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THE LOCAL WELL-POSEDNESS TO THE DENSITY-DEPENDENT MAGNETIC BÉNARD SYSTEM WITH NONNEGATIVE DENSITY*

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Abstract. We study the Cauchy problem of density-dependent magnetic Bénard system with zero density at infinity on the whole two-dimensional (2D) space. Despite the degenerate nature of the problem, we show the local existence of a unique strong solution in weighted Sobolev spaces by energy method.

Keywords. density-dependent magnetic Bénard system; strong solutions; Cauchy problem.

AMS subject classifications. 35Q35; 76D03.

1. Introduction

Consider the following density-dependent incompressible magnetic Bénard system (see [5]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla P = \mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2 + \rho\theta\mathbf{e}_2, \\ \mathbf{b}_t - \nu\Delta\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{u} = 0, \\ (\rho\theta)_t + \operatorname{div}(\rho\theta\mathbf{u}) - \kappa\Delta\theta = \rho\mathbf{u}\cdot\mathbf{e}_2, \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{b} = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is the time, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ is the spatial coordinate, and $\rho = \rho(x,t)$, $\mathbf{u} = (u^1, u^2)(x,t)$, $\mathbf{b} = (b^1, b^2)(x,t)$, $\theta = \theta(x,t)$, and $P = P(x,t)$ denote the density, velocity, magnetic, absolute temperature, and pressure of the fluid, respectively. $\mu > 0$ stands for the viscosity constant. The constant $\nu > 0$ is the resistivity coefficient and $\kappa > 0$ is the heat conductivity coefficient. $\mathbf{e}_2 \triangleq (0,1)$. The forcing term $\rho\theta\mathbf{e}_2$ in the momentum equation describes the action of the buoyancy force on fluid motion and $\rho\mathbf{u}\cdot\mathbf{e}_2$ models the Rayleigh-Bénard convection in a heated inviscid fluid. The magnetic Bénard system illuminates the heat convection phenomenon under the presence of the magnetic field.

Let $\Omega = \mathbb{R}^2$ and we consider the Cauchy problem for (1.1) with $(\rho, \mathbf{u}, \mathbf{b}, \theta)$ vanishing at infinity (in some weak sense) and the initial conditions:

$$\rho(x,0) = \rho_0(x),\ \rho\mathbf{u}(x,0) = \rho_0\mathbf{u}_0(x),\ \mathbf{b}(x,0) = \mathbf{b}_0(x),\ \rho\theta(x,0) = \rho_0\theta_0(x),\ x \in \mathbb{R}^2, \tag{1.2}$$

for given initial data $\rho_0, \mathbf{u}_0, \mathbf{b}_0$, and θ_0.

Recently, a great deal of attention has been focused on studying well-posedness of solutions to the magnetic Bénard system, both from a pure mathematical point of view and for concrete applications. When ρ is a constant, which means the fluid is homogeneous, the magnetic Bénard system has been extensively studied. In particular, many authors investigated the global existence and regularity of 2D homogeneous magnetic Bénard system with partial dissipation. Zhou et al. [18] showed the global

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well-posedness of smooth solutions with zero thermal conductivity. Cheng and Du [3] obtained the global well-posedness without thermal diffusivity and with vertical or horizontal magnetic diffusion. Ye [16] established the global regularity with horizontal dissipation, horizontal magnetic diffusion and with either horizontal or vertical thermal diffusivity. For other studies of homogeneous magnetic Bénard system, please refer to [4, 9–13, 15, 17] and references therein.

When the density is not constant, the system (1.1) is the so-called density-dependent magnetic Bénard system. Imposing a compatibility condition introduced by Choe and Kim [2], Wu [14] showed the local existence of strong solutions with nonnegative density in bounded domains $\Omega\subset\mathbb{R}^n$ $(n=2,3)$. Later on, for the initial density with positive lower bound, with the help of a bootstrap argument, Fan et al. [5] proved global strong solutions of the system (1.1) with $\nu=0$ for the general initial data in a two-dimensional bounded domain. However, even the local existence of strong solutions to the Cauchy problem of (1.1) in $\mathbb{R}^2$ is still unknown. On the one hand, for the initial density allowing vacuum states, the main difficulty lies in the possible degeneracy near vacuum. On the other hand, when the far field density equals zero, it seems difficult to bound the $L^p(\mathbb{R}^2)$-norm of $\mathbf{u}$ by $\|\sqrt{\rho}\mathbf{u}\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^2)}$ for any $p\geq 1$. In this paper, we will investigate the local existence of strong solutions to the Cauchy problem of the density-dependent magnetic Bénard system (1.1) in $\mathbb{R}^2$ with vacuum as far field density. The initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, in particular, the initial density can even have compact support.

Now, we wish to define precisely what we mean by strong solutions.

DEFINITION 1.1. *If all derivatives involved in (1.1) for $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ are regular distributions, and equations (1.1) hold almost everywhere in $\mathbb{R}^2\times(0,T)$ and (1.2) almost everywhere in $\mathbb{R}^2$, then $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ is called a strong solution to (1.1).*

In this section, for $1\leq r\leq\infty$ and $k\geq 1$, we denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r=L^r(\mathbb{R}^2),\quad W^{k,r}=W^{k,r}(\mathbb{R}^2),\quad H^k=W^{k,2}.$$

Our main result can be stated as follows:

THEOREM 1.1. *Let η_0 be a positive constant and*

$$\bar{x}:=(3+|x|^2)^{\frac{1}{2}}\log^{1+\eta_0}(3+|x|^2). \tag{1.3}$$

For constants $q>2$ and $a>1$, assume that the initial data $(\rho_0\geq 0,\mathbf{u}_0,\mathbf{b}_0,\theta_0)$ satisfy

$$\begin{cases}\rho_0\bar{x}^a\in L^1\cap H^1\cap W^{1,q},\ \sqrt{\rho_0}\mathbf{u}_0\in L^2,\ \nabla\mathbf{u}_0\in L^2,\\ \mathbf{b}_0\bar{x}^{\frac{a}{2}}\in L^2,\ \nabla\mathbf{b}_0\in L^2,\ \operatorname{div}\mathbf{u}_0=\operatorname{div}\mathbf{b}_0=0,\\ \sqrt{\rho}\theta_0\in L^2,\ \nabla\theta_0\in L^2.\end{cases} \tag{1.4}$$

Then there exists a positive time $T_0>0$ such that the problem (1.1)–(1.2) has a unique *strong solution* $(\rho\geq 0,\mathbf{u},P,\mathbf{b},\theta)$ *on* $\mathbb{R}^2\times(0,T_0]$ *satisfying*

$$\begin{cases}\rho\in C([0,T_0];L^1\cap H^1\cap W^{1,q}),\\ \rho\bar{x}^a\in L^\infty(0,T_0;L^1\cap H^1\cap W^{1,q}),\\ \sqrt{\rho}\mathbf{u},\,\nabla\mathbf{u},\,\bar{x}^{-1}\mathbf{u},\,\sqrt{t}\sqrt{\rho}\mathbf{u}_t,\,\sqrt{t}\nabla P,\,\sqrt{t}\nabla^2\mathbf{u}\in L^\infty(0,T_0;L^2),\\ \sqrt{\rho}\theta,\,\nabla\theta,\,\bar{x}^{-1}\theta,\,\sqrt{t}\sqrt{\rho}\theta_t,\,\sqrt{t}\nabla^2\theta\in L^\infty(0,T_0;L^2),\\ \mathbf{b},\mathbf{b}\bar{x}^{\frac{a}{2}},\nabla\mathbf{b},\sqrt{t}\mathbf{b}_t,\,\sqrt{t}\nabla^2\mathbf{b}\in L^\infty(0,T_0;L^2),\\ \nabla\mathbf{u}\in L^2(0,T_0;H^1)\cap L^{\frac{q+1}{q}}(0,T_0;W^{1,q}),\\ \nabla P\in L^2(0,T_0;L^2)\cap L^{\frac{q+1}{q}}(0,T_0;L^q),\\ \nabla\mathbf{b}\in L^2(0,T_0;H^1),\,\mathbf{b}_t,\ \nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\in L^2(0,T_0;L^2),\\ \sqrt{t}\nabla\mathbf{u}\in L^2(0,T_0;W^{1,q}),\\ \sqrt{\rho}\mathbf{u}_t,\,\sqrt{t}\nabla\mathbf{b}\bar{x}^{\frac{a}{2}},\,\sqrt{t}\nabla\mathbf{u}_t,\,\sqrt{t}\nabla\mathbf{b}_t\in L^2(\mathbb{R}^2\times(0,T_0)),\end{cases}\tag{1.5}$$

and

$$\inf_{0\leq t\leq T_0}\int_{B_N}\rho(x,t)dx\geq\frac{1}{4}\int_{\mathbb{R}^2}\rho_0(x)dx,\tag{1.6}$$

for some constant $N>0$ *and* $B_N\triangleq\left\{x\in\mathbb{R}^2\middle|\,|x|<N\right\}$.

REMARK 1.1. The a priori estimates in [5,14] for the bounded domain case cannot be applied here. The main reason is that the whole two-dimensional space is the critical case for the standard Sobolev embedding theorem. When the far field density vanishes, it seems difficult to bound the L^p-norm of $\mathbf{u}$ by $\|\sqrt{\rho}\mathbf{u}\|_{L^2}$ and $\|\nabla\mathbf{u}\|_{L^2}$ for any $p\geq 1$.

REMARK 1.2. If $\rho(x,t)\to\tilde{\rho}$ as $|x|\to\infty$ for positive constant $\tilde{\rho}$, that is, the far field density is away from vacuum, then L^p-norm $(p\geq 1)$ of $\mathbf{u}$ can be bounded by $\|\sqrt{\rho}\mathbf{u}\|_{L^2}$ and $\|\nabla\mathbf{u}\|_{L^2}$. Indeed, we obtain from Hölder's inequality and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}\tilde{\rho}\int_{\mathbb{R}^2}|\mathbf{u}|^2dx&=\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2dx+\int_{\mathbb{R}^2}(\tilde{\rho}-\rho)|\mathbf{u}|^2dx\\&\leq\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\rho-\tilde{\rho}\|_{L^2}\|\mathbf{u}\|_{L^4}^2\\&\leq\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+C\|\rho-\tilde{\rho}\|_{L^2}\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2},\end{aligned}$$

which, combined with Young's inequality, gives

$$\|\mathbf{u}\|_{L^2}^2\leq C\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\rho-\tilde{\rho}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\right).$$

Hence, Sobolev's embedding theorem yields the desired result.

We now make some comments on the key ingredients of the analysis in this paper. We take advantage of the method of invading domains to prove Theorem 1.1. Since the local existence of strong solutions of (1.1) with the initial density away from vacuum in bounded domains has been established by Wu [14] (see Lemma 2.1), we thus construct approximate solutions $(\rho_R,\mathbf{u}_R,P_R,\mathbf{b}_R,\theta_R)$ to (1.1), that is, for the density strictly away from vacuum initially, consider (1.1) in any bounded ball B_R with radius $R>0$ and then letting $R\to\infty$ to obtain the solution of (1.1) in $\mathbb{R}^2$. In this limit process, uniform-in-R a priori estimates of $(\rho_R,\mathbf{u}_R,P_R,\mathbf{b}_R,\theta_R)$ play a decisive role in the proof.

As mentioned above, due to criticality of Sobolev's embedding theorem, it seems difficult to bound the $L^p(\mathbb{R}^2)$-norm of $\mathbf{u}$ just in terms of $\|\sqrt{\rho}\mathbf{u}\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^2)}$, some new elaborate estimates are needed to achieve our goal. Motivated by [7], using a Hardy-type inequality (see (2.6)) which is originally due to Lions [8] with some careful estimates on the essential support of the density (see (3.13)), we obtain a key Hardy-type inequality (see (3.14)) to bound the L^p-norm of $\mathbf{u}\bar{x}^{-\eta}$ instead of just the velocity $\mathbf{u}$, and then establish a crucial inequality (see (3.21)) which is used to control the L^p-norm of $\rho\mathbf{u}$. These are the main ones of this paper, in bounding the L^p-norm of $\rho\mathbf{u}$ and $\rho\theta$. Combining some careful estimates on suitable spatially weighted estimates of the magnetic field $\mathbf{b}$ with the time-weighted estimates of the solution $(\rho_R,\mathbf{u}_R,P_R,\mathbf{b}_R,\theta_R)$, we obtain the desired bounds on the solution. These facts are enough to close our arguments.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to the a priori estimates which are needed to obtain the local existence and uniqueness of strong solutions. The main result Theorem 1.1 is proved in Section 4.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be frequently used later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls was shown in [14].

LEMMA 2.1. *For $R>0$ and $B_R=\{x\in\mathbb{R}^2||x|<R\}$, assume that $(\rho_0,\mathbf{u}_0,\mathbf{b}_0,\theta_0)$ satisfies*

$$(\rho_0,\mathbf{u}_0,\mathbf{b}_0,\theta_0)\in H^2(B_R),\ \inf_{x\in B_R}\rho_0(x)>0,\ \operatorname{div}\mathbf{u}_0=\operatorname{div}\mathbf{b}_0=0. \tag{2.1}$$

Then there exists a small time $T_R>0$ such that the Equations (1.1) with the following initial-boundary-value conditions

$$\begin{cases}(\rho,\mathbf{u},\mathbf{b},\theta)(x,t=0)=(\rho_0,\mathbf{u}_0,\mathbf{b}_0,\theta_0), & x\in B_R,\\ \mathbf{u}(x,t)=0,\ \mathbf{b}(x,t)=0,\ \ \theta(x,t)=0, & x\in\partial B_R, t>0,\end{cases} \tag{2.2}$$

has a unique classical solution $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ on $B_R\times(0,T_R]$ satisfying

$$\begin{cases}\rho\in C\left([0,T_R];H^2\right),\\ (\mathbf{u},\mathbf{b},\theta)\in C\left([0,T_R];H^2\right)\cap L^2\left(0,T_R;H^3\right),\\ P\in C\left([0,T_R];H^1\right)\cap L^2\left(0,T_R;H^2\right),\end{cases} \tag{2.3}$$

where we denote $H^k=H^k(B_R)$ for positive integer k.

Next, the following Gagliardo-Nirenberg inequality (see [6, Theorem 10.1, p. 27]) will be useful in the next section.

LEMMA 2.2 (Gagliardo-Nirenberg). *Let $\Omega\subset\mathbb{R}^2$ be a bounded smooth domain. Assume that $1\le q,r\le\infty$, and j,m are arbitrary integers satisfying $0\le j<m$. If $v\in W^{m,r}(\Omega)\cap L^q(\Omega)$, then we have*

$$\|D^jv\|_{L^p}\le C\|v\|_{L^q}^{1-a}\|v\|_{W^{m,r}}^a,$$

where

$$-j+\frac{2}{p}=(1-a)\frac{2}{q}+a\left(-m+\frac{2}{r}\right),$$

and

$$a\in\begin{cases}[\frac{j}{m},1), & \text{if } m-j-\frac{2}{r} \text{ is a nonnegative integer,}\\ [\frac{j}{m},1], & \text{otherwise.}\end{cases}$$

The constant C depends only on $m,j,q,r,a,$ and Ω. In particular, we have

$$\|v\|_{L^4}^4\le C\|v\|_{L^2}^2\|v\|_{H^1}^2.$$

Next, for $\Omega\subset\mathbb{R}^2$, the following weighted L^m-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega)\triangleq\{v\in H^1_{\rm loc}(\Omega)|\nabla v\in L^2(\Omega)\}$ can be found in [8, Theorem B.1].

LEMMA 2.3. *For $m\in[2,\infty)$ and $\theta\in(1+\frac{m}{2},\infty)$, there exists a positive constant C such that for either $\Omega=\mathbb{R}^2$ or $\Omega=B_R$ with $R\ge 1$ and for any $v\in\tilde{D}^{1,2}(\Omega)$,*

$$\left(\int_\Omega\frac{|v|^m}{3+|x|^2}(\log(3+|x|^2))^{-\theta}dx\right)^{\frac{1}{m}}\le C\|v\|_{L^2(B_1)}+C\|\nabla v\|_{L^2(\Omega)}. \tag{2.4}$$

A useful consequence of Lemma 2.3 is the following crucial weighted bounds for elements of $\tilde{D}^{1,2}(\Omega)$, which have been proved in [7, Lemma 2.4].

LEMMA 2.4. *Let $\bar{x}$ and η_0 be as in (1.3) and Ω be as in Lemma 2.3. Assume that $\rho\in L^1(\Omega)\cap L^\infty(\Omega)$ is a non-negative function such that*

$$\int_{B_{N_1}}\rho dx\ge M_1,\quad \|\rho\|_{L^1(\Omega)\cap L^\infty(\Omega)}\le M_2, \tag{2.5}$$

for positive constants $M_1,M_2,$ and $N_1\ge 1$ with $B_{N_1}\subset\Omega$. Then for $\varepsilon>0$ and $\eta>0$, there is a positive constant C depending only on $\varepsilon,\eta,M_1,M_2,N_1,$ and η_0 such that every $v\in\tilde{D}^{1,2}(\Omega)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}(\Omega)}\le C\|\sqrt{\rho}v\|_{L^2(\Omega)}+C\|\nabla v\|_{L^2(\Omega)} \tag{2.6}$$

with $\tilde{\eta}=\min\{1,\eta\}$.

Finally, the following L^p-bound for elliptic systems is a direct result of the combination of the well-known elliptic theory [1] and a standard scaling procedure.

LEMMA 2.5. *For $p>1$ and $k\ge 0$, there exists a positive constant C depending only on p and k such that*

$$\|\nabla^{k+2}v\|_{L^p(B_R)}\le C\|\Delta v\|_{W^{k,p}(B_R)}, \tag{2.7}$$

for every $v\in W^{k+2,p}(B_R)$ satisfying

$$v=0\quad \text{on}\quad B_R.$$

3. A priori estimates

In this section, for $r\in[1,\infty]$ and $k\ge 0$, we denote

$$\int\cdot dx=\int_{B_R}\cdot dx,\quad L^r=L^r(B_R),\quad W^{k,r}=W^{k,r}(B_R),\quad H^k=W^{k,2}.$$

Moreover, for $R>4N_0\geq 4$, assume that $(\rho_0,\mathbf{u}_0,\mathbf{b}_0,\theta_0)$ satisfies, in addition to (2.1), that

$$\frac{1}{2}\leq\int_{B_{N_0}}\rho_0(x)dx\leq\int_{B_R}\rho_0(x)dx\leq\frac{3}{2}. \tag{3.1}$$

Lemma 2.1 thus yields that there exists some $T_R>0$ such that the initial-boundary-value problem (1.1) and (2.2) has a unique classical solution $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ on $B_R\times[0,T_R]$ satisfying (2.3).

Let $\bar{x},\eta_0,a$, and q be as in Theorem 1.1, the main aim of this section is to derive the following key a priori estimate on ψ defined by

$$\begin{aligned}\psi(t):=&1+\|\sqrt{\rho}\mathbf{u}\|_{L^2}+\|\sqrt{\rho}\theta\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\theta\|_{L^2}\\&+\|\nabla\mathbf{b}\|_{L^2}+\|\bar{x}^{\frac{a}{2}}\mathbf{b}\|_{L^2}+\|\bar{x}^a\rho\|_{L^1\cap H^1\cap W^{1,q}}.\end{aligned} \tag{3.2}$$

Proposition 3.1. *Assume that $(\rho_0,\mathbf{u}_0,\mathbf{b}_0,\theta_0)$ satisfies (2.1) and (3.1). Let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ be the solution to the initial-boundary-value problem (1.1) and (2.2) on $B_R\times(0,T_R]$ obtained by Lemma 2.1. Then there exist positive constants T_0 and M both depending only on μ,ν,κ,q, a, η_0, N_0, and E_0 such that*

$$\begin{aligned}&\sup_{0\leq t\leq T_0}\big[\psi(t)+\sqrt{t}\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+\|\sqrt{\rho}\theta_t\|_{L^2}+\|\mathbf{b}_t\|_{L^2}+\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla^2\theta\|_{L^2}+\|\nabla P\|_{L^2}\\&\qquad+\|\nabla^2\mathbf{b}\|_{L^2}\big)\big]\\&+\int_0^{T_0}\big(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\big)\,dt\\&+\int_0^{T_0}\left(\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}+\|\nabla P\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2\mathbf{u}\|_{L^q}^2+t\|\nabla P\|_{L^q}^2\right)dt\\&+\int_0^{T_0}\big(t\|\nabla\mathbf{u}_t\|_{L^2}^2+t\|\nabla\theta_t\|_{L^2}^2+t\|\nabla\mathbf{b}_t\|_{L^2}^2+t\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\big)\,dt\leq M,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}E_0:=&\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}+\|\sqrt{\rho_0}\theta_0\|_{L^2}+\|\nabla\mathbf{u}_0\|_{L^2}+\|\nabla\theta_0\|_{L^2}\\&+\|\bar{x}^a\rho_0\|_{L^1\cap H^1\cap W^{1,q}}+\|\nabla\mathbf{b}_0\|_{L^2}+\|\bar{x}^{\frac{a}{2}}\mathbf{b}_0\|_{L^2}.\end{aligned}$$

To show Proposition 3.1, whose proof will be postponed to the end of this section, we begin with the following standard energy estimate for $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and the estimate on the L^p-norm of the density.

Lemma 3.1. *Under the conditions of Proposition 3.1, let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ be a smooth solution to the initial-boundary-value problem (1.1) and (2.2). Then for any $t\in(0,T_1]$,*

$$\begin{aligned}&\sup_{0\leq s\leq t}\big(\|\rho\|_{L^1\cap L^\infty}+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\big)\\&+\int_0^t\big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\big)\,ds\leq C,\end{aligned} \tag{3.4}$$

where (and in what follows) C denotes a generic positive constant depending only on μ,ν,κ,q,a, η_0, N_0, and E_0. T_1 is as that of Lemma 3.2.

Proof.

(1) Since $\operatorname{div}\mathbf{u}=0$, it is easy to deduce from $(1.1)_1$ that (see [8, Theorem 2.1]),

$$\sup_{0\leq s\leq t}\|\rho\|_{L^1\cap L^\infty}\leq C. \tag{3.5}$$

(2) Multiplying $(1.1)_2$ by $\mathbf{u}$ and integrating by parts, we obtain from $(1.1)_1$ and $(1.1)_5$ that

$$\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}|^2dx+\mu\int|\nabla\mathbf{u}|^2dx=\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx+\int\rho\theta\mathbf{u}\cdot\mathbf{e}_2dx. \tag{3.6}$$

Multiplying $(1.1)_3$ by $\mathbf{b}$ and integrating by parts, we find that

$$\frac{1}{2}\frac{d}{dt}\int|\mathbf{b}|^2dx+\nu\int|\nabla\mathbf{b}|^2dx+\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}dx=0. \tag{3.7}$$

Testing $(1.1)_4$ by θ and using $(1.1)_1$ and $(1.1)_5$, we get

$$\frac{1}{2}\frac{d}{dt}\int\rho\theta^2dx+\kappa\int|\nabla\theta|^2dx=\int\rho\theta\mathbf{u}\cdot\mathbf{e}_2dx. \tag{3.8}$$

Combining (3.6)–(3.8), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)+\left(\mu\|\nabla\mathbf{u}\|_{L^2}^2+\nu\|\nabla\mathbf{b}\|_{L^2}^2+\kappa\|\nabla\theta\|_{L^2}^2\right)\\&=2\int\rho\theta\mathbf{u}\cdot\mathbf{e}_2dx\\&\le2\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\sqrt{\rho}\theta\|_{L^2}\\&\le\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2.\end{aligned} \tag{3.9}$$

Thus, Grönwall's inequality leads to

$$\sup_{0\le s\le t}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\mathbf{b}\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)+\int_0^t\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)ds\le C,$$

which together with (3.2) yields (3.4) and completes the proof of Lemma 3.1.

□

Next, we will give some spatial-weighted estimates on the density and the magnetic field.

LEMMA 3.2. *Under the conditions of Proposition 3.1, let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ be a smooth solution to the initial-boundary-value problem (1.1) and (2.2). Then there exists a $T_1=T_1(N_0,E_0)>0$ such that for all $t\in(0,T_1]$,*

$$\sup_{0\le s\le t}\left(\|\rho\bar{x}^a\|_{L^1}+\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)+\int_0^t\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2ds\le C. \tag{3.10}$$

Proof.

(1) For $N>1$, let $\varphi_N\in C_0^\infty(B_N)$ satisfy

$$0\le\varphi_N\le1,\ \varphi_N(x)=1,\ \text{if } |x|\le\frac{N}{2},\ |\nabla\varphi_N|\le CN^{-1}. \tag{3.11}$$

It follows from $(1.1)_1$ and (3.4) that

$$\begin{aligned}\frac{d}{dt}\int\rho\varphi_{2N_0}dx&=\int\rho\mathbf{u}\cdot\nabla\varphi_{2N_0}dx\\&\ge-CN_0^{-1}\left(\int\rho dx\right)^{\frac{1}{2}}\left(\int\rho|\mathbf{u}|^2dx\right)^{\frac{1}{2}}\ge-\tilde{C}(E_0).\end{aligned} \tag{3.12}$$

Integrating (3.12) and using (3.1) give rise to

$$\inf_{0\le t\le T_1}\int_{B_{2N_0}}\rho dx \ge \inf_{0\le t\le T_1}\int \rho\varphi_{2N_0}dx \ge \int \rho_0\varphi_{2N_0}dx - \tilde{C}T_1 \ge \frac{1}{4}. \tag{3.13}$$

Here, $T_1 \triangleq \min\{1,(4\tilde{C})^{-1}\}$. From now on, we will always assume that $t\le T_1$. The combination of (3.13), (3.4), and (2.6) implies that for $\varepsilon>0$ and $\eta>0$, every $v\in \tilde{D}^{1,2}(B_R)$ satisfies

$$\|v\bar{x}^{-\eta}\|^2_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}} \le C(\varepsilon,\eta)\|\sqrt{\rho}v\|^2_{L^2} + C(\varepsilon,\eta)\|\nabla v\|^2_{L^2}, \tag{3.14}$$

with $\tilde{\eta}=\min\{1,\eta\}$.

(2) Noting that

$$|\nabla\bar{x}| \le (3+2\eta_0)\log^{1+\eta_0}(3+|x|^2) \le C(a,\eta_0)\bar{x}^{\frac{4}{8+a}},$$

multiplying $(1.1)_1$ by $\bar{x}^a$ and integrating by parts imply that

$$\begin{aligned}\frac{d}{dt}\|\rho\bar{x}^a\|_{L^1} &= \int \rho(\mathbf{u}\cdot\nabla)\bar{x}a\bar{x}^{a-1}dx\\ &\le C\int \rho|\mathbf{u}|\bar{x}^{a-1+\frac{4}{8+a}}dx\\ &\le C\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\|_{L^{\frac{8+a}{7+a}}}\|\mathbf{u}\bar{x}^{-\frac{4}{8+a}}\|_{L^{8+a}}\\ &\le C\|\rho\|^{\frac{1}{8+a}}_{L^\infty}\|\rho\bar{x}^a\|^{\frac{7+a}{8+a}}_{L^1}(\|\sqrt{\rho}\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2})\\ &\le C(1+\|\rho\bar{x}^a\|_{L^1})\left(1+\|\nabla\mathbf{u}\|^2_{L^2}\right)\end{aligned}$$

due to (3.4) and (3.14). This combined with Grönwall's inequality and (3.4) leads to

$$\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{L^1} \le C\exp\left\{C\int_0^t\left(1+\|\nabla\mathbf{u}\|^2_{L^2}\right)ds\right\} \le C. \tag{3.15}$$

(3) Multiplying $(1.1)_3$ by $\mathbf{b}\bar{x}^a$ and integrating by parts yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\bar{x}^{a/2}\|^2_{L^2}+\nu\|\nabla\mathbf{b}\bar{x}^{a/2}\|^2_{L^2}\\ &=\frac{\nu}{2}\int|\mathbf{b}|^2\Delta\bar{x}^a dx+\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\bar{x}^a dx+\frac{1}{2}\int|\mathbf{b}|^2\mathbf{u}\cdot\nabla\bar{x}^a dx\\ &=:\bar{I}_1+\bar{I}_2+\bar{I}_3,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}|\bar{I}_1| &\le C\int|\mathbf{b}|^2\bar{x}^a\bar{x}^{-2}\log^{2(1-\eta_0)}(3+|x|^2)dx \le C\int|\mathbf{b}|^2\bar{x}^a dx,\\ |\bar{I}_2| &\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|^2_{L^4}\\ &\le C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}+\|\mathbf{b}\nabla\bar{x}^{\frac{a}{2}}\|_{L^2})\\ &\le C(\|\nabla\mathbf{u}\|^2_{L^2}+1)\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|^2_{L^2}+\frac{\nu}{4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|^2_{L^2},\\ |\bar{I}_3| &\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{u}\bar{x}^{-\frac{3}{4}}\|_{L^4}\end{aligned}$$

$$
\begin{aligned}
&\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}^2+C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2\right)\\
&\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\frac{\nu}{4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,
\end{aligned}
\tag{3.17}
$$

due to the Gagliardo-Nirenberg inequality, (3.4), and (3.14). Putting (3.17) into (3.16), we get after using Grönwall's inequality and (3.4) that

$$
\sup_{0\le s\le t}\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\int_0^t\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2ds\le C\exp\left\{C\int_0^t\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)ds\right\}\le C, \tag{3.18}
$$

which together with (3.15) gives (3.10) and finishes the proof of Lemma 3.2. ∎

LEMMA 3.3. *Let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and T_1 be as in Lemma 3.2. Then there exists a positive constant $\alpha>1$ such that for all $t\in(0,T_1]$,*

$$
\begin{aligned}
&\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\\
&\quad+\int_0^t\left(\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\mathbf{b}_s\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2+\|\sqrt{\rho}\theta_s\|_{L^2}^2\right)ds\\
&\le C+C\int_0^t\psi^\alpha(s)ds.
\end{aligned}
\tag{3.19}
$$

Proof.

(1) It follows from (3.4), (3.10), and (3.14) that for any $\varepsilon>0$ and any $\eta>0$,

$$
\begin{aligned}
\|\rho^\eta v\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}&\le C\|\rho^\eta\bar{x}^{\frac{3\tilde{\eta}a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{3\tilde{\eta}}}}\|v\bar{x}^{-\frac{3\tilde{\eta}a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{\tilde{\eta}}}}\\
&\le C\left(\int\rho^{\frac{4(2+\varepsilon)\eta}{3\tilde{\eta}}-1}\rho\bar{x}^a dx\right)^{\frac{3\tilde{\eta}}{4(2+\varepsilon)}}\|v\bar{x}^{-\frac{3\tilde{\eta}a}{4(2+\varepsilon)}}\|_{L^{\frac{4(2+\varepsilon)}{\tilde{\eta}}}}\\
&\le C\|\rho\|_{L^\infty}^{\frac{4(2+\varepsilon)\eta-3\tilde{\eta}}{4(2+\varepsilon)}}\|\rho\bar{x}^a\|_{L^1}^{\frac{3\tilde{\eta}}{4(2+\varepsilon)}}\left(\|\sqrt{\rho}v\|_{L^2}+\|\nabla v\|_{L^2}\right)\\
&\le C\|\sqrt{\rho}v\|_{L^2}+C\|\nabla v\|_{L^2},
\end{aligned}
\tag{3.20}
$$

where $\tilde{\eta}=\min\{1,\eta\}$ and $v\in\tilde{D}^{1,2}(B_R)$. In particular, this together with (3.4) and (3.14) yields

$$
\|\rho^\eta\mathbf{u}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}+\|\mathbf{u}\bar{x}^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}\right), \tag{3.21}
$$

$$
\|\rho^\eta\theta\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}+\|\theta\bar{x}^{-\eta}\|_{L^{\frac{2+\varepsilon}{\tilde{\eta}}}}\le C\left(1+\|\nabla\theta\|_{L^2}\right). \tag{3.22}
$$

(2) Multiplying $(1.1)_2$ by $\mathbf{u}_t$ and integrating by parts, one has

$$
\mu\frac{d}{dt}\int|\nabla\mathbf{u}|^2dx+\int\rho|\mathbf{u}_t|^2dx\le C\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2dx+\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx+\int\rho\theta|\mathbf{u}_t|dx. \tag{3.23}
$$

We derive from (3.21), Hölder's inequality, and the Gagliardo-Nirenberg inequality that

$$
\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2dx\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^{\frac{8}{3}}}^2
$$

$$
\begin{aligned}
&\leq C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{2}}\|\nabla\mathbf{u}\|_{H^1}^{\frac{1}{2}}\\
&\leq C\psi^\alpha+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2,
\end{aligned}
\tag{3.24}
$$

where (and in what follows) we use $\alpha>1$ to denote a generic constant, which may be different from line to line. For the second term on the right-hand side of (3.23), integration by parts together with $(1.1)_5$ and the Gagliardo-Nirenberg inequality deduces that for any $\varepsilon>0$,

$$
\begin{aligned}
\int\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}_t dx&=-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}_t dx\\
&\leq-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\
&\leq-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\
&\leq-\frac{d}{dt}\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+C\psi^\alpha.
\end{aligned}
\tag{3.25}
$$

From Cauchy-Schwarz inequality and (3.4), we have

$$
\int\rho\theta|\mathbf{u}_t|dx\leq\frac{1}{2}\int\rho|\mathbf{u}_t|^2dx+\frac{1}{2}\int\rho\theta^2dx\leq\frac{1}{2}\int\rho|\mathbf{u}_t|^2dx+C.
\tag{3.26}
$$

Thus, inserting (3.24)–(3.26) into (3.23) gives

$$
\frac{d}{dt}B(t)+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\leq\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha,
\tag{3.27}
$$

where

$$
B(t)\triangleq\mu\|\nabla\mathbf{u}\|_{L^2}^2+\int\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}dx
$$

satisfies

$$
\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2-C_1\|\nabla\mathbf{b}\|_{L^2}^2\leq B(t)\leq C\|\nabla\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^2}^2,
\tag{3.28}
$$

owing to Hölder's inequality, the Gagliardo-Nirenberg inequality, and (3.4).

(3) It follows from $(1.1)_3$ that

$$
\begin{aligned}
&\nu\frac{d}{dt}\|\nabla\mathbf{b}\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\nu^2\|\Delta\mathbf{b}\|_{L^2}^2\\
&\leq C\||\mathbf{b}||\nabla\mathbf{u}|\|_{L^2}^2+C\||\mathbf{u}||\nabla\mathbf{b}|\|_{L^2}^2\\
&\leq C\|\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2+C\|\bar{x}^{-\frac{a}{4}}\mathbf{u}\|_{L^8}^2\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^4}\\
&\leq\frac{\nu^2}{2}\|\Delta\mathbf{b}\|_{L^2}^2+C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2
\end{aligned}
\tag{3.29}
$$

due to (2.7), (3.21), and the Gagliardo-Nirenberg inequality. Multiplying (3.29) by $\nu^{-1}(C_1+1)$ and adding the resulting inequality to (3.27) imply

$$
\begin{aligned}
&\frac{d}{dt}\left(B(t)+(C_1+1)\|\nabla\mathbf{b}\|_{L^2}^2\right)+\frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\frac{\nu}{2}\|\Delta\mathbf{b}\|_{L^2}^2\\
&\leq C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2+\varepsilon\|\nabla^2\mathbf{u}\|_{L^2}^2.
\end{aligned}
\tag{3.30}
$$

Since $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ satisfies the following Stokes system

$$\begin{cases} -\mu\Delta\mathbf{u}+\nabla P=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2+\rho\theta\mathbf{e}_2, & x\in B_R,\\ \operatorname{div}\mathbf{u}=0, & x\in B_R,\\ \mathbf{u}(x)=0, & x\in\partial B_R, \end{cases} \tag{3.31}$$

applying regularity theory of Stokes system to (3.31) yields that for any $p\in[2,\infty)$,

$$\|\nabla^2\mathbf{u}\|_{L^p}+\|\nabla P\|_{L^p}\le C\|\rho\mathbf{u}_t\|_{L^p}+C\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p}+C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^p}+C\|\rho\theta\|_{L^p}. \tag{3.32}$$

Hence, we infer from (3.32), (3.4), (3.21), and the Gagliardo-Nirenberg inequality that

$$\begin{aligned} &\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla P\|_{L^2}^2\\ &\le C\|\rho\mathbf{u}_t\|_{L^2}^2+C\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2+C\||\mathbf{b}||\nabla\mathbf{b}|\|_{L^2}^2+C\|\rho\theta\|_{L^2}^2\\ &\le C\|\rho\|_{L^\infty}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\rho\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2+C\|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{b}\|_{L^4}^2+C\|\rho\|_{L^\infty}\|\sqrt{\rho}\theta\|_{L^2}^2\\ &\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\|\rho\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}+C\|\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{b}\|_{H^1}+C\\ &\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2+\frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2+C\left(1+\|\nabla\mathbf{b}\|_{L^2}^4+\|\nabla\mathbf{u}\|_{L^2}^6\right)\\ &\le C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2+\frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}^2+C\psi^\alpha. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.30) and choosing ε suitably small, one gets

$$\begin{aligned} &\frac{d}{dt}\left(B(t)+(C_1+1)\|\nabla\mathbf{b}\|_{L^2}^2\right)+\frac{1}{4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\frac{\nu^{-1}}{2}\|\mathbf{b}_t\|_{L^2}^2+\frac{\nu}{4}\|\Delta\mathbf{b}\|_{L^2}^2\\ &\le C\psi^\alpha+C\|\bar{x}^{\frac{a}{2}}\nabla\mathbf{b}\|_{L^2}^2. \end{aligned}$$

Integrating the above inequality over $(0,t)$, then we infer from (2.7), (3.28), (3.10), and (3.33) that

$$\begin{aligned} &\sup_{0\le s\le t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^2}^2\right)+\int_0^t\left(\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\mathbf{b}_s\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\ &\le C+C\int_0^t\psi^\alpha(s)ds. \end{aligned} \tag{3.34}$$

(4) Multiplying $(1.1)_4$ by θ_t and integrating by parts, we obtain from (3.21), the Gagliardo-Nirenberg inequality, and (3.4) that for any $\delta>0$,

$$\begin{aligned} &\kappa\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\\ &\le\int\rho|\mathbf{u}||\nabla\theta||\theta_t|dx+\int\rho|\mathbf{u}||\theta_t|dx\\ &\le\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4}\|\sqrt{\rho}\theta_t\|_{L^2}+\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}\\ &\le C\left(1+\|\nabla\mathbf{u}\|_{L^2}\right)\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{H^1}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^2}+\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}\\ &\le\delta\|\nabla^2\theta\|_{L^2}^2+\frac{1}{4}\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\psi^\alpha. \end{aligned} \tag{3.35}$$

It follows from $(1.1)_4$, $(1.1)_1$, (3.4), Hölder's inequality, the Gagliardo-Nirenberg inequality, and (3.21) that

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}^2 &\le C\|\rho\theta_t\|_{L^2}^2 + C\|\rho|\mathbf{u}||\nabla\theta|\|_{L^2}^2 + C\|\rho\mathbf{u}\|_{L^2}^2\\ &\le C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\rho\mathbf{u}\|_{L^8}^2\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^4} + C\\ &\le C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \frac{1}{2}\|\nabla^2\theta\|_{L^2}^2 + C\psi^\alpha,\end{aligned} \tag{3.36}$$

which, combined with (3.35) and choosing δ suitably small, yields

$$\kappa\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \frac{1}{2}\|\sqrt{\rho}\theta_t\|_{L^2}^2 \le C\psi^\alpha.$$

Integrating the above inequality over $(0,t)$ together with (3.36), we have

$$\sup_{0\le s\le t}\|\nabla\theta\|_{L^2}^2 + \int_0^t \left(\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2\right)ds \le C + C\int_0^t \psi^\alpha(s)ds. \tag{3.37}$$

Thus, we derive the desired (3.19) from (3.34) and (3.37). The proof of Lemma 3.3 is finished.

□

Lemma 3.4. *Let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and T_1 be as in Lemma 3.2. Then there exists a positive constant $\alpha>1$ such that for all $t\in(0,T_1]$,*

$$\begin{aligned}&\sup_{0\le s\le t}\left(s\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2 + s\|\sqrt{\rho}\theta_s\|_{L^2}^2 + s\|\mathbf{b}_s\|_{L^2}^2\right) + \int_0^t\left(s\|\nabla\mathbf{u}_s\|_{L^2}^2 + s\|\nabla\theta_s\|_{L^2}^2 + s\|\nabla\mathbf{b}_s\|_{L^2}^2\right)ds\\ &\le C\exp\left\{C\int_0^t\psi^\alpha ds\right\}.\end{aligned} \tag{3.38}$$

Proof.

(1) Differentiating $(1.1)_2$ with respect to t gives

$$\begin{aligned}&\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \mu\Delta\mathbf{u}_t\\ &= -\rho_t(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}_t\cdot\nabla\mathbf{u} - \nabla\left(P + \frac{1}{2}|\mathbf{b}|^2\right)_t + (\mathbf{b}\cdot\nabla\mathbf{b})_t + (\rho\theta)_t\mathbf{e}_2.\end{aligned} \tag{3.39}$$

Multiplying (3.39) by $\mathbf{u}_t$ and integrating by parts the resulting equality over B_R, we obtain after using $(1.1)_1$ and $(1.1)_5$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\mathbf{u}_t|^2dx + \mu\int|\nabla\mathbf{u}_t|^2dx\\ &\le C\int\rho|\mathbf{u}||\mathbf{u}_t|\left(|\nabla\mathbf{u}_t| + |\nabla\mathbf{u}|^2 + |\mathbf{u}||\nabla^2\mathbf{u}|\right)dx + C\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|dx\\ &\quad + C\int\rho|\mathbf{u}_t|^2|\nabla\mathbf{u}|dx + \int\mathbf{b}_t\cdot\nabla\mathbf{b}\cdot\mathbf{u}_tdx + \int\mathbf{b}\cdot\nabla\mathbf{b}_t\cdot\mathbf{u}_tdx + \int(\rho\theta)_t\mathbf{e}_2\cdot\mathbf{u}_tdx\\ &=: \sum_{i=1}^6\hat{I}_i.\end{aligned} \tag{3.40}$$

It follows from (3.20), (3.21), and the Gagliardo-Nirenberg inequality that

$$\hat{I}_1 \le C\|\sqrt{\rho}\mathbf{u}\|_{L^6}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac{1}{2}}\left(\|\nabla\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^4}^2\right)$$

$$
\begin{aligned}
&+C\|\rho^{\frac14}\mathbf{u}\|_{L^{12}}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}\\
\le{}& C(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}_t\|_{L^2})^{\frac12}\\
&\times\left(\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla^2\mathbf{u}\|_{L^2}\right)\\
\le{}&\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{u}\|_{L^2}^2.
\end{aligned}\tag{3.41}
$$

Hölder's inequality combined with (3.20) and (3.21) leads to

$$
\begin{aligned}
\hat{I}_2+\hat{I}_3&\le C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^6}^{\frac32}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\\
&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2}^2\right).
\end{aligned}\tag{3.42}
$$

Integration by parts together with $(1.1)_5$, Hölder's and the Gagliardo-Nirenberg inequalities indicates that

$$
\begin{aligned}
\hat{I}_4+\hat{I}_5&=-\int\mathbf{b}_t\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}dx-\int\mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_tdx\\
&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\mathbf{b}\|_{L^4}^2\|\mathbf{b}_t\|_{L^4}^2\\
&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\mu\nu}{4(C_2+1)}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\psi^\alpha\|\mathbf{b}_t\|_{L^2}^2.
\end{aligned}\tag{3.43}
$$

By virtue of $(1.1)_4$ and integration by parts, we obtain from (3.21), (3.22), and (3.4) that

$$
\begin{aligned}
\hat{I}_6&=\int(\rho\mathbf{u}\cdot\mathbf{e}_2-\operatorname{div}(\rho\theta\mathbf{u})+\kappa\Delta\theta)\mathbf{e}_2\cdot\mathbf{u}_tdx\\
&\le\int\rho|\mathbf{u}||\mathbf{u}_t|dx+\int\rho\theta|\mathbf{u}||\nabla\mathbf{u}_t|dx+\kappa\int|\nabla\theta||\nabla\mathbf{u}_t|dx\\
&\le\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+C(\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\sqrt{\rho}\theta\|_{L^4}+\|\nabla\theta\|_{L^2})\|\nabla\mathbf{u}_t\|_{L^2}\\
&\le\frac{\mu}{8}\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\psi^\alpha.
\end{aligned}\tag{3.44}
$$

Substituting (3.41)–(3.44) into (3.40), we obtain after using (3.33) that

$$
\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\mu\|\nabla\mathbf{u}_t\|_{L^2}^2&\le C\psi^\alpha\left(1+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2\right)\\
&\quad+\frac{\mu\nu}{2(C_2+1)}\|\nabla\mathbf{b}_t\|_{L^2}^2+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{b}\|_{L^2}^2.
\end{aligned}\tag{3.45}
$$

(2) Differentiating $(1.1)_3$ with respect to t shows

$$
\mathbf{b}_{tt}-\mathbf{b}_t\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{u}_t+\mathbf{u}_t\cdot\nabla\mathbf{b}+\mathbf{u}\cdot\nabla\mathbf{b}_t=\nu\Delta\mathbf{b}_t.\tag{3.46}
$$

Multiplying (3.46) by $\mathbf{b}_t$ and integrating the resulting equality over B_R yield that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int|\mathbf{b}_t|^2dx+\nu\int|\nabla\mathbf{b}_t|^2dx\\
={}&\int\mathbf{b}\cdot\nabla\mathbf{u}_t\cdot\mathbf{b}_tdx-\int\mathbf{u}_t\cdot\nabla\mathbf{b}\cdot\mathbf{b}_tdx+\int\mathbf{b}_t\cdot\nabla\mathbf{u}\cdot\mathbf{b}_tdx-\int\mathbf{u}\cdot\nabla\mathbf{b}_t\cdot\mathbf{b}_tdx\\
\triangleq{}&\sum_{i=1}^4S_i.
\end{aligned}\tag{3.47}
$$

On the one hand, we deduce from (3.14) and (3.18) that

$$\begin{aligned}\sum_{i=1}^{2} S_i &\le C\|\nabla \mathbf{u}_t\|_{L^2}\|\mathbf{b}_t\|_{L^4}\|\mathbf{b}\|_{L^4} + C\|\nabla \mathbf{b}_t\|_{L^2}\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}\\ &\le C\|\mathbf{b}_t\|_{L^4}^2 + C\|\nabla \mathbf{u}_t\|_{L^2}^2 + \frac{\nu}{8}\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\||\mathbf{u}_t||\mathbf{b}|\|_{L^2}^2\\ &\le \frac{\nu}{4}\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\mathbf{u}_t\bar{x}^{-\frac{a}{4}}\|_{L^8}^2\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{b}\|_{L^4}\\ &\le \frac{\nu}{4}\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2, \end{aligned}\tag{3.48}$$

where one has used the following estimate

$$\sup_{0\le s\le t}\||\mathbf{b}|^2\|_{L^2}^2 + \int_0^t \||\nabla \mathbf{b}||\mathbf{b}|\|_{L^2}^2 ds \le C. \tag{3.49}$$

Indeed, multiplying $(1.1)_3$ by $\mathbf{b}|\mathbf{b}|^2$ and integrating by parts lead to

$$\begin{aligned}&\frac{1}{4}\left(\||\mathbf{b}|^2\|_{L^2}^2\right)_t + \nu\||\nabla \mathbf{b}||\mathbf{b}|\|_{L^2}^2 + \frac{\nu}{2}\|\nabla|\mathbf{b}|^2\|_{L^2}^2\\ &\le C\|\nabla \mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^4}^2 \le C\|\nabla \mathbf{u}\|_{L^2}\||\mathbf{b}|^2\|_{L^2}\|\nabla|\mathbf{b}|^2\|_{L^2}\\ &\le \frac{\nu}{4}\|\nabla|\mathbf{b}|^2\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2}^2\||\mathbf{b}|^2\|_{L^2}^2, \end{aligned}\tag{3.50}$$

which together with Grönwall's inequality and (3.4) gives (3.49).
On the other hand, integration by parts combined with $(1.1)_4$ and the Gagliardo-Nirenberg inequality yields

$$\sum_{i=3}^{4} S_i = \int \mathbf{b}_t\cdot\nabla \mathbf{u}\cdot \mathbf{b}_t dx \le C\|\mathbf{b}_t\|_{L^2}\|\nabla \mathbf{b}_t\|_{L^2}\|\nabla \mathbf{u}\|_{L^2} \le \frac{\nu}{4}\|\nabla \mathbf{b}_t\|_{L^2}^2 + C\psi^\alpha\|\mathbf{b}_t\|_{L^2}^2. \tag{3.51}$$

Inserting (3.48) and (3.51) into (3.47), one has

$$\frac{d}{dt}\|\mathbf{b}_t\|_{L^2}^2 + \nu\|\nabla \mathbf{b}_t\|_{L^2}^2 \le C\psi^\alpha\left(\|\mathbf{b}_t\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right) + C_2\|\nabla \mathbf{u}_t\|_{L^2}^2. \tag{3.52}$$

(3) Differentiating $(1.1)_4$ with respect to t shows

$$\rho\theta_{tt} + \rho\mathbf{u}\cdot\nabla\theta_t - \kappa\Delta\theta_t = -\rho_t(\theta_t + \mathbf{u}\cdot\nabla\theta) - \rho\mathbf{u}_t\cdot\nabla\theta + (\rho\mathbf{u})_t\cdot\mathbf{e}_2. \tag{3.53}$$

Multiplying (3.53) by θ_t and integrating the resulting equality over B_R yield that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\rho\theta_t^2 dx + \kappa\int|\nabla\theta_t|^2dx =& -\int\rho_t(\theta_t+\mathbf{u}\cdot\nabla\theta)\theta_t dx - \int\rho\theta_t\mathbf{u}_t\cdot\nabla\theta dx\\ &+\int\theta_t(\rho\mathbf{u})_t\cdot\mathbf{e}_2 dx \triangleq I_1 + I_2 + I_3. \end{aligned}\tag{3.54}$$

It follows from $(1.1)_1$, integration by parts, Hölder's inequality, (3.20), (3.21), and the Gagliardo-Nirenberg inequality that

$$|I_1| \le C\int\rho|\mathbf{u}|\left(|\theta_t||\nabla\theta_t| + |\theta_t||\nabla\mathbf{u}||\nabla\theta| + |\theta_t||\mathbf{u}||\nabla^2\theta| + |\nabla\theta_t||\mathbf{u}||\nabla\theta|\right)dx$$

$$
\begin{aligned}
&\le C\|\sqrt{\rho}\mathbf{u}\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}(\|\nabla\theta_t\|_{L^2}+\|\nabla\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4})\\
&\quad+C\|\rho^{\frac{1}{4}}\mathbf{u}\|_{L^{12}}^2\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}+C\|\sqrt{\rho}\mathbf{u}\|_{L^8}^2\|\nabla\theta\|_{L^4}\|\nabla\theta_t\|_{L^2}\\
&\le C(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})^{\frac{1}{2}}\times(\|\nabla\theta_t\|_{L^2}\\
&\qquad+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla\theta\|_{L^2}^2+\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}+\|\nabla^2\theta\|_{L^2})\\
&\quad+C(1+\|\nabla\mathbf{u}\|_{L^2}^2)\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{H^1}\|\nabla\theta_t\|_{L^2}\\
&\le\frac{\kappa}{6}\|\nabla\theta_t\|_{L^2}^2+C\psi^\alpha\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\psi^\alpha+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{u}\|_{L^2}^2\\
&\quad+C\left(1+\|\nabla\theta\|_{L^2}^2\right)\|\nabla^2\theta\|_{L^2}^2. \qquad (3.55)
\end{aligned}
$$

By Hölder's inequality and (3.20), we have

$$
\begin{aligned}
|I_2|&\le C\|\nabla\theta\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{3}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\\
&\le C\|\nabla\theta\|_{L^2}(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})^{\frac{3}{2}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{1}{2}}\\
&\le\frac{\kappa}{6}\|\nabla\theta_t\|_{L^2}^2+C\psi^\alpha\left(\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right). \qquad (3.56)
\end{aligned}
$$

From $(1.1)_1$, integration by parts, Hölder's inequality, (3.20), and (3.21), we get

$$
\begin{aligned}
|I_3|&=\left|\int\rho\mathbf{u}_t\cdot\mathbf{e}_2\theta_t dx+\int\rho_t\mathbf{u}\cdot\mathbf{e}_2\theta_t dx\right|\\
&\le\int\left(\rho|\mathbf{u}_t||\theta_t|+\rho|\mathbf{u}||\nabla\mathbf{u}||\theta_t|+\rho|\mathbf{u}|^2|\nabla\theta_t|\right)dx\\
&\le\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}+\|\sqrt{\rho}\mathbf{u}\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}\\
&\quad+\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\theta_t\|_{L^2}\\
&\le\frac{\kappa}{6}\|\nabla\theta_t\|_{L^2}^2+C\psi^\alpha\left(1+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right). \qquad (3.57)
\end{aligned}
$$

Hence, substituting (3.55)–(3.57) into (3.54), we obtain that

$$
\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\kappa\|\nabla\theta_t\|_{L^2}^2&\le C\psi^\alpha\left(1+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)\\
&\quad+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{u}\|_{L^2}^2+C\left(1+\|\nabla\theta\|_{L^2}^2\right)\|\nabla^2\theta\|_{L^2}^2,
\end{aligned}
$$

which together with (3.36) gives that

$$
\frac{d}{dt}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\kappa\|\nabla\theta_t\|_{L^2}^2\le C\psi^\alpha\left(1+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{u}\|_{L^2}^2. \qquad (3.58)
$$

(4) From (3.45) multiplied by $\mu^{-1}(C_2+1)$, (3.52), and (3.58), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\mu^{-1}(C_2+1)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\|\nabla\mathbf{u}_t\|_{L^2}^2+\frac{\nu}{2}\|\nabla\mathbf{b}_t\|_{L^2}^2+\kappa\|\nabla\theta_t\|_{L^2}^2\\
&\le C\psi^\alpha\left(1+\|\mathbf{b}_t\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+C\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\nabla^2\mathbf{b}\|_{L^2}^2. \qquad (3.59)
\end{aligned}
$$

Multiplying (3.59) by t, we obtain (3.38) after using Grönwall's inequality and (3.19). The proof of Lemma 3.4 is finished.

□

LEMMA 3.5. *Let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and T_1 be as in Lemma 3.2. Then there exists a positive constant $\alpha>1$ such that for all $t\in(0,T_1]$,*

$$\sup_{0\le s\le t}\left(s\|\nabla^2\mathbf{u}\|_{L^2}^2+s\|\nabla^2\theta\|_{L^2}^2+s\|\nabla^2\mathbf{b}\|_{L^2}^2+s\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)+\int_0^t s\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2ds$$
$$\le C\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.60}$$

Proof.

(1) Multiplying $(1.1)_3$ by $\Delta\mathbf{b}\bar{x}^a$ and integrating by parts lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{b}|^2\bar{x}^a dx+\nu\int|\Delta\mathbf{b}|^2\bar{x}^a dx\\&\le C\int|\nabla\mathbf{b}||\mathbf{b}||\nabla\mathbf{u}||\nabla\bar{x}^a|dx+C\int|\nabla\mathbf{b}|^2|\mathbf{u}||\nabla\bar{x}^a|dx+C\int|\nabla\mathbf{b}||\Delta\mathbf{b}||\nabla\bar{x}^a|dx\\&\quad+C\int|\mathbf{b}||\nabla\mathbf{u}||\Delta\mathbf{b}|\bar{x}^a dx+C\int|\nabla\mathbf{u}||\nabla\mathbf{b}|^2\bar{x}^a dx\triangleq\sum_{i=1}^5 J_i.\end{aligned} \tag{3.61}$$

Applying (3.10), (3.14), Hölder's inequality, and the Gagliardo-Nirenberg inequality, one gets by some direct calculations that

$$\begin{aligned}J_1&\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\\&\le C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{1}{2}}\left(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}+\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{H^1}^{\frac{1}{2}}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\\&\le C\psi^\alpha+C\|\nabla^2\mathbf{u}\|_{L^2}^2+C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\J_2&\le C\||\nabla\mathbf{b}|^{2-\frac{2}{3a}}\bar{x}^{a-\frac{1}{3}}\|_{L^{\frac{6a}{6a-2}}}\|\mathbf{u}\bar{x}^{-\frac{1}{3}}\|_{L^{6a}}\||\nabla\mathbf{b}|^{\frac{2}{3a}}\|_{L^{6a}}\\&\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{6a-2}{3a}}\|\nabla\mathbf{b}\|_{L^4}^{\frac{2}{3a}}\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\mathbf{b}\|_{L^4}^2\\&\le C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+\frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2,\\J_3+J_4&\le\frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\&\le\frac{\nu}{4}\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\\&\quad+C\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\left(\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}+\|\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\right)\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\\&\le\varepsilon\|\Delta\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\psi^\alpha\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+C\psi^\alpha+C\|\nabla^2\mathbf{u}\|_{L^2}^2,\\J_5&\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\le C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}\right)\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates into (3.61) and noting the following fact

$$\begin{aligned}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx&=\int|\Delta\mathbf{b}|^2\bar{x}^a dx-\int\partial_i\partial_k\mathbf{b}\cdot\partial_k\mathbf{b}\partial_i\bar{x}^a dx+\int\partial_i\partial_i\mathbf{b}\cdot\partial_k\mathbf{b}\partial_k\bar{x}^a dx\\&\le\int|\Delta\mathbf{b}|^2\bar{x}^a dx+\frac{1}{2}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx+C\int|\nabla\mathbf{b}|^2\bar{x}^a dx,\end{aligned}$$

we derive that

$$\frac{d}{dt}\int|\nabla\mathbf{b}|^2\bar{x}^a dx+\frac{\nu}{2}\int|\nabla^2\mathbf{b}|^2\bar{x}^a dx$$

$$\le C\left(\psi^\alpha + \|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}\right)\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\left(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \psi^\alpha\right). \tag{3.62}$$

(2) We now claim that

$$\int_0^t\left(\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2\mathbf{u}\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2\right)ds \le C\exp\left\{C\int_0^t\psi^\alpha(s)ds\right\}, \tag{3.63}$$

whose proof will be given at the end of this proof. Thus, multiplying (3.62) by t, we infer from (3.10), (3.19), (3.63), and Grönwall's inequality that

$$\sup_{0\le s\le t}\left(s\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right) + \int_0^t s\|\nabla^2\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 ds \le C\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.64}$$

(3) It deduces from $(1.1)_3$, (2.7), (3.4), (3.21), Hölder's inequality, and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
\|\nabla^2\mathbf{b}\|_{L^2}^2 &\le C\|\mathbf{b}_t\|_{L^2}^2 + C\||\mathbf{u}||\nabla\mathbf{b}|\|_{L^2}^2 + C\||\mathbf{b}||\nabla\mathbf{u}|\|_{L^2}^2\\
&\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\mathbf{u}\bar{x}^{-\frac{a}{4}}\|_{L^8}^2\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\nabla\mathbf{b}\|_{L^4} + C\|\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\\
&\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C\|\mathbf{u}\bar{x}^{-\frac{a}{4}}\|_{L^8}^4\|\nabla\mathbf{b}\|_{L^4}^2 + C\|\nabla^2\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\\
&\le C\|\mathbf{b}_t\|_{L^2}^2 + C\|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \frac{1}{4}\|\nabla^2\mathbf{b}\|_{L^2}^2 + C\left(1+\|\nabla\mathbf{u}\|_{L^2}^8\right)\left(1+\|\nabla\mathbf{b}\|_{L^2}^2\right),
\end{aligned} \tag{3.65}$$

which together with (3.33) gives that

$$\begin{aligned}
\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2\mathbf{b}\|_{L^2}^2 &\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2 + \|\nabla\mathbf{b}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)\\
&\quad + C\left(1+\|\nabla\mathbf{u}\|_{L^2}^8\right)\left(1+\|\nabla\mathbf{b}\|_{L^2}^4\right).
\end{aligned} \tag{3.66}$$

Then, multiplying (3.66) by s, one gets from (3.19), (3.38), and (3.64) that

$$\begin{aligned}
&\sup_{0\le s\le t}\left(s\|\nabla^2\mathbf{u}\|_{L^2}^2 + s\|\nabla P\|_{L^2}^2 + s\|\nabla^2\mathbf{b}\|_{L^2}^2\right)\\
&\le C\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\} + C\left(1+\int_0^t\psi^\alpha(s)ds\right)^{12}\\
&\le C\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.
\end{aligned} \tag{3.67}$$

(4) Multiplying (3.36) by s, one gets from (3.19) and (3.38) that

$$\sup_{0\le s\le t} s\|\nabla^2\theta\|_{L^2}^2 \le C\exp\left\{C\int_0^t\psi^\alpha ds\right\}, \tag{3.68}$$

which, combined with (3.64) and (3.67), implies (3.60).

(5) To finish the proof of Lemma 3.5, it suffices to show (3.63). Indeed, choosing $p=q$ in (3.32), we deduce from (3.4), (3.20), and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
&\|\nabla^2\mathbf{u}\|_{L^q} + \|\nabla P\|_{L^q}\\
&\le C\left(\|\rho\mathbf{u}_t\|_{L^q} + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^q} + \||\mathbf{b}||\nabla\mathbf{b}|\|_{L^q} + \|\rho\theta\|_{L^q}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le C(\|\rho\mathbf{u}_t\|_{L^q}+\|\rho\mathbf{u}\|_{L^{2q}}\|\nabla\mathbf{u}\|_{L^{2q}}+\|\mathbf{b}\|_{L^{2q}}\|\nabla\mathbf{b}\|_{L^{2q}}+\|\sqrt{\rho}\theta\|_{L^2}+\|\nabla\theta\|_{L^2})\\
&\le C\|\rho\mathbf{u}_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\rho\mathbf{u}_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}}+C\psi^\alpha\left(1+\|\nabla^2\mathbf{u}\|_{L^2}^{1-\frac1q}+\|\nabla^2\mathbf{b}\|_{L^2}^{1-\frac1q}\right)\\
&\le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}}+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\right)\\
&+C\psi^\alpha\left(1+\|\nabla^2\mathbf{u}\|_{L^2}^{1-\frac1q}+\|\nabla^2\mathbf{b}\|_{L^2}^{1-\frac1q}\right),
\end{aligned}
\tag{3.69}
$$

which together with (3.19) and (3.38) implies that

$$
\begin{aligned}
&\int_0^t\left(\|\nabla^2\mathbf{u}\|_{L^q}^{\frac{q+1}{q}}+\|\nabla P\|_{L^q}^{\frac{q+1}{q}}\right)ds\\
&\le C\int_0^t s^{-\frac{q+1}{2q}}\left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-1}{q(q^2-2)}}\left(s\|\nabla\mathbf{u}_t\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}}ds\\
&\quad+C\int_0^t\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{q+1}{q}}ds+C\int_0^t\psi^\alpha\left(1+\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{q^2-1}{q^2}}+\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{q^2-1}{q^2}}\right)ds\\
&\le C\sup_{0\le s\le t}\left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-1}{q(q^2-2)}}\int_0^t s^{-\frac{q+1}{2q}}\left(s\|\nabla\mathbf{u}_t\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2(q^2-2)}}ds\\
&\quad+C\int_0^t\left(\psi^\alpha+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\le C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\left(1+\int_0^t\left(s^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}}+s\|\nabla\mathbf{u}_t\|_{L^2}^2\right)ds\right)\\
&\le C\exp\left\{C\int_0^t\psi^\alpha ds\right\}
\end{aligned}
\tag{3.70}
$$

and

$$
\begin{aligned}
&\int_0^t\left(s\|\nabla^2\mathbf{u}\|_{L^q}^2+s\|\nabla P\|_{L^q}^2\right)ds\\
&\le C\int_0^t s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2ds+C\int_0^t\left(s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\right)^{\frac{2(q-1)}{q^2-2}}\left(s\|\nabla\mathbf{u}_t\|_{L^2}^2\right)^{\frac{q^2-2q}{q^2-2}}ds\\
&\quad+C\int_0^t s\psi^\alpha\left(1+\|\nabla^2\mathbf{u}\|_{L^2}^{1-\frac1q}+\|\nabla^2\mathbf{b}\|_{L^2}^{1-\frac1q}\right)^2ds\\
&\le C\int_0^t s\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2ds+C\int_0^t s\|\nabla\mathbf{u}_t\|_{L^2}^2ds+C\int_0^t\left(\psi^\alpha+s\|\nabla^2\mathbf{u}\|_{L^2}^2+s\|\nabla^2\mathbf{b}\|_{L^2}^2\right)ds\\
&\le C\exp\left\{C\int_0^t\psi^\alpha ds\right\}.
\end{aligned}
\tag{3.71}
$$

One thus obtains (3.63) from (3.70)–(3.71) and finishes the proof of Lemma 3.5. □

LEMMA 3.6. *Let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and T_1 be as in Lemma 3.2. Then there exists a positive constant $\alpha>1$ such that for all $t\in(0,T_1]$,*

$$
\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{L^1\cap H^1\cap W^{1,q}}\le\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.
\tag{3.72}
$$

Proof.

(1) It follows from Sobolev's inequality and (3.21) that for $0<\delta<1$,

$$\begin{aligned}\|\mathbf{u}\bar{x}^{-\delta}\|_{L^\infty} &\le C(\delta)\left(\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}}+\|\nabla(\mathbf{u}\bar{x}^{-\delta})\|_{L^3}\right)\\ &\le C(\delta)\left(\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}}+\|\nabla\mathbf{u}\|_{L^3}+\|\mathbf{u}\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^{\frac{12}{4-3\delta}}}\right)\\ &\le C(\delta)\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2}\right). \end{aligned}\tag{3.73}$$

(2) One derives from $(1.1)_1$ that $\rho\bar{x}^a$ satisfies

$$\partial_t(\rho\bar{x}^a)+\mathbf{u}\cdot\nabla(\rho\bar{x}^a)-a\rho\bar{x}^a\mathbf{u}\cdot\nabla\log\bar{x}=0,\tag{3.74}$$

which along with (3.73) and (3.15) gives that for any $r\in[2,q]$,

$$\begin{aligned}\frac{d}{dt}\|\nabla(\rho\bar{x}^a)\|_{L^r} \le& C\left(1+\|\nabla\mathbf{u}\|_{L^\infty}+\|\mathbf{u}\cdot\nabla\log\bar{x}\|_{L^\infty}\right)\|\nabla(\rho\bar{x}^a)\|_{L^r}\\ &+C\|\rho\bar{x}^a\|_{L^\infty}\left(\||\nabla\mathbf{u}||\nabla\log\bar{x}|\|_{L^r}+\||\mathbf{u}||\nabla^2\log\bar{x}|\|_{L^r}\right)\\ \le& C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\|\nabla(\rho\bar{x}^a)\|_{L^r}\\ &+C\|\rho\bar{x}^a\|_{L^\infty}\left(\|\nabla\mathbf{u}\|_{L^r}+\|\mathbf{u}\bar{x}^{-\frac{2}{5}}\|_{L^{4r}}\|\bar{x}^{-\frac{3}{2}}\|_{L^{\frac{4r}{3}}}\right)\\ \le& C\left(\psi^\alpha+\|\nabla^2\mathbf{u}\|_{L^2\cap L^q}\right)\left(1+\|\nabla(\rho\bar{x}^a)\|_{L^r}+\|\nabla(\rho\bar{x}^a)\|_{L^q}\right). \end{aligned}\tag{3.75}$$

Hence, we get the desired (3.72) from (3.19), (3.63), (3.10), (3.75), and Grönwall's inequality. This completes the proof of Lemma 3.6.

□

Now, Proposition 3.1 is a direct consequence of Lemmas 3.1–3.6.

Proof. **(Proof of Proposition 3.1.)** It follows from (3.4), (3.10), (3.19), and (3.72) that

$$\psi(t)\le\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.$$

Standard arguments yield that for $M:=e^{Ce}$ and $T_0:=\min\{T_1,(CM^\alpha)^{-1}\}$,

$$\sup_{0\le t\le T_0}\psi(t)\le M,$$

which together with (3.10), (3.19), (3.38), (3.60), (3.63), and (3.67) gives the desired (3.3). The proof of Proposition 3.1 is completed. □

4. Proof of Theorem 1.1

With the a priori estimates in Section 3 at hand, we are in a position to prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Let $(\rho_0,\mathbf{u}_0,\mathbf{b}_0,\theta_0)$ be as in Theorem 1.1. Without loss of generality, we assume that the initial density ρ_0 satisfies

$$\int_{\mathbb{R}^2}\rho_0dx=1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}}\rho_0dx\ge\frac{3}{4}\int_{\mathbb{R}^2}\rho_0dx=\frac{3}{4}.\tag{4.1}$$

We construct $\rho_0^R=\hat\rho_0^R+R^{-1}e^{-|x|^2}$, where $0\le\hat\rho_0^R\in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\begin{cases}\int_{B_{N_0}}\hat\rho_0^R dx\ge\frac12,\\ \bar x^a\hat\rho_0^R\to\bar x^a\rho_0 \quad \text{in } L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2), \text{ as } R\to\infty.\end{cases}\tag{4.2}$$

Noting that $\mathbf{b}_0\bar x^{\frac a2}\in L^2(\mathbb{R}^2)$ and $\nabla\mathbf{b}_0\in L^2(\mathbb{R}^2)$, we choose $\mathbf{b}_0^R\in\{\mathbf{w}\in C_0^\infty(B_R)\,|\,\mathrm{div}\,\mathbf{w}=0\}$ satisfying

$$\mathbf{b}_0^R\bar x^{\frac a2}\to\mathbf{b}_0\bar x^{\frac a2},\quad \nabla\mathbf{b}_0^R\to\nabla\mathbf{b}_0\quad \text{in } L^2(\mathbb{R}^2),\quad \text{as } R\to\infty.\tag{4.3}$$

Since $\nabla\mathbf{u}_0\in L^2(\mathbb{R}^2)$, we select $\mathbf{v}_i^R\in C_0^\infty(B_R)$ $(i=1,2)$ such that for $i=1,2$,

$$\lim_{R\to\infty}\|\mathbf{v}_i^R-\partial_i\mathbf{u}_0\|_{L^2(\mathbb{R}^2)}=0.\tag{4.4}$$

We consider the unique smooth solution $\mathbf{u}_0^R$ of the following elliptic problem:

$$\begin{cases}-\Delta\mathbf{u}_0^R+\rho_0^R\mathbf{u}_0^R+\nabla P_0^R=\sqrt{\rho_0^R}\mathbf{h}^R-\partial_i\mathbf{v}_i^R, & \text{in } B_R,\\ \mathrm{div}\,\mathbf{u}_0^R=0, & \text{in } B_R,\\ \mathbf{u}_0^R=0, & \text{on } \partial B_R,\end{cases}\tag{4.5}$$

where $\mathbf{h}^R=(\sqrt{\rho_0}\mathbf{u}_0)*j_{\frac1R}$ with j_δ being the standard mollifying kernel of width δ.

Extending $\mathbf{u}_0^R$ to $\mathbb{R}^2$ by defining 0 outside B_R and denoting it by $\tilde{\mathbf{u}}_0^R$, we claim that

$$\lim_{R\to\infty}\left(\|\nabla(\tilde{\mathbf{u}}_0^R-\mathbf{u}_0)\|_{L^2(\mathbb{R}^2)}+\left\|\sqrt{\rho_0^R}\tilde{\mathbf{u}}_0^R-\sqrt{\rho_0}\mathbf{u}_0\right\|_{L^2(\mathbb{R}^2)}\right)=0.\tag{4.6}$$

In fact, it is easy to find that $\tilde{\mathbf{u}}_0^R$ is also a solution of (4.5) in $\mathbb{R}^2$. Multiplying (4.5) by $\tilde{\mathbf{u}}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned}&\int_{\mathbb{R}^2}\rho_0^R|\tilde{\mathbf{u}}_0^R|^2dx+\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{u}}_0^R|^2dx\\ &\le\|\sqrt{\rho_0^R}\tilde{\mathbf{u}}_0^R\|_{L^2(B_R)}\|\mathbf{h}^R\|_{L^2(B_R)}+C\|\mathbf{v}_i^R\|_{L^2(B_R)}\|\partial_i\tilde{\mathbf{u}}_0^R\|_{L^2(B_R)}\\ &\le\frac12\|\nabla\tilde{\mathbf{u}}_0^R\|_{L^2(B_R)}^2+\frac12\int_{B_R}\rho_0^R|\tilde{\mathbf{u}}_0^R|^2dx+C\|\mathbf{h}^R\|_{L^2(B_R)}^2+C\|\mathbf{v}_i^R\|_{L^2(B_R)}^2,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^2}\rho_0^R|\tilde{\mathbf{u}}_0^R|^2dx+\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{u}}_0^R|^2dx\le C\tag{4.7}$$

for some C independent of R. This together with (4.2) yields that there exists a subsequence $R_j\to\infty$ and a function $\tilde{\mathbf{u}}_0\in\{\tilde{\mathbf{u}}_0\in H^1_{\mathrm{loc}}(\mathbb{R}^2)|\sqrt{\rho_0}\tilde{\mathbf{u}}_0\in L^2(\mathbb{R}^2),\nabla\tilde{\mathbf{u}}_0\in L^2(\mathbb{R}^2)\}$ such that

$$\begin{cases}\sqrt{\rho_0^{R_j}}\tilde{\mathbf{u}}_0^{R_j}\rightharpoonup\sqrt{\rho_0}\tilde{\mathbf{u}}_0 \text{ weakly in } L^2(\mathbb{R}^2),\\ \nabla\tilde{\mathbf{u}}_0^{R_j}\rightharpoonup\nabla\tilde{\mathbf{u}}_0 \text{ weakly in} L^2(\mathbb{R}^2).\end{cases}\tag{4.8}$$

Next, we will show

$$\tilde{\mathbf{u}}_0=\mathbf{u}_0.\tag{4.9}$$

Indeed, multiplying (4.5) by a test function $\pi\in C_0^\infty(\mathbb{R}^2)$ with $\operatorname{div}\pi=0$, it holds that

$$\int_{\mathbb{R}^2}\partial_i\left(\tilde{\mathbf{u}}_0^{R_j}-\mathbf{v}_i^{R_j}\right)\cdot\partial_i\pi dx+\int_{\mathbb{R}^2}\sqrt{\rho_0^{R_j}}\left(\sqrt{\rho_0^{R_j}}\tilde{\mathbf{u}}_0^{R_j}-\mathbf{h}^{R_j}\right)\cdot\pi dx=0. \tag{4.10}$$

Let $R_j\to\infty$, it follows from (4.2), (4.4), and (4.8) that

$$\int_{\mathbb{R}^2}\partial_i(\tilde{\mathbf{u}}_0-\mathbf{u}_0)\cdot\partial_i\pi dx+\int_{\mathbb{R}^2}\rho_0(\tilde{\mathbf{u}}_0-\mathbf{u}_0)\cdot\pi dx=0, \tag{4.11}$$

which implies (4.9).

Furthermore, multiplying (4.5) by $\tilde{\mathbf{u}}_0^{R_j}$ and integrating the resulting equation over $\mathbb{R}^2$, by the same arguments as (4.11), we have

$$\lim_{R_j\to\infty}\int_{\mathbb{R}^2}\left(|\nabla\tilde{\mathbf{u}}_0^{R_j}|^2+\rho_0^{R_j}|\tilde{\mathbf{u}}_0^{R_j}|^2\right)dx=\int_{\mathbb{R}^2}\left(|\nabla\mathbf{u}_0|^2+\rho_0|\mathbf{u}_0|^2\right)dx,$$

which combined with (4.8) leads to

$$\lim_{R_j\to\infty}\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{u}}_0^{R_j}|^2dx=\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{u}}_0|^2dx,\ \lim_{R_j\to\infty}\int_{\mathbb{R}^2}\rho_0^{R_j}|\tilde{\mathbf{u}}_0^{R_j}|^2dx=\int_{\mathbb{R}^2}\rho_0|\tilde{\mathbf{u}}_0|^2dx.$$

This along with (4.9) and (4.8) gives (4.6).

Hence, by virtue of Lemma 2.1, the initial-boundary-value problem (1.1) and (2.2) with the initial data $(\rho_0^R,\mathbf{u}_0^R,\mathbf{b}_0^R,\theta_0^R)$ has a classical solution $(\rho^R,\mathbf{u}^R,P^R,\mathbf{b}^R,\theta^R)$ on $B_R\times[0,T_R]$. Moreover, Proposition 3.1 shows that there exists a T_0 independent of R such that (3.3) holds for $(\rho^R,\mathbf{u}^R,P^R,\mathbf{b}^R,\theta^R)$.

For simplicity, in what follows, we denote

$$L^p=L^p(\mathbb{R}^2),\quad W^{k,p}=W^{k,p}(\mathbb{R}^2).$$

Extending $(\rho^R,\mathbf{u}^R,P^R,\mathbf{b}^R,\theta^R)$ by zero on $\mathbb{R}^2\setminus B_R$ and denoting it by

$$\left(\tilde\rho^R:=\varphi_R\rho^R,\tilde{\mathbf{u}}^R,\tilde P^R,\tilde{\mathbf{b}}^R,\tilde\theta^R\right)$$

with φ_R satisfying (3.11). First, (3.3) leads to

$$\begin{aligned}&\sup_{0\le t\le T_0}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}^R\|_{L^2}+\|\sqrt{\tilde\rho^R}\tilde\theta^R\|_{L^2}+\|\nabla\tilde{\mathbf{u}}^R\|_{L^2}+\|\nabla\tilde\theta^R\|_{L^2}+\|\nabla\tilde{\mathbf{b}}^R\|_{L^2}+\|\tilde{\mathbf{b}}^R\bar x^{\frac a2}\|_{L^2}\right)\\ &\le\sup_{0\le t\le T_0}\left(\|\sqrt{\rho^R}\mathbf{u}^R\|_{L^2(B_R)}+\|\sqrt{\rho^R}\theta^R\|_{L^2(B_R)}+\|\nabla\mathbf{u}^R\|_{L^2(B_R)}\right.\\ &\qquad\left.+\|\nabla\theta^R\|_{L^2(B_R)}+\|\nabla\mathbf{b}^R\|_{L^2(B_R)}+\|\mathbf{b}^R\bar x^{\frac a2}\|_{L^2(B_R)}\right)\\ &\le C,\end{aligned} \tag{4.12}$$

and

$$\sup_{0\le t\le T_0}\|\tilde\rho^R\bar x^a\|_{L^1\cap L^\infty}\le C.$$

Similarly, it follows from (3.3) that for $q>2$,

$$\sup_{0\le t\le T_0}\sqrt{t}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\|_{L^2}+\|\sqrt{\tilde\rho^R}\tilde\theta_t^R\|_{L^2}+\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^2}+\|\nabla^2\tilde\theta^R\|_{L^2}\right.$$

$$
\begin{aligned}
&\quad +\|\nabla^2\tilde{\mathbf{b}}^R\|_{L^2}+\|\tilde{\mathbf{b}}_t^R\|_{L^2}\Big)\\
&+\int_0^{T_0}\left(\|\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\|_{L^2}^2+\|\sqrt{\tilde\rho^R}\tilde\theta_t^R\|_{L^2}^2+\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^2}^2+\|\nabla^2\tilde\theta^R\|_{L^2}^2\right)dt\\
&+\int_0^{T_0}\left(\|\nabla^2\tilde{\mathbf{b}}^R\|_{L^2}^2+\|\nabla\tilde{\mathbf{b}}^R\bar{x}^{\frac{a}{2}}\|_{L^2}^2\right)dt\\
&+\int_0^{T_0}\left(\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^q}^{\frac{q+1}{q}}+t\|\nabla^2\tilde{\mathbf{u}}^R\|_{L^q}^2+t\|\nabla\tilde{\mathbf{u}}_t^R\|_{L^2}^2+t\|\nabla\tilde\theta_t^R\|_{L^2}^2+t\|\nabla\tilde{\mathbf{b}}_t^R\|_{L^2}^2\right)dt\\
\le& C. \qquad (4.13)
\end{aligned}
$$

Next, for $p\in[2,q]$, we obtain from (3.3) and (3.72) that

$$
\begin{aligned}
\sup_{0\le t\le T_0}\|\nabla(\tilde\rho^R\bar{x}^a)\|_{L^p}&\le C\sup_{0\le t\le T_0}\left(\|\nabla(\rho^R\bar{x}^a)\|_{L^p(B_R)}+R^{-1}\|\rho^R\bar{x}^a\|_{L^p(B_R)}\right)\\
&\le C\sup_{0\le t\le T_0}\|\rho^R\bar{x}^a\|_{H^1(B_R)\cap W^{1,p}(B_R)}\le C, \qquad (4.14)
\end{aligned}
$$

which together with (3.73) and (3.3) yields

$$
\begin{aligned}
\int_0^{T_0}\|\bar{x}\tilde\rho_t^R\|_{L^p}^2dt&\le C\int_0^{T_0}\|\bar{x}|\mathbf{u}^R||\nabla\rho^R|\|_{L^p(B_R)}^2dt\\
&\le C\int_0^{T_0}\|\bar{x}^{1-a}\mathbf{u}^R\|_{L^\infty(B_R)}^2\|\bar{x}^a\nabla\rho^R\|_{L^p(B_R)}^2dt\\
&\le C. \qquad (4.15)
\end{aligned}
$$

By virtue of the same arguments as those of (3.60) and (3.63), one gets

$$
\sup_{0\le t\le T_0}\sqrt{t}\|\nabla\tilde{P}^R\|_{L^2}+\int_0^{T_0}\left(\|\nabla\tilde{P}^R\|_{L^2}^2+\|\nabla\tilde{P}^R\|_{L^q}^{\frac{1+q}{q}}\right)dt\le C. \qquad (4.16)
$$

With the estimates (4.12)–(4.16) at hand, we find that the sequence $(\tilde\rho^R,\tilde{\mathbf{u}}^R,\tilde{P}^R,\tilde{\mathbf{b}}^R,\tilde\theta^R)$ converges, up to the extraction of subsequences, to some limit $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ in the obvious weak sense, that is, as $R\to\infty$, we have

$$\tilde\rho^R\bar{x}\to\rho\bar{x},\ \text{in } C(\overline{B_N}\times[0,T_0]),\ \text{for any } N>0, \qquad (4.17)$$

$$\tilde\rho^R\bar{x}^a\rightharpoonup\rho\bar{x}^a,\ \text{weakly * in } L^\infty(0,T_0;H^1\cap W^{1,q}), \qquad (4.18)$$

$$\tilde{\mathbf{b}}^R\bar{x}^{\frac{a}{2}}\rightharpoonup\mathbf{b}\bar{x}^{\frac{a}{2}},\ \text{weakly * in} L^\infty(0,T_0;L^2), \qquad (4.19)$$

$$\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}^R\rightharpoonup\sqrt{\rho}\mathbf{u},\ \nabla\tilde{\mathbf{u}}^R\rightharpoonup\nabla\mathbf{u},\ \nabla\tilde{\mathbf{b}}^R\rightharpoonup\nabla\mathbf{b},\ \text{weakly * in } L^\infty(0,T_0;L^2), \qquad (4.20)$$

$$\sqrt{\tilde\rho^R}\tilde\theta^R\rightharpoonup\sqrt{\rho}\theta,\ \nabla\tilde\theta^R\rightharpoonup\nabla\theta,\ \text{weakly * in } L^\infty(0,T_0;L^2), \qquad (4.21)$$

$$\nabla^2\tilde{\mathbf{u}}^R\rightharpoonup\nabla^2\mathbf{u},\ \nabla\tilde{P}^R\rightharpoonup\nabla P,\ \text{weakly in } L^{\frac{q+1}{q}}(0,T_0;L^q)\cap L^2(\mathbb{R}^2\times(0,T_0)), \qquad (4.22)$$

$$\tilde{\mathbf{b}}_t^R\rightharpoonup\mathbf{b}_t,\ \nabla\tilde{\mathbf{b}}^R\bar{x}^{\frac{a}{2}}\rightharpoonup\nabla\mathbf{b}\bar{x}^{\frac{a}{2}},\ \nabla^2\tilde{\mathbf{b}}^R\rightharpoonup\nabla^2\mathbf{b},\ \text{weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \qquad (4.23)$$

$$\sqrt{t}\nabla^2\tilde{\mathbf{u}}^R\rightharpoonup\sqrt{t}\nabla^2\mathbf{u},\ \text{weakly in } L^2(0,T_0;L^q),\ \text{weakly * in } L^\infty(0,T_0;L^2), \qquad (4.24)$$

$$\sqrt{t}\sqrt{\tilde\rho^R}\tilde{\mathbf{u}}_t^R\rightharpoonup\sqrt{t}\sqrt{\rho}\mathbf{u}_t,\ \sqrt{t}\nabla\tilde{P}^R\rightharpoonup\sqrt{t}\nabla P,\ \text{weakly * in } L^\infty(0,T_0;L^2), \qquad (4.25)$$

$$\sqrt{t}\tilde{\mathbf{b}}_t^R\rightharpoonup\sqrt{t}\mathbf{b}_t,\ \sqrt{t}\nabla^2\tilde{\mathbf{b}}^R\rightharpoonup\sqrt{t}\nabla^2\mathbf{b},\ \text{weakly * in } L^\infty(0,T_0;L^2), \qquad (4.26)$$

$$\sqrt{t}\sqrt{\tilde\rho^R}\tilde\theta_t^R\rightharpoonup\sqrt{t}\sqrt{\rho}\theta_t,\ \sqrt{t}\nabla^2\tilde\theta^R\rightharpoonup\sqrt{t}\nabla^2\theta,\ \text{weakly * in } L^\infty(0,T_0;L^2), \qquad (4.27)$$

$$\sqrt{t}\nabla\tilde{\mathbf{u}}_t^R \rightharpoonup \sqrt{t}\nabla\mathbf{u}_t, \quad \sqrt{t}\nabla\tilde{\mathbf{b}}_t^R \rightharpoonup \sqrt{t}\nabla\mathbf{b}_t, \text{ weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.28}$$

$$\sqrt{t}\nabla\tilde{\theta}_t^R \rightharpoonup \sqrt{t}\nabla\theta_t, \text{ weakly in } L^2(\mathbb{R}^2\times(0,T_0)), \tag{4.29}$$

with

$$\rho\bar{x}^a \in L^\infty(0,T_0;L^1), \quad \inf_{0\le t\le T_0}\int_{B_{2N_0}}\rho(x,t)dx \ge \frac{1}{4}. \tag{4.30}$$

For any function $\phi\in C_0^\infty(\mathbb{R}^2\times[0,T_0))$, we take $\phi\varphi_R$ as test function in the initial-boundary-value problem (1.1) and (2.2) with the initial data $(\rho_0^R,\mathbf{u}_0^R,\mathbf{b}_0^R,\theta_0^R)$. Then, letting $R\to\infty$, standard arguments together with (4.17)–(4.30) show that $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ is a strong solution of (1.1)–(1.2) on $\mathbb{R}^2\times(0,T_0]$ satisfying (1.5) and (1.6). Indeed, the existence of a pressure P follows immediately from $(1.1)_2$ and $(1.1)_5$ by a classical consideration. The proof of the existence part of Theorem 1.1 is finished. □

It remains only to prove the uniqueness of the strong solutions satisfying (1.5) and (1.6). Let $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and $(\bar\rho,\bar{\mathbf{u}},\bar P,\bar{\mathbf{b}},\bar\theta)$ be two strong solutions satisfying (1.5) and (1.6) with the same initial data, and denote

$$\Theta \triangleq \rho-\bar\rho,\ \mathbf{U}\triangleq \mathbf{u}-\bar{\mathbf{U}},\ \mathbf{\Phi}\triangleq\mathbf{b}-\bar{\mathbf{b}},\ \Psi\triangleq\theta-\bar\theta.$$

First, subtracting the mass equation satisfied by $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and $(\bar\rho,\bar{\mathbf{u}},\bar P,\bar{\mathbf{b}},\bar\theta)$ gives

$$\Theta_t+\bar{\mathbf{u}}\cdot\nabla\Theta+\mathbf{U}\cdot\nabla\rho=0. \tag{4.31}$$

Multiplying (4.31) by $2\Theta\bar{x}^{2r}$ for $r\in(1,\tilde a)$ with $\tilde a=\min\{2,a\}$ and integrating by parts yield

$$\begin{aligned}&\frac{d}{dt}\|\Theta\bar{x}^r\|_{L^2}^2\\ &\le C\|\bar{\mathbf{u}}\bar{x}^{-\frac12}\|_{L^\infty}\|\Theta\bar{x}^r\|_{L^2}^2+C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-(\tilde a-r)}\|_{L^{\frac{2q}{(q-2)(\tilde a-r)}}}\|\bar{x}^{\tilde a}\nabla\rho\|_{L^{\frac{2q}{q-(q-2)(\tilde a-r)}}}\\ &\le C(1+\|\nabla\bar{\mathbf{u}}\|_{W^{1,q}})\|\Theta\bar{x}^r\|_{L^2}^2+C\|\Theta\bar{x}^r\|_{L^2}(\|\nabla\mathbf{U}\|_{L^2}+\|\sqrt{\rho}\mathbf{U}\|_{L^2})\end{aligned}$$

due to Sobolev's inequality, (1.6), (3.14), and (3.73). This combined with Grönwall's inequality shows that for all $0\le t\le T_0$,

$$\|\Theta\bar{x}^r\|_{L^2}\le C\int_0^t(\|\nabla\mathbf{U}\|_{L^2}+\|\sqrt{\rho}\mathbf{U}\|_{L^2})ds. \tag{4.32}$$

Next, subtracting $(1.1)_2$, $(1.1)_3$, and $(1.1)_4$ satisfied by $(\rho,\mathbf{u},P,\mathbf{b},\theta)$ and $(\bar\rho,\bar{\mathbf{u}},\bar P,\bar{\mathbf{b}},\bar\theta)$ leads to

$$\begin{aligned}\rho\mathbf{U}_t+\rho\mathbf{u}\cdot\nabla\mathbf{U}-\mu\Delta\mathbf{U}=&-\rho\mathbf{U}\cdot\nabla\bar{\mathbf{u}}-\Theta(\bar{\mathbf{u}}_t+\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}})-\nabla(P-\bar P)\\ &-\frac12\nabla\left(|\mathbf{b}|^2-|\bar{\mathbf{b}}|^2\right)+\mathbf{b}\cdot\nabla\mathbf{\Phi}+\mathbf{\Phi}\cdot\nabla\bar{\mathbf{b}}+\Theta\theta\mathbf{e}_2+\bar\rho\Psi\mathbf{e}_2,\end{aligned} \tag{4.33}$$

$$\mathbf{\Phi}_t-\nu\Delta\mathbf{\Phi}=\mathbf{b}\cdot\nabla\mathbf{U}+\mathbf{\Phi}\cdot\nabla\bar{\mathbf{u}}-\mathbf{u}\cdot\nabla\mathbf{\Phi}-\mathbf{U}\cdot\nabla\bar{\mathbf{b}}, \tag{4.34}$$

and

$$\rho\Psi_t+\rho\mathbf{u}\cdot\nabla\Psi-\kappa\Delta\Psi=-\rho\Psi\cdot\nabla\bar{\mathbf{u}}-\Theta(\bar{\mathbf{u}}_t+\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}})+\Theta\mathbf{u}\cdot\mathbf{e}_2+\bar\rho\mathbf{U}\cdot\mathbf{e}_2. \tag{4.35}$$

Multiplying (4.33) by $\mathbf{U}$, (4.34) by $\mathbf{\Phi}$, and (4.35) by Ψ, respectively, and adding the resulting equations together, we obtain after integration by parts that

$$\begin{aligned}
&\frac{d}{dt}\int\left(\rho|\mathbf{U}|^2+|\mathbf{\Phi}|^2+\rho|\Psi|^2\right)dx+\int\left(\mu|\nabla\mathbf{U}|^2+\nu|\nabla\mathbf{\Phi}|^2+\kappa|\nabla\Psi|^2\right)dx\\
&\le C\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\int\left(\rho|\mathbf{U}|^2+|\mathbf{\Phi}|^2+\rho|\Psi|^2\right)dx+C\int|\Theta||\mathbf{U}|\left(|\bar{\mathbf{u}}_t|+|\bar{\mathbf{u}}||\nabla\bar{\mathbf{u}}|\right)dx\\
&\quad-\int\mathbf{\Phi}\cdot\nabla\mathbf{U}\cdot\bar{\mathbf{b}}dx-\int\mathbf{U}\cdot\nabla\bar{\mathbf{b}}\cdot\mathbf{\Phi}dx+\int(\rho\Psi+\Theta\bar{\theta})\mathbf{e}_2\cdot\mathbf{U}dx\\
&\quad+C\int|\Theta||\Psi|\left(|\bar{\mathbf{u}}_t|+|\bar{\mathbf{u}}||\nabla\bar{\mathbf{u}}|\right)dx+\int(\Theta\bar{\mathbf{u}}+\rho\mathbf{U})\cdot\mathbf{e}_2\Psi dx\\
&=:C\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\int\left(\rho|\mathbf{U}|^2+|\mathbf{\Phi}|^2+\rho|\Psi|^2\right)dx+\sum_{i=1}^{6}K_i. \qquad (4.36)
\end{aligned}$$

Hölder's inequality combined with (1.6), (2.6), (3.3), and (4.32) yields that for $r\in(1,\tilde{a})$,

$$\begin{aligned}
K_1&\le C\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-\frac{r}{2}}\|_{L^4}\left(\|\bar{\mathbf{u}}_t\bar{x}^{-\frac{r}{2}}\|_{L^4}+\|\nabla\bar{\mathbf{u}}\|_{L^\infty}\|\bar{\mathbf{u}}\bar{x}^{-\frac{r}{2}}\|_{L^4}\right)\\
&\le C(\varepsilon)\left(\|\sqrt{\bar{\rho}}\bar{\mathbf{u}}_t\|_{L^2}^2+\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2+\|\nabla\bar{\mathbf{u}}\|_{L^\infty}^2\right)\|\Theta\bar{x}^r\|_{L^2}^2\\
&\quad+\varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{U}\|_{L^2}^2\right)\\
&\le C(\varepsilon)\left(1+t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2+t\|\nabla^2\bar{\mathbf{u}}\|_{L^q}^2\right)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds\\
&\quad+\varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{U}\|_{L^2}^2\right). \qquad (4.37)
\end{aligned}$$

For the term K_2, we derive from the Gagliardo-Nirenberg inequality and (3.49) that

$$K_2\le C\|\bar{\mathbf{b}}\|_{L^4}\|\mathbf{\Phi}\|_{L^4}\|\nabla\mathbf{U}\|_{L^2}\le\varepsilon\|\nabla\mathbf{U}\|_{L^2}^2+\varepsilon\|\nabla\mathbf{\Phi}\|_{L^2}^2+C(\varepsilon)\|\mathbf{\Phi}\|_{L^2}^2. \qquad (4.38)$$

Owing to (1.6), (2.6), and (3.3), K_3 can be estimated as follows

$$\begin{aligned}
K_3&\le C\|\mathbf{U}\bar{x}^{-a}\|_{L^4}\||\nabla\bar{\mathbf{b}}|^{\frac{1}{2}}\bar{x}^a\|_{L^4}\||\nabla\bar{\mathbf{b}}|^{\frac{1}{2}}\|_{L^4}\|\mathbf{\Phi}\|_{L^4}\\
&\le C\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}+\|\nabla\mathbf{U}\|_{L^2}\right)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^{\frac{1}{2}}\|\mathbf{\Phi}\|_{L^4}\\
&\le\varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{U}\|_{L^2}^2\right)+C(\varepsilon)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}\|\mathbf{\Phi}\|_{L^4}^2\\
&\le\varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{U}\|_{L^2}^2\right)+\varepsilon\|\nabla\mathbf{\Phi}\|_{L^2}^2+C(\varepsilon)\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^2\|\mathbf{\Phi}\|_{L^2}^2. \qquad (4.39)
\end{aligned}$$

We obtain from Hölder's inequality, (1.6), (2.6), (3.3), and (4.32) that for $r\in(1,\tilde{a})$,

$$\begin{aligned}
K_4&\le\|\sqrt{\rho}\Psi\|_{L^2}\|\sqrt{\rho}\mathbf{U}\|_{L^2}+\|\Theta\bar{x}^r\|_{L^2}\|\mathbf{U}\bar{x}^{-\frac{r}{2}}\|_{L^4}\|\bar{\theta}\bar{x}^{-\frac{r}{2}}\|_{L^4}\\
&\le\|\sqrt{\rho}\Psi\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+C(\varepsilon)\left(\|\sqrt{\bar{\rho}}\bar{\theta}\|_{L^2}^2+\|\nabla\bar{\theta}\|_{L^2}^2\right)\|\Theta\bar{x}^r\|_{L^2}^2\\
&\quad+\varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{U}\|_{L^2}^2\right)\\
&\le\|\sqrt{\rho}\Psi\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+C(\varepsilon)\left(1+\|\nabla\bar{\theta}\|_{L^2}^2\right)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds\\
&\quad+\varepsilon\left(\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{U}\|_{L^2}^2\right). \qquad (4.40)
\end{aligned}$$

Similarly to (4.37), one has

$$K_5\le C(\varepsilon)\left(1+t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2+t\|\nabla^2\bar{\mathbf{u}}\|_{L^q}^2\right)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds$$

$$+\varepsilon\left(\|\sqrt{\rho}\Psi\|_{L^2}^2+\|\nabla\Psi\|_{L^2}^2\right). \tag{4.41}$$

The last term K_6 can be bounded similarly as K_4

$$\begin{aligned}K_6 &\le \|\sqrt{\rho}\Psi\|_{L^2}\|\sqrt{\rho}\mathbf{U}\|_{L^2}+\|\Theta\bar{x}^r\|_{L^2}\|\Psi\bar{x}^{-\frac{r}{2}}\|_{L^4}\|\bar{\mathbf{u}}\bar{x}^{-\frac{r}{2}}\|_{L^4}\\ &\le \|\sqrt{\rho}\Psi\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+C(\varepsilon)\left(\|\sqrt{\bar\rho}\bar{\mathbf{u}}\|_{L^2}^2+\|\nabla\bar{\mathbf{u}}\|_{L^2}^2\right)\|\Theta\bar{x}^r\|_{L^2}^2\\ &\quad+\varepsilon\left(\|\sqrt{\rho}\Psi\|_{L^2}^2+\|\nabla\Psi\|_{L^2}^2\right)\\ &\le \|\sqrt{\rho}\Psi\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+C(\varepsilon)\left(1+\|\nabla\bar{\mathbf{u}}\|_{L^2}^2\right)\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2\right)ds\\ &\quad+\varepsilon\left(\|\sqrt{\rho}\Psi\|_{L^2}^2+\|\nabla\Psi\|_{L^2}^2\right).\end{aligned} \tag{4.42}$$

Denoting

$$\begin{aligned}G(t):=&\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\mathbf{\Phi}\|_{L^2}^2+\|\sqrt{\rho}\Psi\|_{L^2}^2\\ &+\int_0^t\left(\|\nabla\mathbf{U}\|_{L^2}^2+\|\nabla\mathbf{\Phi}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{U}\|_{L^2}^2+\|\nabla\Psi\|_{L^2}^2+\|\sqrt{\rho}\Psi\|_{L^2}^2\right)ds,\end{aligned}$$

then substituting (4.37)–(4.42) into (4.36) and choosing ε suitably small lead to

$$G'(t)\le C\left(1+\|\nabla\bar{\mathbf{u}}\|_{L^\infty}+\|\nabla\bar{\mathbf{b}}\bar{x}^{\frac{a}{2}}\|_{L^2}^2+t\|\nabla\bar{\mathbf{u}}_t\|_{L^2}^2+\|\nabla\bar\theta\|_{L^2}^2+\|\nabla\bar{\mathbf{u}}\|_{L^2}^2+t\|\nabla^2\mathbf{u}\|_{L^q}^2\right)G(t),$$

which together with Grönwall's inequality and (1.5) implies $G(t)=0$. Hence, $(\mathbf{U},\mathbf{\Phi},\Psi)(x,t)=(\mathbf{0},\mathbf{0},0)$ for almost everywhere $(x,t)\in\mathbb{R}^2\times(0,T)$. Finally, one can deduce from (4.32) that $\Theta=0$ for almost everywhere $(x,t)\in\mathbb{R}^2\times(0,T)$. The proof of Theorem 1.1 is completed.

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