AN OPEN MICROSCOPIC MODEL OF HEAT CONDUCTION: EVOLUTION AND NON-EQUILIBRIUM STATIONARY STATES*

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Abstract. We consider a one-dimensional chain of coupled oscillators in contact at both ends with heat baths at different temperatures, and subject to an external force at one end. The Hamiltonian dynamics in the bulk is perturbed by random exchanges of the neighbouring momenta such that the energy is locally conserved. We prove that in the stationary state the energy and the volume stretch profiles, in large scale limit, converge to the solutions of a diffusive system with Dirichlet boundary conditions. As a consequence the macroscopic temperature stationary profile presents a maximum inside the chain higher than the thermostats temperatures, as well as the possibility of uphill diffusion (energy current against the temperature gradient). Finally, we are also able to derive the non-stationary macroscopic coupled diffusive equations followed by the energy and volume stretch profiles.

Keywords. Open chain of oscillators; heat conduction; non-equilibrium stationary state; uphill heat diffusion.

AMS subject classifications. 60K35; 82C70.

1. Introduction

Non-equilibrium transport in one dimension presents itself to be an interesting phenomenon and in many models numerical simulations can be easily performed. Most of the attention has been focused on the study of the non-equilibrium stationary states (NESS), where the systems are subject to exterior heat baths at different temperatures and other external forces, so that the invariant measure is not the equilibrium Gibbs measure.

The most interesting models are those with various conserved quantities (energy, momentum, volume stretch...) whose transport is coupled. The densities of these quantities may evolve at different time scales, particularly when the spatial dimension of the system equals one. For example, in the Fermi-Pasta-Ulam (FPU) chain, volume stretch, mechanical energy and momentum all evolve in the hyperbolic time scale. Their evolution is governed by the Euler equations (see [8]) while the thermal energy is expected to evolve at a superdiffusive time scale, with an autonomous evolution described by a fractional heat equation. This has been predicted [22], confirmed by many numerical experiments on the NESS [18,19] and proved analytically for harmonic chains with random exchanges of momenta that conserve energy, momentum and volume stretch, see [12].

In contrast to the situation described above, the present paper deals with a system for which conserved quantities evolve macroscopically in the same *diffusive* time scale, and their macroscopic evolution is governed by a system of *coupled* diffusive equations. One example is given by the chain of coupled rotors, whose dynamics conserves the energy and the angular momentum. In [10] the NESS of this chain is studied numerically, when Langevin thermostats are applied at both ends, while a constant force is applied

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to one end and the position of the rotor on the opposite side is kept fixed. While heat flows from the thermostats, work is performed by the torque, increasing the mechanical energy, which is then transformed into thermal energy by the dynamics of the rotors. The stationary temperature profiles observed numerically in [10] present a maximum inside the chain higher than the temperature of both thermostats. Furthermore, a negative linear response for the energy flux has been observed for certain values of the external parameters. This phenomenon is referred to in the literature as an *uphill diffusion*, see [17] or [7] and references therein. These numerical results have been confirmed in [11], as well as an instability of the system when thermostats are at zero temperature.

The present work aims at describing a similar phenomenon for the NESS, but for a different model. In particular, we are able to show rigorously that the maximum of the temperature profile occurs inside the system. According to our knowledge it is the first theoretical result that rigorously establishes the heating inside the system and uphill diffusion phenomena.

More specifically, we consider a chain of unpinned harmonic oscillators whose dynamics is perturbed by a random mechanism that conserves the energy and volume stretch: any two nearest neighbour particles exchange their momenta randomly in such a way that the total kinetic energy is conserved. Two Langevin thermostats are attached at the opposite ends of the chain and a constant force $\overline{\tau}_+$ acts on the last particle of the chain. This system has only two conserved quantities: total energy and volume. Since the random mechanism does not conserve the total momentum, the macroscopic behaviour of these two quantities is diffusive, and the non-stationary hydrodynamic limit with periodic boundary conditions (no thermostats or exterior force present) has been proven in [16].

The action of this constant force puts the system out of equilibrium, even when the temperatures of the thermostats are equal. As in the rotor chain described above, the exterior force performs positive work on the system, that increases the mechanical energy (concentrated on low frequency modes). The random mechanism, which consists in the kinetic energy exchange between neighbouring atoms, see Definition (2.8) and the following explanations, transforms the mechanical energy into the thermal one (uniformly distributed in all frequencies, when the system is in a local equilibrium), which is eventually dissipated by the thermostats. This transfer of mechanical into thermal energy happens in the bulk of the system and is already completely predicted by the solution of the macroscopic diffusive system of equations obtained in the hydrodynamic limit [16], see also [1] for a similar model without boundary conditions.

In the present article we study the NESS of this dynamics. We prove, see Theorem 3.3 below, that the energy and the volume stretch profiles converge to the stationary solution of the diffusive system, with the boundary conditions imposed by the thermostats and the external tension. It turns out that these stationary equations can be solved explicitly and the stretch profile is linear between 0 and $\overline{\tau}_+$, while the thermal energy (temperature) profile is a concave parabola with the boundary conditions coinciding with the temperatures of the thermostats. The curvature of the parabola is proportional to $\overline{\tau}_+^2$, i.e. the increase of the bulk temperature is not a linear response term. In the case $\overline{\tau}_+ = 0$, the NESS was studied in [4], where the temperature profile is proved to be linear: more details are available in [2, 3]. This *heating inside the system* phenomenon is similar to the ohmic loss, due to the diffusion of electricity in a resistive system (see e.g. [6]).

The NESS for our model also provides a simple example of an *uphill* energy diffusion:

if the force $\overline{\tau}_+$ is large enough and applied on the side, where the coldest thermostat is acting, the sign of the energy current can be equal to that of the temperature gradient, see Theorem 3.2 below. It is not surprising after understanding that this is regulated by a system of two diffusive coupled equations. On the other hand, the model does not work as a *stationary refrigerator*: i.e. a system where the heat on the coldest thermostat flows into it.

Our results suggest that there is a universal behaviour of the temperature profiles in the NESS when there are at least two conserved quantities. This should be tested on a system with three conserved quantities that evolve in the diffusive scale, such as e.g. a non-acoustic harmonic chain with a random exchange of momentum as considered in [15], where the non-stationary hydrodynamic limit is proven. An attempt to describe more generally the systems for which the phenomena of an uphill energy diffusion and heating inside the system occur is made in [21].

Let us add a comment on the proofs of our main results. In proving Theorem 3.3 (on the asymptotics of energy and stretch profile) we need to make an additional hypothesis concerning the strength $\gamma > 0$ of the noisy part of the dynamics, see (2.3) and (2.4). More precisely we suppose that $\gamma = 1$. This assumption is of purely technical nature and allows to discard the term corresponding to the equipartition of the mechanical and kinetic energy in the decomposition (4.8) of the microscopic energy profile of the chain (the term $\mathcal{H}_n^{\mathrm{m}}$). We conjecture that this term vanishes in the stationary macroscopic limit, making the conclusion of the theorem valid for any $\gamma > 0$, but are unable to prove this fact at the moment. We do not need this hypothesis in our proof of Theorem 3.2 (on the uphill diffusion phenomenon).

In the appendix we give a proof for the non-stationary macroscopic evolution of the energy and the volume stretch profiles in the diffusive space-time scaling. As for the NESS, the proof is rigorous only for $\gamma = 1$, for similar reasons. The corresponding result with periodic boundary conditions was contained in [1].

The rest of the paper is organized as follows: in Section 2 we define the microscopic model under investigation and give the expected macroscopic system of equations, showing the phenomenon of uphill diffusion. In Section 3 we state the main results of the paper, namely the convergence of the non-equilibrium stationary profiles of elongation, current and energy. In order to prove them, we need precise computations on the averages and second-order moments taken with respect to the NESS. Section 4 provides elements of the proofs and preliminary computations on the averages, while Section 5 provides all the remaining technical lemmas, concerning the second-order moments.

2. Microscopic dynamics and macroscopic behaviour

2.1. Open chain of oscillators. Let $\mathbb{I}_n := \{1, \ldots, n\}$, $\overline{\mathbb{I}}_n := \mathbb{I}_n \cup \{0\}$ and $\mathbb{I} := [0,1]$. The configuration space $\Omega_n := \mathbb{R}^{\mathbb{I}_n} \times \mathbb{R}^{\overline{\mathbb{I}}_n}$ consists of all sequences $(\mathbf{q}, \mathbf{p}) := \{q_x, p_x\}_{x \in \overline{\mathbb{I}}_n}$, where $p_x \in \mathbb{R}$ stands for the momentum of the oscillator at site x, and $q_x \in \mathbb{R}$ represents its position. The interaction between two particles x and $x+1$ is described by the quadratic potential energy $V(q_x - q_{x+1}) := \frac{1}{2}(q_x - q_{x+1})^2$ of a harmonic spring linking the particles. At the boundaries the system is connected to two Langevin heat baths at temperatures T_- and T_+. Furthermore, on the right boundary acts a force (tension) $\overline{\tau}_+$, possibly slowly changing in time at a scale t/n^2. Note that the system is *unpinned*, i.e. there is no external potential binding the particles. Consequently, the absolute positions q_x do not have a precise meaning, and the dynamics only depends on the interparticle elongations $r_x := q_x - q_{x-1}, x \in \mathbb{I}_n$. The configurations can then be described
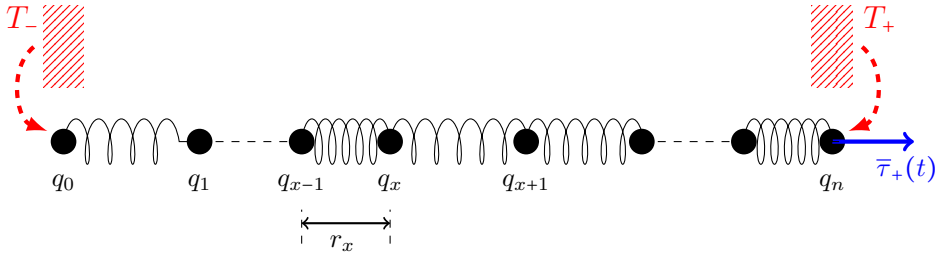
FIG. 2.1. *Oscillator chains with heat baths and one boundary force.*

by

$$(\mathbf{r}, \mathbf{p}) = (r_1, \ldots, r_n, p_0, \ldots, p_n) \in \Omega_n. \tag{2.1}$$

The total energy of the system is defined by the Hamiltonian:

$$\mathcal{H}_n(\mathbf{r}, \mathbf{p}) := \sum_{x \in \mathbb{I}_n} \mathcal{E}_x + \frac{p_0^2}{2}, \tag{2.2}$$

with

$$\mathcal{E}_x := \frac{p_x^2}{2} + \frac{r_x^2}{2}, \quad x \in \mathbb{I}_n.$$

We investigate this system in the diffusive time scale (when the ratio of the microscopic *vs* macroscopic time is n^2), therefore the equations of the microscopic dynamics are given in the bulk by

$$\mathrm{d}r_x(t) = n^2 (p_x(t) - p_{x-1}(t)) \mathrm{d}t, \qquad x \in \mathbb{I}_n \tag{2.3}$$

$$\mathrm{d}p_x(t) = n^2 (r_{x+1}(t) - r_x(t)) \mathrm{d}t - \gamma n^2 p_x(t) \mathrm{d}t$$

$$+ n\sqrt{\gamma}\big(p_{x-1}(t)\mathrm{d}w_{x-1,x}(t) - p_{x+1}(t)\mathrm{d}w_{x,x+1}(t)\big), \quad x \in \{1, \ldots, n-1\} \tag{2.4}$$

and at the boundaries:

$$\mathrm{d}p_0(t) = n^2 r_1(t)\mathrm{d}t - \frac{n^2}{2}(\gamma + \widetilde{\gamma})p_0(t)\mathrm{d}t - n\sqrt{\gamma}p_1(t)\mathrm{d}w_{0,1}(t) + n\sqrt{\widetilde{\gamma}T_-}\mathrm{d}\widetilde{w}_0(t) \tag{2.5}$$

$$\mathrm{d}p_n(t) = -n^2 r_n(t)\mathrm{d}t + n^2 \overline{\tau}_+(t)\mathrm{d}t - \frac{n^2}{2}(\gamma + \widetilde{\gamma})p_n(t)\mathrm{d}t$$

$$+ n\sqrt{\gamma}p_{n-1}(t)\mathrm{d}w_{n-1,n}(t) + n\sqrt{\widetilde{\gamma}T_+}\mathrm{d}\widetilde{w}_n(t) \tag{2.6}$$

where $w_{x,x+1}(t)$, $x \in \{0, \ldots, n-1\}$, $\widetilde{w}_0(t)$ and $\widetilde{w}_n(t)$ are independent, standard one dimensional Wiener processes, and $\gamma > 0$ (resp. $\widetilde{\gamma} > 0$) regulates the intensity of the random perturbation (resp. the Langevin thermostats). See Figure 2.1 for a representation of the chain. Note that the purely Hamiltonian dynamics is perturbed by a stochastic noise which exchanges kinetic energy between the neighbouring atoms, and with the boundary thermostats.

We let

$$(\mathbf{r}_n(t), \mathbf{p}_n(t)) := (r_1(t), \ldots, r_n(t), p_0(t), \ldots, p_n(t)), \quad t \geqslant 0 \tag{2.7}$$

be the Ω_n–valued process whose dynamics is determined by the Equations (2.3)–(2.6). Its generator is given by

$$L := n^2 \Big(A + \frac{\gamma}{2} S + \frac{\widetilde{\gamma}}{2} \widetilde{S} \Big), \tag{2.8}$$

where

$$A := \sum_{x=1}^{n} (p_x - p_{x-1}) \partial_{r_x} + \sum_{x=1}^{n-1} (r_{x+1} - r_x) \partial_{p_x} + r_1 \partial_{p_0} + (\overline{\tau}_+(t) - r_n) \partial_{p_n}$$

$$Sf := \sum_{x=0}^{n-1} \mathcal{X}_x \circ \mathcal{X}_x(f),$$

where $\mathcal{X}_x$ is the momentum exchange operator defined as

$$\mathcal{X}_x := p_{x+1} \partial_{p_x} - p_x \partial_{p_{x+1}},$$

and moreover the generator of the Langevin heat baths at the boundaries is given by

$$\widetilde{S} := T_- \partial_{p_0}^2 - p_0 \partial_{p_0} + T_+ \partial_{p_n}^2 - p_n \partial_{p_n}.$$

From the microscopic energy conservation law there exist microscopic energy currents $j_{x,x+1}$ which satisfy

$$n^{-2} L \mathcal{E}_x = j_{x-1,x} - j_{x,x+1}, \qquad \text{for any } x \in \overline{\overline{\mathbb{I}}}_n \tag{2.9}$$

and are given by

$$j_{x,x+1} := -p_x r_{x+1} + \frac{\gamma}{2} (p_x^2 - p_{x+1}^2), \qquad \text{if } x \in \{0, \ldots, n-1\}, \tag{2.10}$$

while at the boundaries

$$j_{-1,0} := \frac{\widetilde{\gamma}}{2} \big(T_- - p_0^2 \big), \qquad j_{n,n+1} := -\frac{\widetilde{\gamma}}{2} \big(T_+ - p_n^2 \big) - \overline{\tau}_+(t) p_n. \tag{2.11}$$

2.2. Macroscopic equations.   Suppose that $r(t,u)$, $e(t,u)$, $(t,u) \in \mathbb{R}_+ \times \mathbb{I}$, are the macroscopic profiles of *elongation (or stretch)* and *energy* of the macroscopic system, obtained in the diffusive scaling limit. The profiles $r(t,\cdot), e(t,\cdot)$ are the expected limits, as n gets large, of

$$\frac{1}{n} \sum_{x \in \mathbb{I}_n} r_x(t) \delta_{x/n}(\cdot), \quad \text{and} \quad \frac{1}{n} \sum_{x \in \overline{\overline{\mathbb{I}}}_n} \mathcal{E}_x(t) \delta_{x/n}(\cdot),$$

where $\delta_u(\cdot)$ is the Dirac delta function at point u. These convergences are expected to hold in the weak formulation sense: more details will be given in the appendix. If both convergences do hold at time $t = 0$ for some given profiles $r_0(u)$ and $\mathcal{E}_0(u)$, then we expect that they satisfy the following system of equations[1],

$$\partial_t r(t,u) = \gamma^{-1} \partial_{uu}^2 r(t,u) \tag{2.12}$$

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[1] See also Theorems 3.7 and 3.8 of [16] for a similar model which gives a similar coupled diffusive system for every value of γ.

$$\partial_t e(t,u) = \frac{1}{2}\partial_{uu}^2\left\{\left(\gamma^{-1}+\gamma\right)e(t,u)+\frac{1}{2}\left(\gamma^{-1}-\gamma\right)r^2(t,u)\right\}, \quad (t,u)\in\mathbb{R}_+\times\mathbb{I}, \tag{2.13}$$

with the boundary conditions

$$r(t,0)=0, \qquad\qquad r(t,1)=\overline{\tau}_+(t),$$

$$e(t,0)=T_-, \qquad\qquad e(t,1)=T_+ + \frac{\left(\overline{\tau}_+(t)\right)^2}{2}$$

and with the initial condition

$$r(0,u)=r_0(u), \qquad e(0,u)=\mathcal{E}_0(u).$$

In the appendix we will give the proof arguments for a derivation of these macroscopic equations, which are rigorous for $\gamma=1$, and conditioned to a form of local equilibrium result for $\gamma\neq 1$, stated in (6.46).

Define now $e_{\text{mech}}(t,u):=\frac{1}{2}r^2(t,u)$ and $e_{\text{th}}(t,u):=e(t,u)-e_{\text{mech}}(t,u)$ as respectively the *mechanical* and *thermal* components of the macroscopic energy. From (2.12) and (2.13) we conclude that

$$\partial_t e_{\text{mech}}(t,u) = \gamma^{-1}\left(\partial_{uu}^2 e_{\text{mech}}(t,u)-\left(\partial_u r(t,u)\right)^2\right), \quad (t,u)\in\mathbb{R}_+\times\mathbb{I}$$

with

$$e_{\text{mech}}(t,0)=0, \quad e_{\text{mech}}(t,1)=\frac{\left(\overline{\tau}_+(t)\right)^2}{2}, \quad e_{\text{mech}}(0,0)=\frac{r_0^2(u)}{2}, \quad (t,u)\in\mathbb{R}_+\times\mathbb{I}$$

and

$$\partial_t e_{\text{th}}(t,u) = \frac{1}{2}\left(\gamma^{-1}+\gamma\right)\partial_{uu}^2 e_{\text{th}}(t,u)+\gamma^{-1}\left(\partial_u r(t,u)\right)^2, \quad (t,u)\in\mathbb{R}_+\times\mathbb{I}, \tag{2.14}$$

with

$$e_{\text{th}}(t,0)=T_-, \quad e_{\text{th}}(t,1)=T_+, \quad t>0.$$

2.3. Stationary non-equilibrium states. From now on we assume $\overline{\tau}_+(t)\equiv\overline{\tau}_+$ to be constant in time.

When $\overline{\tau}_+=0$ and $T_-=T_+=T$, the system is *in equilibrium* and the stationary probability distribution is given explicitly by the homogeneous Gibbs measure

$$\nu_T(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}):=g_T(\mathbf{r},\mathbf{p})\mathrm{d}p_0\prod_{x\in\mathbb{I}_n}\mathrm{d}p_x\mathrm{d}r_x,$$

where

$$g_T(\mathbf{r},\mathbf{p}):=\frac{e^{-p_0^2/2T}}{\sqrt{2\pi T}}\prod_{x\in\mathbb{I}_n}\frac{e^{-\mathcal{E}_x/T}}{2\pi T}. \tag{2.15}$$

If $\overline{\tau}_+\neq 0$, or $T_-\neq T_+$, the stationary measure exists and is unique, but it is not given explicitly. More precisely, we know that there exists a unique stationary probability distribution μ_{ss} on Ω_n (cf. (2.1)) for the microscopic dynamics described by the Equations (2.3)–(2.6). As a consequence $\langle LF\rangle_{\text{ss}}=0$ for any function F in the domain of the operator L, given by (2.8). Hereafter, we denote

$$\langle F\rangle_{\text{ss}} := \int_{\Omega_n} F\,\mathrm{d}\mu_{\text{ss}}.$$

The proof of the existence and uniqueness of a stationary state follows from the same argument as the one used in [4, Appendix A] for $\overline{\tau}_+ = 0$. The fact that in our case $\overline{\tau}_+$ does not vanish requires only minor modifications. In addition, one can show, see bound (A.1) in [4], that for a fixed n we have $\langle \mathcal{H}_n \rangle_{ss} < +\infty$ (cf. (2.2)).

The corresponding *stationary* profiles, denoted respectively by $r_{ss}(u)$ and $e_{th,ss}(u)$, will solve the stationary version of Equations (2.12) and (2.14), i.e.:

$$r_{ss}(u) = \overline{\tau}_+ u \tag{2.16}$$

and

$$\left(\gamma^{-1} + \gamma\right)\partial_{uu}^2 e_{th,ss}(u) + 2\gamma^{-1}\overline{\tau}_+^2 = 0,$$

with the boundary conditions

$$e_{th,ss}(0) = T_-, \qquad e_{th,ss}(1) = T_+.$$

In other words

$$e_{th,ss}(u) = \frac{\overline{\tau}_+^2}{1+\gamma^2}\, u(1-u) + (T_+ - T_-)u + T_-. \tag{2.17}$$

Taking the average with respect to the stationary state in (2.9), we get the *stationary microscopic energy current*

$$\langle j_{x,x+1}\rangle_{ss} =: \overline{j}_s, \qquad \text{for any } x \in \{-1, \dots, n\}. \tag{2.18}$$

The *macroscopic stationary energy current* is defined as the limit of $n\overline{j}_s$, as $n \to +\infty$. It equals, see Theorem 3.2 below,

$$J_{ss} = -\frac{1}{2}(\gamma^{-1} + \gamma)(T_+ - T_-) - \frac{\overline{\tau}_+^2}{2\gamma}.$$

Observe that the energy current can flow against the temperature gradient if $T_- > T_+$ and $|\overline{\tau}_+|$ is large enough (*uphill diffusion*). Assuming $T_+ \geqslant T_-$ the maximum stationary temperature $e_{th,ss}^{max}$ is reached at

$$u_{max} = \left(\frac{1}{2} + \frac{1+\gamma^2}{\overline{\tau}_+^2}(T_+ - T_-)\right) \wedge 1$$

which implies that, if the condition $2(1+\gamma^2)(T_+ - T_-) \leqslant \overline{\tau}_+^2$ is satisfied, then the maximum temperature of the chain is attained inside, since $u_{max} < 1$ (see Figure 2.2), and it equals

$$e_{th,ss}^{max} = \frac{(T_+ - T_-)}{2} + T_- + \frac{\overline{\tau}_+^2}{4(1+\gamma^2)} \ \geqslant T_+.$$

Note that this does not depend on the sign of $\overline{\tau}_+$.

This phenomenon was observed by dynamical numerical simulations in [10] for the stationary states of the rotor model. It has attracted quite some interest from physicists, see [17] for a review. The present article is devoted to the proof of such a phenomenon, when $\gamma = 1$. This restriction is technical and will be further explained in Section 4.2. According to our knowledge it is the first rigorous proof of this fact in the existing literature.
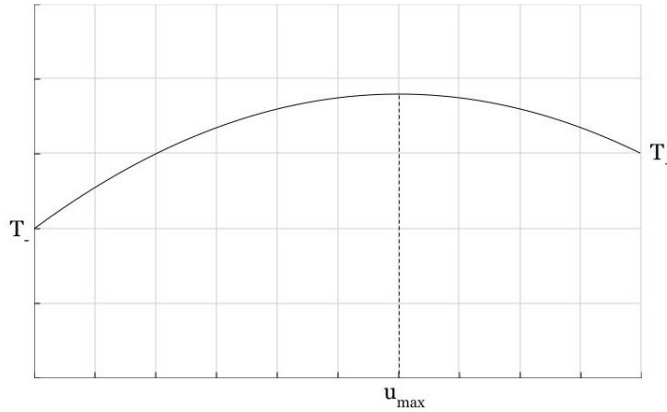
FIG. 2.2. *Temperature profile when* $T_+ - T_- < 2\overline{\tau}_+^2$.

3. Main results

Let us start with the following:

THEOREM 3.1 (Stationary elongation profile). *The following uniform convergence holds:*

$$\sup_{u \in \mathbb{I}} \left| \langle r_{[nu]} \rangle_{\mathrm{ss}} - r_{\mathrm{ss}}(u) \right| \xrightarrow[n \to \infty]{} 0,$$

where $r_{\mathrm{ss}}(u) := \overline{\tau}_+ u$. *In particular, for any continuous test function* $G : \mathbb{I} \to \mathbb{R}$,

$$\frac{1}{n} \sum_{x \in \mathbb{I}_n} G\left(\frac{x}{n}\right) \langle r_x \rangle_{\mathrm{ss}} \xrightarrow[n \to \infty]{} \int_{\mathbb{I}} G(u) r_{\mathrm{ss}}(u) \, \mathrm{d}u.$$

Proof. The averages under the stationary state $\langle r_x \rangle_{\mathrm{ss}}$ and $\langle p_x \rangle_{\mathrm{ss}}$ are computable explicitly, see Proposition 4.1 in the next section. It turns out that $\langle p_x \rangle_{\mathrm{ss}}$ is constant for all $x \in \overline{\mathbb{I}}_n$ and equals $\overline{p}_s := \overline{\tau}_+ / (\gamma n + \widetilde{\gamma})$ (see (4.1)). From (4.2) we also have

$$n \left(\langle r_{x+1} \rangle_{\mathrm{ss}} - \langle r_x \rangle_{\mathrm{ss}} \right) = n \gamma \overline{p}_s \xrightarrow[n \to \infty]{} \overline{\tau}_+, \qquad \text{for } x \in \{1, ..., n-1\}$$

and

$$\langle r_1 \rangle_{\mathrm{ss}} \xrightarrow[n \to \infty]{} 0, \qquad \langle r_n \rangle_{\mathrm{ss}} \xrightarrow[n \to \infty]{} \overline{\tau}_+.$$

Finally, (4.2) directly implies the conclusion of the theorem. □

Concerning the *stationary energy flow* and the validity of the Fourier law we show the following result on the macroscopic stationary energy current.

THEOREM 3.2 (Stationary energy current and Fourier law).

$$n \overline{j}_s \xrightarrow[n \to \infty]{} -\frac{1}{2}(\gamma^{-1} + \gamma)(T_+ - T_-) - \frac{\overline{\tau}_+^2}{2\gamma}. \tag{3.1}$$

Note that Theorems 3.1 and 3.2 are valid for any $\gamma > 0$. We now state our last main result about the stationary energy profile, which we are able to prove only for

$\gamma = 1$. Before stating it we introduce the stationary microscopic mechanical and thermal energy *per particle* as follows

$$\mathcal{E}_x^{\text{mech}} := \frac{1}{2}\langle r_x^2 \rangle_{\text{ss}}$$

$$\mathcal{E}_x^{\text{th}} := \mathcal{E}_x - \mathcal{E}_x^{\text{mech}} = \frac{1}{2}p_x^2 + \frac{1}{2}\big(r_x^2 - \langle r_x^2 \rangle_{\text{ss}}\big), \qquad x \in \mathbb{I}_n.$$

THEOREM 3.3 (Stationary energy profile). *Assume that $\gamma = 1$. For any continuous test function $G : \mathbb{I} \to \mathbb{R}$ we have*

$$\frac{1}{n}\sum_{x \in \mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle \mathcal{E}_x^{\text{mech}} \rangle_{\text{ss}} \xrightarrow[n \to \infty]{} \int_{\mathbb{I}} G(u)\frac{1}{2}r_{\text{ss}}^2(u)\,\mathrm{d}u, \tag{3.2}$$

$$\frac{1}{n}\sum_{x \in \mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle \mathcal{E}_x^{\text{th}} \rangle_{\text{ss}} \xrightarrow[n \to \infty]{} \int_{\mathbb{I}} G(u)e_{\text{th,ss}}(u)\,\mathrm{d}u, \tag{3.3}$$

$$\frac{1}{n}\sum_{x \in \mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle \mathcal{E}_x \rangle_{\text{ss}} \xrightarrow[n \to \infty]{} \int_{\mathbb{I}} G(u)\Big(e_{\text{th,ss}}(u) + \frac{1}{2}r_{\text{ss}}^2(u)\Big)\,\mathrm{d}u, \tag{3.4}$$

where

$$r_{\text{ss}}(u) = \overline{\tau}_+ u,$$

$$e_{\text{th,ss}}(u) = \frac{\overline{\tau}_+^2}{2}u(1-u) + (T_+ - T_-)u + T_+.$$

The remaining part of the paper deals with the proofs of Theorems 3.2 and 3.3.

4. The stationary state

Let us start with explicit computations for the average momenta and elongations with respect to the NESS.

4.1. Elongation and momenta averages.

PROPOSITION 4.1. *The average stationary momenta are equal to*

$$\langle p_x \rangle_{\text{ss}} = \overline{p}_s := \frac{\overline{\tau}_+}{\gamma n + \widetilde{\gamma}}, \qquad \text{for any } x \in \overline{\mathbb{I}}_n. \tag{4.1}$$

The average stationary elongations are equal to

$$\langle r_x \rangle_{\text{ss}} = \frac{\overline{p}_s}{2}(\widetilde{\gamma} - \gamma + 2\gamma x) = \frac{\overline{\tau}_+(2\gamma x + \widetilde{\gamma} - \gamma)}{2(\gamma n + \widetilde{\gamma})} \qquad \text{for any } x \in \mathbb{I}_n. \tag{4.2}$$

Proof. We start with some useful relations that hold for the stationary state:
(1) since $\langle Lr_x \rangle_{\text{ss}} = 0$, applying (2.8), we conclude

$$\langle p_x \rangle_{\text{ss}} = \langle p_{x-1} \rangle_{\text{ss}} = \overline{p}_s, \qquad \text{for any } x \in \mathbb{I}_n;$$

(2) from $\langle Lp_x \rangle_{\text{ss}} = 0$ we get

$$\langle r_{x+1} \rangle_{\text{ss}} - \langle r_x \rangle_{\text{ss}} = \gamma \overline{p}_s, \qquad \text{for any } x \in \{2, \dots, n-2\}$$

$$\langle r_1 \rangle_{\text{ss}} = \frac{1}{2}(\gamma + \widetilde{\gamma})\overline{p}_s,$$

$$\langle r_n \rangle_{\text{ss}} = -\frac{1}{2}(\gamma + \widetilde{\gamma})\overline{p}_s + \overline{\tau}_+.$$

These equations determine the average stationary momentum and elongation as given in formulas (4.1) and (4.2). □

4.2. Elements of the proofs of Theorems 3.2 and 3.3. One of the main characteristics of this model is the existence of an explicit *fluctuation-dissipation relation*, which permits to write the stationary current $\bar{j}_s$ as a discrete gradient of some local function, as given in the following:

PROPOSITION 4.2 (Decomposition of the stationary current). *We can write $\bar{j}_s$ as a discrete gradient, namely*

$$\bar{j}_s = \nabla\phi(x) := \phi(x+1) - \phi(x), \qquad x \in \{1,\ldots,n-1\}, \tag{4.3}$$

with

$$\phi(x) := -\frac{1}{2\gamma}\big(\langle r_x^2\rangle_{\mathrm{ss}} + \langle p_{x-1}p_x\rangle_{\mathrm{ss}}\big) - \frac{\gamma}{4}\big(\langle p_x^2\rangle_{\mathrm{ss}} + \langle p_{x-1}^2\rangle_{\mathrm{ss}}\big), \qquad x \in \mathbb{I}_n. \tag{4.4}$$

REMARK 4.1. Thanks to (4.3), the function $\phi(x)$ is *harmonic*, i.e.:

$$\Delta\phi(x) := \phi(x+1) + \phi(x-1) - 2\phi(x) = 0, \qquad \text{for any } x \in \{2,\ldots,n-1\}. \tag{4.5}$$

Proof. (**Proof of Proposition 4.2.**) By a direct calculation one can easily check that the energy currents $j_{x,x+1}$ (defined in (2.10)) satisfy the following *fluctuation-dissipation relation*:

$$j_{x,x+1} = n^{-2}Lg_x - \frac{1}{2\gamma}\nabla\big(r_x^2 + p_{x-1}p_x\big) - \frac{\gamma}{4}\nabla\big(p_{x-1}^2 + p_x^2\big), \tag{4.6}$$

for any $x \in \{1,\ldots,n-1\}$, with

$$g_x := -\frac{1}{4}p_x^2 + \frac{1}{2\gamma}p_x\big(r_x + r_{x+1}\big).$$

Therefore, (4.3) is obtained by taking the average in (4.6) with respect to the stationary state. □

We can now sketch the proof of Theorem 3.3: straightforward computations, using the Definition (4.4) of ϕ, yield

$$\langle\mathcal{E}_x\rangle_{\mathrm{ss}} = \frac{1}{2}\big(\langle p_x^2\rangle_{\mathrm{ss}} + \langle r_x^2\rangle_{\mathrm{ss}}\big) = -\frac{2\gamma}{1+\gamma^2}\phi(x) + \frac{\gamma^2}{2(1+\gamma^2)}\big(\langle p_x^2\rangle_{\mathrm{ss}} - \langle p_{x-1}^2\rangle_{\mathrm{ss}}\big)$$
$$-\frac{1}{1+\gamma^2}\langle p_x p_{x-1}\rangle_{\mathrm{ss}} + \frac{1-\gamma^2}{2(1+\gamma^2)}\big(\langle p_x^2\rangle_{\mathrm{ss}} - \langle r_x^2\rangle_{\mathrm{ss}}\big). \tag{4.7}$$

Therefore, the microscopic energy profile can be decomposed as the sum of four terms:

$$\mathcal{H}_n(G) := \frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\langle\mathcal{E}_x\rangle_{\mathrm{ss}} = \mathcal{H}_n^\phi(G) + \mathcal{H}_n^\nabla(G) + \mathcal{H}_n^{\mathrm{corr}}(G) + \mathcal{H}_n^{\mathrm{m}}(G), \tag{4.8}$$

where

$$\mathcal{H}_n^\phi(G) := -\frac{2\gamma}{1+\gamma^2}\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\phi(x),$$

$$\mathcal{H}_n^\nabla(G) := \frac{\gamma^2}{2(1+\gamma^2)}\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\big(\langle p_x^2\rangle_{\mathrm{ss}} - \langle p_{x-1}^2\rangle_{\mathrm{ss}}\big),$$

$$\mathcal{H}_n^{\mathrm{corr}}(G) := -\frac{1}{1+\gamma^2}\frac{1}{n}\sum_{x\in\mathbb{I}_n}G\Big(\frac{x}{n}\Big)\langle p_{x-1}p_x\rangle_{\mathrm{ss}},$$

$$\mathcal{H}_n^{\mathrm{m}}(G) := \frac{1-\gamma^2}{2(1+\gamma^2)}\frac{1}{n}\sum_{x\in\mathbb{I}_n}G\Big(\frac{x}{n}\Big)\big(\langle p_x^2\rangle_{\mathrm{ss}} - \langle r_x^2\rangle_{\mathrm{ss}}\big).$$

Note that, if $\gamma = 1$, then $\mathcal{H}_n^{\mathrm{m}} \equiv 0$. If $\gamma \neq 1$, we conjecture that this last term vanishes as $n \to \infty$, but we are not able to prove it at the moment. The limits of the other three terms will be obtained in the next section and are summarized in the following proposition:

PROPOSITION 4.3.   *For any continuous test function* $G:\mathbb{I}\to\mathbb{R}$,

$$\mathcal{H}_n^{\phi}(G)\xrightarrow[n\to\infty]{}\int_{\mathbb{I}}G(u)\Big(\frac{\overline{\tau}_+^2}{1+\gamma^2}u + (T_+ - T_-)u + T_-\Big)\mathrm{d}u, \qquad (4.9)$$

$$\mathcal{H}_n^{\nabla}(G)\xrightarrow[n\to\infty]{}0, \qquad (4.10)$$

$$\mathcal{H}_n^{\mathrm{corr}}(G)\xrightarrow[n\to\infty]{}0. \qquad (4.11)$$

 The complete proof of the proposition will be given in Section 5.4. Let us first comment on the ideas used in the argument. The limit (4.9) will be concluded using the fact that ϕ is harmonic, (4.10) is a consequence of the presence of a discrete gradient $\langle p_x^2\rangle_{\mathrm{ss}} - \langle p_{x-1}^2\rangle_{\mathrm{ss}}$ inside the sum, and (4.11) will be shown thanks to the second-order bounds, which are obtained in the next section.

Proof. (**Proof of Theorem 3.3.**)   With the help of Proposition 4.3 the proof of Theorem 3.3 becomes straightforward. Assume that $\gamma = 1$. From the decomposition (4.7) and Proposition 4.3, we get

$$\frac{1}{n}\sum_{x\in\mathbb{I}_n}G\Big(\frac{x}{n}\Big)\langle\mathcal{E}_x\rangle_{\mathrm{ss}}\xrightarrow[n\to\infty]{}\int_{\mathbb{I}}G(u)\Big(\frac{\overline{\tau}_+^2}{2}u + (T_+ - T_-)u + T_-\Big)\mathrm{d}u$$

$$= \int_{\mathbb{I}}G(u)\Big(\frac{\overline{\tau}_+^2}{2}u(1-u) + (T_+ - T_-)u + T_- + \frac{(\overline{\tau}_+u)^2}{2}\Big)\mathrm{d}u.$$

Recalling (2.16) and (2.17) we conclude that the right-hand side equals

$$\int_{\mathbb{I}}G(u)\Big(e_{\mathrm{th,ss}}(u) + \frac{r_{\mathrm{ss}}^2(u)}{2}\Big)\mathrm{d}u.$$

Thus (3.4) follows. From Theorem 3.1 we immediately conclude (3.2). The convergence of the stationary microscopic thermal energy profile in (3.3) is an immediate consequence of these two statements.   □

5. Moment bounds under the stationary state

In this section we present a complete proof of Proposition 4.3 (see Section 5.4) and we show Theorem 3.2 (see Proposition 5.5). Before presenting the proof, we need a few technical estimates on the entropy production (Section 5.1) and second-order moments (Section 5.2 and Section 5.3). In the whole section we do not assume $\gamma = 1$, since our results hold for any $\gamma > 0$.

5.1. Entropy production of the stationary state.   Recall Definition (2.15). For a given T we will use

$$\nu_T(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p}) = g_T(\mathbf{r},\mathbf{p})\mathrm{d}p_0\prod_{x\in\mathbb{I}_n}\mathrm{d}p_x\mathrm{d}r_x,$$

as a reference measure, and denote its respective expectation by $\langle\!\langle \cdot \rangle\!\rangle_T$.

Stationarity of μ_{ss} under the microscopic dynamics implies that $L^\star \mu_{\text{ss}} = 0$ (in the sense of distributions). The operator $L^\star$ is hypoelliptic, thus by [9, Theorem 1.1, p. 149], the measure μ_{ss} has a smooth density f_s with respect to ν_{T_+}, i.e.

$$\langle F \rangle_{\text{ss}} = \langle\!\langle F f_s \rangle\!\rangle_{T_+} = \int F f_s \, d\nu_{T_+}.$$

PROPOSITION 5.1 (Entropy production). *Denote* $h := g_{T_-}/g_{T_+}$. *The following formula holds*

$$\gamma \sum_{x=0}^{n-1} \mathcal{D}_x(f_s) + \widetilde{\gamma} T_- \left\langle\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)} \right\rangle\!\right\rangle_{T_-} + \widetilde{\gamma} T_+ \left\langle\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s} \right\rangle\!\right\rangle_{T_+}$$
$$= \frac{\overline{\tau}_+^2}{T_+(\gamma n + \widetilde{\gamma})} + \widetilde{\gamma}\left(\frac{1}{T_+} - \frac{1}{T_-}\right)(T_- - \langle p_0^2 \rangle_{\text{ss}}), \tag{5.1}$$

where

$$\mathcal{D}_x(f_s) := \left\langle\!\left\langle \frac{(\mathcal{X}_x f_s)^2}{f_s} \right\rangle\!\right\rangle_{T_+}.$$

Proof.
Integration by parts yields

$$\left\langle\!\left\langle g \widetilde{S} f \right\rangle\!\right\rangle_{T_+} = T_+ \left\langle\!\left\langle \partial_{p_n} f \partial_{p_n} g \right\rangle\!\right\rangle_{T_+} + T_- \left\langle\!\left\langle \partial_{p_0} f \partial_{p_0} g \right\rangle\!\right\rangle_{T_+} + T_- \left(\frac{1}{T_-} - \frac{1}{T_+}\right)\left\langle\!\left\langle g \partial_{p_0} f \right\rangle\!\right\rangle_{T_+}$$

for any $f, g \in C_0^\infty(\Omega_n)$, where $C_0^\infty(\Omega_n)$ is the space of compactly supported smooth functions. As

$$A(\mathcal{H}_n(\mathbf{r}, \mathbf{p})) = \overline{\tau}_+ p_n \quad \text{and} \quad \mathcal{X}_x\left(\sum_{y=0}^n p_y^2\right) = 0, \qquad x \in \{0, \ldots, n-1\}$$

we conclude

$$-\left\langle\!\left\langle g A f \right\rangle\!\right\rangle_{T_+} = \left\langle\!\left\langle f A g \right\rangle\!\right\rangle_{T_+} + \overline{\tau}_+ \left\langle\!\left\langle p_n f g \right\rangle\!\right\rangle_{T_+},$$

$$-\left\langle\!\left\langle g \mathcal{X}_x^2 f \right\rangle\!\right\rangle_{T_+} = \left\langle\!\left\langle \mathcal{X}_x f \mathcal{X}_x g \right\rangle\!\right\rangle_{T_+},$$

for any $x = 0, \ldots, n-1$, and any $f, g \in C_0^\infty(\Omega_n)$. We take the average of $-n^{-2} L(\log f_s)$ with respect to the stationary state μ_{ss}. Taking into account the above identities we obtain

$$0 = -n^{-2} \langle L \log f_s \rangle_{\text{ss}} = -n^{-2} \left\langle\!\left\langle f_s L \log f_s \right\rangle\!\right\rangle_{T_+}$$

$$= \gamma \sum_{x=0}^{n-1} \mathcal{D}_x(f_s) + \widetilde{\gamma} T_+ \left\langle\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s} \right\rangle\!\right\rangle_{T_+} - \overline{\tau}_+ \left\langle\!\left\langle \partial_{p_n} f_s \right\rangle\!\right\rangle_{T_+} - \widetilde{\gamma}\left\langle (T_- \partial_{p_0}^2 - p_0 \partial_{p_0})\log f_s \right\rangle_{\text{ss}}.$$

From the definition $h = g_{T_-}/g_{T_+}$, the last term can be rewritten in the form:

$$-\left\langle (T_- \partial_{p_0}^2 - p_0 \partial_{p_0})\log f_s \right\rangle_{\text{ss}} = -\int \frac{f_s}{h}(T_- \partial_{p_0}^2 - p_0 \partial_{p_0})\left(\log\left(\frac{f_s}{h}\right)\right) g_{T_-} dp_0 \prod_{x=1}^n dp_x dr_x$$

$$- \int f_s \big(T_- \partial_{p_0}^2 - p_0 \partial_{p_0} \big) (\log h) \, g_{T_+} dp_0 \prod_{x=1}^n dp_x dr_x$$
$$= T_- \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)} \right\rangle\!\!\right\rangle_{T_-} + \left(\frac{1}{T_-} - \frac{1}{T_+} \right) \big(T_- - \langle p_0^2 \rangle_{\mathrm{ss}} \big).$$

Moreover, by integration by parts and (4.1), we obtain

$$\langle\!\langle \partial_{p_n} f_s \rangle\!\rangle_{T_+} = T_+^{-1} \langle\!\langle p_n f_s \rangle\!\rangle_{T_+} = \frac{\overline{p}_s}{T_+} = \frac{\overline{\tau}_+}{T_+ (\gamma n + \widetilde{\gamma})}$$

and (5.1) follows. Since f_s need not be compactly supported the above calculation is somewhat formal. A rigorous argument (using variational principles) can be found in [5, Section 3]. □

5.2. Bounds on second moments.   In the present section we obtain some bounds on the covariance functions of momenta and positions, with respect to the stationary states. In particular, we estimate the magnitude of the average current $|\overline{j}_s|$, see (2.18) and investigate the behaviour of $\phi(1)$ and $\phi(n)$ as $n \to \infty$, see (4.4).

Let us first state a rough estimate on the second moments at the boundaries, which is going to be refined further.

PROPOSITION 5.2 (Second moments at the boundaries: Part I).   *The following equality holds*

$$\langle p_0^2 \rangle_{\mathrm{ss}} + \langle p_n^2 \rangle_{\mathrm{ss}} = T_+ + T_- + \frac{2\overline{\tau}_+ \overline{p}_s}{\widetilde{\gamma}} \qquad \textit{for all } n \geqslant 1. \tag{5.2}$$

Moreover, there exists a constant $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, *such that*

$$\langle r_1^2 \rangle_{\mathrm{ss}} + \langle r_n^2 \rangle_{\mathrm{ss}} \leqslant C \qquad \textit{for all } n \geqslant 1. \tag{5.3}$$

REMARK 5.1.   *By convention, the constants appearing in the statements below depend only on the parameters indicated in parentheses in the statement of the proposition.*

Proof. (**Proof of Proposition 5.2.**) The first identity (5.2) is an easy consequence of (2.11) and (2.18), which yields

$$\overline{j}_s = \frac{\widetilde{\gamma}}{2} \big(T_- - \langle p_0^2 \rangle_{\mathrm{ss}} \big), \tag{5.4}$$

$$\overline{j}_s = -\overline{\tau}_+ \overline{p}_s - \frac{\widetilde{\gamma}}{2} \big(T_+ - \langle p_n^2 \rangle_{\mathrm{ss}} \big). \tag{5.5}$$

Identity (5.2) is obtained by adding sideways the above equalities. To show estimate (5.3) note that

$$n^{-2} L(p_0 r_1) = (p_1 - p_0) p_0 + r_1^2 - \frac{1}{2} (\widetilde{\gamma} + \gamma) p_0 r_1 \tag{5.6}$$

$$n^{-2} L(p_n r_n) = p_n (p_n - p_{n-1}) + (\overline{\tau}_+ - r_n) r_n - \frac{1}{2} (\widetilde{\gamma} + \gamma) p_n r_n. \tag{5.7}$$

After taking the average with respect to the stationary state from (5.6) we conclude

$$\langle r_1^2 \rangle_{\mathrm{ss}} = \langle p_0^2 \rangle_{\mathrm{ss}} - \langle p_1 p_0 \rangle_{\mathrm{ss}} + \frac{1}{2} (\widetilde{\gamma} + \gamma) \langle p_0 r_1 \rangle_{\mathrm{ss}}. \tag{5.8}$$

Recalling the definition of the current (2.10) and then invoking (5.4), we get

$$\langle r_1^2 \rangle_{ss} = \langle p_0^2 \rangle_{ss} - \langle p_1 p_0 \rangle_{ss} - \frac{1}{2}(\widetilde{\gamma} + \gamma)\Big(\langle j_{0,1} \rangle_{ss} + \frac{\gamma}{2}\big(\langle p_1^2 \rangle_{ss} - \langle p_0^2 \rangle_{ss}\big)\Big)$$

$$= \langle p_0^2 \rangle_{ss} - \langle p_1 p_0 \rangle_{ss} - \frac{\widetilde{\gamma}}{4}(\widetilde{\gamma} + \gamma)\big(T_- - \langle p_0^2 \rangle_{ss}\big) - \frac{\gamma}{4}(\widetilde{\gamma} + \gamma)\big(\langle p_1^2 \rangle_{ss} - \langle p_0^2 \rangle_{ss}\big).$$

Using Young's inequality

$$\big|\langle p_1 p_0 \rangle_{ss}\big| \leqslant \frac{A}{2}\langle p_1^2 \rangle_{ss} + \frac{1}{2A}\langle p_0^2 \rangle_{ss},$$

with $A = \frac{\gamma}{2}(\gamma + \widetilde{\gamma})$, we get

$$\langle r_1^2 \rangle_{ss} \leqslant \Big(\frac{1}{\gamma(\gamma + \widetilde{\gamma})} + 1 + \frac{1}{4}(\gamma + \widetilde{\gamma})^2\Big)\langle p_0^2 \rangle_{ss}. \tag{5.9}$$

From (5.2) we conclude that $\langle r_1^2 \rangle_{ss}$ is bounded.

To estimate $\langle r_n^2 \rangle_{ss}$, note that from (5.7) we write

$$\langle r_n^2 \rangle_{ss} = \langle p_n^2 \rangle_{ss} - \langle p_n p_{n-1} \rangle_{ss} + \overline{\tau}_+ \langle r_n \rangle_{ss} - \frac{1}{2}(\widetilde{\gamma} + \gamma)\langle p_n r_n \rangle_{ss}. \tag{5.10}$$

We use Young's inequality again

$$\big|\langle p_n p_{n-1} \rangle_{ss}\big| \leqslant \frac{A}{2}\langle p_n^2 \rangle_{ss} + \frac{1}{2A}\langle p_{n-1}^2 \rangle_{ss},$$

with $A = 1/(2\gamma)$ and we get

$$\langle r_n^2 \rangle_{ss} \leqslant \Big(1 + \frac{1}{4\gamma}\Big)\langle p_n^2 \rangle_{ss} + \gamma\langle p_{n-1}^2 \rangle_{ss} + \tau\langle r_n \rangle_{ss} - \frac{1}{2}(\widetilde{\gamma} + \gamma)\langle p_n r_n \rangle_{ss}. \tag{5.11}$$

To replace $\langle p_{n-1}^2 \rangle_{ss}$, note that

$$n^{-2}L(p_n^2) = 2(\overline{\tau}_+ - r_n)p_n + \gamma(p_{n-1}^2 - p_n^2) + \widetilde{\gamma}(T_+ - p_n^2). \tag{5.12}$$

Taking the average with respect to the stationary state, we obtain:

$$\gamma\langle p_{n-1}^2 \rangle_{ss} = 2\langle r_n p_n \rangle_{ss} - 2\overline{\tau}_+ \langle p_n \rangle_{ss} + (\gamma + \widetilde{\gamma})\langle p_n^2 \rangle_{ss} - \widetilde{\gamma}T_+, \tag{5.13}$$

which, in (5.11), gives

$$\langle r_n^2 \rangle_{ss} \leqslant \Big(1 + \frac{1}{4\gamma} + \gamma + \widetilde{\gamma}\Big)\langle p_n^2 \rangle_{ss} + \overline{\tau}_+\big(\langle r_n \rangle_{ss} - 2\langle p_n \rangle_{ss}\big) + \frac{1}{2}(4 - \widetilde{\gamma} - \gamma)\langle p_n r_n \rangle_{ss} - \widetilde{\gamma}T_+.$$

Using Young's inequality again

$$\big|\langle p_n r_n \rangle_{ss}\big| \leqslant \frac{A}{2}\langle p_n^2 \rangle_{ss} + \frac{1}{2A}\langle r_n^2 \rangle_{ss}$$

with $A = \frac{1}{2}|4 - \gamma - \widetilde{\gamma}|$, we finally arrive at

$$\frac{1}{2}\langle r_n^2 \rangle_{ss} \leqslant \Big(1 + \frac{1}{4\gamma} + \gamma + \widetilde{\gamma} + \frac{1}{4}(4 - \gamma - \widetilde{\gamma})^2\Big)\langle p_n^2 \rangle_{ss} + \overline{\tau}_+\big(\langle r_n \rangle_{ss} - 2\langle p_n \rangle_{ss}\big) - \widetilde{\gamma}T_+. \tag{5.14}$$

We now invoke (5.2), (4.1) and (4.2) to conclude the bound on $\langle r_n^2 \rangle_{\mathrm{ss}}$, which combined with the already obtained bound on $\langle r_1^2 \rangle_{\mathrm{ss}}$, yields (5.3). □

COROLLARY 5.1 (Second moments at the boundaries: Part II). *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, *such that*

$$\langle p_1^2 \rangle_{\mathrm{ss}} + \langle p_{n-1}^2 \rangle_{\mathrm{ss}} \leqslant C \qquad \text{for all } n \geqslant 1. \tag{5.15}$$

Proof. To bound $\langle p_{n-1}^2 \rangle_{\mathrm{ss}}$ we use formula (5.13). From an elementary inequality $|\langle r_n p_n \rangle_{\mathrm{ss}}| \leqslant \frac{1}{2}(\langle p_n^2 \rangle_{\mathrm{ss}} + \langle r_n^2 \rangle_{\mathrm{ss}})$ and Proposition 5.2, we easily conclude that $\langle p_{n-1}^2 \rangle_{\mathrm{ss}}$ is bounded.

The bound for $\langle p_1^2 \rangle_{\mathrm{ss}}$ is obtained similarly. First, note that

$$n^{-2} L(p_0^2) = 2 r_1 p_0 + \gamma(p_1^2 - p_0^2) + \widetilde{\gamma}(T_- - p_0^2). \tag{5.16}$$

Taking the average with respect to the stationary state, using the inequality $|\langle r_1 p_0 \rangle_{\mathrm{ss}}| \leqslant \frac{1}{2}(\langle r_1^2 \rangle_{\mathrm{ss}} + \langle p_0^2 \rangle_{\mathrm{ss}})$, and invoking Proposition 5.2, we conclude the desired bound on $\langle p_1^2 \rangle_{\mathrm{ss}}$. Thus (5.15) follows. □

In the next proposition we provide a bound on the energy current under the stationary state, which will be further refined in Proposition 5.5.

PROPOSITION 5.3 (The stationary current: Part I). *There exists a constant* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, *such that the stationary current satisfies*

$$|\overline{j}_s| \leqslant \frac{C}{n} \qquad \text{for all } n \geqslant 1. \tag{5.17}$$

Proof. We sum the identity (4.3) from $x = 1$ to $n-1$ and apply (4.4) to express $\phi(n)$ and $\phi(1)$. In this way we get

$$(n-1)\overline{j}_s = \phi(n) - \phi(1)$$
$$= \left\langle -\frac{p_{n-1}p_n + r_n^2}{2\gamma} - \frac{\gamma(p_{n-1}^2 + p_n^2)}{4} \right\rangle_{\mathrm{ss}} + \left\langle \frac{p_1 p_0 + r_1^2}{2\gamma} + \frac{\gamma(p_1^2 + p_0^2)}{4} \right\rangle_{\mathrm{ss}}. \tag{5.18}$$

Therefore, (5.17) follows from the elementary inequalities

$$|\langle p_{x-1} p_x \rangle_{\mathrm{ss}}| \leqslant \frac{1}{2}\big(\langle p_x^2 \rangle_{\mathrm{ss}} + \langle p_{x-1}^2 \rangle_{\mathrm{ss}}\big) \qquad \text{for } x = n \text{ and } x = 1$$

together with the bounds obtained in Proposition 5.2 and Corollary 5.1. □

Proposition 5.3 permits to get a better estimate on the entropy production. Namely, combining (5.1), (5.4) and (5.17) we conclude the following.

COROLLARY 5.2. *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, *such that*

$$\gamma \sum_{x=0}^{n-1} \mathcal{D}_x(f_s) + \widetilde{\gamma} T_- \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)} \right\rangle\!\!\right\rangle_{T_-} + \widetilde{\gamma} T_+ \left\langle\!\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s} \right\rangle\!\!\right\rangle_{T_+} \leqslant \frac{C}{n}, \qquad n \geqslant 1. \tag{5.19}$$

Thanks to Proposition 5.3 we are now able to estimate the covariances of momenta and stretches at the boundaries as follows:

PROPOSITION 5.4 (Second moment at the boundaries: Part III). *There exists* $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, *such that, at the left boundary point*

$$\left|\langle p_0 p_1\rangle_{\text{ss}}\right| + \left|\langle r_1 p_1\rangle_{\text{ss}}\right| + \left|\langle r_1 p_0\rangle_{\text{ss}}\right| \leqslant \frac{C}{\sqrt{n}}, \qquad n \geqslant 1, \tag{5.20}$$

and at the right boundary point

$$\left|\langle p_n p_{n-1}\rangle_{\text{ss}}\right| + \left|\langle r_n p_n\rangle_{\text{ss}}\right| + \left|\langle r_n p_{n-1}\rangle_{\text{ss}}\right| \leqslant \frac{C}{\sqrt{n}}, \qquad n \geqslant 1. \tag{5.21}$$

Proof. Integration by parts yields

$$\langle p_0 p_1\rangle_{\text{ss}} = -T_- \left\langle\!\!\left\langle p_1 (f_s/g_{T_-})\partial_{p_0}g_{T_-}\right\rangle\!\!\right\rangle_{T_+} = T_- \left\langle\!\!\left\langle p_1 \partial_{p_0}(f_s/h)\right\rangle\!\!\right\rangle_{T_-}.$$

We use the entropy production bound (5.19) and estimate (5.15) on $\langle p_1^2\rangle_{\text{ss}}$, to estimate the right-hand side. As a result we get

$$\left|\langle p_0 p_1\rangle_{\text{ss}}\right| = T_- \left|\left\langle\!\!\left\langle p_1 \partial_{p_0}(f_s/h)\right\rangle\!\!\right\rangle_{T_-}\right| \leqslant T_- \left\langle p_1^2\right\rangle_{\text{ss}}^{\frac{1}{2}} \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)}\right\rangle\!\!\right\rangle_{T_-}^{\frac{1}{2}} \leqslant \frac{C}{\sqrt{n}}.$$

Similarly,

$$\left|\langle p_n p_{n-1}\rangle_{\text{ss}}\right| = T_+ \left|\left\langle\!\!\left\langle p_{n-1} \partial_{p_n} f_s\right\rangle\!\!\right\rangle_{T_+}\right| \leqslant T_+ \left\langle p_{n-1}^2\right\rangle_{\text{ss}}^{\frac{1}{2}} \left\langle\!\!\left\langle \frac{(\partial_{p_n} f_s)^2}{f_s}\right\rangle\!\!\right\rangle_{T_+}^{\frac{1}{2}} \leqslant \frac{C}{\sqrt{n}}.$$

Finally, note that, for any $x \in \mathbb{I}_n$

$$n^{-2} L(r_x^2) = 2(p_x - p_{x-1})r_x. \tag{5.22}$$

Therefore, upon averaging with respect to the NESS, we get

$$\langle p_x r_x\rangle_{\text{ss}} = \langle p_{x-1} r_x\rangle_{\text{ss}}, \quad x \in \mathbb{I}_n. \tag{5.23}$$

In particular, applying (5.23) for $x = 1$ and $x = n$, we remark that the only quantities we need to yet estimate are $\left|\langle r_1 p_0\rangle_{\text{ss}}\right|$ and $\left|\langle r_n p_n\rangle_{\text{ss}}\right|$. This is done using the entropy production bound (5.19) in the same manner as before, namely:

$$\left|\langle r_1 p_0\rangle_{\text{ss}}\right| = T_- \left|\left\langle\!\!\left\langle r_1 \partial_{p_0}(f_s/h)\right\rangle\!\!\right\rangle_{T_-}\right| \leqslant T_- \left\langle r_1^2\right\rangle_{\text{ss}}^{\frac{1}{2}} \left\langle\!\!\left\langle \frac{(\partial_{p_0}(f_s/h))^2}{(f_s/h)}\right\rangle\!\!\right\rangle_{T_-}^{\frac{1}{2}} \leqslant \frac{C}{\sqrt{n}},$$

from (5.3) and (5.1). We leave the last estimate for the reader. $\square$

We now have all the ingredients necessary to prove moments convergences at the boundaries:

COROLLARY 5.3 (Second moments at the boundaries: Part IV). *The following limits hold: at the left boundary point,*

$$\langle p_x^2\rangle_{\text{ss}} \xrightarrow[n\to\infty]{} T_- \qquad for\ x \in \{0,1\}, \tag{5.24}$$

$$\langle r_1^2\rangle_{\text{ss}} \xrightarrow[n\to\infty]{} T_-, \tag{5.25}$$

$$\langle r_1 r_2\rangle_s \xrightarrow[n\to\infty]{} 0, \tag{5.26}$$

and at the right boundary point,

$$\langle p_x^2\rangle_{ss} \xrightarrow[n\to\infty]{} T_+ \qquad for\ x\in\{n-1,n\}, \tag{5.27}$$

$$\langle r_n^2\rangle_{ss} \xrightarrow[n\to\infty]{} T_+ + \overline{\tau}_+^2, \tag{5.28}$$

$$\langle r_{n-1}r_n\rangle_s \xrightarrow[n\to\infty]{} \overline{\tau}_+^2. \tag{5.29}$$

Proof. **(Proofs of (5.24) and (5.27).)** From (5.4) and Proposition 5.3 we get $\langle p_0^2\rangle_{ss}\to T_-$. Thanks to (5.16) and (5.20) we deduce $\langle p_1^2\rangle_{ss}\to T_-$, which in turn proves (5.24). A similar argument proves (5.27). Indeed, from (5.5) and Proposition 5.3, we get $\langle p_n^2\rangle_{ss}\to T_+$ and from (5.13) and (5.21), we deduce $\langle p_{n-1}^2\rangle_{ss}\to T_+$.

Proofs of (5.25) and (5.28). The limit (5.25) follows directly from (5.8) and Proposition 5.4. From (4.2) we conclude that $\langle r_n\rangle_{ss}\to\overline{\tau}_+$. Using then (5.10) together with Proposition 5.4 we conclude (5.28).

Proofs of (5.26) and (5.29). Note that

$$n^{-2}L(r_1 p_1) = (p_1-p_0)p_1 + (r_2-r_1)r_1 - \gamma r_1 p_1 \tag{5.30}$$

$$n^{-2}L(r_n p_{n-1}) = (p_n-p_{n-1})p_{n-1} + (r_n-r_{n-1})r_n - \gamma r_n p_{n-1}. \tag{5.31}$$

Taking the average with respect to the stationary state, and using Proposition 5.4 together with (5.24) proved above, we get

$$\langle r_1^2\rangle_{ss} - \langle r_1 r_2\rangle_s \xrightarrow[n\to\infty]{} T_- \tag{5.32}$$

and

$$\langle r_n^2\rangle_{ss} - \langle r_{n-1}r_n\rangle_s \xrightarrow[n\to\infty]{} T_+. \tag{5.33}$$

Using the already proved limits (5.25) and (5.28) we conclude (5.26) and (5.29). □

PROPOSITION 5.5 (The stationary current: Part II). *The following limits hold:*

$$\phi(1) \xrightarrow[n\to\infty]{} -\frac{1}{2}(\gamma^{-1}+\gamma)T_- \tag{5.34}$$

$$\phi(n) \xrightarrow[n\to\infty]{} -\frac{1}{2}(\gamma^{-1}+\gamma)T_+ - \frac{\overline{\tau}_+^2}{2}. \tag{5.35}$$

In consequence, (3.1) holds and Theorem 3.2 is proved.

Proof. Limits in (5.34) and (5.35) follow from formula (4.4), Proposition 5.4 and the limits computed in Corollary 5.3. The limit (3.1) is a consequence of (5.34), (5.35) and formula (5.18). □

5.3. Energy bounds. We now provide bounds on the total energy under the stationary state:

PROPOSITION 5.6 (Energy bounds). *There exists* $C = C(\gamma,\widetilde{\gamma},\overline{\tau}_+,T_+,T_-) > 0$, *such that*

$$\frac{1}{n}\sum_{x=1}^n \langle p_x^2\rangle_{ss} \leqslant C \qquad and \qquad \frac{1}{n}\sum_{x=1}^n \langle r_x^2\rangle_{ss} \leqslant C, \quad n\geqslant 1. \tag{5.36}$$

Proof.   From the current decomposition given by (4.3), we easily get that

$$\phi(x) = (x-1)\overline{j}_s + \phi(1), \qquad \text{for any } x \in \mathbb{I}_n.$$

Summing over x, this gives

$$\frac{1}{n}\sum_{x=1}^{n}\phi(x) = \frac{1}{n}\sum_{x=2}^{n}(x-1)\overline{j}_s + \phi(1) = \frac{n(n-1)}{2n}\,\overline{j}_s + \phi(1).$$

Therefore, recalling (5.34) and (3.1), we get

$$\frac{1}{n}\sum_{x=1}^{n}\phi(x) \xrightarrow[n\to\infty]{} -\frac{1}{4}\big(\gamma^{-1}+\gamma\big)(T_{+}+T_{-}) - \frac{\overline{\tau}_{+}^{2}}{4\gamma}. \tag{5.37}$$

From (4.4), we have

$$\frac{1}{n}\sum_{x=1}^{n}\phi(x) = -\frac{1}{2\gamma n}\sum_{x=1}^{n}\langle r_x^2\rangle_{\mathrm{ss}} - \frac{1}{2\gamma n}\sum_{x=1}^{n}\langle p_x p_{x-1}\rangle_{\mathrm{ss}} - \frac{\gamma}{2n}\sum_{x=1}^{n}\langle p_x^2\rangle_{\mathrm{ss}} + \frac{\gamma}{4n}\big(\langle p_n^2\rangle_{\mathrm{ss}} - \langle p_0^2\rangle_{\mathrm{ss}}\big). \tag{5.38}$$

To compute the limit of the second sum in the right-hand side of (5.38), we first write:

$$n^{-2}L(p_{x-1}p_x) = (r_{x+1}-r_x)p_{x-1} + (r_x - r_{x-1})p_x - 2\gamma p_x p_{x-1}, \qquad x \in \{2,\ldots,n-1\}. \tag{5.39}$$

Thus, taking the average with respect to the stationary state and subsequently using (5.23), we obtain

$$\begin{aligned}
2\gamma\langle p_x p_{x-1}\rangle_{\mathrm{ss}} &= \langle p_x r_x\rangle_{\mathrm{ss}} + \langle p_{x-1} r_{x+1}\rangle_{\mathrm{ss}} - \langle p_x r_{x-1}\rangle_{\mathrm{ss}} - \langle p_{x-1} r_x\rangle_{\mathrm{ss}} \\
&= \langle p_{x-1} r_{x+1}\rangle_{\mathrm{ss}} - \langle p_x r_{x-1}\rangle_{\mathrm{ss}}.
\end{aligned} \tag{5.40}$$

On the other hand

$$n^{-2}L(r_x r_{x+1}) = (p_x - p_{x-1})r_{x+1} + (p_{x+1} - p_x)r_x, \qquad x \in \{1,\ldots,n-1\}.$$

Hence, taking the average and using (5.23) again, we get

$$\begin{aligned}
0 &= \langle p_{x+1} r_x\rangle_{\mathrm{ss}} + \langle p_x r_{x+1}\rangle_{\mathrm{ss}} - \langle p_x r_x\rangle_{\mathrm{ss}} - \langle p_{x-1} r_{x+1}\rangle_{\mathrm{ss}} \\
&= \langle p_{x+1} r_x\rangle_{\mathrm{ss}} + \langle p_{x+1} r_{x+1}\rangle_{\mathrm{ss}} - \langle p_x r_x\rangle_{\mathrm{ss}} - \langle p_{x-1} r_{x+1}\rangle_{\mathrm{ss}},
\end{aligned}$$

which yields

$$\langle p_{x+1} r_{x+1}\rangle_{\mathrm{ss}} - \langle p_x r_x\rangle_{\mathrm{ss}} = \langle p_{x-1} r_{x+1}\rangle_{\mathrm{ss}} - \langle p_{x+1} r_x\rangle_{\mathrm{ss}}$$

for any $x \in \{2,\ldots,n-1\}$. Combining with (5.40) we get

$$2\gamma\langle p_x p_{x-1}\rangle_{\mathrm{ss}} = \langle p_{x+1} r_{x+1}\rangle_{\mathrm{ss}} - \langle p_x r_x\rangle_{\mathrm{ss}} + \langle p_{x+1} r_x\rangle_{\mathrm{ss}} - \langle p_x r_{x-1}\rangle_{\mathrm{ss}}, \tag{5.41}$$

for any $x \in \{2,\ldots,n-1\}$. Summing over x, one gets:

$$\sum_{x=2}^{n-1}\langle p_x p_{x-1}\rangle_{\mathrm{ss}} = \frac{1}{2\gamma}\big(\langle p_n r_n\rangle_{\mathrm{ss}} - \langle p_2 r_2\rangle_{\mathrm{ss}} + \langle p_n r_{n-1}\rangle_{\mathrm{ss}} - \langle p_2 r_1\rangle_{\mathrm{ss}}\big). \tag{5.42}$$

To compute the limit as $n\to\infty$, we need to estimate the covariances appearing in the right-hand side. The covariance $\langle p_n r_n\rangle_{\mathrm{ss}}$ can be estimated thanks to Proposition 5.4.

We still need the bounds on the covariances $\langle p_2 r_2 \rangle_{\mathrm{ss}}$, $\langle p_n r_{n-1} \rangle_{\mathrm{ss}}$ and $\langle p_2 r_1 \rangle_{\mathrm{ss}}$. To deal with it we write

$$n^{-2} L(p_0 p_1) = (r_2 - r_1) p_0 + r_1 p_1 - \frac{1}{2} (3\gamma + \widetilde{\gamma}) p_0 p_1 \tag{5.43}$$

$$n^{-2} L(r_1 r_2) = (p_1 - p_0) r_2 + (p_2 - p_1) r_1 \tag{5.44}$$

$$n^{-2} L(p_{n-1} p_n) = (\overline{\tau}_+ - r_n) p_{n-1} + (r_n - r_{n-1}) p_n - \frac{1}{2} (3\gamma + \widetilde{\gamma}) p_{n-1} p_n. \tag{5.45}$$

Taking averages with respect to the stationary state and summing (5.43) and (5.44) sideways gives (using $\langle p_2 r_2 \rangle_{\mathrm{ss}} = \langle p_1 r_2 \rangle_{\mathrm{ss}}$ from (5.23))

$$\langle p_2 r_2 \rangle_{\mathrm{ss}} + \langle p_2 r_1 \rangle_{\mathrm{ss}} = \langle p_0 r_1 \rangle_{\mathrm{ss}} + \frac{1}{2} (3\gamma + \widetilde{\gamma}) \langle p_0 p_1 \rangle_{\mathrm{ss}} \xrightarrow[n \to \infty]{} 0, \tag{5.46}$$

from Proposition 5.4. Moreover, (5.45) gives (using $\langle p_n r_n \rangle_{\mathrm{ss}} = \langle p_{n-1} r_n \rangle_{\mathrm{ss}}$)

$$\langle r_{n-1} p_n \rangle_{\mathrm{ss}} = \overline{\tau}_+ \langle p_{n-1} \rangle_{\mathrm{ss}} - \frac{1}{2} (3\gamma + \widetilde{\gamma}) \langle p_{n-1} p_n \rangle_{\mathrm{ss}} \xrightarrow[n \to \infty]{} 0, \tag{5.47}$$

from (4.1) and Proposition 5.4. Therefore, we have proved that (5.42) vanishes as $n \to \infty$. In fact, due to the estimates obtained in Proposition 5.4 we have even proved that there exists a constant $C = C(\gamma, \widetilde{\gamma}, \overline{\tau}_+, T_+, T_-) > 0$, such that

$$\Big| \sum_{x=1}^{n} \langle p_x p_{x-1} \rangle_{\mathrm{ss}} \Big| \leqslant \frac{C}{\sqrt{n}}, \quad n \geqslant 1. \tag{5.48}$$

From (5.38) it follows that

$$\frac{1}{2\gamma n} \sum_{x=1}^{n} \langle r_x^2 \rangle_{\mathrm{ss}} + \frac{\gamma}{2n} \sum_{x=1}^{n} \langle p_x^2 \rangle_{\mathrm{ss}} = -\frac{1}{2\gamma n} \sum_{x=1}^{n} \langle p_x p_{x-1} \rangle_{\mathrm{ss}} - \frac{1}{n} \sum_{x=1}^{n} \phi(x) + \frac{\gamma}{4n} \big(\langle p_n^2 \rangle_{\mathrm{ss}} - \langle p_0^2 \rangle_{\mathrm{ss}} \big)$$

$$\xrightarrow[n \to \infty]{} \frac{1}{4} \big(\gamma^{-1} + \gamma \big) (T_+ + T_-) + \frac{\overline{\tau}_+^2}{4\gamma},$$

due to (5.37) and (5.48). This in particular implies estimate (5.36). □

Thanks to the energy bounds, we are finally able to prove one further convergence, which will be essential in establishing Proposition 4.3.

PROPOSITION 5.7. *For any continuous test function* $G : \mathbb{I} \to \mathbb{R}$ *we have*

$$\frac{1}{n} \sum_{x=1}^{n-1} G\Big(\frac{x}{n} \Big) \langle p_x p_{x+1} \rangle_{\mathrm{ss}} \xrightarrow[n \to \infty]{} 0. \tag{5.49}$$

Proof. Assume first that $G \in C^1(\mathbb{I})$. For the sake of brevity we denote $G_x := G(x/n)$ for any $x \in \mathbb{I}_n$ and $\psi(x) = \langle p_{x+1} r_{x+1} \rangle_{\mathrm{ss}} + \langle p_{x+1} r_x \rangle_{\mathrm{ss}}$. Then (5.41) says that

$$\langle p_x p_{x+1} \rangle_{\mathrm{ss}} = \frac{1}{2\gamma} \big(\psi(x+1) - \psi(x) \big), \qquad \text{for any } x \in \{1, \ldots, n-2\}.$$

Therefore, by an application of summation by parts formula, we get

$$\frac{1}{n} \sum_{x=1}^{n-1} G_x \langle p_x p_{x+1} \rangle_{\mathrm{ss}} = \frac{1}{2\gamma n^2} \sum_{x=2}^{n-2} n (G_{x-1} - G_x) \psi(x) \tag{5.50}$$

$$+\frac{1}{n}\langle p_n p_{n-1}\rangle_{\mathrm{ss}} G_{n-1} + \frac{1}{2\gamma n}\big(\langle p_n r_n\rangle_{\mathrm{ss}} + \langle p_n r_{n-1}\rangle_{\mathrm{ss}}\big) \tag{5.51}$$

$$-\frac{1}{2\gamma n}\big(\langle p_2 r_2\rangle_{\mathrm{ss}} + \langle p_2 r_1\rangle_{\mathrm{ss}}\big) G_1. \tag{5.52}$$

The boundary terms (5.51) and (5.52) vanish, as $n\to\infty$, thanks to (5.21), (5.46) and (5.47). To deal with the sum in the right-hand side of (5.50) note that, since $G\in C^1(\mathbb{I})$, we have

$$\sup_{x\in\{2,\dots,n-2\}} n|G_{x-1}-G_x| \leqslant \|G'\|_\infty. \tag{5.53}$$

Since

$$|\psi(x)| \leqslant 2\big(2\langle p_{x+1}^2\rangle_{\mathrm{ss}} + \langle r_x^2\rangle_{\mathrm{ss}} + \langle r_{x+1}^2\rangle_{\mathrm{ss}}\big), \qquad x\in\{1,\dots,n-1\}$$

we conclude that

$$\frac{1}{2\gamma n^2}\left|\sum_{x=2}^{n-2} n(G_{x-1}-G_x)\psi(x)\right| \leqslant \frac{C}{n}\left\{\frac{1}{n}\sum_{x=1}^{n}\big(\langle p_x^2\rangle_{\mathrm{ss}} + \langle r_x^2\rangle_{\mathrm{ss}}\big)\right\}, \tag{5.54}$$

which vanishes, as $n\to+\infty$, thanks to the energy bound (5.36). This proves (5.49) for any test function $G\in C^1(\mathbb{I})$. The result can be extended to all continuous functions by the standard density argument and the energy bound (5.36). □

5.4. Proof of Proposition 4.3.   We now have at our disposal all components needed to prove Proposition 4.3 and thus conclude the proof of Theorem 3.3. There are three convergences to prove:

Proof of (4.9). From Proposition 4.2 (in particular (4.5)), the function ϕ is linear and is completely determined from the values at the endpoints. More precisely,

$$\phi(x) = \frac{\phi(n)-\phi(1)}{n-1}x + \frac{n\phi(1)-\phi(n)}{n-1}, \qquad \text{for any } x\in\mathbb{I}_n.$$

Since, from Proposition 5.5, the values $\phi(1)$ and $\phi(n)$ are of order 1 as $n\to\infty$, we see that

$$\phi(x) \simeq \big(\phi(n)-\phi(1)\big)\frac{x}{n} + \phi(1), \qquad \text{as } n\to\infty,$$

and therefore we easily obtain

$$\frac{1}{n}\sum_{x\in\mathbb{I}_n} G\Big(\frac{x}{n}\Big)\phi(x) \xrightarrow[n\to\infty]{} \int_{\mathbb{I}} G(u)\left\{-\frac{\bar{\tau}_+^2}{2\gamma}u - \frac{1}{2}(\gamma^{-1}+\gamma)\big[(T_+-T_-)u+T_-\big]\right\}\mathrm{d}u.$$

which proves (4.9) directly.

Proof of (4.10). Concerning $\mathcal{H}_n^\nabla(G)$ we use a summation by parts formula (with the notation $G_x = G(x/n)$), which leads to:

$$\mathcal{H}_n^\nabla(G) = \frac{\gamma^2}{2(1+\gamma^2)}\left(\frac{G_n}{n}\langle p_n^2\rangle_{\mathrm{ss}} - \frac{G_1}{n}\langle p_0^2\rangle_{\mathrm{ss}} + \frac{1}{n^2}\sum_{x=1}^{n-1} n(G_x-G_{x+1})\langle p_x^2\rangle_{\mathrm{ss}}\right).$$

The boundary terms in the right-hand side vanish, as $n\to+\infty$, since $\langle p_n^2\rangle_{\mathrm{ss}}$ and $\langle p_0^2\rangle_{\mathrm{ss}}$ are bounded, due to (5.2). To deal with the limit of the last sum in the right-hand side, we can repeat the argument made in (5.53)-(5.54), which shows that the expression vanishes. Thus (4.10) holds.

Proof of (4.11)**.** This is a consequence of (5.49).

Appendix. Non-Stationary behaviour. In this section we explain how to derive (2.12) and (2.13): while the derivation of (2.12) is rigorous, in order to obtain (2.13) we need to assume a form of local equilibrium that allows for the local equipartition of kinetic and potential energy, see (6.46) below. In the stationary setting this term corresponds to $\mathcal{H}_n^{\mathrm{m}}(G)$ in (4.8) and, similarly, does not appear in the case $\gamma=1$. Unfortunately, quite analogously with the stationary situation, the relative entropy method does not allow us to treat the case $\gamma\neq1$. Throughout the present section we allow $\overline{\tau}_+(t)$ to be a C^1 function.

6.1. Preliminaries. In the present section we establish non-stationary asymptotics corresponding to Corollary 5.3. They will be useful in proving the hydrodynamic limit in Section 6.2. Since ν_{T_+} is not stationary, except for the corresponding equilibrium boundary conditions, the relative entropy $H_n(t)$ defined as

$$H_n(t) := \int f_n(t)\log f_n(t)\mathrm{d}\nu_{T_+}$$

is not strictly decreasing in time, where hereafter $f_n(t)$ is the density of the Ω_n–valued random variable $(\mathbf{r}_n(t),\mathbf{p}_n(t))$ (recall (2.7)), with respect to ν_{T_+}. However, the effect of the boundary condition can be controlled and one can obtain a linear-in-n bound at any time t, i.e. (see the proof of Proposition 4.1 in [20] for the details of the argument)

$$H_n(t) \leqslant C(t)n, \quad n\geqslant 1, t\geqslant 0. \tag{6.1}$$

Both here and throughout the remainder of the paper $C(t)$ shall denote a generic constant, always independent of n and locally bounded in t.

Furthermore, one obtains the bounds on the Dirichlet form controlling the entropy production, similar to (5.19),

$$\int_0^t \mathrm{d}s\left[\gamma\sum_{x=0}^{n-1}\mathcal{D}_x(f_n(s)) + \widetilde{\gamma}T_-\left\langle\!\!\left\langle\frac{(\partial_{p_0}(f_n(s)/h))^2}{(f_n(s)/h)}\right\rangle\!\!\right\rangle_{T_-} + \widetilde{\gamma}T_+\left\langle\!\!\left\langle\frac{(\partial_{p_n}f_n(s))^2}{f_n(s)}\right\rangle\!\!\right\rangle_{T_+}\right]$$
$$\leqslant \frac{C(t)}{n}, \tag{6.2}$$

where $h = g_{T_-}/g_{T_+}$ and $n\geqslant 1$, $t\geqslant 0$, see Proposition 4.1 and Appendix D of [20] for the proof.

Below we list some consequences of the above bounds on the entropy and Dirichlet forms.

LEMMA 6.1. *The following equalities hold:*

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^t\left(p_n^2(s)-T_+\right)\mathrm{d}s\right]=0, \qquad \lim_{n\to\infty}\mathbb{E}\left[\int_0^t\left(p_0^2(s)-T_-\right)\mathrm{d}s\right]=0 \tag{6.3}$$

and

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^t j_{n-1,n}(s)\mathrm{d}s\Big]=0, \qquad \lim_{n\to\infty}\mathbb{E}\Big[\int_0^t j_{0,1}(s)\mathrm{d}s\Big]=0. \tag{6.4}$$

Proof. Note that,

$$\mathbb{E}\Big[\int_0^t \big(p_n^2(s)-T_+\big)\mathrm{d}s\Big]=\int_0^t \mathrm{d}s \int \big(p_n^2-T_+\big)f_n(s)\mathrm{d}\nu_{T_+}$$
$$=T_+\int_0^t \mathrm{d}s \int p_n\partial_{p_n}f_n(s)\mathrm{d}\nu_{T_+}.$$

Thus, by the Cauchy-Schwarz inequality

$$\Big|\mathbb{E}\Big[\int_0^t \big(p_n^2(s)-T_+\big)\mathrm{d}s\Big]\Big|$$
$$\leqslant T_+\Big(\int_0^t \mathrm{d}s \int p_n^2 f_n(s)\mathrm{d}\nu_{T_+}\Big)^{1/2}\Big(\int_0^t \mathrm{d}s \int \frac{(\partial_{p_n}f_n(s))^2}{f_n(s)}\mathrm{d}\nu_{T_+}\Big)^{1/2}. \tag{6.5}$$

By the entropy inequality, see e.g. [14, p. 338], we can write (recall (2.2))

$$\int \mathcal{H}_n(\mathbf{r},\mathbf{p})f_n(s,\mathbf{r},\mathbf{p})\nu_{T_+}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p})$$
$$\leqslant\frac{1}{\alpha}\Big\{\log\Big[\int \exp\{\alpha\mathcal{H}_n(\mathbf{r},\mathbf{p})\}\nu_{T_+}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p})\Big]+H_n(s)\Big\}$$

for any $\alpha>0$. By (6.1), for any $t>0$ and a sufficiently small $\alpha>0$, there exists $C>0$ such that

$$\sup_{s\in[0,t]}\int \mathcal{H}_n(\mathbf{r},\mathbf{p})f_n(s,\mathbf{r},\mathbf{p})\nu_{T_+}(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p})\leqslant Cn, \quad n\geqslant 1. \tag{6.6}$$

Consequently, by (6.2) and (6.5), there exists $C>0$ such that

$$\sup_{s\in[0,t]}\Big|\mathbb{E}\Big[\int_0^t \big(p_n^2(s)-T_+\big)\mathrm{d}s\Big]\Big|\leqslant C, \quad n\geqslant 1.$$

Hence, in particular we obtain

$$\int_0^t \mathrm{d}s \int p_n^2 f_n(s)\mathrm{d}\nu_{T_+}\leqslant C. \tag{6.7}$$

Using this estimate in (6.5) together with (6.2) we conclude that for any $t\geqslant 0$ there exists $C>0$ for which

$$\Big|\mathbb{E}\Big[\int_0^t \big(p_n^2(s)-T_+\big)\mathrm{d}s\Big]\Big|\leqslant\frac{C}{\sqrt{n}}, \quad n\geqslant 1. \tag{6.8}$$

Hence the first equality of (6.3) follows. The proof of the second equality of (6.3) is similar.

Concerning (6.4): from the energy conservation it follows that

$$n^{-2}L\mathcal{E}_n=j_{n-1,n}+\frac{\widetilde{\gamma}}{2}(T_+-p_n^2)+\overline{\tau}_+(t)p_n.$$

To deal with the term $\overline{\tau}_+(t)p_n$, note that

$$
\begin{aligned}
\left| \int_0^t \overline{\tau}_+(s) \mathbb{E}[p_n(s)] \mathrm{d}s \right| &\leqslant \|\overline{\tau}_+\|_\infty \int_0^t \left| \int p_n f_n(s) \mathrm{d}\nu_{T_+} \right| \mathrm{d}s \\
&= \frac{\|\overline{\tau}_+\|_\infty}{T_+} \int_0^t \left| \int \partial_{p_n} f_n(s) \mathrm{d}\nu_{T_+} \right| \mathrm{d}s \\
&\leqslant \frac{\|\overline{\tau}_+\|_\infty}{T_+} \int_0^t \left(\int f_n(s) d\nu_{T_+} \right)^{1/2} \left(\int \frac{(\partial_{p_n} f_n(s))^2}{f_n(s)} \mathrm{d}\nu_{T_+} \right)^{1/2} \mathrm{d}s \\
&\leqslant \frac{\|\overline{\tau}_+\|_\infty}{T_+} \frac{C}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0,
\end{aligned}
\tag{6.9}
$$

by virtue of (6.2).

The first equality of (6.4) is then a direct consequence of (6.3). The same argument works for $j_{0,1}$. □

Using the energy conservation it follows immediately:

COROLLARY 6.1.   *The currents, defined in* (2.10) *and* (2.11), *satisfy*

$$
\lim_{n\to\infty} \mathbb{E}\left[\int_0^t j_{x,x+1}(s) \mathrm{d}s \right] = 0, \quad x = -1, \ldots, n, \quad t \geqslant 0.
\tag{6.10}
$$

Concerning the potential energy at the boundary points we have the following bound.

LEMMA 6.2.   *There exists a constant* $C < \infty$ *such that*

$$
\mathbb{E}\left[\int_0^t \left(r_1^2(s) + r_n^2(s) \right) \mathrm{d}s \right] \leqslant C, \quad n \geqslant 1.
\tag{6.11}
$$

Proof.   Using (5.7) we get

$$
\begin{aligned}
&n^{-2} \mathbb{E}\left[p_n(t) r_n(t) - p_n(0) r_n(0) \right] \\
&= \int_0^t \mathbb{E}\left[p_n(s)(p_n(s) - p_{n-1}(s)) + (\overline{\tau}_+(s) - r_n(s)) r_n(s) - \frac{1}{2}(\widetilde{\gamma} + \gamma) p_n(s) r_n(s) \right] \mathrm{d}s.
\end{aligned}
\tag{6.12}
$$

The term in the left-hand side vanishes, as $n \to +\infty$, due to estimate (6.6). We can repeat then the same arguments as we have used to obtain (5.14) and conclude that there exists $C > 0$ such that

$$
\mathbb{E}\left[\int_0^t r_n^2(s) \mathrm{d}s \right] \leqslant C \left\{ \mathbb{E}\left[\int_0^t p_n^2(s) \mathrm{d}s \right] + 1 \right\}, \quad n \geqslant 1.
\tag{6.13}
$$

Estimate (6.7) can be used to obtain the desired bound for $\mathbb{E}\left[\int_0^t r_n^2(s) \mathrm{d}s \right]$. An analogous estimate on $\mathbb{E}\left[\int_0^t r_1^2(s) \mathrm{d}s \right]$ follows from the same argument, using (5.6) and the second equality in (6.3) instead. □

LEMMA 6.3.   *The following convergences hold: at the left boundary point:*

$$
\mathbb{E}\left[\int_0^t p_0(s) r_1(s) \mathrm{d}s \right] \xrightarrow[n\to\infty]{} 0,
\tag{6.14}
$$

$$\mathbb{E}\Big[\int_0^t \big(p_1^2(s) - p_0^2(s)\big)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0, \tag{6.15}$$

$$\mathbb{E}\Big[\int_0^t r_1(s)r_2(s)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0 \tag{6.16}$$

and at the right boundary point:

$$\mathbb{E}\Big[\int_0^t p_{n-1}(s)r_n(s)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0, \tag{6.17}$$

$$\mathbb{E}\Big[\int_0^t \big(p_{n-1}^2(s) - p_n^2(s)\big)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0, \tag{6.18}$$

$$\mathbb{E}\Big[\int_0^t \big(r_{n-1}(s)r_n(s) - \overline{\tau}_+^2(s)\big)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0. \tag{6.19}$$

Proof. Since $n^{-2}Lr_n^2 = 2r_np_n - 2r_np_{n-1}$, we conclude that (6.17) holds, provided that we can prove

$$\mathbb{E}\Big[\int_0^t r_n(s)p_n(s)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0. \tag{6.20}$$

The latter is a consequence of the following estimate, cf. (6.2),

$$\Big|\mathbb{E}\Big[\int_0^t r_n(s)p_n(s)\mathrm{d}s\Big]\Big| = \Big|\int_0^t \mathrm{d}s \int r_n \partial_{p_n} f_n(s)\mathrm{d}\nu_{T_+}\Big|$$

$$\leqslant \Big(\int_0^t \mathrm{d}s \int r_n^2 f_n(s)\mathrm{d}\nu_{T_+}\Big)^{1/2} \Big(\int_0^t \mathrm{d}s \int \frac{(\partial_{p_n} f_n(s))^2}{f_n(s)}\mathrm{d}\nu_{T_+}\Big)^{1/2}$$

$$\leqslant \Big(\int_0^t \mathrm{d}s \int r_n^2 f_n(s)\mathrm{d}\nu_{T_+}\Big)^{1/2} \frac{C}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0.$$

The proof of (6.14) is similar.

To show (6.18) note that (see (5.12))

$$\gamma \int_0^t \mathbb{E}\big[p_{n-1}^2(s) - p_n^2(s)\big]\mathrm{d}s = 2\int_0^t \overline{\tau}_+(s)\mathbb{E}\big[p_n(s)\big]\mathrm{d}s - 2\int_0^t \mathbb{E}\big[r_n(s)p_n(s)\big]\mathrm{d}s$$

$$+ \widetilde{\gamma}\int_0^t \mathbb{E}\big[T_+ - p_n^2(s)\big]\mathrm{d}s - \frac{1}{n^2}\mathbb{E}\big[p_n^2(t) - p_n^2(0)\big].$$

The second, third and fourth terms in the right-hand side vanish due to (6.20), Lemma 6.1 and (6.6), respectively. The first term has been already treated in (6.9). An analogous argument, starting from (5.16) allows us to prove (6.15).

In addition, we have

$$\mathbb{E}\Big[\int_0^t p_{n-1}(s)p_n(s)\mathrm{d}s\Big]$$

$$= \mathbb{E}\Big[\int_0^t p_{n-1}(s)(p_n(s) - T_+)\mathrm{d}s\Big] + T_+\mathbb{E}\Big[\int_0^t p_{n-1}(s)\mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0. \tag{6.21}$$

The above convergence is proved as follows: the first term in the right-hand side can be estimated by the Cauchy-Schwarz inequality. Then we can use the bound

$$\mathbb{E}\Big[\int_0^t p_{n-1}^2(s)\mathrm{d}s\Big] \leqslant C, \quad n \geqslant 1$$

(it follows from the already proved (6.18)) and Lemma 6.1 to prove that it vanishes, as $n \to +\infty$. To show that the second term vanishes we can use estimates analogous to (6.9). The argument for (6.14) follows essentially along the same lines.

By (5.7) we can now write

$$\mathbb{E}\Big[\int_0^t \big(r_n^2(s) - T_+ - \overline{\tau}_+^2(s)\big)\mathrm{d}s\Big] = \int_0^t \overline{\tau}_+(s)\mathbb{E}\big[r_n(s) - \overline{\tau}_+(s)\big]\mathrm{d}s$$

$$+ \int_0^t \mathbb{E}\big[p_n^2(s) - T_+\big]\mathrm{d}s - \int_0^t \mathbb{E}\big[p_n(s)p_{n-1}(s)\big]\mathrm{d}s$$

$$- \frac{1}{2}(\widetilde{\gamma} + \gamma)\int_0^t \mathbb{E}\big[p_n(s)r_n(s)\big]\mathrm{d}s$$

$$- \frac{1}{n^2}\mathbb{E}\big[p_n(t)r_n(t) - p_n(0)r_n(0)\big]. \qquad (6.22)$$

By the previous results, the last four terms in the right-hand side vanish, as $n \to +\infty$. Concerning the first term we use

$$n^{-2}Lp_n(s) = \overline{\tau}_+(s) - r_n(s) - \frac{1}{2}(\widetilde{\gamma} + \gamma)p_n(s) \qquad (6.23)$$

to conclude that

$$\int_0^t \overline{\tau}_+(s)\mathbb{E}\big[r_n(s) - \overline{\tau}_+(s)\big]\mathrm{d}s = \frac{1}{2}(\widetilde{\gamma} + \gamma)\int_0^t \overline{\tau}_+(s)\mathbb{E}\big[p_n(s)\big]\mathrm{d}s$$

$$- \frac{1}{n^2}\int_0^t \overline{\tau}_+(s)\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}\big[p_n(s)\big]\mathrm{d}s. \qquad (6.24)$$

The first term vanishes, as $n \to +\infty$, by (6.9). By integration by parts the second term equals

$$\frac{1}{n^2}\int_0^t \overline{\tau}_+'(s)\mathbb{E}\big[p_n(s)\big]\mathrm{d}s - \frac{1}{n^2}\Big(\overline{\tau}_+(t)\mathbb{E}\big[p_n(t)\big] - \overline{\tau}_+(0)\mathbb{E}\big[p_n(0)\big]\Big)$$

which also vanishes, thanks to (6.9) and (6.6). Therefore,

$$\mathbb{E}\Big[\int_0^t \big(r_n^2(s) - T_+ - \overline{\tau}_+^2(s)\big)\mathrm{d}s\Big] \longrightarrow 0.$$

To see (6.19) it suffices to show that

$$\mathbb{E}\Big[\int_0^t \big(r_n^2(s) - r_n(s)r_{n-1}(s) - T_+\big)\mathrm{d}s\Big] \longrightarrow 0.$$

For that purpose we invoke (5.31), which permits to write

$$\mathbb{E}\Big[\int_0^t \big((r_{n-1}(s) - r_n(s))r_n(s) + T_+\big)\mathrm{d}s\Big] = \mathbb{E}\Big[\int_0^t (p_n^2(s) - p_{n-1}^2(s))\mathrm{d}s\Big]$$

$$+ \mathbb{E}\Big[\int_0^t p_n(s)p_{n-1}(s)\mathrm{d}s\Big] + \mathbb{E}\Big[\int_0^t (T_+ - p_n^2(s))\mathrm{d}s\Big]$$

$$- \gamma\mathbb{E}\Big[\int_0^t r_n(s)p_{n-1}(s)\mathrm{d}s\Big] - \frac{1}{n^2}\mathbb{E}\big[r_n(t)p_{n-1}(t) - r_n(0)p_{n-1}(0)\big].$$

Each term in the right-hand side of the above equality vanishes, as $n \to +\infty$, by virtue of the already proved estimates. With a similar procedure we obtain (6.16).   □

6.2. Hydrodynamic limit.   Let us now turn to equation (2.12), which can be formulated in a weak form as:

$$\int_0^1 \mathrm{d}u\, G(u)\big(r(t,u)-r(0,u)\big)$$
$$=\frac{1}{\gamma}\int_0^t \mathrm{d}s\int_0^1 \mathrm{d}u\, G''(u) r(s,u)-\frac{1}{\gamma}G'(1)\int_0^t \overline{\tau}_+(s)\mathrm{d}s, \tag{6.25}$$

for any test function $G \in C_{0,1}^2(\mathbb{I})$ – the class of C^2 functions on $[0,1]$ such that $G(0)=G(1)=0$. Existence and uniqueness of such weak solutions in an appropriate space of integrable functions are standard. By the microscopic evolution equations (2.3) we have (recall that $G_x=G(x/n)$)

$$\mathbb{E}\left[\frac{1}{n}\sum_{x=1}^n G_x\left(r_x(t)-r_x(0)\right)\right]=\mathbb{E}\left[\int_0^t \mathrm{d}s\, n\sum_{x=1}^n G_x\left(p_x(s)-p_{x-1}(s)\right)\right]$$
$$=\mathbb{E}\left[\int_0^t \mathrm{d}s\left\{-\sum_{x=1}^{n-1}(\nabla_n G)_x\, p_x(s)-nG_1 p_0(s)\right\}\right], \tag{6.26}$$

where $(\nabla_n G)_x := n(G_{x+1}-G_x)$. Using (2.4) we can write (6.26) as

$$\mathbb{E}\left[-\int_0^t \mathrm{d}s\left\{\sum_{x=1}^{n-1}\frac{1}{\gamma}(\nabla_n G)_x\left(r_{x+1}(s)-r_x(s)\right)+\frac{1}{2(\gamma+\widetilde{\gamma})}nG_1 r_1(s)\right\}\right] \tag{6.27}$$

$$+\mathbb{E}\left[\frac{1}{\gamma n^2}\sum_{x=1}^{n-1}(\nabla_n G)_x\left(p_x(t)-p_x(0)\right)+\frac{1}{2(\gamma+\widetilde{\gamma})n^2}nG_1\left(p_0(t)-p_0(0)\right)\right]. \tag{6.28}$$

Since G is smooth, $nG_1 \to G'(0)$ and $(\nabla_n G)_x \to G'(x)$, as $n\to+\infty$. Using this and (6.6), one shows that the expression (6.28) converges to 0. The only significant term is therefore the first one (6.27). Summing by parts, using the notation

$$(\Delta_n G)_x := n^2(G_{x+1}+G_{x-1}-2G_x)$$

and recalling that $G(0)=0$, it can be rewritten as

$$\mathbb{E}\left[\int_0^t \frac{1}{\gamma}\left\{\frac{1}{n}\sum_{x=2}^{n-1}(\Delta_n G)_x\, r_x(s)-(\nabla_n G)_{n-1}\, r_n(s)\right\}\mathrm{d}s\right]$$
$$-\left(\frac{1}{2(\gamma+\widetilde{\gamma})}(\nabla_n G)_0-\frac{1}{\gamma}(\nabla_n G)_1\right)\mathbb{E}\left[\int_0^t r_1(s)\mathrm{d}s\right]. \tag{6.29}$$

It is easy to see, using (2.5), that

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^t r_1(s)\mathrm{d}s\right]=\frac{\gamma+\widetilde{\gamma}}{2}\lim_{n\to+\infty}\int_0^t \mathrm{d}s\int p_1 f_n(s)\mathrm{d}\nu_{T_+}$$
$$=\frac{\gamma+\widetilde{\gamma}}{2T_+}\lim_{n\to+\infty}\int_0^t \mathrm{d}s\int \partial_{p_1} f_n(s)\mathrm{d}\nu_{T_+}=0,$$

by (6.2). Using (6.23) we also obtain

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^t r_n(s)\mathrm{d}s\right]=\int_0^t \overline{\tau}_+(s)\mathrm{d}s.$$

Therefore, we can rewrite (6.29) as

$$\int_0^t \mathrm{d}s\left\{\frac{1}{\gamma}\int_0^1 G''(u)r^{(n)}(s,u)\mathrm{d}u - G'(1)\overline{\tau}_+(s)\right\} + o_n(t), \tag{6.30}$$

where

$$\overline{r}^{(n)}(t,u) = \mathbb{E}\big[r_x(t)\big] \quad \text{for } u \in \big[\tfrac{x}{n}; \tfrac{x+1}{n}\big), \quad n \geqslant 1 \tag{6.31}$$

and $\lim_{n\to+\infty}\sup_{s\in[0,t]} o_n(s) = 0$. Thanks to (6.6) we know that there exists $R > 0$ such that

$$\sup_{n\geqslant 1}\ \sup_{s\in[0,t]}\left\|\overline{r}^{(n)}(s,\cdot)\right\|_{L^2(\mathbb{I})} =: R < +\infty. \tag{6.32}$$

The above means that for each $s \in [0,t]$ the sequence $\big\{\overline{r}^{(n)}(s,\cdot)\big\}_{n\geqslant 1}$ is contained in $\overline{B}_R$ – the closed ball of radius $R > 0$ in $L^2(\mathbb{I})$, centered at 0. The ball is compact in $L^2_w(\mathbb{I})$ – the space of square integrable functions on $\mathbb{I}$ equipped with the weak L^2 topology. The topology restricted to $\overline{B}_R$ is metrizable. From the above argument it follows in particular that for each $t > 0$ the sequence $\big\{\overline{r}^{(n)}(\cdot)\big\}$ is equicontinuous in $\mathcal{C}\big([0,t],\overline{B}_R\big)$. Thus, according to the Arzela theorem, see e.g. [13, p. 234], it is sequentially pre-compact in the space $\mathcal{C}\big([0,t], L^2_w(\mathbb{I})\big)$ for any $t > 0$. Consequently, any limiting point of the sequence satisfies (6.25).

Concerning equation (2.13) with the respective boundary condition, its weak formulation is as follows: for any $G \in L^1\big([0,+\infty); C^2_{0,1}(\mathbb{I})\big)$ we have

$$\begin{aligned}
0 = {}&\int_0^1 G(0,u)e(0,u)\,\mathrm{d}u \\
&+ \int_0^{+\infty}\mathrm{d}s\int_0^1\left(\partial_s G(s,u) + \frac{1}{2}(\gamma^{-1}+\gamma)\partial_u^2 G(s,u)\right)e(s,u)\,\mathrm{d}u \\
&+ \frac{1}{4}(\gamma^{-1}-\gamma)\int_0^{+\infty}\mathrm{d}s\int_0^1 \partial_u^2 G(s,u)r^2(s,u)\,\mathrm{d}u \\
&- \int_0^{+\infty}\left(\partial_u G(s,1)\left[(\gamma^{-1}+\gamma)T_+ + \frac{1}{4}(\gamma^{-1}-\gamma)\overline{\tau}_+(s)^2\right] - \partial_u G(s,0)T_-\right)\mathrm{d}s. \tag{6.33}
\end{aligned}$$

Given a non-negative initial data $e(0,\cdot) \in L^1(\mathbb{I})$ and the macroscopic stretch $r(\cdot)$ (determined via (6.25)) one can easily show that the respective weak formulation of the boundary value problem for a linear heat equation, resulting from (6.33), admits a unique measure-valued solution.

The averaged thermal energy function is defined as

$$\overline{\mathcal{E}}_n(t,u) := \mathbb{E}\big[\mathcal{E}_x(t)\big], \quad u \in \big[\tfrac{x}{n}, \tfrac{x+1}{n}\big), \quad x = 0,\dots,n-1.$$

It is easy to see, thanks to (6.6), that $\big(\overline{\mathcal{E}}_n(t)\big)_{n\geqslant 1}$ is bounded in the dual to the separable Banach space $L^1\big([0,+\infty); C^2_{0,1}(\mathbb{I})\big)$. Thus it is $\star$-weakly compact. In what follows we identify its limit $e(t)$ by showing that it satisfies (6.33). To achieve this goal we are going to use the microscopic energy currents given in (2.10).

Consider now a smooth test function $G \in C_0^\infty\big([0,+\infty)\times\mathbb{I}\big)$ such that $G(s,0) = G(s,1) \equiv 0$, $s \geqslant 0$. Then, from (2.9), we get

$$-\frac{1}{n}\mathbb{E}\left[\sum_{x=0}^n G_x(0)\mathcal{E}_x(0)\right] = -\frac{1}{n}\mathbb{E}\left[\sum_{x=1}^{n-1}G_x(0)\mathcal{E}_x(0)\right] = -\frac{1}{n}\int_0^t\mathrm{d}s\,\mathbb{E}\left[\sum_{x=0}^{n-1}\partial_s G_x(s)\mathcal{E}_x(s)\right]$$

$$+ \int_0^t \mathrm{d}s \, \mathbb{E}\left[\sum_{x=1}^{n-2} (\nabla_n G)_x(s) \, j_{x,x+1}(s) - n G_{n-1}(s) j_{n-1,n}(s) + n G_1(s) j_{0,1}(s) \right]. \qquad (6.34)$$

Here $G_x(s) := G(s, x/n)$ and we use a similar notation for $\nabla_n G_x(s)$, $\Delta_n G_x(s)$.

Concerning (6.34): by (6.4), the expectation of its last two terms are negligible. In order to treat the first term of (6.34), we use the fluctuation-dissipation relation (4.6), i.e.

$$j_{x,x+1} = n^{-2} L g_x + \nabla V_x, \qquad (6.35)$$

with

$$V_x = -\frac{1}{2\gamma} r_x^2 - \frac{\gamma}{4}(p_x^2 + p_{x-1}^2) - \frac{1}{2\gamma} p_x p_{x-1}.$$

Using this relation we can rewrite the last term (6.34) as

$$\int_0^t \mathrm{d}s \, \mathbb{E}\left[\frac{1}{n} \sum_{x=2}^{n-2} (\Delta_n G)_x(s) V_x(s) - (\nabla_n G)_{n-2}(s) V_{n-1}(s) + (\nabla_n G)_1(s) V_1(s) \right] \qquad (6.36)$$

plus expressions involving the average fluctuating term

$$\int_0^t \mathrm{d}s \, \mathbb{E}\left[\frac{1}{n^2} \sum_{x=1}^{n-2} (\nabla_n G)_x(s) L g_x(s) \right]$$
$$= \mathbb{E}\left[\frac{1}{n^2} \sum_{x=1}^{n-2} \left((\nabla_n G)_x(t) g_x(t) - (\nabla_n G)_x(0) g_x(0) \right) \right] - \int_0^t \mathrm{d}s \, \mathbb{E}\left[\frac{1}{n^2} \sum_{x=1}^{n-2} (\nabla_n \partial_s G)_x(s) g_x(s) \right]$$

which turns out to be small, as $n \to +\infty$, thanks to the energy bound (6.6). By Lemmas 6.1 and 6.3 we have:

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^t \mathrm{d}s \, (\nabla_n G)_1(s) V_1(s) \right] = -\int_0^t \mathrm{d}s \, \partial_u G(s,0) \frac{1}{2}\left(\gamma^{-1} + \gamma\right) T_-, \qquad (6.37)$$

which takes care of the left boundary condition. Concerning the right one we have

$$\mathbb{E}\left[\int_0^t \mathrm{d}s \, (\nabla_n G)_{n-2}(s) V_{n-1}(s) \right] = -\mathbb{E}\left[\int_0^t \mathrm{d}s \, (\nabla_n G)_{n-2}(s) \nabla V_{n-1}(s) \right] \qquad (6.38)$$
$$+ \mathbb{E}\left[\int_0^t \mathrm{d}s \, (\nabla_n G)_{n-2}(s) V_n(s) \right]. \qquad (6.39)$$

Using the results of Lemmas 6.1 and 6.3 again we conclude that the limit of the second term (6.39) equals

$$-\int_0^t \mathrm{d}s \, \partial_u G(s,1)\left(\frac{1}{2}\left(\gamma^{-1} + \gamma\right) T_+ + \frac{1}{2\gamma} \overline{\tau}_+^2(s) \right).$$

On the other hand, using (6.35), the term (6.38) equals

$$\frac{1}{n^2} \mathbb{E}\left[\int_0^t \mathrm{d}s \, (\nabla_n G)_{n-2}(s) L g_{n-1}(s) \right] + \int_0^t \mathrm{d}s \, (\nabla_n G)_{n-2}(s) \mathbb{E}\big[j_{n-1,n}(s) \big]. \qquad (6.40)$$

From (6.10) we conclude that the second term vanishes, with $n \to +\infty$. By integration by parts the first term equals

$$\frac{1}{n^2}\mathbb{E}\Big[(\nabla_n G)_{n-2}(t)g_{n-1}(t) - (\nabla_n G)_{n-2}(0)g_{n-1}(0)\Big]$$
$$-\frac{1}{n^2}\mathbb{E}\Big[\int_0^t \mathrm{d}s\,(\nabla_n \partial_s G)_{n-2}(s)g_{n-1}(s)\Big], \quad (6.41)$$

which vanishes, thanks to (6.6). Summarizing, we have shown that

$$\lim_{n\to+\infty}\mathbb{E}\Big[\int_0^t \mathrm{d}s\,(\nabla_n G)_{n-2}(s)V_{n-1}(s)\Big]$$
$$=-\int_0^t \mathrm{d}s\,\partial_u G(s,1)\Big[\frac{1}{2}\big(\gamma^{-1}+\gamma\big)T_+ + \frac{1}{2\gamma}\overline{\tau}_+^2(s)\Big].$$

Now, for the bulk, it follows from (5.22) and (5.39) that

$$n^{-2}Lh_x = \nabla W_x - 3\gamma p_x p_{x-1}, \quad x=2,\dots,n-1, \quad (6.42)$$

with

$$h_x := \frac{1}{2}(r_x + r_{x-1})^2 + p_x p_{x-1} - r_x^2, \qquad W_x := (r_{x-1}+r_x)p_{x-2}.$$

Therefore by (6.2) and an argument similar to the one used above we conclude that

$$\lim_{n\to\infty}\int_0^t \mathrm{d}s\,\mathbb{E}\Big[\frac{1}{n}\sum_{x=2}^{n-2}(\Delta_n G)_x(s)p_x(s)p_{x-1}(s)\Big]=0. \quad (6.43)$$

In the case $\gamma=1$ we can rewrite

$$V_x = -\mathcal{E}_x + \frac{1}{4}(p_x^2 - p_{x-1}^2) - \frac{1}{2}p_x p_{x-1} \quad (6.44)$$

so that it is easy to see that (6.36) is equivalent to

$$-\int_0^t \mathrm{d}s\,\mathbb{E}\Big[\frac{1}{n}\sum_{x=2}^{n-2}(\Delta_n G)_x(s)\mathcal{E}_x(s)\Big]+o_n(1), \quad (6.45)$$

closing the energy conservation equation and concluding the proof.

Finally, for $\gamma\neq 1$ we expect the following term to vanish as $n\to+\infty$:

$$\int_0^t \mathrm{d}s\,\mathbb{E}\Big[\frac{1}{n}\sum_{x=2}^{n-2}(\Delta_n G)_x\Big(p_x^2(s) - \big(r_x(s) - \mathbb{E}[r_x(s)]\big)^2\Big)\Big], \quad (6.46)$$

as can be guessed by local equilibrium considerations. Unfortunately in order to prove the last limit one needs some higher moment bounds that are not available from relative entropy considerations. One prospective work could be to proceed in an analogous way as in the periodic case [16], by studying the evolution of the Wigner distribution of the thermal energy in Fourier coordinates.

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