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REGULARITY RESULTS FOR THE NAVIER-STOKES-MAXWELL SYSTEM*

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Abstract. In this paper, we study the Cauchy problem of the incompressible Navier-Stokes-Maxwell system with Ohm's law in two and three space-dimensions. On the one hand, we establish an improved regularity criterion based on the velocity for the three-dimensional Navier-Stokes-Maxwell system. On the other hand, we establish the global regularity result for the Navier-Stokes-Maxwell system with the dissipation strength at the logarithmically supercritical level both in two and three space-dimensions.

Keywords. Navier-Stokes-Maxwell system; regularity criterion; global regularity.

AMS subject classifications. 35Q35; 35B65; 35B45; 76W05.

1. Introduction

The incompressible Navier-Stokes-Maxwell system consists of the incompressible Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The coupling comes from the Lorentz force in the fluid equation and the electric current in the Maxwell equations. Specially, the incompressible Navier-Stokes-Maxwell system with Ohms law in two and three space-dimensions reads as follows

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = j\times B, \\ \partial_t E - \nabla\times B = -j, \qquad j = \sigma(E + u\times B), \\ \partial_t B + \nabla\times E = 0, \\ \nabla\cdot u = \nabla\cdot B = 0, \\ u(x,0) = u_0(x), \quad E(x,0) = E_0(x), \quad B(x,0) = B_0(x), \end{cases} \tag{1.1}$$

where $u=(u_1,u_2,u_3)$ is the velocity of the incompressible fluid, $E=(E_1,E_2,E_3)$ and $B=(B_1,B_2,B_3)$ are the electric and magnetic fields respectively. All are three-component vector fields. However, in two-dimensional (2D) case, it is assumed that $u_3=E_3=B_1=B_2=0$. The scalar function p is the pressure, ν is the viscosity, j is the electric current which is given by Ohm's law, σ is the electric conductivity. For simplicity, we may assume $\nu=\sigma=1$. The Navier-Stokes-Maxwell system (1.1) is derived from kinetic Vlasov-Maxwell-Boltzmann system [2], which describes the evolution of a plasma (i.e. a charged fluid) subject to a self-induced electromagnetic Lorentz force $j\times B$. The Navier-Stokes-Maxwell system (1.1) has strong physical background, the reader can refer to [4,5] for more details.

Due to their physical applications and mathematical significance, the well-posedness problem on the Navier-Stokes-Maxwell system has attracted considerable attention recently. Let us review some very related works on (1.1). The well-posedness of (1.1) is a highly nontrivial problem both in the context of weak solutions and more regular frameworks. In fact, due to the hyperbolic nature of the Maxwell equation,

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the existence of global-in-time Leray-type weak solutions are completely open, even for the 2D case. A first breakthrough comes from Masmoudi [14] by imposing more regularity on the initial electric and magnetic fields. More precisely, for the initial data $(u_0,E_0,B_0)\in L^2\times H^s\times H^s$ with $s>0$, Masmoudi in [14] proved the existence and uniqueness of global solutions for the system (1.1) in 2D case. His proof highly relies on a time-space logarithmic inequality that enabled him to upper estimate the L^∞-norm of the velocity field by the energy norm and higher Sobolev norms. Another line of research was carried out by Ibrahim and Keraani [8], who proved a local-in-time strong solution in the borderline space $\dot{B}^0_{2,1}\times L^2_{\log}\times L^2_{\log}$, where the space $L^2_{\log}$ resembles an H^s-space with a logarithmic weight on high frequencies instead of an algebraic weight. Based on this, a global-in-time result for small initial data and a local-in-time result for the large initial data in the borderline space $L^2\times L^2_{\log}\times L^2_{\log}$ were derived in [7]. We refer to several recent interesting works in this direction (see [1,16]), which extend the earlier results in many respects. For the three-dimensional (3D) case, up to now, we know the global-in-time result for small initial data in various functional frameworks [1,7–9] and the regularity criteria [6,11,15]. Actually, in the absence of global regularity of the system (1.1) with general initial data in 3D case, it is natural to investigate the regularity criterion, which is of major importance for both theoretical and practical purposes. Our first goal of this paper is to establish two improved regularity criteria based on the velocity for the system (1.1) in 3D case. More precisely, it can be stated as follows.

THEOREM 1.1. *Suppose that* $(u_0,E_0,B_0)\in H^s(\mathbb{R}^3)$ *with* $s>\frac{1}{2}$ *and* $\nabla\cdot u_0=\nabla\cdot B_0=0$. *Let* (u,E,B) *be a local smooth solution to the corresponding system* (1.1). *If one of the following conditions holds*

$$\int_0^T\left(\|u(t)\|^2_{\dot{B}^0_{\infty,2}}+\frac{\|u(t)\|^2_{\dot{B}^{\frac{3}{r}}_{r,\infty}}}{\ln\big(e+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}\big)}\right)dt<\infty,\quad 2\le r<6, \tag{1.2}$$

$$\int_0^T\left(\frac{\|u(t)\|^2_{L^\infty}+\|u(t)\|^2_{\dot{B}^{\frac{3}{r}}_{r,\infty}}}{\ln\big(e+\|u(t)\|_{L^\infty}+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}\big)}\right)dt<\infty,\quad 2\le r<6, \tag{1.3}$$

then the solution remains smooth on $[0,T]$ *and satisfies*

$$(u,E,B)\in L^\infty([0,T];H^s(\mathbb{R}^3)),\quad (\nabla u,j)\in L^2([0,T];H^s(\mathbb{R}^3)).$$

REMARK 1.1. On the one hand, conditions (1.2) and (1.3) are based only on the velocity field. On the other hand, by the fact $L^3(\mathbb{R}^3)\hookrightarrow\dot{B}^{\frac{3}{r}-1}_{r,\infty}(\mathbb{R}^3)$ with $r\in[3,6)$, (1.3) can be regarded as a further improvement of [15, (2.1)].

REMARK 1.2. Unfortunately, at present we are not able to show whether (1.2) and (1.3) hold true for the case $r\in[6,\infty]$, and this is still an interesting problem.

In the 3D case, when the dissipation $-\Delta u$ is replaced by $(-\Delta)^{\frac{3}{2}}u$, the global regularity result can be proved via the basic energy method (see [10]). Consequently, our last objective is to establish the global regularity result with the dissipation strength at the logarithmically supercritical level for the Navier-Stokes-Maxwell system both in 2D case and 3D case. More precisely, it reads as follows.

THEOREM 1.2. *Consider the following Navier-Stokes-Maxwell system both in two and* *three space-dimensions*

$$\begin{cases}\partial_t u+(u\cdot\nabla)u+\mathcal{L}^2u+\nabla p=j\times B,\\ \partial_t E-\nabla\times B=-j, \qquad j=E+u\times B,\\ \partial_t B+\nabla\times E=0,\\ \nabla\cdot u=\nabla\cdot B=0,\\ u(x,0)=u_0(x), \quad E(x,0)=E_0(x), \quad B(x,0)=B_0(x),\end{cases} \tag{1.4}$$

where the operator $\mathcal{L}$ is defined by

$$\widehat{\mathcal{L}u}(\xi)=\frac{|\xi|^{\frac{n}{2}}}{g(\xi)}\widehat{u}(\xi), \quad n=2,3$$

for some non-decreasing symmetric function $g(\tau)\geq 1$ defined on $\tau\geq 0$. Assume the initial data $(u_0,E_0,B_0)\in H^s(\mathbb{R}^n)$ with $s>0$, and $\nabla\cdot u_0=\nabla\cdot B_0=0$. If g satisfies the following growth condition

$$\int_e^\infty \frac{d\tau}{\tau\ln\tau g^2(\tau)}=\infty, \tag{1.5}$$

then (1.4) has a unique global solution (u,E,B) such that for any given $\widetilde{T}>0$,

$$(u,E,B)\in L^\infty([0,\widetilde{T}];H^s(\mathbb{R}^n)), \quad \mathcal{L}u, j\in L^2([0,\widetilde{T}];H^s(\mathbb{R}^n)).$$

REMARK 1.3. We remark that the typical examples satisfying the condition (1.5) are

$$g(\xi)=\left[\ln(e+\ln(e+|\xi|))\right]^{\frac{1}{2}};$$
$$g(\xi)=\left[\ln(e+\ln(e+|\xi|))\ln(e+\ln(e+\ln(e+|\xi|)))\right]^{\frac{1}{2}}.$$

Finally, we establish the global regularity of the 2D incompressible Navier-Stokes-Maxwell system (1.1) with vertical dissipation in the horizontal velocity equation and horizontal dissipation in the vertical velocity equation. More precisely, we prove the following theorem.

THEOREM 1.3. *Consider the following 2D incompressible Navier-Stokes-Maxwell system with vertical dissipation in the horizontal velocity equation and horizontal dissipation in the vertical velocity equation*

$$\begin{cases}\partial_t u_1+(u\cdot\nabla)u_1-\partial_{x_2}^2u_1+\partial_{x_1}p=(j\times B)_1,\\ \partial_t u_2+(u\cdot\nabla)u_2-\partial_{x_1}^2u_2+\partial_{x_2}p=(j\times B)_2,\\ \partial_t E-\nabla\times B=-j, \qquad j=E+u\times B,\\ \partial_t B+\nabla\times E=0,\\ \nabla\cdot u=\nabla\cdot B=0\end{cases} \tag{1.6}$$

subject to the initial data $u(x,0)=u_0(x), E(x,0)=E_0(x), B(x,0)=B_0(x)$. Assume the initial data $(u_0,E_0,B_0)\in H^s(\mathbb{R}^2)$ with $s>0$, and $\nabla\cdot u_0=\nabla\cdot B_0=0$, then (1.6) has a unique global solution (u,E,B) such that for any given $T>0$,

$$(u,E,B)\in L^\infty([0,T];H^s(\mathbb{R}^n)), \quad (\nabla u, j)\in L^2([0,T];H^s(\mathbb{R}^n)).$$

REMARK 1.4. Making use of the following fact

$$\|\nabla u\|_{L^2}=\|\omega\|_{L^2}\le\|\partial_{x_2}u_1\|_{L^2}+\|\partial_{x_1}u_2\|_{L^2},$$

the proof of Theorem 1.3 can be performed as that of Theorem 1.1 and Theorem 1.2. We thus omit the details.

The paper is organized as follows: Section 2 and Section 3 are devoted to the proof of Theorem 1.1 and Theorem 1.2, respectively. For convenience, we present the Littlewood-Paley theory, the Besov spaces and some useful facts in Appendix A. In Appendix B we sketch the proof of (2.6).

2. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Before the proof, we will state several notations. For simplicity, we always denote $\Lambda=(-\Delta)^{\frac{1}{2}}$. In this paper, we shall use the convention that C denotes a generic constant, which may change from line to line. We shall write $C(\alpha_1,\alpha_2,\cdots,\alpha_k)$ as the constant C depends on the quantities $\alpha_1,\alpha_2,\cdots,\alpha_k$. As the existence and uniqueness of local regular solutions in our functional space can be obtained via the classical Friedrich's method (see for instance [17]), it suffices to establish the *a prior* estimates.

Let us begin with the basic energy estimates.

LEMMA 2.1. *Assume (u_0,E_0,B_0) satisfies the conditions stated in Theorem 1.1. Then for any corresponding smooth solution (u,E,B) of (1.1), we have for any $t>0$*

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|E(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2+\int_0^t(\|\nabla u(\tau)\|_{L^2}^2+\|j(\tau)\|_{L^2}^2)\,d\tau\\&\le\|u_0\|_{L^2}^2+\|E_0\|_{L^2}^2+\|B_0\|_{L^2}^2.\end{aligned}\tag{2.1}$$

Proof. Taking the inner product of (1.1) with (u,E,B) and using $\nabla\cdot u=0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2+\|E(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2)+\|\nabla u\|_{L^2}^2=\int_{\mathbb{R}^3}(j\times B)\cdot u\,dx-\int_{\mathbb{R}^3}j\cdot E\,dx,$$

where we have used the following cancellation identity

$$\int_{\mathbb{R}^3}\nabla\times E\cdot B\,dx-\int_{\mathbb{R}^3}\nabla\times B\cdot E\,dx=0.$$

Utilizing the relation $j=E+u\times B$ leads to

$$\begin{aligned}\int_{\mathbb{R}^3}(j\times B)\cdot u\,dx-\int_{\mathbb{R}^3}j\cdot E\,dx&=\int_{\mathbb{R}^3}(j\times B)\cdot u\,dx-\int_{\mathbb{R}^3}j\cdot(j-u\times B)\,dx\\&=-\int_{\mathbb{R}^3}j\cdot j\,dx\\&=-\|j\|_{L^2}^2,\end{aligned}$$

where we have used the simple fact

$$\int_{\mathbb{R}^3}(j\times B)\cdot u\,dx+\int_{\mathbb{R}^3}j\cdot(u\times B)\,dx=0.$$

As a result, it yields

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2+\|E(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\|j\|_{L^2}^2=0.$$

The desired estimate (2.1) follows by integrating it in time. This completes the proof of Lemma 2.1. ∎

For simplicity, we denote

$$M:=\int_0^T\left(\|u(t)\|_{\dot{B}^0_{\infty,2}}^2+\frac{\|u(t)\|^2_{\dot{B}^{\frac{3}{r}}_{r,\infty}}}{\ln(e+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)dt$$

or

$$M:=\int_0^T\left(\frac{\|u(t)\|_{L^\infty}^2+\|u(t)\|^2_{\dot{B}^{\frac{3}{r}}_{r,\infty}}}{\ln(e+\|u(t)\|_{L^\infty}+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)dt.$$

We are now ready to establish the following crucial estimate.

LEMMA 2.2. *Assume (u_0,E_0,B_0) satisfies the conditions stated in Theorem 1.1. If (1.2) holds, then for any corresponding smooth solution (u,E,B) of (1.1), it holds*

$$\begin{aligned}&\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2+\int_0^t(\|\Lambda^\delta\nabla u(\tau)\|_{L^2}^2+\|\Lambda^\delta j(\tau)\|_{L^2}^2)\,d\tau\\&\le C(t,M,u_0,E_0,B_0),\end{aligned}\tag{2.2}$$

where δ satisfies $0<\delta<\frac{3}{r}$ with $r\in[2,6)$.

REMARK 2.1. We remark that following the proof of Lemma 2.2, it requires $\frac{1}{2}<\delta<\frac{3}{r}$ to ensure (2.2). However, due to the basic L^2-energy estimate (2.1), (2.2) with $\frac{1}{2}<\delta<\frac{3}{r}$ and the simple interpolation inequality, one may conclude that (2.2) is true for all $0<\delta<\frac{3}{r}$ with $r\in[2,6)$.

Proof. **(Proof of Lemma 2.2.)** Applying Λ^δ to (1.1) and taking its inner product with $(\Lambda^\delta u,\Lambda^\delta E,\Lambda^\delta B)$, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2)+\|\Lambda^\delta\nabla u\|_{L^2}^2\\&=\int_{\mathbb{R}^3}\Lambda^\delta(j\times B)\cdot\Lambda^\delta u\,dx-\int_{\mathbb{R}^3}\Lambda^\delta j\cdot\Lambda^\delta E\,dx-\int_{\mathbb{R}^3}\Lambda^\delta(u\cdot\nabla u)\cdot\Lambda^\delta u\,dx\\&=\int_{\mathbb{R}^3}\Lambda^\delta(j\times B)\cdot\Lambda^\delta u\,dx-\int_{\mathbb{R}^3}\Lambda^\delta j\cdot\Lambda^\delta(j-u\times B)\,dx-\int_{\mathbb{R}^3}\Lambda^\delta(u\cdot\nabla u)\cdot\Lambda^\delta u\,dx\\&=-\|\Lambda^\delta j\|_{L^2}^2+\int_{\mathbb{R}^3}\Lambda^\delta(j\times B)\cdot\Lambda^\delta u\,dx+\int_{\mathbb{R}^3}\Lambda^\delta j\cdot\Lambda^\delta(u\times B)\,dx\\&\quad-\int_{\mathbb{R}^3}\Lambda^\delta(u\cdot\nabla u)\cdot\Lambda^\delta u\,dx,\end{aligned}\tag{2.3}$$

where we have used the cancellation property

$$\int_{\mathbb{R}^3}\Lambda^\delta\nabla\times E\cdot\Lambda^\delta B\,dx-\int_{\mathbb{R}^3}\Lambda^\delta\nabla\times B\cdot\Lambda^\delta E\,dx=0.$$

Employing (A.2) and $\nabla\cdot u=0$ implies

$$\begin{aligned}\left|-\int_{\mathbb{R}^3}\Lambda^\delta(u\cdot\nabla u)\cdot\Lambda^\delta u\,dx\right|&=\left|\int_{\mathbb{R}^3}\Lambda^\delta(uu)\cdot\Lambda^\delta\nabla u\,dx\right|\\&\le C\|\Lambda^\delta(uu)\|_{L^2}\|\Lambda^\delta\nabla u\|_{L^2}\\&\le C\|u\|_{L^\infty}\|\Lambda^\delta u\|_{L^2}\|\Lambda^\delta\nabla u\|_{L^2}\\&\le\frac{1}{16}\|\Lambda^\delta\nabla u\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\Lambda^\delta u\|_{L^2}^2.\end{aligned}\tag{2.4}$$

Thanks to (A.3) and the duality of homogeneous Besov spaces (see [3, Proposition 2.29]), one derives

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\Lambda^\delta(j\times B)\cdot\Lambda^\delta u\,dx\right|&\le C\|\Lambda^{2\delta}(j\times B)\|_{\dot B^{-\frac{3}{r}}_{\frac{r}{r-1},1}}\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}\\&\le C\|j\times B\|_{\dot B^{2\delta-\frac{3}{r}}_{\frac{r}{r-1},1}}\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}\\&\le C\|j\times B\|_{\dot B^{\varepsilon}_{\frac{3}{3+\varepsilon-2\delta},1}}\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}\\&\le C(\|B\|_{\dot B^{-\theta}_{\frac{6}{3-2(\delta+\theta)},2}}\|j\|_{\dot B^{\varepsilon+\theta}_{\frac{6}{3-2(\delta-\theta-\varepsilon)},2}}\\&\quad+\|j\|_{\dot B^{-\theta}_{\frac{6}{3-2(\delta+\theta)},2}}\|B\|_{\dot B^{\varepsilon+\theta}_{\frac{6}{3-2(\delta-\theta-\varepsilon)},2}})\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}\\&\le C\|\Lambda^\delta B\|_{L^2}\|\Lambda^\delta j\|_{L^2}\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}\\&\le\frac{1}{16}\|\Lambda^\delta j\|_{L^2}^2+C\|u\|^2_{\dot B^{\frac{3}{r}}_{r,\infty}}\|\Lambda^\delta B\|_{L^2}^2,\end{aligned}\tag{2.5}$$

where $\varepsilon>0$ and $\theta>0$ should satisfy

$$\max\left\{0,\ 2\delta-\frac{3}{r}\right\}<\varepsilon<\delta-\theta,\qquad 0<\theta<\min\left\{\delta,\ \frac{3}{r}-\delta\right\}.$$

Let us admit the following inequality (the proof of which is postponed to Appendix B):

$$\|uB\|_{\dot B^{\delta}_{2,2}(\mathbb{R}^n)}\le C(\|u\|_{L^\infty(\mathbb{R}^n)}+\|u\|_{\dot B^{\frac{n}{r}}_{r,\infty}(\mathbb{R}^n)})\|\Lambda^\delta B\|_{L^2(\mathbb{R}^n)}\tag{2.6}$$

for $0<\delta<\frac{n}{r}$ and $r\in[2,\infty)$. Thus, it follows from (2.6) that

$$\begin{aligned}\left|-\int_{\mathbb{R}^3}\Lambda^\delta j\cdot\Lambda^\delta(u\times B)\,dx\right|&\le C\|\Lambda^\delta(u\times B)\|_{L^2}\|\Lambda^\delta j\|_{L^2}\\&\approx C\|u\times B\|_{\dot B^{\delta}_{2,2}}\|\Lambda^\delta j\|_{L^2}\\&\le C(\|u\|_{L^\infty}+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}})\|\Lambda^\delta B\|_{L^2}\|\Lambda^\delta j\|_{L^2}\\&\le\frac{1}{16}\|\Lambda^\delta j\|_{L^2}^2+C(\|u\|_{L^\infty}^2+\|u\|^2_{\dot B^{\frac{3}{r}}_{r,\infty}})\|\Lambda^\delta B\|_{L^2}^2.\end{aligned}\tag{2.7}$$

Putting (2.4), (2.5) and (2.7) into (2.3) yields

$$\frac{d}{dt}(\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2)+\|\Lambda^\delta\nabla u\|_{L^2}^2+\|\Lambda^\delta j\|_{L^2}^2$$

$$\le C(\|u\|_{L^\infty}^2+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2)(\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2). \tag{2.8}$$

Using the following logarithmic Sobolev embedding inequality

$$\|f\|_{L^\infty(\mathbb{R}^3)}\le C\Big(1+\|f\|_{L^2(\mathbb{R}^3)}+\|f\|_{\dot B^0_{\infty,2}(\mathbb{R}^3)}\sqrt{\ln(e+\|f\|_{\dot H^s(\mathbb{R}^3)})}\Big),\quad s>\frac{3}{2},$$

we deduce from (2.8) that

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2)+\|\Lambda^\delta\nabla u\|_{L^2}^2+\|\Lambda^\delta j\|_{L^2}^2\\
&\le C\Big(1+\|u\|_{L^2}+\|u\|_{\dot B^0_{\infty,2}}^2\Big)\ln(e+\|\Lambda^\delta\nabla u\|_{L^2})(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)\\
&\quad+C\left(\frac{\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot B^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}})(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)\\
&\le C\Big(1+\|u\|_{\dot B^0_{\infty,2}}^2\Big)\ln(e+\|\Lambda^\delta\nabla u\|_{L^2}^2)(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)\\
&\quad+C\left(1+\frac{\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot B^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2)(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)\\
&\le C\left(1+\|u\|_{\dot B^0_{\infty,2}}^2+\frac{\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot B^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2+\|\Lambda^\delta\nabla u\|_{L^2}^2)\\
&\quad\times(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)\\
&\le C\left(1+\|u\|_{\dot B^0_{\infty,2}}^2+\frac{\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot B^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|u\|_{L^2}^2+\|\Lambda^\delta\nabla u\|_{L^2}^2)\\
&\quad\times(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)\\
&\le C\left(1+\|u\|_{\dot B^0_{\infty,2}}^2+\frac{\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot B^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|\Lambda^\delta\nabla u\|_{L^2}^2)\\
&\quad\times(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2),
\end{aligned}\tag{2.9}$$

where we need the restriction $\delta>\frac{1}{2}$ and the interpolation inequality

$$\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}\le C\|u\|_{L^2}^{1-\frac{3}{2(\delta+1)}}\|\Lambda^\delta\nabla u\|_{L^2}^{\frac{3}{2(\delta+1)}}\le C(\|u\|_{L^2}+\|\Lambda^\delta\nabla u\|_{L^2}).$$

Such δ exists since $\delta<\frac{3}{r}$ with $2\le r<6$. We also derive from (2.8) that

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2)+\|\Lambda^\delta\nabla u\|_{L^2}^2+\|\Lambda^\delta j\|_{L^2}^2\\
&\le C\left(1+\frac{\|u\|_{L^\infty}^2+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u\|_{L^\infty}+\|u(t)\|_{\dot B^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|u\|_{\dot B^{\frac{3}{r}}_{r,\infty}}^2+\|u\|_{L^\infty}^2)\\
&\quad\times(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2)
\end{aligned}$$

$$\leq C\left(1+\frac{\|u\|_{L^\infty}^2+\|u\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u\|_{L^\infty}+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)\ln(e+\|\Lambda^\delta\nabla u\|_{L^2}^2)$$
$$\times(e+\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2). \tag{2.10}$$

Denoting

$$A(t):=e+\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2,$$
$$B(t):=e+\|\Lambda^\delta \nabla u(t)\|_{L^2}^2+\|\Lambda^\delta j(t)\|_{L^2}^2,$$

we in particular deduce from (2.9) and (2.10) that

$$\frac{d}{dt}A(t)+B(t)\leq C\left(1+\|u\|_{\dot{B}^0_{\infty,2}}^2+\frac{\|u\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)\ln\big(e+B(t)\big)A(t), \tag{2.11}$$

$$\frac{d}{dt}A(t)+B(t)\leq C\left(1+\frac{\|u\|_{L^\infty}^2+\|u\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u\|_{L^\infty}+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)\ln\big(e+B(t)\big)A(t). \tag{2.12}$$

Due to the following simple facts

$$\|u\|_{\dot{B}^0_{\infty,2}}\leq C\|\Lambda^\delta u\|_{L^2}^{\frac{2\delta-1}{2}}\|\Lambda^\delta\nabla u\|_{L^2}^{\frac{3-2\delta}{2}},$$
$$\|u\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}\leq C\|\Lambda^\delta u\|_{L^2}^{\frac{2\delta-1}{2}}\|\Lambda^\delta\nabla u\|_{L^2}^{\frac{3-2\delta}{2}},$$
$$\|u\|_{L^\infty}\leq C\|\Lambda^\delta u\|_{L^2}^{\frac{2\delta-1}{2}}\|\Lambda^\delta\nabla u\|_{L^2}^{\frac{3-2\delta}{2}},$$

we infer that

$$C\left(1+\|u\|_{\dot{B}^0_{\infty,2}}^2+\frac{\|u\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)\leq CA(t)^{\frac{2\delta-1}{2}}B(t)^{\frac{3-2\delta}{2}},$$

$$C\left(1+\frac{\|u\|_{L^\infty}^2+\|u\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}}^2}{\ln(e+\|u\|_{L^\infty}+\|u(t)\|_{\dot{B}^{\frac{3}{r}}_{r,\infty}})}\right)\leq CA(t)^{\frac{2\delta-1}{2}}B(t)^{\frac{3-2\delta}{2}},$$

where $\frac{3-2\delta}{2}<1$ due to $\delta>\frac{1}{2}$. Applying the refined logarithmic Grönwall inequality (see Lemma A.4) to (2.11) and (2.12) yields

$$\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2+\int_0^t(\|\Lambda^\delta\nabla u(\tau)\|_{L^2}^2+\|\Lambda^\delta j(\tau)\|_{L^2}^2)\,d\tau$$
$$\leq C(t,M,u_0,E_0,B_0).$$

Thus, we complete the proof of Lemma 2.2. □

With (2.2) at our disposal, we are in a position to show the global H^s-estimate for any $s>\frac{1}{2}$.

Proof. **(Proof of Theorem 1.1.)** Applying Λ^s for any $s>\frac{1}{2}$ to (1.1), taking the L^2 inner product with $(\Lambda^s u, \Lambda^s E, \Lambda^s B)$ and adding them up, we thus obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s E(t)\|_{L^2}^2+\|\Lambda^s B(t)\|_{L^2}^2)+\|\Lambda^s\nabla u\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx-\int_{\mathbb{R}^3}\Lambda^s j\cdot\Lambda^s E\,dx-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&=\int_{\mathbb{R}^3}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx-\int_{\mathbb{R}^3}\Lambda^s j\cdot\Lambda^s(j-u\times B)\,dx-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&=-\|\Lambda^s j\|_{L^2}^2+\int_{\mathbb{R}^3}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx-\int_{\mathbb{R}^3}\Lambda^s j\cdot\Lambda^s(u\times B)\,dx-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx.
\end{aligned}\tag{2.13}$$

According to (A.2) and the Young inequality, we conclude

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx\right|\\
\le&C\|\Lambda^s(j\times B)\|_{L^{\frac{3}{3-\delta}}}\|\Lambda^s u\|_{L^{\frac{3}{\delta}}}\\
\le&C(\|j\|_{L^{\frac{6}{3-2\delta}}}\|\Lambda^s B\|_{L^2}+\|B\|_{L^{\frac{6}{3-2\delta}}}\|\Lambda^s j\|_{L^2})\times\|\Lambda^s u\|_{L^2}^{\frac{2\delta-1}{2}}\|\Lambda^s\nabla u\|_{L^2}^{\frac{3-2\delta}{2}}\\
\le&C(\|\Lambda^\delta j\|_{L^2}\|\Lambda^s B\|_{L^2}+\|\Lambda^\delta B\|_{L^2}\|\Lambda^s j\|_{L^2})\times\|\Lambda^s u\|_{L^2}^{\frac{2\delta-1}{2}}\|\Lambda^s\nabla u\|_{L^2}^{\frac{3-2\delta}{2}}\\
\le&\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+\frac{1}{8}\|\Lambda^s j\|_{L^2}^2+C\|\Lambda^\delta j\|_{L^2}^{\frac{4}{2\delta+1}}\|\Lambda^s B\|_{L^2}^{\frac{4}{2\delta+1}}\|\Lambda^s u\|_{L^2}^{\frac{2(2\delta-1)}{2\delta+1}}\\
&+C\|\Lambda^\delta B\|_{L^2}^{\frac{4}{2\delta-1}}\|\Lambda^s u\|_{L^2}^2\\
\le&\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+\frac{1}{8}\|\Lambda^s j\|_{L^2}^2+C(\|\Lambda^\delta B\|_{L^2}^{\frac{4}{2\delta-1}}+\|\Lambda^\delta j\|_{L^2}^{\frac{4}{2\delta+1}})\times(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2)\\
\le&\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+\frac{1}{8}\|\Lambda^s j\|_{L^2}^2+C(1+\|\Lambda^\delta B\|_{L^2}^{\frac{4}{2\delta-1}}+\|\Lambda^\delta j\|_{L^2}^2)\times(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2),
\end{aligned}$$

$$\begin{aligned}
&\left|-\int_{\mathbb{R}^3}\Lambda^s j\cdot\Lambda^s(u\times B)\,dx\right|\\
\le&C\|\Lambda^s j\|_{L^2}\|\Lambda^s(u\times B)\|_{L^2}\\
\le&C\|\Lambda^s j\|_{L^2}(\|u\|_{L^\infty}\|\Lambda^s B\|_{L^2}+\|B\|_{L^{\frac{6}{3-2\delta}}}\|\Lambda^s u\|_{L^{\frac{3}{\delta}}})\\
\le&C\|\Lambda^s j\|_{L^2}(\|u\|_{L^\infty}\|\Lambda^s B\|_{L^2}+\|B\|_{L^{\frac{6}{3-2\delta}}}\|\Lambda^s u\|_{L^2}^{\frac{2\delta-1}{2}}\|\Lambda^s\nabla u\|_{L^2}^{\frac{3-2\delta}{2}})\\
\le&\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+\frac{1}{8}\|\Lambda^s j\|_{L^2}^2+C(\|\Lambda^\delta B\|_{L^2}^{\frac{4}{2\delta-1}}+\|u\|_{L^\infty}^2)\times(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2),
\end{aligned}$$

$$\begin{aligned}
\left|-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\right|\le&C\|\Lambda^s\nabla u\|_{L^2}\|\Lambda^s(uu)\|_{L^2}\\
\le&C\|\Lambda^s\nabla u\|_{L^2}\|u\|_{L^\infty}\|\Lambda^s u\|_{L^2}\\
\le&\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2.
\end{aligned}$$

Substituting all the preceding estimates into (2.13) implies

$$\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s E(t)\|_{L^2}^2+\|\Lambda^s B(t)\|_{L^2}^2)+\|\Lambda^s\nabla u\|_{L^2}^2+\|\Lambda^s j\|_{L^2}^2$$

$$\leq C(1+\|\Lambda^\delta B\|_{L^2}^{\frac{4}{2\delta-1}}+\|\Lambda^\delta j\|_{L^2}^2+\|u\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s E\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2). \tag{2.14}$$

Recalling (2.1) and (2.2), one gets

$$\int_0^t (1+\|\Lambda^\delta B(\tau)\|_{L^2}^{\frac{4}{2\delta-1}}+\|\Lambda^\delta j(\tau)\|_{L^2}^2+\|u(\tau)\|_{L^\infty}^2)\,d\tau<\infty,$$

which along with (2.14) and the Grönwall inequality yields

$$\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s E(t)\|_{L^2}^2+\|\Lambda^s B(t)\|_{L^2}^2+\int_0^t (\|\Lambda^s\nabla u(\tau)\|_{L^2}^2+\|\Lambda^s j(\tau)\|_{L^2}^2)\,d\tau<\infty.$$

Therefore, this completes the proof of Theorem 1.1. ∎

3. The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Similar to Lemma 2.1, we have the basic global L^2-bound.

Lemma 3.1. *Assume (u_0,E_0,B_0) satisfies the conditions stated in Theorem 1.2. Then for any corresponding smooth solution (u,E,B) of (1.4), we have for any $t>0$*

$$\|u(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2+\int_0^t \|\mathcal{L}u(\tau)\|_{L^2}^2\,d\tau\leq C(u_0,E_0,B_0). \tag{3.1}$$

Next we are able to derive the following key bound.

Lemma 3.2. *Assume (u_0,E_0,B_0) satisfies the conditions stated in Theorem 1.2. Then for any corresponding smooth solution (u,E,B) of (1.4), we have for any $t>0$*

$$\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2+\int_0^t (\|\Lambda^\delta\mathcal{L}u(\tau)\|_{L^2}^2+\|\Lambda^\delta j(\tau)\|_{L^2}^2)\,d\tau$$
$$\leq C(t,u_0,E_0,B_0), \tag{3.2}$$

where $\delta\in(0,1)$. In particular, it holds

$$\int_0^t \|u(\tau)\|_{L^\infty}^2\,d\tau\leq C(t,u_0,E_0,B_0). \tag{3.3}$$

Proof. Firstly, for $n=2,3$, we claim

$$\frac{d}{dt}(\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2)+\|\Lambda^\delta\mathcal{L}u\|_{L^2}^2+\|\Lambda^\delta j\|_{L^2}^2$$
$$\leq C(\|u\|_{L^\infty}^2+\|u\|_{\dot{B}^1_{n,\infty}}^2)(\|\Lambda^\delta u\|_{L^2}^2+\|\Lambda^\delta E\|_{L^2}^2+\|\Lambda^\delta B\|_{L^2}^2), \tag{3.4}$$

where $\delta\in(0,1)$. Due to (2.8), it suffices to consider the case $n=2$. According to (2.4), (2.5) and (2.7), it is sufficient to estimate (2.5) differently. As a matter of fact, it can be bounded by

$$\left|\int_{\mathbb{R}^3}\Lambda^\delta(j\times B)\cdot\Lambda^\delta u\,dx\right|$$
$$\leq C\|j\times B\|_{\dot{B}^{2\delta-1}_{2,1}}\|u\|_{\dot{B}^1_{2,\infty}}$$
$$\leq C\|j\times B\|_{\dot{B}^{\varepsilon}_{\frac{2}{2+\varepsilon-2\delta},1}}\|u\|_{\dot{B}^1_{2,\infty}}$$

$$
\begin{aligned}
&\leq C(\|B\|_{\dot{B}^{-\theta}_{\frac{2}{1-(\delta+\theta)},2}}\|j\|_{\dot{B}^{\varepsilon+\theta}_{\frac{2}{1-(\delta-\theta-\varepsilon)},2}}+\|j\|_{\dot{B}^{-\theta}_{\frac{2}{1-(\delta+\theta)},2}}\|B\|_{\dot{B}^{\varepsilon+\theta}_{\frac{2}{1-(\delta-\theta-\varepsilon)},2}})\|u\|_{\dot{B}^{1}_{2,\infty}}\\
&\leq C\|\Lambda^{\delta}B\|_{L^2}\|\Lambda^{\delta}j\|_{L^2}\|u\|_{\dot{B}^{1}_{2,\infty}}\\
&\leq \frac{1}{16}\|\Lambda^{\delta}j\|_{L^2}^2+C\|u\|_{\dot{B}^{1}_{2,\infty}}^2\|\Lambda^{\delta}B\|_{L^2}^2,
\end{aligned}
$$

where $\varepsilon>0$ and $\theta>0$ should satisfy

$$
\max\{0,\ 2\delta-1\}<\varepsilon<\delta-\theta,\qquad 0<\theta<\min\{\delta,\ 1-\delta\}.
$$

Therefore, the desired (3.4) holds true for both $n=2$ and $n=3$. Noticing the assumptions on g (more precisely, g grows logarithmically), we infer that for any fixed $\gamma>0$, there exist $N=N(\gamma)$ and $\widetilde{C}=\widetilde{C}(\gamma)$ such that

$$
g(r)\leq \widetilde{C}r^{\gamma},\quad \forall r\geq N.
$$

Consequently, for any $0<\gamma<\frac{n}{2}$, we conclude

$$
\begin{aligned}
\|\mathcal{L}f\|_{L^2}^2&=\int_{|\xi|<N(\gamma)}\frac{|\xi|^n}{g^2(|\xi|)}|\widehat{f}(\xi)|^2\,d\xi+\int_{|\xi|\geq N(\gamma)}\frac{|\xi|^n}{g^2(|\xi|)}|\widehat{f}(\xi)|^2\,d\xi\\
&\geq \int_{|\xi|\geq N(\gamma)}\frac{|\xi|^n}{\big[\widetilde{C}|\xi|^{\gamma}\big]^2}|\widehat{f}(\xi)|^2\,d\xi\\
&=\int_{\mathbb{R}^n}\frac{|\xi|^n}{\big[\widetilde{C}|\xi|^{\gamma}\big]^2}|\widehat{f}(\xi)|^2\,d\xi-\int_{|\xi|<N(\gamma)}\frac{|\xi|^n}{\big[\widetilde{C}|\xi|^{\gamma}\big]^2}|\widehat{f}(\xi)|^2\,d\xi\\
&\geq C_1\|\Lambda^{\frac{n}{2}-\gamma}f\|_{L^2}^2-C_2\|f\|_{L^2}^2, \qquad (3.5)
\end{aligned}
$$

where C_1 and C_2 depend only on n and γ. It follows from the high-low frequency technique that

$$
\|u\|_{L^\infty}\leq \|\Delta_{-1}u\|_{L^\infty}+\sum_{l=0}^{N-1}\|\Delta_l u\|_{L^\infty}+\sum_{l=N}^{\infty}\|\Delta_l u\|_{L^\infty},
$$

where Δ_l $(l=-1,0,1,\cdots)$ denotes the frequency operator (see Appendix A for details). Thanks to the Bernstein-type inequality (see Lemma A.1), we deduce that for $\frac{n}{2}-\delta<\varrho<\frac{n}{2}$

$$
\|\Delta_{-1}u\|_{L^\infty}\leq C\|u\|_{L^2},
$$

$$
\begin{aligned}
\sum_{l=N}^{\infty}\|\Delta_l u\|_{L^\infty}&\leq C\sum_{l=N}^{\infty}2^{\frac{n}{2}l}\|\Delta_l u\|_{L^2}\\
&=C\sum_{l=N}^{\infty}2^{l(\frac{n}{2}-\delta-\varrho)}\|\Delta_l\Lambda^{\delta+\varrho}u\|_{L^2}\\
&\leq C2^{N(\frac{n}{2}-\delta-\varrho)}\|\Lambda^{\delta+\varrho}u\|_{L^2}\\
&\leq C2^{N(\frac{n}{2}-\delta-\varrho)}(\|\Lambda^{\delta}u\|_{L^2}+\|\mathcal{L}\Lambda^{\delta}u\|_{L^2}),
\end{aligned}
$$

where in the last line we have used the inequality

$$
\|\Lambda^{\delta+\varrho}u\|_{L^2}\leq C(\|\Lambda^{\delta}u\|_{L^2}+\|\mathcal{L}\Lambda^{\delta}u\|_{L^2}).
$$

Actually, the above inequality can be deduced from the proof of (3.5) by replacing f with $\Lambda^\delta u$. Invoking the Bernstein-type inequality and the Plancherel theorem yields

$$\begin{aligned}
\sum_{l=0}^{N-1}\|\Delta_l u\|_{L^\infty} &\le C\sum_{l=0}^{N-1}2^{\frac{n}{2}l}\|\Delta_l u\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}\|\Delta_l\Lambda^{\frac{n}{2}}u\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}\Big\|\varphi(2^{-l}\xi)|\xi|^{\frac{n}{2}}\widehat{u}(\xi)\Big\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}\Big\|\varphi(2^{-l}\xi)g(|\xi|)\frac{|\xi|^{\frac{n}{2}}}{g(|\xi|)}\widehat{u}(\xi)\Big\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}g(2^l)\Big\|\frac{|\xi|^{\frac{n}{2}}}{g(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|_{L^2}\\
&\le C\Big(\sum_{l=0}^{N-1}g^2(2^l)\Big)^{\frac12}\left(\sum_{l=0}^{N-1}\Big\|\frac{|\xi|^{\frac{n}{2}}}{g(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|_{L^2}^2\right)^{\frac12}\\
&\le Cg(2^N)\Big(\sum_{l=1}^{N-1}1\Big)^{\frac12}\Big\|\frac{\Lambda^{\frac{n}{2}}}{g(\Lambda)}u\Big\|_{L^2}\\
&\le Cg(2^N)\sqrt{N}\|\mathcal{L}u\|_{L^2}.
\end{aligned}$$

Hence, we obviously deduce that

$$\|u\|_{L^\infty}\le C\|u\|_{L^2}+Cg(2^N)\sqrt{N}\|\mathcal{L}u\|_{L^2}+C2^{N(\frac{n}{2}-\delta-\varrho)}(\|\Lambda^\delta u\|_{L^2}+\|\mathcal{L}\Lambda^\delta u\|_{L^2}).\tag{3.6}$$

Exactly along the same lines as deriving (3.6), it is not hard to check that

$$\|u\|_{\dot B^{\frac{n}{r}}_{r,\infty}}\le C\|u\|_{L^2}+Cg(2^N)\sqrt{N}\|\mathcal{L}u\|_{L^2}+C2^{N(\frac{n}{2}-\delta-\varrho)}(\|\Lambda^\delta u\|_{L^2}+\|\mathcal{L}\Lambda^\delta u\|_{L^2}).\tag{3.7}$$

As a matter of fact, (3.7) can be improved as

$$\|u\|_{\dot B^{\frac{n}{r}}_{r,\infty}}\le C\|u\|_{L^2}+Cg(2^N)\|\mathcal{L}u\|_{L^2}+C2^{N(\frac{n}{2}-\delta-\varrho)}(\|\Lambda^\delta u\|_{L^2}+\|\mathcal{L}\Lambda^\delta u\|_{L^2}).$$

For simplicity, we denote

$$X(t):=\|\Lambda^\delta u(t)\|_{L^2}^2+\|\Lambda^\delta E(t)\|_{L^2}^2+\|\Lambda^\delta B(t)\|_{L^2}^2,$$

$$Y(t):=\|\Lambda^\delta\mathcal{L}u(t)\|_{L^2}^2+\|\Lambda^\delta j(t)\|_{L^2}^2.$$

It thus follows from (3.4) that

$$\frac{d}{dt}X(t)+Y(t)\le C\Big(1+g^2(2^N)N\|\mathcal{L}u\|_{L^2}^2+2^{2N(\frac{n}{2}-\delta-\varrho)}\big(X(t)+Y(t)\big)\Big)X(t).$$

Taking $N\in\mathbb{N}$ such that

$$2^N\approx[2C(e+X(t))]^\alpha\quad\text{or}\quad N=\left[\frac{\alpha\ln\big(2C(e+X(t))\big)}{\ln 2}\right]+1,\qquad \alpha:=\frac{1}{2\delta+2\varrho-n}>0,$$

we know that

$$\begin{aligned}C2^{2N(\frac{n}{2}-\delta-\varrho)}\big(X(t)+Y(t)\big)X(t)&=C2^{2N(\frac{n}{2}-\delta-\varrho)}X(t)Y(t)+C2^{2N(\frac{n}{2}-\delta-\varrho)}X(t)X(t)\\&=C2^{-\frac{N}{\alpha}}X(t)Y(t)+C2^{-\frac{N}{\alpha}}X(t)X(t)\\&\le\frac{1}{2}Y(t)+\frac{1}{2}X(t).\end{aligned}$$

Therefore, we can deduce

$$\begin{aligned}\frac{d}{dt}X(t)+Y(t)\le{}&C\big(1+g^2([(2C(e+X(t))]^{\alpha})\ln(e+X(t))\|\mathcal{L}u\|_{L^2}^2\big)X(t)\\&+\frac{1}{2}X(t)+\frac{1}{2}Y(t),\end{aligned}$$

which implies

$$\frac{d}{dt}X(t)+Y(t)\le C(e+X(t))\ln(e+X(t))g^2([2C(e+X(t))]^{\alpha})(1+\|\mathcal{L}u\|_{L^2}^2). \quad (3.8)$$

Then, (3.8) allows us to show

$$\int_{e+X(0)}^{e+X(t)}\frac{d\tau}{\tau\ln\tau g^2([2C\tau]^{\alpha})}\le C\int_0^t\big(1+\|\mathcal{L}u(\tau)\|_{L^2}^2\big)\,d\tau. \quad (3.9)$$

On the one hand, thanks to (1.5) and variable substitutions, we deduce

$$\int_e^{\infty}\frac{d\tau}{\tau\ln\tau g^2([2C\tau]^{\alpha})}=\int_{(2Ce)^{\alpha}}^{\infty}\frac{d\tau}{\tau(\ln\tau-\alpha\ln 2C)g^2(\tau)}\ge\int_{(2Ce)^{\alpha}}^{\infty}\frac{d\tau}{\tau\ln\tau g^2(\tau)}=\infty. \quad (3.10)$$

On the other hand, (3.1) implies

$$\int_0^t\big(1+\|\mathcal{L}u(\tau)\|_{L^2}^2\big)\,d\tau\le C(t,u_0,E_0,B_0). \quad (3.11)$$

Combining (3.9), (3.10) with (3.11), we may verify that $X(t)$ is finite for any finite $t>0$, namely,

$$X(t)\le C(t,u_0,E_0,B_0).$$

Coming back to (3.8), one can now conclude that

$$\int_0^t Y(\tau)\,d\tau\le C(t,u_0,E_0,B_0).$$

Consequently, we get the desired estimate (3.2). According to (3.5), we easily find

$$\|\Lambda^{\vartheta}u\|_{L^2}\le C(\|u\|_{L^2}+\|\mathcal{L}\Lambda^{\delta}u\|_{L^2}),\quad \forall\vartheta<\delta+\frac{n}{2}.$$

Taking $\vartheta\in(\frac{n}{2},\delta+\frac{n}{2})$ and making use of the simple interpolation inequality, we get

$$\int_0^t\|u(\tau)\|_{L^\infty}^2\,d\tau\le C\int_0^t(\|u(\tau)\|_{L^2}^2+\|\Lambda^{\vartheta}u(\tau)\|_{L^2}^2)\,d\tau\le C(t,u_0,E_0,B_0),$$

which is the desired bound (3.3). This ends the proof of Lemma 3.2. ∎

We are now ready to complete the proof of Theorem 1.2.

Proof. **(Proof of Theorem 1.2.)** Applying Λ^s for any $s>0$ to (1.4) and taking the L^2 inner product with $(\Lambda^s u, \Lambda^s E, \Lambda^s B)$, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s E(t)\|_{L^2}^2+\|\Lambda^s B(t)\|_{L^2}^2)+\|\Lambda^s \mathcal{L}u\|_{L^2}^2\\
&=\int_{\mathbb{R}^n}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx-\int_{\mathbb{R}^n}\Lambda^s j\cdot\Lambda^s E\,dx-\int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&=\int_{\mathbb{R}^n}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx-\int_{\mathbb{R}^n}\Lambda^s j\cdot\Lambda^s(j-u\times B)\,dx-\int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&=-\|\Lambda^s j\|_{L^2}^2+\int_{\mathbb{R}^n}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx-\int_{\mathbb{R}^n}\Lambda^s j\cdot\Lambda^s(u\times B)\,dx\\
&\quad-\int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx.
\end{aligned}\tag{3.12}$$

Following the argument in obtaining (3.5), it gives

$$\|\Lambda^{s+\gamma}u\|_{L^2}\le C(\|\Lambda^s u\|_{L^2}+\|\Lambda^s\mathcal{L}u\|_{L^2}),\quad \forall\gamma\in\Big[0,\frac{n}{2}\Big),\tag{3.13}$$

$$\|\Lambda^{\delta+\gamma}u\|_{L^2}\le C(\|\Lambda^\delta u\|_{L^2}+\|\Lambda^\delta\mathcal{L}u\|_{L^2}),\quad \forall\gamma\in\Big[0,\frac{n}{2}\Big).\tag{3.14}$$

Again, using (A.2), we conclude by (3.13) that

$$\begin{aligned}
\left|\int_{\mathbb{R}^n}\Lambda^s(j\times B)\cdot\Lambda^s u\,dx\right|&\le C\|\Lambda^s(j\times B)\|_{L^{\frac{n}{n-\delta}}}\|\Lambda^s u\|_{L^{\frac{n}{\delta}}}\\
&\le C(\|j\|_{L^{\frac{2n}{n-2\delta}}}\|\Lambda^s B\|_{L^2}+\|B\|_{L^{\frac{2n}{n-2\delta}}}\|\Lambda^s j\|_{L^2})\\
&\quad\times\|\Lambda^s u\|_{L^2}^{1-\frac{n-2\delta}{2\gamma}}\|\Lambda^{s+\gamma}u\|_{L^2}^{\frac{n-2\delta}{2\gamma}}\\
&\le C(\|j\|_{L^{\frac{2n}{n-2\delta}}}\|\Lambda^s B\|_{L^2}+\|B\|_{L^{\frac{2n}{n-2\delta}}}\|\Lambda^s j\|_{L^2})\\
&\quad\times\|\Lambda^s u\|_{L^2}^{1-\frac{n-2\delta}{2\gamma}}(\|\Lambda^s u\|_{L^2}+\|\Lambda^s\mathcal{L}u\|_{L^2})^{\frac{n-2\delta}{2\gamma}}\\
&\le\frac{1}{8}\|\Lambda^s\mathcal{L}u\|_{L^2}^2+\frac{1}{8}\|\Lambda^s j\|_{L^2}^2+C(1+\|\Lambda^\delta B\|_{L^2}^{\frac{4\gamma}{2\delta+2\gamma-n}}+\|\Lambda^\delta j\|_{L^2}^2)\\
&\quad\times(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2),
\end{aligned}$$

$$\begin{aligned}
\left|-\int_{\mathbb{R}^n}\Lambda^s j\cdot\Lambda^s(u\times B)\,dx\right|&\le C\|\Lambda^s j\|_{L^2}\|\Lambda^s(u\times B)\|_{L^2}\\
&\le C\|\Lambda^s j\|_{L^2}(\|u\|_{L^\infty}\|\Lambda^s B\|_{L^2}+\|B\|_{L^{\frac{2n}{n-2\delta}}}\|\Lambda^s u\|_{L^{\frac{n}{\delta}}})\\
&\le C\|\Lambda^s j\|_{L^2}\Big(\|u\|_{L^\infty}\|\Lambda^s B\|_{L^2}\\
&\quad+\|\Lambda^\delta B\|_{L^2}(\|\Lambda^s u\|_{L^2}+\|\Lambda^s\mathcal{L}u\|_{L^2})^{\frac{n-2\delta}{2\gamma}}\Big)\\
&\le\frac{1}{8}\|\Lambda^s\mathcal{L}u\|_{L^2}^2+\frac{1}{8}\|\Lambda^s j\|_{L^2}^2+C(1+\|\Lambda^\delta B\|_{L^2}^{\frac{4\gamma}{2\delta+2\gamma-n}}+\|u\|_{L^\infty}^2)\\
&\quad\times(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2).
\end{aligned}$$

In view of $\nabla\cdot u=0$, (A.1), (3.14) and (3.13), it follows that

$$\begin{aligned}\Big|-\int_{\mathbb{R}^n}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\Big|=&\Big|-\int_{\mathbb{R}^n}[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u\,dx\Big|\\ \le& C\|[\Lambda^s,u\cdot\nabla]u\|_{L^{\frac{2n}{n+\gamma}}}\|\Lambda^s u\|_{L^{\frac{2n}{n-\gamma}}}\\ \le& C\|\nabla u\|_{L^{\frac{n}{\gamma}}}\|\Lambda^s u\|^2_{L^{\frac{2n}{n-\gamma}}}\\ \le& C(\|u\|_{L^2}+\|\Lambda^{\delta+\gamma}u\|_{L^2})\|\Lambda^s u\|_{L^2}\|\Lambda^{s+\gamma}u\|_{L^2}\\ \le& C(\|u\|_{L^2}+\|\Lambda^{\delta}u\|_{L^2}+\|\Lambda^{\delta}\mathcal{L}u\|_{L^2})\|\Lambda^s u\|_{L^2}\\ &\times(\|\Lambda^s u\|_{L^2}+\|\Lambda^s\mathcal{L}u\|_{L^2})\\ \le&\frac{1}{8}\|\Lambda^s\mathcal{L}u\|^2_{L^2}+C(1+\|u\|^2_{L^2}+\|\Lambda^{\delta}u\|^2_{L^2}+\|\Lambda^{\delta}\mathcal{L}u\|^2_{L^2})\|\Lambda^s u\|^2_{L^2}\\ \le&\frac{1}{8}\|\Lambda^s\mathcal{L}u\|^2_{L^2}+C(1+\|\Lambda^{\delta}u\|^2_{L^2}+\|\Lambda^{\delta}\mathcal{L}u\|^2_{L^2})\|\Lambda^s u\|^2_{L^2}.\end{aligned}$$

Substituting the above estimates into (3.12), we achieve

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^s u(t)\|^2_{L^2}+\|\Lambda^s E(t)\|^2_{L^2}+\|\Lambda^s B(t)\|^2_{L^2})+\|\Lambda^s\mathcal{L}u\|^2_{L^2}+\|\Lambda^s j\|^2_{L^2}\\ \le& H(t)(\|\Lambda^s u\|^2_{L^2}+\|\Lambda^s E\|^2_{L^2}+\|\Lambda^s B\|^2_{L^2}),\end{aligned}\tag{3.15}$$

where $H(t)$ is given by

$$H(t):=C(1+\|\Lambda^{\delta}B(t)\|^{\frac{4\gamma}{2\delta+2\gamma-n}}_{L^2}+\|\Lambda^{\delta}j(t)\|^2_{L^2}+\|u(t)\|^2_{L^\infty}+\|\Lambda^{\delta}u(t)\|^2_{L^2}+\|\Lambda^{\delta}\mathcal{L}u(t)\|^2_{L^2}).$$

Keeping in mind (3.2) and (3.3) implies

$$\int_0^t H(\tau)\,d\tau\le C(t,u_0,E_0,B_0).$$

This together with (3.15) and the Grönwall inequality lead to

$$\begin{aligned}&\|\Lambda^s u(t)\|^2_{L^2}+\|\Lambda^s E(t)\|^2_{L^2}+\|\Lambda^s B(t)\|^2_{L^2}\\ &+\int_0^t(\|\Lambda^s\mathcal{L}u(\tau)\|^2_{L^2}+\|\Lambda^s j(\tau)\|^2_{L^2})\,d\tau\le C(t,u_0,E_0,B_0).\end{aligned}\tag{3.16}$$

Finally, we are going to show the uniqueness. To this end, we consider two solutions $(u^{(1)},E^{(1)},B^{(1)},j^{(1)},p^{(1)})$ and $(u^{(2)},E^{(2)},B^{(2)},j^{(2)},p^{(2)})$ of (1.4), emanating from the same initial data, and fulfilling the above estimates (2.1) and (3.16). Denoting $\widetilde{u}=u^{(1)}-u^{(2)}$, $\widetilde{E}=E^{(1)}-E^{(2)}$, $\widetilde{B}=B^{(1)}-B^{(2)}$, $\widetilde{j}=j^{(1)}-j^{(2)}$ and $\widetilde{p}=p^{(1)}-p^{(2)}$, we deduce

$$\begin{cases}\partial_t\widetilde{u}+(u^{(2)}\cdot\nabla)\widetilde{u}+\mathcal{L}^2\widetilde{u}+\nabla\widetilde{p}=\widetilde{j}\times B^{(2)}+j^{(1)}\times\widetilde{B}-(\widetilde{u}\cdot\nabla)u^{(1)},\\ \partial_t\widetilde{E}-\nabla\times\widetilde{B}=-\widetilde{j},\qquad \widetilde{j}=\widetilde{E}+u^{(1)}\times\widetilde{B}+\widetilde{u}\times B^{(2)},\\ \partial_t\widetilde{B}+\nabla\times\widetilde{E}=0,\\ \nabla\cdot\widetilde{u}=\nabla\cdot\widetilde{B}=0,\\ \widetilde{u}(x,0)=0,\quad \widetilde{E}(x,0)=0,\quad \widetilde{B}(x,0)=0.\end{cases}\tag{3.17}$$

Taking the inner product of (3.17) with $(\widetilde{u}, \widetilde{E}, \widetilde{B})$ yields

$$\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{E}(t)\|_{L^2}^2+\|\widetilde{B}(t)\|_{L^2}^2)+\|\mathcal{L}\widetilde{u}\|_{L^2}^2+\|\widetilde{j}\|_{L^2}^2=J_1+J_2+J_3, \tag{3.18}$$

where

$$J_1=\int_{\mathbb{R}^n} j^{(1)}\times\widetilde{B}\cdot\widetilde{u}\,dx,\quad J_2=\int_{\mathbb{R}^n}(u^{(1)}\times\widetilde{B})\cdot\widetilde{j}\,dx,\quad J_3=-\int_{\mathbb{R}^n}(\widetilde{u}\cdot\nabla)u^{(1)}\cdot\widetilde{u}\,dx.$$

It follows from (3.13) that

$$\|\Lambda^\gamma\widetilde{u}\|_{L^2}\le C(\|\widetilde{u}\|_{L^2}+\|\mathcal{L}\widetilde{u}\|_{L^2}),\quad \forall\gamma\in\Big[0,\frac{n}{2}\Big), \tag{3.19}$$

By several interpolation inequalities and (3.19), we can show that

$$\begin{aligned}
|J_1|\le& C\|j^{(1)}\|_{L^{\frac{n}{\gamma}}}\|\widetilde{B}\|_{L^2}\|\widetilde{u}\|_{L^{\frac{2n}{n-2\gamma}}}\\
\le& C(\|j^{(1)}\|_{L^2}+\|\Lambda^s j^{(1)}\|_{L^2})\|\widetilde{B}\|_{L^2}\|\Lambda^\gamma\widetilde{u}\|_{L^2}\\
\le& C(\|j^{(1)}\|_{L^2}+\|\Lambda^s j^{(1)}\|_{L^2})\|\widetilde{B}\|_{L^2}(\|\widetilde{u}\|_{L^2}+\|\mathcal{L}\widetilde{u}\|_{L^2})\\
\le& \frac{1}{8}\|\mathcal{L}\widetilde{u}\|_{L^2}^2+C(1+\|j^{(1)}\|_{L^2}^2+\|\Lambda^s j^{(1)}\|_{L^2}^2)(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{B}\|_{L^2}^2),
\end{aligned}$$

$$\begin{aligned}
|J_2|\le& C\|u^{(1)}\|_{L^\infty}\|\widetilde{B}\|_{L^2}\|\widetilde{j}\|_{L^2}\\
\le& \frac{1}{8}\|\widetilde{j}\|_{L^2}^2+C\|u^{(1)}\|_{L^\infty}^2\|\widetilde{B}\|_{L^2}^2\\
\le& \frac{1}{8}\|\widetilde{j}\|_{L^2}^2+C(\|u^{(1)}\|_{L^2}^2+\|\Lambda^s\mathcal{L}u^{(1)}\|_{L^2}^2)\|\widetilde{B}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
|J_3|\le& C\|\nabla u^{(1)}\|_{L^{\frac{n}{\gamma}}}\|\widetilde{u}\|_{L^{\frac{2n}{n-\gamma}}}^2\\
\le& C(\|u^{(1)}\|_{L^2}+\|\Lambda^{s+\gamma}u^{(1)}\|_{L^2})\|\widetilde{u}\|_{L^2}\|\Lambda^\gamma\widetilde{u}\|_{L^2}\\
\le& C(\|u^{(1)}\|_{L^2}+\|\Lambda^s u^{(1)}\|_{L^2}+\|\Lambda^s\mathcal{L}u^{(1)}\|_{L^2})\|\widetilde{u}\|_{L^2}\times(\|\widetilde{u}\|_{L^2}+\|\mathcal{L}u\|_{L^2})\\
\le& \frac{1}{8}\|\mathcal{L}\widetilde{u}\|_{L^2}^2+C(1+\|u^{(1)}\|_{L^2}^2+\|\Lambda^s u^{(1)}\|_{L^2}^2+\|\Lambda^s\mathcal{L}u^{(1)}\|_{L^2}^2)\|\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

Putting the above estimates into (3.18) implies

$$\frac{d}{dt}(\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{E}(t)\|_{L^2}^2+\|\widetilde{B}(t)\|_{L^2}^2)\le G(t)(\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{E}(t)\|_{L^2}^2+\|\widetilde{B}(t)\|_{L^2}^2), \tag{3.20}$$

where

$$G(t):=C(1+\|j^{(1)}\|_{L^2}^2+\|\Lambda^s j^{(1)}\|_{L^2}^2+\|u^{(1)}\|_{L^2}^2+\|\Lambda^s u^{(1)}\|_{L^2}^2+\|\Lambda^s\mathcal{L}u^{(1)}\|_{L^2}^2)(t).$$

Thanks to (2.1) and (3.16), we have

$$\int_0^t G(\tau)\,d\tau<\infty,$$

which, along with (3.20) and the Grönwall inequality, gives

$$\widetilde{u}(t)=\widetilde{E}(t)=\widetilde{B}(t)=0.$$

This yields the uniqueness of the solution. Consequently, we complete the proof of Theorem 1.2. ∎

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Appendix A. Besov spaces and some useful facts. This appendix recalls the Littlewood-Paley theory, introduces the Besov spaces and provides some useful facts. We start with the Littlewood-Paley theory. We choose some smooth radial non-increasing function χ with values in $[0,1]$ such that $\chi\in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B}:=\{\xi\in\mathbb{R}^n,|\xi|\le\frac{4}{3}\}$ and with value 1 on $\{\xi\in\mathbb{R}^n,|\xi|\le\frac{3}{4}\}$, then we set $\varphi(\xi)=\chi\big(\frac{\xi}{2}\big)-\chi(\xi)$. One easily verifies that $\varphi\in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C}:=\{\xi\in\mathbb{R}^n,\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ and satisfies

$$\chi(\xi)+\sum_{j\ge0}\varphi(2^{-j}\xi)=1,\quad \forall\xi\in\mathbb{R}^n.$$

Let $h=\mathcal{F}^{-1}(\varphi)$ and $\widetilde{h}=\mathcal{F}^{-1}(\chi)$, then we introduce the dyadic blocks Δ_j of our decomposition by setting

$$\Delta_j u=0,\ \ j\le-2;\qquad \Delta_{-1}u=\chi(D)u=\int_{\mathbb{R}^n}\widetilde{h}(y)u(x-y)\,dy;$$

$$\Delta_j u=\varphi(2^{-j}D)u=2^{jn}\int_{\mathbb{R}^n}h(2^jy)u(x-y)\,dy,\ \ \forall j\in\mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_ju=\chi(2^{-j}D)u=\sum_{-1\le k\le j-1}\Delta_k u=2^{jn}\int_{\mathbb{R}^n}\widetilde{h}(2^jy)u(x-y)\,dy,\ \ \forall j\in\mathbb{N}.$$

Meanwhile, we define the homogeneous dyadic blocks as

$$\dot{\Delta}_j u=\varphi(2^{-j}D)u=2^{jn}\int_{\mathbb{R}^n}h(2^jy)u(x-y)\,dy,\ \ \forall j\in\mathbb{Z}.$$

We denote the function spaces of rapidly decreasing functions by $S(\mathbb{R}^n)$, tempered distributions by $S'(\mathbb{R}^n)$, and polynomials by $\mathcal{P}(\mathbb{R}^n)$. Now we are in a position to define the homogeneous and inhomogeneous Besov spaces through the dyadic decomposition.

Definition A.1. *Let $s\in\mathbb{R},(p,r)\in[1,+\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as a space of $f\in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that*

$$\dot{B}^s_{p,r}=\{f\in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n);\|f\|_{\dot{B}^s_{p,r}}<\infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}}=\begin{cases}\Big(\sum_{j\in\mathbb{Z}}2^{jrs}\|\dot{\Delta}_jf\|_{L^p}^r\Big)^{\frac{1}{r}}, & \forall\ r<\infty,\\ \sup_{j\in\mathbb{Z}}2^{js}\|\dot{\Delta}_jf\|_{L^p}, & \forall\ r=\infty.\end{cases}$$

Definition A.2. *Let $s\in\mathbb{R},(p,r)\in[1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f\in S'(\mathbb{R}^n)$ such that*

$$B^s_{p,r}=\{f\in S'(\mathbb{R}^n);\|f\|_{B^s_{p,r}}<\infty\},$$

where

$$\|f\|_{B^s_{p,r}}=\begin{cases}\Big(\sum\limits_{j\geq -1}2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & \forall\ r<\infty,\\ \sup\limits_{j\geq -1}2^{js}\|\Delta_j f\|_{L^p}, & \forall\ r=\infty.\end{cases}$$

The following lemma is the well-known Bernstein-type inequality (see [3, Lemma 2.1]).

Lemma A.1. *Assume $1\leq a\leq b\leq\infty$. Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball of $\mathbb{R}^n$. Then it holds*

$$\mathrm{Supp}\widehat{f}\subset\lambda\mathcal{B}\quad\Rightarrow\quad \|\Lambda^k f\|_{L^b}\leq C_1\lambda^{k+n(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a},\quad k\geq 0;$$

$$\mathrm{Supp}\widehat{f}\subset\lambda\mathcal{C}\quad\Rightarrow\quad C_2\lambda^k\|f\|_{L^b}\leq\|\Lambda^k f\|_{L^b}\leq C_3\lambda^{k+n(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a},\quad k\in\mathbb{R},$$

where C_1, C_2 and C_3 are constants depending on n,k,a and b only.

We also need the so-called Kato-Ponce-type inequalities (see [12, 13]).

Lemma A.2. *Let $p,p_1,p_3\in(1,\infty)$ and $p_2,p_4\in[1,\infty]$ satisfy*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Then for $s>0$, there exists a positive constant C such that

$$\|[\Lambda^s,f]g\|_{L^p}\leq C\left(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|\Lambda^{s-1}g\|_{L^{p_3}}\|\nabla f\|_{L^{p_4}}\right),\tag{A.1}$$

$$\|\Lambda^s(gf)\|_{L^p}\leq C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|\Lambda^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}).\tag{A.2}$$

Next we recall the following bilinear estimate in the homogeneous Besov spaces (see [19, Lemma1]).

Lemma A.3. *Assume that $1\leq p,r\leq\infty, s>0,\delta_1>0,\delta_2>0$ and $1\leq p_i,r_i\leq\infty (i=1,2,3,4)$ satisfy*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4},\qquad \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}.$$

Then there exists a constant C such that

$$\|fg\|_{\dot{B}^s_{p,r}}\leq C\|f\|_{\dot{B}^{-\delta_1}_{p_1,r_1}}\|g\|_{\dot{B}^{s+\delta_1}_{p_2,r_2}}+C\|g\|_{\dot{B}^{-\delta_2}_{p_3,r_3}}\|f\|_{\dot{B}^{s+\delta_2}_{p_4,r_4}}.\tag{A.3}$$

Finally, we recall the refined logarithmic Grönwall inequality [18, Lemma 2.7].

Lemma A.4. *Let A and B be two absolutely continuous and nonnegative functions on $(0,T)$ for any given $T>0$, satisfying*

$$A'(t)+B(t)\leq[l(t)+m(t)\ln(A+e)+n(t)\ln(A+B+e)](A+e)+f(t),$$

for any $t\in(0,T)$, *where* $l(t), m(t), n(t)$ *and* $f(t)$ *are all nonnegative and integrable functions on* $(0,T)$. *Assume further that there are three constants* $K\in[0,\infty)$, $\alpha\in[0,\infty)$ *and* $\beta\in[0,1)$ *such that for any* $t\in(0,T)$

$$n(t)\le K\big(A(t)+e\big)^{\alpha}\big(A(t)+B(t)+e\big)^{\beta}.$$

Then the following estimate holds

$$A(t)+\int_0^t B(s)\,ds\le \widetilde{C}(l,m,n,f,\alpha,\beta,K,t)<\infty,$$

for any $t\in(0,T)$.

Appendix B. The proof of (2.6).

Thanks to the so-called Bony decomposition, we have

$$uB=\dot{T}_uB+\dot{T}_Bu+\dot{R}(u,B),$$

where the definitions of $\dot{T}_vw$ and $\dot{R}(v,w)$ can be found in [3, Definition 2.45]. By Theorem 2.47 and Theorem 2.52 in [3, Section 2.6], it is not hard to check that

$$\begin{aligned}\|\dot{T}_uB\|_{\dot{B}^{\delta}_{2,2}}&\le C\|u\|_{L^\infty}\|B\|_{\dot{B}^{\delta}_{2,2}}\\&\approx C\|u\|_{L^\infty}\|\Lambda^{\delta}B\|_{L^2},\\ \|\dot{T}_Bu\|_{\dot{B}^{\delta}_{2,2}}&\le C\|B\|_{\dot{B}^{\delta-\frac{n}{r}}_{\frac{2r}{r-2},2}}\|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}}\\&\le C\|B\|_{\dot{B}^{\delta}_{2,2}}\|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}}\\&\approx \|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}}\|\Lambda^{\delta}B\|_{L^2},\end{aligned}\tag{B.1}$$

$$\begin{aligned}\|\dot{R}(u,B)\|_{\dot{B}^{\delta}_{2,2}}&\le C\|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}}\|B\|_{\dot{B}^{\delta-\frac{n}{r}}_{\frac{2r}{r-2},2}}\\&\le C\|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}}\|B\|_{\dot{B}^{\delta}_{2,2}}\\&\approx \|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}}\|\Lambda^{\delta}B\|_{L^2},\end{aligned}\tag{B.2}$$

where we need $\delta<\frac{n}{r}$ in (B.1) and $\delta>0$ in (B.2). Summing them up together gives

$$\|uB\|_{\dot{B}^{\delta}_{2,2}}\le C(\|u\|_{L^\infty}+\|u\|_{\dot{B}^{\frac{n}{r}}_{r,\infty}})\|\Lambda^{\delta}B\|_{L^2},$$

which concludes the proof of (2.6).

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