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STABILITY AND BACK FLOW OF BOUNDARY LAYERS FOR WIND-DRIVEN OCEANIC CURRENT*

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Abstract. The proposal of this paper is to study the well-posedness and properties of solutions to the boundary layer problem for wind-driven oceanic current, which differs from the classical Prandtl boundary layer equations with a nonlocal integral term arising from the Coriolis force. First, under Oleinik's monotonic condition [O.A. Oleinik, Dokl. Akad. Nauk SSSR, 150(4):28–31, 1963] on the tangential velocity field, we obtain the local well-posedness of the boundary layer problem by using the Crocco transformation. Secondly, we show that the back flow point appears at the physical boundary in a finite time under certain constraint on the growth rate of the tangential velocity when both the initial tangential velocity and the upstream velocity are monotonically increasing with respect to the normal variable of the boundary, even if the momentum of the outer flow is favorable for the classical Prandtl equations, in the sense with this favorable condition there will be no back flow in the two-dimensional Prandtl boundary layer. This shows that the factor of the Coriolis force stimulates the appearance of the back flow of the boundary layer.

Keywords. Boundary layer; Navier-Stokes-Coriolis equations; well-posedness; back flow.

AMS subject classifications. 35B40; 35Q30; 76D05; 76D10.

1. Introduction

In this paper, we study the well-posedness and qualitative properties of solutions to the following initial and boundary value problem for the boundary layer equations in $D_T=\{0\le t\le T, 0\le x\le X, y\ge 0\}$,

$$\begin{cases}\partial_t u+u\partial_x u+v\partial_y u-\partial_y^2 u=\partial_t U+U\partial_x U+\int_{+\infty}^y (U-u)dy',\\ \partial_x u+\partial_y v=0,\\ u|_{t=0}=u_0(x,y),\quad u|_{x=0}=u_1(t,y),\\ (u,v)|_{y=0}=(0,0),\quad \lim\limits_{y\to+\infty} u(t,x,y)=U(t,x).\end{cases} \tag{1.1}$$

where (u,v) represents the velocity field in the boundary layer, and $U(t,x)$ is the tangential velocity of the outer flow. The problem (1.1) describes the motion of the oceanic current near the western coast in some regimes, which can be derived from the Navier-Stokes-Coriolis equations in the small viscosity and Coriolis parameter limit under certain constraints. The detailed derivation of the problem (1.1) from the Navier-Stokes-Coriolis equations will be given in Appendix A, it also can be found in [26]. By observation, the equations given in (1.1) differ from the classical Prandtl boundary layer equation studied in [15, 17, 19], only by the additional integral term, which arises from the Coriolis force.

In the boundary layer, it is well known that the tangential velocity changes fast in the normal direction and the vorticity is unbounded in the small viscosity limit. Therefore, in order to study the stability of the two-dimensional boundary layer, a reasonable

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approach is to control the vorticity of the flow under the monotonicity condition on the tangential velocity field,

$$u_y(t,x,y) > 0, \quad \forall\, y \geq 0, \tag{1.2}$$

under which the vorticity in the boundary layer keeps the sign unchanged. In 1963, Oleinik [15] obtained the existence of a classical solution to the two-dimensional classical Prandtl equation in the monotonic class by introducing the Crocco transformation. One motivation of this paper is to study the influence of the integral term on the local well-posedness of (1.1) in the monotonic class by developing the idea given in [15, 17]. In contrast with the classical Prandtl equation, we will see that both, the equation and the boundary condition given in the problem (2.2), derived from (1.1) by using the Crocco transformation, have nonlocal terms with the integral of $\frac{1-\eta}{\omega}$. Thus, we need to estimate this quantity $\frac{1-\eta}{\omega}$ carefully and it also appears when one establishes the maximal principle for this problem.

On the other hand, it is an important problem to study whether the monotonicity of the solution to the problem (1.1) is preserved when the time evolves, this is closely related to the back flow of boundary layer, which is an important physical event, eventually leading to the separation of boundary layer. The first back flow point is defined as $(t_0, x_0, 0)$ at which

$$u_y(t_0, x_0, 0) = 0, \tag{1.3}$$

and

$$u_y(t,x,y) > 0, \quad \text{for all } 0 \leq t < t_0,\ 0 \leq x \leq X,\ 0 \leq y < +\infty. \tag{1.4}$$

The second aim of this paper is to investigate whether there exists a back flow point for the time evolution of the system (1.1) when both the initial tangential velocity and the upstream velocity are in the monotonic class with respect to the normal variable of the boundary. In fact, one can observe that the integration $\int_{+\infty}^{y}(U-u)dy'$ in the equation plays a role in decelerating the flow. In particular, the integral term could be the dominating one near the boundary, which ensures that the total force near the boundary is negative, even if the momentum $U_t + UU_x$ is favorable for the classical Prandtl equation studied in [28]. Consequently, we can obtain the occurrence of the back flow point under certain condition by developing the Lyapunov functional argument given in [27], in which it is interesting to see that the integral term has a damaging effect on the monotonicity of flow.

Now we state the main results of this paper. First, we have the following result on the local well-posedness of the problem (1.1):

THEOREM 1.1. *For any given $X > 0$ and $T_0 > 0$, assume that the initial and boundary data $u_0 \in C^8([0,X] \times \mathbb{R}^+)$, $u_1 \in C^8([0,T_0] \times \mathbb{R}^+)$ and the outer flow $U \in C^8([0,T_0] \times [0,X])$ satisfy the compatibility conditions of the problem (1.1) up to order 6. In addition, for all $y > 0$, $u_{0y}(x,y) > 0$ and $u_{1y}(t,y) > 0$, and there exist positive constants C_1 and C_2 such that*

$$C_1\Big(1 - \frac{u_0}{U(0,x)}\Big) \leq \frac{u_{0y}}{U(0,x)} \leq C_2\Big(1 - \frac{u_0}{U(0,x)}\Big) \tag{1.5}$$

and

$$C_1\Big(1 - \frac{u_1}{U(t,0)}\Big) \leq \frac{u_{1y}}{U(t,0)} \leq C_2\Big(1 - \frac{u_1}{U(t,0)}\Big). \tag{1.6}$$

Then, the problem (1.1) *admits a unique classical solution* (u,v) *in* D_T *for some* $0<T<T_0$. *Moreover, one has*

$$M_1(1-\frac{u}{U}) \leq \frac{u_y}{U} \leq M_2(1-\frac{u}{U}), \quad \forall (t,x,y) \in D_T, \tag{1.7}$$

for some constants $M_2 \geq M_1 > 0$.

The second result concerns the appearance of a back flow point in the evolution of the boundary layer under certain growth rate constraint of the initial and boundary data, even though both the initial tangential velocity and upstream flow are strictly increasing with respect to the normal variable of the boundary.

THEOREM 1.2.
(1) For any fixed $X,T>0$, *and* $\lambda>0$ *being given in* (3.1), *if there exists a positive constant* k *such that the outflow, the initial and boundary data satisfy*

$$\max_{D_T}\left(\frac{u_{0y}}{U(0,x)-u_0}, \frac{u_{1y}}{U(t,0)-u_1}e^{-\lambda t}\right) < k, \tag{1.8}$$

and

$$\frac{U_t + UU_x}{U} \leq \frac{e^{-\lambda t}}{k}(1-2\delta) \tag{1.9}$$

for a positive constant $0<\delta<\frac{1}{2}$. *Then, the first zero point of* $\partial_y u(t,x,y)$, *when the time evolves, should be at the boundary* $\{y=0\}$, *if it exists for some time* $t>0$.

(2) Moreover, when the initial velocity $u_0(x,y)$ *satisfies*

$$\int_0^\infty \int_0^X \frac{(X-x)^{\frac{3}{2}}\partial_y u_0}{\sqrt{(\partial_y u_0)^2 + u_0^2}} dxdy \geq C_*, \tag{1.10}$$

for a positive constant C_* *depending only on* X, T, U, δ, λ *and* k, *then there is a back flow point* $(t^*,x^*) \in (0,T) \times [0,X]$, *such that*

$$\begin{cases} \partial_y u(t^*,x^*,0) = 0, \\ \partial_y u(t,x,y) > 0, \quad \forall 0<t<t^*, x\in[0,X], y\geq 0. \end{cases} \tag{1.11}$$

REMARK 1.1. Compared with the results obtained in [27] for the classical Prandtl equation, due to the additional integral term given in (1.1), here the condition of the outflow momentum satisfying $\partial_t U + U\partial_x U < 0$ has been relaxed to the conditions given in (1.8) and (1.9). It shows that the Coriolis force stimulates the appearance of the back flow of the boundary layer for the problem (1.1).

Before the end of this section, let us briefly review the related work of the classical Prandtl equation. As mentioned before, the local well-posedness of classical solutions to the two-dimensional steady and unsteady Prandtl equation was obtained in the pioneering work [15–17], with the monotonicity assumption on the tangential velocity for the unsteady problem. This local well-posedness result of the two-dimensional unsteady Prandtl equation was re-studied in [1, 13] by developing an energy method. A local well-posedness of classical solutions to the three-dimensional Prandtl equation was given in [12]. Under the assumption of adverse pressure gradient of the outflow, Oleinik ([16, 17]) and Suslov ([24]) proved that the two-dimensional steady Prandtl boundary layer separates from the boundary. The asymptotic behavior of flow near the separation point of the two-dimensional steady Prandtl equation was formally investigated by Goldstein in [10], and improved by Stewartson in [23]. A rigorous analysis of this asymptotic behavior of flow near separation was first given by E and Caffarelli in an unpublished manuscript mentioned in [7], then was studied in detail recently by Dalibard and Masmoudi in [4], and by Shen, Wang and Zhang in [22]. In [28], Xin and Zhang proved that there is a global weak solution to the two-dimensional unsteady Prandtl equation in the monotonic class under the assumption of the favorable pressure gradient. However, Moore ([14]), Rott ([20]) and Sears ([21]) pointed out that the back flow point defined as in (1.3)-(1.4), in general, is not a separation point in unsteady flow, and they concluded that separation occurs at the point of zero shear stress within the boundary layer, rather than on the surface as in the steady case, and it is a singular point of the flow. Since then, many people studied singularities in boundary layers, cf. [2, 9, 25] and references therein. Several rigorous results on the singularity formation of the classical Prandtl equation were given in [3, 8, 11]. Recently, in [27] we have obtained a result on the existence of a back flow point for the unsteady Prandtl equation, when the data satisfy the monotonicity assumption but with an adverse pressure gradient.

The remainder of this paper is arranged as follows: In Section 2, we construct the solution of the problem (1.1) by using the Crocco transformation. In Section 3, we study the existence of the back flow point for the boundary layer described by the problem (1.1), and conclude the results given in Theorem 1.2 by using a Lyapunov functional argument. In Appendix A, by multi-scale analysis we derive the boundary layer problem (1.1) from the Navier-Stokes-Coriolis equations in the small viscosity and Coriolis parameter limit, and in Appendix B, we construct the auxiliary function used in the barrier function in Lemma 3.2 for proving the first zero shear stress point being on the boundary.

2. Local well-posedness in the monotonic class

In this section, we study the local well-posedness of the boundary layer problem (1.1). In the monotonic class (1.2) of solutions, as in [15, 17], we introduce the following Crocco transformation

$$\tau = t, \quad \xi = x, \quad \eta = \frac{u}{U}, \quad \omega = \frac{u_y}{U}, \tag{2.1}$$

then from (1.1), we know that the new unknown $\omega(\tau,\xi,\eta)$ satisfies the following problem for a degenerate parabolic equation with a nonlocal term in $Q_T = \{0<\tau<T, 0<\xi<X, 0<\eta<1\}$:

$$\begin{cases} \omega_\tau + \eta U \omega_\xi + A\omega_\eta + B\omega - \omega^2 \omega_{\eta\eta} = 1-\eta, \\ \omega|_{\tau=0} = \omega_0, \quad \omega|_{\xi=0} = \omega_1, \\ \omega|_{\eta=1} = 0, \quad \left(\omega\omega_\eta - \int_0^1 \frac{1-\eta'}{\omega} d\eta' + C\right)|_{\eta=0} = 0, \end{cases} \tag{2.2}$$

where $\omega_i = \frac{u_{iy}}{U}$ $(i=0,1)$,

$$A = (1-\eta)\frac{U_t}{U} + (1-\eta^2)U_x - \int_\eta^1 \frac{1-\eta'}{\omega} d\eta', \quad B = \frac{U_t}{U} + \eta U_x, \quad C = U_x + \frac{U_t}{U}.$$

Similar to [15, 17], to solve the problem (2.2), we construct the approximation solutions via the following iteration scheme:

$$\begin{cases} \partial_\tau \omega^n + \eta U \partial_\xi \omega^n + A^{n-1} \partial_\eta \omega^n + B\omega^n - (\omega^{n-1})^2 \partial_\eta^2 \omega^n = 1-\eta, \\ \omega^n|_{\tau=0} = \omega_0, \quad \omega^n|_{\xi=0} = \omega_1, \\ \omega^n|_{\eta=1} = 0, \quad \left(\omega^{n-1}\partial_\eta \omega^n - \int_0^1 \frac{1-\eta'}{\omega^{n-1}} d\eta' + C\right)|_{\eta=0} = 0, \end{cases} \tag{2.3}$$

where $A^{n-1} = (1-\eta)\frac{U_t}{U} + (1-\eta^2)U_x - \int_\eta^1 \frac{1-\eta'}{\omega^{n-1}} d\eta'$. The existence of $\omega^n (n \geq 0)$ can be obtained by extending the domain, constructing the zeroth-order approximate solution and approximating the linear degenerate parabolic equation given in (2.3) by a non-degenerate one, which is similar to that given in [17, pp. 201-211], and we omit the details here.

Observing the structure of the problem (2.3), in order to obtain the convergence of $\{\omega^n\}_{n\geq 0}$, we shall study the uniform estimates of $\frac{\omega^n}{1-\eta}$ and its derivatives up to order two, which is different from the study given in [15, 17] on the estimate of ω^n in $C^2(Q_T)$ for the classical Prandtl equations.

To study the upper and lower bounds of the solution ω^n, denote by the lower barrier function

$$\underline{V} = m(2 - e^{-a_0\eta})(1-\eta)e^{-\alpha_0\tau}$$

and the upper barrier function

$$\overline{V} = M(1-\eta)e^{\beta_0\tau}.$$

As in [17, Lemma 4.3.2], we have the following result:

LEMMA 2.1. *Suppose that ω^n is a classical solution to the problem (2.3), then there exist $T>0, m>0, M>0$ and properly large $a_0, \alpha_0, \beta_0 > 0$ such that the inequalities*

$$\underline{V}(\tau,\xi,\eta) \leq \omega^n(\tau,\xi,\eta) \leq \overline{V}(\tau,\xi,\eta) \tag{2.4}$$

hold in $\overline{Q_T}$ for all $n \geq 0$.

This lemma can be proved in a way similar to that given in [17, Lemma 4.3.2] by using the maximal principle for the problem (2.3), we omit the details for simplicity.

To study the estimates of the first and second-order derivatives of $\frac{\omega^n}{1-\eta}$, let $V^n = \omega^n e^{\alpha\eta}$ with a positive constant α to be chosen later, then we know that V^n satisfies the following problem:

$$\begin{cases} L_n^0(V^n) + \widehat{B}^n V^n = -(1-\eta)e^{\alpha\eta}, \\ V^{n-1}(\partial_\eta V^n - \alpha V^n) - \int_0^1 \frac{1-\eta'}{V^{n-1}e^{-\alpha\eta'}} d\eta' + C = 0, \quad \text{on } \eta = 0, \end{cases} \tag{2.5}$$

with $L_n^0(V^n) = (\omega^{n-1})^2 \partial_\eta^2 V^n - \partial_\tau V^n - \eta U \partial_\xi V^n + \widehat{A}^n \partial_\eta V^n$, and

$$\widehat{A}^n = -(A^{n-1} + 2\alpha(\omega^{n-1})^2),$$
$$\widehat{B}^n = \alpha^2(\omega^{n-1})^2 + \alpha A^{n-1} - B.$$

Due to the nonlocal integration in $\widehat{A}^n$ and the boundary condition at $\eta = 0$, we study the first-order derivatives of $\frac{V^n}{1-\eta}$ with the aid of the following auxiliary functional

$$\Phi^n = (\frac{V_\tau^n}{1-\eta})^2 + (\frac{V_\xi^n}{1-\eta})^2 + (V_\eta^n)^2 + K_1^n\eta + K_0,$$

where K_0 and K_1^n are positive constants to be determined later, especially the dependence of K_1^n on n will be specified.

LEMMA 2.2. *For some T, K_0, $K_1^n>0$ and suitably large α, the functional Φ^n satisfies*

$$\begin{cases} L_n^0(\Phi^n)+R^n\Phi^n\geq 0, & \text{in } \ \mathrm{Q_T}, \\ \partial_\eta\Phi^n\geq\alpha\Phi^n-\frac{\alpha}{2}\Phi^{n-1}, & \text{on } \ \eta=0, \end{cases} \tag{2.6}$$

where K_1^n depends only on $\frac{\omega_\tau^{n-1}}{1-\eta}$ and $\frac{\omega_\xi^{n-1}}{1-\eta}$, and R^n depends on $\frac{\omega^{n-1}}{1-\eta}$ and its first-order derivatives.

Proof. Here, we divide the proof into three steps.

Step1. The derivation of equation for Φ^n: For $l\in\{\xi,\tau\}$, applying the operators $2\frac{V_l^n}{(1-\eta)^2}\partial_l$ and $2V_\eta\partial_\eta$ on the first equation given in (2.5), by a direct calculation, we know that the functional Φ^n satisfies

$$L_n^0(\Phi^n)-\widehat{A}^nK_1+2\widehat{B}^n\Phi^n-2\widehat{B}^n(K_1\eta+K_0)+I_1+I_2+I_3=0, \tag{2.7}$$

where

$$I_1=-2(\omega^{n-1})^2(\partial_\eta(\frac{V_\xi^n}{1-\eta}))^2-2(\omega^{n-1})^2(\partial_\eta(\frac{V_\tau^n}{1-\eta}))^2-2(\omega^{n-1})^2(V_{\eta\eta}^n)^2, \tag{2.8}$$

and

$$\begin{aligned} I_2=&-4(\omega^{n-1})^2\frac{V_\xi^n}{(1-\eta)^2}\partial_\eta(\frac{V_\xi^n}{1-\eta})+2\frac{V_\xi^n}{(1-\eta)^2}(\partial_\xi(\omega^{n-1})^2)\partial_\eta^2V^n+2V_\eta^nV_{\eta\eta}^n\partial_\eta(\omega^{n-1})^2\\ &-4(\omega^{n-1})^2\frac{V_\tau^n}{(1-\eta)^2}\partial_\eta(\frac{V_\tau^n}{1-\eta})+2\frac{V_\tau^n}{(1-\eta)^2}(\partial_\tau(\omega^{n-1})^2)\partial_\eta^2V^n,\\ I_3=&-2\frac{V_\xi^n}{(1-\eta)^2}\partial_\xi(\eta U)V_\xi^n+\frac{V_\xi^n}{(1-\eta)^2}\partial_\xi\widehat{A}^nV_\eta^n-2\frac{\widehat{A}^n}{1-\eta}(\frac{V_\xi^n}{1-\eta})^2-2\frac{V_\tau^n}{(1-\eta)^2}\partial_\tau(\eta U)V_\xi^n\\ &+\frac{V_\tau^n}{(1-\eta)^2}\partial_\tau\widehat{A}^nV_\eta^n-2\frac{\widehat{A}^n}{1-\eta}(\frac{V_\tau^n}{1-\eta})^2-2V_\eta^nU\partial_\xi V^n+2V_\eta^n\partial_\eta\widehat{A}^nV_\eta^n+(V_\eta^n)^2\\ &+2\frac{V_\xi^n}{(1-\eta)^2}\partial_\xi\widehat{B}^nV^n+2\frac{V_\tau^n}{(1-\eta)^2}\partial_\tau\widehat{B}^nV^n+2V_\eta^n\partial_\eta\widehat{B}^nV^n+2V_\eta^n\partial_\eta((1-\eta)e^{\alpha\eta}). \end{aligned}$$

Step 2. The estimates of I_2 and I_3. By using the Cauchy inequality, it is easy to know that there is a constant c_1 depending on $\frac{V^{n-1}}{1-\eta}$ and its derivatives such that for any $\delta>0$, we have

$$\begin{aligned} |I_2|\leq&\,\delta((\omega^{n-1})^2(\partial_\eta(\frac{V_\xi^n}{1-\eta}))^2+(\omega^{n-1})^2(\partial_\eta(\frac{V_\tau^n}{1-\eta}))^2+(\omega^{n-1})^2(V_{\eta\eta}^n)^2)\\ &+\frac{c_1}{\delta}((\frac{V_\tau^n}{1-\eta})^2+(\frac{V_\xi^n}{1-\eta})^2+(V_\eta^n)^2). \end{aligned} \tag{2.9}$$

Similarly, I_3 can be bounded by

$$|I_3|\leq c_2((\frac{V_\tau^n}{1-\eta})^2+(\frac{V_\xi^n}{1-\eta})^2+(V_\eta^n)^2)+c_3, \tag{2.10}$$

in which c_2 depends on $\frac{V^{n-1}}{1-\eta}$ and its derivatives, and c_3 depends only on $\frac{V^{n-1}}{1-\eta}$.

Thus, by plugging (2.8), (2.9) and (2.10) into (2.7) it follows that there is a large K_0 such that

$$L_n^0(\Phi^n)+R^n\Phi^n\geq 0, \tag{2.11}$$

holds, where R^n depends on $\frac{V^{n-1}}{1-\eta}$ and its derivatives.

Step 3. The boundary condition for Φ^n at $\eta=0$.

Obviously, one has that on $\{\eta=0\}$,

$$\Phi_\eta^n=2\frac{V_\tau^n}{1-\eta}\partial_\eta\left(\frac{V_\tau^n}{1-\eta}\right)+2\frac{V_\xi^n}{1-\eta}\partial_\eta\left(\frac{V_\xi^n}{1-\eta}\right)+2V_\eta^n\partial_\eta V_\eta^n+K_1^n:=J_1^\tau+J_1^\xi+J_2+K_1^n. \tag{2.12}$$

By using the equation and the boundary condition given in (2.3), there holds

$$\begin{aligned}
J_1^\tau|_{\eta=0}&=2V_\tau^n(\partial_\eta V_\tau^n+V_\tau^n)\\
&=2(V_\tau^n)^2+2V_\tau^n\Big(\alpha V_\tau^n-\frac{V_\tau^{n-1}}{(V^{n-1})^2}\big(\int_0^1\frac{1-\eta'}{\omega^{n-1}}d\eta'+C\big)\\
&\quad-\frac{1}{V^{n-1}}\big(\int_0^1\frac{(1-\eta')\omega_\tau^{n-1}}{(\omega^{n-1})^2}d\eta'-\partial_\tau C\big)\Big)\\
&\geq 2(\alpha+1)(V_\tau^n)^2-\frac{\alpha}{2}(V_\tau^n)^2-\frac{c_4}{\alpha}(V_\tau^{n-1})^2-\frac{c_5}{\alpha}\max\Big|\frac{\omega_\tau^{n-1}}{1-\eta}\Big|^2-c_6,\\
J_2|_{\eta=0}&=2\big(\alpha V^n+\frac{1}{V^{n-1}}\big(\int_0^1\frac{1-\eta'}{\omega^{n-1}}d\eta'-C\big)\big)\frac{1}{(\omega^{n-1})^2}\big(V_\tau^n-\widehat{A}^nV_\eta^n-\widehat{B}V^n-(1-\eta)e^{\alpha\eta}\big)\\
&\geq-\frac{\alpha}{2}(V_\tau^n)^2-c_7,
\end{aligned}$$

and J_1^ξ satisfies the same estimate as that of J_1^τ given above. Choose α large enough such that $\frac{c_4}{\alpha}\leq\frac{\alpha}{2}$ and $\frac{c_5}{\alpha}\leq\frac{1}{4}$, and K_1^n is given in the form

$$K_1^n=\widehat{K}_1+\frac{1}{4}\max\left(\Big|\frac{\omega_\tau^{n-1}}{1-\eta}\Big|^2+\Big|\frac{\omega_\xi^{n-1}}{1-\eta}\Big|^2\right) \tag{2.13}$$

then from (2.12) we get

$$\begin{aligned}
\Phi_\eta^n&\geq\alpha\Phi^n-\frac{\alpha}{2}\Phi^{n-1}-\alpha K_0-c_8+\widehat{K}_1\\
&\geq\alpha\Phi^n-\frac{\alpha}{2}\Phi^{n-1}
\end{aligned}$$

at the boundary $\eta=0$, therefore it implies that (2.6) holds, which completes the proof of this lemma. ∎

Similarly, to study the second-order derivatives of $\frac{V^n}{1-\eta}$, we introduce the following second -order functional Ψ^n:

$$\Psi^n=\sum_{\partial_{l_1},\partial_{l_2}\in\{\partial_\tau,\partial_\xi\}}(\partial_{l_2}\partial_{l_2}\frac{V^n}{1-\eta})^2+\sum_{\partial_l\in\{\partial_\tau,\partial_\xi\}}(\partial_lV_\eta^n)^2+(V_{\eta\eta}^n)^2+N_1^n\eta+N_0,$$

where N_0 and N_1^n are positive constants to be determined later.

LEMMA 2.3. *There exist some $N_0>0$, and $N_1^n>0$ depending on the second-order derivative of $\frac{\omega^{n-1}}{1-\eta}$ with respect to τ and ξ, such that Ψ^n satisfies:*

$$\begin{cases} L_n^0(\Psi^n)+C^n\Psi^n+D^n\geq 0, & \text{in } \mathrm{Q_T}, \\ \partial_\eta\Psi^n\geq\alpha\Psi^n-\frac{\alpha}{2}\Psi^{n-1}, & \text{on } \eta=0, \end{cases} \tag{2.14}$$

where C^n depends on $\frac{\omega^{n-1}}{1-\eta}$ and its first and second-order derivatives, and D^n depends on $\frac{\omega^n}{1-\eta}$, $\frac{\omega^{n-1}}{1-\eta}$, $\frac{\omega^{n-2}}{1-\eta}$ and their first-order derivatives.

Proof. The first inequality given in (2.14) can be obtained in the same way as given in the proof of Lemma 2.2, so we omit the detailed calculation. To verify the second inequality given in (2.14), by definition

$$\partial_\eta\Psi^n=2\frac{V_{l_1l_2}^n}{1-\eta}\partial_\eta\left(\frac{V_{l_1l_2}^n}{1-\eta}\right)+2V_{l\eta}^n\partial_\eta V_{l\eta}^n+2V_{\eta\eta}^n\partial_\eta V_{\eta\eta}^n+N_1^n=:J_3+J_4+J_5+N_1^n.$$

By using the equation and the boundary condition given in (2.3), there are constants c_j $(9\leq j\leq 13)$ depending on Φ^i $(n-2\leq i\leq n)$, such that it holds

$$\begin{aligned} J_3|_{\eta=0}&=2V_{l_1l_2}^n(V_{l_1l_2}^n+\partial_\eta V_{l_1l_2}^n)\\ &\geq 2(\alpha+1)(V_{l_1l_2}^n)^2-\frac{\alpha}{2}(V_{l_1l_2}^n)^2-\frac{c_9}{\alpha}(V_{l_1l_2}^{n-1})^2-\frac{c_{10}}{\alpha}\max\left|\frac{\omega_{l_1l_2}^n}{1-\eta}\right|^2-c_{11},\\ J_4|_{\eta=0}&\geq-\frac{1}{2}(V_{l\tau}^n)^2-c_{12},\\ J_5|_{\eta=0}&\geq-c_{13}. \end{aligned}$$

Next, let α be large enough such that $\frac{c_9}{\alpha}\leq\frac{\alpha}{2}$ and $\frac{c_{10}}{\alpha}\leq\frac{1}{4}$, and choose N_1^n in the form

$$N_1^n=\widehat{N}_1+\frac{1}{4}\max\left|\frac{\omega_{l_1l_2}^{n-1}}{1-\eta}\right|^2$$

with $\widehat{N}_1$ properly large, there holds

$$\begin{aligned} \partial_\eta\Psi^n&\geq\alpha\Psi^n-\frac{\alpha}{2}\Psi^{n-1}-\alpha N_0-c_{14}+\widehat{N}_1\\ &\geq\alpha\Psi^n-\frac{\alpha}{2}\Psi^{n-1}, \end{aligned}$$

from which the second inequality of (2.14) follows. □

LEMMA 2.4. *For some $T>0$ and the solution ω^n of the problem (2.3) in Q_T, we have that $\frac{\omega^n}{1-\eta}$ and its derivatives up to order two are bounded uniformly in $n\geq 1$.*

Proof. It suffices to show that there exist M_1, M_2 and $T>0$ such that $\Phi^k\leq M_1$, $\Psi^k\leq M_2$ for all $k\geq 0$. We prove it by induction, assume that it holds for all $0\leq k\leq n-1$.

Let $\Phi_1^n=\Phi^n e^{-\gamma\tau}$ with γ large enough such that $R^n-\gamma\leq -1$ in Q_T. According to Lemma 2.2, it follows that Φ_1^n satisfies

$$L_n^0(\Phi_1^n)+(R^n-\gamma)\Phi_1^n\geq 0, \quad \text{in } \mathrm{Q_T}, \tag{2.15}$$

and

$$\partial_\eta\Phi_1^n\geq\alpha\Phi_1^n-\frac{\alpha}{2}\Phi_1^{n-1}, \quad \text{on } \eta=0.$$

Therefore, by applying the maximal principle to the Equation (2.15), Φ_1^n can only attain its positive maximum at either $\tau=0$, $\xi=0$ or $\eta=0$.

If it happens at either $\tau=0$ or $\xi=0$, then we know from (2.13) that

$$\Phi_1^n=\Phi^n e^{-\gamma\tau}\le\Phi^n<k_1+\frac{1}{4}M_1,$$

where the constant k_1 is determined only by the initial and boundary data.

If Φ_1^n attains its positive maximum on $\overline{Q_T}$ at $\eta=0$, it follows from the boundary condition in (2.6) that at this point,

$$\Phi_1^n\le\frac{1}{2}\Phi_1^{n-1}\le\frac{1}{2}M_1.$$

In summary, we conclude that for all $(\tau,\xi,\eta)\in\overline{Q_T}$,

$$\Phi_1^n(\tau,\xi,\eta)\le\max\{\frac{1}{2}M_1,k_1+\frac{1}{4}M_1\}.$$

Thus, $\Phi^n\le M_1$ holds on $\overline{Q_{T_1}}$ if we initially choose $M_1=4k_1$ and $e^{\gamma T_1}=2$.

Similarly, by using Lemma 2.3 we can obtain that there is $T_2>0$ such that $\Psi^n\le M_2$ holds on $\overline{Q_{T_2}}$. Finally, by taking $T=\min\{T_1,T_2\}>0$, then $\Phi^n\le M_1$ and $\Psi^n\le M_2$ hold on $\overline{Q_T}$ for all $n\ge1$. □

Based on the above result, we are ready to give the proof of Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** Since the Crocco transformation is invertible in the monotonic class (1.2), it suffices to prove the well-posedness of the problem (2.2), the reverse from ω to u refers to [17].

To show the convergence of ω^n, let us introduce

$$\theta^n=\frac{\omega^n}{1-\eta},\quad\text{and}\quad P^n=(\theta^n-\theta^{n-1})e^{-\alpha\tau+\beta\eta}$$

with the parameters α and β to be chosen later.

Denote $\widehat{A}^{n-1}=A^{n-1}+2\frac{(\omega^{n-1})^2}{1-\eta}$ and $\widehat{B}^{n-1}=B-\frac{A^{n-1}}{1-\eta}$, then from (2.3), we know that P^n satisfies

$$(\omega^{n-1})^2\partial_\eta^2P^n-\partial_\tau P^n-\eta U\partial_\xi P^n-(2\beta(\omega^{n-1})^2+\widehat{A}^{n-1})\partial_\eta P^n-\Lambda P^n+R(P^{n-1})=0, \tag{2.16}$$

where

$$\begin{aligned}\Lambda=&-\beta^2(\omega^{n-1})^2+\alpha-\beta\widehat{A}^{n-1}+\widehat{B}^{n-1},\\ R(P^{n-1})=&(1-\eta)(\omega^{n-1}+\omega^{n-2})\partial_\eta^2\theta^{n-1}P^{n-1}+\frac{\theta^{n-1}}{1-\eta}\int_\eta^1\frac{(1-\eta')^2}{\omega^{n-1}\omega^{n-2}}P^{n-1}e^{-\beta\eta'+\beta\eta}d\eta'\\ &-(\int_\eta^1\frac{(1-\eta')^2}{\omega^{n-1}\omega^{n-2}}P^{n-1}e^{-\beta\eta'+\beta\eta}d\eta'+2(\omega^{n-1}+\omega^{n-2})P^{n-1})\partial_\eta\theta^{n-1}.\end{aligned}$$

In addition, at $\eta=0$, we also have that

$$\partial_\eta P^n-SP^n+K(P^{n-1})=0, \tag{2.17}$$

where $S=\frac{\beta\omega^{n-1}+\theta^{n-1}}{\omega^{n-1}}$ and $K(P^{n-1})=\frac{(1-\eta)\partial_\eta\theta^{n-1}-\theta^{n-1}}{\omega^{n-1}}P^{n-1}+\int_0^1\frac{P^{n-1}e^{-\beta\eta'}}{\theta^{n-1}\theta^{n-2}}d\eta'$.

Now we choose proper constants α and β, such that the inequalities

$$\Big|\frac{R(P^{n-1})}{\Lambda}\Big|\leq q\max_{\overline{Q_T}}|P^{n-1}|,\qquad \Big|\frac{K(P^{n-1})}{S}\Big|\leq q\max_{\overline{Q_T}}|P^{n-1}|$$

hold for some positive constant $0<q<1$. Therefore, by noting the above estimates and the fact that $P^n=0$ at $\tau=0$ and $\xi=0$, we apply the maximal principle for the problem (2.16)-(2.17) to obtain

$$\max_{\overline{Q}_T}|P^n|\leq q\max_{\overline{Q}_T}|P^{n-1}|.$$

This implies that the series $\sum\limits_{n=1}^{\infty}P^n$ is uniformly convergent.

On the other hand, due to the uniform boundedness of the first and second-order derivatives of $\frac{\omega^n}{1-\eta}$, one has

$$\omega^n\to\omega \ \text{ uniformly in } \ C^1(\overline{Q}_T), \ \text{ as } \ \mathrm{n}\to+\infty.$$

Meanwhile, it follows from the equation given in (2.3) that $\omega^n_{\eta\eta}$ also converges uniformly to $\omega_{\eta\eta}$ for any $\eta<1$. Hence, ω is a classical solution to (2.2).

Similarly we can obtain the uniqueness of the solution to the problem (2.2) by considering the difference $\theta=\frac{\omega-\tilde{\omega}}{1-\eta}$ of two solutions ω and $\tilde{\omega}$. It ends the proof of this theorem. ∎

3. Back flow of the boundary layer

In this section, we investigate the effect of the integral term $\int_{+\infty}^{y}(U-u)dy'$ on the formation of the vanishing shear stress point during the evolution of the solution of (1.1) in the monotonic class, which leads to the back flow phenomenon ([27]) after this point.

For given $X>0$ and $T>0$, we assume that the boundary layer problem admits a classical solution in the monotonic class $u_y(t,x,y)>0$ for all $(t,x,y)\in\{0\leq t\leq T,0\leq x\leq X,0\leq y<+\infty\}$, otherwise the vanishing shear stress point has already occurred.

Comparing the system (2.2) with the model derived from the Prandtl equation, the integrations in the equation and the boundary condition bring us new difficulties in exploring the features of the solution, especially when using the maximal principle. In addition, there is a non-negative source term $1-\eta$ in the equation, formally, it may prevent the appearance of the back flow. However, if we take $V=C(t,x)(1-\eta)$ as a barrier function, then this source term can be exactly balanced by the term

$$-\int_{\eta}^{1}\frac{1-\eta'}{V}d\eta' V_\eta.$$

Hence, one could not simply view it as a source term. On the other hand, formally from the boundary condition given in (2.2) one may have $\omega\omega_\eta(\tau,\xi,0)>0$ if $\frac{\omega}{1-\eta}$ is sufficiently small, this could lead to the appearance of back flow in boundary layer. For this reason, let us first give an upper bound result of ω in $\overline{Q_T}=\{0\leq\tau\leq T,0\leq\xi\leq X,0\leq\eta\leq 1\}$.

LEMMA 3.1. *For given X, $T>0$, and a fixed*

$$\lambda>\|\frac{U_\tau}{U}+\eta U_\xi\|_{L^\infty([0,T]\times[0,X])}+\|\frac{U_\tau}{U}+(1+\eta)U_\xi\|_{L^\infty([0,T]\times[0,X])},\tag{3.1}$$

if there is positive constant k satisfying

$$\omega_0(\xi,\eta)<k(1-\eta),\quad \omega_1(\tau,\eta)e^{-\lambda\tau}<k(1-\eta) \tag{3.2}$$

and

$$\frac{U_\tau+UU_\xi}{U}<\frac{e^{-\lambda\tau}}{k} \tag{3.3}$$

for all $0\leq\eta<1$, $0\leq\tau\leq T$ and $0\leq\xi\leq X$, then the inequalities

$$\omega(\tau,\xi.\eta)e^{-\lambda\tau}\leq k(1-\eta)\quad\text{and}\quad \omega\omega_\eta(\tau,\xi,0)>0 \tag{3.4}$$

hold for the classical solution ω of (2.2) in Q_T.

Proof.

(1) If the first assertion given in (3.4) is true, then from the boundary condition given in (2.2) we have that

$$\omega\omega_\eta(\tau,\xi,0)=\int_0^1\frac{1-\eta'}{\omega(\tau,\xi,\eta')}d\eta'-\frac{UU_\xi+U_\tau}{U}\geq\frac{e^{-\lambda\tau}}{k}-\frac{UU_\xi+U_\tau}{U}>0 \tag{3.5}$$

holds by using (3.3). Thus, the second assertion given in (3.4) holds.

(2) We shall use the maximal principle to prove the first inequality of (3.4). Setting $\tilde\omega=\omega e^{-\lambda\tau}$ with λ being given in (3.1), from (2.2) we know that $\tilde\omega$ satisfies

$$\tilde\omega_\tau+\eta U\tilde\omega_\xi+(\tilde A-\int_\eta^1\frac{1-\eta'}{\tilde\omega}d\eta'e^{-\lambda\tau})\tilde\omega_\eta+(B+\lambda)\tilde\omega-\omega^2\tilde\omega_{\eta\eta}=(1-\eta)e^{-\lambda\tau}, \tag{3.6}$$

where $\tilde A=(1-\eta)\frac{U_\tau}{U}+(1-\eta^2)U_\xi$.

Denote by

$$L_1(\Phi):=\Phi_\tau+\eta U\Phi_\xi+(\tilde A-\int_\eta^1\frac{1-\eta'}{\Phi}d\eta'e^{-\lambda\tau})\Phi_\eta+(B+\lambda)\Phi-\omega^2\Phi_{\eta\eta}. \tag{3.7}$$

Setting $F(\tau,\xi,\eta)=k(1-\eta)$, with k being given in (3.2), it is easy to have

$$L_1(F)=(1-\eta)e^{-\lambda\tau}+k(B+\lambda-\frac{\tilde A}{1-\eta})(1-\eta)>(1-\eta)e^{-\lambda\tau}=L_1(\tilde\omega) \tag{3.8}$$

for $0\leq\eta<1$, by noting that $\lambda>|B|_{L^\infty}+|\frac{\tilde A}{1-\eta}|_{L^\infty}$.

If the first inequality of (3.4) does not hold in Q_T, then $\tilde\omega-F>0$ at some point in the domain Q_T. From (3.2), we know that

$$(\tilde\omega-F)\big|_{\eta=1}=0,\quad (\tilde\omega-F)\big|_{\tau=0}<0,\quad (\tilde\omega-F)\big|_{\xi=0}<0,$$

then by continuity, there must be a first time $\tau_0>0$ and the smallest $\xi_0>0$, and $0\leq\eta_0<1$ such that

$$(F-\tilde\omega)(\tau_0,\xi_0,\eta_0)=0,\quad F-\tilde\omega\geq0 \tag{3.9}$$

for all points in $\{0\leq\tau\leq\tau_0,0\leq\xi\leq\xi_0,0\leq\eta<1\}$. On the other hand, from the computation given in (3.5) and using (3.3), (3.9) we have $\omega_\eta(\tau_0,\xi_0,0)>0$. Obviously, $F_\eta=-k<0$, so $F-\tilde\omega$ is strictly decreasing in η at $(\tau_0,\xi_0,0)$. Thus, from (3.9) we get $0<\eta_0<1$, and

$$\begin{cases}\eta_0 \text{ is a minimal point of } (F-\tilde\omega)(\tau_0,\xi_0,\eta),\\ \partial_\tau(F-\tilde\omega)(\tau_0,\xi_0,\eta_0)\leq0,\quad \partial_\xi(F-\tilde\omega)(\tau_0,\xi_0,\eta_0)\leq0.\end{cases} \tag{3.10}$$

Consequently, it follows that

$$L_1(F)-L_1(\tilde{\omega})=(F-\tilde{\omega})_\tau+\eta U(F-\tilde{\omega})_\xi+(A-\int_\eta^1\frac{1-\eta'}{\tilde{\omega}}d\eta' e^{-\lambda\tau})(F-\tilde{\omega})_\eta$$
$$+(B+\lambda)(F-\tilde{\omega})+\int_\eta^1\frac{(1-\eta')(F-\tilde{\omega})}{\tilde{\omega}F}d\eta' e^{-\lambda\tau}F_\eta-\omega^2(F-\tilde{\omega})_{\eta\eta}\leq 0 \qquad (3.11)$$

at (τ_0,ξ_0,η_0), which is a contradiction to (3.8). Hence $F\geq\tilde{\omega}$ in Q_T, which ends the proof of this lemma. ∎

Remark 3.1.

(1) The conditions given in (3.2) and (3.3) immediately follow from the assumptions given in Theorem 1.2.

(2) If the outer flow satisfies

$$U(\tau,\xi)>0,\quad U_\tau+UU_\xi\leq 0,\quad \forall 0\leq\tau\leq T, 0\leq x\leq X, \qquad (3.12)$$

then (3.3) does not give any constraint on $k>0$. Otherwise, if (3.12) is changed as

$$U(\tau,\xi)>0,\quad \max_{0\leq\tau\leq T,0\leq x\leq X}(U_\tau+UU_\xi)>0, \qquad (3.13)$$

then the conditions given in (3.2)-(3.3) imply that the initial, boundary data and the outer flow of the problem (2.2) should satisfy the constraint

$$\max_{\overline{Q}_T}\left(\frac{\omega_0(\xi,\eta)}{1-\eta},\frac{\omega_1(\tau,\eta)}{1-\eta}e^{-\lambda\tau}\right)<\frac{Ue^{-\lambda\tau}}{\max\limits_{[0,T]\times[0,X]}(U_\tau+UU_\xi)}. \qquad (3.14)$$

Next, we will make use of the smallness of ω in Q_T and the reverse force $\int_0^1\frac{1-\eta}{\omega}d\eta$ to prove the existence of the vanishing shear stress point. First of all, by developing the idea given in [27], we have the following result.

Lemma 3.2. *Under the same assumption as given in Lemma 3.1, the first vanishing shear stress point $(\tau_\star,\xi_\star,\eta_\star)$, i.e.*

$$\omega(\tau_\star,\xi_\star,\eta_\star)=0,\quad \omega(\tau,\xi,\eta)>0,\quad \forall 0\leq\tau<\tau_*, 0\leq\xi\leq X, 0\leq\eta<1, \qquad (3.15)$$

can only occur at the physical boundary, i.e. $\eta_\star=0$.

Proof.

(1) If the conclusion is not true, it means there is an interior point, $0<\eta_\star<1$, such that (3.15) holds. Then, the Crocco transformation (2.1) is invertible in the region $Q_\star=\{0<\tau<\tau_\star,0<\xi<X,0<\eta<1\}$.

(2) In order to use the comparison principle for the problem (2.2) after the Crocco transformation, fix three different points η_i (i=1, 2, 3) with $0<\eta_1<\eta_2<\eta_3<1$, and choose a non-negative function $\phi(\eta)$ satisfying

1) $\phi(\eta)=\alpha\eta$ for $0\leq\eta\leq\eta_1$,

2) $\phi'(\eta)\geq 0$ for $\eta\in[\eta_1,\eta_2]$,

3) $\int_\eta^1\frac{1-\eta'}{\phi(\eta')}d\eta'\phi'(\eta)=\sigma(\eta)(\eta-1)$ for $\eta\in[\eta_2,\eta_3]$,

4) $\phi(\eta)=1-\eta$ for $\eta\in[\eta_3,1]$,

for a positive constant $\alpha>0$, where $\sigma(\eta)$ is a smooth function defined on $[\eta_2,\eta_3]$ satisfying $0\le\sigma(\eta)\le 1$ and $\sigma(\eta_2)=0$. The construction of $\phi(\eta)$ satisfying the above third condition shall be given in Appendix B.

We choose a barrier function $F(\tau,\eta)=\varepsilon\phi(\eta)e^{-M\tau}$, with the parameters ε small and $M>0$ large to be determined later, such that

$$\omega_0>F(0,\eta),\quad \omega_1 e^{-\lambda\tau}>F(\tau,\eta),\quad \text{for } 0\le\tau\le\tau_\star,\eta\in[0,1). \tag{3.16}$$

We are going to prove that there is $M>0$ large such that

$$\tilde\omega\ge F \tag{3.17}$$

in $Q_\star$, with $\tilde\omega=\omega e^{-\lambda\tau}$. Letting $\tau\to\tau_\star^-$, $\xi\to\xi_\star$ and $\eta\to\eta_\star$, from (3.17) it yields

$$\tilde\omega(\tau_\star,\xi_\star,\eta_\star)\ge\varepsilon\phi(\eta_\star)e^{-M\tau_*}>0. \tag{3.18}$$

This is a contradiction to the assumption $\omega(\tau_\star,\xi_\star,\eta_\star)=0$.

(3) Define the operator L_2 by

$$\begin{aligned}L_2\psi:=&\psi_\tau+\eta U\psi_\xi+\tilde A\psi_\eta+(B+\lambda)\psi-\omega^2\psi_{\eta\eta}\\&-\Big(\chi_{[0,\eta_2]}(\eta)\int_\eta^1\frac{1-\eta'}{\tilde\omega}d\eta'+\chi_{(\eta_2,1]}(\eta)\int_\eta^1\frac{1-\eta'}{\psi}d\eta'\Big)e^{-\lambda\tau}\psi_\eta\end{aligned}$$

with $\tilde A$ being given in (3.6), $\chi_{[0,\eta_2]}(\eta)$ and $\chi_{[\eta_2,1]}(\eta)$ being the characteristic functions on $[0,\eta_2]$ and $(\eta_2,1]$, respectively.

In the next point (4), we shall determine a large $M>0$ such that

$$\begin{aligned}L_2F-L_2\tilde\omega=&(F-\tilde\omega)_\tau+\eta U(F-\tilde\omega)_\xi+A(F-\tilde\omega)_\eta+(B+\lambda)(F-\tilde\omega)-\omega^2(F-\tilde\omega)_{\eta\eta}\\&-[\chi_{[0,\eta_2]}(\eta)\int_\eta^1\frac{1-\eta'}{\tilde\omega}d\eta'+\chi_{(\eta_2,1]}(\eta)\int_\eta^1\frac{1-\eta'}{\tilde\omega}d\eta']e^{-\lambda\tau}(F-\tilde\omega)_\eta\\&+\chi_{(\eta_2,1]}(\eta)\int_\eta^1\frac{(1-\eta')(F-\tilde\omega)}{F\tilde\omega}d\eta' e^{-\lambda\tau}F_\eta\\<&0\end{aligned} \tag{3.19}$$

in Q_*.

Now, assume that the assertion (3.17) fails. From (3.16), we suppose that the first zero point of $\tilde\omega-F$ in Q_* is (τ_0,ξ_0,η_0), and

$$\begin{cases}\tilde\omega(\tau_0,\xi_0,\eta)-F(\tau_0,\eta)\ge0,\quad \forall 0\le\eta\le1,\\ \tilde\omega(\tau,\xi,\eta)-F(\tau,\eta)>0,\quad \forall 0\le\tau<\tau_0,0\le\xi\le X,0\le\eta<1.\end{cases}$$

By noting from (3.4) that $\tilde\omega(\tau,\xi,0)>0$, one has $0<\eta_0<1$. Thus, η_0 is the minimal point of $\tilde\omega(\tau_0,\xi_0,\eta)-F(\tau_0,\eta)$ on $[0,1]$.

Noting $F_\eta\le0$ in $[\eta_2,1]$, from the expression of $L_2F-L_2\tilde\omega$ given in (3.19) we have

$$L_2F-L_2\tilde\omega\ge0,\quad \text{at } (\tau_0,\xi_0,\eta_0)$$

which is a contradiction to (3.19). Thus, the inequality (3.17) holds in Q_* when (3.19) is true.

(4) It remains to verify the inequality (3.19) by choosing M properly large.

First, from (3.6) we know that $\tilde{\omega}=\omega e^{-\lambda\tau}$ satisfies:

$$L_2\tilde{\omega}=(1-\eta)e^{-\lambda\tau}, \tag{3.20}$$

For $\eta\in[0,\eta_1]$, by a direct calculation, one has

$$\begin{aligned} L_2F &= \varepsilon\alpha e^{-M\tau}(\tilde{A}+(B+\lambda-M)\eta-\int_\eta^1\frac{1-\eta'}{\tilde{\omega}}d\eta' e^{-\lambda\tau}) \\ &= \varepsilon\alpha e^{-M\tau}(C+(\lambda-M)\eta-\int_\eta^1\frac{1-\eta'}{\tilde{\omega}}d\eta' e^{-\lambda\tau})<0 \end{aligned} \tag{3.21}$$

when $M\geq\lambda$, by using (3.2) and (3.4).

For $\eta\in[\eta_1,\eta_2]$, by definition, one has

$$\begin{aligned} L_2F &= \varepsilon e^{-M\tau}(\tilde{A}\phi'(\eta)+(B+\lambda-M)\phi-\omega^2\phi''-\int_\eta^1\frac{1-\eta'}{\tilde{\omega}}d\eta' e^{-\lambda\tau}\phi'(\eta)) \\ &\leq \varepsilon e^{-M\tau}[\tilde{A}\phi'(\eta)+(B+\lambda-M)\phi-\omega^2\phi'']<0, \end{aligned} \tag{3.22}$$

when M is large enough, by using $\phi'(\eta)\geq 0$ and $\phi(\eta)\geq\alpha\eta_1>0$ on $[\eta_1,\eta_2]$.

For $\eta\in[\eta_2,\eta_3]$, by a direct calculation it follows that

$$\begin{aligned} L_2F &= \varepsilon e^{-M\tau}[\tilde{A}\phi'(\eta)+(B+\lambda-M)\phi-\omega^2\phi'']-\int_\eta^1\frac{1-\eta'}{\phi}d\eta'\phi'(\eta)e^{-\lambda\tau} \\ &< -\int_\eta^1\frac{1-\eta'}{\phi}d\eta'\phi'(\eta)e^{-\lambda\tau}=\sigma(\eta)(1-\eta)e^{-\lambda\tau}\leq L_2\tilde{\omega}, \end{aligned} \tag{3.23}$$

when M is large enough, by using that ϕ is bounded from below and $0\leq\sigma(\eta)\leq 1$ on $[\eta_2,\eta_3]$.

When $\eta\in[\eta_3,1)$, since $\phi(\eta)=1-\eta$, we have

$$\begin{aligned} L_2F &= \varepsilon e^{-M\tau}[-\tilde{A}+(B+\lambda-M)(1-\eta)]+(1-\eta)e^{-\lambda\tau} \\ &< L_2\tilde{\omega}, \end{aligned} \tag{3.24}$$

when M is large enough.

Summarizing (3.20)-(3.24), we get the conclusion (3.19). It ends the proof of this lemma. □

Based on the above two lemmas, we are ready to prove the back flow result given in Theorem 1.2. The proof idea is inspired from the Lyapunov functional approach given in [27] for the classical Prandtl equation.

Proof. **(Proof of Theorem 1.2.)** Assume that ω is a classical solution of the problem (2.2) in Q_T, introduce the Lyapunov functional for $0\leq\tau\leq T$,

$$G(\tau)=\int_0^X\int_0^1\frac{\psi(\xi)}{\sqrt{\omega^2(\tau,\xi,\eta)+\eta^2}}d\xi d\eta \tag{3.25}$$

where $\psi(\xi)=(X-\xi)^{\frac{3}{2}}$. Due to the invertibility of the Crocco transformation, it suffices to prove that the functional $G(\tau)$ will blow up in a finite time under the assumptions.

Denoting $W(\tau,\xi,\eta)=\frac{1}{\sqrt{\omega^2+\eta^2}}$, from (2.2) we know that W satisfies

$$W_\tau+\eta UW_\xi+(\tilde{A}-\int_\eta^1\frac{1-\eta'}{\omega}d\eta')W_\eta=BW+\eta(\int_\eta^1\frac{1-\eta'}{\omega}d\eta'-\frac{U_\tau+UU_\xi}{U})W^3 \\ -\frac{\omega^3}{(\omega^2+\eta^2)^{\frac{3}{2}}}\omega_{\eta\eta}-\frac{(1-\eta)\omega}{(\omega^2+\eta^2)^{\frac{3}{2}}}. \tag{3.26}$$

Therefore, one has

$$\frac{dG(\tau)}{d\tau}=\int_0^X\int_0^1\Big\{-\eta UW_\xi-[(\tilde{A}-\int_\eta^1\frac{1-\eta'}{\omega}d\eta')W_\eta-BW]+\eta(\int_\eta^1\frac{1-\eta'}{\omega}d\eta' \\ -\frac{U_\tau+UU_\xi}{U})W^3-\frac{\omega^3}{(\omega^2+\eta^2)^{\frac{3}{2}}}\omega_{\eta\eta}-\frac{(1-\eta)\omega}{(\omega^2+\eta^2)^{\frac{3}{2}}}\Big\}\psi(\xi)d\xi d\eta \\ :=\sum_{i=0}^5 I_i. \tag{3.27}$$

Now, it remains to estimate the five terms given on the right-hand side of (3.27) one by one. The main idea is to prove the blowup of G in virtue of the cubic term W^3 in the equation. Note that the coefficient of the cubic term is only positive for η away from the upper bound $\eta=1$. However, by the definition of W, it is controllable for $\eta\geq\delta$ for any $\delta>0$.

For this reason, we can estimate the cubic term in the following way:

$$I_3=\int_0^X(\int_\delta^1+\int_0^\delta)\eta(\int_\eta^1\frac{1-\eta'}{\omega}d\eta'-\frac{U_\tau+UU_\xi}{U})W^3\psi(\xi)d\xi d\eta \\ =\int_0^X\int_0^\delta\eta(\int_\eta^1\frac{1-\eta'}{\omega}d\eta'-\frac{U_\tau+UU_\xi}{U})W^3\psi(\xi)d\xi d\eta-c_1(\tau,X) \\ \geq\frac{e^{-\lambda\tau}}{k}\int_0^X\int_0^\delta\eta(2\delta-\eta)W^3\psi(\xi)d\xi d\eta-c_1(\tau,X), \tag{3.28}$$

by using (1.9), with

$$c_1(\tau,X)=\int_0^X\int_\delta^1\eta(\int_\eta^1\frac{1-\eta'}{\omega}d\eta'-\frac{U_\tau+UU_\xi}{U})W^3\psi(\xi)d\xi d\eta.$$

For the term I_1 given in (3.27), by using integrating by parts,

$$I_1=\int_0^1\eta U(\tau,0)W(\tau,0,\eta)\psi(0)d\eta+\int_0^X\int_0^1\eta W(U_\xi\psi(\xi)+U\psi'(\xi))d\xi d\eta \\ \geq-c_2(\tau,X)G(\tau)+c_3(\tau,X)+\int_0^X\int_0^\delta\eta WU\psi'(\xi)d\xi d\eta \\ \geq-(\int_0^X\int_0^\delta\eta(2\delta-\eta)W^3\psi(\xi)d\xi d\eta)^{\frac{1}{3}}(\int_0^X\int_0^\delta\eta(2\delta-\eta)^{-\frac{1}{2}}U^{\frac{3}{2}}\frac{|\psi'(\xi)|^{\frac{3}{2}}}{\psi^{\frac{1}{2}}}d\xi d\eta)^{\frac{2}{3}} \\ -c_2(\tau,X)G(\tau)+c_3(\tau,X) \\ \geq-\frac{e^{-\lambda\tau}}{2k}\int_0^X\int_0^\delta\eta(2\delta-\eta)W^3\psi(\xi)d\xi d\eta-c_2(\tau,X)G(\tau)+c_3(\tau,X)-c_4(\tau,X),$$

with $c_2(\tau,X)=\max_{0\le\eta\le1,0\le\xi\le X}|\eta U_\xi|$,

$$c_3(\tau,X)=\int_0^X\int_\delta^1 \eta WU\psi'(\xi)d\xi d\eta$$

and

$$c_4(\tau,X)=(\frac{8ke^{\lambda\tau}}{27})^{\frac{1}{2}}\int_0^X\int_0^\delta \eta(2\delta-\eta)^{-\frac{1}{2}}U^{\frac{3}{2}}\frac{|\psi'(\xi)|^{\frac{3}{2}}}{\psi^{\frac{1}{2}}}d\xi d\eta.$$

For the term I_2 of (3.27), one has

$$\begin{aligned}I_2=&\int_0^X\big((\tilde{A}-\int_\eta^1\frac{1-\eta'}{\omega}d\eta')W\big)\big|_{\eta=0}\psi(\xi)d\xi+\int_0^X\int_0^1(\tilde{A}_\eta+B+\frac{1-\eta}{\omega})W\psi(\xi)d\xi d\eta\\ =&-\int_0^X\omega_\eta(\tau,\xi,0)\psi(\xi)d\xi+\int_0^X\int_0^1(\frac{1-\eta}{\omega}-\eta U_\xi)W\psi(\xi)d\xi d\eta\\ \ge&-\int_0^X\omega_\eta(\tau,\xi,0)\psi(\xi)d\xi+\int_0^X\int_0^1\frac{1-\eta}{\omega}W\psi(\xi)d\xi d\eta-c_2(\tau,X)G\end{aligned}$$

and

$$\begin{aligned}I_4=&\int_0^X\omega_\eta(\tau,\xi,0)\psi(\xi)d\xi+3\int_0^X\int_0^1\frac{\eta^2\omega^2\omega_\eta^2-\eta\omega^3\omega_\eta}{(\omega^2+\eta^2)^{\frac{5}{2}}}\psi(\xi)d\xi d\eta\\ \ge&\int_0^X\omega_\eta(\tau,\xi,0)\psi(\xi)d\xi+\frac{3}{2}\int_0^X\int_0^1\frac{\eta^2\omega^2\omega_\eta^2-\omega^4}{(\omega^2+\eta^2)^{\frac{5}{2}}}\psi(\xi)d\xi d\eta\\ \ge&\int_0^X\omega_\eta(\tau,\xi,0)\psi(\xi)d\xi+\frac{3}{2}\int_0^X\int_0^1\frac{\eta^2\omega^2\omega_\eta^2}{(\omega^2+\eta^2)^{\frac{5}{2}}}\psi(\xi)d\xi d\eta-\frac{3}{2}G(\tau).\end{aligned}$$

Combining the above four estimates for I_i (i=1, 2, 3, 4) and (3.27), it leads to

$$\begin{aligned}\frac{dG}{d\tau}\ge&\frac{e^{-\lambda\tau}}{2k}\int_0^X\int_0^\delta\eta(2\delta-\eta)W^3\psi(\xi)d\xi d\eta+\int_0^X\int_0^1\frac{1-\eta}{\omega}W\psi(\xi)d\xi d\eta\\ &-\int_0^X\int_0^1\frac{(1-\eta)\omega}{(\omega^2+\eta^2)^{\frac{3}{2}}}\psi(\xi)d\xi d\eta-(2c_2(\tau)+\frac{3}{2})G-(c_1(\tau)-c_3(\tau)+c_4(\tau))\\ =&\frac{e^{-\lambda\tau}}{2k}\int_0^X\int_0^\delta\eta(2\delta-\eta)W^3\psi(\xi)d\xi d\eta+\int_0^X\int_0^1\frac{(1-\eta)\eta^2}{\omega(\omega^2+\eta^2)^{\frac{3}{2}}}\psi(\xi)d\xi d\eta\\ &-(2c_2(\tau)+\frac{3}{2})G-(c_1(\tau)-c_3(\tau)+c_4(\tau))\\ \ge&\frac{e^{-\lambda\tau}}{2k}\int_0^X\int_0^\delta\eta(2\delta-\eta)W^3\psi(\xi)d\xi d\eta-(2c_2(\tau)+\frac{3}{2})G-(c_1(\tau)-c_3(\tau)+c_4(\tau)).\end{aligned} \tag{3.29}$$

By using Cauchy-Schwartz's inequality, then

$$\begin{aligned}\frac{dG}{d\tau}\ge&\frac{e^{-\lambda\tau}}{2k}(\int_0^X\int_0^\delta W\psi(\xi)d\xi d\eta)^3(\int_0^X\int_0^\delta[\eta(2\delta-\eta)]^{-\frac{1}{2}}\psi(\xi)d\xi d\eta)^{-2}\\ &-(2c_2(\tau)+\frac{3}{2})G-(c_1(\tau)-c_3(\tau)+c_4(\tau))\end{aligned}$$

$$\geq \frac{c_5 e^{-\lambda\tau}}{k}(G - \int_0^X \int_\delta^1 W\psi(\xi)d\xi d\eta)^3 - (2c_2(\tau)+\frac{3}{2})G - (c_1(\tau)-c_3(\tau)+c_4(\tau)), \tag{3.30}$$

with $c_5 = \frac{1}{2}(\int_0^X \int_0^\delta [\eta(2\delta-\eta)]^{-\frac{1}{2}}\psi(\xi)d\xi d\eta)^{-2}$.

Since $\int_0^X \int_\delta^1 W\psi(\xi)d\xi d\eta \leq c_6(X)$, it is reduced as

$$\frac{dG}{d\tau} \geq \frac{c_5 e^{-\lambda\tau}}{k}(G - c_6(X))^3 - (2c_2(\tau)+\frac{3}{2})G - (c_1(\tau)-c_3(\tau)+c_4(\tau)), \tag{3.31}$$

which implies that $\tilde{G}(\tau) = G(\tau) - c_6(X)$ satisfies

$$\frac{d\tilde{G}}{d\tau} \geq \frac{c_5 e^{-\lambda\tau}}{k}\tilde{G}^3 - (2c_2(\tau)+\frac{3}{2})\tilde{G} - c_7(\tau) \tag{3.32}$$

with $c_7(\tau) = c_1(\tau) - c_3(\tau) + c_4(\tau) + c_6(\tau)(2c_2(\tau)+\frac{3}{2})$.

Thus, $\tilde{G}(\tau)$ will go to infinity at some time $\bar{\tau} \in (0,T]$ if the initial data $\tilde{G}(0)$ is large enough, which is equivalent to that $\lim_{\tau\to\bar{\tau}} G(\tau) = +\infty$ as $G(0)$ is large. This implies that $\omega(\bar{\tau},\bar{\xi},0) = 0$ for some $0 < \bar{\tau} \leq T$ and $0 < \bar{\xi} \leq X$, which is the vanishing shear stress point at the physical boundary $\eta = 0$. It ends the proof of this theorem. □

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Appendix A. Derivation of the problem (1.1). To derive (1.1), we consider the vanishing viscosity limit for the incompressible geophysical equation with rotation, it is governed by the following problem for the Navier-Stokes-Coriolis equations (see [5,18]) in $\{t>0, x\in\mathbf{R}, Y>0\}$:

$$\begin{cases} \beta(\partial_t u^{\beta,\nu} + u^{\beta,\nu}\partial_x u^{\beta,\nu} + v^{\beta,\nu}\partial_Y u^{\beta,\nu}) + xv^{\beta,\nu} + \partial_x P^{\beta,\nu} = \kappa_1 + \nu\Delta u^{\beta,\nu}, \\ \beta(\partial_t v^{\beta,\nu} + u^{\beta,\nu}\partial_x v^{\beta,\nu} + v^{\beta,\nu}\partial_Y v^{\beta,\nu}) - xu^{\beta,\nu} + \partial_Y P^{\beta,\nu} = \kappa_2 + \nu\Delta v^{\beta,\nu}, \\ u_x^{\beta,\nu} + v_Y^{\beta,\nu} = 0, \\ (u^{\beta,\nu}, v^{\beta,\nu})|_{Y=0} = (0,0), \\ (u^{\beta,\nu}, v^{\beta,\nu})|_{t=0} = (u_0^{\beta,\nu}, v_0^{\beta,\nu}), \end{cases} \tag{A.1}$$

where $\kappa = (\kappa_1,\kappa_2)^T$ represents the shear tensor created by wind, the term $x(v^{\beta,\nu}, -u^{\beta,\nu})$ is due to the Coriolis force. The parameters β and ν are relatively small due to the large scale effect.

First, as $(\beta,\nu) \to (0,0)$, from (A.1) it is easy to deduce formally that

$$\begin{cases} u^{0,0} = -\mathrm{curl}\kappa \\ u_x^{0,0} + v_Y^{0,0} = 0, \quad v^{0,0}|_{Y=0} = 0 \end{cases}$$

It is easy to know that $u^{0,0}$ defined above does not satisfy the no-slip boundary condition given in (A.1) in general. Therefore, as in [15,17,19], boundary layers should be involved to describe the limit of $u^{\beta,\nu}$ as β, ν go to zero.

For the special case of $\beta = O(\nu^{\frac{1}{3}})$, we have a balance between convection, Coriolis force and viscosity of the fluid near the boundary by choosing the thickness of boundary layer as $\varepsilon = \nu^{\frac{1}{3}}$. Applying the classical boundary layer theory ([19]), we take the following ansatz for the solutions $(u^{\beta,\nu}, v^{\beta,\nu})$ and $P^{\beta,\nu}$ of (A.1):

$$\begin{cases} u^{\beta,\nu}(t,x,Y) = \sum_{j=0}^{\infty} \nu^{\frac{j}{3}} (u^{I,j}(t,x,Y) + u^{B,j}(t,x,\frac{Y}{\nu^{\frac{1}{3}}})), \\ v^{\beta,\nu}(t,x,Y) = \sum_{j=0}^{\infty} \nu^{\frac{j}{3}} (v^{I,j}(t,x,Y) + v^{B,j}(t,x,\frac{Y}{\nu^{\frac{1}{3}}})), \\ P^{\beta,\nu}(t,x,Y) = \sum_{j=0}^{\infty} \nu^{\frac{j}{3}} (P^{I,j}(t,x,Y) + P^{B,j}(t,x,\frac{Y}{\nu^{\frac{1}{3}}})). \end{cases}$$

with the terms like $u^{B,j} (j = 0,1,\cdots)$ exponentially decaying in the fast variable $y = \frac{Y}{\nu^{\frac{1}{3}}}$.

Plugging the above expansion into the problem (A.1), for points away from $Y = 0$, gathering the terms of the order of $O(1)$ leads to

$$\begin{cases} xv^{I,0} + \partial_x P^{I,0} = \kappa_1, \\ -xu^{I,0} + \partial_Y P^{I,0} = \kappa_2, \\ u_x^{I,0} + v_Y^{I,0} = 0. \end{cases} \tag{A.2}$$

It implies that $u^{I,0} = -\text{curl}\kappa$, and then

$$u^{B,0}|_{y=0} = \text{curl}\kappa(t,x,0). \tag{A.3}$$

Meanwhile, to solve the problem (A.1) up to the order of $O(\varepsilon^2)$, we also have

$$\begin{cases} xv^{I,1} + \partial_x P^{I,1} = -\partial_t u^{I,0} - u^{I,0}\partial_x u^{I,0} - v^{I,0}\partial_Y u^{I,0}, \\ -xu^{I,1} + \partial_Y P^{I,1} = -\partial_t v^{I,0} - u^{I,0}\partial_x v^{I,0} - v^{I,0}\partial_Y v^{I,0}, \\ u_x^{I,1} + v_Y^{I,1} = 0. \end{cases} \tag{A.4}$$

For points near the boundary, it follows from the divergence-free condition and the matching condition at infinity that

$$v^{B,0}(t,x,y) \equiv 0. \tag{A.5}$$

As a result, we have $v^{I,0}|_{Y=0} = -v^{B,0}|_{y=0} = 0$, which is the boundary condition for solving the problem (A.2).

Similarly, gathering the terms of the order $O(\varepsilon^{-1})$ in the equation of $v^{\beta,\nu}$ in (A.1), we obtain

$$P^{B,0}(t,x,y) \equiv 0. \tag{A.6}$$

On the other hand, setting $Y = \nu^{\frac{1}{3}} y$ for the term $u^{I,j}$ etc., taking the Taylor expansions of terms varying in $Y = \nu^{\frac{1}{3}} y$ at $y = 0$ and collecting the terms of $O(\varepsilon)$ and $O(1)$ in the equations of $u^{\beta,\nu}$ and $v^{\beta,\nu}$, respectively, we have

$$\begin{cases} \partial_t(\overline{u^{I,0}} + u^{B,0}) + (\overline{u^{I,0}} + u^{B,0})\partial_x(\overline{u^{I,0}} + u^{B,0}) + (v^{B,1} + \overline{v^{I,1}} + y\overline{\partial_Y v^{I,0}})\partial_y u^{B,0} \\ +x(v^{B,1} + \overline{v^{I,1}} + y\overline{\partial_Y v^{I,0}}) + \partial_x(P^{B,1} + \overline{P^{I,1}} + y\overline{\partial_Y P^{I,0}}) = y\overline{\partial_Y \kappa_1} + \partial_y^2 u^{B0}, \\ -x(\overline{u^{I,0}} + u^{B,0}) + \overline{\partial_Y P^{I,0}} + \partial_y P^{B,1} = 0, \\ \partial_x(\overline{u^{I,0}} + u^{B,0}) + \partial_y v^{B,1} + \overline{\partial_Y v^{I,0}} = 0, \end{cases} \tag{A.7}$$

where $\overline{f}$ denotes the trace $f(t,x,0)$ of $f(t,x,Y)$ at $Y = 0$. Combining (A.7) with (A.2) and (A.4), it follows that

$$\begin{cases} \partial_t u^{B,0} + (\overline{u^{I,0}} + u^{B,0})\partial_x u^{B,0} + u^{B,0}\overline{\partial_x u^{I,0}} + (v^{B,1} + \overline{v^{I,1}} + y\overline{\partial_Y v^{I,0}})\partial_y u^{B,0} \\ +xv^{B,1} + \partial_x P^{B,1} = \partial_y^2 u^{B0}, \\ -xu^{B,0} + \partial_y P^{B,1} = 0, \\ \partial_x u^{B,0} + \partial_y v^{B,1} = 0. \end{cases} \tag{A.8}$$

From the second equation of (A.8), we know that

$$\partial_x P^{B,1}(t,x,y) = \int_\infty^y u^{B,0}(t,x,z)dz - xv^{B,1}.$$

Setting $(u,v) = (u^{B,0}, v^{B,1} + \overline{v^{I,1}} + y\overline{\partial_Y v^{I,0}})$, from (A.8) we get that

$$\begin{cases} \partial_t u + (\overline{u^{I,0}} + u)\partial_x u + u\overline{\partial_x u^{I,0}} + v\partial_y u - \int_y^{+\infty} u dz = \partial_y^2 u, \\ \partial_x(u + \overline{u^{I,0}}) + \partial_y v = 0, \end{cases} \tag{A.9}$$

which are exactly the equations we considered in (1.1) if we introduce new u for $u + \overline{u^{I,0}}$.

Appendix B. Construction of the auxiliary function $\phi(\eta)$. In this appendix, we construct the function $\phi(\eta)$ introduced in the proof of Lemma 3.2. To satisfy the condition for $\phi(\eta)$ given in the proof of Lemma 3.2, we require that $\phi(\eta)$ satisfies

$$\int_{\eta'}^1 \frac{1-\tilde{\eta}}{\phi(\tilde{\eta})} d\tilde{\eta} \phi'(\eta') = \sigma(\eta')(\eta' - 1) \tag{B.1}$$

as $\eta' \in [\eta_2, 1]$, for any given smooth function $\sigma(\eta)$ satisfying that $0 \le \sigma(\eta) \le 1$, $\sigma(\eta_2) = 0$ and $\sigma(\eta) = 1$ when $\eta_3 \le \eta \le 1$. Integrating (B.1) with respect to η' over $[\eta, 1]$ for $\eta_2 < \eta < 1$, one gets that

$$\int_\eta^1 \frac{1-\eta'}{\phi(\eta')} d\eta' \phi(\eta) = \int_\eta^1 (\sigma(\eta') + 1)(1 - \eta')d\eta'. \tag{B.2}$$

Letting $F(\eta) = \int_\eta^1 \frac{1-\eta'}{\phi(\eta')} d\eta'$, we know from (B.2) that

$$\frac{F'(\eta)}{F(\eta)} = \gamma(\eta), \tag{B.3}$$

with

$$\gamma(\eta) = -\frac{1-\eta}{\int_\eta^1 (\sigma(\eta') + 1)(1 - \eta')d\eta'}. \tag{B.4}$$

For any $\eta_2 \le \eta < \eta_3$, integrating (B.3) over $[\eta, \eta_3]$ it follows that

$$\frac{1-\eta_3}{F(\eta)} = \exp\left(\int_\eta^{\eta_3} \gamma(\eta')d\eta'\right)$$

which implies

$$\int_\eta^1 \frac{1-\eta'}{\phi(\eta')} d\eta' = (1-\eta_3)\exp\left(-\int_\eta^{\eta_3} \gamma(\eta')d\eta'\right). \tag{B.5}$$

By differentiating the identity (B.5), we get

$$\phi(\eta) = \frac{1}{1-\eta_3} \int_\eta^1 (\sigma(\eta') + 1)(1 - \eta')d\eta' \cdot \exp\left(\int_\eta^{\eta_3} \gamma(\eta')d\eta'\right). \tag{B.6}$$

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