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GLOBAL STABILITY OF LARGE SOLUTIONS TO THE 3–D COMPRESSIBLE FLOW OF LIQUID CRYSTALS*

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Abstract. The current paper is devoted to the investigation of the global-in-time stability of large solutions to the compressible liquid crystal equations in the whole space. Suppose that the density is bounded from above uniformly in time in the Höder space C^α with α sufficiently small and in L^∞ space respectively. Then we prove two results: (1) Such kind of the solution will converge to its associated equilibrium with a rate which is the same as that for the heat equation. (2) Such kind of the solution is stable, which means any perturbed solution will remain close to the reference solution if initially they are close to each other. This implies that the set of the smooth and bounded solutions is open.

Keywords. compressible liquid crystal; large solutions; stability.

AMS subject classifications. 35A01; 35B45; 35Q35; 76A05; 76D03.

1. Introduction

In this paper, we are concerned with the global stability of the large solutions to 3−D compressible liquid crystal equations:

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p=\operatorname{div}T-\nabla d\cdot\triangle d, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \partial_t d+u\cdot\nabla d=\triangle d+|\nabla d|^2 d, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \lim\limits_{|x|\to\infty}(\rho,u,d)=(1,0,\underline{d}), \qquad t\in\mathbb{R}^+, \end{cases} \tag{1.1}$$

where $\rho=\rho(t,x)\in\mathbb{R}^+$ denotes the density, $u=u(t,x)\in\mathbb{R}^3$ denotes the velocity, $d(t,x)\in\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, represents the macroscopic average of the nematic liquid crystal orientation field, and the pressure p is given by smooth function $p=p(\rho)$. Here we take $p(\rho)=\rho^\gamma$ with the adiabatic exponent $\gamma\geq1$. And T is the stress tensor given by $T=\mu(\nabla u+(\nabla u)^t)+\lambda(\operatorname{div}u)I$ with I the identity matrix. The viscosity coefficients of the flow satisfy $\mu>0$ and $2\mu+3\lambda>0$. And $\underline{d}$ is a unit constant vector in $\mathbb{S}^2$.

The above system is basically a coupling of compressible Navier-Stokes equations and parabolic heat flow. When $d\equiv\underline{d}$, the system (1.1) reduces to the well-known Navier-Stokes equations for compressible isentropic flows which have been studied by many researchers, see [3,4,10] and the references therein. When considering the compressible nematic liquid crystal flow under the assumption that the director d has variable degrees of orientations, the global existence of weak solutions in $\mathbb{R}^3$ has been obtained by [14] and [19]. Inspired by the work of [12] for parabolic incompressible flow, the corresponding global finite-energy weak solutions to (1.1) was proved in [13]. The local existence of strong solutions in $\mathbb{R}^3$ has been studied by [8] and [9]. Recently, Chen and Zhai [2]

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established the global solutions and incompressible limit supplemented with arbitrary large initial velocity and almost constant density, for large volume (bulk) viscosity.

In the present work, we are interested in the following two problems for the system (1.1): (i) What is the long-time behavior of the solution to (1.1)? (ii) Which kind of the solution to (1.1) is stable?

Obviously these two problems are fundamental for (1.1). However both of them are not solved well. The main obstruction comes from the existence of global smooth solution. So far, there has been a large amount of literature on this issue, but most of the results are restricted to the perturbation framework. In other words, the global solution and its long-time behavior is considered near the equilibrium. We refer readers to [11]. Because of this restriction, the method on the global dynamics and the stability of (1.1) relies heavily on the analysis of the linearization of the system. The interested reader is referred to [1,5,15,16,20] and the references therein for details. These results can be summarized as follows. Assume that the initial data (ρ_0,u_0,d_0) is a small perturbation of equilibrium $(1,0,\underline{d})$ in $\left(L^1\cap H^3\right)\times\left(L^1\cap H^3\right)\times\left(L^1\cap H^3\right)$. Then it can be proved that

$$\|(\rho-1,u,\nabla(d-\underline{d}))(t)\|_{L^2}\le C(1+t)^{-\frac{3}{4}}. \tag{1.2}$$

This shows that in the close-to-equilibrium setting the rate of the convergence of the solution is the same as that for the heat equations if we put the same condition on the initial data. In this sense, (1.2) can be regarded as the optimal decay estimate for system (1.1).

The aim of the paper is to investigate the long-time behavior and the global-in-time stability of the solution to (1.1) for general initial data. The global existence of this kind of solution is not the purpose of this paper. We refer readers to [2] and [6].

To obtain the long-time behavior and the global-in-time stability, we need to impose some assumptions on the solution itself which at first looks unsatisfactory. Our key observation is as follows. The basic energy identity shows that the system has a dissipation structure which is not complete. In the case of liquid crystal system, there is no dissipation for the density. However the coupling effect behind the system helps us to obtain the dissipation for the density. Then the system will look like a heat equation. By time-frequency splitting method, we can get the global dynamics: the propagation of the smoothness and the convergence to equilibrium with the same rate as the one in the result obtained by the linearization method. More explanation is given below.

Inspired by [6,7], we separate the process to get the stability of liquid crystal system into three steps. The first step is to get the uniform-in-time bounds for the propagation of the regularity. Because of the induction equation, we need to involve some new methods which come from the corresponding blow-up results (see [9,18]). Because of the definition of the effective viscosity flux G which contains the orientation field now, some new terms from $(1.1)_3$ come out, see Lemma 2.4. To overcome that, we not only need to apply methods from [6,7], but also use the structure of the system.

Each time when the uniform-in-time bounds for the regularity of the solution are improved, the dissipation inequality can also be improved correspondingly. Thanks to this observation, finally we obtain that

$$\frac{d}{dt}\|(\rho-1,u,\nabla(d-\underline{d}))\|^2+\|\nabla(\rho-1,u,\nabla(d-\underline{d}))\|^2\le 0,$$

which enjoys the same structure as that for the heat equation. Now the time-frequency splitting method (see [17]) can be applied to get that

$$\frac{d}{dt}\|(\rho-1,u,\nabla(d-\underline{d}))\|^2+\frac{1}{1+t}\|(\rho-1,u,\nabla(d-\underline{d}))\|^2$$
$$\leq\frac{1}{1+t}\int_{|\xi|\leq(1+t)^{-\frac{1}{2}}}\left|(\widehat{\rho-1},\hat{u},\widehat{\nabla(d-\underline{d})})(\xi)\right|^2d\xi. \tag{1.3}$$

The problem of convergence is reduced to the estimate of the low-frequency part of the solution, which is easy to do for the linear equation. By making full use of the cancellation and the coupling effect of the system (1.1), we get the control of the right-hand side of (1.3). Then the optimal decay estimate (1.2) follows.

Once the global dynamics of the equations is clear, we can prove the global-in-time stability for the system (1.1) as follows:

(1) By the local well-posedness for the system (1.1), we show that the perturbed solution will remain close to the reference solution for a long time if initially they are close.
(2) The long-time behavior of the solution suggests that the reference solution is close to the equilibrium after a long time.
(3) Combining these two facts, we find a time t_0 such that t_0 is far away from the initial time and at this moment the solution is close to the equilibrium. Then the desired result follows from the standard perturbation framework.

Before we state our results, let us introduce the notations which are used throughout the paper. We use the notation $a\sim b$ whenever $a\leq C_1b$ and $b\leq C_2a$ where C_1 and C_2 are universal constants. We denote $C(\lambda_1,\lambda_2,\cdots,\lambda_n)$ by a constant depending on parameters $\lambda_1,\lambda_2,\cdots,\lambda_n$.

Now we are in a position to state our main results on the system (1.1). Our first result is concerned with the global dynamics of the system.

THEOREM 1.1. *Let $\mu>\frac{1}{2}\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1) with $0\leq\rho\leq M$, and initial data (ρ_0,u_0,d_0) verifying that $\rho_0\geq c>0$ and the admissible condition*

$$\begin{cases} u_t\big|_{t=0}=-u_0\cdot\nabla u_0+\frac{1}{\rho_0}(\mathrm{div}T_0-\nabla d_0\cdot\triangle d_0-\nabla\rho_0^\gamma),\\ d_t\big|_{t=0}=\triangle d_0+|\nabla d_0|^2d_0-u_0\cdot\nabla d_0,\end{cases} \tag{1.4}$$

and $\sup_{t\in\mathbb{R}^+}\|\nabla d(t,\cdot)\|_{L^\infty}+\sup_{t\in\mathbb{R}^+}\|\rho(t,\cdot)\|_{C^\alpha}\leq\mathcal{M}$ for some $0<\alpha<1$. Then if $(\rho_0-1,u_0,\nabla(d_0-\underline{d}))\in L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$, then there exists a constant $\underline{\rho}=\underline{\rho}(c,M,\mathcal{M})>0$ such that for all $t\geq0$, we have

$$\rho(t,x)\geq\underline{\rho}. \tag{1.5}$$

We have the uniform-in-time bounds for the regularity of the solutions, assuming that $n=\rho-1$, $m=d-\underline{d}$,

$$\|n\|_{H^2}^2+\|u\|_{H^2}^2+\|\nabla m\|_{H^2}^2+\int_0^\infty(\|\nabla n(\tau)\|_{H^1}^2+\|\nabla u(\tau)\|_{H^2}^2+\|\nabla^2m(\tau)\|_{H^2}^2)d\tau$$
$$\leq C(\underline{\rho},\mathcal{M},\|(n_0,u_0,\nabla m_0)\|_{L^1\cap H^2},\|d_0\|_{L^2}). \tag{1.6}$$

Moreover, we have the decay estimate for the solution

$$\|n\|_{H^1}+\|u\|_{H^1}+\|\nabla m\|_{H^1}\leq C(\underline{\rho},\mathcal{M},\|n_0\|_{L^1\cap H^1},\|(u_0,\nabla m_0)\|_{L^1\cap H^2},\|m_0\|_{L^2})(1+t)^{-\frac{3}{4}}. \tag{1.7}$$

Remark 1.1. Once the constants $\underline{\rho}$ and $\mathcal{M}$ are fixed, our theorem shows that all the upper bounds obtained in the theorem depend only on the initial data.

Remark 1.2. Since (1.7) implies (1.2), our decay estimate is optimal in some sense.

Remark 1.3. It is easy to verify the additional condition that the density belongs to a Hölder space, if the initial data (ρ_0,u_0,d_0) is a small perturbation of equilibrium $(1,0,\bar{d})$.

Remark 1.4. Here we don't consider the global existence of the solution with initial data which is far away from equilibrium. One can refer this in [2] and [6].

Our second result can be stated as follows:

Theorem 1.2. *Let $(\tilde\rho,\tilde u,\tilde d)$ be a global solution for (1.1) with the initial data $(\tilde\rho_0,\tilde u_0,\tilde d_0)$ verifying that*

$$\|\frac{1}{\tilde\rho},\tilde\rho,\nabla\tilde\rho\|_{H^4}+\|\tilde u_t,\tilde u,\nabla\tilde u\|_{H^4}+\|\tilde d,\nabla\tilde d\|_{H^4}\le C. \tag{1.8}$$

Assume that $(\rho_0-1,u_0,\nabla(d_0-\underline{d}))\in L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$. Then there exists an ϵ depending only on C such that

$$\|\rho_0-\tilde\rho_0\|_{H^4}+\|u_0-\tilde u_0\|_{H^4}+\|d_0-\tilde d_0\|_{H^4}\le\epsilon, \tag{1.9}$$

then (1.1) admits a global and unique solution (ρ,u,d) with the initial data (ρ_0,u_0,d_0). Moreover, for any $t>0$, we have

$$\|(\rho-\tilde\rho)(t)\|_{H^4}+\|(u-\tilde u)(t)\|_{H^4}+\|(d-\tilde d)(t)\|_{H^4}\le C\min\{(1+c|\ln\epsilon|)^{-\frac{3}{4}},\epsilon+(1+t)^{-\frac{3}{4}}\}, \tag{1.10}$$

where c is a constant independent of ϵ.

2. Global dynamics of the liquid crystal equations

2.1. Uniform-in-time bounds. In what follows, we will set $n=\rho-1$, $\mathfrak{n}=p-1$ and $m=d-\underline{d}$.

We first recall the basic energy identity for (1.1).

Lemma 2.1. *Let (ρ,u,d) be a global and smooth solution of (1.1), then the following equality holds*

$$\frac{d}{dt}\left(\int F(\rho|1)dx+\frac{1}{2}\int\rho|u|^2dx+\frac{1}{2}\int|\nabla m|^2dx\right)$$
$$+\mu\|\nabla u\|_{L^2}^2+(\mu+\lambda)\|\operatorname{div}u\|_{L^2}^2+\left\|\triangle m+|\nabla m|^2(m+\underline{d})\right\|_{L^2}^2=0, \tag{2.1}$$

where

$$F(\rho|1)=\begin{cases}\frac{1}{\gamma-1}(\rho^\gamma-1-\gamma(\rho-1)) & \text{for}\quad \gamma>1,\\ \rho\ln\rho-\rho+1, & \text{for}\quad \gamma=1.\end{cases} \tag{2.2}$$

Remark 2.1. By Taylor expansion, it is not difficult to check that $F(\rho|1)\ge C(M)(\rho-1)^2$ if $\rho\le M$.

LEMMA 2.2. *Let $\mu > \frac{1}{2}\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1), then the following inequality holds*

$$\begin{aligned}
&\frac{d}{dt}\left(\int \rho|u|^4dx+\int|\nabla m|^4dx\right)+\left\||u||\nabla u|\right\|_{L^2}^2+\left\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\right\|_{L^2}^2\\
&\quad+\left\||\nabla m||\nabla^2 m+|\nabla m|^2(m+\underline{d})|\right\|_{L^2}^2+\left\||\nabla m||\nabla|\nabla m||\right\|_{L^2}^2\\
&\le C\left(1+\|\nabla m\|_{L^\infty}\right)^2\left(\|\nabla u\|_{L^2}^2+\left\|\triangle m+|\nabla m|^2(m+\underline{d})\right\|_{L^2}^2\right)\\
&\quad\times\left(\|\mathfrak{n}\|_{L^6}^2+\|\nabla u\|_{L^2}^2+\|\nabla m\|_{H^1}^2\right),
\end{aligned}\tag{2.3}$$

where C is a positive constant depending on μ,λ.

Proof. Multiplying the second equation of (1.1) by $4|u|^2u$, we apply operator ∇ to the third equation of (1.1) and multiplying the resulting equation by $4|\nabla m|^2\nabla m$, summing up and integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(\int \rho|u|^4dx+\int|\nabla m|^4dx\right)-4\int|u|^2(\mu\triangle u+(\mu+\lambda)\nabla\operatorname{div}u)\cdot udx\\
&\qquad-4\int|\nabla m|^2\nabla(\triangle m+|\nabla m|^2(m+\underline{d}))\cdot(\nabla m)dx\\
&=-4\int|u|^2\nabla p\cdot udx-4\int|u|^2(\nabla m\cdot\triangle m)\cdot udx\\
&\qquad-4\int|\nabla m|^2\nabla(u\cdot\nabla m)\cdot\nabla mdx\overset{def}{=}\sum_{i=1}^3 I_i.
\end{aligned}\tag{2.4}$$

Using the inequality $\big|\nabla|u|\big|\le|\nabla u|$, and $\mu>\frac{1}{2}\lambda$, we have

$$\begin{aligned}
&-4\int|u|^2(\mu\triangle u+(\mu+\lambda)\nabla\operatorname{div}u)\cdot udx\\
&=4\int|u|^2\left(\mu|\nabla u|^2+(\mu+\lambda)(|\operatorname{div}u|-\big|\nabla|u|\big|)^2\right)+(\mu-\lambda)|u|^2\big|\nabla|u|\big|^2dx\\
&\ge C\int|u|^2|\nabla u|^2dx.
\end{aligned}$$

Using the relation $|m+\underline{d}|=|d|=1$, we have

$$\begin{aligned}
&-4\int|\nabla m|^2\nabla(\triangle m+|\nabla m|^2(m+\underline{d}))\cdot(\nabla m)dx\\
&=4\int|\nabla m|^2|\nabla^2m|^2dx+8\int|\nabla m|^2\big|\nabla|\nabla m|\big|^2dx\\
&\quad-2\int|\nabla m|^2\nabla|\nabla m|^2\cdot\nabla|m+\underline{d}|^2dx-4\int|\nabla m|^6dx\\
&\ge2\int|\nabla m|^2|\triangle m+|\nabla m|^2(m+\underline{d})|^2dx+2\int|\nabla m|^2|\nabla^2m+|\nabla m|^2(m+\underline{d})|^2dx\\
&\quad+8\int|\nabla m|^2\big|\nabla|\nabla m|\big|^2dx.
\end{aligned}$$

It is easy to check that

$$I_1\le C\int|\mathfrak{n}||u|^2|\nabla u|dx\le C\|\mathfrak{n}\|_{L^6}\|u\|_{L^6}^2\|\nabla u\|_{L^2}\le C\|\nabla u\|_{L^2}^2\left(\|\mathfrak{n}\|_{L^6}^2+\|\nabla u\|_{L^2}^2\right).$$

Similarly,

$$I_2 \le C\|\nabla m\|_{L^\infty}\|\nabla^2 m\|_{L^2}\|u\|_{L^6}^3 \le C\|\nabla m\|_{L^\infty}\|\nabla u\|_{L^2}^2\Big(\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2\Big),$$

and

$$\begin{aligned}
I_3 =& 4\int |\nabla m|^2(u\cdot\nabla m)\cdot(\triangle m+|\nabla m|^2(m+\underline{d}))dx+4\int \nabla(|\nabla m|^2)(u\cdot\nabla m)\cdot\nabla m dx\\
\le& C\|\nabla m\|_{L^\infty}\|u\|_{L^6}\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}\|\nabla m\|_{L^6}^2\\
&+C\Big\||\nabla m||\nabla|\nabla m|\Big\|_{L^2}\|\nabla m\|_{L^\infty}\|u\|_{L^6}\|\nabla m\|_{L^3}\\
\le& \eta\Big\||\nabla m||\nabla|\nabla m|\Big\|_{L^2}^2+C_\eta\Big(1+\|\nabla m\|_{L^\infty}\Big)^2\\
&\times\Big(\|\nabla u\|_{L^2}^2+\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2\Big)\|\nabla m\|_{H^1}^2.
\end{aligned}$$

Combining these above estimates, we can prove Lemma 2.2. ∎

LEMMA 2.3. *Let (ρ,u,d) be a global and smooth solution of (1.1), then the following inequality holds for any positive number η,*

$$\begin{aligned}
&\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\\
\le& \eta\|\nabla m_t\|_{L^2}^2+C\Big(1+\|\nabla m\|_{L^\infty}^{\frac{2}{3}}\|\nabla m\|_{L^2}^{\frac{4}{3}}\Big)\Big(\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m|\Big\|_{L^2}^2\Big)\\
&+C\Big(\|\nabla m\|_{L^\infty}+\|\nabla m\|_{L^\infty}^2\|\nabla m\|_{L^2}^2\Big)\|\nabla u\|_{L^2}^2\Big(\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2\Big),
\end{aligned}\tag{2.5}$$

where C is a positive constant depending on η.

Proof. We apply operator ∇ to the third equation of (1.1) and multiplying the resulting equation by $|u|^2\nabla m$, and integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\\
=&-\int |u|^2\nabla m\cdot\nabla m_t dx-\frac{1}{2}\int \nabla(|u|^2)\cdot\nabla(|\nabla m|^2)dx\\
&-\int |u|^2\nabla(u\cdot\nabla m)\cdot\nabla m dx+\int |u|^2\nabla(|\nabla m|^2(m+\underline{d}))\cdot\nabla m dx \overset{def}{=} \sum_{i=1}^4 J_i.
\end{aligned}\tag{2.6}$$

It is easy to check that

$$J_1 \le C\|\nabla m\|_{L^3}\|u^2\|_{L^6}\|\nabla m_t\|_{L^2} \le \eta\|\nabla m_t\|_{L^2}^2+C_\eta\|\nabla m\|_{L^\infty}^{\frac{2}{3}}\|\nabla m\|_{L^2}^{\frac{4}{3}}\Big\||u||\nabla u|\Big\|_{L^2}^2,$$

$$J_2 \le C\Big(\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m|\Big\|_{L^2}^2\Big),$$

$$\begin{aligned}
J_3 \le& C\|\nabla m\|_{L^\infty}(\|u\|_{L^6}^3\|\nabla^2 m\|_{L^2}+\|u\|_{L^6}^2\|\nabla m\|_{L^6}\|\nabla u\|_{L^2})\\
\le& C\|\nabla m\|_{L^\infty}\|\nabla u\|_{L^2}^2\Big(\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2\Big),
\end{aligned}$$

$$J_4=\int|u|^2|\nabla m|^4dx\le C\||\nabla m|^2\|_{L^6}\|\nabla m\|_{L^\infty}\|\nabla m\|_{L^2}\|u\|_{L^6}^2$$
$$\le C\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2+C\|\nabla m\|_{L^\infty}^2\|\nabla m\|_{L^2}^2\|\nabla u\|_{L^2}^4.$$

Combining these above estimates, we complete the proof of Lemma 2.3. ∎

Now we need to estimate ∇u and $\nabla^2 m$. First we denote the effective viscosity flux as $G=\mathrm{div}\,u-\frac{1}{2\mu+\lambda}(\mathfrak{n}-\frac{1}{2}|\nabla m|^2)$, which plays a crucial role in the proof of the main theorem.

Lemma 2.4. *Let (ρ,u,d) be a global and smooth solution of (1.1) with $0\le\rho\le M$, then the following inequality holds*

$$\begin{aligned}&\frac{d}{dt}\Big(\|\nabla u\|_{L^2}^2+\|\mathrm{div}\,u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2+\|\triangle m+|\nabla m|^2(m+\underline{d})\|_{L^2}^2-\int\mathfrak{n}\,\mathrm{div}\,udx\\&\quad+\int F(\rho)dx+\int u\cdot(\nabla m\cdot\triangle m)dx\Big)+\|\rho^{\frac12}u_t\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2\\&\le\eta\|G\|_{\dot H^1}^2+C_\eta\Big(\big(1+\|\nabla m\|_{L^\infty}\big)^2\|\nabla u\|_{L^2}^2+\Big\||u||\nabla u|\Big\|_{L^2}^2\\&\quad+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2\Big),\end{aligned}\tag{2.7}$$

where η is a small constant and C_η depends on the initial data, M and $F(\rho)$ is defined as (2.9) below.

Proof. Multiplying the second equation of (1.1) by u_t, we apply operator ∇ to the third equation of (1.1) and multiplying the resulting equation by ∇m_t, summing up and integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\frac{\mu+\lambda}{2}\|\mathrm{div}\,u\|_{L^2}^2+\frac14\|\nabla^2 m\|_{L^2}^2+\frac14\|\triangle m+|\nabla m|^2(m+\underline{d})\|_{L^2}^2\Big)\\&\quad+\|\rho^{\frac12}u_t\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2\\&=-\int\nabla p\cdot u_tdx-\int(\rho u\cdot\nabla u)\cdot u_tdx-\int(\nabla m\cdot\triangle m)\cdot u_tdx\\&\quad-\int\nabla(u\cdot\nabla m)\nabla m_tdx+\int\nabla(|\nabla m|^2)(m+\underline{d})\cdot\nabla m_tdx\overset{def}{=}\sum_{i=1}^5K_i.\end{aligned}\tag{2.8}$$

Observe that

$$\begin{aligned}K_1&=\frac{d}{dt}\int\mathfrak{n}\,\mathrm{div}\,udx+\int\gamma\rho^\gamma|\mathrm{div}\,u|^2dx-\int\mathfrak{n}|\mathrm{div}\,u|^2dx+\int\mathrm{div}(\mathfrak{n}u)\,\mathrm{div}\,udx\\&\le\frac{d}{dt}\int\mathfrak{n}\,\mathrm{div}\,udx-\frac{1}{2\mu+\lambda}\int\mathfrak{n}u\cdot\nabla\mathfrak{n}dx+C\|\nabla u\|_{L^2}^2\\&\quad+C\|\mathfrak{n}\|_{L^3}\|u\|_{L^6}\|\nabla(\mathrm{div}\,u-\frac{1}{2\mu+\lambda}\mathfrak{n})\|_{L^2}\\&\le\frac{d}{dt}\int\mathfrak{n}\,\mathrm{div}\,udx-\frac{1}{2\mu+\lambda}\frac{d}{dt}\int F(\rho)dx+C\|\nabla u\|_{L^2}^2+C\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2+\eta\|G\|_{\dot H^1}^2,\end{aligned}$$

where

$$F(\rho)=\begin{cases}\frac{\gamma^2}{2(2\gamma-1)}(\rho-1)^2-\Big(\frac{\gamma-1}{2(2\gamma-1)}\rho^\gamma+\frac{\gamma(\gamma-1)}{2(2\gamma-1)}\rho-\frac{\gamma^2+2\gamma-1}{2(2\gamma-1)}\Big)F(\rho|1),&\text{for}\quad\gamma>1,\\\frac12(\rho-1)^2-(\rho\ln\rho-\rho+1),&\text{for}\quad\gamma=1.\end{cases}\tag{2.9}$$

By Taylor expansion, it is not difficult to check that $F(\rho|1)\geq C(M)(\rho-1)^2$ if $\rho\leq M$, and $|F(\rho)|\leq CF(\rho|1)$ if $\rho\leq M$.
It is easy to check that

$$K_2\leq C\|\rho^{\frac{1}{2}}\|_{L^\infty}\|\rho^{\frac{1}{2}}u_t\|_{L^2}\Big\||u||\nabla u|\Big\|_{L^2}\leq\eta\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2+C_\eta\Big\||u||\nabla u|\Big\|_{L^2}^2.$$

Observe that

$$\begin{aligned}K_3+K_4=&-\frac{d}{dt}\int u\cdot(\nabla m\cdot\triangle m)dx-2\int(\nabla u\cdot\nabla m)\cdot\nabla m_t dx\\&+\int(u\cdot\nabla m_t)\cdot\triangle m dx-2\int u\cdot(\nabla^2 m\cdot\nabla m_t)dx\\\leq&-\frac{d}{dt}\int u\cdot(\nabla m\cdot\triangle m)dx+\eta\|\nabla m_t\|_{L^2}+C_\eta\Big(\|\nabla m\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\Big),\end{aligned}$$

and

$$K_5\leq\eta\|\nabla m_t\|_{L^2}^2+C_\eta\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2.$$

Plugging these estimates, we complete the proof, where η is a small constant and C_η depends on the initial data and M. ∎

Next, we improve the estimates by using the elliptic system.

LEMMA 2.5. *Let (ρ,u,d) be a global and smooth solution of (1.1) with $0\leq\rho\leq M$, then the following inequality holds*

$$\begin{aligned}\|\triangle\mathcal{P}u\|_{L^2}^2+\|G\|_{\dot H^1}^2\leq& C\Big(\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2+\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2\\&+\Big(1+\|\nabla m\|_{L^\infty}^{\frac{2}{3}}\|\nabla m\|_{L^2}^{\frac{4}{3}}\Big)\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2\Big),\\\|\nabla\triangle m\|_{L^2}^2\leq& C\Big(\|\nabla m_t\|_{L^2}^2+\|\nabla m\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\\&+\Big(1+\|\nabla m\|_{L^\infty}^{\frac{2}{3}}\|\nabla m\|_{L^2}^{\frac{4}{3}}\Big)\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2\Big),\end{aligned}\tag{2.10}$$

where the constant C depends on M.

Proof. We apply operators $\mathcal{P}$ and $\Lambda^{-1}\mathrm{div}$ to the second equation of (1.1) where $\Lambda=\sqrt{-\triangle}$, then we have

$$\begin{aligned}&-\mu\triangle\mathcal{P}u=\mathcal{P}(-\rho(u_t+u\cdot\nabla u)-\nabla m\cdot\triangle m),\\&-(2\mu+\lambda)\Lambda\,\mathrm{div}\,u-\Lambda(\mathfrak{n}-\frac{1}{2}|\nabla m|^2)=\Lambda^{-1}\mathrm{div}(-\rho(u_t+u\cdot\nabla u)-\nabla(\nabla m\cdot\nabla m)).\end{aligned}\tag{2.11}$$

By the standard elliptic estimates, we can get that

$$\begin{aligned}&\mu\|\triangle\mathcal{P}u\|_{L^2}^2+(2\mu+\lambda)\|G\|_{\dot H^1}^2\\\leq& C\Big(\|\rho u_t\|_{L^2}^2+\|\rho u\cdot\nabla u\|_{L^2}^2+\|\nabla m\cdot\triangle m\|_{L^2}^2+\|\nabla|\nabla m|^2\|_{L^2}^2\Big)\\\leq& C\Big(\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2+\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2\\&+\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2+\|\nabla m\|_{L^6}^6\Big)\end{aligned}$$

$$\leq C\Big(\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2+\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2$$
$$+\Big(1+\|\nabla m\|_{L^\infty}^{\frac{2}{3}}\|\nabla m\|_{L^2}^{\frac{4}{3}}\Big)\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2\Big).$$

We also have

$$\|\nabla\triangle m\|_{L^2}^2$$
$$\leq C\Big(\|\nabla m_t\|_{L^2}^2+\|\nabla m\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2+\|\nabla m\|_{L^6}^6\Big),$$

thus we complete the proof of this Lemma. □

Now we are in a position to estimate $\mathfrak{n}$, for which we have the following Lemma.

LEMMA 2.6. *Let (ρ,u,d) be a global and smooth solution of (1.1), then the following inequality holds*

$$\frac{d}{dt}\|\mathfrak{n}\|_{L^6}^2+\|\mathfrak{n}\|_{L^6}^2\leq C\|G\|_{\dot{H}^1}^2+C\Big\||\nabla m|\big|\nabla|\nabla m|\big|\Big\|_{L^2}^2, \tag{2.12}$$

where C is a positive constant depending on γ.

Proof. The first equation of (1.1) can be rewritten as

$$\mathfrak{n}_t+u\cdot\nabla\mathfrak{n}+\frac{\gamma}{2\mu+\lambda}\mathfrak{n}+\gamma\mathfrak{n}\operatorname{div}u=-\gamma(\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n}). \tag{2.13}$$

Multiplying the above equation by $|\mathfrak{n}|^4\mathfrak{n}$ and integrating on $\mathbb{R}^3$, we have

$$\frac{1}{6}\frac{d}{dt}\|\mathfrak{n}\|_{L^6}^6+\frac{1}{2\mu+\lambda}\int[\gamma+(\gamma-\frac{1}{6})\mathfrak{n}]|\mathfrak{n}|^6dx\leq C\|\mathfrak{n}\|_{L^6}^5\|\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n}\|_{L^6}.$$

Dividing the above estimate by $\|\mathfrak{n}\|_{L^6}^4$, and using $\gamma+(\gamma-\frac{1}{6})\mathfrak{n}\geq\frac{1}{6}$, we can obtain that

$$\frac{d}{dt}\|\mathfrak{n}\|_{L^6}^2+\|\mathfrak{n}\|_{L^6}^2\leq C\|\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n}\|_{L^6}^2$$
$$\leq C\|G\|_{L^6}^2+C\big\||\nabla m|^2\big\|_{L^6}^2,$$

thus we complete the proof. □

PROPOSITION 2.1. *Let $\mu>\frac{1}{2}\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1) with $0\leq\rho\leq M$ and $\sup_{t\in\mathbb{R}^+}\|\nabla m\|_{L^\infty}\leq\mathcal{M}$, then we have $\rho^{\frac{1}{2}}u\in L^\infty(\mathbb{R}^+;L^2\cap L^4)$, $u\in L^\infty(\mathbb{R}^+;\dot{H}^1)\cap L^2(\mathbb{R}^+;\dot{H}^1)$, $\rho^{\frac{1}{2}}u_t\in L^2(\mathbb{R}^+;L^2)$, $\nabla m\in L^\infty(\mathbb{R}^+;H^1\cap L^4)$, $(\triangle m+|\nabla m|^2(m+\bar{d}))\in L^2(\mathbb{R}^+;L^2)$, $\nabla m_t\in L^2(\mathbb{R}^+;L^2)$, $\mathfrak{n}\in L^\infty(\mathbb{R}^+;L^2\cap L^6)\cap L^2(\mathbb{R}^+;L^6)$, $|u||\nabla u|,|\nabla m|\big|\nabla|\nabla m|\big|,|\nabla m||\triangle m+|\nabla m|^2(m+\bar{d})|,|u||\nabla^2 m|\in L^2(\mathbb{R}^+;L^2)$. Furthermore, the following inequality holds*

$$\frac{d}{dt}\Big\{\int F(\rho|1)dx+\|\rho^{\frac{1}{2}}u\|_{L^2}^2+\|\nabla m\|_{L^2}^2+\|\rho^{\frac{1}{4}}u\|_{L^4}^4+\|\nabla m\|_{L^4}^4+\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2$$
$$+\|\mathfrak{n}\|_{L^6}^2\Big\}+C\Big(\|\mathfrak{n}\|_{L^6}+\|\nabla u\|_{L^2}^2+\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2+\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2$$
$$+\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\bar{d})|\Big\|_{L^2}^2$$

$$+\Big\||\nabla m||\nabla^2 m+|\nabla m|^2(m+\bar{d})|\Big\|_{L^2}^2+\|\triangle\mathcal{P}u\|_{L^2}^2+\|G\|_{\dot{H}^1}^2+\|\nabla\triangle m\|_{L^2}^2\Big)\leq 0, \quad (2.14)$$

where C is a positive constant depending on the initial data and $\mu,\lambda,M,\mathcal{M}$.

Proof. First, from Lemma 2.1, we have $\nabla m\in L^\infty(\mathbb{R}^+;L^2)$. Combining the Lemmas 2.1-2.6, and choosing η small enough and choosing $0<A_5\ll A_7\ll A_4\ll A_3\ll A_2\ll A_1$, where A_i are positive constants depending on the initial data and $\mu,\lambda,\bar{\rho},\mathcal{M}$, we can obtain that

$$\begin{aligned}
&\frac{d}{dt}\Big\{A_1\Big(\int F(\rho|1)dx+\|\rho^{\frac12}u\|_{L^2}^2+\|\nabla m\|_{L^2}^2\Big)+A_2\Big(\|\rho^{\frac14}u\|_{L^4}^4+\|\nabla m\|_{L^4}^4\Big)\\
&\quad+A_4\Big(\|\nabla u\|_{L^2}^2+\|\operatorname{div}u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2+\|\triangle m+|\nabla m|^2(m+\underline{d})\|_{L^2}^2-\int\mathfrak{n}\operatorname{div}u dx\\
&\quad+\int F(\rho)dx+\int u\cdot(\nabla m\cdot\triangle m)dx\Big)+A_5\|\mathfrak{n}\|_{L^6}^2\Big\}\\
&\quad+A_6\Big(\|\mathfrak{n}\|_{L^6}^2+\|\nabla u\|_{L^2}^2+\|\operatorname{div}u\|_{L^2}^2+\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2+\|\rho^{\frac12}u_t\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2\\
&\quad+\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla^2 m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2\\
&\quad+\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2\Big)+A_3\Big\||u||\nabla^2 m|\Big\|_{L^2}^2+A_7\left(\|\triangle\mathcal{P}u\|_{L^2}^2+\|G\|_{\dot{H}^1}^2+\|\nabla\triangle m\|_{L^2}^2\right)\\
&\leq A_8\Big(\|\nabla u\|_{L^2}^2+\Big\|\triangle m+|\nabla m|^2(m+\bar{d})\Big\|_{L^2}^2\Big)\Big(\|\mathfrak{n}\|_{L^6}^2+\|\nabla u\|_{L^2}^2+\|\nabla m\|_{H^1}^2\Big), \qquad (2.15)
\end{aligned}$$

which ensure that the term $A_4\int\mathfrak{n}\operatorname{div}u dx$ can be controlled by $A_4\|\operatorname{div}u\|_{L^2}^2$ and $A_1\int F(\rho|1)dx$, the term $A_4\int u\cdot(\nabla m\cdot\triangle m)dx$ can be controlled by $A_4\|\nabla^2 m\|_{L^2}^2$ and $A_1\|\nabla m\|_{L^\infty}^2\|u\|_{L^2}^2$. By Grönwall's inequality, using Lemma (2.1), we can obtain that $\rho^{\frac12}u\in L^\infty(\mathbb{R}^+;L^2\cap L^4)$, $u\in L^\infty(\mathbb{R}^+;\dot{H}^1)\cap L^2(\mathbb{R}^+;\dot{H}^1)$, $\rho^{\frac12}u_t\in L^2(\mathbb{R}^+;L^2)$, $\nabla m\in L^\infty(\mathbb{R}^+;H^1\cap L^4)$, $(\triangle m+|\nabla m|^2(m+\underline{d}))\in L^2(\mathbb{R}^+;L^2)$, $\nabla m_t\in L^2(\mathbb{R}^+;L^2)$, $\mathfrak{n}\in L^\infty(\mathbb{R}^+;L^2\cap L^6)\cap L^2(\mathbb{R}^+;L^6)$, $|u||\nabla u|,|\nabla m|\big|\nabla|\nabla m|\big|,|\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|,|u||\nabla^2 m|\in L^2(\mathbb{R}^+;L^2)$. □

2.2. Improving regularity estimate for u and ∇m. In order to get the dissipation estimate for n, we first improve regularity estimate for u and ∇m in this subsection. We set up some notations. For a function or vector field (or even a 3×3 matrix) $f(t,x)$, the material derivative $\dot{f}$ is defined by

$$\dot{f}=f_t+u\cdot\nabla f,$$

and $\operatorname{div}(f\otimes u)=\sum_{j=1}^3\partial_j(fu_j)$. For two matrices $A=(a_{ij})_{3\times3}$ and $B=(b_{ij})_{3\times3}$, we use the notation $A:B=\sum_{i,j=1}^3 a_{ij}b_{ij}$ and AB is as usual the multiplication of matrices.

LEMMA 2.7. *Let $\mu>\frac12\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1) with $0\leq\rho\leq M$ and $\sup_{t\in\mathbb{R}^+}\|\nabla m\|_{L^\infty}\leq\mathcal{M}$, then the following inequality holds*

$$\begin{aligned}
&\frac{d}{dt}\int\left(\rho\dot{u}^2+(\nabla m_t)^2\right)dx+\|\nabla\dot{u}\|_{L^2}^2+\|\operatorname{div}\dot{u}\|_{L^2}^2+\|\nabla^2 m_t\|_{L^2}^2\\
&\leq C\Big(\|\nabla u\|_{L^4}^4+\Big(\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2+\|\nabla u\|_{L^6}^2+\|\nabla^2 m\|_{L^6}^2\Big)\Big(1+\|\nabla m_t\|_{L^2}^2\Big)\Big), \qquad (2.16)
\end{aligned}$$

where $\dot u = u_t + u\cdot\nabla u$, and C is a positive constant depending on the initial data and $\mu,\lambda,M,\mathcal{M}$.

Proof. We rewrite the second equation and the third equation of (1.1) as

$$\begin{cases} \rho\dot u_t + \rho u\cdot\nabla\dot u - \mu(\triangle u_t + \operatorname{div}(u\otimes\triangle u)) - (\mu+\lambda)(\nabla\operatorname{div}u_t + \operatorname{div}(u\otimes\nabla\operatorname{div}u)) \\ \quad +\nabla\mathfrak{n}_t + \operatorname{div}(u\otimes\nabla\mathfrak{n}) = -(\nabla m\cdot\triangle m)_t - \operatorname{div}(u\otimes(\nabla m\cdot\triangle m)), \\ \nabla m_{tt} + \nabla(u\cdot\nabla m)_t = \nabla(\triangle m + |\nabla m|^2(m+\underline{d}))_t. \end{cases} \tag{2.17}$$

Multiplying above the first equation by $\dot u$, the second equation by ∇m_t, summing up and integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned} &\frac12\frac{d}{dt}\int\left(\rho\dot u^2 + (\nabla m_t)^2\right)dx + \|\nabla^2 m_t\|_{L^2}^2 \\ =&\int(\mathfrak{n}_t\operatorname{div}\dot u + (u\cdot\nabla\dot u)\cdot\nabla\mathfrak{n})\,dx - \mu\int(\nabla\dot u : \nabla u_t + \nabla\dot u : (u\otimes\triangle u))\,dx \\ &+(\mu+\lambda)\int(\nabla\operatorname{div}u_t + \operatorname{div}(u\otimes\nabla\operatorname{div}u))\cdot\dot u dx \\ &-\int\operatorname{div}(u\otimes(\nabla m\cdot\triangle m))\cdot\dot u dx + \int\nabla(|\nabla m|^2(m+\underline{d}))_t\cdot\nabla m_t dx \\ &-\int\Big((\nabla m\cdot\triangle m)_t\cdot\dot u + \nabla(u\cdot\nabla m)_t\nabla m_t\Big)dx \overset{def}{=} \sum_{i=1}^6 L_i. \end{aligned} \tag{2.18}$$

It is not difficult to derive that

$$L_1 = -\gamma\int\rho^\gamma\operatorname{div}u\operatorname{div}\dot u dx + \int\mathfrak{n}\left(\operatorname{div}u\operatorname{div}\dot u - (\nabla u)^t : \nabla\dot u\right)dx \le \eta\|\nabla\dot u\|_{L^2}^2 + C_\eta\|\nabla u\|_{L^2}^2,$$

$$L_2 \le -\frac{3\mu}{4}\|\nabla\dot u\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4,$$

$$L_3 \le -\frac{\mu+\lambda}{2}\|\operatorname{div}\dot u\|_{L^2}^2 + \frac{\mu}{4}\|\nabla\dot u\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4,$$

$$L_4 \le C\|\nabla\dot u\|_{L^2}\|u\|_{L^6}\|\nabla m\|_{L^6}\|\nabla^2 m\|_{L^6} \le \eta\|\nabla\dot u\|_{L^2}^2 + C_\eta\left(\|\nabla u\|_{L^2}^4 + \|\nabla^2 m\|_{L^2}^4\right)\|\nabla^2 m\|_{L^6}^2,$$

$$\begin{aligned} L_5 \le& C\|\nabla m\|_{L^6}^2\|m_t\|_{L^6}\|\nabla^2 m_t\|_{L^2} + \|\nabla m\|_{L^6}\|\nabla m_t\|_{L^3}\|\nabla^2 m_t\|_{L^2} \\ \le&\eta\|\nabla^2 m_t\|_{L^2}^2 + C_\eta\|\nabla^2 m\|_{L^2}^4\|\nabla m_t\|_{L^2}^2, \end{aligned}$$

As for the last term L_6, observe that

$$L_6 = -\int(\nabla m_t\cdot\triangle m)\cdot\dot u dx - \int(u\cdot\nabla u)\cdot\nabla m\cdot\triangle m_t dx + \int(u\cdot\nabla m_t)\cdot\triangle m_t dx \overset{def}{=} \sum_{i=1}^3 L_{6i}.$$

In a similar calculation, we also have

$$L_{61} \le C\|\nabla m_t\|_{L^3}\|\triangle m\|_{L^2}\|\dot u\|_{L^6} \le \eta\|\nabla\dot u\|_{L^2}^2 + \eta\|\nabla^2 m_t\|_{L^2}^2 + C_\eta\|\nabla^2 m\|_{L^2}^4\|\nabla m_t\|_{L^2}^2,$$

$$L_{62} \le \eta \|\nabla^2 m_t\|_{L^2}^2 + C_\eta \Big(\|\nabla u\|_{L^2}^4 + \|\nabla^2 m\|_{L^2}^4 \Big) \|\nabla u\|_{L^6}^2,$$

$$L_{63} \le \|u\|_{L^\infty} \|\nabla m_t\|_{L^2} \|\nabla^2 m_t\|_{L^2} \le \eta \|\nabla^2 m_t\|_{L^2}^2 + C_\eta (\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^6}^2) \|\nabla m_t\|_{L^2}^2.$$

Substituting these estimates, choosing η small enough and applying the Proposition 2.1, we proof the Lemma. □

LEMMA 2.8. *Let $\mu > \frac{1}{2}\lambda$, and (ρ, u, d) be a global and smooth solution of (1.1) with $0 \le \rho \le M$ and $\sup_{t \in \mathbb{R}^+} \|\nabla m\|_{L^\infty} \le \mathcal{M}$, then the following inequality holds*

$$\begin{aligned} &\frac{d}{dt} \int |\nabla m|^2 |\nabla^2 m|^2 dx + \Big\| |\nabla m| |\nabla \triangle m| \Big\|_{L^2}^2 \\ &\le C \Big(\|\nabla^2 m\|_{L^2}^2 + \|\nabla u\|_{L^6}^2 + \|\nabla^2 m\|_{L^6}^2 \Big) \Big(1 + \|\nabla m_t\|_{L^2}^2 \Big), \end{aligned} \tag{2.19}$$

where C is a positive constant depending on the initial data and $\mu, \lambda, M, \mathcal{M}$.

Proof. We apply the ∇ operator to the third equation of (1.1), multiplying by $-|\nabla m|^2 \nabla \triangle m$, and integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\nabla m|^2 |\nabla^2 m|^2 dx + \int |\nabla m|^2 |\nabla \triangle m|^2 dx \\ = &-2 \int (\nabla^2 m \cdot \nabla m_t) : (\nabla^2 m \cdot \nabla m) dx + \int |\nabla^2 m|^2 \nabla m : \nabla m_t dx \\ &+ \int |\nabla m|^2 \nabla (u \cdot \nabla m) \cdot \nabla \triangle m dx - \int |\nabla m|^2 \nabla (|\nabla m|^2 (m + \underline{d})) \cdot \nabla \triangle m dx \\ \overset{def}{=} &\sum_{i=1}^4 M_i. \end{aligned} \tag{2.20}$$

It is not difficult to derive that

$$M_1 + M_2 \le C \|\nabla m\|_{L^6} \|\nabla m_t\|_{L^2} \|\nabla^2 m\|_{L^6}^2 \le C \Big(\|\nabla^2 m\|_{L^2}^2 + \|\nabla m_t\|_{L^2}^2 \Big) \|\nabla^2 m\|_{L^6}^2,$$

$$M_3 \le \eta \Big\| |\nabla m| |\nabla \triangle m| \Big\|_{L^2}^2 + C_\eta (\|\nabla u\|_{L^6}^2 + \|\nabla^2 m\|_{L^6}^2),$$

$$M_4 \le \eta \Big\| |\nabla m| |\nabla \triangle m| \Big\|_{L^2}^2 + C_\eta (\|\nabla^2 m\|_{L^2}^2 + \|\nabla^2 m\|_{L^6}^2).$$

Substituting these estimates, choosing η small enough, we prove the Lemma. □

LEMMA 2.9. *Let $\mu > \frac{1}{2}\lambda$, and (ρ, u, d) be a global and smooth solution of (1.1) with $0 \le \rho \le M$ and $\sup_{t \in \mathbb{R}^+} \|\nabla m\|_{L^\infty} \le \mathcal{M}$, then the following inequality holds*

$$\begin{aligned} &\frac{d}{dt} \int |u|^2 |\nabla^2 m|^2 dx + \Big\| |u| |\nabla \triangle m| \Big\|_{L^2}^2 \\ &\le \eta \|\nabla \dot{u}\|_{L^2}^2 + C \left(\|\nabla^2 m\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\nabla^2 m\|_{L^4}^4 + \|\nabla u\|_{L^6}^2 + \|\nabla^2 m\|_{L^6}^2 \right), \end{aligned} \tag{2.21}$$

where C is a positive constant depending on the initial data and $\mu, \lambda, M, \mathcal{M}$.

Proof. Applying ∇^2 operator to the third equation of (1.1), multiplying by $|u|^2\nabla^2 m$, and integrating on $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |u|^2|\nabla^2 m|^2dx+\int |u|^2|\nabla\triangle m|^2dx\\
=&\int |\nabla^2 m|^2u\cdot u_tdx-2\int(\nabla\triangle m\cdot\nabla^2 m):(\nabla u\cdot u)dx\\
&-\int |u|^2\nabla^2(u\cdot\nabla m)\cdot\nabla^2 mdx+\int |u|^2\nabla^2(|\nabla m|^2(m+\underline{d}))\cdot\nabla^2 mdx\overset{def}{=}\sum_{i=1}^4 N_i.
\end{aligned}\tag{2.22}$$

It is not difficult to derive that

$$\begin{aligned}
N_1=&\int |\nabla^2 m|^2u\cdot\dot{u}dx-\int |\nabla^2 m|^2u\cdot(u\cdot\nabla u)dx\\
\le&\eta\|\nabla\dot{u}\|_{L^2}^2+C_\eta\Big(\|\nabla^2 m\|_{L^4}^4+\|\nabla u\|_{L^6}^2+\|\nabla^2 m\|_{L^6}^2\Big),
\end{aligned}$$

$$N_2\le\eta\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2+C_\eta\Big(\|\nabla u\|_{L^4}^4+\|\nabla^2 m\|_{L^4}^4\Big),$$

$$N_3\le\eta\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2+C_\eta\Big(\|\nabla u\|_{L^4}^4+\|\nabla^2 m\|_{L^4}^4+\|\nabla u\|_{L^6}^2+\|\nabla^2 m\|_{L^6}^2\Big),$$

$$N_4\le\eta\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2+C_\eta\Big(\|\nabla^2 m\|_{L^2}^2+\|\nabla u\|_{L^4}^4+\|\nabla^2 m\|_{L^4}^4+\|\nabla u\|_{L^6}^2+\|\nabla^2 m\|_{L^6}^2\Big).$$

Substituting these estimates, choosing η small enough, we prove the Lemma. □

PROPOSITION 2.2. *Let $\mu>\frac{1}{2}\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1) with $0\le\rho\le M$ and $\sup_{t\in\mathbb{R}^+}\|\nabla m\|_{L^\infty}\le\mathcal{M}$, and the admissible condition (1.4), then the following inequality holds*

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int F(\rho|1)dx+\|\rho^{\frac{1}{2}}u\|_{L^2}^2+\|\nabla m\|_{L^2}^2+\|\rho^{\frac{1}{4}}u\|_{L^4}^4+\|\nabla m\|_{L^4}^4+\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2\\
&+\|\mathfrak{n}\|_{L^6}^2+\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\Big\}\\
&+C\Big(\|\mathfrak{n}\|_{L^6}+\|\nabla u\|_{L^2}^2+\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2+\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2\\
&+\Big\||u||\nabla u|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m|\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2\\
&+\|\triangle\mathcal{P}u\|_{L^2}^2+\|G\|_{\dot{H}^1}^2+\|\nabla\triangle m\|_{L^2}^2+\|\nabla\mathcal{P}u\|_{W^{1,6}}^2+\|G\|_{W^{1,6}}^2+\|\nabla u\|_{L^6}^2\\
&+\|\nabla^2 m\|_{L^6}^2+\|\nabla\dot{u}\|_{L^2}^2+\|\nabla^2 m_t\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2\Big)\le 0,
\end{aligned}\tag{2.23}$$

where C is a positive constant depending on the initial data and $\mu,\lambda,M,\mathcal{M}$.

Proof. Together with (2.16), (2.19) and (2.21), we deduce that

$$\frac{d}{dt}\Big(\|\rho^{\frac{1}{2}}\dot{u}\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\Big)$$

$$
\begin{aligned}
&\quad+\|\nabla\dot u\|_{L^2}^2+\|\nabla^2 m_t\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2\\
&\leq C\Big(\|\nabla u\|_{L^4}^4+\|\nabla^2 m\|_{L^4}^4+\Big(\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2+\|\nabla u\|_{L^6}^2+\|\nabla^2 m\|_{L^6}^2\Big)\\
&\quad\times\Big(1+\|\nabla m_t\|_{L^2}^2\Big)\Big).
\end{aligned}\tag{2.24}
$$

Then the Lemma 2.5 implies that

$$
\begin{aligned}
\|\nabla u\|_{L^4}^4\leq&C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3\\
\leq&C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^2\Big(\|\nabla\mathcal{P}u\|_{L^6}+\|G\|_{L^6}+\||\nabla m|^2\|_{L^6}+\|\mathfrak{n}\|_{L^6}\Big)\\
\leq&C\|\nabla u\|_{L^6}^2\Big(1+\|\rho^{\frac12}\dot u\|_{L^2}^2+\Big\||\nabla m||\triangle m+|\nabla m|^2(m+\underline{d})|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla|\nabla m||\Big\|_{L^2}^2\Big)\\
\leq&C\|\nabla u\|_{L^6}^2\Big(1+\|\rho^{\frac12}\dot u\|_{L^2}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2\Big),
\end{aligned}
$$

and we also have

$$
\begin{aligned}
\|\nabla^2 m\|_{L^4}^4\leq&C\|\nabla^2 m\|_{L^6}^2\Big(\|\nabla m_t\|_{L^2}+\Big\||u||\nabla^2 m|\Big\|_{L^2}+\Big\||\nabla u||\nabla m|\Big\|_{L^2}\\
&+\Big\||\nabla m||\nabla|\nabla m|\Big\|_{L^2}+\||\nabla m\|_{L^6}^6\Big)\\
\leq&C\|\nabla^2 m\|_{L^6}^2\Big(1+\|\nabla m_t\|_{L^2}+\Big\||u||\nabla^2 m|\Big\|_{L^2}+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}\Big).
\end{aligned}
$$

By using these∇^2 operator estimates, we can obtain that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\rho^{\frac12}\dot u\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\Big)\\
&\quad+\|\nabla\dot u\|_{L^2}^2+\|\nabla^2 m_t\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2\\
&\leq C\Big(\|\nabla u\|_{L^2}^2+\|\nabla^2 m\|_{L^2}^2+\|\nabla u\|_{L^6}^2+\|\nabla^2 m\|_{L^6}^2\Big)\\
&\quad\times\Big(1+\|\rho^{\frac12}\dot u\|_{L^2}^2+\|\nabla m_t\|_{L^2}+\Big\||u||\nabla^2 m|\Big\|_{L^2}+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}\Big).
\end{aligned}\tag{2.25}
$$

Noting that $\nabla u\in L^2(\mathbb{R}^+;L^2\cap L^6)$ and $\nabla^2 m\in L^2(\mathbb{R}^+;L^2\cap L^6)$, by Proposition 2.1, it follows that

$$
\begin{aligned}
&\|\nabla u\|_{L^6}\leq\|\nabla\mathcal{P}u\|_{L^6}+\|G\|_{L^6}+\frac{1}{2\mu+\lambda}\|\mathfrak{n}\|_{L^6}+\frac{1}{2(2\mu+\lambda)}\||\nabla m|^2\|_{L^6},\\
&\|\nabla^2 m\|_{L^6}\leq C\|\nabla^3 m\|_{L^2}\leq C\|\nabla\triangle m\|_{L^2}.
\end{aligned}
$$

Then we get by Grönwall's inequality that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\rho^{\frac12}\dot u\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2\Big)\\
&\quad+\|\nabla\dot u\|_{L^2}^2+\|\nabla^2 m_t\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2\leq 0.
\end{aligned}\tag{2.26}
$$

On the other hand, from Lemma (2.5), we have

$$
\begin{aligned}
\|\triangle\mathcal{P}u\|_{L^6}^2+\|\nabla G\|_{L^6}^2\leq&C\Big(\|\nabla\dot u\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+\|\nabla^2 m\|_{L^4}^4\Big)\\
\leq&C\Big(\|\nabla\dot u\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+\|\nabla^2 m\|_{L^6}^2\Big),
\end{aligned}
$$

which, together with Proposition 2.1, completes the proof of this proposition. ∎

2.3. Estimate for the propagation of $\nabla\mathfrak{n}$. In this subsection, we want to give the proof of the upper bound of $\|\nabla u\|_{L^2(\mathbb{R}^+;L^\infty)}$ which in turn gives the estimates for the propagation of $\nabla\mathfrak{n}$.

PROPOSITION 2.3. *Let $\mu>\frac{1}{2}\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1) with $0\le\rho\le\bar{\rho}$, $\sup_{t\in\mathbb{R}^+}\|\nabla m\|_{L^\infty}\le\mathcal{M}$, and initial data (ρ_0,u_0,d_0) verifying that $\rho_0\ge c>0$, the admissible condition (1.4) and*

$$\sup_{t\in\mathbb{R}^+}\|\rho(t,\cdot)\|_{C^\alpha}\le\mathcal{M},\quad \text{with}\quad 0<\alpha<1. \tag{2.27}$$

As a consequence, there exists a constant $\underline{\rho}=\underline{\rho}(c,M,\mathcal{M})>0$ such that for all $t\ge0$, we have $\rho(t,x)\ge\underline{\rho}$. Then we have $\nabla u\in L^2(\mathbb{R}^+;L^\infty)$, moreover, the following inequality holds

$$\begin{aligned}&\frac{d}{dt}\|\nabla\mathfrak{n}\|_{L^2}^2+\|\nabla\mathfrak{n}\|_{L^2}^2\\ &\le C\Big(\|\mathfrak{n}\|_{L^6}^2+\|G\|_{\dot{H}^1}^2+\|G\|_{W^{1,6}}^2+\|\nabla\mathcal{P}u\|_{W^{1,6}}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2\Big),\end{aligned} \tag{2.28}$$

where C is a constant depending on the initial data, M and $\mathcal{M}$.

Proof. First of all, by using the interpolation inequality, we can get $\|\nabla\Lambda^{-1}\mathfrak{n}\|_{L^\infty}\le\|\mathfrak{n}\|_{L^6}^{\beta}|\mathfrak{n}\|_{C^\alpha}^{1-\beta}$, with $\beta=1-\frac{1}{1+2\alpha}\in(0,1)$, then we have

$$\begin{aligned}\|\nabla u\|_{L^\infty}^2&\le C\Big(\|\nabla\mathcal{P}u\|_{L^\infty}^2+\|\nabla\Lambda^{-1}(\operatorname{div}u-\frac{1}{2\mu+\lambda}(\mathfrak{n}-\frac{1}{2}|\nabla m|^2))\|_{L^\infty}^2\\ &\qquad+\|\nabla\Lambda^{-1}\mathfrak{n}\|_{L^\infty}^2+\|\nabla\Lambda^{-1}|\nabla m|^2\|_{L^\infty}^2\Big)\\ &\le C\Big(\|\nabla\mathcal{P}u\|_{W^{1,6}}^2+\|\nabla\Lambda^{-1}G\|_{W^{1,6}}^2+\|\mathfrak{n}\|_{L^6}^{2\beta}\|\mathfrak{n}\|_{C^\alpha}^{2(1-\beta)}+\||\nabla m|^2\|_{W^{1,6}}^2\Big)\\ &\le\eta+C_\eta\Big(\|\nabla\mathcal{P}u\|_{W^{1,6}}^2+\|G\|_{W^{1,6}}^2+\|\mathfrak{n}\|_{L^6}^2+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2\\ &\qquad+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2\Big).\end{aligned} \tag{2.29}$$

On the other hand, it is not difficult to derive that

$$\partial_t\nabla\mathfrak{n}+(u\cdot\nabla)\nabla\mathfrak{n}+\frac{\gamma\rho^\gamma}{2\mu+\lambda}\nabla\mathfrak{n}=-\nabla u\nabla\mathfrak{n}-\gamma\operatorname{div}u\nabla\mathfrak{n}-\gamma\rho^\gamma\nabla(\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n}).$$

By energy estimates, we can derive that for any $q\ge2$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathfrak{n}\|_{L^q}^2+\frac{1}{(2\mu+\lambda)q}\|\nabla\mathfrak{n}\|_{L^q}^2\le C\Big(&\|\nabla u\|_{L^\infty}\|\nabla\mathfrak{n}\|_{L^q}^2+\|\nabla(\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n})\|_{L^q}\|\nabla\mathfrak{n}\|_{L^q}\\ &+\|\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n}\|_{L^\infty}\|\nabla\mathfrak{n}\|_{L^q}^2\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{d}{dt}\|\nabla\mathfrak{n}\|_{L^q}^2+\|\nabla\mathfrak{n}\|_{L^q}^2\le C\Big(&\|\nabla u\|_{L^\infty}^2\|\nabla\mathfrak{n}\|_{L^q}^2+\|\nabla(\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n})\|_{L^q}^2\\ &+\|\operatorname{div}u-\frac{1}{2\mu+\lambda}\mathfrak{n}\|_{W^{1,6}}^2\|\nabla\mathfrak{n}\|_{L^q}^2\Big)\end{aligned}$$

$$\leq C\Big(\|\nabla u\|_{L^\infty}^2\|\nabla\mathfrak{n}\|_{L^q}^2+\|\nabla G\|_{L^q}^2+\|G\|_{W^{1,6}}^2\|\nabla\mathfrak{n}\|_{L^q}^2\Big)$$
$$+C\Big(\|\nabla(|\nabla m|^2)\|_{L^q}^2+\big\||\nabla m|^2\big\|_{W^{1,6}}^2\|\nabla\mathfrak{n}\|_{L^q}^2\Big). \tag{2.30}$$

By taking $q=6$ in (2.30) and using (2.29) and Propostion 2.2, we obtain from Grönwall's inequality that $\|\mathfrak{n}\|_{L^\infty(\mathbb{R}^+;W^{1,6})\cap L^2(\mathbb{R}^+;W^{1,6})}\leq C$; from which, together with (2.29), it implies that $\|\nabla u\|_{L^2(\mathbb{R}^+;L^\infty)}\leq C$.

Now we go back to (2.30) with $q=2$. By Grönwall's inequality, we obtain that $\nabla\mathfrak{n}\in L^\infty(\mathbb{R}^+;L^2)\cap L^2(\mathbb{R}^+;L^2)$. Thanks to the uniform-in-time bounds obtained above, (2.30) with $q=2$ will yield that

$$\frac{d}{dt}\|\nabla\mathfrak{n}\|_{L^2}^2+\|\nabla\mathfrak{n}\|_{L^2}^2\leq C\Big(\|\nabla G\|_{L^2}^2+\|G\|_{W^{1,6}}^2+\|\nabla\mathcal{P}u\|_{W^{1,6}}^2$$
$$+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2\Big)+C\|\mathfrak{n}\|_{L^6}^2,$$

then we obtain (2.28).

Now using the first equation of (1.1) and above inequality, we have

$$\rho(t,x)\geq\rho_0(x)e^{-\int_0^t\|\mathrm{div}\,u(\tau)\|_{L^\infty}d\tau}\geq\rho_0(x)e^{-Ct^{\frac12}}. \tag{2.31}$$

On the other hand, thanks to Lemma 2.6, we derive that $\lim_{t\to\infty}\|n(t)\|_{L^6}=\lim_{t\to\infty}\|\mathfrak{n}(t)\|_{L^6}=0$, from which, together with upper bounds for ρ in C^α, we derive that $\lim_{t\to\infty}\|n(t)\|_{L^\infty}=0$. These two facts imply that there exists a constant $\underline{\rho}=\underline{\rho}(c,M,\mathcal{M})>0$ such that for all $t\geq0$, $\rho(t,x)\geq\underline{\rho}$. We complete the proof of the proposition. □

2.4. Deriving the dissipation inequality. Thanks to Propositions 2.1-2.3, we obtain that:

PROPOSITION 2.4. *Let $\mu>\frac12\lambda$, and (ρ,u,d) be a global and smooth solution of (1.1) with $0\leq\rho\leq M$ and $\sup_{t\in\mathbb{R}^+}\|\nabla d(t,\cdot)\|_{L^\infty}+\sup_{t\in\mathbb{R}^+}\|\rho(t,\cdot)\|_{C^\alpha}\leq\mathcal{M}$ with $0<\alpha<1$, initial data (ρ_0,u_0,d_0) verifying that $\rho_0\geq c>0$ and the admissible condition (1.4). Set*

$$X(t)=\|n\|_{H^1}^2+\|u\|_{H^1}^2+\|\nabla m\|_{H^1}^2+\|\dot u\|_{L^2}^2+\|\nabla m_t\|_{L^2}^2$$
$$+\Big\||\nabla m||\nabla^2 m|\Big\|_{L^2}^2+\Big\||u||\nabla^2 m|\Big\|_{L^2}^2. \tag{2.32}$$

It is easy to check that

$$\|\nabla m\|_{L^4}\leq\|\nabla m\|_{L^2}^{\frac14}\|\nabla^2 m\|_{L^2}^{\frac34},\qquad \|\nabla m\|_{L^4}\leq\|\nabla m\|_{L^2}^{\frac58}\|\nabla^2 m\|_{L^6}^{\frac38},$$

so that

$$\begin{aligned}\|\nabla^2 m\|_{L^2}^2&\leq\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2+\|\nabla m\|_{L^4}^4\\&\leq\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2+C\|\nabla m\|_{L^2}^2\|\nabla^2 m\|_{L^2}\|\nabla^2 m\|_{L^6}\\&\leq\Big\|\triangle m+|\nabla m|^2(m+\underline{d})\Big\|_{L^2}^2+\eta\|\nabla^2 m\|_{L^2}^2+C\|\nabla m\|_{L^2}^4\|\nabla^2 m\|_{L^6}^2,\end{aligned}$$

$$\begin{aligned}\Big\|\nabla(|\nabla m||\nabla^2 m|)\Big\|_{L^2}^2&\leq C\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+C\|\nabla^2 m\|_{L^4}^4\\&\leq C\Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2+C\|\nabla^2 m\|_{L^6}^2,\end{aligned}$$

and

$$\Big\|\nabla(|u||\nabla^2 m|)\Big\|_{L^2}^2 \le C\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|\nabla^2 m\|_{L^4}^4$$
$$\le C\Big\||u||\nabla\triangle m|\Big\|_{L^2}^2 + C\|\nabla u\|_{L^6}^2 + C\|\nabla^2 m\|_{L^6}^2.$$

Then the following inequality holds

$$\frac{d}{dt}X(t) + C\Big(\|\nabla \mathfrak{n}\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla^2 m\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla^2 m_t\|_{L^2}^2 + \Big\||u||\nabla\triangle m|\Big\|_{L^2}^2 + \Big\||\nabla m||\nabla\triangle m|\Big\|_{L^2}^2 + \|\nabla u\|_{L^6}^2 + \|\nabla^2 m\|_{L^6}^2\Big) \le 0, \quad (2.33)$$

where C is a constant depending on the initial data (ρ_0, u_0, d_0) and $\mu, \lambda, c, M, \mathcal{M}$.

3. Convergence to the equilibrium

The aim of this section is to show the convergence of the solution to the equilibrium.

PROPOSITION 3.1. *Let $\mu > \frac{1}{2}\lambda$, and (ρ, u, d) be a global and smooth solution of (1.1) with $0 \le \rho \le M$ and $\sup_{t\in\mathbb{R}^+}\|\nabla d(t,\cdot)\|_{L^\infty} + \sup_{t\in\mathbb{R}^+}\|\rho(t,\cdot)\|_{C^\alpha} \le \mathcal{M}$ with $0<\alpha<1$, initial data $n_0 \in L^1(\mathbb{R}^3)\cap H^1(\mathbb{R}^3), u_0 \in L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3), \nabla m_0 \in L^1(\mathbb{R}^3), m_0 \in H^3(\mathbb{R}^3)$ verifying that $\rho_0 \ge c > 0$ and the admissible condition (1.4). Then we have*

$$\|n(t)\|_{H^1} + \|u(t)\|_{H^1} + \|\nabla m(t)\|_{H^1} \le C(1+t)^{-\frac{3}{4}}, \quad (3.1)$$

where C is a constant depending on the initial data (ρ_0, u_0, m_0) and $\mu, \lambda, c, M, \mathcal{M}$.

Proof. We take the Fourier transform of the first and second equations in (1.1), we apply the ∇ operator to the third equation in (1.1) then apply the Fourier transform, and multiplying the resulting first equation by $\gamma\bar{\hat{n}}$, the resulting second equation by $\bar{\widehat{\rho u}}$, the resulting third equation by $\bar{\widehat{\nabla m}}$, summing up and integrating on $S(t) = \Big\{\xi \big| |\xi| \le C(1+t)^{-\frac{1}{2}}\Big\}$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{S(t)}\Big(\gamma|\hat{n}(t,\xi)|^2 + |\widehat{\rho u}(t,\xi)|^2 + |\widehat{\nabla m}(t,\xi)|^2\Big)d\xi \\
&\quad + \int_0^t\int_{S(t)}\Big(\mu|\xi|^2|\hat{u}(s,\xi)|^2 + (\mu+\lambda)|\xi\cdot\hat{u}(s,\xi)|^2 + |\xi|^2|\widehat{\nabla m}(s,\xi)|^2\Big)d\xi ds \\
&= \frac{1}{2}\int_{S(t)}\Big(\gamma|\hat{n}(0,\xi)|^2 + |\widehat{\rho u}(0,\xi)|^2 + |\widehat{\nabla m}(0,\xi)|^2\Big)d\xi + Re\int_0^t\int_{S(t)}\Big\{-\widehat{\mathrm{div}(\rho u\otimes u)}\cdot\bar{\widehat{\rho u}} \\
&\quad + (\mu\widehat{\triangle u} + (\mu+\lambda)\widehat{\nabla \mathrm{div} u})\cdot\bar{\widehat{nu}} + i(\gamma-1)\xi\widehat{F(\rho|1)}\cdot\bar{\widehat{\rho u}} \\
&\quad - (\widehat{\nabla m\cdot\triangle m})\cdot\bar{\widehat{\rho u}} - (\nabla\widehat{(u\cdot\nabla m)})\cdot\bar{\widehat{\nabla m}} + (\nabla\widehat{(|\nabla m|^2(m+\underline{d}))})\cdot\bar{\widehat{\nabla m}}\Big\}d\xi ds \\
&= \frac{1}{2}\int_{S(t)}\Big(\gamma|\hat{n}(0,\xi)|^2 + |\widehat{\rho u}(0,\xi)|^2 + |\widehat{\nabla m}(0,\xi)|^2\Big)d\xi + \sum_{i=1}^{6}O_i.
\end{aligned} \quad (3.2)$$

Applying the Proposition 2.3 and Proposition 2.4, we have

$$\begin{aligned}O_1\leq&\eta\int_0^t\int_{S(t)}\mu|\xi|^2|\hat{u}|^2d\xi ds+C_\eta\int_0^t\int_{S(t)}|\widehat{\rho u\otimes u}|^2d\xi ds+\int_0^t\int_{S(t)}|\xi||\widehat{\rho u\otimes u}||\widehat{nu}|d\xi ds\\ \leq&\eta\int_0^t\int_{S(t)}\mu|\xi|^2|\hat{u}|^2d\xi ds+C_\eta\int_0^t\|\widehat{\rho u\otimes u}\|_{L^\infty}^2\int_{S(t)}d\xi ds\\ &+\int_0^t\|\widehat{\rho u\otimes u}\|_{L^\infty}\|\widehat{nu}\|_{L^\infty}\int_{S(t)}|\xi|d\xi ds\\ \leq&\eta\int_0^t\int_{S(t)}\mu|\xi|^2|\hat{u}|^2d\xi ds+C_\eta(1+t)^{-\frac{3}{2}}\int_0^t\|u\|_{L^2}^4ds+C(1+t)^{-2}\int_0^t\|n\|_{L^2}\|u\|_{L^2}^3ds,\end{aligned}$$

in a similar fashion, we have

$$O_2\leq\eta\int_0^t\int_{S(t)}\mu|\xi|^2|\hat{u}|^2d\xi ds+C_\eta(1+t)^{-\frac{5}{2}}\int_0^t\|n\|_{L^2}^2\|u\|_{L^2}^2ds,$$

$$O_3\leq\eta\int_0^t\int_{S(t)}\mu|\xi|^2|\hat{u}|^2d\xi ds+C_\eta(1+t)^{-\frac{3}{2}}\int_0^t\|n\|_{L^2}^4ds+C_\eta(1+t)^{-\frac{5}{2}}\int_0^t\|n\|_{L^2}^2\|u\|_{L^2}^2ds,$$

$$\begin{aligned}O_4\leq&\eta\int_0^t\int_{S(t)}\mu|\xi|^2|\hat{u}|^2d\xi ds+C_\eta(1+t)^{-\frac{3}{2}}\int_0^t\|\nabla m\|_{L^2}^4ds\\ &+C(1+t)^{-2}\int_0^t\|n\|_{L^2}\|u\|_{L^2}\|\nabla m\|_{L^2}^2ds,\end{aligned}$$

$$O_5\leq\eta\int_0^t\int_{S(t)}|\xi|^2|\widehat{\nabla m}|^2d\xi ds+C_\eta(1+t)^{-\frac{3}{2}}\int_0^t\|u\|_{L^2}^2\|\nabla m\|_{L^2}^2ds,$$

$$O_6\leq\eta\int_0^t\int_{S(t)}|\xi|^2|\widehat{\nabla m}|^2d\xi ds+C_\eta(1+t)^{-\frac{3}{2}}\int_0^t\|m+\underline{d}\|_{L^\infty}^2\|\nabla m\|_{L^2}^4ds.$$

Note that $(n_0,\rho_0u_0,\nabla m_0)\in L^1(\mathbb{R}^3)$, then we have

$$\frac{1}{2}\int_{S(t)}\Big(\gamma|\hat{n}(0,\xi)|^2+|\widehat{\rho u}(0,\xi)|^2+|\widehat{\nabla m}(0,\xi)|^2\Big)d\xi\leq C\left(\|n_0\|_{L^1}^2+\|\rho_0u_0\|_{L^1}^2+\|\nabla m_0\|_{L^1}^2\right)(1+t)^{-\frac{3}{2}}.$$

Plugging above estimates, and choosing η small enough, we arrive at

$$\begin{aligned}&\int_{S(t)}\Big(|\hat{n}(t,\xi)|^2+|\widehat{\rho u}(t,\xi)|^2+|\widehat{\nabla m}(t,\xi)|^2\Big)d\xi\\ \leq&C(1+t)^{-\frac{3}{2}}+C(1+t)^{-\frac{3}{2}}\int_0^t\left(\|n\|_{L^2}^4+\|u\|_{L^2}^4+\|\nabla m\|_{L^2}^4\right)ds\leq C(1+t)^{-\frac{1}{2}}.\end{aligned}\tag{3.3}$$

We recall the dissipation inequality from Proposition 2.4, by frequency splitting method, it is not difficult to derive that

$$\begin{aligned}\frac{d}{dt}X(t)+\frac{1}{1+t}X(t)\leq&\frac{1}{1+t}\int_{S(t)}\Big(|\hat{n}(t,\xi)|^2+|\hat{u}(t,\xi)|^2+|\widehat{\nabla m}(t,\xi)|^2+|\hat{u}(t,\xi)|^2+|\widehat{\nabla m_t}(t,\xi)|^2\\ &+|\widehat{|\nabla m||\nabla^2 m|}(t,\xi)|^2+|\widehat{|u||\nabla^2 m|}(t,\xi)|^2\Big)d\xi,\end{aligned}$$

due to the fact $u=\rho u-nu$, we have

$$\int_{S(t)}|\widehat{u}(t,\xi)|^2 d\xi \le \int_{S(t)}|\widehat{\rho u}(t,\xi)|^2 d\xi + \int_{S(t)}|\widehat{nu}(t,\xi)|^2 d\xi \le C(1+t)^{-\frac{1}{2}},$$

following the same argument, we can obtain that

$$\int_{S(t)}|\widehat{\rho u}(t,\xi)|^2 d\xi \le C\int_{S(t)}\Big|\mu\widehat{\triangle u}(t,\xi)+(\mu+\lambda)\widehat{\nabla \mathrm{div}\, u}(t,\xi)-(\gamma-1)\widehat{\nabla F(\rho|1)}(t,\xi) \\ -\gamma\widehat{\nabla n}(t,\xi)+\widehat{\nabla m\cdot\triangle m}(t,\xi)\Big|^2 d\xi \le C(1+t)^{-\frac{3}{2}},$$

$$\int_{S(t)}|\widehat{\nabla m_t}(t,\xi)|^2 d\xi \le C\int_{S(t)}\Big|\widehat{\nabla\triangle m}(t,\xi)+\nabla(\widehat{|\nabla m|^2(m+\underline{d})})(t,\xi)-\nabla\widehat{(u\cdot\nabla m)}(t,\xi)\le C(1+t)^{-\frac{5}{2}},$$

$$\int_{S(t)}|\widehat{|\nabla m||\nabla^2 m|}(t,\xi)|^2 d\xi + \int_{S(t)}|\widehat{|u||\nabla^2 m|}(t,\xi)|^2 d\xi \le C(1+t)^{-\frac{3}{2}},$$

which implies that

$$\frac{d}{dt}X(t)+\frac{1}{1+t}X(t)\le C(1+t)^{-\frac{3}{2}}, \quad \|(n,u,\nabla m)(t)\|_{H^1}\le C(1+t)^{-\frac{1}{4}}.$$

We need to improve the decay estimate. Now following a similar argument used in the previous proof, we conclude that

$$\int_{S(t)}\Big(|\hat{n}(t,\xi)|^2+|\widehat{\rho u}(t,\xi)|^2+|\widehat{\nabla m}(t,\xi)|^2\Big)d\xi \le C(1+t)^{-\frac{3}{2}}+C(1+t)^{-\frac{3}{2}}\log(1+t),$$

which implies that

$$\frac{d}{dt}X(t)+\frac{1}{1+t}X(t)\le C(1+t)^{-\frac{5}{2}}\log(1+t), \quad \|(n,u,\nabla m)(t)\|_{H^1}\le C(1+t)^{-\frac{3}{4}}\log^{\frac{1}{2}}(1+t).$$

Now we repeat the same process as above to get that

$$\int_{S(t)}\Big(|\hat{n}(t,\xi)|^2+|\widehat{\rho u}(t,\xi)|^2+|\widehat{\nabla m}(t,\xi)|^2\Big)d\xi \le C(1+t)^{-\frac{3}{2}},$$

which implies that

$$\frac{d}{dt}X(t)+\frac{1}{1+t}X(t)\le C(1+t)^{-\frac{5}{2}}.$$

And this completes the proof of the proposition. □

4. Global-in-time stability for liquid crystal system

In this section, we want to prove Theorem 1.2. By the local well-posedness for the system (1.1), we can show that the perturbed solutions will remain close to the reference solutions for a long time if their initial values are close. Then the convergence result implies that the reference solutions are close to the equilibrium after a long time, and the perturbed solutions are also close to the equilibrium. Then we can prove the global existence in the equilibrium framework.

Let $(\tilde{\rho},\tilde{u},\tilde{d})$ be a global solution for (1.1) with the initial data $(\tilde{\rho}_0,\tilde{u}_0,\tilde{d}_0)$, and let (ρ,u,d) be a global solution for (1.1) with the initial data (ρ_0,u_0,d_0). We denote $h=\rho-\tilde{\rho}$, $v=u-\tilde{u}$, $b=d-\tilde{d}$ which satisfy the following error equations:

$$\begin{cases} \partial_t h+(v+\tilde{u})\cdot\nabla h=-v\cdot\nabla\tilde{\rho}-(h+\tilde{\rho})\operatorname{div}v-h\operatorname{div}\tilde{u}, \qquad (t,x)\in \mathrm{R}^+\times\mathrm{R}^3,\\ \partial_t v+v\cdot\nabla v-\mu\operatorname{div}(\frac{1}{\rho}\nabla v)-(\mu+\lambda)\nabla(\frac{1}{\rho}\operatorname{div}v)=F,\\ \partial_t b-\triangle b=G, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} F=&-\frac{1}{\rho}\Big(h\tilde{u}_t+hv\cdot\nabla\tilde{u}+h\tilde{u}\cdot\nabla v+h\tilde{u}\cdot\nabla\tilde{u}+\tilde{\rho}v\cdot\nabla\tilde{u}+\gamma(\rho^{\gamma-1}-\tilde{\rho}^{\gamma-1})\nabla\tilde{\rho}\\ &+\nabla b\cdot\triangle b+\nabla b\cdot\triangle\tilde{d}\Big)-\gamma(\rho^{\gamma-2}-\tilde{\rho}^{\gamma-2})\nabla h-(\frac{\nabla\tilde{d}}{\rho}-\frac{\nabla\tilde{d}}{\tilde{\rho}})\cdot\triangle b-\frac{\nabla\tilde{d}}{\tilde{\rho}}\cdot\triangle b\\ &-\tilde{\rho}^{\gamma-2}\nabla h-(\frac{\tilde{\rho}\tilde{u}}{\rho}-\frac{\tilde{\rho}\tilde{u}}{\tilde{\rho}})\cdot\nabla v-\frac{\tilde{\rho}\tilde{u}}{\tilde{\rho}}\cdot\nabla v+\mu\frac{\nabla h}{\rho^2}\nabla v+(\mu+\lambda)\frac{\nabla h}{\rho^2}\operatorname{div}v\\ &+\mu(\frac{\nabla\tilde{\rho}}{\rho^2}-\frac{\nabla\tilde{\rho}}{\tilde{\rho}^2})\nabla v+\mu\frac{\nabla\tilde{\rho}}{\tilde{\rho}^2}\nabla v+(\mu+\lambda)(\frac{\nabla\tilde{\rho}}{\rho^2}-\frac{\nabla\tilde{\rho}}{\tilde{\rho}^2})\operatorname{div}v+(\mu+\lambda)\frac{\nabla\tilde{\rho}}{\tilde{\rho}^2}\operatorname{div}v,\\ G=&-(v+\tilde{u})\cdot\nabla b-v\cdot\nabla\tilde{d}+|\nabla b|^2b+|\nabla b|^2\tilde{d}+|\nabla\tilde{d}|^2b+2\nabla b:\nabla\tilde{d}(b+\tilde{d}). \end{aligned}$$

PROPOSITION 4.1. *Let $(\tilde{\rho},\tilde{u},\tilde{d})$ be the smooth solution for (1.1) satisfying assumption of Theorem 1.2. Given an $\epsilon>0$, if the initial data (h_0,v_0,b_0) of (4.1) satisfy that*

$$\|h_0\|_{H^4}+\|v_0\|_{H^4}+\|b_0\|_{H^4}\le\epsilon, \tag{4.2}$$

then there exists a constant c independent of ϵ, such that for any $t\in[0,c|\ln\epsilon|]$, there holds

$$\|h(t)\|_{H^4}+\|v(t)\|_{H^4}+\|b(t)\|_{H^4}\le\epsilon^{\frac{1}{2}}. \tag{4.3}$$

Proof. We use the continuity argument to prove the desired results. Let $\mathcal{T}$ be the maximum time such that for any $t\in[0,\mathcal{T}]$, there holds

$$\|h(t)\|_{H^4}+\|v(t)\|_{H^4}+\|b(t)\|_{H^4}\le\epsilon^{\frac{1}{2}}.$$

The existence of $\mathcal{T}$ can be obtained by the local well-posedness, then we need to prove that $\mathcal{T}\ge c|\ln\epsilon|$, where c is a constant independent of ϵ.

Now for $0\le k\le 4$, applying ∇^k to (4.1) and then multiplying the first equation by $\nabla^k h$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} \frac{d}{dt}\|h(t)\|_{H^4}^2\le& C(\|\tilde{u}\|_{H^5}+\|v\|_{H^4})\|h\|_{H^4}^2+C(1+\|\tilde{\rho}-1\|_{H^5}+\|h\|_{H^4})\|\nabla v\|_{H^4}\|h\|_{H^4}\\ \le& C_\eta\|h\|_{H^4}^2+\eta\|\nabla v\|_{H^4}^2, \end{aligned}$$

then the second equation by $\nabla^k u$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{d}{dt}\|v(t)\|_{H^4}^2+\|\nabla v(t)\|_{H^4}^2\\ \le& C(\|h\|_{H^4}^2+\|v\|_{H^4}^2+\|b\|_{H^4}^2)+C\epsilon^{\frac{1}{2}}(\|\nabla v\|_{H^4}^2+\|\nabla b\|_{H^4}^2)+\eta\|\nabla v\|_{H^4}^2+C_\eta\|\nabla b\|_{H^4}^2, \end{aligned}$$

then the third equation by $\nabla^k b$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{d}{dt}\|b(t)\|_{H^4}^2+\|\nabla b(t)\|_{H^4}^2\\ \leq& C(\|\tilde{u}\|_{H^4}+\|v\|_{H^4})\|b\|_{H^4}^2+C(1+\|\tilde{d}-\bar{d}\|_{H^4})\|v\|_{H^4}\|\nabla b\|_{H^4}\\ &+C(\|b\|_{H^4}^3+(1+\|\tilde{d}-\bar{d}\|_{H^4})\|b\|_{H^4}^2+(1+\|\tilde{d}-\bar{d}\|_{H^4}^2)\|b\|_{H^4})\|\nabla b\|_{H^4}\\ \leq& C\|v\|_{H^4}^2+C\|b\|_{H^4}^2+\eta\|\nabla b\|_{H^4}^2.\end{aligned}$$

Summing up above estimates, choosing η small enough, we obtain that

$$\begin{aligned}&\|h(t)\|_{H^4}^2+\|v(t)\|_{H^4}^2+\|b(t)\|_{H^4}^2\\ \leq&\|h_0\|_{H^4}^2+\|v_0\|_{H^4}^2+\|b_0\|_{H^4}^2+C\int_0^t(\|h(\tau)\|_{H^4}^2+\|v(\tau)\|_{H^4}^2+\|b(\tau)\|_{H^4}^2)d\tau,\end{aligned}$$

for any $t\in[0,\mathcal{T}]$. By Grönwall's inequality, we get that

$$\|h(t)\|_{H^4}^2+\|v(t)\|_{H^4}^2+\|b(t)\|_{H^4}^2\leq C\left(\|h_0\|_{H^4}^2+\|v_0\|_{H^4}^2+\|b_0\|_{H^4}^2\right)e^{Ct},$$

for any $t\in[0,\mathcal{T}]$. According to the definition of $\mathcal{T}$, which implies that $\mathcal{T}\geq c|\ln\epsilon|$ for a suitable c independent of ϵ. Then we complete the proof. □

Proof. (**Proof of Theorem 1.2.**) Thanks to Theorem 1.1, we can choose $t_0=\frac{1}{2}(1+c|\ln\epsilon|)$, then we have

$$\|(\tilde{\rho}-1)(t_0)\|_{H^4}+\|\tilde{u}(t_0)\|_{H^4}+\|(\tilde{d}-\bar{d})(t_0)\|_{H^4}\leq C(1+c|\ln\epsilon|)^{-\frac{3}{4}},$$

then we derive that

$$\|(\rho-1)(t_0)\|_{H^4}+\|u(t_0)\|_{H^4}+\|(d-\bar{d})(t_0)\|_{H^4}\leq\epsilon^{\frac{1}{2}}+C(1+c|\ln\epsilon|)^{-\frac{3}{4}}\leq C(1+c|\ln\epsilon|)^{-\frac{3}{4}}. \tag{4.4}$$

This means that at t_0, the system (1.1) is in the close-to-equilibrium regime. Then we can obtain the global existence for $(\rho-1,u,d-\bar{d})$. Moreover due to the definition of $\mathcal{T}$, we conclude that for any $t>0$

$$\|h(t)\|_{H^4}+\|v(t)\|_{H^4}+\|b(t)\|_{H^4}\leq C\min\{(1+c|\ln\epsilon|)^{-\frac{3}{4}},\epsilon^{\frac{1}{2}}+(1+t)^{-\frac{3}{4}}\}. \tag{4.5}$$

It completes the proof to Theorem 1.2. □

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