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EXISTENCE OF LARGE SOLUTIONS TO THE LANDAU-LIFSHITZ-BLOCH EQUATION*

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Abstract. In this work we discuss existence of solutions of the Landau-Lifshitz-Bloch equation describing the dynamics of the magnetization for the whole range of the temperature. By using energy method we prove global existence of strong solutions for given initial data, existence of time-periodic solutions as well as existence of steady state solutions of the equation.

Keywords. Landau-Lifshitz-Bloch equation; magnetization; anisotropic energy, magnetostatic equations; regularization method; Galerkin approximation, fixed-point theorems

AMS subject classifications. 35A01; 35G31; 35Q60; 82D40.

1. Introduction

A macroscopic description of the dynamics of the magnetization of ferromagnets at low temperature as well as at elevated temperature is described by the Landau-Lifshitz-Bloch (LLB) equation. This equation interpolates between the Landau-Lifshitz (LL) equation which is valid for temperatures below the Curie point θ_c and the Bloch equation when the temperatures exceed θ_c. (LLB) equation involves the longitudinal variation of the magnetization so the magnetization length is not conserved as in (LL) equation. The (LLB) model, first introduced in [8], has been discussed from the physical point of view in many recent papers, see [13,17] for example, this growing interest is sparked by the many applications of the model which concern among others, the magnetic write head and the recording medium.

To state the equations of this model, we consider an open bounded domain $D\subset\mathbb{R}^3$ which is simply connected and regular with boundary Γ and we denote ν the unit outward normal to Γ. Let $T>0$ be a fixed final time, we set $D_T=(0,T)\times D$ and $\Gamma_T=(0,T)\times\Gamma$. The (LLB) equation, satisfied by the magnetization $m=(m_1,m_2,m_3)$, takes the form

$$\partial_t m=-\gamma\Big(m\times\mathcal{H}_{llb}+\alpha_{tr}\,\omega(m)\times(\omega(m)\times\mathcal{H}_{llb})-\alpha_l\,(\omega(m)\cdot\mathcal{H}_{llb})\omega(m)\Big)\ \text{ in } D_T, \quad (1.1)$$

where the effective magnetic field $\mathcal{H}_{llb}$ is given, see [8] for example, by

$$\mathcal{H}_{llb}=a\Delta m+H-\frac{\widehat{m}}{\chi_{tr}}-\zeta(m), \quad (1.2)$$

and $\omega(m)=\frac{m}{|m|}$. Of course Equation (1.1) makes sense as long as $m\neq 0$. The demagnetizing field $H=(H_1,H_2,H_3)$ satisfies the magnetostatic equations

$$\operatorname{div}(H+m)=F,\quad H=\nabla\varphi\ \text{ in } D_T, \quad (1.3)$$

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$\widehat{m}=(m_1,m_2,0)$ is related to the anisotropic energy and $\zeta(m)$ is the internal exchange term defined according to the temperature θ by

$$\zeta(m)=\begin{cases}\zeta_1(m):=\dfrac{1}{2\chi_l}(\dfrac{|m|^2}{m_e^2}-1)m, & \text{for } 0<\theta<\theta_c,\\ \zeta_2(m):=\dfrac{1}{\chi_l}(\mu|m|^2+1)m, & \text{for } \theta>\theta_c.\end{cases} \tag{1.4}$$

Equations (1.1) and (1.3) are completed with the following boundary and initial conditions

$$\nabla m\cdot\nu=0, \quad (H+m)\cdot\nu=0 \quad \text{on}\,\Gamma_T, \tag{1.5}$$
$$m(0)=m_0 \ \text{ in } D. \tag{1.6}$$

Problem (1.1)-(1.3)-(1.5)-(1.6) will be labeled $(\mathcal{P})$, without distinction between the cases below or above the Curie temperature θ_c. In this problem, $|.|$ and $\times$ denote respectively the Euclidean norm and the cross product in $\mathbb{R}^3$, φ is the magnetic potential and $\gamma>0$ is the gyromagnetic parameter. Without loss of generality we set in the sequel $\gamma=1$ and other physical parameters which are not relevant to our work are also equated to 1 like a and m_e. The significant parameters are μ, χ_{tr}, $\chi_l>0$ together with the transverse and longitudinal damping parameters α_{tr} and α_l, given for fixed temperature by the laws, see [8],

$$\alpha_{tr}=\lambda\chi(\theta)(1-\frac{\theta}{3\theta_c})+\lambda(1-\chi(\theta))\frac{2\theta}{3\theta_c}, \quad \alpha_l=\lambda\frac{2\theta}{3\theta_c}, \tag{1.7}$$

where $\lambda>0$ is a physical parameter that we take equal to 1 and χ is the function defined by $\chi(\theta)=1$ if $0<\theta<\theta_c$ and $\chi(\theta)=0$ if $\theta>\theta_c$. This means that

$$\alpha_{tr}=(1-\frac{\theta}{3\theta_c}), \quad \alpha_l=\frac{2\theta}{3\theta_c}, \quad \text{if } 0<\theta<\theta_c,$$
$$\alpha_{tr}=\alpha_l=\frac{2\theta}{3\theta_c}, \quad \text{if } \theta>\theta_c.$$

We define the parameter β by

$$\beta=\alpha_{tr}-\alpha_l, \tag{1.8}$$

so that

$$\beta=(1-\frac{\theta}{\theta_c})>0, \quad \text{for } 0<\theta<\theta_c, \quad \text{and} \quad \beta=0, \quad \text{for } \theta>\theta_c. \tag{1.9}$$

Using the relation

$$\mathcal{H}_{llb}=(\mathcal{H}_{llb}\cdot\omega(m))\omega(m)-\omega(m)\times(\omega(m)\times\mathcal{H}_{llb}), \tag{1.10}$$

the magnetization Equation (1.1) can be rewritten as

$$\partial_t m-\alpha_{tr}\mathcal{H}_{llb}=-m\times\mathcal{H}_{llb}-\beta(\mathcal{H}_{llb}\cdot\omega(m))\omega(m), \tag{1.11}$$

or equivalently as follows, observing that $(\omega(m)\cdot\zeta(m))\omega(m)=\zeta(m)$

$$\partial_t m-\alpha_{tr}\Delta m+m\times\Delta m+\alpha_l\zeta(m)+\beta(\omega(m)\cdot\Delta m)\omega(m)=$$

$$\alpha_{tr}\,(H-\frac{\widehat{m}}{\chi_{tr}})-m\times(H-\frac{\widehat{m}}{\chi_{tr}})-\beta\big(\omega(m)\cdot(H-\frac{\widehat{m}}{\chi_{tr}})\big)\,\omega(m). \tag{1.12}$$

We notice that for $\theta>\theta_c$, since $\beta=0$ the (LLB) equation reduces to

$$\partial_t m-\alpha_{tr}\,\Delta m+m\times\Delta m+\alpha_l\zeta(m)=\alpha_{tr}\,(H-\frac{\widehat{m}}{\chi_{tr}})-m\times(H-\frac{\widehat{m}}{\chi_{tr}}), \tag{1.13}$$

which is well defined even if $m=0$. Moreover this equation contains the term $m\times\Delta m$ which appears in the (LL) equation and the terms $m\times H$ and Δm which are in the Bloch-Torrey equation used in the theory of ferrofluid flows, see [13, 14] for example.

The (LL) equation is well understood since the pioneering work by Alouges-Soyeur [1] where the global existence of weak solutions is proved as well as nonuniqueness of the solutions. Regularity results on the solutions were obtained by Carbou [2] and Carbou-Fabrie [3] and recently by Feischt and Tran [6]. A generalization called Landau-Lifshitz-Maxwell equation was discussed in Ding-Guo-Lin-Zeng [4] and Dumas-Sueur [5]. Finally Hubert [10] proved in particular the existence of time-periodic solutions to (LL) equation. We also quote the book of Guo-Ding [9] for useful presentation and results on the (LL) equation.

The study of (LLB) equation in its deterministic form is very recent, so the literature on this subject is not abundant. In [12], the author considered the model (1.13) in the absence of both the magnetic field H and the anisotropy field $\frac{1}{\chi_{tr}}\widehat{m}$. She proved by using Galerkin approximation, the existence of global weak solutions.

An interesting question that arises is the following. Since the (LLB) equation is built by interpolating between the (LL) equation and the Bloch equation, what should be the asymptotic behaviors of the (LLB) equation when $\theta\to 0$ and when $\theta\to+\infty$. It is expected to get (LL) equation when $\theta\to 0$ and Bloch equation when $\theta\to+\infty$ see [18, 19], the rigorous proofs being open.

To conclude this paragraph, we mention that we will consider also the existence of time-periodic solutions as well as the existence of steady state solutions, assuming the source term F first time-periodic then time independent. The periodic problem named $(\mathcal{P}_{per})$ is defined by the Equations (1.1)-(1.3)-(1.5) with the periodicity condition

$$m(0)=m(T), \tag{1.14}$$

and the stationary problem $(\mathcal{S})$ is stated as follows

$$\begin{aligned}
&-\alpha_{tr}\,\mathcal{H}_{llb}+m\times\mathcal{H}_{llb}+\beta\,(\mathcal{H}_{llb}\cdot\omega(m))\omega(m)=0\ \text{ in } D,\qquad \nabla m\cdot\nu=0\quad\text{on}\quad\Gamma,\\
&\operatorname{div}(H+m)=F,\qquad H=\nabla\varphi\quad\text{in } D,\quad (H+m)\cdot\nu=0\quad\text{on }\Gamma.
\end{aligned} \tag{1.15}$$

2. Main results

To start, let us precise the functional framework and the notations used in this work. We will employ the standard notations for the Lebesgue spaces $L^p(D)$ and Sobolev spaces $H^s(D), W^{s,p}(D)$ of real-valued functions and we introduce the notations $\mathbb{L}^p(D)=(L^p(D))^3$, $\mathbb{H}^s(D)=(H^s(D))^3$ and $\mathbb{W}^{s,p}(D)=(W^{s,p}(D))^3$ for vectorial fields functional spaces. $\|\cdot\|$ and $(;)$ denote respectively the norm and scalar product of $L^2(D)$ and $\mathbb{L}^2(D)$ whereas $\|\cdot\|_p$ denotes the norm in the other $L^p(D)$ and $\mathbb{L}^p(D)$ spaces. To deal with the magnetostatic equation, we define the spaces

$$L^2_\sharp(D)=\{\psi\in L^2(D);\int_D\psi(x)\;dx=0\},$$

$$H^1_\sharp(D)=H^1(D)\cap L^2_\sharp(D),$$

where $L^2_\sharp(D)$ is equipped with the L^2- norm and $H^1_\sharp(D)$ with the L^2- norm of the gradient, since by Poincaré-Wirtinger inequality, there exists $C>0$ depending on D such that

$$\|\psi\| \leq C\|\nabla\psi\|, \quad \forall \psi \in H^1_\sharp(D). \tag{2.1}$$

More generally, if V is a Banach space, we denote its norm by $\|\cdot\|_V$ and the bracket notation $\langle\cdot;\cdot\rangle_{V'\times V}$ (or simply $\langle\cdot;\cdot\rangle$ if no confusion arises) will be reserved for pairings between V and its dual V'.
In the sequel $C>0$ denotes various constants which depend on the domain D and the physical parameters appearing in the equations. Sometimes we denote by C_T or $C(T,..)$ positive constants depending in addition on the terms indicated as subscripts or arguments.

To simplify the presentation of our results, we introduce further notations. Let the nonlinear partial differential operator $\mathcal{A}^\beta$ be defined by

$$\begin{aligned}
&\mathcal{A}^\beta = A + \beta P,\\
&A(m) = -\alpha_{tr}\Delta m + m\times\Delta m + \alpha_l\zeta(m),\\
&P(m) = (\omega(m)\cdot\Delta m)\omega(m),
\end{aligned} \tag{2.2}$$

and we set

$$\begin{aligned}
&\mathcal{L}^\beta = L - \beta K,\\
&L(m,H) = \alpha_{tr}(H - \frac{\widehat{m}}{\chi_{tr}}) - m\times(H - \frac{\widehat{m}}{\chi_{tr}}),\\
&K(m,H) = \big(\omega(m)\cdot(H - \frac{\widehat{m}}{\chi_{tr}})\big)\omega(m),
\end{aligned} \tag{2.3}$$

so that the magnetization Equation (1.12) writes as

$$\partial_t m + \mathcal{A}^\beta(m) = \mathcal{L}^\beta(m,H) \quad \text{in} \quad D_T. \tag{2.4}$$

The superscript is relative to the parameter β of the equation so that $\mathcal{A}^0 = A$ and $\mathcal{L}^0 = L$ correspond to the elevated temperature case $\theta > \theta_c$.

Let $m \in \mathbb{H}^1(D)$ be such that $\Delta m \in \mathbb{L}^2(D)$ and $\nabla m\cdot\nu = 0$ on Γ then for all $\Phi \in \mathbb{H}^1(D)$ we can write

$$\int_D \mathcal{A}^\beta(m)\cdot\Phi\, dx = \langle\mathcal{B}^\beta(m),\Phi\rangle, \tag{2.5}$$

where operator $\mathcal{B}^\beta(m)$ is defined by

$$\langle\mathcal{B}^\beta(m),\Phi\rangle = \alpha_{tr}\int_D \nabla m\cdot\nabla\Phi\, dx + \int_D \big(m\times\Delta m + \alpha_l\zeta(m) + \beta P(m)\big)\cdot\Phi\, dx. \tag{2.6}$$

Note that the linear form $\mathcal{B}^\beta(m)$ is well defined and continuous on $\mathbb{H}^1(D)$ because of the Sobolev embedding $\mathbb{H}^1(D)\subset\mathbb{L}^6(D)$, that is to say $\mathcal{B}^\beta(m)\in(\mathbb{H}^1(D))'$.

Finally to express the energy estimates satisfied by any solution of our problem, we set $\mathcal{E} := \mathcal{E}(m,H)$ with

$$\mathcal{E}(m,H) = \begin{cases} \|\nabla m\|^2 + \dfrac{\|H\|^2}{2} + \dfrac{\|\widehat{m}\|^2}{\chi_{tr}} + \dfrac{\|m\|^2 + \big\||m|^2-1\big\|^2}{2\chi_l} & \text{if } 0<\theta<\theta_c,\\[2ex] \|\nabla m\|^2 + \dfrac{\|H\|^2}{2} + \dfrac{\|\widehat{m}\|^2}{\chi_{tr}} + \dfrac{\|m\|^2 + \mu\|m\|_4^4}{\chi_l} & \text{if } \theta>\theta_c. \end{cases} \tag{2.7}$$

As already mentioned, the magnetization m may vanish, so the meaning of the magnetization equation should be clarified. It actually makes sense when $\theta > \theta_c$ since in this case $\beta = 0$ so the terms involving the undefined function $\omega(m)$ disappear and the equation simplifies into Equation (1.13). However when $\theta < \theta_c$, in order to avoid the indetermination when $|m(t,x)| = 0$, we will replace the magnetization equation by the following one

$$|m|^2 \big(\partial_t m + A(m)\big) + \beta (m \cdot \Delta m) m = |m|^2 L(m,H) - \beta m \cdot (H - \frac{\widehat{m}}{\chi_{tr}}) m \quad \text{in } D_T, \tag{2.8}$$

see (2.2) and (2.3) for the definitions of $A(m)$ and $L(m,H)$. This equation will be completed in case of problem $(\mathcal{P})$ by the following boundary and initial conditions

$$|m|^2 \nabla m \cdot \nu = 0 \quad \text{on } \Gamma_T, \quad |m(0)|^2 m(0) = |m_0|^2 m_0 \quad \text{in } D, \tag{2.9}$$

and we notice that the problem at hand is equivalent to the initial one as long as $|m(t,x)| \neq 0$. Of course the same transformation will be operated in case of time-periodic and steady problems.

Now we are in the position to define the solutions of our problems and formulate the main results of this paper.

Definition 2.1. *We say that (m,H) is a global solution of problem $(\mathcal{P})$ if for all $T > 0$,*

$$\begin{aligned} m \in \mathcal{C}([0,T];\mathbb{H}^1(D)) \cap L^2(0,T;\mathbb{H}^2(D)), \quad \partial_t m \in L^2(0,T;\mathbb{L}^{3/2}(D)), \\ H \in \mathcal{C}([0,T];\mathbb{H}^1(D)), \end{aligned} \tag{2.10}$$

and (m,H) satisfies almost everywhere the equations (1.13)-(1.3)-(1.5)-(1.6) for $\theta > \theta_c$ and (2.8)-(1.3)-(2.9) for $0 < \theta < \theta_c$.

This definition would be adapted to the time-periodic as well as for the stationary problems.

Theorem 2.1. *Assume that $m_0 \in \mathbb{H}^1(D)$ and $F \in \mathcal{C}([0,T];L^2_\sharp(D))$. Then problem $(\mathcal{P})$ admits a global-in-time solution (m,H). Moreover there exists $C > 0$ such that for all $t \in [0,T]$, the following estimates hold*

$$\|m(t)\|^2 + 2\alpha_l \int_0^t \mathcal{E}(s)\,ds \le \|m_0\|^2 + C(\chi(\theta)T + \|F\|^2_{L^2(D_T)}),$$
$$\|\nabla m(t)\|^2 + \|H(t)\|^2_{\mathbb{H}^1(D)} + \alpha_l \int_0^t \|\Delta m(s)\|^2\,ds \le C(\chi(\theta)T + \|m_0\|^2_{\mathbb{H}^1(D)} + \|F\|^2_{L^2(D_T)}),$$

the energy $\mathcal{E}$ being defined by (2.7).

Theorem 2.2. *Assume that F is time-periodic with period $T > 0$ and $F \in \mathcal{C}([0,T];L^2_\sharp(D))$. Then there exists a time-periodic solution (m,H) with period T of problem $(\mathcal{P}_{per})$ satisfying the regularity (2.10). Moreover there exists $C, C_T > 0$ such that*

$$\|m(t)\|^2 + \|H(t)\|^2 \le C_T C(\chi(\theta)T + \|F\|^2_{L^2(D_T)}), \forall t \in [0,T],$$
$$\int_0^T (\mathcal{E}(t) + \|\Delta m(t)\|^2 + \|H(t)\|^2_{\mathbb{H}^1(D)})\,dt \le C(\chi(\theta)T + \|F\|^2_{L^2(D_T)}),$$

where $C_T=(2-e^{-\frac{\alpha_l}{\chi_l}T})(1-e^{-\frac{\alpha_l}{\chi_l}T})^{-1}$.

THEOREM 2.3. *Let* $F\in L^2_\sharp(D)$. *There exists a solution* $(m,H)\in\mathbb{H}^2(D)\times\mathbb{H}^1(D)$ *of the stationary problem* $(\mathcal{S})$ *satisfying*

$$\|m\|^2_{\mathbb{H}^1(D)}+\|H\|^2_{\mathbb{H}^1(D)}+\|\Delta m\|^2_{\mathbb{L}^2(D)}\leq C(\chi(\theta)+\|F\|^2). \tag{2.11}$$

We observe that all the estimates of the solutions simplify in the case $\theta>\theta_c$ since $\chi(\theta)=0$. Moreover if $F\in L^2(\mathbb{R}^+;L^2(D))$, then the global solutions of the initial boundary value problem $(\mathcal{P})$ remain bounded for all time by the norms of the data m_0 and F.

Theorem 2.1 extends the results obtained in [12] for the case $\theta>\theta_c$, since in solving the problem, we have taken into account the magnetic field H as well as the contribution of the anisotropic field $\widehat{m}$. Moreover we have improved the solution's regularity. The question of existence of periodic solutions has been discussed for the problem of (LL), see for instance the work already mentioned [10] in which the author considered that the ferromagnetic particles are small and used operator theory and spectral analysis. But, to the best of our knowledge, there is no result available in the literature for the problem of (LLB), so we provide an answer to this question by proving Theorem 2.2. The methods used to prove Theorems 2.1 and 2.2 also allow to establish the existence of stationary solutions for problem $(\mathcal{S})$.

The rest of the paper is organized as follows. In Section 3, we prove some preliminary results and the formal energy estimates for the problems. In Section 4, we prove the global existence of solutions (m,H) of problem $(\mathcal{P})$ at any temperature. The proof is based on the energy method, regularization and approximations and regarding the difficulties of the problem, it will be done in several steps, all being clearly justified. In section 5, we are interested in the existence of periodic solutions. We rely on the Galerkin approximation obtained previously and using Brouwer's fixed-point theorem, we get approximated time-periodic solutions. Then we conclude following globally the same procedure as in the proof of Theorem 2.1, so we will avoid the fastidious details. Section 6 is devoted to the stationary problem $(\mathcal{S})$, existence of solutions is proved applying energy method and a fixed-point theorem.

3. Preliminary results and formal energy estimates

We give some results that will be needed later. We start by reviewing some properties satisfied by the solution of the magnetostatic equations.

3.1. The magnetostatic equations. Let $F\in L^2_\sharp(D)$ and $m\in\mathbb{H}^1(D)$ be fixed and let $\varphi\in H^1_\sharp(D)\cap H^2(D)$ be the unique solution of the problem

$$\nabla\varphi=H,\ \mathrm{div}(H+m)=F\ \text{ in } D,\quad (H+m)\cdot\nu=0\ \text{ on }\Gamma. \tag{3.1}$$

Multiplying this equation by φ and integrating by parts, we get the identity

$$\|H\|^2=-\int_D H\cdot m\,dx-\int_D F\varphi\,dx, \tag{3.2}$$

and since the Poincaré-Wirtinger inequality (2.1) leads to

$$\left|\int_D F\varphi\,dx\right|\leq C\|H\|\,\|F\|, \tag{3.3}$$

we easily get the estimate

$$\|H\|\leq C(\|m\|+\|F\|). \tag{3.4}$$

We observe that using Young's inequality, we deduce from (3.3) the following bound of $\int_D F\varphi dx$ which will be useful later

$$|\int_D F\varphi dx| \leq \frac{1}{d}\|H\|^2 + C(d)\|F\|^2, \tag{3.5}$$

taking any arbitrary constant $d>0$. Next we write the equation (3.1) in the form

$$\Delta\varphi = -\operatorname{div} m + F \text{ in } D, \quad \nabla\varphi\cdot\nu = -m\cdot\nu \text{ on } \Gamma, \tag{3.6}$$

and apply elliptic regularity results. Since $\operatorname{div} m \in L^2(D)$ and $m\cdot\nu \in H^{1/2}(\Gamma)$, then $\varphi \in H^2(D)$ and

$$\|\varphi\|_{H^2(D)} \leq C(\|m\|_{\mathbb{H}^1(D)} + \|F\|),$$

which means that

$$\|H\|_{\mathbb{H}^1(D)} \leq C(\|m\|_{\mathbb{H}^1(D)} + \|F\|). \tag{3.7}$$

In particular the linear mapping

$$\mathcal{H} : (m,F) \longmapsto H \tag{3.8}$$

is continuous from $\mathbb{L}^2(D)\times L^2_\sharp(D)$ to $\mathbb{L}^2(D)$ and from $\mathbb{H}^1(D)\times L^2_\sharp(D)$ to $\mathbb{H}^1(D)$. Similarly if $m\in\mathcal{C}([0,T];\mathbb{H}^1(D))$ and $F\in\mathcal{C}([0,T];L^2_\sharp(D))$, then there exists a unique solution $\varphi\in\mathcal{C}([0,T];H^1_\sharp(D)\cap H^2(D))$ satisfying

$$\operatorname{div}(\nabla\varphi + m) = F \text{ in } D_T, \quad (\nabla\varphi+m)\cdot\nu = 0 \text{ on } \Gamma_T. \tag{3.9}$$

Furthermore $H=\nabla\varphi$ fulfills for $p=2$ and $p=\infty$ the following estimates

$$\|H\|_{L^p(0,T;\mathbb{L}^2(D))} \leq C(\|m\|_{L^p(0,T;\mathbb{L}^2(D))} + \|F\|_{L^p(0,T;L^2(D))}), \tag{3.10}$$

$$\|H\|_{L^p(0,T;\mathbb{H}^1(D))} \leq C(\|m\|_{L^p(0,T;\mathbb{H}^1(D))} + \|F\|_{L^p(0,T;L^2(D))}), \tag{3.11}$$

and the mapping $\mathcal{H}$ is continuous from $L^2(0,T;\mathbb{L}^2(D)\times L^2_\sharp(D))$ into $\mathbb{L}^2(D_T)$ and from $L^2(0,T;\mathbb{H}^1(D)\times L^2_\sharp(D))$ into $L^2(0,T;\mathbb{H}^1(D))$.

3.2. The magnetization equation. Let $F\in L^2_\sharp(D)$, m be a regular function such that $\omega(m)$ is well defined and $H=\nabla\varphi$ the solution of problem (3.1). We consider the operator $\mathcal{L}^\beta(m,H)$ defined by (2.3). We easily see that the following estimate holds true

$$\|\mathcal{L}^\beta(m,H) + m\times H\|^2 \leq C(\|H\|^2 + \|\widehat{m}\|^2 + \|m\|_4^4), \tag{3.12}$$

and using (3.2) to express the term $\int_D H\cdot m\,dx$, we get the identity

$$\int_D \mathcal{L}^\beta(m,H)\cdot m\,dx = -\alpha_l\Big(\frac{1}{\chi_{tr}}\|\widehat{m}\|^2 + \|H\|^2 + \int_D F\varphi dx\Big). \tag{3.13}$$

Next we assume also that $\nabla m\cdot\nu = 0$ on Γ and we consider operator $\mathcal{B}^\beta$ defined by (2.6). The identity $(\omega(m)\cdot\Delta m)\cdot m = \Delta m\cdot m$ allows to write

$$\langle\mathcal{B}^\beta(m),m\rangle = \alpha_{tr}\|\nabla m\|^2 - \alpha_l\int_D \zeta(m)\cdot m\,dx - \beta\int_D \Delta m\cdot m\,dx,$$

and using Green's formula in the last integral, we arrive at

$$\langle \mathcal{B}^\beta(m), m\rangle = \alpha_l(\|\nabla m\|^2 + \int_D \zeta(m)\cdot m\,dx), \tag{3.14}$$

with

$$\int_D \zeta(m)\cdot m\,dx = \frac{1}{2\chi_l}(\||m|^2-1\|^2 + \|m\|^2 - |D|), \quad \text{if } 0<\theta<\theta_c,$$
$$\int_D \zeta(m)\cdot m\,dx = \frac{1}{\chi_l}(\mu\|m\|_4^4 + \|m\|^2), \quad \text{if } \theta>\theta_c, \tag{3.15}$$

$|D|$ being the Lebesgue measure of D. Moreover since

$$-\langle \mathcal{B}^\beta(m), \Delta m\rangle = \alpha_{tr}\|\Delta m\|^2 - \alpha_l\int_D \zeta(m)\cdot\Delta m\,dx - \beta\int_D (\omega(m)\cdot\Delta m)^2\,dx,$$

and $|\omega(m)|\le 1$, we deduce the inequality below

$$-\langle \mathcal{B}^\beta(m), \Delta m\rangle \ge \alpha_l(\|\Delta m\|^2 - \int_D \zeta(m)\cdot\Delta m\,dx), \tag{3.16}$$

with

$$-\int_D \zeta(m)\cdot\Delta m\,dx = \frac{1}{2\chi_l}\int_D (|m|^2|\nabla m|^2\,dx + 2(m\cdot\nabla m)^2 - |\nabla m|^2)\,dx, \quad \text{if } 0<\theta<\theta_c,$$
$$-\int_D \zeta(m)\cdot\Delta m\,dx = \frac{1}{\chi_l}\int_D (\mu|m|^2+1)|\nabla m|^2 + 2\mu(m\cdot\nabla m)^2)\,dx, \quad \text{if } \theta>\theta_c. \tag{3.17}$$

Similarly an integration by parts leads to

$$|\int_D \Delta m\cdot m\times H\,dx| = |\int_D \nabla m\cdot m\times\nabla H\,dx|,$$

and we get by Young's inequality the bound

$$|\int_D \Delta m\cdot m\times H\,dx| \le d'\int_D |m|^2|\nabla m|^2\,dx + C(d')\|\nabla H\|^2, \tag{3.18}$$

for an arbitrary constant $d'>0$.

3.3. Formal energy estimates. We will establish some useful estimates satisfied by the solutions (m,H) of problem $(\mathcal{P})$ by using the results of the previous subsections.

PROPOSITION 3.1. *Under hypotheses of Theorem 2.1, any strong solution (m,H) of problem $(\mathcal{P})$ satisfies the estimates*

$$\|m(t)\|^2 + 2\alpha_l\int_0^t \mathcal{E}(s)\,ds \le \|m_0\|^2 + C(\chi(\theta)T + \|F\|^2_{L^2(D_T)}), \quad t\in[0,T], \tag{3.19}$$
$$\|m\|^4_{L^4(D_T)} \le C(\chi(\theta)T + \|m_0\|^2 + \|F\|^2_{L^2(D_T)}), \tag{3.20}$$
$$\|\nabla m(t)\|^2 + \alpha_l\int_0^t \|\Delta m(s)\|^2\,ds \le \|\nabla m_0\|^2 + C(\chi(\theta)T + \|m_0\|^2 + \|F\|^2_{L^2(D_T)}), \tag{3.21}$$

$$\|H\|^2_{L^\infty(0,T;\mathbb{H}^1(D))} \leq C(\chi(\theta)T + \|m_0\|^2_{\mathbb{H}^1(D)} + \|F\|^2_{L^2(D_T)}). \tag{3.22}$$

Proof. We multiply the magnetization Equation (2.4) by m and integrate over D to write

$$\frac{1}{2}\frac{d}{dt}\|m\|^2 + \langle \mathcal{B}^\beta(m), m\rangle = \int_D \mathcal{L}^\beta(m,H)\cdot m\, dx. \tag{3.23}$$

Therefore using (3.13), (3.14), (3.15) and inequality (3.5), we get

$$\frac{1}{2}\frac{d}{dt}\|m(t)\|^2 + \alpha_l \mathcal{E}(t) \leq C(\chi(\theta) + \|F(t)\|^2), \tag{3.24}$$

where the energy $\mathcal{E}$ is defined in (2.7). Hence we obtain the first estimate (3.19) and we deduce directly inequality (3.20) for $\theta > \theta_c$ since in this case $\|m\|^4_{L^4(D_T)} \leq C\int_0^T \mathcal{E}(t)\, dt$. In the case $\theta < \theta_c$, we get a similar inequality using the relation $|m|^4 \leq 2((|m|^2-1)^2 + |m|^2)$.

Now we multiply Equation (2.4) by $-\Delta m$ and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla m\|^2 - \langle \mathcal{B}^\beta(m), \Delta m\rangle = -\int_D \mathcal{L}^\beta(m,H)\cdot \Delta m\, dx. \tag{3.25}$$

Using inequalities (3.12) and (3.18) together with the bounds (3.4) and (3.7) of H, we deduce that

$$\begin{aligned}|\int_D \mathcal{L}^\beta(m,H)\cdot\Delta m\, dx| \leq& \frac{\alpha_l}{2}\|\Delta m\|^2 + d'\int_D |m|^2|\nabla m|^2\, dx\\ &+ C(d')(\|F\|^2 + \|m\|^2_{\mathbb{H}^1(D)} + \|m\|^4_4).\end{aligned} \tag{3.26}$$

Therefore inequality (3.16) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla m\|^2 + \frac{\alpha_l}{2}\|\Delta m\|^2 - \alpha_l\int_D \zeta(m)\cdot\Delta m\, dx\\ \leq& d'\int_D |m|^2|\nabla m|^2\, dx + C(d')(\|F\|^2 + \|m\|^2_{\mathbb{H}^1(D)} + \|m\|^4_4).\end{aligned} \tag{3.27}$$

In view of the identity (3.17), we choose $d' = \frac{\alpha_l}{4\chi_l}$ in the case $\theta < \theta_c$ and $d' = \frac{\mu\alpha_l}{2\chi_l}$ if $\theta > \theta_c$ to get for all temperatures the following inequality

$$\frac{d}{dt}\|\nabla m\|^2 + \alpha_l\|\Delta m\|^2 \leq C(\|F\|^2 + \|m\|^2_{\mathbb{H}^1(D)} + \|m\|^4_4). \tag{3.28}$$

Integrating (3.28) with respect to time and using the previous results, we get estimate (3.21) and we deduce (3.22) using the bound (3.11) and estimate (3.19). ∎

Hence we see that the estimates given in Theorem 2.1 are fulfilled. It is easy to deduce from the previous calculations that the smooth solutions (m,H) of the periodic problem $(\mathcal{P}_{per})$ satisfy the estimates given in Theorem 2.2. In the same way, any strong solution (m,H) of the stationary problem $(\mathcal{S})$ satisfies the bound given in Theorem 2.3.

4. Solving problem $(\mathcal{P})$

The resolution of problem $(\mathcal{P})$ involves several stages. First, in order to overcome some difficulties related to the unit magnetization vector $\omega(m)$, we define a regularized problem named $(\mathcal{P}_\delta)$ depending on a small parameter $\delta > 0$. Secondly, we introduce the Galerkin approximations of problem $(\mathcal{P}_\delta)$ and look for solutions (m^n,H^n) of the approximated problem $(\mathcal{P}_\delta^n)$. To end up to a solution of our problem, we will perform the limit in the approximated solutions. First, for each fixed δ, letting $n\to\infty$ in (m^n,H^n), we get at the limit solutions (m^δ,H^δ), then letting $\delta\to 0$ in (m^δ,H^δ) evolves a solution of $(\mathcal{P})$. This is not easy to do and requires uniform estimations on the approximated solutions together with some compactness results to deal with the nonlinear terms and the difficulties inherent to the unit magnetization vector $\omega(m)$. Before going on, we recall that the terms containing $\omega(m)$ in problem $(\mathcal{P})$ are all factored by β and then vanish when $\theta>\theta_c$ rendering the first step unnecessary in this case.

4.1. The regularized problem $(\mathcal{P}_\delta)$. Let $\delta>0$ be a small parameter, we introduce the regularized vectorial field

$$\omega_\delta(m)=\frac{m}{\sqrt{|m|^2+\delta^2}},\quad m\in\mathbb{R}^3,$$

having the features of being infinitely differentiable on $\mathbb{R}^3$ and verifying

$$|\omega_\delta(m)|\le 1,\quad m\in\mathbb{R}^3 \qquad \text{and} \qquad \lim_{\delta\to 0}\omega_\delta(m)=\omega(m),\quad \forall m\in\mathbb{R}^3\backslash\{0\}.$$

In addition, for all m and V in $\mathbb{R}^3$, the following properties are satisfied

$$(\omega_\delta(m)\cdot V)(\omega_\delta(m)\cdot m)=\frac{|m|^2}{|m|^2+\delta^2}\,m\cdot V=m\cdot V-\frac{\delta^2}{|m|^2+\delta^2}\,m\cdot V, \tag{4.1}$$

$$(\omega_\delta(m)\cdot V)^2=\frac{1}{|m|^2+\delta^2}\,(m\cdot V)^2. \tag{4.2}$$

Now we consider the magnetization Equation (2.4) and modify it, both by replacing $\omega(m)$ by $\omega_\delta(m)$ and by adding the term $\beta\frac{\delta^2}{|m|^2+\delta^2}\Delta m$ which will be useful to our purpose. The transformed equation reads as

$$\partial_t m+\mathcal{A}_\delta^\beta(m)=\mathcal{L}_\delta^\beta(m,H)\ \text{ in } D_T, \tag{4.3}$$

where we set

$$\begin{aligned}
&\mathcal{A}_\delta^\beta=A+\beta P_\delta,\quad \mathcal{L}_\delta^\beta=L-\beta K_\delta,\\
&P_\delta(m)=\big(\omega_\delta(m)\cdot\Delta m\big)\,\omega_\delta(m)+\frac{\delta^2}{|m|^2+\delta^2}\Delta m,\\
&K_\delta(m,H)=\big(\omega_\delta(m)\cdot(H-\frac{\widehat{m}}{\chi_{tr}})\big)\,\omega_\delta(m),
\end{aligned} \tag{4.4}$$

A and L being defined in (2.2) and (2.3) and we observe that formally $P_0=P$ and $K_0=K$. The regularized problem $(\mathcal{P}_\delta)$ is given by the set of equations (4.3)-(1.3)-(1.5)-(1.6). Before going on, we define on $\mathbb{H}^1(D)$ the linear form $\mathcal{B}_\delta^\beta(m)$ associated to $\mathcal{A}_\delta^\beta(m)$ as we did for $\mathcal{B}^\beta$ in (2.6). If $m\in\mathbb{H}^1(D)$, $\Delta m\in\mathbb{L}^2(D)$ and $\nabla m\cdot\nu=0$ on Γ, we have the identity

$$\int_D \mathcal{A}_\delta^\beta(m)\cdot\Phi\,dx=\langle\mathcal{B}_\delta^\beta(m),\Phi\rangle,\quad \forall\Phi\in\mathbb{H}^1(D), \tag{4.5}$$

with

$$\langle \mathcal{B}_\delta^\beta(m), \Phi\rangle = \alpha_{tr}\int_D \nabla m\cdot\nabla\Phi\,dx + \int_D \big(m\times\Delta m + \alpha_l\zeta(m) + \beta P_\delta(m)\big)\cdot\Phi\,dx, \tag{4.6}$$

for all $\Phi\in\mathbb{H}^1(D)$. Therefore we observe that by adding the term $\beta\frac{\delta^2}{|m|^2+\delta^2}\Delta m$ to the equation, property (4.1) leads to

$$\langle \mathcal{B}_\delta^\beta(m), m\rangle = \langle \mathcal{B}^\beta(m), m\rangle, \tag{4.7}$$

for all $\delta>0$. Moreover by properties (4.1) and (4.2) we get the identity

$$\mathcal{L}_\delta^\beta(m,H)\cdot m = \mathcal{L}^\beta(m,H)\cdot m + \beta\frac{\delta^2}{|m|^2+\delta^2}m\cdot(H - \frac{\widehat{m}}{\chi_{tr}}), \tag{4.8}$$

together with the inequality

$$P_\delta(m)\cdot\Delta m = \frac{(m\cdot\Delta m)^2 - |m|^2|\Delta m|^2}{|m|^2+\delta^2} + |\Delta m|^2 \le |\Delta m|^2. \tag{4.9}$$

Consequently, one can easily establish the results given below, see Subsection 3.2, under the same assumptions

$$\int_D \mathcal{L}_\delta^\beta(m,H)\cdot m\,dx$$
$$= -\alpha_l(\frac{1}{\chi_{tr}}\|\widehat{m}\|^2 + \|H\|^2 + \int_D F\varphi\,dx) + \beta\int_D \frac{\delta^2}{|m|^2+\delta^2}(m\cdot H - \frac{|\widehat{m}|^2}{\chi_{tr}})\,dx, \tag{4.10}$$

$$\|\mathcal{L}_\delta^\beta(m,H) + m\times H\|^2 \le C(\|H\|^2 + \|\widehat{m}\|^2 + \|m\|_4^4), \tag{4.11}$$

$$\langle \mathcal{B}_\delta^\beta(m), m\rangle = \alpha_l(\|\nabla m\|^2 + \int_D \zeta(m)\cdot m\,dx), \tag{4.12}$$

$$-\langle \mathcal{B}_\delta^\beta(m), \Delta m\rangle \ge \alpha_l(\|\Delta m\|^2 - \int_D \zeta(m)\cdot\Delta m\,dx). \tag{4.13}$$

We shall prove the following result

THEOREM 4.1. *Let $\delta>0$ be fixed. Under the hypotheses of Theorem 2.1, there exists a global solution (m^δ, H^δ) of problem $(\mathcal{P}^\delta)$ such that $m^\delta\in L^\infty(0,T;\mathbb{H}^1(D))\cap L^2(0,T;\mathbb{H}^2(D))$, $H^\delta = \nabla\varphi^\delta\in L^\infty(0,T;\mathbb{H}^1(D))$ and (m^δ,H^δ) satisfies the estimates given in Theorem 2.1. Therefore m^δ and H^δ are uniformly bounded in $L^\infty(0,T;\mathbb{H}^1(D))\cap L^2(0,T;\mathbb{H}^2(D))$ and $L^\infty(0,T;\mathbb{H}^1(D))$ respectively with respect to δ. Moreover $\partial_t m^\delta$ is uniformly bounded in $L^2(0,T;\mathbb{L}^{3/2}(D))$.*

The next two subsections are devoted to the proof of this theorem.

4.2. Approximate solutions for $(\mathcal{P}_\delta)$. Let $(\Phi_k)_{k\ge1}$ be the Hilbert basis of $\mathbb{H}^1(D)$ defined by

$$-\Delta\Phi_k + \Phi_k = \lambda_k\Phi_k \ \text{ in } D, \quad \nabla\Phi_k\cdot\nu = 0 \ \text{ on }\Gamma, \tag{4.14}$$

then $(\Phi_k)\subset\mathbb{H}^2(D)$ and we will assume this basis to be orthonormal in $\mathbb{L}^2(D)$. For $n\in\mathbb{N}^\star$, we set $\mathcal{V}^n :=$ span$\{\Phi_1,\Phi_2,\cdots,\Phi_n\}$. Let m_0^n be the approximations of the initial data m_0 with

$$m_0^n(x) = \sum_{k=1}^n a_{0k}^n\Phi_k(x),$$

$$m_0^n \to m_0 \text{ strongly in } \mathbb{H}^1(D) \text{ as } n \to +\infty, \tag{4.15}$$

$$\|m_0^n\| \le \|m_0\|, \quad \|\nabla m_0^n\| \le C\|\nabla m_0\|, \quad \forall n \ge 1, \tag{4.16}$$

where $C > 0$ is independent of n.

We seek for approximated solutions $(m^n, H^n)_n$ where $m^n \in \mathcal{V}^n$ is of the form

$$m^n(t,x) = \sum_{k=1}^{n} a_k^n(t)\,\Phi_k(x), \tag{4.17}$$

and H^n is related to m^n through the magnetostatic equations, see (3.8), by

$$H^n = \nabla\varphi^n = \mathcal{H}(m^n, F). \tag{4.18}$$

For each $n \ge 1$, m^n has to satisfy the system labeled $(\mathcal{P}_\delta^n)$ given below

$$\frac{d}{dt}\int_D m^n \cdot \Phi_k\,dx + \langle \mathcal{B}_\delta^\beta(m), \Phi_k\rangle = \int_D \mathcal{L}_\delta^\beta(m^n, H^n)\cdot \Phi_k\,dx, \quad t \in]0,T[, \tag{4.19}$$

for all $k = 1, \cdots, n$ with the initial condition

$$m^n(0) = m_0^n. \tag{4.20}$$

We set $a^n = (a_1^n, a_2^n, \cdots, a_n^n)$ and $a_0^n = (a_{01}^n, a_{02}^n, \cdots, a_{0n}^n)$. Then problem $(\mathcal{P}_\delta^n)$ is a system of ordinary differential equations satisfied by a^n of the following type

$$\frac{da^n}{dt} + \mathcal{M}(t, a^n) = 0, \quad t \in]0,T[, \tag{4.21}$$

$$a^n(0) = a_0^n. \tag{4.22}$$

In solving this system, we get

Proposition 4.1. *Let assumptions of Theorem 2.1 hold, for all $a_0^n \in \mathbb{R}^n$, system (4.21)-(4.22) admits a unique solution $a^n \in \mathcal{C}([0,T]) \cap \mathcal{C}^1(]0,T[)$. Therefore for all $n \ge 1$ and $m_0^n \in \mathcal{V}^n$, problem $(\mathcal{P}_\delta^n)$ admits a unique solution (m^n, H^n) such that $m^n \in \mathcal{C}^1(]0,T[, \mathcal{V}^n) \cap \mathcal{C}([0,T], \mathcal{V}^n)$, $H^n \in \mathcal{C}([0,T], \mathbb{H}^1(D))$.*

Proof. Considering that the functions $\nabla m^n, \Delta m^n, \dfrac{1}{|m^n|^2 + \delta^2}, \omega_\delta(m^m), \zeta(m^n)$ are all of class $\mathcal{C}^1$ with respect to a^n and since the magnetic field H^n depends linearly on m^n and F, we see that $\mathcal{L}_\delta^\beta(m^n, H^n)$ is continuous with respect to (t, a^n) and of class $\mathcal{C}^1$ with respect to a^n and so is $\mathcal{M}(t, a^n)$. By using Cauchy-Lipschitz theorem, the system $(4.21) - (4.22)$ admits a unique local solution a^n. In other words, there exists a time $T_n \in]0,T]$ such that $a^n \in \mathcal{C}^1(]0,T_n[) \cap \mathcal{C}([0,T_n])$ and (m^n, H^n) defined by (4.17) and (4.18) satisfies problem $(\mathcal{P}_\delta^n)$ on $(0, T_n)$. The following uniform bounds which are similar to those of Proposition 3.1, will enable us to extend the solution until time T for all n and end the proof of the proposition. ∎

Lemma 4.1. *There exists $C > 0$ independent of n and δ such that for all $n \ge 1$ the approximated solutions (m^n, H^n) satisfy the bounds*

$$\|m^n(t)\|^2 + 2\alpha_l \int_0^t \mathcal{E}^n(s)\,ds \le \|m_0\|^2 + C(\chi(\theta)T + \|F\|^2_{L^2(D_T)}), \quad t \in [0, T_n], \tag{4.23}$$

$$\int_0^{T_n} \|m^n(s)\|_4^4\,ds \le C(\chi(\theta)T + \|m_0\|^2 + \|F\|^2_{L^2(D_T)}), \tag{4.24}$$

$$\|\nabla m^n(t)\|^2 + \alpha_l \int_0^t \|\Delta m^n(s)\|^2\,ds \le C(\theta,T,m_0,F), \quad t\in[0,T_n], \tag{4.25}$$

$$\|m^n\|^2_{\mathcal{C}([0,T_n];\mathbb{H}^1(D))} + \|H^n\|^2_{\mathcal{C}([0,T_n];\mathbb{H}^1(D))} \le C(\theta,T,m_0,F), \tag{4.26}$$

where $C(\theta,T,m_0,F) = C\Big(\chi(\theta)T + \|m_0\|^2_{\mathbb{H}^1(D)} + \|F\|^2_{L^2(D_T)}\Big)$, $\mathcal{E}^n = \mathcal{E}(m^n,H^n)$ *and the energy* $\mathcal{E}$ *is defined by (2.7).*

Proof. First, multiplying (4.19) by a_k^n, integrating by parts and taking the sum over $k=1,\cdots,n$ we arrive at

$$\frac{1}{2}\frac{d}{dt}\|m^n(t)\|^2 + \langle \mathcal{B}_\delta^\beta(m^n), m^n\rangle = \int_D \mathcal{L}_\beta^\beta(m^n,H^n)\cdot m^n\,dx. \tag{4.27}$$

To deal with the right-hand side of this identity, we use relation (4.10) and the inequalities

$$-\frac{\beta}{\chi_{tr}}\int_D \frac{\delta^2}{|m^n|^2+\delta^2} m^n\cdot \widehat{m}^n\,dx \le 0,$$
$$\beta\Big|\int_D \frac{\delta^2}{|m^n|^2+\delta^2} m^n\cdot H^n\,dx\Big| \le \beta\delta\int_D |H^n|\,dx \le \frac{\alpha_l}{4}\|H^n\|^2 + \frac{\beta^2}{\alpha_l}|D|,$$

for $\delta>0$ small. We observe that we can replace the term $\frac{\beta^2}{\alpha_l}|D|$ of this inequality by $C\chi(\theta)$ in view of the relations (1.9). Hence using (3.5) (with $d=4$), we obtain the inequality

$$\int_D \mathcal{L}_\beta^\beta(m^n,H^n)\cdot m^n\,dx \le -\alpha_l\Big(\int_D \frac{1}{\chi_{tr}}\|\widehat{m^n}\|^2 + \frac{1}{2}\|H^n\|^2\Big) + C\alpha_l\|F\|^2 + C\chi(\theta),$$

and (4.12) allows to get

$$\frac{1}{2}\frac{d}{dt}\|m^n(t)\|^2 + \alpha_l\Big(\|\nabla m^n\|^2 + \int_D \zeta(m^n)\cdot m^n\,dx + \frac{1}{\chi_{tr}}\|\widehat{m^n}\|^2 + \frac{1}{2}\|H^n\|^2\Big)$$
$$\le C(\chi(\theta) + \|F\|^2). \tag{4.28}$$

Upon substituting the expression of $\int_D \zeta(m^n)\cdot m^n\,dx$ given in (3.15), we conclude that (m^n,H^n) satisfies for all $t\in[0,T_n]$, the following bound

$$\|m^n(t)\|^2 + 2\alpha_l\int_0^t \mathcal{E}^n(s)\,ds \le \|m_0^n\|^2 + C(\chi(\theta)t + C\int_0^t \|F(s)\|^2\,ds), \tag{4.29}$$

which leads to (4.23) thanks to (4.16). The second estimate (4.24) is derived from the first one as in the proof of Proposition 3.1.

To prove the next estimates, first we use Green's formula to rewrite the term $\alpha_{tr}\int_D \nabla m^n\cdot\nabla\Phi_k\,dx$ of Equation (4.19) as $-\alpha_{tr}\int_D \Delta m^n\cdot\Phi_k\,dx$. Then multiplying the equation by $(\lambda_k-1)a_k^n$, using (4.14) and taking the sum over $k=1,\cdots,n$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla m^n\|^2 - \langle \mathcal{B}_\delta^\beta(m^n),\Delta m^n\rangle = -\int_D \mathcal{L}_\delta^\beta(m^n,H^n)\cdot\Delta m^n\,dx. \tag{4.30}$$

Therefore thanks to the results given at the end of subSection 4.1, the proof of (3.21) remains valid so with the help of the bounds (4.16) satisfied by m_0^n we get estimate (4.25). At last the bound of H^n in $L^\infty(0,T_n;\mathbb{H}^1(D))$ is derived as for estimate (3.22) given in Proposition 3.1. This ends the proof of the Lemma. ∎

With the estimates given in Lemma 4.1, we may extend the solution (m^n,H^n) over all the interval $[0,T]$ and the uniform bounds given therein are satisfied for all $t\in[0,T]$. We summarize these results and complete them in the following proposition.

PROPOSITION 4.2. *The sequences (m^n) and (H^n) are uniformly bounded with respect to n and δ in $L^\infty(0,T;\mathbb{H}^1(D))\cap L^2(0,T;\mathbb{H}^2(D))\cap W^{1,1}(0,T;\mathbb{L}^2(D))$ and in $L^\infty(0,T;\mathbb{H}^1(D))$ respectively.*

Proof. Since m^n is uniformly bounded with respect to n and δ in $L^2(0,T;\mathbb{H}^1(D))$ and Δm^n is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$ with $\nabla m^n\cdot\nu=0$ on Γ_T, then we deduce that (m^n) is bounded in $L^2(0,T;\mathbb{H}^2(D))$. Now we deal with the uniform bound of $\partial_t m^n$. We have for all $k=1,\cdots,n$

$$\int_D \partial_t m^n\cdot\Phi_k\,dx+\int_D \mathcal{A}_\delta^\beta(m^n)\cdot\Phi_k\,dx=\int_D \mathcal{L}_\delta^\beta(m^n,H^n)\cdot\Phi_k\,dx. \tag{4.31}$$

Since $\partial_t m^n(t)\in\mathcal{V}^n$, the above equation leads to

$$\|\partial_t m^n(t)\|^2=\int_D\left(\mathcal{A}_\delta^\beta(m^n)-\mathcal{L}_\delta^\beta(m^n,H^n)\right)\cdot\partial_t m^n\,dx\le\|\partial_t m^n\|\,\|\mathcal{A}_\delta^\beta(m^n)-\mathcal{L}_\delta^\beta(m^n,H^n)\|,$$

and therefore

$$\|\partial_t m^n(t)\|\le\|\mathcal{A}_\delta^\beta(m^n(t))-\mathcal{L}_\delta^\beta(m^n(t),H^n(t))\|,\quad t\in]0,T[.$$

We will estimate $\mathcal{L}_\delta^\beta(m^n,H^n)$ and $m^n\times\Delta m^n$. First the inequality

$$\|m^n\times H^n\|^2\le C(\|m^n\|_4^4+\|H^n\|_4^4),$$

and the bound (4.11) allow to get the following inequality

$$\|\mathcal{L}_\delta^\beta(m^n,H^n)\|^2\le C\left(\|H^n\|^2+\|\widehat{m}^n\|^2+\|m^n\|_4^4+\|H^n\|_4^4\right), \tag{4.32}$$

so we deduce that $\mathcal{L}_\delta^\beta(m^n,H^n)$ is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$ with respect to n. Using the embedding $H^2(D)\subset L^\infty(D)$ and the inequality $\|m^n\times\Delta m^n\|\le\|m^n\|_\infty\|\Delta m^n\|$, we get that $m^n\times\Delta m^n$ is uniformly bounded in $L^1(0,T;\mathbb{L}^2(D))$ and we conclude that $\partial_t m^n$ is uniformly bounded in $L^1(0,T;\mathbb{L}^2(D))$. ∎

4.3. Convergence as $n\to+\infty$. In this paragraph, we aim to pass to the limit as $n\to\infty$ in problem $(\mathcal{P}_n^\delta)$. In view of the estimates obtained in Proposition 4.2, we infer that

PROPOSITION 4.3. *Let $\delta>0$ be fixed. There exists a subsequence still labeled (m^n,H^n) and (m^δ,H^δ) such that as $n\to+\infty$, $m^n\rightharpoonup m^\delta$ weakly-$\star$ in $L^\infty(0,T;\mathbb{H}^1(D))$ and weakly in $L^2(0,T;\mathbb{H}^2(D))$ whereas $H^n\rightharpoonup H^\delta$ weakly-$\star$ in $L^\infty(0,T;\mathbb{H}^1(D))$. Moreover, we have*

$$m^n\to m^\delta\ \text{ strongly in } L^2(0,T;\mathbb{H}^s(D)),\ s<2, \tag{4.33}$$

$$H^n\to H^\delta=\mathcal{H}(m^\delta,F)\ \text{ strongly in } L^2(0,T;\mathbb{H}^1(D)), \tag{4.34}$$

where $\mathcal{H}$ is the linear mapping defined in (3.8).

Proof. The weak convergences ensue directly from the bounds of Proposition 4.2 and the strong convergence result (4.33) of m^n is obtained using Aubin's lemma (see [15, 16]) and a Sobolev embedding. Therefore, the continuity of operator $\mathcal{H}$ leads to (4.34). □

To perform the limit in the magnetization equation of the problem as $n \to \infty$, we need further convergence results. Let us prove the following ones.

LEMMA 4.2. *Up to a subsequence, we have*

$$\mathcal{A}_\delta^\beta(m^n) \rightharpoonup \mathcal{A}_\delta^\beta(m^\delta) \ \text{ weakly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \tag{4.35}$$

$$\mathcal{L}_\delta^\beta(m^n,H^n) \to \mathcal{L}_\delta^\beta(m^\delta,H^\delta) \ \text{ strongly in } L^2(0,T;\mathbb{L}^{3/2}(D)). \tag{4.36}$$

Proof. We will examine the limit of each nonlinear term involved in $\mathcal{A}_\delta^\beta(m^n)$ and $\mathcal{L}_\delta^\beta(m^n,H^n)$. Let us prove the following convergences

$$m^n \times \Delta m^n \rightharpoonup m^\delta \times \Delta m^\delta \ \text{ weakly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \tag{4.37}$$

$$m^n \times H^n \to m^\delta \times H^\delta \ \text{ strongly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \tag{4.38}$$

$$m^n \times \widehat{m^n} \to m^\delta \times \widehat{m^\delta} \ \text{ strongly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \tag{4.39}$$

$$\zeta(m^n) \rightharpoonup \zeta(m^\delta) \quad \text{weakly in} \quad L^2(0,T;\mathbb{L}^2(D)), \tag{4.40}$$

$$(\omega_\delta(m^n)\cdot\Delta m^n)\omega_\delta(m^n) \rightharpoonup (\omega_\delta(m^\delta)\cdot\Delta m^\delta)\omega_\delta(m^\delta) \ \text{ weakly in } L^2(0,T;\mathbb{L}^2(D)), \tag{4.41}$$

$$(\omega_\delta(m^n)\cdot H^n)\omega_\delta(m^n) \to (\omega_\delta(m^\delta)\cdot H^\delta)\omega_\delta(m^\delta) \ \text{ strongly in } L^2(0,T;\mathbb{L}^2(D)), \tag{4.42}$$

$$(\omega_\delta(m^n)\cdot\widehat{m^n})\omega_\delta(m^n) \to (\omega_\delta(m^\delta)\cdot\widehat{m^\delta})\omega_\delta(m^\delta) \ \text{ strongly in } L^2(0,T;\mathbb{L}^2(D)), \tag{4.43}$$

$$\frac{\delta^2}{|m^n|^2+\delta^2}\Delta m^n \rightharpoonup \frac{\delta^2}{|m^\delta|^2+\delta^2}\Delta m^\delta \ \text{ weakly in } L^2(0,T;\mathbb{L}^2(D)). \tag{4.44}$$

Proof of (4.37). Using Proposition 4.2, Sobolev embeddings and writing

$$\begin{aligned}\|m^n\times\Delta m^n\|_{L^2(0,T;\mathbb{L}^{3/2}(D))} &\le \|m^n\|_{L^\infty(0,T;\mathbb{L}^6(D))}\|\Delta m^n\|_{L^2(0,T;\mathbb{L}^2(D))}\\ &\le C\|m^n\|_{L^\infty(0,T;\mathbb{H}^1(D))}\|\Delta m^n\|_{L^2(0,T;\mathbb{L}^2(D))},\end{aligned}$$

we see that $m^n \times \Delta m^n$ is uniformly bounded in $L^2(0,T;\mathbb{L}^{3/2}(D))$. It follows that there exists a subsequence and Λ^δ such that

$$m^n \times \Delta m^n \rightharpoonup \Lambda^\delta \ \text{ weakly in } L^2(0,T;\mathbb{L}^{3/2}(D)).$$

Since $m^n \to m^\delta$ strongly in $L^2(0,T;\mathbb{H}^1(D))$ and $\Delta m^n \rightharpoonup \Delta m^\delta$ weakly in $L^2(0,T;\mathbb{L}^2(D))$ then $m^n \times \Delta m^n \rightharpoonup m^\delta \times \Delta m^\delta$ at least in the sense of distributions. Hence $\Lambda^\delta = m^\delta \times \Delta m^\delta$.

Proof of (4.38) and (4.39). We write

$$\begin{aligned}&\|m^n\times H^n - m^\delta\times H^\delta\|_{L^2(0,T;\mathbb{L}^{3/2}(D))}\\ &\le \|(m^n-m^\delta)\times H^n\|_{L^2(0,T;\mathbb{L}^{3/2}(D))} + \|m^\delta\times(H^n-H^\delta)\|_{L^2(0,T;\mathbb{L}^{3/2}(D))},\end{aligned}$$

with

$$\|(m^n-m^\delta)\times H^n\|_{L^2(0,T;\mathbb{L}^{3/2}(D))} \le \|H^n\|_{L^\infty(0,T;\mathbb{L}^2(D))}\|(m^n-m^\delta)\|_{L^2(0,T;\mathbb{L}^6(D))},$$

$$\|m^\delta\times(H^n-H^\delta)\|_{L^2(0,T;\mathbb{L}^{3/2}(D))}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^2(D))}\|(H^n-H^\delta)\|_{L^2(0,T;\mathbb{L}^6(D))},$$

so using (4.33) and (4.34) we get the convergence stated in (4.38). The same argument holds true to prove (4.39).

Proof of (4.40). We start with the inequality

$$|\zeta(m^n)|\le C(|m^n|^3+|m^n|)\text{ a.e. in }\ D_T,$$

where $C>0$ is independent of n so we see that $(\zeta(m^n))$ is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$. Therefore there exists a subsequence weakly convergent in this space and since $\zeta(m^n)\to\zeta(m^\delta)$ a.e. in D_T, we conclude that the limit is $\zeta(m^\delta)$.

Proof of (4.41), (4.42) and (4.43). Let us first prove a strong convergence of $(\omega_\delta(m^n))$. We know that $\omega_\delta(m^n)\to\omega_\delta(m^\delta)$ a.e. in D_T and since $|\omega_\delta(m^n)|\le 1$ a.e. in D_T then, by means of Lebesgue dominated convergence theorem

$$\omega_\delta(m^n)\to\omega_\delta(m^\delta)\quad\text{strongly in}\quad L^p(0,T;\mathbb{L}^q(D)),\quad 1\le p,q<+\infty.$$

Similarly $\omega_\delta(m^n)\otimes\omega_\delta(m^n)\to\omega_\delta(m^\delta)\otimes\omega_\delta(m^\delta)$ strongly in $L^p(0,T;\mathbb{L}^q(D))$ for $1\le p,q<+\infty$ where the symbol $\otimes$ denotes the tensorial product of vectors ie $(v\otimes v)_{ij}=v_iv_j$, $1\le i,j\le 3$, $v\in\mathbb{R}^3$. So we have

$$(\omega_\delta(m^n)\cdot\Delta m^n)\omega_\delta(m^n)\to(\omega_\delta(m^\delta)\cdot\Delta m^\delta)\omega_\delta(m^\delta),$$

at least in the sense of distributions. As this sequence is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$, we conclude that $(\omega_\delta(m^n)\cdot\Delta m^n)\omega_\delta(m^n)\rightharpoonup(\omega_\delta(m^\delta)\cdot\Delta m^\delta)\omega_\delta(m^\delta)$ weakly in $L^2(0,T;\mathbb{L}^2(D))$. In the same way, the strong convergence of H^n and $\widehat{m^n}$ in $L^2(0,T;\mathbb{L}^2(D))$ leads to (4.42) and (4.43).

Proof of (4.44). We proceed as above, observing that

$$\frac{\delta^2}{|m^n|^2+\delta^2}\to\frac{\delta^2}{|m^\delta|^2+\delta^2}\quad\text{strongly in}\quad L^p(0,T;\mathbb{L}^q(D)),\quad 1\le p,q<+\infty.$$

This ends the proof of the lemma. □

Now we are in position to perform the limit as $n\to\infty$ in problem $(\mathcal{P}_\delta^n)$ for $\delta>0$ fixed and achieve the proof of Theorem 4.1.

PROPOSITION 4.4. *Let (m^δ,H^δ) be the functions provided by Proposition 4.3 for $\delta>0$. Then (m^δ,H^δ) solves problem $(\mathcal{P}_\delta)$ and satisfies the properties given in Theorem 4.1.*

Proof. First (4.34) means that the magnetostatic equation of problem $(\mathcal{P}_\delta)$ is satisfied. Next inasmuch as the estimates satisfied by the sequence (m^n,H^n) are not only uniform with respect to n but also with respect to δ, we deduce the bounds of (m^δ,H^δ). Let us pass to the limit in the magnetization equation.

Let $\psi=\psi(t)\in\mathcal{D}([0,T[)$, $\Phi=\Phi(x)\in\mathbb{H}^1(D)$ and let $\Phi^n=\sum_{k=1}^n\alpha_k^n\Phi_k\in\mathcal{V}^n$ such that $\Phi^n\to\Phi$ strongly in $\mathbb{H}^1(D)$. Considering the weak formulation (4.19), we have for each $n\ge 1$

$$\begin{aligned}&-\int_{D_T}m^n\cdot\Phi^n\psi'(t)\,dxdt-\psi(0)\int_D m_0^n\cdot\Phi^n\,dx+\int_0^T\langle\mathcal{B}_\delta^\beta(m^n),\Phi^n\rangle\psi(t)\,dt\\&=\int_{D_T}\mathcal{L}_\delta^\beta(m^n,H^n)\cdot\Phi^n\psi(t)\,dxdt.\end{aligned}\tag{4.45}$$

Writing

$$\int_0^T \langle \mathcal{B}_\delta^\beta(m^n),\Phi^n\rangle\psi(t)dt$$
$$=\int_{D_T}(\mathcal{A}_\delta^\beta(m^n)+\alpha_{tr}\Delta m^n)\cdot\Phi^n\psi(t)dt+\alpha_{tr}\int_{D_T}\nabla m^n\cdot\nabla\Phi^n\,\psi\,dxdt,$$

and using the previous lemmas, we deduce that as $n\to\infty$ the limit of the right-hand side is

$$\int_{D_T}(\mathcal{A}_\delta^\beta(m^\delta)+\alpha_{tr}\Delta m^\delta)\cdot\Phi\psi(t)dt+\alpha_{tr}\int_{D_T}\nabla m^\delta\cdot\nabla\Phi\,\psi\,dxdt$$

which is nothing else than $\int_0^T \langle \mathcal{B}_\delta^\beta(m^\delta),\Phi\rangle\psi(t)\,dt$, therefore

$$\lim_{n\to\infty}\int_0^T \langle \mathcal{B}_\delta^\beta(m^n),\Phi^n\rangle\psi(t)\,dt=\int_0^T \langle \mathcal{B}_\delta^\beta(m^\delta),\Phi\rangle\psi(t)\,dt.$$

Consequently we infer that the limit (m^δ,H^δ) satisfies the equation

$$-\int_{D_T} m^\delta\cdot\Phi\,\psi'(t)\,dxdt-\psi(0)\int_D m_0\cdot\Phi\,dx+\int_0^T\langle \mathcal{B}_\delta^\beta(m^\delta),\Phi\rangle\psi(t)dt$$
$$=\int_{D_T}\mathcal{L}_\delta^\beta(m^\delta,H^\delta)\cdot\Phi\,\psi(t)\,dxdt, \tag{4.46}$$

for all $\Phi\in\mathbb{H}^1(D)$ and $\psi\in\mathcal{D}([0,T[)$. In particular we get in $\mathcal{D}'(]0,T[)$ the equation

$$\frac{d}{dt}\int_D m^\delta\cdot\Phi\,dx+\langle \mathcal{B}_\delta^\beta(m^\delta),\Phi\rangle=\int_D \mathcal{L}_\delta^\beta(m^\delta,H^\delta)\cdot\Phi\,dx, \tag{4.47}$$

for all $\Phi\in\mathbb{H}^1(D)$ which leads to

$$\partial_t m^\delta=-\mathcal{A}_\delta^\beta(m^\delta)+\mathcal{L}_\delta^\beta(m^\delta,H^\delta)\quad\text{in } D_T,\quad \nabla m^\delta\cdot\nu=0\quad\text{on }\Gamma_T. \tag{4.48}$$

Hence we deduce that $\partial_t m^\delta\in L^2(0,T;\mathbb{L}^{3/2}(D))\subset L^2(0,T;(\mathbb{H}^2(D))')$ so $m^\delta\in\mathcal{C}([0,T];\mathbb{H}^1(D))$ and then the trace $m^\delta(0)$ is well defined in $\mathbb{H}^1(D)$. Multiplying Equation (4.47) by $\psi\in\mathcal{D}([0,T[)$ and integrating by parts, we derive using (4.46) that $m^\delta(0)=m_0$.

It remains to verify the bound of $\partial_t m^\delta$. We consider Equation (4.48), we easily see that each term of the right-hand side is uniformly bounded in $L^2(0,T;\mathbb{L}^{3/2}(D))$ with respect to δ, the bound of the term $m^\delta\times\Delta m^\delta$ being a consequence of the inequality

$$\|m^\delta\times\Delta m^\delta\|_{L^2(0,T;\mathbb{L}^{3/2}(D))}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^6(D))}\,\|\Delta m^\delta\|_{L^2(0,T;\mathbb{L}^2(D))}.$$

This ends proofs of Proposition 4.4 and Theorem 4.1. □

4.4. End of proof of Theorem 2.1. From Proposition 4.4, we easily deduce the following convergence results.

COROLLARY 4.1. *There exists a subsequence still denoted (m^δ,H^δ) and (m,H) such that as $\delta\to 0$, we have the following weak convergences*

$$m^\delta\rightharpoonup m \textit{ weakly -}\star \textit{ in } L^\infty(0,T;\mathbb{H}^1(D))\quad \textit{and weakly in } L^2(0,T;\mathbb{H}^2(D)),$$

$$\partial_t m^\delta \rightharpoonup \partial_t m \textit{ weakly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \quad H^\delta \rightharpoonup H \textit{ weakly-} \star \textit{ in } L^\infty(0,T;\mathbb{H}^1(D)),$$

as well as the strong convergences stated below

$$\begin{aligned} &m^\delta \to m \textit{ strongly in } L^2(0,T;\mathbb{H}^s(D)), s<2, \\ &H^\delta \to H \textit{ strongly in } L^2(0,T;\mathbb{H}^1(D)), \quad H = \mathcal{H}(m,F), \end{aligned}$$

where $\mathcal{H}$ is defined in (3.8).

In the sequel, we will prove that the limit (m,H) provided by Corollary 4.1 is a solution of problem $(\mathcal{P})$ according to Theorem 2.1. Clearly, the magnetostatic equation is satisfied and to pass to the limit as $\delta\to 0$ in the magnetization equation of problem $(\mathcal{P}^\delta)$, we may distinguish two cases according to whether $\theta>\theta_c$ or $\theta<\theta_c$.

The case $\theta>\theta_c$. It means that $\beta=0$ and we can easily pass to the limit as $\delta\to 0$ in equation (4.46) and proceeding as for the proof of Proposition 4.4, we get that the limit (m,H) satisfies almost everywhere the Equations (2.4)-(1.3) with the boundary and initial conditions (1.5)-(1.6).

The case $\theta<\theta_c$. This case is more complicated. To proceed with, it is useful to introduce the notation

$$p^\delta(v) = |v|^2 + \delta^2, \quad v\in\mathbb{R}^3,$$

and to rewrite Equation (4.48) in the equivalent form

$$\begin{aligned} &p^\delta(m^\delta)\big(\partial_t m^\delta + A(m^\delta)\big) + \beta(m^\delta\cdot\Delta m^\delta)m^\delta + \beta\delta^2\Delta m^\delta \\ =&p^\delta(m^\delta)L(m^\delta,H^\delta) - \beta\big(m^\delta\cdot(H^\delta - \frac{\widehat{m^\delta}}{\chi_{tr}})\big)m^\delta. \end{aligned} \tag{4.49}$$

Further results are needed to be able to pass to the limit as $\delta\to 0$ in this new formulation. Let us prove that

LEMMA 4.3. *The sequence $(p^\delta(m^\delta))$ is uniformly bounded in $L^2(0,T;W^{2,3/2}(D))$ and in $H^1(0,T;L^{6/5}(D))$ and up to a subsequence, the following convergences hold*

$$p^\delta(m^\delta) \rightharpoonup |m|^2 \quad \textit{weakly in } \ L^2(0,T;W^{2,3/2}(D))\cap H^1(0,T;L^{6/5}(D)), \tag{4.50}$$

$$p^\delta(m^\delta) \to |m|^2 \quad \textit{strongly in } \ L^2(0,T;W^{1,3}(D)), \tag{4.51}$$

$$p^\delta(m^\delta)m^\delta \to |m|^2 m \quad \textit{strongly in } L^2(0,T;\mathbb{L}^2(D)). \tag{4.52}$$

Proof. The uniform estimates are a consequence of the bounds of m^δ in $L^\infty(0,T;\mathbb{H}^1(D))\cap L^2(0,T;\,\mathbb{H}^2(D))\cap H^1(0,T;\mathbb{L}^{3/2}(D))$ and the Sobolev embeddings. Indeed m^δ being uniformly bounded in $L^\infty(0,T;\mathbb{L}^6(D))$ means that $p^\delta(m^\delta)$ is uniformly bounded in $L^\infty(0,T;L^3(D))$ and so in $L^2(0,T;\,L^3(D))$. Moreover the uniform bounds of m^δ, ∇m^δ and $\partial_t m^\delta$ in $L^\infty(0,T;\mathbb{L}^6(D))$, $L^2(0,T;\mathbb{L}^6(D))$ and $L^2(0,T;\mathbb{L}^{3/2}(D))$ respectively imply that the first derivatives $\nabla p^\delta(m^\delta)$ and $\partial_t p^\delta(m^\delta)$ of $p^\delta(m^\delta)$ are uniformly bounded in $L^2(0,T;\mathbb{L}^3(D))$ and $L^2(0,T;\mathbb{L}^{6/5}(D))$ respectively.

It remains to estimate the second derivatives $\partial^2_{ij}p^\delta(m^\delta)$ for $1\le i,j\le 3$. We have $\partial^2_{ij}p^\delta(m^\delta) = 2[\partial^2_{ij}m^\delta.m^\delta + \partial_i m^\delta.\partial_j m^\delta]$ so it is uniformly bounded in $L^2(0,T;L^{3/2}(D))$ due to the uniform bounds of m^δ, $\partial^2_{ij}m^\delta$ and $\partial_i m^\delta$ in $L^\infty(0,T;\mathbb{L}^6(D))$, $L^2(0,T;\mathbb{L}^2(D))$ and $L^\infty(0,T;\mathbb{L}^2(D))\cap L^2(0,T;\mathbb{L}^6(D))$ respectively.

Therefore the weak convergence (4.50) follows, the limit being $|m|^2$ since $p^\delta(m^\delta)\to|m|^2$ a.e. in D_T. To obtain the strong convergence (4.51), we apply Aubin's compactness lemma and use the compact embedding $W^{2,3/2}(D)\subset W^{1,3}(D)$. Note that this implies the strong convergence of $p^\delta(m^\delta)$ in $L^2(0,T;L^p(D))$ for all $1\le p<\infty$. The last convergence (4.52) results from the previous ones and the inequality below

$$\Big\|p^\delta(m^\delta)m^\delta-|m|^2m\Big\|_{L^2(0,T;\mathbb{L}^2(D))}\le\|p^\delta(m^\delta)-|m|^2\|_{L^2(0,T;\mathbb{L}^3(D))}\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^6(D))}+\|m\|^2_{L^\infty(0,T;\mathbb{L}^6(D))}\|m^\delta-m\|_{L^2(0,T;\mathbb{L}^6(D))}.$$

□

Let Φ and ψ be test functions in $(\mathcal{D}(\overline{D}))^3$ and $\mathcal{D}([0,T[)$ respectively. We write the weak formulation of equation (4.49) as follows

$$\begin{aligned}&\int_{D_T}p^\delta(m^\delta)\big(\partial_t m^\delta+A(m^\delta)\big)\cdot\Phi\psi\,dxdt\\&\qquad+\beta\int_{D_T}(m^\delta\cdot\Delta m^\delta)(m^\delta\cdot\Phi)\psi\,dxdt+\beta\delta^2\int_{D_T}\Delta m^\delta\cdot\Phi\psi\,dxdt\\&=\int_{D_T}\Big(p^\delta(m^\delta)L(m^\delta,H^\delta)-\beta(m^\delta\cdot(H^\delta-\frac{\widehat{m^\delta}}{\chi_{tr}}))m^\delta\Big)\cdot\Phi\psi\,dxdt.\end{aligned}\tag{4.53}$$

We pass to the limit as $\delta\to0$ in each integral of (4.53) by means of the weak-strong convergence principle, using the convergence results of (m^δ,H^δ) given in Corollary 4.1 and the strong convergences of $p^\delta(m^\delta)$ and $p^\delta(m^\delta)m^\delta$ provided by Lemma 4.3. Henceforh we arrive at

$$\begin{aligned}&\int_{D_T}|m|^2(\partial_t m+A(m))\cdot\Phi\psi\,dxdt+\beta\int_{D_T}(m\cdot\Delta m)(m\cdot\Phi)\psi\,dxdt\\&=\int_{D_T}\Big(|m|^2L(m,H)-\beta(m\cdot(H-\frac{\widehat{m}}{\chi_{tr}}))m\Big)\cdot\Phi\psi\,dxdt.\end{aligned}\tag{4.54}$$

Therefore the magnetization equation given in (2.8) is satisfied almost everywhere in D_T.

It remains to verify the initial and boundary conditions. An integration by parts with respect to the variable t leads to

$$\begin{aligned}\int_{D_T}p^\delta(m^\delta)\partial_t m^\delta\cdot\Phi\psi\,dxdt&=-\int_{D_T}p^\delta(m^\delta)m^\delta\cdot\Phi\psi'\,dxdt\\&\quad-\int_{D_T}\partial_t p^\delta(m^\delta)m^\delta\cdot\Phi\psi\,dxdt-\psi(0)\int_D p^\delta(m_0)m_0\cdot\Phi\,dx,\end{aligned}\tag{4.55}$$

and as $\delta\to0$, exploiting the results of Lemma 4.3 we get

$$\begin{aligned}&\int_{D_T}|m|^2\partial_t m\cdot\Phi\psi\,dxds\\&=-\int_{D_T}|m|^2m\cdot\Phi\psi'\,dxdt-\int_{D_T}\partial_t|m|^2m\cdot\Phi\psi\,dxdt-\psi(0)\int_D|m_0|^2m_0\cdot\Phi\,dx.\end{aligned}\tag{4.56}$$

Therefore integrating again by parts the left-hand side of this equality, we conclude that

$$\psi(0)\int_D|m(0)|^2m(0)\cdot\Phi\,dx=\psi(0)\int_D|m_0|^2m_0\cdot\Phi\,dx,$$

so $|m(0)|^2 m(0) = |m_0|^2 m_0$ almost everywhere in D. Similarly the equality

$$-\int_{D_T} p^\delta(m^\delta)\Delta m^\delta \cdot \Phi\psi\, dxdt$$
$$= \int_{D_T} \nabla(p^\delta(m^\delta)) \otimes \Phi \cdot \nabla m^\delta \psi\, dxdt + \int_{D_T} p^\delta(m^\delta)\nabla m^\delta \cdot \nabla\Phi\psi\, dxdt$$

leads to

$$-\int_{D_T} |m|^2 \Delta m \cdot \Phi\psi\, dxdt$$
$$= \int_{D_T} \nabla(|m|^2) \otimes \Phi \cdot \nabla m \psi\, dxdt + \int_{D_T} |m|^2 \nabla m \cdot \nabla\Phi\psi\, dxdt$$
$$= \int_{D_T} \nabla(|m|^2) \otimes \Phi \cdot \nabla m \psi\, dxdt + \int_{D_T} |m|^2 \nabla m \cdot \nabla\Phi\psi\, dxdt - \int_{\Gamma_T} |m|^2 \nabla m \cdot \nu \Phi\psi\, d\Gamma dt,$$

which means that $|m|^2 \nabla m \cdot \nu = 0$ on Γ_T.

Finally letting $\delta \to 0$ in the estimates satisfied by (m^δ, H^δ) we deduce that (m, H) verifies them too, achieving the proof of Theorem 2.1.

5. The time-periodic problem $(\mathcal{P}_{per})$

In this section we are interested with the existence of time-periodic solutions of (LLB) when F is assumed to be time-periodic with period $T > 0$. This problem labeled problem $(\mathcal{P}_{per})$ is defined by the set of Equations (1.1)-(1.3)-(1.5)-(1.14) and we aim to prove Theorem 2.2.

We will proceed along the lines of proof of Theorem 2.1 and, to avoid repetitions we summarize below the most important steps.

First, we introduce problem $(\mathcal{P}^\delta_{per})$ defined by replacing in problem $(\mathcal{P}_\delta)$ the initial condition with the periodic one

$$m^\delta(0) = m^\delta(T). \tag{5.1}$$

We will prove the existence result stated below, following in some sense, the proof of the existence of time-periodic solutions to the compressible Navier-Stokes equation given in [7, 11, 16].

THEOREM 5.1. *Let $\delta > 0$ be fixed. Under hypotheses of Theorem 2.2, there exists a time-periodic solution (m^δ, H^δ) of problem $(\mathcal{P}^\delta_{per})$ such that $m^\delta \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^2(D))$ and $H^\delta \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1(D))$ with $H^\delta = \mathcal{H}(m^\delta, F)$. Moreover (M^δ, H^δ) satisfies (uniformly with respect to δ) the bounds given in Theorem 2.2.*

To prove Theorem 5.1, we use the Galerkin method so we consider the basis $(\Phi_k)_{k\geq 1}$ introduced in (4.14) and look for approximated solutions (m^n, H^n) of the form (4.17)-(4.18) and satisfying system (4.19) with the periodicity condition

$$m^n(0) = m^n(T). \tag{5.2}$$

This problem named $(\mathcal{P}^{n,\delta}_{per})$ will be solved by resolving the system of ODEs (4.21) subjected to the condition

$$a^n(0) = a^n(T). \tag{5.3}$$

For this purpose, we rely on the results of Section 4 and use a fixed-point procedure, as specified hereafter.

For $a_0^n=(a_{01}^n,a_{02}^n,\cdots,a_{0n}^n)\in\mathbb{R}^n$ and $m_0^n=\sum_{j=1}^n a_{0j}^n\Phi_j\in\mathcal{V}^n$, let $a^n=a^n(t)=(a_1^n,a_2^n,...,a_n^n)\in\mathcal{C}([0,T])\cap\mathcal{C}^1(]0,T[)$ be the solution of (4.21)-(4.22) provided by Proposition 4.1 and let $(m^n,H^n)\in(\mathcal{C}^1(]0,T[,\mathcal{V}^n)\cap\mathcal{C}([0,T],\mathcal{V}^n))\times\mathcal{C}([0,T],\mathbb{H}^1(D))$ be the corresponding solution of problem $(\mathcal{P}_\delta^n)$ with the initial data m_0^n, given in Section 4, that is to say that

$$m^n=\sum_{j=1}^n a_j^n(t)\Phi_j,\quad H^n=\nabla\varphi^n=\mathcal{H}(m^n,F). \tag{5.4}$$

Now we define the mapping

$$S:\mathbb{R}^n\longrightarrow\mathbb{R}^n;\quad S(a_0^n)=a^n(T), \tag{5.5}$$

with the aim to prove that it admits a fixed point a_0^n so that $a_0^n=a^n(T)$. In this case, the corresponding solution m^n satisfies the condition (5.2) giving rise to a time-periodic solution to problem $(\mathcal{P}_{per}^{\delta,n})$. From there it will only remain to pass to the limit as $n\to\infty$ to get a solution (m^δ,H^δ) of problem $(\mathcal{P}_{per}^\delta)$ and then let $\delta\to 0$ to obtain a solution (m,H) of problem $(\mathcal{P}_{per})$.

5.1. Solution of problem $(\mathcal{P}_{per}^{\delta,n})$. We will use the Brouwer fixed-point theorem so the starting point is to find a closed ball of $\mathbb{R}^n$ which is left stable by the mapping S.

LEMMA 5.1. *Assume F to be time-periodic with period T and $F\in\mathcal{C}([0,T];L_\sharp^2(D))$. Let (m^n,H^n) be the solution of $(\mathcal{P}_\delta^n)$ with initial data $m_0^n\in\mathcal{V}^n$ given by (5.4). For $\delta>0$ small enough, we have for all $t\in[0,T]$*

$$\|m^n(t)\|^2\le e^{-\frac{\alpha_l}{\chi_l}t}\|m_0^n\|^2+C(\chi(\theta)t+\int_0^t\|F(s)\|^2\,ds), \tag{5.6}$$

where $C>0$ is independent of n,δ and T. Therefore the closed ball $B(0,\kappa)\subset\mathbb{R}^n$ centered at 0 is stable by the mapping S, the radius $\kappa=\kappa(T,F)>0$ being defined by

$$\kappa^2=(1-e^{-\frac{\alpha_l}{\chi_l}T})^{-1}C(\chi(\theta)T+\|F\|_{L^2(D_T)}^2). \tag{5.7}$$

Proof. We start with the estimate (4.28) satisfied by (m^n,H^n) which results in the following inequality

$$\frac{1}{2}\frac{d}{dt}\|m^n(t)\|^2+\alpha_l\mathcal{E}^n(t)\le C(\chi(\theta)+\|F(t)\|^2), \tag{5.8}$$

where $\mathcal{E}^n=\mathcal{E}(m^n,H^n)$, $\mathcal{E}$ being defined by (2.7) and $C>0$ is independent of n, δ and T. In particular

$$\frac{1}{2}\frac{d}{dt}\|m^n(t)\|^2+\frac{\alpha_l}{2\chi_l}\|m^n(t)\|^2\le C(\chi(\theta)+\|F(t)\|^2), \tag{5.9}$$

which leads to (5.6). Consequently

$$\|m^n(T)\|^2\le e^{-\frac{\alpha_l}{\chi_l}T}\|m_0^n\|^2+C(\chi(\theta)T+\|F\|_{L^2(D_T)}^2). \tag{5.10}$$

Therefore assuming $\|m_0^n\|^2 \le \kappa^2$, we get $\|m^n(T)\|^2 \le e^{-\frac{\alpha_l}{\chi_l}T}\kappa^2 + C(\chi(\theta)T + \|F\|^2_{L^2(D_T)})$ so that

$$\|m^n(T)\|^2 \le C\left(\frac{e^{-\frac{\alpha_l}{\chi_l}T}}{1-e^{-\frac{\alpha_l}{\chi_l}T}} + 1\right)(\chi(\theta)T + \|F\|^2_{L^2(D_T)}) = \kappa^2.$$

Since the basis $(\Phi_k)_{k\ge 1}$ is orthonormal in $\mathbb{L}^2(D)$, this result means that if $|a_0^n|^2 \le \kappa^2$ then $|a^n(T)|^2 = |S(a_0^n)|^2 \le \kappa^2$ or in other words $S(B(0,\kappa)) \subset B(0,\kappa)$. ∎

Lemma 5.1 enables us to get a solution of problem $(\mathcal{P}^{\delta,n}_{per})$.

PROPOSITION 5.1. *Under the hypothesis of Lemma 5.1, the mapping S defined by (5.5) has a fixed point a_0^n in $B(0,\kappa)$. Therefore for all $\delta > 0$ small enough and all $n \ge 1$, problem $(\mathcal{P}^{\delta,n}_{per})$ admits a time-periodic solution (m^n, H^n) of period T satisfying the following bounds*

$$\|m^n(t)\|^2 \le C_T C(\theta,T,F), \quad \forall t \in [0,T], \tag{5.11}$$

$$\int_0^T \mathcal{E}^n(t)\,dt + \|m^n\|^4_{\mathbb{L}^4(D_T)} + \|\Delta m^n\|^2_{\mathbb{L}^2(D_T)} \le C(\theta,T,F), \tag{5.12}$$

$$\|H^n(t)\|^2 \le C_T C(\theta,T,F), \quad \forall t \in [0,T], \tag{5.13}$$

$$\|H^n\|^2_{L^2(0,T;\mathbb{H}^1(D))} \le C(\theta,T,F), \tag{5.14}$$

*with $C(\theta,T,F) = C(\chi(\theta)T + \|F\|^2_{L^2(D_T)})$, $C > 0$ independent of n and δ and $C_T = (2 - e^{-\frac{\alpha_l}{\chi_l}T})(1 - e^{-\frac{\alpha_l}{\chi_l}T})^{-1}$.
Therefore the sequence $(m^n, H^n)_n$ is uniformly bounded in $L^2(0,T;\mathbb{H}^2(D)\times\mathbb{H}^1(D)) \cap L^\infty(0,T;\mathbb{L}^2(D)\times\mathbb{L}^2(D))$ with respect to n and δ and $(\partial_t m^n)_n$ so is in $L^1(0,T;\mathbb{L}^2(D))$.*

Proof. Using the dependence results of the solution of an ode upon the initial data, it is easy to conclude that the application S is continuous over $\mathbb{R}^n$. According to Brouwer's fixed-point theorem the map S admits a fixed point $a_0^n \in B(0,\kappa)$ which means that $a_0^n = S(a_0^n) = a^n(T)$ and therefore $m^n(t) = \sum_{j=1}^n a_j^n(t)\Phi_j$ solves system (4.19)-(5.2). The magnetic field $H^n = \mathcal{H}(m^n, F)$ is also time-periodic with period T so the coupling (m^n, H^n) is a solution of problem $(\mathcal{P}^{\delta,n}_{per})$.

Furthermore from (5.6), we see that $\|m^n(t)\|^2 \le \kappa^2(1 + e^{-\frac{\alpha_l}{\chi_l}t} - e^{-\frac{\alpha_l}{\chi_l}T})$ so m^n satisfies estimate (5.11) for all $t \in [0,T]$ and integrating (5.8) between 0 and T, since $m^n(T) = m^n(0)$ we get the bound of $\int_0^T \mathcal{E}^n(t)\,dt$. Therefore we deduce, using the results (3.4) and (3.7), the estimates of H^n. The $\mathbb{L}^4$-estimate of m^n is derived as we have done in the proof of Proposition 3.1, and to prove the bound of Δm^n, we rely on the results of Section 4 see (3.28), to get

$$\alpha_l \int_0^T \|\Delta m^n(t)\|^2\,dt \le C\int_0^T (\|F(t)\|^2 + \|m(t)\|^2_{\mathbb{H}^1(D)} + \|m(t)\|_4^4)\,dt,$$

so using the previous bounds, we obtain the result. Finally proceeding as in proof of Proposition 4.2, we deduce that $\partial_t m^n$ is uniformly bounded in $L^1(0,T;\mathbb{L}^2(D))$. ∎

5.2. Solution to problem $(\mathcal{P}^\delta_{per})$. We aim to perform the limit when $n \to +\infty$. Using Proposition 5.1, we will prove that

PROPOSITION 5.2. *Let $\delta > 0$ be fixed, there exists a subsequence still labeled (m^n, H^n) and (m^δ, H^δ) which is time-periodic with period T such that the following convergence* *results hold true*

$$(m^n,H^n)\rightharpoonup(m^\delta,H^\delta) \quad \textit{weakly-}\star \textit{ in} \quad L^\infty(0,T;\mathbb{L}^2(D)\times\mathbb{L}^2(D)),$$
$$m^n\rightharpoonup m^\delta \quad \textit{weakly in} \quad L^2(0,T;\mathbb{H}^2(D)),$$
$$(m^n,H^n)\to(m^\delta,H^\delta) \quad \textit{strongly in} \quad L^2(0,T;\mathbb{H}^s(D)\times\mathbb{H}^1(D)),\ s<2.$$

In addition $H^\delta=\mathcal{H}(m^\delta,F)$, (m^δ,H^δ) *is a solution of problem* $(\mathcal{P}^\delta_{per})$ *and the sequences* (m^δ) *and* (H^δ) *are uniformly bounded with respect to* δ *in* $L^\infty(0,T;\mathbb{L}^2(D))\cap L^2(0,T;\mathbb{H}^2(D))\cap\mathbb{L}^4(D_T)$ *and* $L^\infty(0,T;\mathbb{L}^2(D))\cap L^2(0,T;\mathbb{H}^1(D))$ *respectively.*

Proof. We will pass to the limit as $n\to+\infty$ in the weak formulation of problem $(\mathcal{P}^{\delta,n}_{per})$ as we have done in Section 4, this time we use test functions $\Phi\in\mathbb{H}^1(D)$ and $\psi\in\mathcal{D}_{per}$ where

$$\mathcal{D}_{per}=\{v\in\mathcal{D}([0,T]);\ v(T)=v(0)\}.$$

We point out that presently m^n and H^n are not bounded in $L^\infty(0,T;\mathbb{H}^1(D))$. Nevertheless except for the convergence (4.37) of $m^n\times\Delta m^n$, all the other convergences given in the proof of Lemma 4.2 remain valid in the present context because they derive from the boundedness of m^n and H^n in $L^\infty(0,T;\mathbb{L}^2(D))\cap L^2(0,T;\mathbb{H}^1(D))$. For the sequence $m^n\times\Delta m^n$, we prove that up to a subsequence, we have

$$m^n\times\Delta m^n\rightharpoonup m^\delta\times\Delta m^\delta \ \text{ weakly in } L^{4/3}(0,T;\mathbb{L}^{4/3}(D)). \tag{5.15}$$

Indeed we come back to the proof given in Lemma 4.2 and use the boundedness of m^n in $L^4(D_T)$ instead of that in $L^\infty(0,T;\mathbb{L}^6(D))$ to get

$$\|m^n\times\Delta m^n\|_{L^{4/3}(0,T;\mathbb{L}^{4/3}(D))}\le\|m^n\|_{L^4(0,T;\mathbb{L}^4(D))}\|\Delta m^n\|_{L^2(0,T;\mathbb{L}^2(D))}.$$

Hence we get all the convergences stated in Proposition 5.2 and the uniform bounds of Proposition 5.1 satisfied by (m^n,H^n) allow to deduce the same ones for (m^δ,H^δ).

Now we are in position to achieve the limit as $n\to\infty$ and we get

$$-\int_{D_T}m^\delta\cdot\Phi\psi'(t)\,dxdt+\int_0^T\langle\mathcal{B}^\beta_\delta(m^\delta),\Phi\rangle\psi(t)dt=\int_{D_T}\mathcal{L}^\beta_\delta(m^\delta,H^\delta)\cdot\Phi\psi(t)\,dxdt, \tag{5.16}$$

for all $\Phi\in\mathbb{H}^1(D)$ and $\psi\in\mathcal{D}_{per}$. Therefore taking $\psi\in\mathcal{D}(]0,T[)$ we obtain that for all $\Phi\in\mathbb{H}^1(D)$

$$\frac{d}{dt}\int_D m^\delta\cdot\Phi\,dx+\langle\mathcal{B}^\beta_\delta(m^\delta),\Phi\rangle=\int_D\mathcal{L}^\beta_\delta(m^\delta,H^\delta)\cdot\Phi\,dx, \tag{5.17}$$

first in the sense of distributions then almost everywhere in $(0,T)$. Then we see that m^δ satisfies the regularized magnetization equation (4.3) and it is not difficult to show that H^δ verifies the magnetostatic equation. Next we derive that $\partial_t m^\delta\in\mathbb{L}^{4/3}(D_T)$ so $m^\delta\in C([0,T];\mathbb{H}^1(D))$. It remains to verify that $m^\delta(0)=m^\delta(T)$. Let $\Phi\in\mathbb{H}^1(D)$, we multiply (5.17) by $\psi\in\mathcal{D}_{per}$, integrating by parts and using the periodicity of m^n, we get

$$\psi(0)\int_D(m^\delta(T)-m^\delta(0))\cdot\Phi\,dx=0, \tag{5.18}$$

which shows that $m^\delta(T)=m^\delta(0)$ and this implies that $H^\delta(0)=H^\delta(T)$. This ends the proofs of Proposition 5.2 and Theorem 5.1. □

5.3. End of proof of Theorem 2.2. Additional estimates on (m^δ,H^δ) are needed to perform the limit as $\delta\to 0$ in problem $(\mathcal{P}^\delta_{per})$. Let us prove the following results.

PROPOSITION 5.3. *(m^δ,H^δ) satisfies the uniform estimate*

$$\|m^\delta\|_{L^\infty(0,T;\mathbb{H}^1(D))}+\|H^\delta\|_{L^\infty(0,T;\mathbb{H}^1(D))}+\|\partial_t m^\delta\|_{L^2(0,T;\mathbb{L}^{3/2}(D))}\le C(\theta,T,F), \quad (5.19)$$

where $C(\theta,T,F)>0$ is independent of δ.

Proof. We have to prove the bounds of m^δ and $\partial_t m^\delta$ then we derive the estimate of H^δ using (3.11). We begin by establishing the following bound of the time derivative $\partial_t m^\delta$

$$\|\partial_t m^\delta\|_{L^2(0,T;(\mathbb{H}^2(D))')}\le C(\theta,T,F). \quad (5.20)$$

We write $\partial_t m^\delta=-\mathcal{A}^\beta_\delta(m^\delta)+\mathcal{L}^\beta_\delta(m^\delta,H^\delta)$ in D_T, then using the bounds of Theorem 5.1, the embedding $\mathbb{H}^2(D)\subset\mathbb{L}^\infty(D)$ and the following inequalities

$$\|m^\delta\times\widehat{m^\delta}\|^2_{L^2(0,T;\mathbb{L}^2(D))}\le\|m^\delta\|^4_{\mathbb{L}^4(D_T)},$$
$$\|m^\delta\times H^\delta\|_{L^2(0,T;\mathbb{L}^2(D))}\le\|m^\delta\|_{L^2(0,T;\mathbb{L}^\infty(D))}\|H^\delta\|_{L^\infty(0,T;\mathbb{L}^2(D))},$$

we see that all the terms defining $\mathcal{A}^\beta_\delta(m^\delta)$ and $\mathcal{L}^\beta_\delta(m^\delta,H^\delta)$ are bounded uniformly with respect to δ in $L^2(0,T;\mathbb{L}^2(D))$ except for $m^\delta\times\Delta m^\delta$ and $\zeta(m^\delta)$ which are uniformly bounded in $L^2(0,T;\mathbb{L}^1(D))$. Indeed we have

$$\|m^\delta\times\Delta m^\delta\|_{L^2(0,T;\mathbb{L}^1(D))}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^2(D))}\|\Delta m^\delta\|_{L^2(0,T;\mathbb{L}^2(D))}\le C(\theta,T,F),$$

and the bound of $\zeta(m^\delta)$ is a consequence of the inequality

$$\big\||m^\delta(t)|^3\big\|_{\mathbb{L}^1(D)}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^2(D))}\|m^\delta(t)\|^2_4,$$

which holds a.e. $t\in(0,T)$ and implies that

$$\big\||m^\delta|^3\big\|_{L^2(0,T;\mathbb{L}^1(D))}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^2(D))}\|m^\delta\|^2_{\mathbb{L}^4(D_T)}\le C(\theta,T,F).$$

So one concludes that $\|\partial_t m^\delta\|_{L^2(0,T;\mathbb{L}^1(D))}\le C(\theta,T,F)$ and since $\mathbb{L}^1(D)\subset(\mathbb{H}^2(D))'$ we obtain the intermediate result (5.20). To end up to the bound of m^δ stated in (5.19), we introduce the following notations. We set $\mathbb{V}=\mathbb{H}^2(D)$ and $\mathbb{H}=\mathbb{H}^1(D)$, by identifying $\mathbb{H}$ and $\mathbb{H}'$, we get $\mathbb{V}\subset\mathbb{H}\subset\mathbb{V}'$. We have proved that (m^δ) is uniformly bounded in the space $\mathbb{W}=\{v\in L^2(0,T;\mathbb{V});\frac{dv}{dt}\in L^2(0,T;\mathbb{V}')\}$ and since $\mathbb{W}$ is continuously embedded in $\mathcal{C}([0,T];\mathbb{H})$, we conclude that (m^δ) is uniformly bounded in $\mathcal{C}([0,T];\mathbb{H}^1(D))$. To get the bound of $\partial_t m^\delta$, it is enough to see that the following inequalities are satisfied

$$\|m^\delta\times\Delta m^\delta\|_{L^2(0,T;\mathbb{L}^{3/2}(D))}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^6(D))}\|\Delta m^\delta\|_{L^2(0,T;\mathbb{L}^2(D))}\le C(\theta,T,F),$$
$$\big\||m^\delta|^3\big\|_{L^2(0,T;\mathbb{L}^{3/2}(D))}\le\|m^\delta\|_{L^\infty(0,T;\mathbb{L}^6(D))}\|m^\delta\|^2_{L^4(D_T)}\le C(\theta,T,F).$$

□

The estimates of Theorem 5.1 and Proposition 5.3 show that the solutions (m^δ,H^δ) satisfy the same uniform bounds as the solutions of problem $(\mathcal{P}^\delta)$, see Theorem 4.1, therefore the convergence results given in Corollary 4.1 and Lemma 4.3 given in Section 4 remain valid in this case. Then we may pass to the limit as $\delta\to 0$ in problem $(\mathcal{P}^\delta_{per})$ as we have done to end the proof of Theorem 2.1 and we obtain a solution of problem $(\mathcal{P}_{per})$. Once again, the estimates satisfied by (m^δ,H^δ) hold true for (m,H), which ends the proof of Theorem 2.2.

6. The stationary problem $(\mathcal{S})$

In this section we consider the problem $(\mathcal{S})$ defined in (1.15). Our aim is to prove the existence result stated in Theorem 2.3, following globally the same approach as for the unsteady case.

Let $\delta>0$, we define the regularized problem $(\mathcal{S}^\delta)$ as for the unsteady case, more precisely problem $(\mathcal{S}^\delta)$ writes as

$$\begin{aligned}&\mathcal{A}_\delta^\beta(m)-\mathcal{L}_\delta^\beta(m,H)=0 \text{ in } D, \quad \nabla m\cdot\nu=0 \text{ on }\Gamma,\\&\operatorname{div}(H+m)=F, \quad H=\nabla\varphi \text{ in } D, \quad (H+m)\cdot\nu=0 \text{ on }\Gamma,\end{aligned}\tag{6.1}$$

where $\mathcal{A}_\delta^\beta$ and $\mathcal{L}_\delta^\beta$ are defined by (4.4). We will prove the following result

Theorem 6.1. *Let $\delta>0$ be fixed. Under the hypothesis of Theorem 2.3, there exists a solution $(m^\delta,H^\delta)\in\mathbb{H}^2(D)\times\mathbb{H}^1(D)$ of $(\mathcal{S}^\delta)$ verifying uniformly with respect to δ the following inequality*

$$\|m^\delta\|^2_{\mathbb{H}^1(D)}+\|\Delta m^\delta\|^2+\|H^\delta\|^2_{\mathbb{H}^1(D)}\le C(\chi(\theta)+\|F\|^2).\tag{6.2}$$

To prove this result we use Galerkin method to construct approximated solutions (m^n,H^n) then we pass to the limit as $n\to+\infty$. In this context, we consider the Hilbert basis $(\Phi_k)_{k\ge1}$ of $\mathbb{H}^1(D)$ used in the previous sections, assuming this time that the basis is orthonormal in $\mathbb{H}^1(D)$. We set $m^n(x)=\sum_{k=1}^n a_k^n\Phi_k(x)\in\mathcal{V}^n$ and $H^n=\nabla\varphi^n=\mathcal{H}(m^n,F)$ and we look for $a^n=(a_1^n,\dots,a_n^n)\in\mathbb{R}^n$ satisfying the following equations for $k=1,\cdots,n$

$$\langle\mathcal{B}_\delta^\beta(m),\Phi_k\rangle-\int_D\mathcal{L}_\delta^\beta(m^n,H^n)\cdot\Phi_k\,dx=0.\tag{6.3}$$

System (6.3) is a nonlinear algebraic equation of the form $\mathcal{F}(a^n)=0$ where

$$\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2,\cdots,\mathcal{F}_n):\mathbb{R}^n\to\mathbb{R}^n$$

is defined by

$$\begin{aligned}&\forall a=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n,\quad\forall k=1,\cdots,n,\\&\mathcal{F}_k(a)=\langle\mathcal{B}_\delta^\beta(m),\Phi_k\rangle-\int_D\mathcal{L}_\delta^\beta(m,H)\cdot\Phi_k\,dx,\end{aligned}\tag{6.4}$$

where we set $m=\sum_{k=1}^n a_k\Phi_k$ and $H=\mathcal{H}(m,F)$. Let us prove the following result.

Lemma 6.1. *The map $\mathcal{F}$ is continuous on $\mathbb{R}^n$ and there exists $C>0$ independent of δ and n such that for $\delta>0$ small enough,*

$$\mathcal{F}(a)\cdot a\ge0 \textit{ for } a\in\mathbb{R}^n \textit{ such that } |a|^2=C(\chi(\theta)+\|F\|^2).\tag{6.5}$$

Proof. Based on the arguments already developed in Section 4, we can conclude that $\mathcal{F}$ is continuous over $\mathbb{R}^n$. Let us verify (6.5). Let $a\in\mathbb{R}^n$, multiplying each component $\mathcal{F}_k(a)$ in (6.4) by a_k, integrating by parts and summing up the results from $k=1,\dots,n$, we get, following the computations of the proof of Lemma 4.1 (see (4.28)), that for $\delta>0$ small enough

$$\mathcal{F}(a)\cdot a=\langle\mathcal{B}_\delta^\beta(m),m\rangle-\int_D\mathcal{L}_\delta^\beta(m,H)\cdot m\,dx$$

$$\geq \alpha_l\,(\|\nabla m\|^2 + \int_D \zeta(m)\cdot m\,dx + \frac{1}{\chi_{tr}}\|\widehat{m}\|^2 + \frac{1}{2}\|H\|^2) - C(\chi(\theta) + \|F\|^2), \tag{6.6}$$

so using the expression of $\int_D \zeta(m)\cdot m\,dx$ given in (3.15), we get

$$\mathcal{F}(a)\cdot a \geq \alpha_l(\|\nabla m\|^2 + \frac{1}{2\chi_l}\|m\|^2) - C(\chi(\theta) + \|F\|^2). \tag{6.7}$$

Since $\|m\|^2_{\mathbb{H}^1(D)} = \sum_{k=1}^n |a_k|^2 = |a|^2$, this inequality means that

$$\mathcal{F}(a)\cdot a \geq \alpha_l \min(1, \frac{1}{2\chi_l})\,|a|^2 - C(\chi(\theta) + \|F\|^2), \quad \forall a \in \mathbb{R}^n, \tag{6.8}$$

which ends the proof of the lemma. ∎

According to Brouwer fixed-point theorem (see Lemma 4.3 of [11]) one deduces that

PROPOSITION 6.1. *For $\delta > 0$ small enough, there exists $a^n = (a_1^n, a_2^n, \ldots, a_n^n) \in \mathbb{R}^n$ such that*

$$|a^n|^2 \leq C(\chi(\theta) + \|F\|^2) \quad \text{and} \quad \mathcal{F}(a^n) = 0. \tag{6.9}$$

Hence $m^n = \sum_{k=1}^n a_k^n \Phi_k$ satisfies (6.3).

Moreover (m^n, H^n) fulfils the following uniform estimates.

LEMMA 6.2. *There exists a constant $C > 0$ independent of n and δ such that it holds*

$$\|m^n\|^2_{\mathbb{H}^1(D)} + \|H^n\|^2_{\mathbb{H}^1(D)} \leq C(\chi(\theta) + \|F\|^2), \tag{6.10}$$

$$\|\Delta m^n\|^2 \leq C(\chi(\theta) + \|F\|^2). \tag{6.11}$$

Therefore (m^n, H^n) is uniformly bounded in $\mathbb{H}^2(D) \times \mathbb{H}^1(D)$ with respect to n and δ.

Proof. From the inequality given in (6.9), we deduce that $\|m^n\|^2_{\mathbb{H}^1(D)} \leq C(\chi(\theta) + \|F\|^2)$ so we can derive (6.10) by using (3.7). Next, testing Equation (6.3) by $\Delta\Phi_k$, we get estimate (6.11) observing the same calculations as in proving (4.25) in Lemma 4.1 and the bound of m^n in $\mathbb{H}^2(D)$ follows since $\nabla m^n \cdot \nu = 0$ on Γ. ∎

End of proof of Theorem 6.1. We let $n \to \infty$, for δ fixed. From the previous results we deduce, see Section 4 for the details, that if $\delta > 0$ is small enough, then there exists a subsequence still denoted (m^n, H^n) and (m^δ, H^δ) such that

$$m^n \rightharpoonup m^\delta \text{ weakly in } \mathbb{H}^2(D), \quad m^n \to m^\delta \text{ strongly in } \mathbb{H}^s(D),\ s < 2, \tag{6.12}$$

$$H^n \to H^\delta = \mathcal{H}(m^\delta, F) \text{ strongly in } \mathbb{H}^1(D), \tag{6.13}$$

and (m^δ, H^δ) is a solution of problem $(\mathcal{S}^\delta)$. Moreover the estimates of Lemma 6.2 imply that the sequence (m^δ, H^δ) is bounded in $\mathbb{H}^2(D) \times \mathbb{H}^1(D)$ uniformly with respect to δ.

End of proof of Theorem 2.3. From the previous results we deduce that there exists a subsequence still denoted (m^δ, H^δ) and (m, H) such that

$$m^\delta \rightharpoonup m \text{ weakly in } \mathbb{H}^2(D) \quad \text{and} \quad m^\delta \to m \text{ strongly in } \mathbb{H}^s(D),\ s < 2, \tag{6.14}$$

$$H^\delta \to H = \mathcal{H}(m, F) \text{ strongly in } \mathbb{H}^1(D). \tag{6.15}$$

This result allows one to perform the limit when $\delta\to 0$ in problem $(\mathcal{S}^\delta)$ as it was done in the unsteady cases and we conclude that the limit (m,H) satisfies the problem $(\mathcal{S})$ according to the result announced in Theorem 2.3.

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