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LOCAL WELL AND ILL POSEDNESS FOR THE MODIFIED KDV EQUATIONS IN SUBCRITICAL MODULATION SPACES*

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Abstract. We consider the Cauchy problem of the modified KdV (mKdV) equation. Local well-posedness of this problem is obtained in modulation spaces $M_{2,q}^{1/4}(\mathbb{R})$ $(2\le q\le\infty)$. Moreover, we show that the data-to-solution map fails to be C^3 continuous in $M_{2,q}^{s}(\mathbb{R})$ when $s<1/4$. We notice that $H^{1/4}$ is the critical Sobolev space for mKdV such that it is well-posed in H^s for $s\ge 1/4$ and ill-posed (in the sense of uniform continuity) in $H^{s'}$ with $s'<1/4$. Recalling that $M_{2,q}^{1/4}\subset B_{2,q}^{1/q-1/4}$ is a sharp embedding and $H^{-1/4}\subset B_{2,\infty}^{-1/4}$, our results contain all of the subcritical data in $M_{2,q}^{1/4}$, which contains a class of functions in $H^{-1/4}\setminus H^{1/4}$.

Keywords. Local well-posedness; Ill-posedness; Modified KdV equations; Modulation spaces.

AMS subject classifications. 35A01; 35Q53.

1. Introduction

In this paper we study the Cauchy problem of the modified Korteweg-de Vries (mKdV) equation on the real line $\mathbb{R}$:

$$u_t+u_{xxx}\pm(u^3)_x=0,\quad u(0,x)=u_0(x),\quad x\in\mathbb{R}, \tag{1.1}$$

where $u=u(x,t)\in\mathbb{R}$ with $(x,t)\in\mathbb{R}^{1+1}$.

The scale invariant homogeneous Sobolev space for mKdV is $\dot{H}^{-1/2}$. That is to say, for any solution $u(x,t)$ of (1.1) with initial data $u_0(x)$, the scaling function $u_\lambda(x,t):=\lambda u(\lambda x,\lambda^3 t)$ is also a solution of (1.1) with initial data $u_{0,\lambda}:=\lambda u_0(\lambda x)$, and satisfies

$$\|u_{0,\lambda}\|_{\dot{H}^{-1/2}}=\|u_0\|_{\dot{H}^{-1/2}}. \tag{1.2}$$

On the other hand, $H^{1/4}$ is the critical Sobolev space of mKdV so that it is globally well-posed in H^s for $s\ge 1/4$ and ill-posed in $H^{s'}$ with $s'<1/4$. The ill-posed result is in the sense that the data-to-solution map fails to be uniformly continuous on a fixed ball in $H^{s'}$ with $s'<1/4$. The local well-posed result for $s\ge 1/4$ by using the contraction method and ill-posed result for the focusing equation (+ sign in front of the nonlinearity) were proved by Kenig, Ponce and Vega, see [22] and [23], respectively. The local well-posed result was extended to a global one for $s>1/4$ due to Colliander, Keel, Staffilani, Takaoka and Tao by using I-method, see [10]. The global result for $s=1/4$ was obtained by Guo in [16]. In addition, the ill-posed result for the defocusing equation (− sign in front of the nonlinearity) was obtained by Christ, Colliander and Tao [6].

Therefore, there is 3/4 derivative gap between $H^{-1/2}$ and $H^{1/4}$ for the well-posedness result of mKdV. In order to discover the behavior of the solution out of $H^{1/4}$, Grünrock brought in the $\widehat{H_s^{q'}}$ spaces for which the norm is defined by

$$\|u\|_{\widehat{H_s^{q'}}}:=\|\langle\xi\rangle^s\hat{u}\|_{L^q},\qquad 1/q+1/q'=1,$$

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and he obtained the local well-posedness of (1.1) for data $u_0 \in \widehat{H^{q'}_s}(\mathbb{R}), 2 \le q < 4, s \ge s(q) := 1/2q$ in [12]. In 2009, Grünrock and Vega broadened the range of q to $2 \le q < \infty$ by using the trilinear estimates in [13]. From the scaling point, the spaces $\widehat{H^{q'}_s}$ behave like the Sobolev spaces H^σ, if $s - 1/2 + 1/q = \sigma$. Thus, they can lower the regularity to $-1/2$ by taking q tending to infinity, but there is no result for $q = \infty$. In this paper we consider the initial data in more general modulation spaces $M^s_{2,q}$, $2 \le q \le \infty$ (Indeed, $\widehat{H^{q'}_s} \subset M^s_{2,q}$). It should be noted that a similar result, which does not contain the case $q = \infty$, was proved by Oh and Wang [28] using different methods, where they aim for the global well-posedness.

Modulation space $M^s_{p,q}$ was introduced by Feichtinger [11] in 1983 and equivalently defined in the following way (cf. [31–34]):

$$\|f\|_{M^s_{p,q}(\mathbb{R})} = \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\square_k f\|^q_{L^p(\mathbb{R})} \right)^{1/q}, \tag{1.3}$$

where $\square_k = \mathscr{F}^{-1} \chi_{[k-1/2, k+1/2]} \mathscr{F}$, $\mathscr{F}$ ($\mathscr{F}^{-1}$) denotes the (inverse) Fourier transform on $\mathbb{R}$, χ_E denotes the characteristic function on E and $\langle k \rangle = (1+|k|^2)^{1/2}$. From Plancherel theorem and Hölder's inequality, we know that $\widehat{H^{q'}_s} \subset M^s_{2,q}$ $(2 \le q \le \infty)$. Moreover, combining the sharp inclusions between Besov and modulation spaces, we have (cf. [29, 33])

$$\widehat{H^{q'}_{1/4}} \subset M^{1/4}_{2,q} \subset B^{1/q-1/4}_{2,q}, \quad 2 \le q \le \infty,$$

where the inclusions are optimal. Therefore, our result in which the initial data belong to $M^{1/4}_{2,\infty}$ can be certainly seen as an improvement. Our main theorem is as follows.

THEOREM 1.1. *Let $2 \le q \le \infty$, $u_0 \in M^{1/4}_{2,q}$. Then there exists $T > 0$ such that mKdV (1.1) is locally well posed in $C([0,T]; M^{1/4}_{2,q}) \cap X^{1/4}_{q,A}([0,T])$, where $X^{1/4}_{q,A}$ is defined in the next section. Moreover, the regularity index $1/4$ in $M^{1/4}_{2,q}$ is optimal. Specifically, if $s < 1/4$, the data-to-solution map in $M^s_{2,q}(\mathbb{R})$ is not C^3 continuous at origin.*

Modulation spaces contain a class of initial data out of the critical Sobolev spaces H^{s_c}, for which the nonlinear PDE is well-posed for $s > s_c$ and ill-posed for $s < s_c$. Therefore, solving the nonlinear PDE in modulation spaces has absorbed some researchers' attention, see [1–4, 7–9, 18–21, 30, 35]. We will use U^p and V^p spaces in our discussion, since the dual relation and other important properties are ideally to deal with the nonlinearity. U^p and V^p spaces are introduced to solving PDEs by Koch and Tataru, see [5, 17, 25, 26]. Combining U^p, V^p and modulation spaces, Guo, Ren and Wang have considered the cubic and derivative nonlinear Schrödinger equation, respectively, see [14, 15].

Let us list some notations. Let $c < 1$, $C > 1$ denote positive universal constants, which can be different at different places; $a \lesssim b$ stands for $a \le Cb$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$; $a \approx b$ means that $|a - b| \le C$, $a \gg b$ means that $a > b + C$; We write $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$; p' is the dual number of $p \in [1,\infty]$, i.e., $1/p + 1/p' = 1$.

2. Function spaces

2.1. Definitions. In this subsection, we review some function spaces used to obtain the well-posedness theory for non-linear dispersive equations. U^p spaces were first applied by Koch and Tataru [5, 25–27], and V^p spaces are due to Wiener [36].

Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_{K-1} < t_K = \infty$. In the following, we consider functions taking values in $L^2 := L^2(\mathbb{R}^d;\mathbb{C})$, but in general L^2 may be replaced by an arbitrary Hilbert space or general Banach space.

DEFINITION 2.1. *Let $1 \le p < \infty$. For any $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_2^p = 1$, $\phi_0 = 0$. A step function $a : \mathbb{R} \to L^2$ given by*

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)} \phi_{k-1}$$

is said to be a U^p-atom. All of the U^p atoms are denoted by $\mathcal{A}(U^p)$. The U^p space is

$$U^p := \left\{ u = \sum_{j=1}^{\infty} c_j a_j : \ a_j \in \mathcal{A}(U^p), \quad c_j \in \mathbb{C}, \quad \sum_{j=1}^{\infty} |c_j| < \infty \right\}$$

for which the norm is given by

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |c_j| : \quad u = \sum_{j=1}^{\infty} c_j a_j, \quad a_j \in \mathcal{A}(U^p), \quad c_j \in \mathbb{C} \right\}.$$

DEFINITION 2.2. *Let $1 \le p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L^2$ such that $\lim_{t \to \pm\infty} v(t)$ exist and for which the norm*

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}$$

is finite, where we use the convention that $v(-\infty) = \lim_{t \to \infty} v(t)$ and $v(\infty) = 0$ (here $v(\infty)$ and $\lim_{t \to \infty} v(t)$ are different notations). Likewise, we denote by V_-^p the subspace of all $v \in V^p$ so that $v(-\infty) = 0$. Moreover, we define the closed subspace V_{rc}^p ($V_{-,rc}^p$) as all of the right continuous functions in V^p (V_-^p).

DEFINITION 2.3. *We define*

$$U_A^p := e^{-\cdot \partial_x^3} U^p, \quad \|u\|_{U_A^p} = \|e^{t\partial_x^3} u\|_{U^p},$$

$$V_A^p := e^{-\cdot \partial_x^3} V^p, \quad \|u\|_{V_A^p} = \|e^{t\partial_x^3} u\|_{V^p},$$

and similarly for the definition of $V_{rc,A}^p$, $V_{-,A}^p$, $V_{-,rc,A}^p$.

DEFINITION 2.4. *Besov-type Bourgain's spaces $\dot{X}^{s,b,q}$ are defined by*

$$\|u\|_{\dot{X}^{s,b,q}} := \left\| \|\chi_{|\tau - \xi^3| \in [2^{j-1}, 2^j)} |\xi|^s |\tau - \xi^3|^b \widehat{u}(\tau,\xi)\|_{L^2_{\xi,\tau}} \right\|_{\ell^q_{j \in \mathbb{Z}}}.$$

DEFINITION 2.5. *The frequency-uniform localized U^2-spaces $X_q^s(I)$ and V^2-spaces $Y_q^s(I)$ are defined by*

$$\|u\|_{X_q^s(I)} = \left(\sum_{\lambda \in I \cap \mathbb{Z}} \langle \lambda \rangle^{sq} \|\Box_\lambda u\|_{U^2}^q \right)^{1/q}, \quad X_q^s := X_q^s(\mathbb{R}), \tag{2.1}$$

$$\|v\|_{Y^s_q(I)} = \left(\sum_{\lambda\in I\cap\mathbb{Z}} \langle\lambda\rangle^{sq}\|\Box_\lambda v\|^q_{V^2}\right)^{1/q}, \quad Y^s_q := Y^s_q(\mathbb{R}), \tag{2.2}$$

$$\|u\|_{X^s_{q,A}} := \|e^{t\partial_x^3}u\|_{X^s_q}, \quad \|v\|_{Y^s_{q,A}} := \|e^{t\partial_x^3}v\|_{Y^s_q}. \tag{2.3}$$

2.2. Known results. The following known results about U^p and V^p can be found in [14,17,25,27].

PROPOSITION 2.1. (Embedding) *Let $1\le p<q<\infty$. We have the following results.*

(1) *U^p and V^p, V^p_{rc}, V^p_-, $V^p_{rc,-}$ are Banach spaces.*

(2) *$U^p\subset V^p_{rc,-}\subset U^q\subset L^\infty(\mathbb{R},L^2)$. Every $u\in U^p$ is right continuous on $t\in\mathbb{R}$.*

(3) *$V^p\subset V^q$, $V^p_-\subset V^q_-$, $V^p_{rc}\subset V^q_{rc}$, $V^p_{rc,-}\subset V^q_{rc,-}$.*

(4) *$\dot X^{0,1/2,1}\subset U^2_A\subset V^2_A\subset \dot X^{0,1/2,\infty}$.*

Similar to the Schrödinger equation, whose dispersive modulation is $|\tau+\xi^2|$, the mKdV equation's dispersive modulation is $|\tau-\xi^3|$. By the last inclusion of (4) in Proposition 2.1, we see that

LEMMA 2.1 (Dispersion Modulation Decay). *Suppose that the dispersion modulation $|\tau-\xi^3|\gtrsim\mu$ for a function $u\in L^2_{x,t}$, then we have*

$$\|u\|_{L^2_{x,t}}\lesssim \mu^{-1/2}\|u\|_{V^2_A}. \tag{2.4}$$

PROPOSITION 2.2 (Interpolation). *Let $1\le p<q<\infty$. There exists a positive constant $\epsilon(p,q)>0$, such that for any $u\in V^p$ and $M>1$, there exists a decomposition $u=u_1+u_2$ satisfying*

$$\frac{1}{M}\|u_1\|_{U^p}+e^{\epsilon M}\|u_2\|_{U^q}\lesssim\|u\|_{V^p}. \tag{2.5}$$

Let $I\subset\mathbb{R}$ be an interval with finite length. For the sake of simplicity, we denote

$$u_\lambda=\Box_\lambda u,\quad u_I=\sum_{\lambda\in I\cap\mathbb{Z}}u_\lambda.$$

PROPOSITION 2.3 (orthogonality in V^2). *Take an interval $I\subset\mathbb{R}$, then for $u\in V^2$ the following orthogonality holds:*

$$\|u_I\|_{V^2}\le\left(\sum_{\lambda\in I\cap\mathbb{Z}}\|u_\lambda\|^2_{V^2}\right)^{1/2}. \tag{2.6}$$

PROPOSITION 2.4 (Duality). *Let $1\le p<\infty$, $1/p+1/p'=1$. Then $(U^p)^*=V^{p'}$ in the sense that*

$$T:V^{p'}\to(U^p)^*;\quad T(v)=B(\cdot,v), \tag{2.7}$$

is an isometric mapping. The bilinear form $B:U^p\times V^{p'}$ is defined in the following way: For a partition $\mathrm{t}:=\{t_k\}_{k=0}^K\in\mathcal{Z}$, we define

$$B_{\mathrm{t}}(u,v)=\sum_{k=1}^K\langle u(t_{k-1}),\ v(t_k)-v(t_{k-1})\rangle. \tag{2.8}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product on L^2. For any $u\in U^p$, $v\in V^{p'}$, there exists a unique number $B(u,v)$ satisfying the following property. For any $\varepsilon>0$, there exists a partition t *such that*

$$|B(u,v)-B_{\mathrm{t}'}(u,v)|<\varepsilon,\quad \forall\ \mathrm{t}'\supset\mathrm{t}.$$

Moreover,

$$|B(u,v)|\le\|u\|_{U^p}\|v\|_{V^{p'}}.$$

In particular, let $u\in V_-^1$ be absolutely continuous on a compact interval, then for any $v\in V^{p'}$,

$$B(u,v)=\int\langle u'(t),v(t)\rangle dt.$$

Proposition 2.5 ([15] Duality). *Let $1\le q<\infty$. Then $(X_q^s)^*=Y_{q'}^{-s}$ in the sense that*

$$T:Y_{q'}^{-s}\to(X_q^s)^*;\quad T(v)=B(\cdot,v),\tag{2.9}$$

is an isometric mapping, where the bilinear form $B(\cdot,\cdot)$ is defined in Proposition 2.4. Moreover, we have

$$|B(u,v)|\le\|u\|_{X_q^s}\|v\|_{Y_{q'}^{-s}}.$$

3. Basic estimates

Lemma 3.1 ([24] Strichartz Estimates). *Let (p,q) satisfy the admissibility condition*

$$\frac{2}{p}+\frac{1}{q}=\frac{1}{2},\qquad 4\le p\le\infty,\quad 2\le q\le\infty.\tag{3.1}$$

Then

$$\|D_x^{1/p}e^{-t\partial_x^3}\phi\|_{L_t^pL_x^q}\lesssim\|\phi\|_{L^2}.\tag{3.2}$$

In particular, for $N\ge1$,

$$\|P_Ne^{-t\partial_x^3}\phi\|_{L_t^8L_x^4}\lesssim\langle N\rangle^{-1/8}\|\phi\|_{L^2}.\tag{3.3}$$

By testing atoms in U_A^8 space, we obtain

$$\|P_Nu\|_{L_t^8L_x^4}\lesssim\langle N\rangle^{-1/8}\|u\|_{U_A^8}.\tag{3.4}$$

Lemma 3.2 (Bilinear Estimate). *Suppose that $\widehat{u_0},\widehat{v_0}$ are localized in some compact intervals I_1,I_2 with $dist(I_1,I_2)\gtrsim\lambda$, $dist(I_1,-I_2)\gtrsim\mu$. Then,*

$$\|e^{-t\partial_x^3}u_0e^{-t\partial_x^3}v_0\|_{L_{x,t}^2}\lesssim(\lambda\mu)^{-1/2}\|u_0\|_{L^2}\|v_0\|_{L^2}.\tag{3.5}$$

By testing atoms in U_A^2 space, we obtain

$$\|uv\|_{L_{x,t}^2}\lesssim(\lambda\mu)^{-1/2}\|u\|_{U_A^2}\|v\|_{U_A^2}.\tag{3.6}$$

Applying the interpolation in Proposition 2.2, for any $0<\varepsilon\ll 1$ and $0<T\le 1$, we get

$$\|uv\|_{L^2_{x,t\in[0,T]}}\lesssim T^{\varepsilon/4}(\lambda\mu)^{-1/2+\varepsilon}\|u\|_{V^2_A}\|v\|_{V^2_A}. \tag{3.7}$$

Proof. Taking the Fourier transform in space, we have

$$\mathscr{F}_x\left(e^{-t\partial_x^3}u_0e^{-t\partial_x^3}v_0\right)(\xi,t)=\int e^{it(\xi^3+3\xi\xi_1^2-3\xi^2\xi_1)}\widehat{u}_0(\xi-\xi_1)\widehat{v}_0(\xi_1)d\xi_1. \tag{3.8}$$

Then taking the Fourier transform in time, we obtain

$$\mathscr{F}_{x,t}\left(e^{-t\partial_x^3}u_0e^{-t\partial_x^3}v_0\right)(\xi,\tau)=\int\delta(\tau+3\xi^2\xi_1-\xi^3-3\xi\xi_1^2)\widehat{u}_0(\xi-\xi_1)\widehat{v}_0(\xi_1)d\xi_1. \tag{3.9}$$

Denote

$$g(\xi_1)=\tau+3\xi^2\xi_1-\xi^3-3\xi\xi_1^2,$$

we see that the zeros and the derivative are

$$\xi_1^{\pm}=\frac{\xi}{2}\pm\sqrt{\frac{\xi^2}{4}-\frac{\xi^3-\tau}{3\xi}}:=\frac{\xi}{2}\pm y,\quad g'(\xi_1)=3\xi^2-6\xi\xi_1.$$

Recalling that $\delta(g(\xi_1))=\delta(\xi_1-\xi_1^+)/|g'(\xi_1^+)|+\delta(\xi_1-\xi_1^-)/|g'(\xi_1^-)|=\delta(\xi_1-\xi^+)/6|\xi|y+\delta(\xi_1-\xi_1^-)/6|\xi|y$, we have

$$\begin{aligned}&\mathscr{F}_{x,t}\left(e^{-t\partial_x^3}u_0e^{-t\partial_x^3}v_0\right)(\xi,\tau)\\&=\frac{1}{6|\xi|y}\widehat{u}_0\left(\frac{\xi}{2}-y\right)\widehat{v}_0\left(\frac{\xi}{2}+y\right)+\frac{1}{6|\xi|y}\widehat{u}_0\left(\frac{\xi}{2}+y\right)\widehat{v}_0\left(\frac{\xi}{2}-y\right).\end{aligned} \tag{3.10}$$

By symmetry, it suffices to estimate the first term in (3.10). Changing of variables $y=\sqrt{\frac{\xi^2}{4}-\frac{\xi^3-\tau}{3\xi}}$ and considering $d\tau=c|\xi||y|dy$, we see that

$$\begin{aligned}\left\|e^{-t\partial_x^3}u_0e^{-t\partial_x^3}v_0\right\|^2_{L^2_{x,t}}&\le\int_{\mathbb{R}^2}\frac{c}{|\xi||y|}\left|\widehat{u}_0\left(\frac{\xi}{2}-y\right)\right|^2\left|\widehat{v}_0\left(\frac{\xi}{2}+y\right)\right|^2dyd\xi\\&\lesssim\int_{\mathbb{R}^2}\frac{1}{|\xi_1-\xi_2||\xi_1+\xi_2|}|\widehat{u}_0(\xi_1)|^2|\widehat{v}_0(\xi_2)|^2d\xi_1d\xi_2\\&\lesssim\lambda^{-1}\mu^{-1}\int_{\mathbb{R}^2}|\widehat{u}_0(\xi_1)|^2|\widehat{v}_0(\xi_2)|^2d\xi_1d\xi_2\\&\lesssim\lambda^{-1}\mu^{-1}\|u_0\|_2^2\|v_0\|_2^2,\end{aligned} \tag{3.11}$$

where in the last inequality, we have applied $dist(I_1,I_2)\ge\lambda$ and $dist(I_1,-I_2)\ge\mu$. □

LEMMA 3.3 (L^4 Estimates). *Let $I\subset[0,+\infty)$ or $(-\infty,0]$ with $|I|<\infty$. For any $\theta\in(0,1)$, $\beta>0$, we have*

$$\|u_I\|^2_{L^4_{x,t\in[0,T]}}\lesssim(T^{1/4}+T^{(1-\theta)/4}|I|^{2\beta+(1-\theta)/2})\|u\|^2_{X^{-1/8}_{4,A}(I)}. \tag{3.12}$$

In particular, if $1\lesssim|I|<\infty$, $0<T<1$, then for any $0<\varepsilon\ll 1$, $4\le q\le\infty$

$$\|u_I\|_{L^4_{x,t\in[0,T]}}\lesssim T^{\varepsilon/4}|I|^{1/4-1/q+\varepsilon}\max_{\lambda\in I}\langle\lambda\rangle^{-3/8}\|u\|_{X^{1/4}_{q,A}(I)}. \tag{3.13}$$

Proof. Without loss of generality, we assume $I\subset[0,+\infty)$.

$$\begin{aligned}\|u_I\|^2_{L^4([0,T]\times\mathbb{R})} &= \|(u_I)^2\|_{L^2([0,T]\times\mathbb{R})}\\ &= \Big\|\sum_{m,n\in I\cap\mathbb{Z}} u_m u_n\Big\|_{L^2([0,T]\times\mathbb{R})}\\ &\le \sum_{k\in\mathbb{N}}\Big\|\sum_{m-n\sim 2^k} u_m u_n\Big\|_{L^2([0,T]\times\mathbb{R})}.\end{aligned}$$

Case $k=0$, i.e. $m\approx n$:

$$\begin{aligned}&\Big\|\sum_{n\in I\cap\mathbb{Z}} u_n^2\Big\|_{L^2([0,T]\times\mathbb{R})} \le \Big(\sum_{n\in I\cap\mathbb{Z}} \|u_n^2\|^2_{L^2([0,T]\times\mathbb{R})}\Big)^{1/2}\\ &\le \Big(\sum_{n\in I\cap\mathbb{Z}}\|u_n\|^4_{L^4([0,T]\times\mathbb{R})}\Big)^{1/2} \le T^{1/4}\Big(\sum_{n\in I\cap\mathbb{Z}}\|u_n\|^4_{L^8_{t\in[0,T]}L^4_x}\Big)^{1/2}\\ &\lesssim T^{1/4}\Big(\sum_{n\in I\cap\mathbb{Z}}\Big(\langle n\rangle^{-1/8}\|u_n\|_{U^8_A}\Big)^4\Big)^{1/2} \lesssim T^{1/4}\|u\|^2_{X^{-1/8}_{4,A}},\end{aligned}$$

where the first step is by the orthogonality in L^2 and the last step follows from the Strichartz estimate.

Case $k>0$: Notice that k is summed for $\ln|I|$ times, we have $\sum_{k\in\mathbb{N}}\lesssim\ln|I|$. We split the other sum as follows

$$\sum_{m-n\sim 2^k} u_m u_n = \sum_{n\in I\cap\mathbb{Z}}\ \sum_{\substack{m\in I\cap\mathbb{Z},\\ m-n\sim 2^k}} u_m u_n = \sum_{j\in\mathbb{N}+}\ \sum_{\substack{n\in I\cap\mathbb{Z},\\ n\sim j2^k}}\ \sum_{\substack{m\in I\cap\mathbb{Z},\\ m-n\sim 2^k}} u_m u_n,$$

where j is chosen such that $j2^k,(j+1)2^k\in I$. Hence for u_n with $n\sim j2^k$ and u_m with $m-n\sim 2^k$, we have that the frequency of the function $u_m u_n$ will be close to $(2j+1)2^k$, which implies by orthogonality that

$$\begin{aligned}&\sum_{k\in\mathbb{N}}\Big\|\sum_{m-n\sim 2^k} u_m u_n\Big\|_{L^2([0,T]\times\mathbb{R})}\\ &=\sum_{k\in\mathbb{N}}\Big\|\sum_{j\in\mathbb{N}+}\ \sum_{\substack{n\in I\cap\mathbb{Z},\\ n\sim j2^k}}\ \sum_{\substack{m\in I\cap\mathbb{Z},\\ m-n\sim 2^k}} u_m u_n\Big\|_{L^2([0,T]\times\mathbb{R})}\\ &\lesssim\sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}+}\Big\|\sum_{\substack{n\in I\cap\mathbb{Z},\\ n\sim j2^k}}\ \sum_{\substack{m\in I\cap\mathbb{Z},\\ m\sim(j+1)2^k}} u_m u_n\Big\|^2_{L^2([0,T]\times\mathbb{R})}\Big)^{1/2}.\end{aligned}\tag{3.14}$$

Denote $u_{j,k}:=\sum_{\substack{n\in I,\\ n\sim j2^k}} u_n$, from proposition 2.2 we can write as a sum $u_{j,k}=u_{1,j,k}+u_{2,j,k}$ with the estimate

$$\frac{1}{|I|^\beta}\|u_{1,j,k}\|_{U^2_A}+e^{\epsilon|I|^\beta}\|u_{2,j,k}\|_{U^8_A}\lesssim\|u_{j,k}\|_{V^2_A}.\tag{3.15}$$

Then the estimate (3.14) will be continued by four terms. For the term containing $u_{1,j,k}$ and $u_{1,j+1,k}$, which will be denoted as I_1.

$$\begin{aligned} I_1 &\lesssim \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} \|u_{1,j,k}u_{1,j+1,k}\|_{L^2}^{2\theta} \|u_{1,j,k}u_{1,j+1,k}\|_{L^2([0,T]\times\mathbb{R})}^{2(1-\theta)} \Big)^{1/2} \\ &\lesssim \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} \|u_{1,j,k}u_{1,j+1,k}\|_{L^2}^{2\theta} \|u_{1,j,k}\|_{L^4([0,T]\times\mathbb{R})}^{2(1-\theta)} \|u_{1,j+1,k}\|_{L^4([0,T]\times\mathbb{R})}^{2(1-\theta)} \Big)^{1/2} \\ &\lesssim T^{(1-\theta)/4} \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} \|u_{1,j,k}u_{1,j+1,k}\|_{L^2}^{2\theta} \|u_{1,j,k}\|_{L^8_{t\in[0,T]}L^4_x}^{2(1-\theta)} \|u_{1,j+1,k}\|_{L^8_{t\in[0,T]}L^4_x}^{2(1-\theta)} \Big)^{1/2}. \end{aligned} \tag{3.16}$$

Since $|m-n|\sim 2^k$, $|m+n|\sim(2j+1)2^k\gtrsim\sqrt{j(j+1)}2^k$, we have the bilinear estimates

$$\|u_{1,j,k}u_{1,j+1,k}\|_{L^2} \lesssim (j(j+1))^{-1/4}2^{-k}\|u_{1,j,k}\|_{U_A^2}\|u_{1,j+1,k}\|_{U_A^2}. \tag{3.17}$$

Combining with Strichartz estimate, (3.16) is dominated by

$$\begin{aligned} &\lesssim T^{(1-\theta)/4} \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} (j(j+1))^{-\theta/2} 2^{-2k\theta} \|u_{1,j,k}\|_{U_A^2}^{2\theta} \|u_{1,j+1,k}\|_{U_A^2}^{2\theta} \\ &\qquad (j2^k)^{-(1-\theta)/4} \|u_{1,j,k}\|_{U_A^8}^{2(1-\theta)} ((j+1)2^k)^{-(1-\theta)/4} \|u_{1,j+1,k}\|_{U_A^8}^{2(1-\theta)} \Big)^{1/2} \\ &\lesssim T^{(1-\theta)/4} \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} (j2^k)^{-(1+\theta)/4} ((j+1)2^k)^{-(1+\theta)/4} 2^{-k\theta} \|u_{1,j,k}\|_{U_A^2}^2 \|u_{1,j+1,k}\|_{U_A^2}^2 \Big)^{1/2}. \end{aligned}$$

By applying (3.15) and the orthogonality in V^2, it follows that

$$\begin{aligned} &\lesssim T^{(1-\theta)/4}|I|^{2\beta} \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} (j2^k)^{-1/4}((j+1)2^k)^{-1/4}2^{-k\theta} \|u_{1,j,k}\|_{V_A^2}^2 \|u_{1,j+1,k}\|_{V_A^2}^2 \Big)^{1/2} \\ &\lesssim T^{(1-\theta)/4}|I|^{2\beta} \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} (j2^k)^{-1/4}((j+1)2^k)^{-1/4}2^{-k\theta} \\ &\qquad \times \Big(\sum_{\substack{n\in I,\\ n\sim j2^k}} \|u_n\|_{V_A^2}^2 \Big) \Big(\sum_{\substack{m\in I,\\ m\sim (j+1)2^k}} \|u_m\|_{V_A^2}^2 \Big) \Big)^{1/2} \\ &\lesssim T^{(1-\theta)/4}|I|^{2\beta} \sum_{k\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}^+} (j2^k)^{-1/4}((j+1)2^k)^{-1/4}2^{-k\theta}2^k \\ &\qquad \times \Big(\sum_{\substack{n\in I,\\ n\sim j2^k}} \|u_n\|_{V_A^2}^4 \Big)^{1/2} \Big(\sum_{\substack{m\in I,\\ m\sim (j+1)2^k}} \|u_m\|_{V_A^2}^4 \Big)^{1/2} \Big)^{1/2} \\ &\lesssim T^{(1-\theta)/4}|I|^{2\beta+(1-\theta)/2}\ln|I|\|u\|_{X_{4,A}^{-1/8}}^2 \\ &\lesssim T^{(1-\theta)/4}|I|^{2\beta+(1-\theta)/2}\|u\|_{X_{4,A}^{-1/8}}^2, \end{aligned} \tag{3.18}$$

where the last inequality is obtained by using $2^{k(1-\theta)/2}\lesssim|I|^{(1-\theta)/2}$ and Hölder's inequality. For the rest three terms we will do in a uniform way. We take the term containing $u_{2,j,k}$ and $u_{2,j+1,k}$ for example, and denote it as I_2.

$$\begin{aligned}
I_2 &\lesssim \sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}^+}\|u_{2,j,k}u_{2,j+1,k}\|^2_{L^2([0,T]\times\mathbb{R})}\Big)^{1/2}\\
&\lesssim T^{1/4}\sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}^+}\|u_{2,j,k}\|^2_{L^8_{t\in[0,T]}L^4_x}\|u_{2,j+1,k}\|^2_{L^8_{t\in[0,T]}L^4_x}\Big)^{1/2}\\
&\lesssim T^{1/4}\sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}^+}(j2^k)^{-1/4}\|u_{2,j,k}\|^2_{U^8_A}((j+1)2^k)^{-1/4}\|u_{2,j+1,k}\|^2_{U^8_A}\Big)^{1/2}. \qquad (3.19)
\end{aligned}$$

By applying (3.15) and the orthogonality in V^2 again, it follows from (3.19) that

$$\begin{aligned}
&\lesssim T^{1/4}e^{-2\epsilon|I|^\beta}\sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}^+}(j2^k)^{-1/4}\|u_{j,k}\|^2_{V^2_A}((j+1)2^k)^{-1/4}2^{-k\theta}\|u_{j+1,k}\|^2_{V^2_A}\Big)^{1/2}\\
&\lesssim T^{1/4}e^{-2\epsilon|I|^\beta}\sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}^+}(j2^k)^{-1/4}\Big(\sum_{\substack{n\in I,\\ n\sim j2^k}}\|u_n\|^2_{V^2_A}\Big)((j+1)2^k)^{-1/4}\Big(\sum_{\substack{m\in I,\\ m\sim (j+1)2^k}}\|u_m\|^2_{V^2_A}\Big)\Big)^{1/2}\\
&\lesssim T^{1/4}e^{-2\epsilon|I|^\beta}\sum_{k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}^+}(j2^k)^{-1/4}((j+1)2^k)^{-1/4}2^k\Big(\sum_{\substack{n\in I,\\ n\sim j2^k}}\|u_n\|^4_{V^2_A}\Big)^{1/2}\\
&\qquad\times\Big(\sum_{\substack{m\in I,\\ m\sim (j+1)2^k}}\|u_m\|^4_{V^2_A}\Big)^{1/2}\Big)^{1/2}\\
&\lesssim T^{1/4}e^{-2\epsilon|I|^\beta}|I|^{1/2}\ln|I|\|u\|^2_{X^{-1/8}_{4,A}}\\
&\lesssim T^{1/4}\|u\|^2_{X^{-1/8}_{4,A}}.
\end{aligned}$$

Thus we complete the proof of (3.12). In particular, for $1\lesssim|I|<\infty$ and $0<T<1$, taking β and $1-\theta$ sufficiently small, we have

$$\|u_I\|_{L^4_{x,t\in[0,T]}}\lesssim T^{\varepsilon/4}|I|^{\varepsilon}\|u\|_{X^{-1/8}_{4,A}(I)}. \qquad (3.20)$$

In the end we can obtain (3.13) by Hölder's inequality. ∎

LEMMA 3.4. *Let $I\subset\mathbb{R}$ with $1\lesssim|I|<\infty$, $2\le q\le\infty$, we have*

$$\|u_I\|_{L^\infty_tL^2_x\cap V^2_A}\lesssim|I|^{1/2-1/q}\max_{\lambda\in I}\langle\lambda\rangle^{-1/4}\|u\|_{X^{1/4}_{q,A}(I)}. \qquad (3.21)$$

Proof. Using $V^2_A\subset L^\infty_tL^2_x$, the orthogonality in V^2 and Hölder's inequality one by one, we have

$$\begin{aligned}
\|u_I\|_{L^\infty_tL^2_x\cap V^2_A}&\lesssim\Big(\sum_{\lambda\in I}\|u_\lambda\|^2_{V^2_A}\Big)^{1/2}\lesssim\max_{\lambda\in I}\langle\lambda\rangle^{-1/4}\Big(\sum_{\lambda\in I}(\langle\lambda\rangle^{1/4}\|u_\lambda\|)^2_{V^2_A}\Big)^{1/2}\\
&\lesssim|I|^{1/2-1/q}\max_{\lambda\in I}\langle\lambda\rangle^{-1/4}\|u\|_{X^{1/4}_{q,A}(I)}. \qquad (3.22)
\end{aligned}$$

∎

4. Trilinear estimates

At first, we apply the duality to the norm calculation (Proposition 2.5) to the inhomogeneous part of the solution of mKdV in $X^s_{q,A}$. It is known that (1.1) is equivalent to the following integral equation:

$$u(x,t) = e^{-t\partial_x^3} u_0 - \mathcal{A}((u^3)_x), \tag{4.1}$$

where

$$e^{-t\partial_x^3} = \mathscr{F}^{-1} e^{it\xi^3} \mathscr{F}, \quad \mathcal{A}(f) = \int_0^t e^{-(t-\tau)\partial_x^3} f(\tau) d\tau.$$

By Propositions 2.4 and 2.5, we see that, for supp $v \subset \mathbb{R}\times[0,T]$, $1 \le q < \infty$,

$$\begin{aligned}
\|\mathcal{A}(f)\|_{X^{1/4}_{q,A}} &= \|e^{t\partial_x^3}\mathcal{A}(f)\|_{X^{1/4}_q} \\
&= \sup\left\{ \left| B\left(\int_0^t e^{\tau\partial_x^3} f(\tau) d\tau, v\right)\right| : \ \|v\|_{Y^{-1/4}_{q'}} \le 1 \right\} \\
&\le \sup_{\|v\|_{Y^{-1/4}_{q'}} \le 1} \left| \int_{[0,T]} \langle e^{t\partial_x^3} f(t), \ v(t)\rangle dt \right| \\
&\le \sup_{\|v\|_{Y^{-1/4}_{q'}} \le 1} \left| \int_{[0,T]} \langle f(t), \ e^{-t\partial_x^3} v(t)\rangle dt \right| \\
&\le \sup_{\|v\|_{Y^{-1/4}_{q',A}} \le 1} \left| \int_{[0,T]} \langle f(t), \ v(t)\rangle dt \right|.
\end{aligned} \tag{4.2}$$

For $q = \infty$, we have

$$\begin{aligned}
\|\mathcal{A}(f)\|_{X^{1/4}_{\infty,A}} &= \|e^{t\partial_x^3}\mathcal{A}(f)\|_{X^{1/4}_\infty} \\
&= \sup_{\lambda\in\mathbb{Z}} \langle\lambda\rangle^{1/4} \left\| \square_\lambda \int_0^t e^{\tau\partial_x^3} f(\tau) d\tau \right\|_{U^2} \\
&\le \sup_{\lambda\in\mathbb{Z}} \langle\lambda\rangle^{1/4} \sup_{\|v^{(\lambda)}\|_{V^2}\le 1} \left| \int_{[0,T]} \langle \square_\lambda e^{t\partial_x^3} f(t), \ v^{(\lambda)}(t)\rangle dt \right| \\
&\le \sup_{\lambda\in\mathbb{Z}} \langle\lambda\rangle^{1/4} \sup_{\|v^{(\lambda)}\|_{V^2}\le 1} \left| \int_{[0,T]} \langle f(t), \ \square_\lambda e^{-t\partial_x^3} v^{(\lambda)}(t)\rangle dt \right| \\
&\le \sup_{\lambda\in\mathbb{Z}} \langle\lambda\rangle^{1/4} \sup_{\|v^{(\lambda)}\|_{V^2_A}\le 1} \left| \int_{[0,T]} \langle f(t), \ \square_\lambda v^{(\lambda)}(t)\rangle dt \right|.
\end{aligned} \tag{4.3}$$

To prove Theorem 1.1, we need to control the second term of the integral Equation (4.1) in $X^{1/4}_{q,A}(2 \le q \le \infty)$. More precisely, we want to prove the following lemma.

LEMMA 4.1. *For $2 \le q \le \infty$, there exists $\varepsilon > 0$ such that*

$$\left\| \int_0^t e^{-(t-\tau)\partial_x^3} (u^3)_x(\tau)\, d\tau \right\|_{X^{1/4}_{q,A}} \lesssim T^\varepsilon \|u\|^3_{X^{1/4}_{q,A}}. \tag{4.4}$$

Proof. When $2 \le q < \infty$, in view of (4.2), it suffices to show that

$$\left| \int_{\mathbb{R}\times[0,T]} \overline{v} u^2 \partial_x u \, dx dt \right| \lesssim T^{\varepsilon} \|u\|^3_{X^{1/4}_{q,A}} \|v\|_{Y^{-1/4}_{q',A}}. \tag{4.5}$$

We perform a uniform decomposition with u, v in the left-hand side of (4.5), it suffices to prove that

$$\left| \sum_{\lambda_0,\dots,\lambda_3} \langle \lambda_0 \rangle^{1/4} \int_{[0,T]\times\mathbb{R}} \overline{v}_{\lambda_0} u_{\lambda_1} u_{\lambda_2} \partial_x u_{\lambda_3} \, dx dt \right| \lesssim T^{\varepsilon} \|u\|^3_{X^{1/4}_{q,A}} \|v\|_{Y^0_{q',A}}. \tag{4.6}$$

When $q = \infty$, in view of (4.3), it suffices to show that, for any fixed $\lambda \in \mathbb{Z}$,

$$\left| \sum_{\lambda_1,\lambda_2,\lambda_3} \langle \lambda \rangle^{1/4} \int_{[0,T]\times\mathbb{R}} \overline{\square_\lambda v^{(\lambda)}} u_{\lambda_1} u_{\lambda_2} \partial_x u_{\lambda_3} \, dx dt \right| \lesssim T^{\varepsilon} \|u\|^3_{X^{1/4}_{\infty,A}} \|v^{(\lambda)}\|_{V^2_A}. \tag{4.7}$$

4.1. $q = \infty$, Proof of (4.7). For convenience, denote λ as λ_0, $\square_\lambda v^{(\lambda)} =: v_\lambda = v_{\lambda_0}$. In order to keep the left-hand side of (4.7) nonzero, we have the frequency constraint condition (FCC)

$$\lambda_1 + \lambda_2 + \lambda_3 \approx \lambda_0 \tag{4.8}$$

and dispersion modulation constraint condition (DMCC)

$$\max_{0\le k\le 3} |\xi_k^3 - \tau_k| \gtrsim \left| (\xi_0^3 - \tau_0) - \sum_{1\le k\le 3} (\xi_k^3 - \tau_k) \right| \gtrsim |(\xi_0 - \xi_1)(\xi_0 - \xi_2)(\xi_0 - \xi_3)|. \tag{4.9}$$

It suffices to consider the cases that λ_0 is maximal or second-most maximal number in $\lambda_0, \dots, \lambda_3$ (In the opposite case, one can replace $\lambda_0, \dots, \lambda_3$ with $-\lambda_0, \dots, -\lambda_3$).

Step 1. We assume that $\lambda_0 = \max_{0\le k\le 3} \lambda_k$. From the frequency constraint condition (FCC) $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$, we know that the non-trivial case is that $\lambda_0 \gg 0$ (The case $\lambda_0 \ll 0$ never happens due to the condition (FCC). In addition, the case $|\lambda_0| \lesssim 1$, which leads to $\max_{0\le k\le 3} |\lambda_k| \lesssim 1$, implies that the summation in (4.7) has at most finite terms). Furthermore, in view of $\lambda_0 = \max_{0\le k\le 3} \lambda_k$, $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$, and $\lambda_0 \gg 0$, we see that $\lambda_0 \approx \max_{0\le k\le 3} |\lambda_k| \gg 0$. For convenience, we can take

$$\lambda_0 = \max_{0\le k\le 3} |\lambda_k| \gg 0. \tag{4.10}$$

By symmetry, we can assume $\lambda_1 \ge \lambda_2$. Then $\lambda_0, \dots, \lambda_3$ have the following three orders:

$$\begin{aligned} &\text{Order 1}: \quad \lambda_0 \ge \lambda_3 \ge \lambda_1 \ge \lambda_2; \\ &\text{Order 2}: \quad \lambda_0 \ge \lambda_1 \ge \lambda_3 \ge \lambda_2; \\ &\text{Order 3}: \quad \lambda_0 \ge \lambda_1 \ge \lambda_2 \ge \lambda_3. \end{aligned}$$

We just take Order 1 for example because the other two orders are similar and even more easier (notice that the derivative is located in u_{λ_3}).

Order 1: $\lambda_0 \geq \lambda_3 \geq \lambda_1 \geq \lambda_2$. For short, considering the higher and lower frequencies of λ_k, we use the following notations:

$$\begin{cases} \lambda_k \in h \Leftrightarrow \lambda_k \in [3\lambda_0/4, \lambda_0]; \\ \lambda_k \in h_- \Leftrightarrow \lambda_k \in [-\lambda_0, -3\lambda_0/4]; \\ \lambda_k \in l \Leftrightarrow \lambda_k \in [0, 3\lambda_0/4]; \\ \lambda_k \in l_- \Leftrightarrow \lambda_k \in [-3\lambda_0/4, 0]. \end{cases}$$

We denote by $(\lambda_k) \in hhhh_-$ that all $\lambda_0, \ldots, \lambda_3$ satisfy the conditions (4.8), (4.10) and

$$\lambda_3, \lambda_1 \in h, \quad \lambda_2 \in h_-.$$

According to such kind of notations, we divide the proof into several cases as in Table 4.1.

Case	$\lambda_3 \in$	$\lambda_1 \in$	$\lambda_2 \in$
$hhhh_-$	$[3\lambda_0/4, \lambda_0]$	$[3\lambda_0/4, \lambda_0]$	$[-\lambda_0, -3\lambda_0/4]$
$hhll$	$[3\lambda_0/4, \lambda_0]$	$[0, 3\lambda_0/4]$	$[0, 3\lambda_0/4]$
$hhll_-$	$[3\lambda_0/4, \lambda_0]$	$[0, 3\lambda_0/4]$	$[-3\lambda_0/4, 0]$
hhl_-l_-	$[3\lambda_0/4, \lambda_0]$	$[-3\lambda_0/4, 0]$	$[-3\lambda_0/4, 0]$
$hlll$	$[0, 3\lambda_0/4]$	$[0, 3\lambda_0/4]$	$[0, 3\lambda_0/4]$
hll_-l_-	$[0, 3\lambda_0/4]$	$[-3\lambda_0/4, 0]$	$[-3\lambda_0/4, 0]$

TABLE 4.1. $\lambda_0 = \max_{0 \leq k \leq 3} |\lambda_k|$; $\quad \lambda_0 \geq \lambda_3 \geq \lambda_1 \geq \lambda_2$

Case 1 (Case $hhhh_-$): $\lambda_3 \in h$ and $\lambda_1 \in h$. In consideration of (FCC), we easily see that $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ satisfy the following frequency constraint condition:

$$\lambda_0 = \lambda_1 + \lambda_2 + \lambda_3 + l, \quad |l| \leq 10. \tag{4.11}$$

We know that this case implies that $\lambda_2 \in [-\lambda_0, -\lambda_0/2 - l]$, i.e. $\lambda_2 \in h_-$. We do dyadic decomposition for u_{λ_1}, u_{λ_2} and u_{λ_3}, and keep using uniform decomposition for v_{λ_0}. Let us denote $I_0 = [0,1)$, $I_j = [2^{j-1}, 2^j)$, $j \geq 1$. We decompose $\lambda_1, \lambda_2, \lambda_3$ by:

$$\lambda_k \in \lambda_0 - [0, \lambda_0/4] = \bigcup_{j_k \geq 0} \lambda_0 - I_{j_k}, \ k = 1, 3; \quad \lambda_2 \in -\lambda_0 + [0, \lambda_0/2 - l] = \bigcup_{j_2 \geq 0} -\lambda_0 + I_{j_2},$$

where $[0, \lambda_0/4] = \bigcup_{j \geq 0} I_j$ means that one can use some dyadic intervals $I_j \cap [0, \lambda_0/4]$ to cover the interval $[0, \lambda_0/4]$. We will often use such kind of notations below. From $\lambda_3 \geq \lambda_1$ we know that $j_3 \leq j_1$. In view of condition (FCC), we see that $j_2 \approx j_1$. It follows that

$$0 \leq j_3 \leq j_1 \approx j_2 \leq \log_2 \lambda_0.$$

In the following discussion, we shall omit the condition $j_k \in [0, \log_2 \lambda_0]$, $k = 1,2,3$, for convenience, but it is always satisfied in Step 1. We denote the left-hand side of (4.7) as $\mathscr{L}_{hhhh_-}(u,v)$, and divide it into three parts:

$$\begin{aligned} \mathscr{L}_{hhhh_-}(u,v) &:= \sum_{j_3 \leq j_1 \approx j_2} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} |\overline{v_{\lambda_0}} u_{\lambda_0 - I_{j_1}} u_{-\lambda_0 + I_{j_2}} \partial_x u_{\lambda_0 - I_{j_3}}| dx dt \\ &= \Bigg(\sum_{j_3 \leq j_1 \approx j_2 \lesssim 1} + \sum_{j_3 \lesssim 1 \ll j_1 \approx j_2} + \sum_{1 \ll j_3 \leq j_1 \approx j_2} \Bigg) \end{aligned}$$

$$\begin{aligned}&\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0-I_{j_3}}|dxdt\\ =:&\mathscr{L}^l_{hhhh\_}(u,v)+\mathscr{L}^m_{hhhh\_}(u,v)+\mathscr{L}^h_{hhhh\_}(u,v).\end{aligned}$$

It is easy to see that in $\mathscr{L}^l_{hhhh\_}(u,v)$, $\lambda_0\approx\lambda_3\approx\lambda_1\approx-\lambda_2$ holds. Therefore, by Hölder's inequality and Strichartz estimate, we have

$$\begin{aligned}\mathscr{L}^l_{hhhh\_}(u,v)&\lesssim\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^4_{x,t}}\|u_{\lambda_0}\|^2_{L^4_{x,t}}\|u_{-\lambda_0}\|_{L^4_{x,t}}\\&\lesssim T^{1/2}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^8_tL^4_x}\|u_{\lambda_0}\|^2_{L^8_tL^4_x}\|u_{-\lambda_0}\|_{L^8_tL^4_x}\\&\lesssim T^{1/2}\langle\lambda_0\rangle^{3/4}\|\overline{v_{\lambda_0}}\|_{U^8_A}\|u_{\lambda_0}\|^2_{U^8_A}\|u_{-\lambda_0}\|_{U^8_A}\\&\lesssim T^{1/2}\langle\lambda_0\rangle^{3/4}\|\overline{v_{\lambda_0}}\|_{V^2_A}\|u_{\lambda_0}\|^2_{U^2_A}\|u_{-\lambda_0}\|_{U^2_A}\\&\lesssim T^{1/2}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.\end{aligned}$$

In $\mathscr{L}^m_{hhhh\_}(u,v)$, we see that the frequencies of v_{λ_0} and $u_{\lambda_0-I_{j_3}}$ are localized near λ_0, which are far away from the frequency of $u_{\lambda_0-I_{j_1}}$ and the reciprocal frequency of $u_{-\lambda_0+I_{j_2}}$. Thus we can use bilinear estimate (3.7), Lemma 3.4, and Hölder's inequality to obtain that

$$\begin{aligned}\mathscr{L}^m_{hhhh\_}(u,v)\lesssim&\sum_{j_3\lesssim1\ll j_1\approx j_2}\langle\lambda_0\rangle^{1/4}\|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}\|_{L^2_{x,t}}\|u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\\ \lesssim&T^{\varepsilon/2}\sum_{j_3\lesssim1\ll j_1\approx j_2}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_1})^{-1/2+\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u_{\lambda_0-I_{j_1}}\|_{V^2_A}\\&\times\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_2})^{-1/2+\varepsilon}\|u_{-\lambda_0+I_{j_2}}\|_{V^2_A}\|u_{\lambda_0-I_{j_3}}\|_{V^2_A}\\ \lesssim&T^{\varepsilon/2}\sum_{j_3\lesssim1\ll j_1\approx j_2}\langle\lambda_0\rangle^{1/4+2\varepsilon}(2^{j_1})^{-1/2+\varepsilon}(2^{j_2})^{-1/2+\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\\&\times(2^{j_1})^{1/2}(2^{j_2})^{1/2}(2^{j_3})^{1/2}\langle\lambda_0\rangle^{-3/4}\|u\|^3_{X^{1/4}_{\infty,A}}\\ \lesssim&T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/2+4\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\ \lesssim&T^{\varepsilon/2}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}},\end{aligned}$$

where the last inequality is obtained by taking $\varepsilon\le1/8$.

Now we estimate $\mathscr{L}^h_{hhhh\_}(u,v)$. In view of (DMCC) (4.9), we have the highest dispersion modulation satisfying

$$\max_{0\le k\le3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle\cdot2^{j_1}\cdot2^{j_3}.$$

If v_{λ_0} has the highest dispersion modulation, we divide $\mathscr{L}^h_{hhhh\_}(u,v)$ into two parts.

$$\begin{aligned}&\mathscr{L}^{h1}_{hhhh\_}(u,v)+\mathscr{L}^{h2}_{hhhh\_}(u,v)\\ :=&\Big(\sum_{1\ll j_3\ll j_1\approx j_2}+\sum_{1\ll j_3\approx j_1\approx j_2}\Big)\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0-I_{j_3}}|dxdt.\end{aligned}\tag{4.12}$$

For $\mathscr{L}^{h1}_{hhhh_-}(u,v)$, we can use the bilinear estimate due to $j_2 \gg j_3$. By Hölder's inequality we have

$$\mathscr{L}^{h1}_{hhhh_-}(u,v) \lesssim \sum_{1\ll j_3 \ll j_1 \approx j_2} \langle\lambda_0\rangle^{1/4} \|\overline{v_{\lambda_0}}\|_{L^2_t L^\infty_x} \|u_{\lambda_0 - I_{j_1}}\|_{L^\infty_t L^2_x} \|u_{-\lambda_0+I_{j_2}} \partial_x u_{\lambda_0 - I_{j_3}}\|_{L^2_{x,t}}.$$

Using $\|v_{\lambda_0}\|_{L^\infty_x} \lesssim \|v_{\lambda_0}\|_{L^2_x}$, $V^2_A \subset L^\infty_t L^2_x$, the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4, we have

$$\begin{aligned}
\mathscr{L}^{h1}_{hhhh_-}(u,v) \lesssim & \sum_{1\ll j_3 \ll j_1 \approx j_2} \langle\lambda_0\rangle^{5/4} \langle\lambda_0\rangle^{-1/2} (2^{j_1})^{-1/2} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V^2_A} \|u_{\lambda_0-I_{j_1}}\|_{V^2_A} \\
& \times T^{\varepsilon/4} \langle\lambda_0\rangle^{-1/2+\varepsilon} (2^{j_2})^{-1/2+\varepsilon} \|u_{-\lambda_0+I_{j_2}}\|_{V^2_A} \|u_{\lambda_0-I_{j_3}}\|_{V^2_A} \\
\lesssim & T^{\varepsilon/4} \sum_{1\ll j_3 \ll j_1 \approx j_2} \langle\lambda_0\rangle^{1/4+\varepsilon} (2^{j_1})^{-1/2} (2^{j_2})^{-1/2+\varepsilon} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V^2_A} \\
& \times (2^{j_1})^{1/2} (2^{j_2})^{1/2} (2^{j_3})^{1/2} \langle\lambda_0\rangle^{-3/4} \|u\|^3_{X^{1/4}_{\infty,A}} \\
\lesssim & T^{\varepsilon/4} \langle\lambda_0\rangle^{-1/2+\varepsilon} \left(\sum_{j_2} (2^{j_2})^{\varepsilon} \cdot j_2 \right) \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}}. \qquad (4.13)
\end{aligned}$$

Noticing that $2^{j_2} \lesssim \langle\lambda_0\rangle$ and $j_2 \lesssim (2^{j_2})^\varepsilon$, we can take $\varepsilon \le 1/6$ such that

$$\begin{aligned}
\mathscr{L}^{h1}_{hhhh_-}(u,v) &\lesssim T^{\varepsilon/4} \langle\lambda_0\rangle^{-1/2+3\varepsilon} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/4} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}$$

For $\mathscr{L}^{h2}_{hhhh_-}(u,v)$, by Hölder's inequality, $\|v_{\lambda_0}\|_{L^\infty_x} \lesssim \|v_{\lambda_0}\|_{L^2_x}$, $V^2_A \subset L^\infty_t L^2_x$, the dispersion modulation decay (2.4), the L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}
\mathscr{L}^{h2}_{hhhh_-}(u,v) \lesssim & \sum_{1\ll j_3 \approx j_1 \approx j_2} \langle\lambda_0\rangle^{5/4} \|\overline{v_{\lambda_0}}\|_{L^2_t L^\infty_x} \|u_{\lambda_0-I_{j_1}}\|_{L^\infty_t L^2_x} \|u_{-\lambda_0+I_{j_2}}\|_{L^4_{x,t}} \|u_{\lambda_0-I_{j_3}}\|_{L^4_{x,t}} \\
\lesssim & \sum_{1\ll j_3 \approx j_1 \approx j_2} \langle\lambda_0\rangle^{5/4} \langle\lambda_0\rangle^{-1/2} (2^{j_1})^{-1/2} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V^2_A} \|u_{\lambda_0-I_{j_1}}\|_{V^2_A} \\
& \times T^{\varepsilon/2} \langle\lambda_0\rangle^{-3/4} (2^{j_2})^{1/4+\varepsilon} (2^{j_3})^{1/4+\varepsilon} \|u\|^2_{X^{1/4}_{\infty,A}} \\
\lesssim & T^{\varepsilon/2} \sum_{j_3 \approx j_2} \langle\lambda_0\rangle^{-1/4} (2^{j_2})^{1/4+\varepsilon} (2^{j_3})^{-1/4+\varepsilon} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} \\
\lesssim & T^{\varepsilon/2} \langle\lambda_0\rangle^{-1/4+2\varepsilon} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} \\
\lesssim & T^{\varepsilon/2} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}}, \qquad (4.14)
\end{aligned}$$

where the last inequality is obtained by taking $\varepsilon \le 1/8$.

If $u_{\lambda_0 - I_{j_1}}$ has the highest dispersion modulation, we just take $L^\infty_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0} and $u_{\lambda_0-I_{j_1}}$, respectively, then

$$\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}} \|u_{\lambda_0-I_{j_1}}\|_{L^2_{x,t}} \lesssim \|v_{\lambda_0}\|_{V^2_A} \langle\lambda_0\rangle^{-1/2} (2^{j_1})^{-1/2} (2^{j_3})^{-1/2} \|u_{\lambda_0-I_{j_1}}\|_{V^2_A}, \qquad (4.15)$$

where we use the fact $\|v_{\lambda_0}\|_{L^\infty_{x,t}} \lesssim \|v_{\lambda_0}\|_{L^\infty_t L^2_x} \lesssim \|v_{\lambda_0}\|_{V^2_A}$ and the dispersion modulation decay (2.4) to $u_{\lambda_0-I_{j_1}}$. Then this case reduces to the same estimate as that when v_{λ_0} has the highest dispersion modulation.

If $u_{-\lambda_0+I_{j_2}}$ has the highest dispersion modulation, we still divide $\mathscr{L}^h_{hhhh_-}(u,v)$ into two parts as (4.12). For $\mathscr{L}^{h1}_{hhhh_-}(u,v)$, we can take $L^\infty_{x,t}$, $L^2_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0}, $u_{-\lambda_0+I_{j_2}}$ and $u_{\lambda_0-I_{j_1}}\partial_x u_{\lambda_0-I_{j_3}}$, respectively. By Hölder's inequality, the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4, we have

$$\begin{aligned}\mathscr{L}^{h1}_{hhhh_-}(u,v) &\lesssim \sum_{1\ll j_3\ll j_1\approx j_2} \langle\lambda_0\rangle^{1/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{-\lambda_0+I_{j_2}}\|_{L^2_{x,t}}\|u_{\lambda_0-I_{j_1}}\partial_x u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\\ &\lesssim \sum_{1\ll j_3\ll j_1\approx j_2} \langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V^2_A}\|u_{-\lambda_0+I_{j_2}}\|_{V^2_A}\\ &\qquad \times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_1})^{-1/2+\varepsilon}\|u_{\lambda_0-I_{j_1}}\|_{V^2_A}\|u_{\lambda_0-I_{j_3}}\|_{V^2_A},\end{aligned}$$

which is the same as the right-hand side of the first inequality in (4.13) (noticing that $j_1\approx j_2$).

For $\mathscr{L}^{h2}_{hhhh_-}(u,v)$, we take $L^\infty_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$ and $L^4_{x,t}$ norms to v_{λ_0}, $u_{-\lambda_0+I_{j_2}}$, $u_{\lambda_0-I_{j_1}}$ and $u_{\lambda_0-I_{j_3}}$, respectively, then

$$\begin{aligned}\mathscr{L}^{h2}_{hhhh_-}(u,v) &\lesssim \sum_{1\ll j_3\approx j_1\approx j_2} \langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{-\lambda_0+I_{j_2}}\|_{L^2_{x,t}}\|u_{\lambda_0-I_{j_1}}\|_{L^4_{x,t}}\|u_{\lambda_0-I_{j_3}}\|_{L^4_{x,t}}\\ &\lesssim \sum_{1\ll j_3\approx j_1\approx j_2} \langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V^2_A}\|u_{-\lambda_0+I_{j_2}}\|_{V^2_A}\\ &\qquad \times T^{\varepsilon/2}\langle\lambda_0\rangle^{-3/4}(2^{j_1})^{1/4+\varepsilon}(2^{j_3})^{1/4+\varepsilon}\|u\|^2_{X^{1/4}_{\infty,A}},\end{aligned}$$

which is the same as the right-hand side of the second inequality in (4.14).

If $u_{\lambda_0-I_{j_3}}$ has the highest dispersion modulation, we don't need to divide $\mathscr{L}^h_{hhhh_-}(u,v)$. By Hölder's inequality, we obtain that

$$\begin{aligned}\mathscr{L}^{h}_{hhhh_-}(u,v) &\lesssim \sum_{1\ll j_3\le j_1\approx j_2} \langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\|u_{\lambda_0-I_{j_1}}\|_{L^4_{x,t}}\|u_{-\lambda_0+I_{j_2}}\|_{L^4_{x,t}}\\ &\lesssim \sum_{1\ll j_3\le j_1\approx j_2} \langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V^2_A}\|u_{\lambda_0-I_{j_3}}\|_{V^2_A}\\ &\qquad \times T^{\varepsilon/2}\langle\lambda_0\rangle^{-3/4}(2^{j_1})^{1/4+\varepsilon}(2^{j_2})^{1/4+\varepsilon}\|u\|^2_{X^{1/4}_{\infty,A}}\\ &\lesssim T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/4}\left(\sum_{j_1}(2^{j_1})^{2\varepsilon}\cdot j_1\right)\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\ &\lesssim T^{\varepsilon/2}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}},\end{aligned}$$

where the last inequality is obtained by using $2^{j_1}\lesssim\langle\lambda_0\rangle$, $j_1\lesssim(2^{j_1})^\varepsilon$, and taking $\varepsilon\le 1/12$.

Case 2 (Case *hhll* and Case *hhll*$_-$): $\lambda_3\in h$ and $\lambda_1\in l$. In view of (4.11), we see that $\lambda_2\in[-3\lambda_0/4-l,\lambda_0/4-l]$, i.e., $\lambda_2\in l$ or $\lambda_2\in l_-$. Then we divide Case 2 into two subcases.

Case *hhll*. We decompose $\lambda_1,\lambda_2,\lambda_3$ by:

$$\lambda_k\in[0,3\lambda_0/4]=\bigcup_{j_k\ge 0} I_{j_k},\ k=1,2;\quad \lambda_3\in[3\lambda_0/4,\lambda_0]=\bigcup_{j_3\ge 0}\lambda_0-I_{j_3},\quad j_1,j_2,j_3\le\log_2\lambda_0.$$

In view of the condition (FCC) $\lambda_1+\lambda_2+\lambda_3\approx\lambda_0$, we see that $2^{j_3}\approx 2^{j_1}+2^{j_2}$. Moreover, we can get $j_1\ge j_2$ from $\lambda_1\ge\lambda_2$. Therefore, we know that $j_3\approx j_1\ge j_2$. It means that we need to estimate

$$\mathscr{L}_{hhll}(u,v):=\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{I_{j_2}}\partial_x u_{\lambda_0-I_{j_3}}|dxdt.$$

In view of (DMCC) (4.9), we have the highest dispersion modulation

$$\max_{0\le k\le 3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^2\cdot 2^{j_3}.$$

If v_{λ_0} has the highest dispersion modulation, by Hölder's inequality we have

$$\mathscr{L}_{hhll}(u,v)\lesssim\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{I_{j_2}}\|_{L_{x,t}^4}\|u_{\lambda_0-I_{j_3}}\|_{L_{x,t}^4}.$$

Using $\|v_{\lambda_0}\|_{L_x^\infty}\lesssim\|v_{\lambda_0}\|_{L_x^2}$, $V_A^2\subset L_t^\infty L_x^2$, the dispersion modulation decay (2.4), the L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}\mathscr{L}_{hhll}(u,v)\lesssim&\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{I_{j_1}}\|_{V_A^2}\\&\times T^{\varepsilon/2}(2^{j_2})^{-1/8+\varepsilon}(2^{j_3})^{1/4+\varepsilon}\langle\lambda_0\rangle^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}}\\\lesssim&\, T^{\varepsilon/2}\sum_{0\le j_2\le j_1\le\log_2\lambda_0}\langle\lambda_0\rangle^{-1/8}(2^{j_1})^{\varepsilon}(2^{j_2})^{-1/8+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\\lesssim&\, T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/8+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\\lesssim&\, T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},\end{aligned}\tag{4.16}$$

where the last but one inequality is obtained by summarizing over j_2, j_1 and taking $\varepsilon<1/8$.

If $u_{I_{j_1}}$ has the highest dispersion modulation, we have

$$\begin{aligned}\mathscr{L}_{hhll}(u,v)\lesssim&\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_{x,t}^\infty}\|u_{I_{j_1}}\|_{L_{x,t}^2}\|u_{I_{j_2}}\|_{L_{x,t}^4}\|u_{\lambda_0-I_{j_3}}\|_{L_{x,t}^4}\\\lesssim&\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|u_{I_{j_1}}\|_{V_A^2}\\&\times T^{\varepsilon/2}(2^{j_2})^{-1/8+\varepsilon}(2^{j_3})^{1/4+\varepsilon}\langle\lambda_0\rangle^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}},\end{aligned}$$

which is the same as the right-hand side of the first inequality in (4.16).

If $u_{I_{j_2}}$ has the highest dispersion modulation, we take $L_{x,t}^\infty$, $L_{x,t}^2$, $L_{x,t}^4$ and $L_{x,t}^4$ norms to v_{λ_0}, $u_{I_{j_2}}$, $u_{I_{j_1}}$ and $u_{\lambda_0-I_{j_3}}$, respectively. Then applying the dispersion modulation decay (2.4) to $u_{I_{j_2}}$, we have

$$\begin{aligned}\mathscr{L}_{hhll}(u,v)\lesssim&\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_{x,t}^\infty}\|u_{I_{j_2}}\|_{L_{x,t}^2}\|u_{I_{j_1}}\|_{L_{x,t}^4}\|u_{\lambda_0-I_{j_3}}\|_{L_{x,t}^4}\\\lesssim&\sum_{0\le j_2\le j_1\approx j_3\le \log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|u_{I_{j_2}}\|_{V_A^2}\\&\times T^{\varepsilon/2}(2^{j_1})^{-1/8+\varepsilon}(2^{j_3})^{1/4+\varepsilon}\langle\lambda_0\rangle^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}}\end{aligned}$$

$$\lesssim T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/8}\sum_{0\le j_2\le j_1\le\log_2\lambda_0}(2^{j_1})^{-3/8+2\varepsilon}(2^{j_2})^{1/4}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.$$

Making the summation on j_2, j_1 in order, and taking $\varepsilon<1/16$, we can obtain the desired estimate.

If $u_{\lambda_0-I_{j_3}}$ has the highest dispersion modulation, by Hölder's inequality we have

$$\mathscr{L}_{hhll}(u,v)\lesssim\sum_{0\le j_2\le j_1\approx j_3\le\log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\|u_{I_{j_1}}\|_{L^4_{x,t}}\|u_{I_{j_2}}\|_{L^4_{x,t}}.$$

Using $\|v_{\lambda_0}\|_{L^\infty_x}\lesssim\|v_{\lambda_0}\|_{L^2_x}$, $V_A^2\subset L_t^\infty L_x^2$, the dispersion modulation decay (2.4), the L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}\mathscr{L}_{hhll}(u,v)\lesssim&\sum_{0\le j_2\le j_1\approx j_3\le\log_2\lambda_0}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|u_{\lambda_0-I_{j_3}}\|_{V_A^2}\\&\times T^{\varepsilon/2}(2^{j_1})^{-1/8+\varepsilon}(2^{j_2})^{-1/8+\varepsilon}\|u\|^2_{X^{1/4}_{\infty,A}}\\\lesssim&T^{\varepsilon/2}\|v_{\lambda_0}\|_{V_A^2}\sum_{0\le j_2\le j_1\le\log_2\lambda_0}(2^{j_1})^{-1/8+\varepsilon}(2^{j_2})^{-1/8+\varepsilon}\|u\|^3_{X^{1/4}_{\infty,A}}\\\lesssim&T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},\end{aligned}\tag{4.17}$$

where the last inequality is obtained by taking $\varepsilon<1/8$.

Case $hhll_-$. We decompose $\lambda_1,\lambda_2,\lambda_3$ by:

$$\lambda_1\in[0,3\lambda_0/4]=\bigcup_{j_1\ge0}I_{j_1};\quad\lambda_2\in[-3\lambda_0/4,0]=\bigcup_{j_2\ge0}-I_{j_2};\quad\lambda_3\in[3\lambda_0/4,\lambda_0]=\bigcup_{j_3\ge0}\lambda_0-I_{j_3}.$$

In view of the condition (FCC) $\lambda_1+\lambda_2+\lambda_3\approx\lambda_0$, we see that $2^{j_1}\approx2^{j_2}+2^{j_3}$. Thus, we know that $j_1\approx j_2\vee j_3$. If $j_1\approx j_3\ge j_2$ or $j_1\approx j_2\approx j_3$, it is the same as Case $hhll$ to get the conclusion. So we only need to consider $j_1\approx j_2\gg j_3$, which means that we need to estimate

$$\mathscr{L}_{hhll_-}(u,v):=\sum_{j_3\ll j_2\approx j_1}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-I_{j_2}}\partial_x u_{\lambda_0-I_{j_3}}|\,dxdt.$$

In view of (DMCC) (4.9), we have the highest dispersion modulation satisfying

$$\max_{0\le k\le3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^2\cdot2^{j_3}.$$

If v_{λ_0} has the highest dispersion modulation, we can easily see that $|\lambda_0-2^{j_3}+2^{j_2}|\gtrsim\langle\lambda_0\rangle$ and $|\lambda_0-2^{j_3}-2^{j_2}|\approx|\lambda_0-2^{j_1}|\gtrsim\langle\lambda_0\rangle$, then we shall use the bilinear estimate to $u_{-I_{j_2}}u_{\lambda_0-I_{j_3}}$,

$$\mathscr{L}_{hhll_-}(u,v)\lesssim\sum_{j_3\ll j_2\approx j_1}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{-I_{j_2}}u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}.\tag{4.18}$$

Using $\|v_{\lambda_0}\|_{L^\infty_x}\lesssim\|v_{\lambda_0}\|_{L^2_x}$, $V_A^2\subset L_t^\infty L_x^2$, the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4, we have

$$\mathscr{L}_{hhll_-}(u,v)\lesssim\sum_{j_3\ll j_2\approx j_1}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{I_{j_1}}\|_{V_A^2}$$

$$\begin{aligned}
&\times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1+2\varepsilon}\|u_{-I_{j_2}}\|_{V_A^2}\|u_{\lambda_0-I_{j_3}}\|_{V_A^2}\\
\lesssim& T^{\varepsilon/4}\sum_{j_3\ll j_2\approx j_1}\langle\lambda_0\rangle^{-3/4+2\varepsilon}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\\
&\times(2^{j_1})^{1/4}(2^{j_2})^{1/4}(2^{j_3})^{1/2}\langle\lambda_0\rangle^{-1/4}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/4}\langle\lambda_0\rangle^{-1+2\varepsilon}\left(\sum_{j_1}(2^{j_1})^{1/2}\cdot j_1\right)\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},
\end{aligned}\tag{4.19}$$

where the last but one inequality is obtained by taking $\varepsilon<1/4$.

If $u_{I_{j_1}}$ has the highest dispersion modulation, we have

$$\begin{aligned}
\mathscr{L}_{hhll_-}(u,v)\lesssim&\sum_{j_3\ll j_2\approx j_1}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{I_{j_1}}\|_{L^2_{x,t}}\|u_{-I_{j_2}}u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim&\sum_{j_3\ll j_2\approx j_1}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|u_{I_{j_1}}\|_{V_A^2}\\
&\times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1+2\varepsilon}\|u_{-I_{j_2}}\|_{V_A^2}\|u_{\lambda_0-I_{j_3}}\|_{V_A^2},
\end{aligned}$$

which is the same as the right-hand side of the first inequality in (4.19).

If $u_{-I_{j_2}}$ has the highest dispersion modulation, noticing that $j_1\approx j_2$, we can take $L^\infty_{x,t}$, $L^2_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0}, $u_{-I_{j_2}}$ and $u_{I_{j_1}}u_{\lambda_0-I_{j_3}}$, respectively. Then we can repeat the above proof to obtain the desired estimates.

If $u_{\lambda_0-I_{j_3}}$ has the highest dispersion modulation, comparing with Case *hhll*, the difference is the summation in (4.17) (taking $\varepsilon<1/8$)

$$\sum_{j_3\ll j_2\approx j_1}(2^{j_1})^{-1/8+\varepsilon}(2^{j_2})^{-1/8+\varepsilon}\lesssim\sum_{0\le j_1\le\log_2\lambda_0}(2^{j_1})^{-1/4+2\varepsilon}\times j_1\lesssim 1.\tag{4.20}$$

Case 3 (Case hhl_-l_-): $\lambda_3\in h$ and $\lambda_1\in l_-$. It is easy to see that $\lambda_2\in l_-$. We decompose $\lambda_1,\lambda_2,\lambda_3$ by:

$$\lambda_k\in[-c\lambda_0,0]=\bigcup_{j_k\ge 0}-I_{j_k},k=1,2;\quad \lambda_3\in[c\lambda_0,\lambda_0]=\bigcup_{j_3\ge 0}\lambda_0-I_{j_3}.$$

In view of the condition (FCC) $\lambda_1+\lambda_2+\lambda_3\approx\lambda_0$, we see that $2^{j_1}+2^{j_2}+2^{j_3}\approx 0$. It means that $0\le j_1,j_2,j_3\lesssim 1$. Then using the dispersion modulation decay (2.4), the L^4 estimate (3.13) and Lemma 3.4, and noticing that the summation about j_1, j_2, j_3 is finite, we can get the result and the details are omitted.

Case 4 (Case *hlll* and Case hll_-l_-): $\lambda_3\in l$. This case is easy to estimate because the derivative locates in the low frequency, $\lambda_1,\lambda_2\in\{l,l_-\}$ and the highest dispersion modulation satisfies

$$\max_{0\le k\le 3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^3.$$

We take Case *hlll* ($\lambda_1,\lambda_2,\lambda_3\in l$) as an example. When v_{λ_0} attains the highest dispersion modulation, using a similar way as above, we have

$$\mathscr{L}_{hlll}(u,v)\lesssim\sum_{j_1,j_2,j_3}\langle\lambda_0\rangle^{1/4}2^{j_3}\|\overline{v_{\lambda_0}}\|_{L^2_tL^\infty_x}\|u_{I_{j_1}}\|_{L^\infty_tL^2_x}\|u_{I_{j_2}}\|_{L^4_{x,t}}\|u_{I_{j_3}}\|_{L^4_{x,t}}$$

$$\lesssim \sum_{j_1,j_2,j_3} \langle\lambda_0\rangle^{1/4} 2^{j_3} \langle\lambda_0\rangle^{-3/2} \|v_{\lambda_0}\|_{V_A^2} (2^{j_1})^{1/4} T^{\varepsilon/2} (2^{j_2})^{-1/8+\varepsilon} (2^{j_3})^{-1/8+\varepsilon} \|u\|^3_{X^{1/4}_{\infty,A}}$$
$$\lesssim T^{\varepsilon/2} \langle\lambda_0\rangle^{-5/4} \sum_{j_1,j_2,j_3} (2^{j_1})^{1/4} (2^{j_2})^{-1/8+\varepsilon} (2^{j_3})^{7/8+\varepsilon} \|v_{\lambda_0}\|_{V_A^2} \|u\|^3_{X^{1/4}_{\infty,A}}$$
$$\lesssim T^{\varepsilon/2} \|v_{\lambda_0}\|_{V_A^2} \|u\|^3_{X^{1/4}_{\infty,A}}.$$

When $u_{I_{j_1}}$, $u_{I_{j_2}}$, or $u_{I_{j_3}}$ attains the highest dispersion modulation, we can use an analogous way to get the result. In fact, we just need to take $L^\infty_{x,t}$ norm to v_{λ_0}, $L^2_{x,t}$ norm to the item which has the highest dispersion modulation, and $L^4_{x,t}$ norm to the other two items.

Step 2. We consider the case that λ_0 is the second-most maximal integer in $\lambda_0,\cdots,\lambda_3$. By symmetry, we can assume $\lambda_1 \geq \lambda_2$. Then $\lambda_0,\cdots,\lambda_3$ have the following three orders:

$$\begin{aligned} &\text{Order 1}: && \lambda_3 \geq \lambda_0 \geq \lambda_1 \geq \lambda_2;\\ &\text{Order 2}: && \lambda_1 \geq \lambda_0 \geq \lambda_3 \geq \lambda_2;\\ &\text{Order 3}: && \lambda_1 \geq \lambda_0 \geq \lambda_2 \geq \lambda_3. \end{aligned}$$

Considering the derivative is located in u_{λ_3}, we take the Order 1 for example in the following proof (the other orders are similar). We divide the proof into three cases $|\lambda_0| \lesssim 1$, $\lambda_0 \ll 0$ and $\lambda_0 \gg 0$.

Case 1: $|\lambda_0| \lesssim 1$. We decompose $\lambda_1,\lambda_2,\lambda_3$ by:

$$\lambda_k \in (-\infty,\lambda_0] = \bigcup_{j_k \geq -1} -I_{j_k}, k=1,2; \quad \lambda_3 \in [\lambda_0,+\infty) = \bigcup_{j_3 \geq -1} I_{j_3}, \quad I_{-1} = [-|\lambda_0|,0).$$

In view of $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 \geq \lambda_2$, we have $j_3 \approx j_2 \geq j_1 \geq -1$. By DMCC (4.9) the highest dispersion modulation satisfies

$$\max_{0\leq k\leq 3} |\xi_k^3 - \tau_k| \gtrsim 2^{j_1} \cdot 2^{j_2} \cdot 2^{j_3}. \tag{4.21}$$

If v_{λ_0} gains the highest dispersion modulation, we have

$$\begin{aligned} &\sum_{j_3\approx j_2 \geq j_1 \geq -1} \langle\lambda_0\rangle^{1/4} \int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{-I_{j_1}} u_{-I_{j_2}} \partial_x u_{I_{j_3}}| dxdt \\ \lesssim &\sum_{j_3\approx j_2 \geq j_1 \geq -1} 2^{j_3} \|\overline{v_{\lambda_0}}\|_{L^2_t L^\infty_x} \|u_{-I_{j_1}}\|_{L^\infty_t L^2_x} \|u_{-I_{j_2}}\|_{L^4_{x,t}} \|u_{I_{j_3}}\|_{L^4_{x,t}} \\ \lesssim &\sum_{j_3\approx j_2 \geq j_1 \geq -1} 2^{j_3} (2^{j_1})^{-1/2} (2^{j_2})^{-1/2} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V_A^2} \|u_{-I_{j_1}}\|_{V_A^2} \\ &\qquad \times T^{\varepsilon/4} (2^{j_2})^{-1/8+\varepsilon} T^{\varepsilon/4} (2^{j_3})^{-1/8+\varepsilon} \|u\|^2_{X^{1/4}_{\infty,A}} \\ \lesssim &T^{\varepsilon/2} \sum_{j_3 \geq j_1 \geq -1} (2^{j_3})^{-1/4+2\varepsilon} (2^{j_1})^{-1/4} \|v_{\lambda_0}\|_{V_A^2} \|u\|^3_{X^{1/4}_{\infty,A}} \\ \lesssim &T^{\varepsilon/2} \|v^{(\lambda_0)}\|_{V_A^2} \|u\|^3_{X^{1/4}_{\infty,A}}. \end{aligned} \tag{4.22}$$

If $u_{-I_{j_1}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$ and $L^4_{x,t}$ norms to v_{λ_0}, $u_{-I_{j_1}}$, $u_{-I_{j_2}}$ and $u_{I_{j_3}}$, respectively. Then applying the dispersion modulation decay (2.4) and the L^4 estimate Lemma 3.3, we can get the desired conclusion.

If $u_{I_{j_3}}$ has the highest dispersion modulation, we divide the left-hand side of (4.7) into two terms.

$$\begin{aligned}&\sum_{j_3\approx j_2\geq j_1\geq -1}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{-I_{j_1}}u_{-I_{j_2}}\partial_x u_{I_{j_3}}|dxdt\\ \leq&\Big(\sum_{j_3\approx j_2\approx j_1\geq -1}+\sum_{j_3\approx j_2\gg j_1\geq -1}\Big)\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{-I_{j_1}}u_{-I_{j_2}}\partial_x u_{I_{j_3}}|dxdt\\ :=&I_1(u,v)+I_2(u,v).\end{aligned}\tag{4.23}$$

For $I_1(u,v)$, L^4 estimate (3.13) is enough.

$$\begin{aligned}I_1(u,v)\lesssim&\sum_{j_3\approx j_2\approx j_1\geq -1}2^{j_3}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{I_{j_3}}\|_{L^2_{x,t}}\|u_{-I_{j_1}}\|_{L^4_{x,t}}\|u_{-I_{j_2}}\|_{L^4_{x,t}}\\ \lesssim&\sum_{j_3\approx j_2\approx j_1\geq -1}2^{j_3}\|v_{\lambda_0}\|_{V^2_A}(2^{j_1})^{-1/2}(2^{j_2})^{-1/2}(2^{j_3})^{-1/2}(2^{j_3})^{1/4}\|u\|_{X^{1/4}_{\infty,A}}\\ &\times T^{\varepsilon/4}(2^{j_1})^{-1/8+\varepsilon}T^{\varepsilon/4}(2^{j_2})^{-1/8+\varepsilon}\|u\|^2_{X^{1/4}_{\infty,A}}\\ \lesssim&T^{\varepsilon/2}\sum_{j_3\geq -1}(2^{j_3})^{-1/2+2\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\ \lesssim&T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.\end{aligned}\tag{4.24}$$

For $I_2(u,v)$, we need to use the bilinear estimate (3.7).

$$\begin{aligned}I_2(u,v)\lesssim&\sum_{j_3\approx j_2\gg j_1\geq -1}2^{j_3}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{I_{j_3}}\|_{L^2_{x,t}}\|u_{-I_{j_1}}u_{-I_{j_2}}\|_{L^2_{x,t}}\\ \lesssim&\sum_{j_3\approx j_2\gg j_1\geq -1}2^{j_3}\|v_{\lambda_0}\|_{V^2_A}(2^{j_1})^{-1/2}(2^{j_2})^{-1/2}(2^{j_3})^{-1/2}(2^{j_3})^{1/4}\|u\|_{X^{1/4}_{\infty,A}}\\ &\times T^{\varepsilon/4}(2^{j_2})^{-1+2\varepsilon}(2^{j_1})^{1/4}(2^{j_2})^{1/4}\|u\|^2_{X^{1/4}_{\infty,A}}\\ \lesssim&T^{\varepsilon/4}\sum_{j_3\gg j_1\geq -1}(2^{j_3})^{-1/2+2\varepsilon}(2^{j_1})^{-1/4}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\ \lesssim&T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.\end{aligned}\tag{4.25}$$

If $u_{-I_{j_2}}$ has the highest dispersion modulation, we can get the desired estimate by exchanging the positions of $u_{I_{j_3}}$ and $u_{-I_{j_2}}$ in the above discussion (noticing that $j_2\approx j_3$).

Case 2: $\lambda_0\ll 0$. We decompose λ_1 and λ_2 by:

$$\lambda_k\in(-\infty,\lambda_0]=\bigcup_{j_k\geq 0}\lambda_0-I_{j_k},\quad k=1,2.$$

From the following frequency constraint condition

$$\lambda_0=\lambda_1+\lambda_2+\lambda_3+l,\quad |l|\leq 10,\tag{4.26}$$

we can decompose λ_3 as follows.

$$\lambda_3\in[-\lambda_0-l,+\infty]=\bigcup_{j_3\geq -1}-\lambda_0+I_{j_3},\quad I_{-1}=[-|l|,0).$$

In view of $\lambda_0 \approx \lambda_1+\lambda_2+\lambda_3$ and $\lambda_1 \geq \lambda_2$, we have $j_3 \approx j_2 \geq j_1$. By DMCC (4.9), we can see that the highest dispersion modulation satisfies

$$\max_{0\leq k\leq 3} |\xi_k^3 - \tau_k| \gtrsim 2^{j_1}\cdot 2^{j_2}\cdot (\langle\lambda_0\rangle + 2^{j_3}). \tag{4.27}$$

If the highest dispersion modulation is located in v_{λ_0}, from the dispersion modulation decay (2.4), L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}
&\sum_{j_3\approx j_2\geq j_1} \langle\lambda_0\rangle^{1/4} \int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{\lambda_0-I_{j_1}} u_{\lambda_0-I_{j_2}} \partial_x u_{-\lambda_0+I_{j_3}}|\,dxdt \\
&\lesssim \sum_{j_3\approx j_2\geq j_1} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L_t^2 L_x^\infty} \|u_{\lambda_0-I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{\lambda_0-I_{j_2}}\|_{L^4_{x,t}}\|u_{-\lambda_0+I_{j_3}}\|_{L^4_{x,t}} \\
&\lesssim \sum_{j_3\approx j_2\geq j_1} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})(2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}(\langle\lambda_0\rangle+2^{j_1})^{-1/4} \\
&\qquad \times T^{\varepsilon/4}(2^{j_2})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/8}T^{\varepsilon/4}(2^{j_3})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_3})^{-3/8}\|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2}\sum_{j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})^{-1/4}(2^{j_3})^{2\varepsilon}\sum_{0\leq j_1\leq j_3}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_1})^{-1/4}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned} \tag{4.28}$$

Making the summation on j_1, we see that the summation is controlled by j_3. Then one has that for $0<\varepsilon<1/8$,

$$\begin{aligned}
(4.28) &\lesssim T^{\varepsilon/2}\left(\sum_{j_3\geq 0}(2^{j_3})^{-1/4+2\varepsilon}\cdot j_3\right)\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}$$

If the highest dispersion modulation is located in $u_{\lambda_0-I_{j_1}}$, we take $L^\infty_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$ and $L^4_{x,t}$ norms to v_{λ_0}, $u_{\lambda_0-I_{j_1}}$, $u_{\lambda_0-I_{j_2}}$ and $u_{-\lambda_0+I_{j_3}}$, respectively. Then we can reduce the desired estimate as the above case, so the details are omitted.

If the highest dispersion modulation is located in $u_{\lambda_0-I_{j_2}}$, we divide the left-hand side of (4.7) into two terms.

$$\begin{aligned}
&\sum_{j_3\approx j_2\geq j_1\geq 0} \langle\lambda_0\rangle^{1/4} \int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{\lambda_0-I_{j_1}} u_{\lambda_0-I_{j_2}} \partial_x u_{-\lambda_0+I_{j_3}}|\,dxdt \\
&\leq \left(\sum_{j_3\approx j_2\approx j_1\geq 0} + \sum_{j_3\approx j_2\gg j_1\geq 0}\right)\langle\lambda_0\rangle^{1/4} \int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{\lambda_0-I_{j_1}} u_{\lambda_0-I_{j_2}} \partial_x u_{-\lambda_0+I_{j_3}}|\,dxdt \\
&:= I_1(u,v)+I_2(u,v).
\end{aligned} \tag{4.29}$$

For $I_1(u,v)$, from the dispersion modulation decay (2.4), L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}
&I_1(u,v) \\
&\lesssim \sum_{j_3\approx j_2\approx j_1\geq 0} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{\lambda_0-I_{j_2}}\|_{L^2_{x,t}}\|u_{\lambda_0-I_{j_1}}\|_{L^4_{x,t}}\|u_{-\lambda_0+I_{j_3}}\|_{L^4_{x,t}} \\
&\lesssim \sum_{j_3\approx j_2\approx j_1\geq 0} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|v_{\lambda_0}\|_{V_A^2}(2^{j_1})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/4}
\end{aligned}$$

$$\times T^{\varepsilon/4}(2^{j_1})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_1})^{-3/8}T^{\varepsilon/4}(2^{j_3})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_3})^{-3/8}\|u\|^3_{X^{1/4}_{\infty,A}}$$
$$\lesssim T^{\varepsilon/2}\sum_{j_3\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-1/2}(2^{j_3})^{2\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}. \tag{4.30}$$

Noticing that

$$\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-1/4}\leq 1,\quad (\langle\lambda_0\rangle+2^{j_3})^{-1/4}(2^{j_3})^{2\varepsilon}\leq (2^{j_3})^{-1/4+2\varepsilon},$$

for $0<\varepsilon<1/8$, (4.30) is dominated by

$$\lesssim T^{\varepsilon/2}\sum_{j_3\geq 0}(2^{j_3})^{-1/4+2\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\lesssim T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.$$

For $I_2(u,v)$, from the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4, we have

$$\begin{aligned}I_2(u,v)\lesssim&\sum_{j_3\approx j_2\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{\lambda_0-I_{j_2}}\|_{L^2_{x,t}}\|u_{\lambda_0-I_{j_1}}u_{-\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim&\sum_{j_3\approx j_2\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(2^{j_1})^{-1/2}(2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}\|v_{\lambda_0}\|_{V^2_A}\|u_{\lambda_0-I_{j_2}}\|_{V^2_A}\\
&\times T^{\varepsilon/4}(\langle\lambda_0\rangle+2^{j_3})^{-1/2+\varepsilon}(2^{j_3})^{-1/2+\varepsilon}\|u_{\lambda_0-I_{j_1}}\|_{V^2_A}\|u_{-\lambda_0+I_{j_3}}\|_{V^2_A}\\
\lesssim& T^{\varepsilon/4}\sum_{j_3\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-1/2+\varepsilon}(2^{j_3})^{\varepsilon}(\langle\lambda_0\rangle+2^{j_1})^{-1/4}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/4}\left(\sum_{j_3\geq 0}(2^{j_3})^{-1/2+2\varepsilon}\cdot j_3\right)\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.\end{aligned}$$

If the highest dispersion modulation is located in $u_{-\lambda_0+I_{j_3}}$, noticing that $j_2\approx j_3$ and $|-\lambda_0+I_{j_3}|\sim|\lambda_0-I_{j_2}|\sim(\langle\lambda_0\rangle+2^{j_3})$, we can get the desired estimate by exchanging the positions of $u_{-\lambda_0+I_{j_3}}$ and $u_{\lambda_0-I_{j_2}}$ in the above discussion.

Case 3: $\lambda_0\gg 0$. From the frequency constraint condition $\lambda_0=\lambda_1+\lambda_2+\lambda_3+l$, $|l|\leq 10$, we know that λ_2 must be less than zero. Furthermore, one can divide this case into three subcases: $\lambda_2\in[-c\lambda_0,0]$, $\lambda_2\in[-\lambda_0,-c\lambda_0]$ and $\lambda_2\in(-\infty,-\lambda_0]$.

Case 3.1: $\lambda_2\in[-c\lambda_0,0]$. From the frequency constraint condition we find that $\lambda_1\in[-c\lambda_0,0]$ or $[0,c\lambda_0]$ ($\lambda_1\in[c\lambda_0,\lambda_0]$ will never happen), and λ_3 satisfies Table 4.2.

Case	$\lambda_2\in$	$\lambda_1\in$	$\lambda_3\in$
l_l_h	$[-c\lambda_0,0]$	$[-c\lambda_0,0]$	$[\lambda_0,\lambda_0+2c\lambda_0-l]$
l_lh	$[-c\lambda_0,0]$	$[0,c\lambda_0]$	$[\lambda_0,\lambda_0+c\lambda_0-l]$

TABLE 4.2. $\lambda_2\in[-c\lambda_0,0]$

Case l_l_h. One can use the dyadic decomposition:

$$\lambda_k\in[-c\lambda_0,0]=\bigcup_{j_k\geq 0}-I_{j_k},\ k=1,2;$$
$$\lambda_3\in[\lambda_0,(1+2c)\lambda_0-l]=\bigcup_{j_3\geq 0}\lambda_0+I_{j_3},\quad j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle+1.$$

From the frequency constraint condition (4.8), we know

$$2^{j_1}+2^{j_2}\approx 2^{j_3} \ \Rightarrow\ j_3\approx j_1\vee j_2. \tag{4.31}$$

We can easily get that the highest dispersion modulation satisfies

$$\max_{0\le k\le 3}|\xi_k^3-\tau_k|\gtrsim \langle\lambda_0\rangle^2\cdot 2^{j_3}. \tag{4.32}$$

If v_{λ_0} attains the highest dispersion modulation, from the dispersion modulation decay (2.4), L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}
&\sum_{j_3\approx j_1\vee j_2}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{-I_{j_1}}u_{-I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_3\approx j_1\vee j_2}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{-I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{-I_{j_2}}\|_{L_{x,t}^4}\|u_{\lambda_0+I_{j_3}}\|_{L_{x,t}^4}\\
\lesssim& T^{\varepsilon/2}\sum_{j_3\approx j_1\vee j_2}\langle\lambda_0\rangle^{-1/8}(2^{j_3})^{-1/4+\varepsilon}(2^{j_1})^{1/4}(2^{j_2})^{-1/8+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/8+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},
\end{aligned}$$

where the last but one inequality is gained by summarizing over j_2, j_1 and j_3 in order. One just needs to note that $j_1\le j_3\le \log_2\langle\lambda_0\rangle+1$ and take $\varepsilon<1/8$.

If $u_{-I_{j_1}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$ and $L^4_{x,t}$ norms to v_{λ_0}, $u_{-I_{j_1}}$, $u_{-I_{j_2}}$ and $u_{\lambda_0+I_{j_3}}$, respectively. Then we can get the desired conclusion by the same way as above. If $u_{-I_{j_2}}$ gains the highest dispersion modulation, one can exchange the positions of j_1 and j_2 to obtain the desired estimate.

If $u_{\lambda_0+I_{j_3}}$ has the highest dispersion modulation, we have

$$\begin{aligned}
&\sum_{j_3\approx j_1\vee j_2}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{-I_{j_1}}u_{-I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_3\approx j_1\vee j_2}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_{x,t}^\infty}\|u_{\lambda_0+I_{j_3}}\|_{L_{x,t}^2}\|u_{-I_{j_1}}\|_{L_{x,t}^4}\|u_{-I_{j_2}}\|_{L_{x,t}^4}\\
\lesssim& T^{\varepsilon/2}\sum_{j_3\approx j_1\vee j_2}(2^{j_1})^{-1/8+\varepsilon}(2^{j_2})^{-1/8+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned} \tag{4.33}$$

Case l_lh. One can use the dyadic decomposition:

$$\lambda_1\in[0,c\lambda_0]=\bigcup_{j_1\ge 0}I_{j_1},\ \lambda_2\in[-c\lambda_0,0]=\bigcup_{j_2\ge 0}-I_{j_2},$$
$$\lambda_3\in[\lambda_0,(1+c)\lambda_0-l]=\bigcup_{j_3\ge 0}\lambda_0+I_{j_3},\quad j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle.$$

From the frequency constraint condition (4.8), we get

$$2^{j_1}+2^{j_3}\approx 2^{j_2} \ \Rightarrow\ j_2\approx j_1\vee j_3. \tag{4.34}$$

One can get that the highest dispersion modulation satisfies

$$\max_{0\le k\le 3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^2\cdot 2^{j_3}. \tag{4.35}$$

If v_{λ_0} attains the highest dispersion modulation, from the dispersion modulation decay (2.4), the bilinear estimate and Lemma 3.4, we have

$$\begin{aligned}
&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{-I_{j_2}}u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim& T^{\varepsilon/4}\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{I_{j_1}}\|_{V_A^2}\langle\lambda_0\rangle^{-1+2\varepsilon}\|u_{-I_{j_2}}\|_{V_A^2}\|u_{\lambda_0+I_{j_3}}\|_{V_A^2}\\
\lesssim& T^{\varepsilon/4}\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{-1+2\varepsilon}(2^{j_1})^{1/4}(2^{j_2})^{1/4}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim& T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}$$

If $u_{I_{j_1}}$ has the highest dispersion modulation, we just take $L^\infty_{x,t}$, $L^2_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0}, $u_{I_{j_1}}$ and $u_{-I_{j_2}}u_{\lambda_0+I_{j_3}}$, respectively. If $u_{-I_{j_2}}$ attains the highest dispersion modulation, one can further exchange the positions of j_1 and j_2 to obtain the desired estimate. If $u_{\lambda_0+I_{j_3}}$ has the highest dispersion modulation, we can get the result by the same way as (4.33) in Case l_l_h.

Case 3.2: $\lambda_2\in[-\lambda_0,-c\lambda_0]$. We consider $\lambda_1\in[c\lambda_0,\lambda_0]$, $[0,c\lambda_0]$, $[-c\lambda_0,0]$ and $[-\lambda_0,-c\lambda_0]$, respectively. From the frequency constraint condition we can obtain the corresponding range of λ_3 (see Table 4.3).

Case	$\lambda_2\in$	$\lambda_1\in$	$\lambda_3\in$
h_hh	$[-\lambda_0,-c\lambda_0]$	$[c\lambda_0,\lambda_0]$	$[\lambda_0,2\lambda_0-c\lambda_0-l]$
h_lh	$[-\lambda_0,-c\lambda_0]$	$[0,c\lambda_0]$	$[\lambda_0,2\lambda_0-l]$
h_l_h	$[-\lambda_0,-c\lambda_0]$	$[-c\lambda_0,0]$	$[\lambda_0+c\lambda_0-l,2\lambda_0+c\lambda_0-l]$
h_h_h	$[-\lambda_0,-c\lambda_0]$	$[-\lambda_0,-c\lambda_0]$	$[\lambda_0+2c\lambda_0-l,3\lambda_0-l]$

TABLE 4.3. $\lambda_2\in[-\lambda_0,-c\lambda_0]$

Case h_hh. We decompose $\lambda_1,\lambda_2,\lambda_3$ by:

$$\lambda_1\in[c\lambda_0,\lambda_0]=\bigcup_{j_1\ge0}\lambda_0-I_{j_1},\quad \lambda_2\in[-\lambda_0,-c\lambda_0]=\bigcup_{j_2\ge0}-\lambda_0+I_{j_2},$$
$$\lambda_3\in[\lambda_0,(2-c)\lambda_0-l]=\bigcup_{j_3\ge0}\lambda_0+I_{j_3},\quad j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle.$$

From the frequency constraint condition (4.8), we have

$$2^{j_1}\approx 2^{j_2}+2^{j_3}. \tag{4.36}$$

It follows that $j_1\approx j_2\vee j_3$. When $j_1\approx j_2\ge j_3$, we can get the result by using the similar technique as that used in Case 1 of Step 1. When $j_1\approx j_3\ge j_2$, we just need to exchange the positions of j_2 and j_3 and use the same way to obtain our conclusion. We omit the details.

Case h_lh. We decompose $\lambda_1,\lambda_2,\lambda_3$ by:

$$\lambda_1\in[0,c\lambda_0]=\bigcup_{j_1\geq 0} I_{j_1},\quad \lambda_2\in[-\lambda_0,-c\lambda_0]=\bigcup_{j_2\geq 0} -\lambda_0+I_{j_2},$$
$$\lambda_3\in[\lambda_0,2\lambda_0-l]=\bigcup_{j_3\geq 0}\lambda_0+I_{j_3},\quad j_1,j_2,j_3\leq \log_2\langle\lambda_0\rangle.$$

From the frequency constraint condition (4.8), we have

$$2^{j_1}+2^{j_2}+2^{j_3}\approx\lambda_0. \tag{4.37}$$

By DMCC (4.9) the highest dispersion modulation satisfies

$$\max_{0\leq k\leq 3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^2\cdot 2^{j_3}. \tag{4.38}$$

If v_{λ_0} has the highest dispersion modulation, we have dispersion modulation decay to v_{λ_0}. For $u_{I_{j_1}}$ and $u_{\lambda_0+I_{j_3}}$, we have $|\lambda_0+2^{j_3}+2^{j_1}|\gtrsim\langle\lambda_0\rangle$ and $|\lambda_0+2^{j_3}-2^{j_1}|\gtrsim\langle\lambda_0\rangle$. Thus we can use bilinear estimate (3.7) to $u_{I_{j_1}}u_{\lambda_0+I_{j_3}}$. To be specific, we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{-\lambda_0+I_{j_2}}\|_{L_t^\infty L_x^2}\|u_{I_{j_1}}u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim&\sum_{j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{-\lambda_0+I_{j_2}}\|_{V_A^2}\\
&\quad\times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1+2\varepsilon}\|u_{I_{j_1}}\|_{V_A^2}\|u_{\lambda_0+I_{j_3}}\|_{V_A^2}\\
\lesssim&T^{\varepsilon/4}\sum_{j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{-5/4+2\varepsilon}(2^{j_1})^{1/4}(2^{j_2})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/4}\langle\lambda_0\rangle^{-1/2+2\varepsilon}\log_2\langle\lambda_0\rangle\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}\tag{4.39}$$

If $u_{-\lambda_0+I_{j_2}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0}, $u_{-\lambda_0+I_{j_2}}$, and $u_{I_{j_1}}u_{\lambda_0+I_{j_3}}$, respectively. Then applying the dispersion modulation decay estimate (2.4) to $u_{-\lambda_0+I_{j_2}}$ and the bilinear estimate (3.7) to $u_{I_{j_1}}u_{\lambda_0+I_{j_3}}$, we can get the desired conclusion.

If $u_{I_{j_1}}$ has the highest dispersion modulation, we have dispersion modulation decay to $u_{I_{j_1}}$. For $u_{-\lambda_0+I_{j_2}}$ and $u_{\lambda_0+I_{j_3}}$, we have $|\lambda_0+2^{j_3}-\lambda_0+2^{j_2}|\gtrsim(2^{j_3}+2^{j_2})\approx\lambda_0-2^{j_1}\gtrsim\langle\lambda_0\rangle$ and $|\lambda_0+2^{j_3}+\lambda_0-2^{j_2}|\gtrsim\langle\lambda_0\rangle$. Thus we can use bilinear estimate (3.7) to $u_{-\lambda_0+I_{j_2}}u_{\lambda_0+I_{j_3}}$. Therefore, we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{I_{j_1}}\|_{L^2_{x,t}}\|u_{-\lambda_0+I_{j_2}}u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}
\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|u_{I_{j_1}}\|_{V_A^2}\\&\quad\times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1+2\varepsilon}\|u_{-\lambda_0+I_{j_2}}\|_{V_A^2}\|u_{\lambda_0+I_{j_3}}\|_{V_A^2},\end{aligned}\tag{4.40}$$

which is the same as the third line of (4.39), so we omit the details.

If $u_{\lambda_0+I_{j_3}}$ has the highest dispersion modulation, from the dispersion modulation decay (2.4) and L^4 estimate (3.13), we have

$$\begin{aligned}&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\&\lesssim \sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\|u_{I_{j_1}}\|_{L^4_{x,t}}\|u_{-\lambda_0+I_{j_2}}\|_{L^4_{x,t}}\\&\lesssim \sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}(2^{j_3})^{1/2}\langle\lambda_0\rangle^{-1/4}\|u\|_{X^{1/4}_{\infty,A}}\\&\qquad\times T^{\varepsilon/4}(2^{j_1})^{-1/8+\varepsilon}T^{\varepsilon/4}(2^{j_2})^{1/4+\varepsilon}\langle\lambda_0\rangle^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}}\\&\lesssim T^{\varepsilon/2}\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{-3/8}(2^{j_1})^{-1/8+\varepsilon}(2^{j_2})^{1/4+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.\end{aligned}\tag{4.41}$$

Taking $0<\varepsilon<1/8$, the summation over j_1 is finite. The summation over j_2 and j_3 can be controlled, so (4.41) is continued by

$$\lesssim T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/8+\varepsilon}\log_2\langle\lambda_0\rangle\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\lesssim T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.\tag{4.42}$$

Case h_l_h. We decompose $\lambda_1,\lambda_2,\lambda_3$ in the following way:

$$\lambda_1\in[-c\lambda_0,0]=\bigcup_{j_1\ge0}-I_{j_1},\quad \lambda_2\in[-\lambda_0,-c\lambda_0]=\bigcup_{j_2\ge0}-\lambda_0+I_{j_2},\quad j_1,j_2\le\log_2\langle\lambda_0\rangle,$$
$$\lambda_3\in[\lambda_0+c\lambda_0-l,2\lambda_0+c\lambda_0-l]=\bigcup_{j_3\ge\log_2\langle\lambda_0\rangle-C}\lambda_0+I_{j_3},\quad j_3\le\log_2\langle\lambda_0\rangle+1.$$

From the frequency constraint condition (4.8), we have

$$2^{j_2}+2^{j_3}\approx\lambda_0+2^{j_1}.\tag{4.43}$$

It is easy to see that this case is similar to the above Case h_lh, so the details are omitted.

Case h_h_h. We decompose $\lambda_1,\lambda_2,\lambda_3$ as follows:

$$\lambda_k\in[-\lambda_0,-c\lambda_0]=\bigcup_{j_k\ge0}-\lambda_0+I_{j_k},\quad j_k\le\log_2\langle\lambda_0\rangle,\quad k=1,2;\tag{4.44}$$

$$\lambda_3\in[\lambda_0+2c\lambda_0-l,3\lambda_0-l]=\bigcup_{j_3\ge\log_2(2c\lambda_0-l)}\lambda_0+I_{j_3},\quad j_3\le\log_2\langle\lambda_0\rangle+1.\tag{4.45}$$

From the dispersion modulation constraint condition (4.9), we know the highest dispersion modulation satisfies

$$\max_{0\le k\le3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^2\cdot2^{j_3}.\tag{4.46}$$

If v_{λ_0} has the highest dispersion modulation, we take dispersion modulation decay to v_{λ_0}. For $u_{-\lambda_0+I_{j_1}}$ and $u_{\lambda_0+I_{j_3}}$, we have $|\lambda_0+2^{j_3}-\lambda_0+2^{j_1}|\gtrsim 2^{j_1}$ and $|\lambda_0+2^{j_3}+\lambda_0-2^{j_1}|\gtrsim\langle\lambda_0\rangle$. Thus we can use bilinear estimate (3.7) to $u_{-\lambda_0+I_{j_1}}u_{\lambda_0+I_{j_3}}$. Thus we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{-\lambda_0+I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^2_tL^\infty_x}\|u_{-\lambda_0+I_{j_2}}\|_{L^\infty_tL^2_x}\|u_{-\lambda_0+I_{j_1}}u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V^2_A}\|u_{-\lambda_0+I_{j_2}}\|_{V^2_A}\\
&\quad\times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_1})^{-1/2+\varepsilon}\|u_{-\lambda_0+I_{j_1}}\|_{V^2_A}\|u_{\lambda_0+I_{j_3}}\|_{V^2_A}\\
\lesssim&T^{\varepsilon/4}\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{-1+\varepsilon}(2^{j_1})^{\varepsilon}(2^{j_2})^{1/2}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/4}\langle\lambda_0\rangle^{-1/2+2\varepsilon}\log_2\langle\lambda_0\rangle\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}\tag{4.47}$$

If $u_{-\lambda_0+I_{j_2}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0}, $u_{-\lambda_0+I_{j_2}}$, and $u_{-\lambda_0+I_{j_1}}u_{\lambda_0+I_{j_3}}$, respectively. Then we can get the desired estimate by using an analogous technique. If $u_{-\lambda_0+I_{j_1}}$ has the highest dispersion modulation, due to the symmetry between $u_{-\lambda_0+I_{j_1}}$ and $u_{-\lambda_0+I_{j_2}}$, the estimate is similar so we omit the details.

If $u_{\lambda_0+I_{j_3}}$ has the highest dispersion modulation, from the dispersion modulation decay (2.4) and L^4 estimate (3.13), we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{-\lambda_0+I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\|u_{-\lambda_0+I_{j_1}}\|_{L^4_{x,t}}\|u_{-\lambda_0+I_{j_2}}\|_{L^4_{x,t}}\\
\lesssim&\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{5/4}\|v_{\lambda_0}\|_{V^2_A}\langle\lambda_0\rangle^{-1}(2^{j_3})^{-1/2}(2^{j_3})^{1/2}\langle\lambda_0\rangle^{-1/4}\|u\|_{X^{1/4}_{\infty,A}}\\
&\quad\times T^{\varepsilon/4}(2^{j_1})^{1/4+\varepsilon}\langle\lambda_0\rangle^{-3/8}T^{\varepsilon/4}(2^{j_2})^{1/4+\varepsilon}\langle\lambda_0\rangle^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/2}\sum_{j_1,j_2,j_3\le\log_2\langle\lambda_0\rangle+1}\langle\lambda_0\rangle^{-3/4}(2^{j_1})^{1/4+\varepsilon}(2^{j_2})^{1/4+\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/2}\langle\lambda_0\rangle^{-1/4+2\varepsilon}\log_2\langle\lambda_0\rangle\|v_{\lambda_0}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V^2_A}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}$$

Case 3.3: $\lambda_2\in(-\infty,-\lambda_0]$. We consider $\lambda_1\in[c\lambda_0,\lambda_0]$, $[0,c\lambda_0]$, $[-\lambda_0,0]$ and $(-\infty,-\lambda_0]$, respectively. From the frequency constraint condition we can obtain the corresponding range of λ_3 (see Table 4.4).

Case $2h\_hh$. We decompose $\lambda_1,\lambda_2,\lambda_3$ in the following way:

$$\lambda_1\in[c\lambda_0,\lambda_0]=\bigcup_{j_1\ge 0}\lambda_0-I_{j_1},\quad j_1\le\log_2\langle\lambda_0\rangle;$$

Case	$\lambda_2 \in$	$\lambda_1 \in$	$\lambda_3 \in$
$2h\_hh$	$(-\infty,-\lambda_0]$	$[c\lambda_0,\lambda_0]$	$[\lambda_0,+\infty]$
$2h\_lh$	$(-\infty,-\lambda_0]$	$[0,c\lambda_0]$	$[2\lambda_0-c\lambda_0-l,2\lambda_0]$
$2h\_lh2$	$(-\infty,-\lambda_0]$	$[0,c\lambda_0]$	$[2\lambda_0,\infty]$
$2h\_l\_h$	$(-\infty,-\lambda_0]$	$[-\lambda_0,0]$	$[2\lambda_0-l,+\infty]$
$2h\_h\_h$	$(-\infty,-\lambda_0]$	$(-\infty,-\lambda_0]$	$[3\lambda_0-l,+\infty]$

TABLE 4.4. $\lambda_2\in(-\infty,-\lambda_0]$

$$\lambda_2\in(-\infty,-\lambda_0]=\bigcup_{j_2\geq 0}-\lambda_0-I_{j_2},\quad \lambda_3\in[\lambda_0,+\infty)=\bigcup_{j_3\geq 0}\lambda_0+I_{j_3}.$$

From the frequency constraint condition (4.8), we know that

$$2^{j_3}\approx 2^{j_1}+2^{j_2}\Rightarrow j_3\approx j_1\vee j_2.$$

From the dispersion modulation constraint condition (4.9), we have that the highest dispersion modulation satisfies

$$\max_{0\leq k\leq 3}|\xi_k^3-\tau_k|\gtrsim(\langle\lambda_0\rangle+2^{j_2})\cdot 2^{j_1}\cdot 2^{j_3}. \tag{4.48}$$

If v_{λ_0} has the highest dispersion modulation, from the dispersion modulation decay (2.4) and L^4 estimate (3.13), we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3\geq 0}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}u_{-\lambda_0-I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_1,j_2,j_3\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{\lambda_0-I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{-\lambda_0-I_{j_2}}\|_{L^4_{x,t}}\|u_{\lambda_0+I_{j_3}}\|_{L^4_{x,t}}\\
\lesssim&\sum_{j_1,j_2,j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|_{X^{1/4}_{\infty,A}}\\
&\quad\times T^{\varepsilon/2}(2^{j_2})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/8}(2^{j_3})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_3})^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/2}\sum_{j_1,j_2,j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})^{5/8}(\langle\lambda_0\rangle+2^{j_2})^{-7/8}(2^{j_3})^{-1/4+\varepsilon}(2^{j_2})^{1/4+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}$$

When $0\leq j_1\leq j_2\approx j_3$, the summation in above inequality becomes

$$\sum_{j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})^{-1/4}(2^{j_3})^{2\varepsilon}\cdot j_3\lesssim 1. \tag{4.49}$$

When $0\leq j_2\leq j_1\approx j_3$, noticing that $j_1\leq\log_2\langle\lambda_0\rangle$, we can know that the summation satisfies

$$\sum_{0\leq j_2\leq j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{-1/4}(2^{j_3})^{-1/4+\varepsilon}(2^{j_2})^{1/4+\varepsilon}\cdot j_3\lesssim 1. \tag{4.50}$$

If $u_{\lambda_0-I_{j_1}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$ and $L^4_{x,t}$ norms to v_{λ_0}, $u_{\lambda_0-I_{j_1}}$, $u_{-\lambda_0-I_{j_2}}$ and $u_{\lambda_0+I_{j_3}}$, respectively. Then we can get the desired estimate by a similar way.

If $u_{-\lambda_0-I_{j_2}}$ has the highest dispersion modulation, we take the dispersion modulation decay estimate to $u_{-\lambda_0-I_{j_2}}$. For $u_{\lambda_0-I_{j_1}}$ and $u_{\lambda_0+I_{j_3}}$, we have $|\lambda_0+2^{j_3}-\lambda_0+2^{j_1}|\gtrsim 2^{j_3}$ and $|\lambda_0+2^{j_3}+\lambda_0-2^{j_1}|\gtrsim(\langle\lambda_0\rangle+2^{j_3})$. Thus we can use the bilinear estimate (3.7) to $u_{\lambda_0-I_{j_1}}u_{\lambda_0+I_{j_3}}$. Thus we have

$$\begin{aligned}
&\sum_{j_1,j_2,j_3\geq 0}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}u_{-\lambda_0-I_{j_2}}\partial_x u_{\lambda_0+I_{j_3}}|dxdt\\
\lesssim&\sum_{j_1,j_2,j_3\geq 0}\langle\lambda_0\rangle^{1/4}\|\overline{v_{\lambda_0}}\|_{L_t^\infty L_x^\infty}\|u_{-\lambda_0-I_{j_2}}\|_{L_t^2L_x^2}\|u_{\lambda_0-I_{j_1}}\partial_x u_{\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim&\sum_{j_1,j_2,j_3\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{-\lambda_0-I_{j_2}}\|_{V_A^2}\\
&\quad\times T^{\varepsilon/4}(\langle\lambda_0\rangle+2^{j_3})^{-1/2+\varepsilon}(2^{j_3})^{-1/2+\varepsilon}\|u_{\lambda_0-I_{j_1}}\|_{V_A^2}\|u_{\lambda_0+I_{j_3}}\|_{V_A^2}\\
\lesssim& T^{\varepsilon/4}\sum_{j_1,j_2,j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/4}(2^{j_3})^{-1/2+\varepsilon}(2^{j_2})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned}$$

If $0\leq j_1\leq j_2\approx j_3$, the summation in above inequality becomes

$$\sum_{j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})^{-1/2+\varepsilon}(2^{j_3})^{\varepsilon}\cdot j_3\lesssim 1.$$

If $0\leq j_2\leq j_1\approx j_3$, recalling that $j_1\leq\log_2\langle\lambda_0\rangle$, we can get the summation satisfying

$$\sum_{0\leq j_2\leq j_3\leq\log_2\langle\lambda_0\rangle}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_3})^{-1/2+\varepsilon}(2^{j_2})^{1/2}\cdot j_3\lesssim 1.$$

If $u_{\lambda_0+I_{j_3}}$ attains the highest dispersion modulation, noticing that for $u_{\lambda_0-I_{j_1}}$ and $u_{-\lambda_0-I_{j_2}}$, we have $|\lambda_0+2^{j_2}-\lambda_0+2^{j_1}|\gtrsim 2^{j_2}$ and $|\lambda_0+2^{j_2}+\lambda_0-2^{j_1}|\gtrsim(\langle\lambda_0\rangle+2^{j_2})$. We can use the bilinear estimate (3.7) to $u_{\lambda_0-I_{j_1}}u_{-\lambda_0-I_{j_3}}$ to get our result by using the same way as above.

Case $2h\_lh$. From (FCC) (4.8), we see that $\lambda_2\in[-\lambda_0-c\lambda_0-l,-\lambda_0]$. We decompose $\lambda_1,\lambda_2,\lambda_3$ in a dyadic way:

$$\lambda_1\in[0,c\lambda_0]=\bigcup_{j_1\geq 0}I_{j_1},\quad \lambda_2\in[-\lambda_0-c\lambda_0-l,-\lambda_0]=\bigcup_{j_2\geq 0}-\lambda_0-I_{j_2},$$
$$\lambda_3\in[2\lambda_0-c\lambda_0-l,2\lambda_0]=\bigcup_{j_3\geq 0}\lambda_0+I_{j_3},\quad j_1,j_2,j_3\leq\log_2\langle\lambda_0\rangle.$$

From the frequency constraint condition (4.8), we have

$$2^{j_1}+2^{j_3}-2^{j_2}\approx\lambda_0.$$

By DMCC (4.9) the highest dispersion modulation satisfies

$$\max_{0\leq k\leq 3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle^2\cdot 2^{j_3}.$$

Therefore, the approach to this case is similar to Case h_lh, and we omit it.

Case $2h\_lh2$. We decompose $\lambda_1,\lambda_2,\lambda_3$ in the following way:

$$\lambda_1\in[0,c\lambda_0]=\bigcup_{j_1\geq 0}I_{j_1},\quad j_1\leq\log_2\langle\lambda_0\rangle;$$

$$\lambda_2\in[-\infty,-\lambda_0]=\bigcup_{j_2\geq 0}-\lambda_0-I_{j_2},\quad \lambda_3\in[2\lambda_0,+\infty]=\bigcup_{j_3\geq 0}2\lambda_0+I_{j_3}.$$

From the frequency constraint condition (4.8), we have

$$2^{j_2}\approx 2^{j_1}+2^{j_3},\quad \text{i.e.}\quad j_2\approx j_1\vee j_3. \tag{4.51}$$

By DMCC (4.9) the highest dispersion modulation satisfies

$$\max_{0\leq k\leq 3}|\xi_k^3-\tau_k|\gtrsim\langle\lambda_0\rangle\cdot(\langle\lambda_0\rangle+2^{j_2})\cdot(\langle\lambda_0\rangle+2^{j_3}). \tag{4.52}$$

If v_{λ_0} has the highest dispersion modulation, we take dispersion modulation decay to v_{λ_0}. For $u_{I_{j_1}}$ and $u_{2\lambda_0+I_{j_3}}$, we have $|2\lambda_0+2^{j_3}\pm 2^{j_1}|\gtrsim(\langle\lambda_0\rangle+2^{j_3})$. Thus we can use bilinear estimate (3.7) to $u_{I_{j_1}}u_{2\lambda_0+I_{j_3}}$. Specifically, we have

$$\begin{aligned}
&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-\lambda_0-I_{j_2}}\partial_x u_{2\lambda_0+I_{j_3}}|\,dxdt\\
\lesssim&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{-\lambda_0-I_{j_2}}\|_{L_t^\infty L_x^2}\|u_{I_{j_1}}u_{2\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
\lesssim&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}\langle\lambda_0\rangle^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{-\lambda_0-I_{j_2}}\|_{V_A^2}\\
&\quad\times T^{\varepsilon/4}(\langle\lambda_0\rangle+2^{j_3})^{-1+2\varepsilon}\|u_{I_{j_1}}\|_{V_A^2}\|u_{2\lambda_0+I_{j_3}}\|_{V_A^2}\\
\lesssim&T^{\varepsilon/4}\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{-1/4}(\langle\lambda_0\rangle+2^{j_3})^{-3/4+2\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/4}\\
&\qquad\times(2^{j_1})^{1/4}(2^{j_2})^{1/2}(2^{j_3})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
\lesssim&T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},
\end{aligned}\tag{4.53}$$

where the last inequality is obtained by summing over j_1, j_2 and j_3. Indeed we have the following estimates:

$$\sum_{j_1\leq\log_2\langle\lambda_0\rangle}(2^{j_1})^{1/4}\lesssim\langle\lambda_0\rangle^{1/4};\quad\sum_{j_2\geq 0}(\langle\lambda_0\rangle+2^{j_2})^{-3/4}(2^{j_2})^{1/2}\leq\sum_{j_2\geq 0}(2^{j_2})^{-1/4}\lesssim 1;$$
$$\sum_{j_3\geq 0}(\langle\lambda_0\rangle+2^{j_3})^{-3/4+2\varepsilon}(2^{j_3})^{1/2}\leq\sum_{j_3\geq 0}(2^{j_3})^{-1/4+2\varepsilon}\lesssim 1,\quad 0<\varepsilon<1/8.$$

If $u_{-\lambda_0-I_{j_2}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$ and $L^2_{x,t}$ norms to v_{λ_0}, $u_{-\lambda_0-I_{j_2}}$, and $u_{I_{j_1}}u_{2\lambda_0+I_{j_3}}$, respectively. Then it will be same as (4.53).

If $u_{2\lambda_0+I_{j_3}}$ has the highest dispersion modulation, we take dispersion modulation decay to $u_{2\lambda_0+I_{j_3}}$. For $u_{I_{j_1}}$ and $u_{-\lambda_0-I_{j_2}}$, we have $|\lambda_0+2^{j_2}+2^{j_1}|\gtrsim(\langle\lambda_0\rangle+2^{j_2})$ and $|\lambda_0+2^{j_2}-2^{j_1}|\gtrsim 2^{j_2}$. Thus we can use bilinear estimate (3.7) to $u_{I_{j_1}}u_{-\lambda_0-I_{j_3}}$. To be specific, we have

$$\begin{aligned}
&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-\lambda_0-I_{j_2}}\partial_x u_{2\lambda_0+I_{j_3}}|\,dxdt\\
\lesssim&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{2\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\|u_{I_{j_1}}u_{-\lambda_0-I_{j_2}}\|_{L^2_{x,t}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{j_2\approx j_1\vee j_3} \langle\lambda_0\rangle^{1/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}\|u_{2\lambda_0+I_{j_3}}\|_{V_A^2}\\
&\qquad\times T^{\varepsilon/4}(\langle\lambda_0\rangle+2^{j_2})^{-1/2+\varepsilon}(2^{j_2})^{-1/2+\varepsilon}\|u_{I_{j_1}}\|_{V_A^2}\|u_{-\lambda_0-I_{j_2}}\|_{V_A^2}\\
&\lesssim T^{\varepsilon/4}\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{-1/4}(\langle\lambda_0\rangle+2^{j_3})^{1/4}(\langle\lambda_0\rangle+2^{j_2})^{-5/4+\varepsilon}\\
&\qquad\times(2^{j_1})^{1/4}(2^{j_2})^{\varepsilon}(2^{j_3})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},
\end{aligned}
$$

where the last inequality is obtained by summing over j_1, j_2 and j_3 in order.

If $u_{I_{j_1}}$ has the highest dispersion modulation, from the dispersion modulation decay (2.4) and L^4 estimate (3.13), we have

$$
\begin{aligned}
&\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{I_{j_1}}u_{-\lambda_0-I_{j_2}}\partial_x u_{2\lambda_0+I_{j_3}}|dxdt\\
&\lesssim \sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{I_{j_1}}\|_{L^2_{x,t}}\|u_{-\lambda_0-I_{j_2}}\|_{L^4_{x,t}}\|u_{2\lambda_0+I_{j_3}}\|_{L^4_{x,t}}\\
&\lesssim \sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{1/4}\|v_{\lambda_0}\|_{V_A^2}\langle\lambda_0\rangle^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}(2^{j_1})^{1/4}\|u\|_{X^{1/4}_{\infty,A}}\\
&\qquad\times T^{\varepsilon/4}(2^{j_2})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/8}T^{\varepsilon/4}(2^{j_3})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_3})^{-3/8}\|u\|^2_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/2}\sum_{j_2\approx j_1\vee j_3}\langle\lambda_0\rangle^{-1/4}(2^{j_1})^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{1/8}(\langle\lambda_0\rangle+2^{j_2})^{-7/8}\\
&\qquad\times(2^{j_2})^{1/4+\varepsilon}(2^{j_3})^{1/4+\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}},
\end{aligned}
$$

where the last inequality is obtained by summing over j_1, j_2 and j_3 in order and noticing the condition $j_1\le\log_2\langle\lambda_0\rangle$, $j_3\le j_2$.

Case $2h\_l\_h$. We decompose $\lambda_1,\lambda_2,\lambda_3$ as follows:

$$
\lambda_1\in[-\lambda_0,0]=\bigcup_{j_1\ge 0}-I_{j_1},\quad j_1\le\log_2\langle\lambda_0\rangle;
$$

$$
\lambda_2\in[-\infty,-\lambda_0]=\bigcup_{j_2\ge 0}-\lambda_0-I_{j_2},\quad \lambda_3\in[2\lambda_0-l,+\infty]=\bigcup_{j_3\ge -1}2\lambda_0+I_{j_3}.
$$

From the frequency constraint condition (4.8), we have

$$
2^{j_3}\approx 2^{j_1}+2^{j_2},\quad \text{i.e.}\quad j_3\approx j_1\vee j_2. \tag{4.54}
$$

If $j_3\approx j_2\ge j_1$, the method of this case will be same with Case $2h\_lh2$. If $j_3\approx j_1\ge j_2$, it is to say that $0\le j_2\le j_3\approx j_1\le\log_2\langle\lambda_0\rangle$ holds, which can also ensure the convergence of the summation in Case $2h\_lh2$. Therefore, the details are omitted.

Case $2h\_h\_h$. We decompose $\lambda_1,\lambda_2,\lambda_3$ in the following way:

$$
\lambda_k\in[-\infty,-\lambda_0]=\bigcup_{j_k\ge 0}-\lambda_0-I_{j_k}, k=1,2;\quad \lambda_3\in[3\lambda_0-l,+\infty]=\bigcup_{j_3\ge -1}3\lambda_0+I_{j_3}.
$$

From the frequency constraint condition (4.8) and $\lambda_1 \ge \lambda_2$, we have

$$2^{j_3} \approx 2^{j_1} + 2^{j_2}, \ j_1 \le j_2 \quad \text{i.e.} \quad j_3 \approx j_2 \ge j_1. \tag{4.55}$$

By DMCC (4.9) the highest dispersion modulation satisfies

$$\max_{0\le k\le 3} |\xi_k^3 - \tau_k| \gtrsim (\langle\lambda_0\rangle + 2^{j_1})\cdot(\langle\lambda_0\rangle + 2^{j_2})\cdot(\langle\lambda_0\rangle + 2^{j_3}). \tag{4.56}$$

If v_{λ_0} has the highest dispersion modulation, from the dispersion modulation decay (2.4), L^4 estimate (3.13), and Lemma 3.4, we have

$$\begin{aligned}
&\sum_{j_3\approx j_2\ge j_1} \langle\lambda_0\rangle^{1/4} \int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{-\lambda_0-I_{j_1}} u_{-\lambda_0-I_{j_2}} \partial_x u_{3\lambda_0+I_{j_3}}| dxdt \\
&\lesssim \sum_{j_3\approx j_2\ge j_1} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle + 2^{j_3}) \|\overline{v_{\lambda_0}}\|_{L_t^2 L_x^\infty} \|u_{-\lambda_0-I_{j_1}}\|_{L_t^\infty L_x^2} \|u_{-\lambda_0-I_{j_2}}\|_{L^4_{x,t}} \|u_{3\lambda_0+I_{j_3}}\|_{L^4_{x,t}} \\
&\lesssim \sum_{j_3\approx j_2\ge j_1} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_1})^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}\|v_{\lambda_0}\|_{V_A^2}(2^{j_1})^{1/2} \\
&\qquad \times (\langle\lambda_0\rangle+2^{j_1})^{-1/4} T^{\varepsilon/2}(2^{j_2})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/8}(2^{j_3})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_3})^{-3/8}\|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2} \sum_{j_3\ge j_1} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-3/4}(2^{j_3})^{1/2+2\varepsilon}(\langle\lambda_0\rangle+2^{j_1})^{-3/4}(2^{j_1})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2}\left(\sum_{j_3\ge 0}(2^{j_3})^{-1/4+2\varepsilon}\cdot j_3\right)\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}.
\end{aligned} \tag{4.57}$$

If $u_{-\lambda_0-I_{j_1}}$ has the highest dispersion modulation, we take $L^\infty_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$ and $L^4_{x,t}$ norms to v_{λ_0}, $u_{-\lambda_0-I_{j_1}}$, $u_{-\lambda_0-I_{j_2}}$ and $u_{3\lambda_0+I_{j_3}}$, respectively. Then it will be same as (4.57).

If $u_{3\lambda_0+I_{j_3}}$ has the highest dispersion modulation, we divide the left-hand side of (4.7) into two terms.

$$\begin{aligned}
&\sum_{j_3\approx j_2\ge j_1} \langle\lambda_0\rangle^{1/4} \int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{-\lambda_0-I_{j_1}} u_{-\lambda_0-I_{j_2}} \partial_x u_{3\lambda_0+I_{j_3}}| dxdt \\
&\le \left(\sum_{j_3\approx j_2\approx j_1\ge 0} + \sum_{j_3\approx j_2\gg j_1\ge 0}\right)\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}} |\overline{v_{\lambda_0}} u_{-\lambda_0-I_{j_1}} u_{-\lambda_0-I_{j_2}} \partial_x u_{3\lambda_0+I_{j_3}}| dxdt \\
&:= I_1(u,v) + I_2(u,v).
\end{aligned} \tag{4.58}$$

For $I_1(u,v)$, from the dispersion modulation decay (2.4), L^4 estimate (3.13) and Lemma 3.4, we have

$$\begin{aligned}
&I_1(u,v) \\
&\lesssim \sum_{j_3\approx j_2\approx j_1\ge 0} \langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{3\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\|u_{-\lambda_0-I_{j_1}}\|_{L^4_{x,t}}\|u_{-\lambda_0-I_{j_2}}\|_{L^4_{x,t}} \\
&\lesssim \sum_{j_3\approx j_2\approx j_1\ge 0} \langle\lambda_0\rangle^{1/4}\|v_{\lambda_0}\|_{V_A^2}(\langle\lambda_0\rangle+2^{j_1})^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/4}(2^{j_3})^{1/2} \\
&\qquad \times T^{\varepsilon/2}(2^{j_1})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_1})^{-3/8}(2^{j_2})^{1/4+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-3/8}\|u\|^3_{X^{1/4}_{\infty,A}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim T^{\varepsilon/2}\sum_{j_3\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-3/2}(2^{j_3})^{1+2\varepsilon}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/2}\left(\sum_{j_3\geq 0}(2^{j_3})^{-1/4+2\varepsilon}\right)\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/2}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}. \qquad (4.59)
\end{aligned}
$$

For $I_2(u,v)$, due to $j_2\gg j_1$, we have $|-\lambda_0-2^{j_2}-\lambda_0-2^{j_1}|\gtrsim(\langle\lambda_0\rangle+2^{j_2})$ and $|-\lambda_0-2^{j_2}+\lambda_0+2^{j_1}|\gtrsim 2^{j_2}$, so we can use bilinear estimate (3.7) to $u_{-\lambda_0-I_{j_1}}$ and $u_{-\lambda_0-I_{j_2}}$. To be specific,

$$
\begin{aligned}
I_2(u,v)&\lesssim\sum_{j_3\approx j_2\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{3\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\|u_{-\lambda_0-I_{j_1}}u_{-\lambda_0-I_{j_2}}\|_{L^2_{x,t}}\\
&\lesssim\sum_{j_3\approx j_2\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}\|v_{\lambda_0}\|_{V_A^2}(\langle\lambda_0\rangle+2^{j_1})^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}\\
&\qquad\times(2^{j_3})^{1/2}(\langle\lambda_0\rangle+2^{j_3})^{-1/4}\|u\|_{X^{1/4}_{\infty,A}}T^{\varepsilon/4}(2^{j_2})^{-1/2+\varepsilon}(\langle\lambda_0\rangle+2^{j_2})^{-1/2+\varepsilon}\\
&\qquad\times(2^{j_1})^{1/2}(\langle\lambda_0\rangle+2^{j_1})^{-1/4}(2^{j_2})^{1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/4}\|u\|^2_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\sum_{j_3\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-1+\varepsilon}(2^{j_3})^{1/2+\varepsilon}(\langle\lambda_0\rangle+2^{j_1})^{-1/4}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\left(\sum_{j_3\geq 0}(2^{j_3})^{-1/2+2\varepsilon}\cdot j_3\right)\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}. \qquad (4.60)
\end{aligned}
$$

If $u_{-\lambda_0-I_{j_2}}$ has the highest dispersion modulation, we still divide the left-hand side of (4.7) into two terms as (4.58). For $I_1(u,v)$, because of $j_3\approx j_2\approx j_1$, the estimate is exactly same as (4.59). For $I_2(u,v)$, we use the bilinear estimate (3.7) to $u_{-\lambda_0-I_{j_1}}$ and $u_{3\lambda_0+I_{j_3}}$. Noticing $|3\lambda_0+2^{j_3}\pm(\lambda_0+2^{j_1})|\gtrsim(\langle\lambda_0\rangle+2^{j_3})$, we have

$$
\begin{aligned}
&I_2(u,v)\\
&\lesssim\sum_{j_3\approx j_2\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})\|\overline{v_{\lambda_0}}\|_{L^\infty_{x,t}}\|u_{-\lambda_0-I_{j_2}}\|_{L^2_{x,t}}\|u_{-\lambda_0-I_{j_1}}u_{3\lambda_0+I_{j_3}}\|_{L^2_{x,t}}\\
&\lesssim\sum_{j_3\approx j_2\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}\|v_{\lambda_0}\|_{V_A^2}(\langle\lambda_0\rangle+2^{j_1})^{-1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/2}(\langle\lambda_0\rangle+2^{j_3})^{1/2}\\
&\qquad\times(2^{j_2})^{1/2}(\langle\lambda_0\rangle+2^{j_2})^{-1/4}\|u\|_{X^{1/4}_{\infty,A}}T^{\varepsilon/4}(\langle\lambda_0\rangle+2^{j_3})^{-1+2\varepsilon}\\
&\qquad\times(2^{j_1})^{1/2}(\langle\lambda_0\rangle+2^{j_1})^{-1/4}(2^{j_3})^{1/2}(\langle\lambda_0\rangle+2^{j_3})^{-1/4}\|u\|^2_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\sum_{j_3\gg j_1\geq 0}\langle\lambda_0\rangle^{1/4}(\langle\lambda_0\rangle+2^{j_3})^{-3/2+2\varepsilon}2^{j_3}(\langle\lambda_0\rangle+2^{j_1})^{-3/4}(2^{j_1})^{1/2}\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\left(\sum_{j_3\geq 0}(2^{j_3})^{-1/2+2\varepsilon}\cdot j_3\right)\|v_{\lambda_0}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}\\
&\lesssim T^{\varepsilon/4}\|v^{(\lambda_0)}\|_{V_A^2}\|u\|^3_{X^{1/4}_{\infty,A}}. \qquad (4.61)
\end{aligned}
$$

4.2. $q<\infty$**, Proof of** (4.6)**.** This subsection $q<\infty$ is similar to the last subsection $q=\infty$, the only difference is to deal with the summation of λ_0. The frequency constraint condition (FCC) and dispersion modulation constraint condition (DMCC) are same. Thus, we can use the exactly same assortment to $\lambda_0,\cdots,\lambda_3$. Next we take the Case 1 of Step 1 in last subsection for example.

We just denote the left-hand side of (4.6) as $\mathscr{L}_{hhhh_-}(u,v)$, and divide it into three parts like the last subsection. For $\mathscr{L}^l_{hhhh_-}(u,v)$, $\lambda_0\approx\lambda_3\approx\lambda_1\approx-\lambda_2$ holds. Thus, from Hölder's inequality and Strichartz estimate, we have

$$\begin{aligned}\mathscr{L}^l_{hhhh_-}(u,v)&\lesssim\sum_{\lambda_0}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^4_{x,t}}\|u_{\lambda_0}\|^2_{L^4_{x,t}}\|u_{-\lambda_0}\|_{L^4_{x,t}}\\&\lesssim T^{1/2}\sum_{\lambda_0}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L^8_tL^4_x}\|u_{\lambda_0}\|^2_{L^8_tL^4_x}\|u_{-\lambda_0}\|_{L^8_tL^4_x}\\&\lesssim T^{1/2}\sum_{\lambda_0}\langle\lambda_0\rangle^{3/4}\|\overline{v_{\lambda_0}}\|_{V^2_A}\|u_{\lambda_0}\|^2_{U^2_A}\|u_{-\lambda_0}\|_{U^2_A}\\&\lesssim T^{1/2}\|v\|_{Y^0_{q',A}}\|u\|^3_{X^{1/4}_{q,A}}.\end{aligned}$$

For $\mathscr{L}^m_{hhhh_-}(u,v)$, we still use bilinear estimate (3.7), Lemma 3.4, and Hölder's inequality to obtain that for $0<\varepsilon<1/4q$,

$$\begin{aligned}\mathscr{L}^m_{hhhh_-}(u,v)\lesssim&\sum_{\lambda_0,j_3\lesssim1\ll j_1\approx j_2}\langle\lambda_0\rangle^{1/4}\|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}\|_{L^2_{x,t}}\|u_{-\lambda_0+I_{j_2}}\partial_xu_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\\\lesssim&T^{\varepsilon/2}\sum_{\lambda_0,j_3\lesssim1\ll j_1\approx j_2}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_1})^{-1/2+\varepsilon}\|v_{\lambda_0}\|_{V^2_A}\|u_{\lambda_0-I_{j_1}}\|_{V^2_A}\\&\times\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_2})^{-1/2+\varepsilon}\|u_{-\lambda_0+I_{j_2}}\|_{V^2_A}\|u_{\lambda_0-I_{j_3}}\|_{V^2_A}\\\lesssim&T^{\varepsilon/2}\sum_{\lambda_0,j_3\lesssim1\ll j_1\approx j_2}\langle\lambda_0\rangle^{-1/2+2\varepsilon}(2^{j_1})^{\varepsilon-1/q}(2^{j_2})^{\varepsilon-1/q}(2^{j_3})^{1/2-1/q}\\&\times\|v_{\lambda_0}\|_{V^2_A}\|u\|_{X^{1/4}_{q,A}(\lambda_0-I_{j_3})}\|u\|^2_{X^{1/4}_{q,A}}.\end{aligned}$$

Making the summation on j_1,j_2, then applying Hölder's inequality on λ_0, and finally summing on j_3, we obtain

$$\begin{aligned}\mathscr{L}^m_{hhhh_-}(u,v)\lesssim&T^{\varepsilon/2}\sum_{\lambda_0,j_3\le\log_2\langle\lambda_0\rangle}(2^{j_3})^{4\varepsilon-3/q}\|v_{\lambda_0}\|_{V^2_A}\|u\|_{X^{1/4}_{q,A}(\lambda_0-I_{j_3})}\|u\|^2_{X^{1/4}_{q,A}}\\\lesssim&T^{\varepsilon/2}\|v\|_{Y^0_{q',A}}\|u\|^3_{X^{1/4}_{q,A}}.\end{aligned}$$

For $\mathscr{L}^h_{hhhh_-}(u,v)$, we just take the case that v_{λ_0} has the highest dispersion modulation for example and divide $\mathscr{L}^h_{hhhh_-}(u,v)$ into two parts:

$$\begin{aligned}&\mathscr{L}^{h1}_{hhhh_-}(u,v)+\mathscr{L}^{h2}_{hhhh_-}(u,v)\\:=&\Bigg(\sum_{\lambda_0,1\ll j_3\ll j_1\approx j_2}+\sum_{\lambda_0,1\ll j_3\approx j_1\approx j_2}\Bigg)\langle\lambda_0\rangle^{1/4}\int_{[0,T]\times\mathbb{R}}|\overline{v_{\lambda_0}}u_{\lambda_0-I_{j_1}}u_{-\lambda_0+I_{j_2}}\partial_xu_{\lambda_0-I_{j_3}}|\,dxdt.\end{aligned}$$

For $\mathscr{L}^{h1}_{hhhh_-}(u,v)$, we have for $0<\varepsilon<1/4q$,

$$\begin{aligned}&\mathscr{L}^{h1}_{hhhh_-}(u,v)\\\lesssim&\sum_{\lambda_0,1\ll j_3\ll j_1\approx j_2}\langle\lambda_0\rangle^{1/4}\|\overline{v_{\lambda_0}}\|_{L^2_tL^\infty_x}\|u_{\lambda_0-I_{j_1}}\|_{L^\infty_tL^2_x}\|u_{-\lambda_0+I_{j_2}}\partial_xu_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{\lambda_0,1\ll j_3\ll j_1\approx j_2} \langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{\lambda_0-I_{j_1}}\|_{V_A^2}\\
&\qquad \times T^{\varepsilon/4}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_2})^{-1/2+\varepsilon}\|u_{-\lambda_0+I_{j_2}}\|_{V_A^2}\|u_{\lambda_0-I_{j_3}}\|_{V_A^2}\\
&\lesssim T^{\varepsilon/4}\sum_{\lambda_0,1\ll j_3\ll j_1\approx j_2}\langle\lambda_0\rangle^{-1/2+\varepsilon}(2^{j_2})^{\varepsilon-2/q}(2^{j_3})^{-1/q}\|v_{\lambda_0}\|_{V_A^2}\|u\|_{X^{1/4}_{q,A}(\lambda_0-I_{j_3})}\|u\|^2_{X^{1/4}_{q,A}}\\
&\lesssim T^{\varepsilon/4}\sum_{\lambda_0,1\ll j_3\le \log_2\langle\lambda_0\rangle}(2^{j_3})^{-1/2+2\varepsilon-3/q}\|v_{\lambda_0}\|_{V_A^2}\|u\|_{X^{1/4}_{q,A}(\lambda_0-I_{j_3})}\|u\|^2_{X^{1/4}_{q,A}}\\
&\lesssim T^{\varepsilon/4}\|v\|_{Y^0_{q',A}}\|u\|^3_{X^{1/4}_{q,A}}.
\end{aligned}
$$

For $\mathscr{L}^{h2}_{hhhh_-}(u,v)$, we know that for $0<\varepsilon<1/4q$,

$$
\begin{aligned}
&\mathscr{L}^{h2}_{hhhh_-}(u,v)\\
&\lesssim \sum_{\lambda_0,1\ll j_3\approx j_1\approx j_2}\langle\lambda_0\rangle^{5/4}\|\overline{v_{\lambda_0}}\|_{L_t^2L_x^\infty}\|u_{\lambda_0-I_{j_1}}\|_{L_t^\infty L_x^2}\|u_{-\lambda_0+I_{j_2}}\|_{L^4_{x,t}}\|u_{\lambda_0-I_{j_3}}\|_{L^4_{x,t}}\\
&\lesssim \sum_{\lambda_0,1\ll j_3\approx j_1\approx j_2}\langle\lambda_0\rangle^{5/4}\langle\lambda_0\rangle^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}\|v_{\lambda_0}\|_{V_A^2}\|u_{\lambda_0-I_{j_1}}\|_{V_A^2}\\
&\qquad \times T^{\varepsilon/2}\langle\lambda_0\rangle^{-3/4}(2^{j_2})^{1/4-1/q+\varepsilon}(2^{j_3})^{1/4-1/q+\varepsilon}\|u\|^2_{X^{1/4}_{q,A}}\\
&\lesssim T^{\varepsilon/2}\sum_{\lambda_0,j_1}(2^{j_1})^{2\varepsilon-3/q}\|v_{\lambda_0}\|_{V_A^2}\|u\|_{X^{1/4}_{q,A}(\lambda_0-I_{j_1})}\|u\|^2_{X^{1/4}_{q,A}}\\
&\lesssim T^{\varepsilon/2}\|v\|_{Y^0_{q',A}}\|u\|^3_{X^{1/4}_{q,A}}.
\end{aligned}
$$

Where the last inequality is obtained by applying Hölder's inequality on λ_0. For other cases, we can take a similar calculation to get the desired estimates, thus we omit it. □

5. Ill-posedness result

In this section we study the Cauchy problem of the defocusing mKdV equation (the focusing case can also be treated by our method):

$$u_t+u_{xxx}-(u^3)_x=0,\quad u(0,x)=\delta u_0. \tag{5.1}$$

We have the ill-posedness result as follows.

THEOREM 5.1. *Let $s<1/4$, $2\le q\le\infty$, $0<\delta\ll 1$. Then for the mKdV Equation (5.1), the solution map $\delta u_0\to u(\delta,t)$ in $M^s_{2,q}$ is not C^3 continuous at the origin.*

Proof. From (4.1) we can define the solution map as follows:

$$\mathcal{T}:\quad \delta u_0\to u(\delta,t)=e^{-t\partial_x^3}\delta u_0+\int_0^t e^{-(t-\tau)\partial_x^3}(u^3)_x(\tau)\,d\tau. \tag{5.2}$$

By straightforward calculations, we get

$$u(\delta,t)|_{\delta=0}=0;\quad u_1:=\frac{\partial u}{\partial\delta}\Big|_{\delta=0}=e^{-t\partial_x^3}u_0;\quad u_2:=\frac{\partial^2 u}{\partial\delta^2}\Big|_{\delta=0}=0; \tag{5.3}$$

$$u_3:=\frac{\partial^3 u}{\partial\delta^3}\Big|_{\delta=0}=6\int_0^t e^{-(t-\tau)\partial_x^3}\partial_x(e^{-\tau\partial_x^3}u_0)^3\,d\tau. \tag{5.4}$$

It is well known that if the map $\delta u_0 \to u(\delta)$ is of class C^3 at the origin, the necessary condition is

$$\sup_{t\in[0,T]} \|u_3\|_{M^s_{2,q}} \le C\|u_0\|^3_{M^s_{2,q}}. \tag{5.5}$$

We choose a suitable $u_0 \in M^s_{2,q}$, $s<1/4$ defined by

$$\widehat{u_0}(\xi) = N^{-s+1/4}\big(\chi_{[N,N+\frac{1}{\sqrt{N}}]}(\xi)+\chi_{[-N-\frac{1}{\sqrt{N}},-N]}(\xi)\big).$$

Note that $\|u_0\|_{M^s_{2,q}} \sim 1$.

We estimate the Fourier transform of u_3 in (5.4) as follows

$$\begin{aligned}
\widehat{u_3}(\xi) &\simeq \int_0^t e^{\mathrm{i}(t-\tau)\xi^3}(\mathrm{i}\xi)\int_{\mathbb{R}^2} e^{\mathrm{i}\tau(\xi-\xi_1-\xi_2)^3}\widehat{u_0}(\xi-\xi_1-\xi_2)e^{\mathrm{i}\tau\xi_1^3}\widehat{u_0}(\xi_1)e^{\mathrm{i}\tau\xi_2^3}\widehat{u_0}(\xi_2)d\xi_1 d\xi_2 d\tau \\
&\simeq e^{\mathrm{i}t\xi^3}(\mathrm{i}\xi)\int_{\mathbb{R}^2}\int_0^t e^{\mathrm{i}\tau\Phi(\xi,\xi_1,\xi_2)}d\tau\, \widehat{u_0}(\xi-\xi_1-\xi_2)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi_2)d\xi_1 d\xi_2 \\
&\simeq e^{\mathrm{i}t\xi^3}(\mathrm{i}\xi)\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}t\Phi(\xi,\xi_1,\xi_2)}-1}{\mathrm{i}\Phi(\xi,\xi_1,\xi_2)}\widehat{u_0}(\xi-\xi_1-\xi_2)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi_2)d\xi_1 d\xi_2,
\end{aligned} \tag{5.6}$$

where $\Phi(\xi,\xi_1,\xi_2) = -3(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)$. Noticing that for $\xi-\xi_1-\xi_2$, ξ_1, and ξ_2, if one or two of these items are located in $[N, N+1/\sqrt{N}]$, we have $|\Phi(\xi,\xi_1,\xi_2)| \lesssim 1$; if all the three items are located in $[N,N+1/\sqrt{N}]$ (or $[-N-1/\sqrt{N},-N]$), we have $|\Phi(\xi,\xi_1,\xi_2)| \sim N^3$, then (5.6) shall be much smaller. Therefore,

$$\begin{aligned}
\widehat{u_3}(\xi) &\simeq N^{-3s+3/4}e^{\mathrm{i}t\xi^3}(\mathrm{i}\xi)\int_{\mathbb{R}^2}\frac{e^{\mathrm{i}t\Phi(\xi,\xi_1,\xi_2)}-1}{\mathrm{i}\Phi(\xi,\xi_1,\xi_2)} \\
&\qquad \times \chi_{[N,N+\frac{1}{\sqrt{N}}]}(\xi-\xi_1-\xi_2)\chi_{[N,N+\frac{1}{\sqrt{N}}]}(\xi_1)\chi_{[-N-\frac{1}{\sqrt{N}},-N]}(\xi_2)d\xi_1 d\xi_2 \\
&\simeq N^{-3s+3/4}e^{\mathrm{i}t\xi^3}\xi\int_N^{N+\frac{1}{\sqrt{N}}}\int_{-N-\frac{1}{\sqrt{N}}}^{-N} te^{\mathrm{i}t\theta}\cdot\chi_{[N,N+\frac{1}{\sqrt{N}}]}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2,
\end{aligned}$$

where $\theta\in[0,\Phi]$ or $[\Phi,0]$, $|\Phi(\xi,\xi_1,\xi_2)| = O(1)$, $\xi\in[N-1/\sqrt{N}, N+2/\sqrt{N}]$. Thus there exists a small and fixed constant t such that

$$\|u_3\|_{M^s_{2,q}} \ge CN^{(-2s+1/2)} \quad (2\le q\le\infty).$$

We find that (5.5) leads to

$$-2s+1/2\le 0 \quad \text{i.e.} \quad s\ge 1/4.$$

Now we complete the proof. ∎

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