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ON THE SMOOTH SOLUTIONS OF LANDAU-LIFSHITZ-BLOCH EQUATIONS OF ANTIFERROMAGNETS*

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Abstract. In this paper, we investigate the smooth solutions for the antiferromagnets Landau-Lifshitz-Bloch (LLB) equation with periodic initial value, which can describe the dynamics of micromagnets under high temperature. The existence and uniqueness of smooth solutions for LLB equation in $\mathbb{R}^2$ and $\mathbb{R}^3$ are proved.

Keywords. Smooth solutions; Landau-Lifshitz-Bloch equation.

AMS subject classifications. 35A01; 635A02; 35B65.

1. Introduction

Most crystals have magnetically ordered structures. This means that in the absence of an external magnetic field, the mean magnetic moment of at least one of atoms in each unit cell of the crystal is non-zero.

As is well known, the Landau-Lifshitz equation describes the magnetization dynamics of ferromagnets at low temperature. It is famous and many important results has been obtained [1]. The Landau-Lifshitz-Gilbert (LLG) equation is described as follows [2,3]

$$\mathbf{L}_t=\mathbf{L}\times\Delta\mathbf{L}-\lambda\mathbf{L}\times(\mathbf{L}\times\Delta\mathbf{L}), \tag{1.1}$$

where $\mathbf{L}(x,t)=(L_1(x,t),L_2(x,t),L_3(x,t))$ is magnetization functional vector. $\lambda>0$ is a Gerbert constant. "$\times$" denotes the vector outer product. In order to describe the dynamics of magnetization vector $\mathbf{L}$ in a ferromagnetic body for a wide range of temperatures, Garanin [4–6] derived the Landau-Lifshitz-Bloch (LLB) equation from statistical mechanis with the mean field approximation. At high temperatures ($\theta\geq\theta_c$, θ_c being the Curie value), LLB equation is satisfactory. In Ref. [7], Berti also pointed that from the paramagnetic to the ferromagnetic state is modeled as a second order phase transition.

The LLB equation is given as follows

$$\mathbf{L}_t=-\gamma\mathbf{L}\times\mathbf{H}^{eff}+\frac{a_1}{|\mathbf{L}|^2}(\mathbf{L}\cdot\mathbf{H}^{eff})\mathbf{L}-\frac{a_2}{|\mathbf{L}|^2}\mathbf{L}\times(\mathbf{L}\times\mathbf{H}^{eff}), \tag{1.2}$$

where γ, a_1 and a_2 are constants, $\mathbf{H}^{eff}$ is the effective field. We can also rewrite Equation (1.2) as follows

$$\mathbf{L}_t=-\gamma\mathbf{L}\times\mathbf{H}^{eff}+\frac{\gamma a_{\|}}{|\mathbf{L}|^2}(\mathbf{L}\cdot\mathbf{H}^{eff})\mathbf{L}-\frac{\gamma a_{\perp}}{|\mathbf{L}|^2}\mathbf{L}\times(\mathbf{L}\times\mathbf{H}^{eff}),$$

with $\gamma a_{\|}=a_1$, $\gamma a_{\perp}=a_2$. Here $a_{\|}$ and $a_{\perp}$ are dimensionless damping parameters whose

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dependence on the temperature is assumed as follows [8]

$$a_{\|}(\theta)=\frac{2\theta}{3\theta_c}\lambda,\qquad a_{\perp}(\theta)=\begin{cases}\lambda\left(1-\dfrac{\theta}{3\theta_c}\right), & \text{if } \theta<\theta_c,\\ a_{\|}(\theta), & \text{if } \theta\geq\theta_c,\end{cases}$$

where $\lambda>0$ is a constant. In Ref. [9], the author points that if $a_1=a_2$, Equation (1.2) can be reduced as

$$\frac{\partial\mathbf{u}}{\partial t}=\kappa_1\Delta\mathbf{u}+\gamma\mathbf{u}\times\Delta\mathbf{u}-\kappa_2(1+\mu|\mathbf{u}|^2)\mathbf{u},$$

here all the coefficients κ_1, κ_2, γ, μ are positive, $\mathbf{u}\in C^{\alpha}([0,T],L^{\frac{3}{2}})$, $\sup\limits_{t\in[0,T]}\|\mathbf{u}(\cdot,t)\|_{H^1}<\infty$.

Results in several aspects have been obtained for the above magnetic equations, due to their wide applications. For the LLG equation, the proofs of the existence of the global weak solutions have been given [10–12]. Furthermore, some works about the weak solutions for its stochastic version also have been done [13–15]. The existence of weak solutions and regularity properties for LLB equation have been discussed by Le [9]. The global existence of martingale weak solutions for the stochastic LLB equation have been analyzed [16]. Ref. [17] investigated the global weak solutions to a spatio-temporal fractional LLB equation. Jia [18] brought in LLB equation on m-dimensional closed Riemannian manifold and proved that a unique local solution is admitted.

In antiferromagnets [19, 20], the mean atomic magnetic moments compensate each other within each unit cell (in zero external magnetic field). In other words, antiferromagnet consists of a set of sublattices (called magnetic sublattices), each of which has a non-zero mean magnetic moment provided the temperature of antiferromagnet is higher than a critical temperature $\theta\geq\theta_c$.

To the best of our knowledge, the existence and uniqueness of smooth solutions for the LLB equation with periodic initial value have not been analyzed yet. Motivated by the above, in this paper, we intend to establish the existence of smooth solutions of the following periodic initial value problem for the magnetizations $\mathbf{m}$ and $\mathbf{n}$ of the two magnetic sublattices for the antiferromagnets LLB equation

$$\mathbf{m}_t=\Delta\mathbf{m}+2k_1\mathbf{m}\times\Delta\mathbf{m}+k_{11}\mathbf{m}\times\Delta\mathbf{n}-k_0(1+\mu_0|\mathbf{m}|^2)\mathbf{m},\qquad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.3a}$$

$$\mathbf{n}_t=\Delta\mathbf{n}+2k_2\mathbf{n}\times\Delta\mathbf{n}+k_{22}\mathbf{n}\times\Delta\mathbf{m}-k_0(1+\mu_1|\mathbf{n}|^2)\mathbf{n},\qquad (x,t)\in\Omega\times\mathbb{R}_+, \tag{1.3b}$$

$$\mathbf{m}(x,0)=\mathbf{m}_0(x),\qquad \mathbf{n}(x,0)=\mathbf{n}_0(x),\qquad x\in\Omega, \tag{1.3c}$$

$$\mathbf{m}(x+2De_i,t)=\mathbf{m}(x,t),\qquad \mathbf{n}(x+2De_i,t)=\mathbf{n}(x,t),\qquad (x,t)\in\mathbb{R}^2\times\mathbb{R}_+, \tag{1.3d}$$

where $x+2De_i=(x_1,\cdots,x_{i-1},x_i+2D,x_{i+1},\cdots,x_2)$, $D>0$, $\Omega\subset\mathbb{R}^d$ $(d\geq2)$ is a d-dimensional cube with width $2D$, $\mathbf{m}_0(x+2De_i)=\mathbf{m}_0(x)$, $\mathbf{n}_0(x+2De_i)=\mathbf{n}_0(x)$, $i=1,2$. k_0, k_1, k_2, k_{11}, k_{22}, μ_0 and μ_1 are positive constants, which satisfy the constraints $\frac{k_{11}}{2k_1}=\frac{k_{22}}{2k_2}=a$ and $|a|<1$.

This paper is organized as follows. In Section 2, we state the main results for the above model, i.e., Theorems 2.1-2.6. In Section 3, we give the priori estimate for Problem (1.3) with $\Omega\subset\mathbb{R}^2$. The detailed proofs of Theorems 2.1, 2.4-2.6 will be listed in Section 4. Section 5 will be our conclusions.

2. Main resluts

The main results of this paper are:

THEOREM 2.1 (local existence). *Assume the periodic initial data* $\mathbf{m}_0(x),\mathbf{n}_0(x)\in H^2(\Omega)$, $\Omega\in\mathbb{R}^2$. *Then there exists* $T=T(\|\mathbf{m}_0\|_{H^2}\|\mathbf{n}_0\|_{H^2})>0$ *and a unique solution* $\mathbf{m}(x,t)$, $\mathbf{n}(x,t)$ *for Problem (1.3) in the time interval* $[0,T]$ *satisfying* $\mathbf{m}(x,t)\in C([0,T];H^2(\Omega))$, $\mathbf{n}(x,t)\in C([0,T];H^2(\Omega))$.

THEOREM 2.2. *Let dimension* $d=2$. *Under the assumption of Lemma 3.3, for any* $T>0$, *there exists a global unique solution for Problem (1.3) in the time* $[0,T]$ *satisfying* $\mathbf{m}(x,t)\in C([0,T];H^2(\Omega))$, $\mathbf{n}(x,t)\in C([0,T];H^2(\Omega))$.

THEOREM 2.3. *Let dimension* $d=2$, *with the initial data* $\nabla\mathbf{m}_0,\nabla\mathbf{n}_0\in H^m(m\geq 2)$. *Then for any* $T>0$, *there exists a unique solution for Problem (1.3), which satisfies*

$$\partial_t^j\partial_x^\alpha\mathbf{m}\in L^\infty([0,T];L^2(\mathbb{R}^2)),\ \ \partial_t^j\partial_x^\alpha\mathbf{n}\in L^\infty([0,T];L^2(\mathbb{R}^2)),$$
$$\partial_t^k\partial_x^\beta\mathbf{m}\in L^\infty([0,T];L^2(\mathbb{R}^2)),\ \ \partial_t^k\partial_x^\beta\mathbf{n}\in L^\infty([0,T];L^2(\mathbb{R}^2)),$$

with $2j+|\alpha|\leq m$, $2k+|\beta|\leq m+1$.

THEOREM 2.4 (unique theorem). *Let* $(\mathbf{m}_1,\mathbf{n}_1)$ *and* $(\mathbf{m}_2,\mathbf{n}_2)$ *be two smooth solutions for Problem (1.3) with the same initial data* $\mathbf{m}_1(0)=\mathbf{m}_2(0)\in H^\infty(\Omega)$, $\mathbf{n}_1(0)=\mathbf{n}_2(0)\in H^\infty(\Omega)$, *then* $\mathbf{m}_1\equiv\mathbf{m}_2$, $\mathbf{n}_1\equiv\mathbf{n}_2$.

THEOREM 2.5. *Let dimension* $d\geq 3$ *and the initial data* $\mathbf{m}_0(x)\in H^m(\Omega)$, $\mathbf{n}_0(x)\in H^m(\Omega)$, $\Omega\subset\mathbb{R}^d$, $m\geq 2$. *If the conditions of Theorem 2.3 and* $\|\mathbf{m}_0(x)\|_{H^2(\Omega)}\ll 1$, $\|\mathbf{n}_0(x)\|_{H^2(\Omega)}\ll 1$, *then there exists a unique global smooth solution for Problem (1.3), which satisfies*

$$\partial_t^j\partial_x^\alpha\mathbf{m}\in L^\infty([0,T];L^2(\Omega)),\ \ \partial_t^j\partial_x^\alpha\mathbf{n}\in L^\infty([0,T];L^2(\Omega)),$$
$$\partial_t^k\partial_x^\beta\mathbf{m}\in L^\infty([0,T];L^2(\Omega)),\ \ \partial_t^k\partial_x^\beta\mathbf{n}\in L^\infty([0,T];L^2(\Omega)),$$

with $2j+|\alpha|\leq m$, $2k+|\beta|\leq m+1$.

THEOREM 2.6. *Theorems 2.1-2.5 are held for the initial value Problem (1.3), where* $(x,t)\in\mathbb{R}^d\times\mathbb{R}_+$ $(d\geq 2)$.

3. A priori estimate for Problem (1.3) with $\Omega\subset\mathbb{R}^2$

LEMMA 3.1 (Grönwall's inequality). *Let* I *denote an interval of the real line of the form* $[a,\infty)$ *or* $[a,b]$ *or* $[a,b)$ *with* $a<b$. *Let* β *and* u *be real-valued continuous functions defined on* I. *If* u *is differentiable in the interior* I° *of* I *(the interval* I *without the end points* a *and possibly* b*) and satisfies the differential inequality* $u'(t)\leq\beta(t)u(t)$, $t\in I^\circ$, *then* u *is bounded by the solution of the corresponding differential equation*

$$u(t)\leq u(a)\exp\left(\int_a^t\beta(s)ds\right)\text{for all } t\in I^\circ.$$

The generalized Grönwall's inequality reads as:

If $f'\leq C(f\cdot g)+C$, *then* $f\leq C\exp\left(\int_0^t g dt\right)+C$.

LEMMA 3.2 (Gagliardo-Nirenberg's inequality). u: $\mathbb{R}^n\to\mathbb{R}$, *fix* $1\leq q,r\leq\infty$ *and a natural number* m. *Suppose also that a real number* α *and a natural number* j *are such that* $\frac{1}{p}=\frac{j}{n}+(\frac{1}{r}-\frac{m}{n})\alpha+\frac{1-\alpha}{q}$ *and* $\frac{j}{m}\leq\alpha\leq 1$, *then for every function* u: $\mathbb{R}^n\to\mathbb{R}$

that lies in $L^q(\mathbb{R}^n)$ *with* m*-th derivative in* $L^r(\mathbb{R}^n)$ *also has* j*-th derivative in* $L^p(\mathbb{R}^n)$*. Furthermore, there exists* $\| D^j u\|_{L^p}\leq C\| D^m u\|_{L^r}^{\alpha}\| u\|_{L^q}^{1-\alpha}$.

LEMMA 3.3. *Let dimension* $d=2,3$ *and initial data* $\mathbf{m}_0\in H^m(\Omega)$, $\mathbf{n}_0\in H^m(\Omega)$ $(m\geq 2)$*. For the smooth solution of Problem (1.3), we have*

$$\| \mathbf{m}(\cdot,t)\|_{L^2}^2+2\int_0^t \| \nabla\mathbf{m}(\cdot,s)\|_{L^2}^2 ds+2k_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}^2 ds=\| \mathbf{m}_0\|_{L^2}^2, \tag{3.1a}$$

$$\| \mathbf{n}(\cdot,t)\|_{L^2}^2+2\int_0^t \| \nabla\mathbf{n}(\cdot,s)\|_{L^2}^2 ds+2k_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}^2 ds=\| \mathbf{n}_0\|_{L^2}^2, \tag{3.1b}$$

$$\| \mathbf{m}(\cdot,t)\|_{L^\infty(\Omega)}\leq C\| \mathbf{m}_0\|_{H^2(\Omega)}, \quad t\geq 0, \tag{3.1c}$$

$$\| \mathbf{n}(\cdot,t)\|_{L^\infty(\Omega)}\leq C\| \mathbf{n}_0\|_{H^2(\Omega)}, \quad t\geq 0. \tag{3.1d}$$

Proof. Taking the scalar product of 3-dimensional function $\mathbf{m}$ with (1.3a), $\mathbf{n}$ with (1.3b), respectively; and then integrating the result over Ω for the space variable x, we have

$$\frac{1}{2}\frac{d}{dt}\| \mathbf{m}(\cdot,t)\|_{L^2(\Omega)}^2=\int_\Omega \mathbf{m}\cdot\mathbf{m}_t dx=\int_\Omega \mathbf{m}\cdot\Delta\mathbf{m}dx-k_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}^2 dx.$$

Integrating the above equation over $[0,t]$ for the temporal variable, we have (3.1a) and (3.1b).

Now taking the scalar product of $|\mathbf{m}|^{p-2}\mathbf{m}$ $(p\geq 2)$ with (1.3a), and $|\mathbf{n}|^{p-2}\mathbf{n}$ $(p\geq 2)$ with (1.3b), then integrating the results over Ω for the space variable x, respectively, we get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\| \mathbf{m}(\cdot,t)\|_{L^p(\Omega)}^p&=\int_\Omega |\mathbf{m}|^{p-2}\mathbf{m}\cdot\mathbf{m}_t dx\\ &=\int_\Omega |\mathbf{m}|^{p-2}\mathbf{m}\cdot\Delta\mathbf{m}dx-k\int|\mathbf{m}|^{p-2}(1+\mu_0|\mathbf{m}|^2)\mathbf{m}^2 dx\\ &\leq -\int_\Omega |\mathbf{m}|^{p-2}\nabla\mathbf{m}\cdot\nabla\mathbf{m}dx-(p-2)\int_\Omega|\mathbf{m}|^{p-4}(\mathbf{m}\cdot\nabla\mathbf{m})^2 dx\\ &\leq 0.\end{aligned}$$

This inequality implies that

$$\| \mathbf{m}(\cdot,t)\|_{L^p(\Omega)}\leq\| \mathbf{m}_0\|_{H^2(\Omega)}, \quad \forall\ p\geq 2,\ t\geq 0,$$

where we have used the embedding theorem of Sobolev space. Noting that $\| \mathbf{m}_0(x)\|_{H^2}$ is independent of p and letting $p\to\infty$, estimate (3.1c) is obtained.

Estimate (3.1d) holds too. ∎

LEMMA 3.4. *Assuming that* $\frac{k_{11}}{2k_1}=\frac{k_{22}}{2k_2}=a$, $|a|<1$ *and the initial data* $\mathbf{m}_0(x)\in H^2(\Omega)$, $\mathbf{n}_0(x)\in H^2(\Omega)$*, then for the periodic initial value problem (1.3), we have*

$$\sup_{0\leq t<T}\left(\| \nabla\mathbf{m}(\cdot,t)\|_{L^2(\Omega)}+\| \nabla\mathbf{n}(\cdot,t)\|_{L^2(\Omega)}\right)+\int_0^t\left(\| \Delta\mathbf{m}(\cdot,t)\|_{L^2(\Omega)}^2+\| \Delta\mathbf{n}(\cdot,t)\|_{L^2(\Omega)}^2\right)dx$$
$$\leq K,$$

where the constant K *is only dependent on* $\| \mathbf{m}_0\|_{H^2(\Omega)}$ *and* $\| \mathbf{n}_0\|_{H^2(\Omega)}$.

Proof. Multiplying Equation (1.3a) by $\Delta\mathbf{m}$, Equation (1.3b) by $\Delta\mathbf{n}$ respectively, we have

$$\int_\Omega \mathbf{m}_t\Delta\mathbf{m}dx=\int_\Omega|\Delta\mathbf{m}|^2dx+k_{11}\int_\Omega(\mathbf{m}\times\Delta\mathbf{n})\cdot\Delta\mathbf{m}dx-k_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{m}dx, \tag{3.2a}$$

$$\int_\Omega \mathbf{n}_t\Delta\mathbf{n}dx=\int_\Omega|\Delta\mathbf{n}|^2dx+k_{22}\int_\Omega(\mathbf{n}\times\Delta\mathbf{m})\cdot\Delta\mathbf{n}dx-k_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{n}dx. \tag{3.2b}$$

Integrating above by parts, we get

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\mathbf{m}|^2dx=\int_\Omega|\Delta\mathbf{m}|^2dx-k_{11}\int_\Omega(\mathbf{m}\times\Delta\mathbf{m})\cdot\Delta\mathbf{n}dx-k_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{m}dx, \tag{3.3a}$$

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\mathbf{n}|^2dx=\int_\Omega|\Delta\mathbf{n}|^2dx-k_{22}\int_\Omega(\mathbf{n}\times\Delta\mathbf{n})\cdot\Delta\mathbf{m}dx-k_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{n}dx. \tag{3.3b}$$

Multiplying Equation (1.3a) by $\Delta\mathbf{n}$, Equation (1.3b) by $\Delta\mathbf{m}$, and integrating by parts, we have

$$\int_\Omega \mathbf{m}_t\Delta\mathbf{n}dx=\int_\Omega\Delta\mathbf{m}\cdot\Delta\mathbf{n}dx+2k_1\int_\Omega(\mathbf{m}\times\Delta\mathbf{m})\cdot\Delta\mathbf{n}dx-k_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{n}dx, \tag{3.4a}$$

$$\int_\Omega \mathbf{n}_t\Delta\mathbf{m}dx=\int_\Omega\Delta\mathbf{n}\cdot\Delta\mathbf{m}dx+2k_2\int_\Omega(\mathbf{n}\times\Delta\mathbf{n})\cdot\Delta\mathbf{m}dx-k_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{m}dx, \tag{3.4b}$$

then we obtain

$$\begin{aligned}&k_{11}\int_\Omega(\mathbf{m}\times\Delta\mathbf{m})\cdot\Delta\mathbf{n}dx\\ =&a\left[\int_\Omega\mathbf{m}_t\cdot\Delta\mathbf{n}dx-\int_\Omega\Delta\mathbf{m}\cdot\Delta\mathbf{n}dx+k_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{n}dx\right],\end{aligned} \tag{3.5a}$$

$$\begin{aligned}&k_{22}\int_\Omega(\mathbf{n}\times\Delta\mathbf{n})\cdot\Delta\mathbf{m}dx\\ =&a\left[\int_\Omega\mathbf{n}_t\cdot\Delta\mathbf{m}dx-\int_\Omega\Delta\mathbf{n}\cdot\Delta\mathbf{m}dx+k_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{m}dx\right].\end{aligned} \tag{3.5b}$$

Substituting Equation (3.5) into Equation (3.3), we have

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\mathbf{m}|^2dx=&\int_\Omega|\Delta\mathbf{m}|^2dx+a\int_\Omega\Delta\mathbf{m}\cdot\Delta\mathbf{n}dx-a\int_\Omega\mathbf{m}_t\cdot\Delta\mathbf{n}dx\\ &+ak_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{m}dx,\end{aligned} \tag{3.6a}$$

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\mathbf{n}|^2dx=&\int_\Omega|\Delta\mathbf{n}|^2dx+a\int_\Omega\Delta\mathbf{m}\cdot\Delta\mathbf{n}dx-a\int_\Omega\mathbf{n}_t\cdot\Delta\mathbf{m}dx\\ &+ak_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{n}dx.\end{aligned} \tag{3.6b}$$

Adding these two equations together, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(|\nabla\mathbf{m}|^2+|\nabla\mathbf{n}|^2)dx+\int_\Omega(|\Delta\mathbf{m}|^2+|\Delta\mathbf{n}|^2)dx$$

$$=-a\frac{d}{dt}\int(\nabla\mathbf{m}\cdot\nabla\mathbf{n})dx-2a\int_\Omega\Delta\mathbf{m}\cdot\Delta\mathbf{n}dx$$
$$-ak_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{m}dx-ak_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{n}dx, \tag{3.7}$$

where

$$\left|-ak_0\int(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{m}dx\right|\le C_1(1+\|\mathbf{m}\|_{L^\infty}^2)\int|\nabla\mathbf{m}|^2dx\le C_1\|\nabla\mathbf{m}\|_{L^2}^2, \tag{3.8a}$$
$$\left|-ak_0\int(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{n}dx\right|\le C_2(1+\|\mathbf{n}\|_{L^\infty}^2)\int|\nabla\mathbf{n}|^2dx\le C_2\|\nabla\mathbf{n}\|_{L^2}^2. \tag{3.8b}$$

Thus, from Equations (3.7) and (3.8), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\mathbf{m}(\cdot,t)\|_{L^2(\Omega)}^2+\|\nabla\mathbf{n}(\cdot,t)\|_{L^2(\Omega)}^2\right)+\frac{1}{2}(1-|a|)\int_\Omega(|\nabla\mathbf{m}|^2+|\nabla\mathbf{n}|^2)dx$$
$$+(1-|a|)\left(\|\Delta\mathbf{m}(\cdot,t)\|_{L^2}^2+\|\Delta\mathbf{n}(\cdot,t)\|_{L^2}^2\right)$$
$$\le C_3\left(\|\nabla\mathbf{m}\|_{L^2(\Omega)}^2+\|\nabla\mathbf{n}\|_{L^2(\Omega)}^2\right),$$

by using Grönwall's inequality and $|a|<1$, the lemma is proved. □

Lemma 3.5. *Let dimension $d=2$ and assume $k_{11}\le 5\|\mathbf{m}_0(x)\|_{H^2(\Omega)}$, $k_{22}\le 5\|\mathbf{n}_0(x)\|_{H^2(\Omega)}$. Then for the smooth solution of Problem (1.3), we get*

$$\sup_{0\le t<T}\|\Delta\mathbf{m}(\cdot,t)\|_{L^2(\Omega)}+\sup_{0\le t<T}\|\Delta\mathbf{n}(\cdot,t)\|_{L^2(\Omega)}\le K,$$
$$\int_0^t\left(\|\Delta\nabla\mathbf{m}\|_{L^2(\Omega)}^2+\|\Delta\nabla\mathbf{n}\|_{L^2(\Omega)}^2\right)dt\le K,$$

where the constant K is only dependent on $\|\mathbf{m}_0\|_{H^2(\Omega)}$ and $\|\mathbf{n}_0\|_{H^2(\Omega)}$.

Proof. Taking the Laplace operator Δ to Equations (1.3a) and (1.3b), we get

$$\Delta\mathbf{m}_t=\Delta^2\mathbf{m}+2k_1\Delta(\mathbf{m}\times\Delta\mathbf{m})+k_{11}\Delta(\mathbf{m}\times\Delta\mathbf{n})-k_0\Delta(1+\mu_0|\mathbf{m}|^2)\mathbf{m}, \tag{3.9a}$$
$$\Delta\mathbf{n}_t=\Delta^2\mathbf{n}+2k_2\Delta(\mathbf{n}\times\Delta\mathbf{n})+k_{22}\Delta(\mathbf{n}\times\Delta\mathbf{m})-k_0\Delta(1+\mu_1|\mathbf{n}|^2)\mathbf{n}. \tag{3.9b}$$

Taking the scalar product Equation (3.9a) with $\Delta\mathbf{m}$, Equation (3.9b) with $\Delta\mathbf{n}$, respectively, and integrating for the variable x over $x\in\Omega$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\mathbf{m}\|_{L^2(\Omega)}^2+\|\Delta\nabla\mathbf{m}\|_{L^2(\Omega)}^2=2k_1\int\Delta(\mathbf{m}\times\Delta\mathbf{m})\cdot\Delta\mathbf{m}dx$$
$$+k_{11}\int\Delta(\mathbf{m}\times\Delta\mathbf{n})\cdot\Delta\mathbf{m}dx-k_0\int\Delta(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot\Delta\mathbf{m}dx, \tag{3.10a}$$
$$\frac{1}{2}\frac{d}{dt}\|\Delta\mathbf{n}\|_{L^2(\Omega)}^2+\|\Delta\nabla\mathbf{n}\|_{L^2(\Omega)}^2=2k_2\int\Delta(\mathbf{n}\times\Delta\mathbf{n})\cdot\Delta\mathbf{n}dx$$
$$+k_{22}\int\Delta(\mathbf{n}\times\Delta\mathbf{m})\cdot\Delta\mathbf{n}dx-k_0\int\Delta(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot\Delta\mathbf{n}dx. \tag{3.10b}$$

By Gagliardi-Nirenberg's inequality, we have

$$\|\nabla u\|_{L^4}\le C\|\nabla u\|_{H^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{3}{4}},$$

$$\| \Delta u \|_{L^4} \leq C \| \Delta u \|_{H^2}^{\frac{1}{2}} \| \Delta u \|_{L^2}^{\frac{1}{2}},$$

(i)

$$\begin{aligned}\left| 2k_1 \int_\Omega \Delta(\mathbf{m} \times \Delta \mathbf{m}) \Delta \mathbf{m} dx \right| &\leq 2k_1 \left| \int_\Omega \nabla(\mathbf{m} \times \Delta \mathbf{m}) \nabla^3 \mathbf{m} dx \right| \\ &\leq 2k_1 \| \nabla \mathbf{m} \|_{L^4} \| \Delta \mathbf{m} \|_{L^4} \| \nabla^3 \mathbf{m} \|_{L^2} \\ &\leq C \| \Delta \mathbf{m} \|_{L^2}^{\frac{1}{2}} \| \Delta \nabla \mathbf{m} \|_{L^2}^{\frac{7}{4}} \\ &\leq \frac{1}{5} \| \Delta \nabla \mathbf{m} \|_{L^2}^2 + C(1 + \| \Delta \mathbf{m} \|_{L^2}^4),\end{aligned}$$

(ii)

$$\begin{aligned}&\left| k_{11} \int_\Omega \Delta(\mathbf{m} \times \Delta \mathbf{n}) \cdot \Delta \mathbf{m} dx \right| = \left| k_{11} \int_\Omega \nabla(\mathbf{m} \times \Delta \mathbf{n}) \cdot \nabla^3 \mathbf{m} dx \right| \\ &\leq k_{11} \| \nabla \mathbf{m} \|_{L^4} \| \Delta \mathbf{n} \|_{L^4} \| \nabla^3 \mathbf{m} \|_{L^2} + k_{11} \| \mathbf{m} \|_{L^\infty} \| \nabla^3 \mathbf{n} \|_{L^2} \| \nabla^3 \mathbf{m} \|_{L^2} \\ &\leq C \| \nabla \mathbf{m} \|_{H^2}^{\frac{1}{4}} \| \nabla \mathbf{m} \|_{L^2}^{\frac{3}{4}} \| \Delta \mathbf{n} \|_{H^2}^{\frac{1}{2}} \| \Delta \mathbf{n} \|_{L^2}^{\frac{1}{2}} \| \nabla^3 \mathbf{m} \|_{L^2} + k_{11} \| \mathbf{m}_0 \|_{H^2} \| \nabla^3 \mathbf{n} \|_{L^2} \| \nabla^3 \mathbf{m} \|_{L^2} \\ &\leq C \| \nabla^3 \mathbf{m} \|_{L^2}^{\frac{5}{4}} \| \Delta^3 \mathbf{n} \|_{L^2}^{\frac{1}{2}} \| \Delta \mathbf{n} \|_{L^2}^{\frac{1}{2}} + k_{11} \| \mathbf{m}_0 \|_{H^2} \| \nabla^3 \mathbf{n} \|_{L^2} \| \nabla^3 \mathbf{m} \|_{L^2} \\ &\leq \frac{1}{5} \| \nabla^3 \mathbf{m} \|_{L^2}^2 + C(\| \nabla^3 \mathbf{n} \|_{L^2}^{\frac{4}{3}} \| \Delta \mathbf{n} \|_{L^2}^{\frac{4}{3}}) + k_{11} \| \mathbf{m}_0 \|_{H^2} \| \nabla^3 \mathbf{n} \|_{L^2} \| \nabla^3 \mathbf{m} \|_{L^2} \\ &\leq \frac{1}{5} \| \nabla^3 \mathbf{m} \|_{L^2}^2 + C \| \Delta \mathbf{n} \|_{L^2}^4 + \frac{1}{5}(\| \nabla^3 \mathbf{m} \|_{L^2}^2 + \| \nabla^3 \mathbf{n} \|_{L^2}^2),\end{aligned}$$

if $k_{11} \| \mathbf{m}_0 \|_{H^2} < \frac{2}{5}$.

(iii)

$$\begin{aligned}\left| k_0 \int \Delta(1 + \mu_0 |\mathbf{m}|^2) \mathbf{m} \cdot \Delta \mathbf{m} dx \right| &\leq k_0 \| \mathbf{m} \|_{L^\infty}^2 (\| \nabla \mathbf{m} \|_{L^2} \| \nabla^3 \mathbf{m} \|_{L^2}) \\ &\leq \frac{1}{5} \| \nabla^3 \mathbf{m} \|_{L^2}^2 + C.\end{aligned}$$

The estimate for the right side of Equation (3.10b) is similar to the above (i), (ii), (iii). So we can get

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} (\| \Delta \mathbf{m} \|_{L^2(\Omega)}^2 + \| \Delta \mathbf{n} \|_{L^2(\Omega)}^2) + \frac{4}{5} (\| \Delta \nabla \mathbf{m} \|_{L^2}^2 + \| \Delta \nabla \mathbf{n} \|_{L^2}^2) \\ &\leq C(1 + \| \Delta \mathbf{m} \|_{L^2}^4 + \| \Delta \mathbf{n} \|_{L^2}^4).\end{aligned}$$

By using the generalized Grönwall's inequality and

$$\int_0^t (\| \Delta \mathbf{m} \|_{L^2}^2 + \| \Delta \mathbf{n} \|_{L^2}^2) dt < K,$$

we have

$$\| \Delta \mathbf{m}(\cdot, t) \|_{L^2}^2 + \| \Delta \mathbf{n}(\cdot, t) \|_{L^2}^2 \leq K \tag{3.12}$$

and

$$\int_0^t \left(\| \nabla \Delta \mathbf{m}(\cdot, t) \|_{L^2}^2 + \| \nabla \Delta \mathbf{n}(\cdot, t) \|_{L^2}^2 \right) dt \leq K,$$

where the constant K is only dependent on $\|\mathbf{m}_0\|_{H^2(\Omega)}$, $\|\mathbf{m}_0\|_{H^2(\Omega)}$. ∎

Similarly, we can prove that $\|\nabla^3\mathbf{m}\|_{L^2}$ is bounded, which means $\|\nabla\mathbf{m}\|_{L^\infty}$ is bounded.

Proof. **(Proof of Theorem 2.3.)** To prove Theorem 2.3, applying the differential operator D^{m+1} to both the sides of Equations (1.3a) and (1.3b), and taking the scalar product with $D^{m+1}\mathbf{m}$ and $D^{m+1}\mathbf{n}$, then integrating with respect to x over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|D^{m+1}\mathbf{m}\|_{L^2}^2+\|\nabla D^{m+1}\mathbf{m}\|_{L^2}^2=2k_1\int_\Omega D^{m+1}(\mathbf{m}\times\Delta\mathbf{m})\cdot D^{m+1}\mathbf{m}dx$$
$$+k_{11}\int_\Omega D^{m+1}(\mathbf{m}\times\Delta\mathbf{n})\cdot D^{m+1}\mathbf{m}dx-k_0\int_\Omega D^{m+1}(1+\mu_0|\mathbf{m}|^2)\mathbf{m}\cdot D^{m+1}\mathbf{m}dx, \tag{3.13}$$

$$\frac{1}{2}\frac{d}{dt}\|D^{m+1}\mathbf{n}\|_{L^2}^2+\|\nabla D^{m+1}\mathbf{n}\|_{L^2}^2=2k_1\int_\Omega D^{m+1}(\mathbf{n}\times\Delta\mathbf{n})\cdot D^{m+1}\mathbf{n}dx$$
$$+k_{22}\int_\Omega D^{m+1}(\mathbf{n}\times\Delta\mathbf{m})\cdot D^{m+1}\mathbf{n}dx-k_0\int_\Omega D^{m+1}(1+\mu_1|\mathbf{n}|^2)\mathbf{n}\cdot D^{m+1}\mathbf{n}dx. \tag{3.14}$$

By the assumption of induction, the last term on the right side of Equations (3.13) and (3.14) can be controlled by $\|D^{m+1}\mathbf{m}\|_{L^2}^2$ and $\|D^{m+1}\mathbf{n}\|_{L^2}^2$ respectively. For the first term on the right side of Equations (3.13) and (3.14), we have the following derivations,

$$\int D^{m+1}(\mathbf{m}\times\Delta\mathbf{m})\cdot D^{m+1}\mathbf{m}dx=-\int D^{m+1}(\mathbf{m}\times\nabla\mathbf{m})\cdot\nabla D^{m+1}\mathbf{m}dx,$$
$$\int D^{m+1}(\mathbf{n}\times\Delta\mathbf{n})\cdot D^{m+1}\mathbf{n}dx=-\int D^{m+1}(\mathbf{n}\times\nabla\mathbf{n})\cdot\nabla D^{m+1}\mathbf{n}dx,$$

and

$$D^{m+1}(\mathbf{m}\times\Delta\mathbf{m})=D^{m+1}\mathbf{m}\times\nabla\mathbf{m}+\mathbf{m}\times D^{m+1}\nabla\mathbf{m}+\sum_{h=1}^{m}C_h(D^h\mathbf{m}\times D^{m+1-h}\nabla\mathbf{m}),$$
$$D^{m+1}(\mathbf{n}\times\Delta\mathbf{n})=D^{m+1}\mathbf{n}\times\nabla\mathbf{n}+\mathbf{n}\times D^{m+1}\nabla\mathbf{n}+\sum_{h=1}^{m}C_h(D^h\mathbf{n}\times D^{m+1-h}\nabla\mathbf{n}),$$

thus,

$$\left|\int D^{m+1}(\mathbf{m}\times\Delta\mathbf{m})\cdot D^{m+1}\mathbf{m}dx\right|$$
$$\le\left|\int D^{m+1}\mathbf{m}\times\nabla\mathbf{m}\cdot\nabla D^{m+1}\mathbf{m}dx\right|+\left|\int\sum_{h=1}^{m}C_h(D^h\mathbf{m}\times D^{m+1-h}\nabla\mathbf{m})\cdot\nabla D^{m+1}\mathbf{m}dx\right|$$
$$\le C\|\nabla\mathbf{m}\|_{L^\infty}\|D^{m+1}\mathbf{m}\|_{L^2}\|\nabla D^{m+1}\mathbf{m}\|_{L^2},$$
$$\left|\int D^{m+1}(\mathbf{n}\times\Delta\mathbf{n})\cdot D^{m+1}\mathbf{n}dx\right|$$
$$\le\left|\int D^{m+1}\mathbf{n}\times\nabla\mathbf{n}\cdot\nabla D^{m+1}\mathbf{n}dx\right|+\left|\int\sum_{h=1}^{m}C_h(D^h\mathbf{n}\times D^{m+1-h}\nabla\mathbf{n})\cdot\nabla D^{m+1}\mathbf{n}dx\right|$$
$$\le C\|\nabla\mathbf{n}\|_{L^\infty}\|D^{m+1}\mathbf{n}\|_{L^2}\|\nabla D^{m+1}\mathbf{n}\|_{L^2}.$$

For the second term on the right side of Equation (3.13), we are going to deal with the highest-order term

$$\int_\Omega (\mathbf{m}\times D^{m+1}\Delta\mathbf{n})\cdot D^{m+1}\mathbf{m}dx,$$

while the other terms can be handled with the same method.

In fact, the above term can be written as

$$\int_\Omega (\nabla\mathbf{m}\times D^{m}\Delta\mathbf{n})\cdot D^{m+1}\mathbf{m}dx+\int_\Omega (\mathbf{m}\times D^{m}\Delta\mathbf{n})\cdot D^{m+2}\mathbf{m}dx$$

which can be controlled by

$$\|\nabla\mathbf{m}\|_{L^\infty}\|D^{m+2}\mathbf{n}\|_{L^2}\|D^{m+1}\mathbf{m}\|_{L^2}+\|\mathbf{m}\|_{L^\infty}\|D^{m+2}\mathbf{n}\|_{L^2}\|D^{m+2}\mathbf{m}\|_{L^2}.$$

It has been proved that $\|\nabla\mathbf{m}\|_{L^\infty}$ is bounded, furthermore, $\|\mathbf{m}\|_{L^\infty}$ is sufficiently small. Thus the second term on the right side of Equation (3.13) can be controlled by

$$C\|D^{m+1}\mathbf{m}\|_{L^2}^2+\frac{1}{4}\|\nabla D^{m+1}\mathbf{m}\|_{L^2}^2+\frac{1}{4}\|\nabla D^{m+1}\mathbf{m}\|_{L^2}^2.$$

Similarly, the second term on the right side of Equation (3.14) can be controlled by

$$C\|D^{m+1}\mathbf{n}\|_{L^2}^2+\frac{1}{4}\|\nabla D^{m+1}\mathbf{m}\|_{L^2}^2+\frac{1}{4}\|\nabla D^{m+1}\mathbf{m}\|_{L^2}^2.$$

Combing the above equations, we have

$$\frac{d}{dt}\|D^{m+1}\mathbf{m}\|_{L^2}^2+\frac{d}{dt}\|D^{m+1}\mathbf{n}\|_{L^2}^2+\|\nabla D^{m+1}\mathbf{m}\|_{L^2}^2+\|\nabla D^{m+1}\mathbf{n}\|_{L^2}^2$$

$$\le\|D^{m+1}\mathbf{m}\|_{L^2}^2+\|D^{m+1}\mathbf{n}\|_{L^2}^2,$$

using Grönwall's inequality, we conclude the theorem. □

4. Proof of Theorems 2.1, 2.4-2.6

4.1. Proof of Theorem 2.1.

Proof. By using the Galerkin method, we can easily obtain Theorem 2.1. Assume the approximate solutions for Problem (1.3) as

$$\mathbf{m}_N=\sum_{j=1}^{N}\alpha_{jN}(t)\omega_j(x), \tag{4.1a}$$

$$\mathbf{n}_N=\sum_{j=1}^{N}\beta_{jN}(t)\omega_j(x), \qquad j=1,2,\cdots,N, \tag{4.1b}$$

where $\omega_j(x)$ is the base function, $-\Delta\omega_j(x)=\lambda_j\omega_j(x)$, $x\in\Omega$. Equations (4.1) need to satisfy

$$\big((\mathbf{m}_{Nt}-\Delta\mathbf{m}_N-2k_1\mathbf{m}_N\times\Delta\mathbf{m}_N-k_{11}\mathbf{m}_N\times\Delta\mathbf{n}_N+k_0(1+\mu_0|\mathbf{m}_N|^2)\mathbf{m}_N),\omega_j(x)\big)=0, \tag{4.2a}$$

$$\big((\mathbf{n}_{Nt}-\Delta\mathbf{n}_N-2k_2\mathbf{n}_N\times\Delta\mathbf{n}_N-k_{22}\mathbf{n}_N\times\Delta\mathbf{m}_N+k_0(1+\mu_1|\mathbf{n}_N|^2)\mathbf{n}_N),\omega_j(x)\big)=0, \tag{4.2b}$$

where (,) means the inner product in L^2. Multiplying Equation (4.2a) by $\alpha_{jN}(t)$, Equation (4.2b) by $\beta_{jN}(t)$, and making the summation for j from 1 to N, we get

$$\big((\mathbf{m}_{Nt}-\Delta\mathbf{m}_N-2k_1\mathbf{m}_N\times\Delta\mathbf{m}_N-k_{11}\mathbf{m}_N\times\Delta\mathbf{n}_N+k_0(1+\mu_0|\mathbf{m}_N|^2)\mathbf{m}_N),\mathbf{m}_N\big)=0, \tag{4.3a}$$

$$\big((\mathbf{n}_{Nt}-\Delta\mathbf{n}_N-2k_2\mathbf{n}_N\times\Delta\mathbf{n}_N-k_{22}\mathbf{n}_N\times\Delta\mathbf{m}_N+k_0(1+\mu_1|\mathbf{n}_N|^2)\mathbf{n}_N),\mathbf{n}_N\big)=0. \tag{4.3b}$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}|\mathbf{m}_N|^2+\|\nabla\mathbf{m}_N\|_{L^2}^2+k_0|\mathbf{m}_N|^2+k_0\mu_0|\mathbf{m}_N|^4=0, \tag{4.4a}$$

$$\frac{1}{2}\frac{d}{dt}|\mathbf{n}_N|^2+\|\nabla\mathbf{n}_N\|_{L^2}^2+k_0|\mathbf{n}_N|^2+k_0\mu_1|\mathbf{n}_N|^4=0. \tag{4.4b}$$

Via the Grönwall's inequality, we conclude

$$\|\mathbf{m}_N(t)\|_{L^2}^2\le C, \tag{4.5a}$$

$$\|\mathbf{n}_N(t)\|_{L^2}^2\le C, \tag{4.5b}$$

where C is independent of N. Let $N\to\infty$ and through the priori estimates, we can easily get the existence of the approximate solutions. Thereby, the local existence of the solutions for Problem (1.3) can be obtained. ∎

4.2. Proof of Theorem 2.4.

Proof. The proof is standard. Setting $\mathbf{w}_1=\mathbf{m}_1-\mathbf{m}_2$ and $\mathbf{w}_2=\mathbf{n}_1-\mathbf{n}_2$, we will prove $\mathbf{w}_1=0$ and $\mathbf{w}_2=0$. Since $(\mathbf{m}_1,\mathbf{n}_1)$ and $(\mathbf{m}_2,\mathbf{n}_2)$ satisfy Equations (1.3a) and (1.3b),

$$\mathbf{m}_{1,t}=\Delta\mathbf{m}_1+2k_1\mathbf{m}_1\times\Delta\mathbf{m}_1+k_{11}\mathbf{m}_1\times\Delta\mathbf{n}_1-k_0(1+\mu_0|\mathbf{m}_1|^2)\mathbf{m}_1, \tag{4.6a}$$

$$\mathbf{m}_{2,t}=\Delta\mathbf{m}_2+2k_1\mathbf{m}_2\times\Delta\mathbf{m}_2+k_{11}\mathbf{m}_2\times\Delta\mathbf{n}_2-k_0(1+\mu_0|\mathbf{m}_2|^2)\mathbf{m}_2, \tag{4.6b}$$

$$\mathbf{n}_{1,t}=\Delta\mathbf{n}_1+2k_2\mathbf{n}_1\times\Delta\mathbf{n}_1+k_{22}\mathbf{n}_1\times\Delta\mathbf{m}_1-k_0(1+\mu_1|\mathbf{n}_1|^2)\mathbf{n}_1, \tag{4.6c}$$

$$\mathbf{n}_{2,t}=\Delta\mathbf{n}_2+2k_2\mathbf{n}_2\times\Delta\mathbf{n}_2+k_{22}\mathbf{n}_2\times\Delta\mathbf{m}_2-k_0(1+\mu_1|\mathbf{n}_2|^2)\mathbf{n}_2, \tag{4.6d}$$

the subtraction of the first two equations gives

$$\begin{aligned}\mathbf{w}_{1,t}=&\Delta\mathbf{w}_1+2k_1(\mathbf{m}_1\times\Delta\mathbf{m}_1-\mathbf{m}_2\times\Delta\mathbf{m}_2)\\&+k_{11}(\mathbf{m}_1\times\Delta\mathbf{n}_1-\mathbf{m}_2\times\Delta\mathbf{n}_2)-k_0\mathbf{w}_1-k_0\mu_0(|\mathbf{m}_1|^2\mathbf{m}_1-|\mathbf{m}_2|^2\mathbf{m}_2),\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}\mathbf{m}_1\times\Delta\mathbf{m}_1-\mathbf{m}_2\times\Delta\mathbf{m}_2&=\mathbf{m}_1\times\Delta\mathbf{m}_1-\mathbf{m}_2\times\Delta\mathbf{m}_1+\mathbf{m}_2\times\Delta\mathbf{m}_1-\mathbf{m}_2\times\Delta\mathbf{m}_2\\&=(\mathbf{m}_1-\mathbf{m}_2)\times\Delta\mathbf{m}_1+\mathbf{m}_2\times(\Delta\mathbf{m}_1-\Delta\mathbf{m}_2)\\&=\mathbf{w}_1\times\Delta\mathbf{m}_1+\mathbf{m}_2\times\Delta\mathbf{w}_1,\end{aligned}$$

$$|\mathbf{m}_1|^2\mathbf{m}_1-|\mathbf{m}_2|^2\mathbf{m}_2=(|\mathbf{m}_1|^2+|\mathbf{m}_2|^2+\mathbf{m}_1\cdot\mathbf{m}_2)\mathbf{w}_1,$$

$$\begin{aligned}\mathbf{m}_1\times\Delta\mathbf{n}_1-\mathbf{m}_2\times\Delta\mathbf{n}_2&=\mathbf{m}_1\times\Delta\mathbf{n}_1-\mathbf{m}_2\times\Delta\mathbf{n}_1+\mathbf{m}_2\times\Delta\mathbf{n}_1-\mathbf{m}_2\times\Delta\mathbf{n}_2\\&=(\mathbf{m}_1-\mathbf{m}_2)\times\Delta\mathbf{n}_1+\mathbf{m}_2\times(\Delta\mathbf{n}_1-\Delta\mathbf{n}_2)\\&=\mathbf{w}_1\times\Delta\mathbf{n}_1+\mathbf{m}_2\times\mathbf{w}_2.\end{aligned}$$

Then Equation (4.7) becomes

$$\begin{aligned}\mathbf{w}_{1,t} = &\Delta \mathbf{w}_1 + 2k_1(\mathbf{w}_1 \times \Delta \mathbf{m}_1 + \mathbf{m}_2 \times \Delta \mathbf{w}_1)\\ &+k_{11}(\mathbf{w}_1 \times \Delta \mathbf{n}_1 + \mathbf{m}_2 \times \Delta \mathbf{w}_2) - k_0 \mathbf{w}_1 - k_0\mu_0(|\mathbf{m}_1|^2 + |\mathbf{m}_2|^2 + \mathbf{m}_1 \cdot \mathbf{m}_2)\mathbf{w}_1. \end{aligned} \tag{4.8}$$

Taking the inner product with $\mathbf{w}_1$ on both sides of Equation (4.8), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\mathbf{w}_1|^2 dx = &-\int |\nabla \mathbf{w}_1|^2 dx + 2k_1 \int (\mathbf{m}_2 \times \Delta \mathbf{w}_1)\cdot \mathbf{w}_1 dx + k_{11}\int (\mathbf{m}_2 \times \Delta \mathbf{w}_2)\cdot \mathbf{w}_1 dx\\ &-k_0 \int |\mathbf{w}_1|^2 dx - k_0\mu_0 \int (|\mathbf{m}_1|^2 + |\mathbf{m}_2|^2 + \mathbf{m}_1\cdot \mathbf{m}_2)|\mathbf{w}_1|^2 dx,\end{aligned}$$

where

$$\left|\int (\mathbf{m}_2 \times \Delta \mathbf{w}_1)\cdot \mathbf{w}_1 dx\right| = \left|\int (\nabla \mathbf{m}_2 \times \nabla \mathbf{w}_1)\cdot \mathbf{w}_1 dx\right| \le 2\| \nabla \mathbf{m}_2 \|_{L^\infty}^2 \| \mathbf{w}_1 \|_{L^2}^2 + \frac{1}{2}\| \nabla \mathbf{w}_1 \|_{L^2}^2,$$

$$\left|\int (\mathbf{m}_2 \times \Delta \mathbf{w}_2)\cdot \mathbf{w}_1 dx\right| \le 2\| \nabla \mathbf{m}_2 \|_{L^\infty}^2 \| \mathbf{w}_1 \|_{L^2}^2 + \| \nabla \mathbf{w}_2 \|_{L^2}^2 + 2\| \mathbf{m}_2 \|_{L^\infty}^2 \| \nabla \mathbf{w}_1 \|_{L^2}^2,$$

$$\left|\int (|\mathbf{m}_1|^2 + |\mathbf{m}_2|^2 + \mathbf{m}_1\cdot \mathbf{m}_2)|\mathbf{w}_1|^2 dx\right| \le 2\left(\| \mathbf{m}_2 \|_{L^\infty}^2 + \| \mathbf{m}_1 \|_{L^\infty}^2\right)\| \mathbf{w}_1 \|_{L^2}^2 .$$

Since $(\mathbf{m}_1,\mathbf{n}_1)$ and $(\mathbf{m}_2,\mathbf{n}_2)$ are smooth, the norm $\| \nabla \mathbf{m}_2 \|_{L^\infty}^2$, $\| \mathbf{m}_1 \|_{L^\infty}^2$ and $\| \mathbf{m}_2 \|_{L^\infty}^2$ can be replaced by a constant C, thus it can be concluded that

$$\frac{d}{dt}\int |\mathbf{w}_1|^2 dx \le C\int |\mathbf{w}_1|^2 dx.$$

By Grönwall's inequality and the fact that $\mathbf{w}_1(x,0)\equiv 0$, $\mathbf{w}_1 \equiv 0$ will be obtained.

$\mathbf{w}_2 \equiv 0$ holds too. □

4.3. Proof of Theorem 2.5. We have the following lemma.

LEMMA 4.1. *Let dimension $d=3$ with initial data $\mathbf{m}_0 \in H^m$ $(m\ge 2)$, $\mathbf{n}_0 \in H^m$ $(m \ge 2)$ and $\| \mathbf{m}_0 \|_{H^2} \ll 1$, $\| \mathbf{n}_0 \|_{H^2} \ll 1$, then for the smooth solutions for Problem (1.3), one has the following estimates.*

$$\| \Delta \mathbf{m}(\cdot,t) \|_{L^2}^2 + \int_0^t \| \Delta\nabla \mathbf{m}(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{m}_0 \|_{H^2}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\| \mathbf{m}_t(\cdot,t) \|_{L^2}^2 + \int_0^t \| \nabla \mathbf{m}_t(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{m}_0 \|_{H^2}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\| \Delta\nabla \mathbf{m}(\cdot,t) \|_{L^2}^2 + \int_0^t \| \Delta^2 \mathbf{m}(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{m}_0 \|_{H^3}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\| \nabla \mathbf{m}_t(\cdot,t) \|_{L^2}^2 + \int_0^t \| \Delta \mathbf{m}_t(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{m}_0 \|_{H^3}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\| \Delta \mathbf{n}(\cdot,t) \|_{L^2}^2 + \int_0^t \| \Delta\nabla \mathbf{n}(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{n}_0 \|_{H^2}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\| \mathbf{n}_t(\cdot,t) \|_{L^2}^2 + \int_0^t \| \nabla \mathbf{n}_t(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{n}_0 \|_{H^2}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\| \Delta\nabla \mathbf{n}(\cdot,t) \|_{L^2}^2 + \int_0^t \| \Delta^2 \mathbf{n}(\cdot,s) \|_{L^2}^2 ds \le C(T, \| \mathbf{n}_0 \|_{H^3}), \qquad \forall\ T>0,\ t\in[0,T],$$

$$\|\nabla\mathbf{n}_t(\cdot,t)\|_{L^2}^2+\int_0^t\|\Delta\mathbf{n}_t(\cdot,s)\|_{L^2}^2\,ds\le C(T,\|\mathbf{n}_0\|_{H^3}),\qquad\forall\ T>0,\ \ t\in[0,T].$$

Proof. Only noticing that

$$\|\nabla\mathbf{m}\|_{L^6}\le C\|\mathbf{m}\|_{L^\infty}\|\nabla^3\mathbf{m}\|_{L^2}^{\frac{1}{2}},$$
$$\|\Delta\mathbf{m}\|_{L^3}\le C\|\mathbf{m}\|_{L^\infty}\|\nabla^3\mathbf{m}\|_{L^2}^{\frac{1}{2}},$$

we have

$$\begin{aligned}\left|\int\nabla(\mathbf{m}\times\nabla\mathbf{m})\Delta\mathbf{m}dx\right|&\le C\|\nabla\mathbf{m}\|_{L^6}\|\Delta\mathbf{m}\|_{L^3}\|\nabla^3\mathbf{m}\|_{L^2}\\&\le C\|\mathbf{m}\|_{L^\infty}^2\|\nabla^3\mathbf{m}\|_{L^2}^2\\&\le C\|\mathbf{m}_0\|_{H^2}^2\|\nabla^3\mathbf{m}\|_{L^2}^2.\end{aligned}$$

Then, we can prove this lemma. □

4.4. Proof of Theorem 2.6.

Proof. As the above estimates are independent of D, letting $D\to\infty$, the theorem is proved. □

5. Conclusions

In this paper, we have studied the antiferromagnets Landau-Lifshitz-Bloch equation, i.e., Problem (1.3), which is recommended as a model to describe the dynamics of micromagnets under high temperature. The existence and uniqueness of smooth solutions for Problem (1.3) with periodic initial value have been proved in $\mathbb{R}^2$ and $\mathbb{R}^3$. The main results can be seen in Theorems 2.1-2.6.

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