)$ with $\phi(\cdot,T)=0$. Noticing

$$\begin{aligned}&\iint_{Q_T}\left(\rho^\varepsilon(u^\varepsilon)^2-\rho u^2\right)\xi'\eta\\&\le\iint_{Q_T}\left(|\rho^\varepsilon-\rho||u^\varepsilon|^2|\xi'||\eta|+|\rho||u^\varepsilon-u||u^\varepsilon+u||\xi'||\eta|\right)\\&\le C\|\rho^\varepsilon-\rho\|_{L^\infty(Q_T)}\|u^\varepsilon\|_{L^2(Q_T)}^2+C\|\rho\|_{L^\infty(Q_T)}\|u^\varepsilon-u\|_{L^2(Q_T)}\left(\|u^\varepsilon\|_{L^2(Q_T)}+\|u\|_{L^2(Q_T)}\right)\\&\longrightarrow 0,\qquad\text{as }\varepsilon\to 0\end{aligned}$$

and

$$
\begin{aligned}
&\iint_{Q_T} (\rho^\varepsilon u^\varepsilon \chi^\varepsilon - \rho u \chi)\phi_x \\
\leq &\iint_{Q_T} (|\rho^\varepsilon - \rho||u^\varepsilon||\chi^\varepsilon||\phi_x| + |\rho||u^\varepsilon - u||\chi^\varepsilon||\phi_x| + |\rho||u||\chi^\varepsilon - \chi||\phi_x|) \\
\leq & C\|\rho^\varepsilon - \rho\|_{L^\infty(Q_T)}\|u^\varepsilon\|_{L^2 Q_T}\|\chi^\varepsilon\|_{L^2(Q_T)} + C\|\rho\|_{L^\infty(Q_T)}\|u^\varepsilon - u\|_{L^2(Q_T)}\|\chi^\varepsilon\|_{L^2(Q_T)} \\
&+ C\|\rho\|_{L^\infty(Q_T)}\|u\|_{L^2(Q_T)}\|\chi^\varepsilon - \chi\|_{L^2(Q_T)} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0,
\end{aligned}
$$

it is easy to check that (ρ, u, χ) is a weak solution of the problem (1.1)–(1.3) in the sense of Definition 1.2. The proof of Theorem 1.3 is complete. ∎

In terms of Lemmas 3.1–3.8, we can prove the existence and uniqueness of strong solutions to the problem (1.1)–(1.3) under the assumptions $\rho_0 \in H^2(I)$ with $0 < c_0^{-1} \leq \rho_0 \leq c_0$ and $u_0 \in H_0^1(I)$, $\chi_0 \in H^4(I)$.

Proof. (**Proof of Theorem 1.2.**) Observe that $\rho_0 \in H^2(I)$, $u_0 \in H_0^1(I)$ and $\chi_0 \in H^4(I)$. We assume

$$
\lim_{\varepsilon \to 0} (\|\rho_0^\varepsilon - \rho_0\|_{H^2} + \|u_0^\varepsilon - u_0\|_{H^1} + \|\chi_0^\varepsilon - \chi_0\|_{H^4}) = 0.
$$

Lemmas 3.7 and 3.8 imply

$$
\begin{aligned}
\sup_{0 \leq t \leq T} \int_I & \left(|\rho_t^\varepsilon|^2 + |u_x^\varepsilon|^2 + |\chi_{xxx}^\varepsilon|^2 + |\chi_t^\varepsilon|^2 + |\mu_{xx}|^2\right) \\
&+ \iint_{Q_T} \left(|u_{xx}^\varepsilon|^2 + |u_t^\varepsilon|^2 + |\chi_{xxt}^\varepsilon|^2 + |\mu_{xxx}|^2 + |\mu_t|^2\right) \leq C(T).
\end{aligned}
$$

By the proof of Theorem 1.3 and the weak lower semi-continuity of the norm, we can easily derive that $\rho_t \in L^\infty(0,T;L^2)$, $u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2)$, $u_t \in L^2(0,T;L^2)$, $\chi \in L^\infty(0,T;H^3)$, $\chi_t \in L^\infty(0,T;L^2) \cap L^2(0,T;H^2)$, $\mu \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3)$ and $\mu_t \in L^2(0,T;L^2)$. Moreover, by trace theorem $H^1(U) \hookrightarrow L^2(\partial U)$ for bounded U with $\partial U \in C^1$, there hold that $(\chi_x, \mu_x)|_{\partial I} = (0,0)$ in the sense of trace. Thus, we obtain the existence of the strong solutions to the problem (1.1)–(1.3) in the sense of Definition 1.1.

It remains for us to prove the uniqueness of the strong solutions. Let (ρ_i, u_i, χ_i) be two solutions to the problem (1.1)–(1.3) obtained above. Denote $\widetilde{\rho} = \rho_1 - \rho_2$, $\widetilde{u} = u_1 - u_2$, $\widetilde{\chi} = \chi_1 - \chi_2$ and $\widetilde{\mu} = \mu_1 - \mu_2$. Then

$$
\begin{cases}
\widetilde{\rho}_t + (\widetilde{\rho} u_1)_x + (\rho_2 \widetilde{u})_x = 0, \\
\rho_1 \widetilde{u}_t - \widetilde{u}_{xx} = -\widetilde{\rho} u_{2t} - \rho_1 u_1 \widetilde{u}_x - \rho_1 \widetilde{u} u_{2x} - \widetilde{\rho} u_2 u_{2x} \\
\qquad\qquad - (\rho_1^\gamma - \rho_2^\gamma)_x - \chi_{1x} \widetilde{\chi}_{xx} - \widetilde{\chi}_x \chi_{2xx}, \\
\rho_1 \widetilde{\chi}_t + \rho_1 u_1 \widetilde{\chi}_x = \widetilde{\mu}_{xx} - \widetilde{\rho} \chi_{2t} - \widetilde{\rho} u_2 \chi_{2x} - \rho_1 \widetilde{u} \chi_{2x}, \\
\rho_1 \widetilde{\mu} = -\widetilde{\chi}_{xx} + \rho_1 (\chi_1^2 + \chi_1 \chi_2 + \chi_2^2 - 1)\widetilde{\chi} + \widetilde{\rho}(\chi_2^3 - \chi_2) - \widetilde{\rho} \mu_2
\end{cases}
\tag{4.9}
$$

for $(x,t) \times (0,T)$, subject to the initial boundary value conditions

$$
(\widetilde{\rho},\ \widetilde{u},\ \widetilde{\chi})\Big|_{t=0} = 0 \ \text{ in } I, \qquad (\widetilde{u},\ \widetilde{\chi}_x,\ \widetilde{\mu}_x)\Big|_{\partial I} = 0 \ \text{ for } 0 < t < T.
$$

Multiplying $(4.9)_1$ by $\widetilde{\rho}$, integrating the result over I, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \widetilde{\rho}^2 &= \int_I \widetilde{\rho} u_1 \widetilde{\rho}_x - \int_I (\rho_{2x}\widetilde{u} + \rho_2 \widetilde{u}_x)\widetilde{\rho} \\ &= -\frac{1}{2}\int_I \widetilde{\rho}^2 u_{1x} - \int_I (\rho_{2x}\widetilde{u} + \rho_2\widetilde{u}_x)\widetilde{\rho} \\ &\le \frac{1}{2}\|u_{1x}\|_{L^\infty}\int_I \widetilde{\rho}^2 + \|\widetilde{u}\|_{L^\infty}\|\rho_{2x}\|_{L^2}\|\widetilde{\rho}\|_{L^2} + \|\rho_2\|_{L^\infty}\|\widetilde{u}_x\|_{L^2}\|\widetilde{\rho}\|_{L^2}.\end{aligned}$$

Since $\widetilde{u}(0,t)=0$, we have $\widetilde{u}(y,t)=\int_0^y \widetilde{u}_x(x,t)\mathrm{d}x$ for $(y,t)\in Q_T$. Hence,

$$\|\widetilde{u}\|_{L^\infty} \le \|\widetilde{u}_x\|_{L^2}, \quad t\in[0,T]. \tag{4.10}$$

From (4.10) and the regularities of ρ_i, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \widetilde{\rho}^2 \le C\|u_1\|_{H^2}\int_I \widetilde{\rho}^2 + C\|\widetilde{u}_x\|_{L^2}\|\widetilde{\rho}\|_{L^2} \le C(\|u_1\|_{H^2}+1)\int_I \widetilde{\rho}^2 + \int_I \widetilde{u}_x^2. \tag{4.11}$$

Multiplying $(4.9)_2$ by $\widetilde{u}$ and integrating the result over I, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_1\widetilde{u}^2 + \int_I \widetilde{u}_x^2 \\ =&\frac{1}{2}\int_I \rho_{1t}\widetilde{u}^2 - \int_I \rho_1 u_1 \widetilde{u}_x\widetilde{u} - \int_I \widetilde{\rho}\widetilde{u}u_{2t} - \int_I \widetilde{\rho}\widetilde{u}u_2u_{2x} - \int_I \rho_1\widetilde{u}^2 u_{2x} \\ &+ \int_I (\rho_1^\gamma - \rho_2^\gamma)\widetilde{u}_x + \int_I \chi_{1x}\widetilde{\chi}_x\widetilde{u}_x + \int_I \chi_{1xx}\widetilde{\chi}_x\widetilde{u} - \int_I \chi_{2xx}\widetilde{\chi}_x\widetilde{u}.\end{aligned}$$

Recalling $\rho_{1t}+(\rho_1u_1)_x=0$, we have

$$\frac{1}{2}\int_I \rho_{1t}\widetilde{u}^2 - \int_I \rho_1u_1\widetilde{u}_x\widetilde{u} = \frac{1}{2}\int_I \rho_{1t}\widetilde{u}^2 + \frac{1}{2}\int_I \widetilde{u}^2(\rho_1u_1)_x = 0.$$

Using the above equality, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_1\widetilde{u}^2 + \int_I \widetilde{u}_x^2 \\ \le& \|\widetilde{u}\|_{L^\infty}\|\widetilde{\rho}\|_{L^2}\|u_{2t}\|_{L^2} + \|\widetilde{\rho}\|_{L^2}\|\widetilde{u}\|_{L^\infty}\|u_2\|_{L^\infty}\|u_{2x}\|_{L^2} + \|u_{2x}\|_{L^\infty}\int_I \rho_1\widetilde{u}^2 \\ &+ c\|\widetilde{\rho}\|_{L^2}\|\widetilde{u}_x\|_{L^2} + c\|\widetilde{u}_x\|_{L^2}\|\widetilde{\chi}_x\|_{L^2}\|\chi_{1x}\|_{L^\infty} + c\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}\|\widetilde{\chi}_x\|_{L^2}\left\|\frac{\chi_{1xx}+\chi_{2xx}}{\sqrt{\rho_1}}\right\|_{L^\infty}.\end{aligned}$$

From the regularities of (ρ_i, u_i, χ_i) and (4.10), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_1\widetilde{u}^2 + \int_I \widetilde{u}_x^2 \le& C\|\widetilde{u}_x\|_{L^2}(\|\widetilde{\rho}\|_{L^2}\|u_{2t}\|_{L^2} + \|\widetilde{\rho}\|_{L^2} + \|\widetilde{\chi}_x\|_{L^2}) \\ &+ C\|u_2\|_{H^2}\int_I \rho_1\widetilde{u}^2 + C(\|\chi_1\|_{H^3} + \|\chi_2\|_{H^3})\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}\|\widetilde{\chi}_x\|_{L^2} \\ \le& \frac{1}{2}\|\widetilde{u}_x\|_{L^2}^2 + C\|\widetilde{\rho}\|_{L^2}^2\left(1+\|u_{2t}\|_{L^2}^2\right) + C\|\widetilde{\chi}_x\|_{L^2}^2 \\ &+ C\left(\|u_2\|_{H^2} + \|\chi_1\|_{H^3}^2 + \|\chi_2\|_{H^3}^2\right)\int_I \rho_1\widetilde{u}^2.\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_I \rho_1\widetilde{u}^2+\int_I \widetilde{u}_x^2\\
\le&C(1+\|u_{2t}\|_{L^2}^2)\int_I\widetilde{\rho}^2+C\int_I\widetilde{\chi}_x^2+C(\|u_2\|_{H^2}+\|\chi_1\|_{H^3}^2+\|\chi_2\|_{H^3}^2)\int_I\rho_1\widetilde{u}^2\\
\le&C(1+\|u_{2t}\|_{L^2}^2)\int_I\widetilde{\rho}^2+C\int_I\rho_1\widetilde{\chi}^2+\int_I\frac{1}{\rho_1}\widetilde{\chi}_{xx}^2+C(\|u_2\|_{H^2}+\|\chi_1\|_{H^3}^2+\|\chi_2\|_{H^3}^2)\int_I\rho_1\widetilde{u}^2.
\end{aligned}\tag{4.12}$$

Multiplying $(4.9)_3$ by $\widetilde{\chi}$, integrating the result over I, by $(4.9)_4$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho_1\widetilde{\chi}^2\\
=&\int_I\widetilde{\mu}\widetilde{\chi}_{xx}-\int_I\widetilde{\rho}\chi_{2t}\widetilde{\chi}-\int_I\rho_1\widetilde{u}\chi_{2x}\widetilde{\chi}-\int_I\widetilde{\rho}u_2\chi_{2x}\widetilde{\chi}\\
=&-\int_I\frac{1}{\rho_1}\widetilde{\chi}_{xx}^2+\int_I\frac{\widetilde{\rho}}{\rho_1}(\chi_2^3-\chi_2)\widetilde{\chi}_{xx}+\int_I(\chi_1^2+\chi_1\chi_2+\chi_2^2-1)\widetilde{\chi}\widetilde{\chi}_{xx}\\
&-\int_I\frac{\widetilde{\rho}}{\rho_1}\mu_2\widetilde{\chi}_{xx}-\int_I\widetilde{\rho}\chi_{2t}\widetilde{\chi}-\int_I\rho_1\widetilde{u}\chi_{2x}\widetilde{\chi}-\int_I\widetilde{\rho}u_2\chi_{2x}\widetilde{\chi}\\
\le&-\frac{1}{2}\int_I\frac{1}{\rho_1}\widetilde{\chi}_{xx}^2+c(\|\chi_1^2+\chi_1\chi_2+\chi_2^2\|_{L^\infty}^2+1)\int_I\rho_1\widetilde{\chi}^2+c\|\chi_{2x}\|_{L^\infty}^2\int_I\rho_1\widetilde{u}^2\\
&+c\left\|\frac{1}{\rho}\right\|_{L^\infty}(\|\chi_2^3-\chi_2\|_{L^\infty}^2+\|\mu_2\|_{L^\infty}^2+\|\chi_{2t}\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2\|\chi_{2x}\|_{L^\infty}^2)\int_I\widetilde{\rho}^2.
\end{aligned}$$

Using the embedding theorem and the regularities for (ρ_i,u_i,χ_i), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int_I\rho_1\widetilde{\chi}^2+\int_I\frac{1}{\rho_1}\widetilde{\chi}_{xx}^2\\
\le&C\int_I\rho_1\widetilde{\chi}^2+C\|\chi_2\|_{H^2}^2\int_I\rho_1\widetilde{u}^2+C(1+\|\mu_2\|_{H^1}^2+\|\chi_{2t}\|_{H^1}^2+\|u_2\|_{H^1}^2\|\chi_2\|_{H^2}^2)\int_I\widetilde{\rho}^2\\
\le&C\int_I\rho_1\widetilde{\chi}^2+C\int_I\rho_1\widetilde{u}^2+C(1+\|\chi_{2t}\|_{H^1}^2)\int_I\widetilde{\rho}^2.
\end{aligned}\tag{4.13}$$

Putting (4.11), (4.12) and (4.13) together, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_I(\widetilde{\rho}^2+\rho_1\widetilde{u}^2+\rho_1\widetilde{\chi}^2)\le CE(t)\int_I(\widetilde{\rho}^2+\rho_1\widetilde{u}^2+\rho_1\widetilde{\chi}^2),\tag{4.14}$$

where $E(t)=1+\|u_1\|_{H^2}^2+\|u_{2t}\|_{L^2}^2+\|u_2\|_{H^2}^2+\|\chi_1\|_{H^3}^2+\|\chi_2\|_{H^3}^2+\|\chi_{2t}\|_{H^1}^2$ satisfying $\int_0^T E(t)dt\le C$. Then Grönwall's inequality implies

$$\int_I(\widetilde{\rho}^2+\rho_1\widetilde{u}^2+\rho_1\widetilde{\chi}^2)\le 0.$$

Because of the positivity of ρ_1, we have $(\widetilde{\rho},\widetilde{u},\widetilde{\chi})=0$. The proof of Theorem 1.2 is complete. ∎

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