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THE KINETIC FOKKER-PLANCK EQUATION WITH WEAK CONFINEMENT FORCE*

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Abstract. We consider the kinetic Fokker-Planck equation with weak confinement force. We prove some (polynomial and sub-exponential) rate of convergence to the equilibrium (depending on the space to which the initial datum belongs). Our results generalize some results known for strong confinement to the weak confinement case.

Keywords. weak hypocoercivity; weak hypodissipativity; Fokker-Planck equation; semigroup; weak Poincaré inequality; rate of convergence.

AMS subject classifications. 47D06; 35P15; 35B40; 35Q84.

1. Introduction

In this paper, we consider the weak hypocoercivity issue for a solution f to the kinetic Fokker-Planck (KFP for short) equation

$$\partial_t f = \mathcal{L}f := -v\cdot\nabla_x f + \nabla_x V(x)\cdot\nabla_v f + \Delta_v f + \mathrm{div}_v(vf), \tag{1.1}$$

on a function $f=f(t,x,v)$, with $t\geq 0$, $x\in\mathbb{R}^d$, $v\in\mathbb{R}^d$. The evolution Equation (1.1) is complemented with an initial datum

$$f(0,\cdot)=f_0 \text{ on } \mathbb{R}^{2d}.$$

Throughout the paper, we make the assumption on the confinement potential V

$$V(x)=\langle x\rangle^\gamma,\quad \gamma\in(0,1),$$

where $\langle x\rangle^2 := 1+|x|^2$. Let us make some elementary but fundamental observations. First, the equation is mass conservative, that is

$$\mathcal{M}(f(t,\cdot))=\mathcal{M}(f_0),\quad \forall t\geq 0,$$

where we define the mass of f by

$$\mathcal{M}(f)=\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,dx\,dv.$$

Next, we observe that the function

$$G=Z^{-1}e^{-W},\quad W=\frac{|v|^2}{2}+V(x),\quad Z\in\mathbb{R}_+ \tag{1.2}$$

is a positive normalized steady state of the KFP model, precisely

$$\mathcal{L}G=0,\quad G>0,\quad \mathcal{M}(G)=1,$$

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by choosing the normalizing constant $Z>0$ appropriately. Finally we observe that, contrary to the case $\gamma\geq 1$, a Poincaré inequality of the type

$$\exists c>0, \quad \int_{\mathbb{R}^d} |\phi(x)|^2 \exp(-V(x))dx \leq c\int_{\mathbb{R}^d} |\nabla\phi(x)|^2 \exp(-V(x))dx,$$

for any smooth function $\phi:\mathbb{R}^d\to\mathbb{R}$ such that

$$\int_{\mathbb{R}^d} \phi(x)\exp(-V(x))dx=0,$$

does not hold. Only a weaker version of this inequality remains true (see [11, 18], or Section 2 below). In particular, there is no spectral gap for the associated operator $\mathcal{L}$, nor is there an exponential trend to the equilibrium for the associated semigroup.

For a given weight function m, we will denote $L^p(m)=\{f|fm\in L^p\}$ the associated Lebesgue space and $\|f\|_{L^p(m)}=\|fm\|_{L^p}$ the associated norm. The notation $A\lesssim B$ means $A\leq CB$ for some constant $C>0$.

The main result of this paper writes as follows.

Theorem 1.1.

(1) For any initial datum $f_0\in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p\in[1,\infty)$, $\epsilon>0$ small, the associated solution $f(t,\cdot)$ to the kinetic Fokker-Planck Equation (1.1) satisfies

$$\|f(t,\cdot)-\mathcal{M}(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b}\|f_0-\mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})},$$

for any $b\in(0,\frac{\gamma}{2-\gamma})$ and some constant $C>0$.

(2) For any initial datum $f_0\in L^1(m)$, $m=H^k$, $H=|x|^2+|v|^2+1$, $k\geq 1$, the associated solution $f(t,\cdot)$ to the kinetic Fokker-Planck Equation (1.1) satisfies

$$\|f(t,\cdot)-\mathcal{M}(f_0)G\|_{L^1} \lesssim (1+t)^{-\frac{k}{1-\frac{\gamma}{2}}}\|f_0-\mathcal{M}(f_0)G\|_{L^1(m)}.$$

The constants in the estimates only depend on γ,d,ϵ,p,k.

Remark 1.1. Let us emphasize the loss of tail control in both estimates in Theorem 1.1, which is reminiscent of decay estimates in sub-geometric contexts.

Remark 1.2. In the results above the constants can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 1.3. Theorem 1.1 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$, that is for any $V(x)$ satisfying

$$C_1\langle x\rangle^\gamma \leq V(x)\leq C_2\langle x\rangle^\gamma, \quad \forall x\in\mathbb{R}^d,$$
$$C_3|x|\langle x\rangle^{\gamma-1}\leq x\cdot\nabla_x V(x)\leq C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x\in B_R^c,$$

with B_R denoting the ball centered at origin with radius R and $B_R^c=\mathbb{R}^d\setminus B_R$, and

$$|D_x^2V(x)|\leq C_5\langle x\rangle^{\gamma-2}, \quad \forall x\in\mathbb{R}^d,$$

for some constants $C_i>0$, $R>0$.

REMARK 1.4. There are many classical results about the strong confinement framework corresponding to $\gamma \geq 1$. In this case there is a spectral gap on the operator $\mathcal{L}$ and exponential decay estimates on the associated semigroup $S_{\mathcal{L}}$, we refer the interested readers to [2–4, 7–10, 15, 22].

REMARK 1.5. For the Fokker-Planck equation with weak confinement force, a sub-geometric convergence to equilibrium is established in [11, 18, 20]

REMARK 1.6. There are already some convergence results for the KFP equation with weak confinement case considered in the present paper proved by probability method. In [1], a polynomial rate of convergence to the equilibrium is established, in total variation distance with some weight norm, and in [5], a sub-geometric rate of convergence in total variation distance with some weight norm is established. Both papers use Harris' theorem, which is first introduced in [21] and pushed forward in [19] to show exponential convergence to equilibrium. This paper extends the result to L^p spaces and larger spaces.

One advantage of the method in this paper is that it can yield convergence on a wider range of initial conditions and L^p spaces, while previous proofs of convergence to equilibrium mainly using some strong L^1 norms (probability method) or L^2 norms (PDE methods). Also the method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative-type argument such as the consequence of Krein-Rutman theorem. While our method also has some disadvantages, it requires that the equation has an explicit steady state.

Let us briefly explain the main ideas behind our method of proof.

We introduce four spaces $E_1 = L^2(G^{-1/2})$, $E_2 = L^2(G^{-1/2}e^{\epsilon_1 V(x)})$, $E_3 = L^2(G^{-(1+\epsilon_2)/2})$ and $E_0 = L^2(G^{-1/2}\langle x\rangle^{\gamma-1})$, with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small such that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset L^2$. Thus E_1 is an "interpolation" space between E_0 and E_2. We first use a hypocoercivity argument as in [3, 4] to prove that, for any $f_0 \in E_3$, the solution f to the KFP Equation (1.1) satisfies

$$\frac{d}{dt}\|f(t)\|_{\tilde{E}_1} \leq -\lambda \|f(t)\|_{E_0},$$

for some constant $\lambda > 0$, where the norm of $\tilde{E}_1$ is equivalent to the norm of E_1. We use this and the Duhamel formula to prove

$$\|f(t)\|_{E_2} \lesssim \|f_0\|_{E_3}.$$

Combining the two inequalities and using an interpolation argument as in [11], we get

$$\|f(t)\|_{E_1} \lesssim e^{-at^b}\|f_0\|_{E_3}, \tag{1.3}$$

for some $a > 0, b \in (0,1)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [16] and developed in [6, 13]. For any operator $\mathcal{L}$, denote $S_{\mathcal{L}}(t)$ the associated semigroup. We introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is an appropriately defined bounded operator so that $\mathcal{B}$ becomes a dissipative operator. Moreover we prove that $S_{\mathcal{B}}$ satisfies some regularization estimate

$$\|S_{\mathcal{B}}(t)\|_{L^p(m_1)\to L^2(m_2)} \lesssim t^{-\alpha}, \quad \forall t \in [0,\eta],$$

for any $p\in[1,2)$, some weight function m_1, m_2 and some $\alpha,\eta>0$, and using the iterated Duhamel's formula

$$S_{\mathcal{L}}=S_{\mathcal{B}}+\sum_{l=1}^{n-1}(S_{\mathcal{B}})*(\mathcal{A}S_{\mathcal{B}})^{(*l)}+S_{\mathcal{L}}*(\mathcal{A}S_{\mathcal{B}})^{(*n)}, \tag{1.4}$$

we deduce the L^p convergence on $S_{\mathcal{L}}$ as stated in Theorem 1.1. Here and below $\mathcal{U}*\mathcal{V}$ denotes the convolution of two operator-valued functions $\mathcal{U}$, $\mathcal{V}$ defined by

$$(\mathcal{U}*\mathcal{V})(t)=\int_0^t \mathcal{U}(s)\mathcal{V}(t-s)ds,$$

and we set $\mathcal{U}^{(*0)}=I,\mathcal{U}^{(*1)}=\mathcal{U}$ and for any $k\geq 2$, $\mathcal{U}^{(*k)}=\mathcal{U}^{(*(k-1))}*\mathcal{U}$.

Let us end the introduction by describing the plan of the paper. In Section 2, we will develop a hypocoercivity argument to prove a weighted L^2 estimate for the KFP model. In Section 3, we introduce a splitting $\mathcal{L}=\mathcal{A}+\mathcal{B}$ and using the L^2 estimate, we prove a L^2 convergence. In Section 4, we present the proof of a regularization estimate on $S_{\mathcal{B}}$ from L^p to L^2. In Section 5, we prove some L^1 estimate on the semigroup $S_{\mathcal{B}}$. Finally in Section 6 we use the above regularization estimate to conclude the L^p convergence for the KFP equation.

2. L^2 framework: Dirichlet form and rate of convergence estimate

For later discussion, we introduce some notations for the whole paper.

We split the KFP operator as

$$\mathcal{L}=\mathcal{T}+\mathcal{S},$$

where $\mathcal{T}$ stands for the transport part

$$\mathcal{T}f=-v\cdot\nabla_x f+\nabla_x V(x)\cdot\nabla_v f,$$

and $\mathcal{S}$ stands for the collision part

$$\mathcal{S}f=\Delta_v f+div_v(vf).$$

We will denote the cut-off function χ such that $\chi(x,v)\in[0,1]$, $\chi(x,v)\in C^\infty$, $\chi(x,v)=1$ when $|x|^2+|v|^2\leq 1$, $\chi(x,v)=0$ when $|x|^2+|v|^2\geq 2$, and then denote $\chi_R=\chi(x/R,v/R)$. We may also define another splitting of the KFP operator $\mathcal{L}$ by

$$\mathcal{L}=\mathcal{A}+\mathcal{B},\quad \mathcal{A}=K\chi_R(x,v). \tag{2.1}$$

with $K,R>0$ to be chosen later.

For $f=f(x)$, we use $\int f$ to replace $\int_{\mathbb{R}^d} f dx$, and for $f=f(x,v)$, we use $\int f$ in place of $\int_{\mathbb{R}^d\times\mathbb{R}^d} f dxdv$ for short, for $f=f(x,v)$, $\int f dx$ means $\int_{\mathbb{R}^d} f dx$, $\int f dv$ means $\int_{\mathbb{R}^d} f dv$. $B_{|x|\leq\rho}$ is used to denote the ball such that $\{x\in\mathbb{R}^d||x|\leq\rho\}$, similarly B_ρ means the ball such that $\{x,v\in\mathbb{R}^d||x|^2+|v|^2\leq\rho\}$.

For $V(x)=\langle x\rangle^\gamma$, $0<\gamma<1$, we also denote $\langle\nabla V\rangle$ for $\langle x\rangle^{\gamma-1}$, and $\langle\nabla V\rangle^{-1}$ for $\langle x\rangle^{1-\gamma}$.

With these notations we introduce the Dirichlet form adapted to our problem. We define the 0 order and first order moments

$$\rho_f=\rho[f]=\int f dv,\quad j_f=j[f]=\int vf dv,$$

then we define a projection operator π by

$$\pi f = M\rho_f, \quad M = Ce^{-|v|^2/2}, \quad \int M dv = 1,$$

and the complement of π by

$$\pi^\perp = I - \pi, \quad f^\perp = \pi^\perp f.$$

We define an elliptic operator Δ_V and its dual Δ_V^* by

$$\Delta_V u := div_x(\nabla_x u + \nabla_x V u), \quad \Delta_V^* u := \Delta_x u - \nabla_x V \cdot \nabla_x u,$$

let $u = (\Delta_V^*)^{-1}\xi$ be the solution to the above elliptic equation

$$\Delta_V^* u = \xi \text{ } on \text{ } \mathbb{R}^d,$$

and satisfies

$$\int u e^{-V} \langle \nabla V \rangle^{-2} = 0.$$

We will prove the existence and uniqueness to this elliptic equation in Lemma 2.3 below, we then define $\mathcal{H} = L^2(G^{-1/2})$, $\mathcal{H}_1 = L^2(G^{-1/2}\langle \nabla V \rangle)$ and

$$\mathcal{H}_0 = \{h \in \mathcal{H}, \int f dx dv = 0\}$$

where we recall that G has been introduced in (1.2). Using these notations, define a scalar product by

$$\begin{aligned}((f,g)) &:= (f,g)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j_f, (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} + \epsilon((\rho_f e^V \langle \nabla V \rangle^2), \Delta_V^{-1}\nabla_x j_g)_{L^2} \\ &= (f,g)_{\mathcal{H}} + \epsilon(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V \rangle^2))_{L^2} + \epsilon(\nabla_x(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2), j_g)_{L^2},\end{aligned}$$

for some $\epsilon > 0$ to be specified later.
Finally we define the Dirichlet form

$$\begin{aligned}D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} + \epsilon((\rho[-\mathcal{L}f] e^V \langle \nabla V \rangle^2), \Delta_V^{-1}\nabla_x j_f)_{L^2}.\end{aligned}$$

With these notations we can come to our first theorem.

THEOREM 2.1. *There exists $\epsilon > 0$ small enough, such that on $\mathcal{H}_0$ the norm $((f,f))^{\frac{1}{2}}$ defined above is equivalent to the norm of $\mathcal{H}$, moreover there exists $\lambda > 0$, such that*

$$D[f] \geq \lambda \|f\|_{\mathcal{H}_1}^2, \quad \forall f \in \mathcal{H}_0.$$

As a consequence, for any $f_0 \in \mathcal{H}_0$, the associated solution $f(t,\cdot)$ of the kinetic Fokker-Planck Equation (1.1) satisfies

$$\frac{d}{dt}((f,f)) \leq -C \int f^2 G^{-1} \langle x \rangle^{2(\gamma-1)}, \tag{2.2}$$

for some constant $C > 0$. In particular for any $f_0 \in \mathcal{H}_0$, we have

$$\|f(t,\cdot)\|_{L^2(G^{-\frac{1}{2}})} \leq C \|f_0\|_{L^2(G^{-\frac{1}{2}})}, \tag{2.3}$$

for some constant $C>0$.

REMARK 2.1. In $\mathcal{H}_0$ we have

$$\int \rho_f e^V \langle \nabla V\rangle^2 e^{-V} \langle \nabla V\rangle^{-2} dx = \int \rho_f dx = \int f dx dv = 0,$$

so the term $(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V\rangle^2)$ is well defined in $\mathcal{H}_0$.

REMARK 2.2. Our statement is a generalization of [3, 4].

Before proving the theorem, we need some lemmas.
We say that W satisfies a local Poincaré inequality on a bounded open set Ω if there exists some constant $\kappa_\Omega > 0$ such that:

$$\int_\Omega h^2 W \le \kappa_\Omega \int_\Omega |\nabla h|^2 W + \frac{1}{W(\Omega)} \left(\int_\Omega hW \right)^2,$$

for any nice function $h:\mathbb{R}^d \to \mathbb{R}$ and where we denote $W(\Omega) := \langle W 1_\Omega\rangle$.

LEMMA 2.1. *Under the assumption* $W, W^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$, *the function* W *satisfies the local Poincaré inequality for any ball* $\Omega \subset \mathbb{R}^d$.

For the proof of Lemma 2.1 we refer to [17, Lemma 2.3].

LEMMA 2.2 (weak Poincaré inequality). *There exists a constant* $\lambda > 0$ *such that*

$$\|u\|_{L^2(\langle \nabla V\rangle e^{-V/2})} \le \lambda \|\nabla u\|_{L^2(e^{-V/2})}$$

for any $u \in \mathcal{D}(\mathbb{R}^d)$ *such that*

$$\int u e^{-V} \langle \nabla V\rangle^{-2} = 0$$

Proof. We prove for any $h \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int h e^{-V} \langle \nabla V\rangle^{-2} = 0, \tag{2.4}$$

we have

$$\int |\nabla h|^2 e^{-V} \ge \frac{1}{\lambda} \int h^2 e^{-V} \langle x\rangle^{2(\gamma-1)},$$

for some $\lambda > 0$. Taking $g = h e^{-\frac{1}{2}V}$, we have $\nabla g = \nabla h e^{-\frac{1}{2}V} - \frac{1}{2}\nabla V h e^{-\frac{1}{2}V}$, so that

$$\begin{aligned} 0 \le \int |\nabla g|^2 &= \int |\nabla h|^2 e^{-V} + \int h^2 \frac{1}{4} |\nabla V|^2 e^{-V} - \int \frac{1}{2} \nabla(h^2)\cdot \nabla V e^{-V} \\ &= \int |\nabla h|^2 e^{-V} + \int h^2 \left(\frac{1}{2}\Delta V - \frac{1}{4}|\nabla V|^2 \right) e^{-V}. \end{aligned}$$

Since

$$\frac{1}{4}|\nabla V|^2 - \Delta V \ge \frac{1}{8}\langle \nabla V\rangle^2 - K \mathbb{1}_{B_{R_0}} \langle \nabla V\rangle^{-2}$$

for some $K,R_0>0$. We deduce for some $K,R_0>0$

$$\int |\nabla h|^2 e^{-V} \geq \int \frac{1}{8} h^2 \langle \nabla V\rangle^2 e^{-V} - K\int_{B_{R_0}} h^2 e^{-V}\langle \nabla V\rangle^{-2}.$$

Defining

$$\epsilon_R := \int_{B_R^c} e^{-V}\langle \nabla V\rangle^{-6}, \quad Z_R := \int_{B_R} e^{-V}\langle \nabla V\rangle^{-2},$$

and using (2.4), we get

$$\begin{aligned}\left(\int_{B_R} he^{-V}\langle \nabla V\rangle^{-2}\right)^2 &= \left(\int_{B_R^c} he^{-V}\langle \nabla V\rangle^{-2}\right)^2\\ &\leq \int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2 \int_{B_R^c} e^{-V}\langle \nabla V\rangle^{-6}\\ &\leq \epsilon_R \int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2.\end{aligned}$$

Using the local Poincaré inequality in Lemma 2.1, we deduce

$$\begin{aligned}\int_{B_R} h^2 e^{-V}\langle \nabla V\rangle^{-2} &\leq C_R \int_{B_R} |\nabla h|^2 e^{-V}\langle \nabla V\rangle^{-2} + \frac{1}{Z_R}\left(\int_{B_R} he^{-V}\langle \nabla V\rangle^{-2}\right)^2\\ &\leq C_R^{'}\int_{B_R} |\nabla h|^2 e^{-V} + \frac{\epsilon_R}{Z_R}\int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2.\end{aligned}$$

Putting all the inequalities together and taking $R>R_0$, we finally get

$$\begin{aligned}\int h^2 e^{-V}\langle \nabla V\rangle^2 &\leq 8\int |\nabla h|^2 e^{-V} + 8K\int_{B_{R_0}} h^2 e^{-V}\langle \nabla V\rangle^{-2}\\ &\leq 8(1+KC_R^{'})\int |\nabla h|^2 e^{-V} + \frac{8K\epsilon_R}{Z_R}\int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2,\end{aligned}$$

and we conclude by taking R large such that: $\frac{8K\epsilon_R}{Z_R}\leq \frac{1}{2}$. □

LEMMA 2.3 (Elliptic Estimate). *For any $\xi_1\in L^2(\langle \nabla V\rangle^{-1}e^{-V/2})$ and $\xi_2\in L^2(e^{-V/2})$, there exists a unique solution u to the elliptic equation*

$$-\Delta_V^* u = \xi_1 + \nabla\cdot\xi_2, \quad \int ue^{-V}\langle \nabla V\rangle^{-2} = 0, \tag{2.5}$$

and satisfies

$$\|u\|_{L^2(\langle \nabla V\rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})} \lesssim \|\xi_1\|_{L^2(\langle \nabla V\rangle^{-1}e^{-V/2})} + \|\xi_2\|_{L^2(e^{-V/2})}. \tag{2.6}$$

In addition for any $\xi\in L^2(\langle \nabla V\rangle^{-1}e^{-V/2})$, the solution u to the elliptic problem

$$-\Delta_V^* u = \xi, \quad \int ue^{-V}\langle \nabla V\rangle^{-2} = 0,$$

and satisfies

$$\|u\|_{L^2(\langle \nabla V\rangle^2 e^{-V/2})} + \|\nabla u\|_{L^2(\langle \nabla V\rangle e^{-V/2})} + \|D^2 u\|_{L^2(e^{-V/2})} \lesssim \|\xi\|_{L^2(e^{-V/2}\langle \nabla V\rangle^{-1})}. \tag{2.7}$$

Proof. Multiply (2.5) by ue^{-V} and observe that

$$e^V \mathrm{div}_x[e^{-V}\nabla_x u] = \Delta_x u - \nabla_x V \cdot \nabla_x u = \Delta_V^* u, \tag{2.8}$$

we have after integration

$$-\int e^V \mathrm{div}_x[e^{-V}\nabla_x u] u e^{-V} = \int (\xi_1 + \nabla\cdot\xi_2) u e^{-V}.$$

Performing one integration by parts, we deduce

$$\int e^{-V}|\nabla_x u|^2 = \int (\xi_1 u - \xi_2\cdot\nabla u + \xi_2\cdot\nabla V u)e^{-V},$$

using Lemma 2.2 and Lax-Milgram theorem we obtain (2.6), the existence and thus the uniqueness follows. In inequality (2.7), the first two terms are easily bounded by (2.6) and $\langle\nabla V\rangle \le 1$, we then only need to prove the bound for the third term. By integration by parts, we have

$$\begin{aligned}
\int |D^2u|^2 e^{-V} &= \sum_{i,j=1}^d \int (\partial_{ij}^2 u)^2 e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_i u(\partial_{ij}^2 u \partial_j V - \partial_{ijj}^3 u)e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_{jj}^2 u \partial_i(\partial_i u e^{-V}) - \frac{1}{2}\int (\partial_i u)^2 \partial_j(\partial_j V e^{-V}) \\
&= \int (\Delta u)(-\Delta_V^* u)e^{-V} + \frac{1}{2}\int |\nabla u|^2(|\nabla V|^2 - \Delta V)e^{-V} \\
&\lesssim \|D^2 u\|_{L^2(e^{-V/2})}\|\xi\|_{L^2(e^{-V/2})} + \|\langle\nabla V\rangle\nabla u\|_{L^2(e^{-V/2})},
\end{aligned}$$

where in the third equality we have used

$$\begin{aligned}
\int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} &= -\int \partial_i u \partial_j(\partial_i u \partial_j V e^{-V}) \\
&= -\int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} - \int (\partial_i u)^2 \partial_j(\partial_j V e^{-V}),
\end{aligned}$$

which implies

$$\int \partial_{ij}^2 u \partial_i u \partial_j V e^{-V} = -\frac{1}{2}\int (\partial_i u)^2 \partial_j(\partial_j V e^{-V}),$$

and in the fourth equality we have used (2.8). That concludes the proof. □

Now we turn to the proof of Theorem 2.1.

Proof. **(Proof of Theorem 2.1.)** First we prove the equivalence of the norms associated to $((,))$ and $(,)_{\mathcal{H}}$. By Cauchy-Schwarz inequality and Lemma 2.3, we have

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V\langle\nabla V\rangle^2))_{L^2} \le \|j_f\|_{L^2(e^{V/2})}\|\rho_g e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})},$$

and obviously

$$\|\rho_g e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})} = \|\rho_g\|_{L^2(\langle\nabla V\rangle e^{V/2})} \le \|\rho_g\|_{L^2(e^{V/2})} \lesssim \|g\|_{\mathcal{H}}.$$

Using the elementary observations

$$|j_f| \lesssim \|f\|_{L^2_v(e^{|v|^2/4})} \quad |\rho_f| \lesssim \|f\|_{L^2_v(e^{|v|^2/4})},$$

we deduce

$$(j_f, \nabla_x (\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \lesssim \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}},$$

The third term in the definition of $((,))$ can be estimated in the same way and that ends the proof of equivalence of norms.

Now we prove the main estimate of the theorem. We split the Dirichlet term $D[f]$ into 3 parts

$$D[f] = T_1 + \epsilon T_2 + \epsilon T_3,$$

with

$$\begin{aligned} T_1 &:= (-\mathcal{L}f, f)_{\mathcal{H}} \\ T_2 &:= (\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ T_3 &:= ((\Delta_V)^{-1}\nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2}\langle \nabla V\rangle)}\ , \end{aligned}$$

and compute them separately.

For the T_1 term, using the classical Poincaré inequality, we have

$$\begin{aligned} T_1 &:= (-\mathcal{T}f - \mathcal{S}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}} \\ &= -\int [\Delta_v f + div_v(vf)] f M^{-1} e^V = \int |\nabla_v (f/M)|^2 M e^V \\ &\geq k_p \int |f/M - \rho_f|^2 M e^V = k_p \|f - \rho_f M\|_{\mathcal{H}}^2 = k_p \|f^\perp\|_{\mathcal{H}}^2, \end{aligned}$$

for some $k_p > 0$. We split the T_2 term as

$$\begin{aligned} T_2 &:= (\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &= (\Delta_V^{-1}\nabla_x j[-\mathcal{T}\pi f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &\quad + (\Delta_V^{-1}\nabla_x j[-\mathcal{T}f^\perp], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} + (\Delta_V^{-1}\nabla_x j[-\mathcal{S}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &:= T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned}$$

First observe

$$\mathcal{T}\pi f = -v\cdot\nabla_x \rho_f M - \nabla_x V \cdot v \rho_f M = -e^{-V} M v \cdot \nabla_x (\rho_f / e^{-V}),$$

so that we have

$$j[-\mathcal{T}\pi f] = \sum_{k=1}^{d} \langle v v_k M\rangle e^{-V} \partial_{x_k}(\rho_f/e^{-V}) = e^{-V}\nabla_x(\rho_f/e^{-V}).$$

Next by (2.8), we have

$$\begin{aligned} T_{2,1} &= (j[-\mathcal{T}\pi f], \nabla (\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V\rangle^2))_{L^2} \\ &= (\rho_f, [e^V div_x(e^{-V}\nabla)][(\Delta_V^*)^{-1}(\rho_f e^V\langle \nabla V\rangle^2)])_{L^2} \end{aligned}$$

$$= \|\rho_f e^{V/2}\langle\nabla V\rangle\|_{L^2}^2 = \|\pi f\|_{\mathcal{H}_1}^2.$$

Using the notation $\eta_1 = \langle v\otimes v f^\perp\rangle$ and $\eta_{2,\alpha\beta} = \langle v_\alpha \partial_{v_\beta} f^\perp\rangle$, and observing that

$$|\eta_1| \lesssim \|f^\perp\|_{L^2_v(e^{|v|^2/4})},\ |\eta_2| \lesssim \|f^\perp\|_{L^2_v(e^{|v|^2/4})},$$

we compute

$$\begin{aligned}
T_{2,2} &= (j[-\mathcal{T} f^\perp], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2}\\
&= (D\eta_1 + \eta_2\nabla V, \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2}\\
&= (\eta_1, D^2(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} + (\eta_2, \nabla V\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2}\\
&\le \|\eta_1\|_{L^2(e^{V/2})}\|D^2(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})}\\
&\quad + \|\eta_2\|_{L^2(e^{V/2})}\|\nabla V\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})}.
\end{aligned}$$

By Lemma 2.3, we estimate

$$\begin{aligned}
T_{2,2} &\lesssim \|\eta_1\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})}\\
&\quad + \|\eta_2\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})}\\
&\lesssim \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}.
\end{aligned}$$

Using

$$j[-\mathcal{S} f] = j[-\mathcal{S} f^\perp] = -\int v[\Delta_v f^\perp + div_v(v f^\perp)]dv = d\int f^\perp v dv \lesssim \|f^\perp\|_{L^2_v(e^{|v|^2/4})},$$

and Lemma 2.3, we have

$$\begin{aligned}
T_{2,3} &= (j[-Sf], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2}\\
&\le \|j[-Sf]\|_{L^2(e^{V/2})}\|\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})}\\
&\lesssim \|f^\perp\|_{\mathcal{H}}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})}\\
&= \|f^\perp\|_{\mathcal{H}}\|\rho_f\|_{L^2(\langle\nabla V\rangle e^{V/2})}\\
&= \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}.
\end{aligned}$$

Finally we come to the T_3 term. Using

$$\rho[-Sf] = \int \nabla_v\cdot(\nabla_v f + v f)dv = 0,$$

and

$$\begin{aligned}
\rho[-Tf] &= \rho[v\cdot\nabla_x f - \nabla_x V(x)\cdot\nabla_v f]\\
&= \int v\cdot\nabla_x f - \nabla_x V(x)\cdot\nabla_v f dv\\
&= \nabla_x j[f],
\end{aligned}$$

because $\nabla(\langle\nabla V\rangle^2) \lesssim \langle\nabla V\rangle^2$ and $\langle\nabla V\rangle^2 \lesssim \langle\nabla V\rangle$, we get

$$\begin{aligned}
T_3 &= ((\Delta_V)^{-1}\nabla_x j_f, \rho[-\mathcal{L} f])_{L^2(e^{V/2}\langle\nabla V\rangle)}\\
&= ((\Delta_V)^{-1}\nabla_x j[f^\perp], \rho[-\mathcal{T} f])_{L^2(e^{V/2}\langle\nabla V\rangle)}
\end{aligned}$$

$$
\begin{aligned}
&= (j[f^\perp], \nabla(\Delta_V^*)^{-1}(\nabla_x j[f] e^V \langle \nabla V\rangle^2))_{L^2}\\
&\le \|j[f^\perp]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}[\nabla_x(j_f e^V\langle\nabla V\rangle^2)\\
&\quad -\nabla V j_f e^V\langle \nabla V\rangle^2 - \nabla(\langle\nabla V\rangle^2) j_f e^V]\|_{L^2(e^{-V/2})},
\end{aligned}
$$

using Lemma 2.3 again, we have

$$
\begin{aligned}
T_3 &\lesssim \|j[f^\perp]\|_{L^2(e^{V/2})}(\|j_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} + \|j_f e^V \nabla(\langle\nabla V\rangle^2)\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})})\\
&\lesssim \|f^\perp\|_{\mathcal{H}}\|f\|_{\mathcal{H}_1}.
\end{aligned}
$$

Putting all the terms together and choosing $\epsilon > 0$ small enough, we can deduce

$$
\begin{aligned}
D[f] &\ge k_p\|f^\perp\|_{\mathcal{H}}^2 + \epsilon\|\pi f\|_{\mathcal{H}_1}^2 - \epsilon 2K\|f^\perp\|_{\mathcal{H}}\|f\|_{\mathcal{H}_1} - \epsilon 2K\|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}\\
&\ge k_p\|f^\perp\|_{\mathcal{H}}^2 + \epsilon\|\pi f\|_{\mathcal{H}_1}^2 - (2\epsilon + 4\epsilon^{1/2})K\|f^\perp\|_{\mathcal{H}}^2 - \epsilon^{3/2}4K\|\pi f\|_{\mathcal{H}_1}^2\\
&\ge \frac{k_p}{2}(\|f^\perp\|_{\mathcal{H}}^2 + \epsilon\|\pi f\|_{\mathcal{H}_1}^2) \ge \lambda\|f\|_{\mathcal{H}_1},
\end{aligned}
$$

for some $\lambda > 0$. ∎

3. L^2 sub-exponential decay for the kinetic Fokker-Planck equation based on a splitting trick

In this section we establish a first decay estimate on $S_{\mathcal{L}}$ which is a particular case in the result of Theorem 1.1.

THEOREM 3.1. *Using the notation and results in Theorem 2.1, we have*

$$
\|S_{\mathcal{L}}(t)f_0\|_{L^2(G^{-\frac12})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}}\|f_0\|_{L^2(G^{-(\frac12+\epsilon)})},
$$

for any $f_0 \in L^2(G^{-(\frac12+\epsilon)}) \cap \mathcal{H}_0$, $\epsilon > 0$ small enough.

REMARK 3.1. It's worth emphasizing that we deduce immediately part (1) of Theorem 1.1 for the case $p = 2$ by considering the initial datum $f_0 - \mathcal{M}(f_0)G$ for any $f_0 \in L^2(G^{-\frac12+\epsilon})$.

Recall the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ introduced in (2.1), we first prove some decay estimate on the semigroup $S_{\mathcal{B}}$.

LEMMA 3.1. *Let us fix $p \in [1,\infty)$.*

(1) For any given smooth weight function m, we have

$$
\int |f|^{p-1}\operatorname{sign} f(\mathcal{L}f)G^{-(p-1)}m \le \frac{1}{p}\int |f|^p G^{-(p-1)}\tilde{m}, \tag{3.1}
$$

with

$$
\tilde{m} = \Delta_v m - \nabla_v m\cdot v - \nabla_x V(x)\cdot\nabla_v m + v\cdot\nabla_x m.
$$

(2) Taking $m = e^{\epsilon H^\delta}$, $\epsilon > 0$ if $0 < \delta < \frac{\gamma}{2}$, ϵ small enough if $\delta = \frac{\gamma}{2}$, $H = 3|v|^2 + 2x\cdot v + |x|^2 + 1$, we have

$$
\int |f|^{p-1}\operatorname{sign} f(\mathcal{B}f)G^{-(p-1)}e^{\epsilon H^\delta} \le -C\int |f|^p G^{-(p-1)}e^{\epsilon H^\delta}H^{\frac{\delta}{2}+\gamma-1}, \tag{3.2}
$$

for some K and R large.

(3) With the same notation as above, there holds

$$\|S_{\mathcal{B}}(t)\|_{L^p(e^{2\epsilon H^\delta}G^{-\frac{p-1}{p}})\to L^p(e^{\epsilon H^\delta}G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \tag{3.3}$$

for some $a>0$. In particular, this implies

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})\to L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Proof.
Step 1. Recalling (1.2), we write

$$\begin{aligned}&\int |f|^{p-1}\operatorname{sign} f(\mathcal{L}f)G^{-(p-1)}m\\ &=\int |f|^{p-1}\operatorname{sign} f(\mathcal{T}f)G^{-(p-1)}m+\int |f|^{p-1}\operatorname{sign} f(\mathcal{S}f)G^{-(p-1)}m.\end{aligned}$$

We first compute the contribution of the term with operator $\mathcal{T}$

$$\begin{aligned}\int |f|^{p-1}\operatorname{sign} f(\mathcal{T}f)G^{-(p-1)}m &= \frac{1}{p}\int \mathcal{T}(|f|^p)G^{-(p-1)}m\\ &= -\frac{1}{p}\int |f|^p\mathcal{T}(G^{-(p-1)}m)\\ &= \frac{1}{p}\int |f|^pG^{-(p-1)}(v\cdot\nabla_x m-\nabla V(x)\cdot\nabla_v m).\end{aligned}$$

For the term with operator $\mathcal{S}$, we use one integration by parts, and we get

$$\begin{aligned}&\int |f|^{p-1}\operatorname{sign} f(\mathcal{S}f)G^{-(p-1)}m\\ &=\int |f|^{p-1}\operatorname{sign} f(\Delta_v f+\operatorname{div}_v(vf))G^{-(p-1)}m\\ &=-\int \nabla_v(\operatorname{sign} f(|f|G^{-1})^{p-1}m)\cdot\nabla_v(fG^{-1})G\\ &=-\int (p-1)|\nabla_v(fG^{-1})|^2(|f|G^{-1})^{p-2}Gm-\frac{1}{p}\nabla_v((|f|G^{-1})^p)\cdot(\nabla_v m)G.\end{aligned}$$

Performing another integration by parts on the latter term, we have

$$\begin{aligned}&\int f^{p-1}\operatorname{sign} f(\mathcal{S}f)G^{-(p-1)}m\\ &=\int -(p-1)|\nabla_v(fG^{-1})|^2(|f|G^{-1})^{p-2}Gm+\frac{1}{p}\nabla_v\cdot(G\nabla_v m)(|f|G^{-1})^p\\ &=\int -(p-1)|\nabla_v(fG^{-1})|^2(|f|G^{-1})^{p-2}Gm+\frac{1}{p}(\Delta_v m-v\cdot\nabla_v m)|f|^pG^{-(p-1)}.\end{aligned}$$

Inequality (3.1) follows by putting together the two identities.

Step 2. We particular use $m=e^{\epsilon H^\delta}$ and we easily compute

$$\frac{\nabla_v m}{m}=\delta\epsilon\frac{\nabla_v H}{H^{1-\delta}},\quad \frac{\nabla_x m}{m}=\delta\epsilon\frac{\nabla_x H}{H^{1-\delta}},$$

and

$$\frac{\Delta_v m}{m} \le \delta\epsilon \frac{\Delta_v H}{H^{1-\delta}} + (\delta\epsilon)^2 \frac{|\nabla_v H|^2}{H^{2(1-\delta)}}.$$

We deduce that $\phi = \frac{\tilde{m}}{m}$ satisfies

$$\frac{\phi H^{1-\delta}}{\epsilon\delta} \le \Delta_v H + \epsilon\delta \frac{|\nabla_v H|^2}{H^{1-\delta}} - v\cdot\nabla_v H + v\cdot\nabla_x H - \nabla_x V(x)\cdot\nabla_v H.$$

From the very definition of H, we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

Choosing $\epsilon > 0$ arbitrarily if $0 < 2\delta < \gamma$, ϵ small enough if $2\delta = \gamma$, we deduce

$$\begin{aligned}
&\Delta_v H + 2\epsilon\delta \frac{|\nabla_v H|^2}{H^{1-\delta}} + v\cdot\nabla_x H - v\cdot\nabla_v H - \nabla_x V(x)\cdot\nabla_v H \\
&= 6 + \epsilon\delta \frac{|6v+2x|^2}{H^{1-\delta}} + 2|v|^2 + 2x\cdot v - 6|v|^2 - 2x\cdot v - 6v\cdot\nabla_x V(x) - 2x\cdot\nabla_x V(x) \\
&\le (2|v|^2 + C_1|v| + C_2|v|^{2\delta} - 6|v|^2) + (C_3\epsilon\delta|x|^{2\delta} - 2x\cdot\nabla_x V(x)) + C \\
&\le -C_4|v|^2 - C_5 x\cdot\nabla_x V(x) + C_6 \\
&\le -C_7 H^{\frac{\gamma}{2}} + K\chi_R,
\end{aligned}$$

for some constants $C_i, K, R > 0$. As a consequence, we have proved

$$\phi - K\chi_R \le \frac{-C}{H^{1-\delta-\frac{\gamma}{2}}} \le 0,$$

which is nothing but (3.2).

Step 3. In the following, we use the "interpolation" argument from [11], denote $f_t = S_{\mathcal{B}}(t) f_0$ the solution to the evolution equation $\partial_t f = \mathcal{B}f, f(0) = f_0$. On the one hand, by (3.2) we have

$$\frac{d}{dt}\int |f_t|^p G^{-(p-1)} e^{2\epsilon H^\delta} = \int |f_t|^{p-1} \operatorname{sign} f_t (\mathcal{B} f_t) G^{-(p-1)} e^{2\epsilon H^\delta} \le 0,$$

which implies

$$\int |f_t|^p G^{-(p-1)} e^{2\epsilon H^\delta} \le \int |f_0|^p G^{-(p-1)} e^{2\epsilon H^\delta} := Y_1, \quad \forall t \ge 0$$

On the other hand, defining

$$Y(t) := \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta},$$

using (3.2) again, we have

$$\begin{aligned}
\frac{d}{dt} Y &= p\int |f_t|^{p-1} \operatorname{sign} f_t (\mathcal{B} f_t) G^{-(p-1)} e^{\epsilon H^\delta} \\
&\le -a \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} H^{\delta + \frac{\gamma}{2} - 1}
\end{aligned}$$

$$\begin{aligned} &\le -a \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x\rangle^{2\delta+\gamma-2} \\ &\le -a \int_{B_{|x|\le\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x\rangle^{2\delta+\gamma-2}, \end{aligned}$$

for any $\rho>0$ and for some $a>0$. As $2\delta+\gamma<2$, $0\le|x|\le\rho$ implies $\langle x\rangle^{2\delta+\gamma-2}\ge\langle\rho\rangle^{2\delta+\gamma-2}$, we deduce

$$\begin{aligned} \frac{d}{dt}Y &\le -a\langle\rho\rangle^{2\delta+\gamma-2} \int_{B_{|x|\le\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} \\ &\le -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} \int_{B_{|x|\ge\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta}, \end{aligned}$$

Using that $e^{\epsilon\langle x\rangle^{2\delta}}\ge e^{\epsilon\langle\rho\rangle^{2\delta}}$ on $|x|\ge\rho$, we get

$$\begin{aligned} \frac{d}{dt}Y &\le -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} e^{-\epsilon\langle\rho\rangle^{2\delta}} \int_{B_{|x|\ge\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon\langle x\rangle^{2\delta}} \\ &\le -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} e^{-\epsilon\langle\rho\rangle^{2\delta}} \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon\langle x\rangle^{2\delta}} \\ &\le -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} e^{-\epsilon\langle\rho\rangle^{2\delta}} C Y_1. \end{aligned}$$

Thanks to Grönwall's Lemma

$$\frac{d}{dt}X(t) \le -\alpha X(t) + b \Rightarrow X(t) \le e^{-\alpha t}X(0) + \frac{b}{\alpha}(1-e^{-\alpha t}) \le e^{-\alpha t}X(0) + \frac{b}{\alpha},$$

we obtain

$$\begin{aligned} Y(t) &\le e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t} Y(0) + Ce^{-\epsilon\langle\rho\rangle^{2\delta}} Y_1 \\ &\lesssim (e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t} + e^{-\epsilon\langle\rho\rangle^{2\delta}}) Y_1, \end{aligned}$$

Finally, choosing ρ such that $a\langle\rho\rangle^{2\delta+\gamma-2}t=\epsilon\langle\rho\rangle^{2\delta}$, that is $\langle\rho\rangle^{2-\gamma}=Ct$, we deduce

$$Y(t) \le C_1 e^{-C_2 t^{\frac{2\delta}{2-\gamma}}} Y_1,$$

for some $C_i>0$, and we deduce the proof of (3.3). ∎

Now we come to prove Theorem 3.1.

Proof. **(Proof of Theorem 3.1.)** We recall that from (2.3), we have

$$\|S_{\mathcal{L}}(t)\|_{L^2(G^{-\frac12})\to L^2(G^{-\frac12})} \lesssim 1, \quad \forall t\ge 0.$$

From the very definition of $\mathcal{A}$ we have

$$\|\mathcal{A}\|_{L^2(G^{-\frac12})\to L^2(e^{2\epsilon H^\delta}G^{-\frac12})} \lesssim 1.$$

From Lemma 3.1 case $p=2$, we have

$$\|S_{\mathcal{B}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-\frac12})\to L^2(e^{\epsilon H^\delta}G^{-\frac12})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \quad \forall t\ge 0.$$

Gathering the three estimates and using Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

we deduce

$$\|S_{\mathcal{L}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-\frac{1}{2}}) \to L^2(e^{\epsilon H^\delta}G^{-\frac{1}{2}})} \lesssim 1, \quad \forall t \geq 0.$$

In the following, we denote $f_t = S_{\mathcal{L}}(t)f_0$ the solution to the evolution equation $\partial_t f = \mathcal{L}f, f(0,\cdot) = f_0$. Taking $2\delta = \gamma$, ϵ small enough, we have in particular

$$\int |f_t|^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \leq C \int |f_0|^2 G^{-1} e^{2\epsilon H^{\frac{\gamma}{2}}} =: Y_3.$$

We define

$$Y_2(t) := ((f,f)),$$

with $((,))$ defined in Theorem 2.1. Thanks to the result in (2.2), we have

$$\begin{aligned} \frac{d}{dt} Y_2 &\leq -a \int |f_t|^2 G^{-1} \langle x\rangle^{2(\gamma-1)} \\ &\leq -a \int_{B_{|x|\leq \rho}} |f_t|^2 G^{-1} \langle x\rangle^{2(\gamma-1)}, \end{aligned}$$

for any $\rho \geq 0$, using the same argument as Lemma 3.1, we deduce

$$\begin{aligned} Y_2(t) &\leq C e^{-a\langle \rho\rangle^{2(\gamma-1)}t} Y_2(0) + C e^{-\epsilon_2 \langle \rho\rangle^\gamma} Y_3 \\ &\lesssim (e^{-a\langle \rho\rangle^{2(\gamma-1)}t} + e^{-\epsilon_2\langle \rho\rangle^\gamma}) Y_3. \end{aligned}$$

Choosing ρ such that $a\langle \rho\rangle^{2(\gamma-1)}t = \epsilon_2 \langle \rho\rangle^\gamma$, that is $\langle \rho\rangle^{2-\gamma} = Ct$, we conclude

$$Y_2(t) \leq C_1 e^{-C_2 t^{\gamma/(2-\gamma)}} Y_3,$$

for some constants $C_i > 0$. As $H^{\frac{\gamma}{2}} \lesssim C(\frac{|v|^2}{2} + V(x))$, we have

$$e^{\epsilon H^{\frac{\gamma}{2}}} \leq G^{-C\epsilon},$$

Taking ϵ small, the proof of Theorem 3.1 is done. ∎

4. Regularization property of $S_{\mathcal{B}}$

In this section we will denote $\mathcal{L}^* = \mathcal{L}^*_{G^{-1/2}} = \mathcal{S} - \mathcal{T}$ be the dual operator of $\mathcal{L}$ on $L^2(G^{-\frac{1}{2}})$. In other words, $\mathcal{L}^*$ is defined by the identity

$$\int (\mathcal{L}f) g G^{-1} = \int (\mathcal{L}^* g) f G^{-1}.$$

for any smooth function f,g. We also denote $\mathcal{B}^* = \mathcal{L}^* - K\chi_R$. The aim of this section is to establish the following regularization property. The proof closely follows the proof of similar results in [7, 13, 22].

THEOREM 4.1. *For any $0 \leq \delta < 1$, denote $m_1 = G^{-\frac{1}{2}(1+\delta)}$, there exists $\eta > 0$ such that*

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(m_1)} \lesssim \frac{1}{t^{\frac{3d+2}{4}}} \|f\|_{L^1(m_1)}, \quad \forall t \in (0, \eta].$$

Similarly, for any $0\le\delta<1$*, there exists* $\eta>0$ *such that*

$$\|\mathcal{S}_{\mathcal{B}^*}(t)f\|_{L^2(m_1)}\lesssim\frac{1}{t^{\frac{3d+2}{4}}}\|f\|_{L^1(m_1)},\quad\forall t\in(0,\eta].$$

As a consequence, there exists $\eta>0$ *such that*

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^\infty(G^{-\frac{1}{2}})}\lesssim\frac{1}{t^{\frac{3d+2}{4}}}\|f\|_{L^2(G^{-\frac{1}{2}})},\quad\forall t\in(0,\eta].$$

We start with some elementary lemmas.

LEMMA 4.1. *For any* $0\le\delta<1$*, we have*

$$\begin{aligned}&\int(f(\mathcal{L}g)+g(\mathcal{L}f))G^{-(1+\delta)}\\&=-2\int\nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta}+\int(\delta d-\delta(1-\delta)|v|^2)fgG^{-(1+\delta)}\end{aligned}\tag{4.1}$$

in particular, this implies

$$\begin{aligned}&\int f(\mathcal{L}f)G^{-(1+\delta)}\\&=-\int|\nabla_v(fG^{-1})|^2G^{1-\delta}+\frac{\delta d}{2}\int|f|^2G^{-(1+\delta)}-\frac{\delta(1-\delta)}{2}\int|v|^2|f|^2G^{-(1+\delta)},\end{aligned}\tag{4.2}$$

similarly, for any $0\le\delta<1$*, we have*

$$\begin{aligned}&\int f(\mathcal{L}f)G^{-(1+\delta)}\\&=-\int|\nabla_vf|^2G^{-(1+\delta)}+\frac{\delta(1+\delta)}{2}\int|v|^2|f|^2G^{-(1+\delta)}+\frac{(2+\delta)d}{2}\int|f|^2G^{-(1+\delta)}.\end{aligned}\tag{4.3}$$

All the equalities remain true when $\mathcal{L}$ *is replaced by* $\mathcal{L}^*$*.*

Proof. Recalling $\mathcal{T}(G^{-(1+\delta)})=0$, we have

$$\int f(\mathcal{T}g)G^{-(1+\delta)}=\int\mathcal{T}(fG^{-(1+\delta)})g=-\int(\mathcal{T}f)gG^{-(1+\delta)},$$

which implies

$$\int f(\mathcal{T}g)G^{-(1+\delta)}+\int(\mathcal{T}f)gG^{-(1+\delta)}=0.$$

for the term with operator $\mathcal{S}$ we have

$$\begin{aligned}\int f(\mathcal{S}g)G^{-(1+\delta)}&=-\int\nabla_v(fG^{-(1+\delta)})\cdot(\nabla_vg+vg)\\&=-\int(\nabla_vf+(1+\delta)vf)\cdot(\nabla_vg+vg)G^{-(1+\delta)}\\&=-\int\nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta}-\int(\delta|v|^2fg+\delta fv\cdot\nabla_vg)G^{-(1+\delta)},\end{aligned}$$

using integration by parts

$$\int\delta fv\cdot\nabla_vgG^{-(1+\delta)}=-\int\delta g\nabla_v\cdot(vfG^{-(1+\delta)})$$

$$= -\int \delta g v\cdot\nabla_v f G^{-(1+\delta)} - \int (\delta d+\delta(1+\delta)|v|^2) fgG^{-(1+\delta)},$$

so we deduce

$$\begin{aligned}&\int (f(\mathcal{S}g)+g(\mathcal{S}f))G^{-(1+\delta)}\\ &= -2\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta}+\int(\delta d-\delta(1-\delta)|v|^2)fgG^{-(1+\delta)},\end{aligned}$$

so (4.1) and (4.2) are thus proved by combining the two terms above. Finally, we compute

$$\begin{aligned}&\int f\mathcal{S}fG^{-(1+\delta)}\\ &= -\int (\nabla_v f+(1+\delta)vf)\cdot(\nabla_v f+vf)G^{-(1+\delta)}\\ &= -\int |\nabla_v f|^2G^{-(1+\delta)} - \int (1+\delta)|v|^2|f|^2G^{-(1+\delta)} - \int (2+\delta) fv\cdot\nabla_v fG^{-(1+\delta)}\\ &= -\int |\nabla_v f|^2G^{-(1+\delta)} - \int (1+\delta)|v|^2|f|^2G^{-(1+\delta)} + \frac{2+\delta}{2}\int \nabla_v\cdot(vG^{-(1+\delta)})|f|^2\\ &= -\int |\nabla_v f|^2G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2}\int |v|^2|f|^2G^{-(1+\delta)} + \frac{(2+\delta)d}{2}\int |f|^2G^{-(1+\delta)},\end{aligned}$$

so (4.3) follows by putting together the above equality with

$$\int f\mathcal{T}fG^{-(1+\delta)}=0.$$

Since the term associated with $\mathcal{T}$ is 0, by $\mathcal{L}=\mathcal{S}+\mathcal{T},\mathcal{L}^*=\mathcal{S}-\mathcal{T}$, we know the same equalities will remain true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$. ∎

LEMMA 4.2. *When $f_t=S_{\mathcal{B}}(t)f_0$, denote $m_1=G^{-\frac{1}{2}(1+\delta)}$, define an energy functional*

$$\begin{aligned}\mathcal{F}(t,f_t):=&A\|f_t\|^2_{L^2(m_1)}+at\|\nabla_v f_t\|^2_{L^2(m_1)}\\ &+2ct^2(\nabla_v f_t,\nabla_x f_t)_{L^2(m_1)}+bt^3\|\nabla_x f_t\|^2_{L^2(m_1)},\end{aligned}\tag{4.4}$$

when $f_t=S_{\mathcal{B}^*}(t)f_0$, define another energy functional

$$\begin{aligned}\mathcal{F}^*(t,f_t):=&A\|f_t\|^2_{L^2(m_1)}+at\|\nabla_v f_t\|^2_{L^2(m_1)}\\ &-2ct^2(\nabla_v f_t,\nabla_x f_t)_{L^2(m_1)}+bt^3\|\nabla_x f_t\|^2_{L^2(m_1)},\end{aligned}\tag{4.5}$$

with some constant $a,b,c>0,c\le\sqrt{ab}$ and A large enough. Then for both cases, there exists $\eta>0$ such that

$$\frac{d}{dt}F(t,f_t)\le -L\left(\|\nabla_v f_t\|^2_{L^2(m_1)}+t^2\|\nabla_x f_t\|^2_{L^2(m_1)}\right)+C\|f_t\|^2_{L^2(m_1)},$$

for all $t\in(0,\eta]$, for some $L>0$, $C>0$ and $F=\mathcal{F}$ or $\mathcal{F}^*$.

Proof. We only prove the case $F=\mathcal{F}$, the proof for $F=\mathcal{F}^*$ is the same. We split the computation into several parts and then put them together. First using (4.2) and (4.3) we have

$$\frac{d}{dt}\|f_t\|^2_{L^2(m_1)}=(f_t,(\mathcal{L}-K\chi_R)f_t)_{L^2(m_1)}$$

$$
\begin{aligned}
&= \frac{1-\delta}{2}(f_t, \mathcal{L} f_t)_{L^2(m_1)} + \frac{1+\delta}{2}(f_t, \mathcal{L} f_t)_{L^2(m_1)} - (f_t, K\chi_R f_t)_{L^2(m_1)} \\
&\le \frac{1-\delta}{2}\|\nabla_v f_t\|^2_{L^2(m_1)} - \frac{1+\delta}{2}\|f_t G^{-1}\|^2_{L^2(m_1 G^{1/2})} + C\|f_t\|^2_{L^2(m_1)} \\
&\le \frac{1-\delta}{2}\|\nabla_v f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}.
\end{aligned}
$$

By

$$
\partial_{x_i}\mathcal{L} f = \mathcal{L}\partial_{x_i} f + \sum_{j=1}^d \partial^2_{x_i x_j} V \partial_{v_j} f, \tag{4.6}
$$

and (4.2) we have

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{x_i} f_t\|^2_{L^2(m_1)} \\
&= (\partial_{x_i} f_t, \partial_{x_i}(\mathcal{L} - K\chi_R) f_t)_{L^2(m_1)} \\
&\le \|\nabla_v(\partial_{x_i} f_t G^{-1})\|^2_{L^2(m_1 G)} + \frac{\delta d}{2}\|\partial_{x_i} f_t\|^2_{L^2(m_1)} - \frac{\delta(1-\delta)}{2}\|\partial_{x_i} f_t\|^2_{L^2(m_1|v|)} \\
&\quad + (\partial_{x_i} f_t, \sum_{j=1}^d \partial^2_{x_i x_j} V \partial_{v_j} f_t)_{L^2(m_1)} - (\partial_{x_i} f_t, K\partial_{x_i}\chi_R f_t)_{L^2(m_1)}.
\end{aligned}
$$

Using Cauchy-Schwarz inequality and summing up over i, we get

$$
\begin{aligned}
\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(m_1)} \le& \sum_{i=1}^d \|\nabla_v(\partial_{x_i} f_t G^{-1})\|^2_{L^2(m_1 G)} - \frac{\delta(1-\delta)}{2}\|\nabla_x f_t\|^2_{L^2(m_1|v|)} \\
&+ C\|\nabla_v f_t\|^2_{L^2(m_1)} + C\|\nabla_x f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}
\end{aligned}
$$

for some $C > 0$. Similarly using

$$
\partial_{v_i}\mathcal{L} f = \mathcal{L}\partial_{v_i} f - \partial_{x_i} f + \partial_{v_i} f, \tag{4.7}
$$

and (4.2), we have

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{v_i} f_t\|^2_{L^2(m_1)} \\
&= (\partial_{v_i} f_t, \partial_{v_i}(\mathcal{L} - K\chi_R) f_t)_{L^2(m_1)} \\
&\le \|\nabla_v(\partial_{v_i} f_t G^{-1})\|^2_{L^2(m_1 G)} + \frac{\delta d}{2}\|\partial_{v_i} f_t\|^2_{L^2(m_1)} - \frac{\delta(1-\delta)}{2}\|\partial_{v_i} f_t\|^2_{L^2(m_1|v|)} \\
&\quad - (\partial_{x_i} f_t, \partial_{v_i} f_t)_{L^2(m_1)} + \|\partial_{v_i} f_t\|^2_{L^2(m_1)} - (\partial_{v_i} f_t, K\partial_{v_i}\chi_R f_t)_{L^2(m_1)}.
\end{aligned}
$$

Using Cauchy-Schwarz inequality and summing up over i we get

$$
\begin{aligned}
\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(m_1)} \le& \sum_{i=1}^d \|\nabla_v(\partial_{v_i} f_t G^{-1})\|^2_{L^2(m_1 G)} - \frac{\delta(1-\delta)}{2}\|\nabla_v f_t\|^2_{L^2(m_1|v|)} \\
&+ C\|\nabla_v f_t\|^2_{L^2(m_1)} + C(|\nabla_x f_t|, |\nabla_v f_t|)_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}.
\end{aligned}
$$

For the crossing term, we also split it into two parts

$$
\frac{d}{dt}2(\partial_{x_i} f_t, \partial_{v_i} f_t)_{L^2(m_1)} = (\partial_{x_i} f_t, \partial_{v_i}\mathcal{L} f_t)_{L^2(m_1)} + (\partial_{v_i} f_t, \partial_{x_i}\mathcal{L} f_t)_{L^2(m_1)}
$$

$$
\begin{aligned}
&-(\partial_{x_i} f_t, \partial_{v_i}(K\chi_R f_t))_{L^2(m_1)} - (\partial_{v_i} f_t, \partial_{x_i}(K\chi_R f_t))_{L^2(m_1)} \\
:=& W_1 + W_2.
\end{aligned}
$$

Using (4.6) and (4.7) we have

$$
\begin{aligned}
W_1 =& (\partial_{x_i} f_t, \mathcal{L}(\partial_{v_i} f_t))_{L^2(m_1)} + (\partial_{v_i} f_t, \mathcal{L}(\partial_{x_i} f_t))_{L^2(m_1)} \\
&+ (\partial_{v_i} f_t, \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f_t)_{L^2(m_1)} - \|\partial_{x_i} f_t\|^2_{L^2(m_1)} + (\partial_{x_i} f_t, \partial_{v_i} f_t)_{L^2(m_1)}.
\end{aligned}
$$

By (4.1), we deduce

$$
\begin{aligned}
W_1 \le& (\nabla_v(\partial_{x_i} f_t G^{-1}), \nabla_v(\partial_{v_i} f_t G^{-1}))_{L^2(m_1 G)} + \delta d(\partial_{v_i} f_t, \partial_{x_i} f_t)_{L^2(m_1)} \\
&- \delta(1-\delta)(\partial_{v_i} f_t, \partial_{x_i} f_t)_{L^2(m_1|v|)} + (\partial_{v_i} f_t, \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f_t)_{L^2(m_1)} \\
&- \|\partial_{x_i} f_t\|^2_{L^2(m_1)} + (\partial_{x_i} f_t, \partial_{v_i} f_t)_{L^2(m_1)}.
\end{aligned}
$$

For the W_2 term we have

$$
\begin{aligned}
W_2 =& -2(\partial_{x_i} f_t, K\chi_R \partial_{v_i} f_t)_{L^2(m_1)} - (\partial_{x_i} f_t, K\partial_{v_i}\chi_R f_t)_{L^2(m_1)} - (\partial_{v_i} f_t, K\partial_{x_i}\chi_R f_t)_{L^2(m_1)} \\
\le& C(|\partial_{x_i} f_t|, |f_t|)_{L^2(m_1)} + C(|f_t|, |\partial_{v_i} f_t|)_{L^2(m_1)} + C(|\partial_{x_i} f_t|, |\partial_{v_i} f_t|)_{L^2(m_1)}.
\end{aligned}
$$

Combining the two parts, using Cauchy-Schwarz inequality, and summing up over i we get

$$
\begin{aligned}
&\frac{d}{dt} 2(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} \\
\le& 2\sum_{i=1}^{d} (\nabla_v(\partial_{x_i} f_t G^{-1}), \nabla_v(\partial_{v_i} f_t G^{-1}))_{L^2(m_1 G)} - \delta(1-\delta)(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1|v|)} \\
&- \frac{1}{2}\|\nabla_x f_t\|^2_{L^2(m_1)} + C\|\nabla_v f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}.
\end{aligned}
$$

From the very definition of $\mathcal{F}$ in (4.4), we easily compute

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}(t, f_t) =& A\frac{d}{dt}\|f_t\|^2_{L^2(m_1)} + at\frac{d}{dt}\|\nabla_v f_t\|^2_{L^2(m_1)} + 2ct^2\frac{d}{dt}(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} \\
&+ bt^3\frac{d}{dt}\|\nabla_x f_t\|^2_{L^2(m_1)} + a\|\nabla_v f_t\|^2_{L^2(m_1)} + 4ct(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} \\
&+ 3bt^2\|\nabla_x f_t\|^2_{L^2(m_1)}.
\end{aligned}
$$

Gathering all the inequalities above together, we have

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}(t, f_t) \\
\le& \left(2a - \frac{A(1-\delta)}{2} + Cat + 2Ct^2c + Cbt^3\right)\|\nabla_v f_t\|^2_{L^2(m_1)} \\
&+ (3bt^2 - \frac{c}{2}t^2 + Cbt^3)\|\nabla_x f_t\|^2_{L^2(m_1)} + (4ct + Cat)(|\nabla_v f_t|, |\nabla_x f_t|)_{L^2(m_1)} \\
&- \sum_{i=1}^{d}[at\|\nabla_v(\partial_{v_i} f_t G^{-1})\|^2_{L^2(m_1 G)} + bt^3\|\nabla_v(\partial_{x_i} f_t G^{-1})\|^2_{L^2(m_1 G)}
\end{aligned}
$$

$$+2ct^2(\nabla_v(\partial_{x_i}f_tG^{-1}),\nabla_v(\partial_{v_i}f_tG^{-1}))_{L^2(m_1G)}]-\frac{\delta(1-\delta)}{2}[at\|\nabla_vf_t\|^2_{L^2(m_1|v|)}$$
$$+bt^3\|\nabla_xf_t\|^2_{L^2(m_1|v|)}+2ct^2(\nabla_vf_t,\nabla_xf_t)_{L^2(m_1|v|)}]+C\|f_t\|^2_{L^2(m_1)},$$

for some $C>0$. We observe that

$$|2ct^2(\nabla_vf_t,\nabla_xf_t)_{L^2(m_1|v|)}|\le at\|\nabla_vf_t\|^2_{L^2(m_1|v|)}+bt^3\|\nabla_xf_t\|^2_{L^2(m_1|v|)},$$

and

$$|2ct^2(\nabla_v(\partial_{x_i}f_tG^{-1}),\nabla_v(\partial_{v_i}f_tG^{-1}))_{L^2(m_1G)}|$$
$$\le at\|\nabla_v(\partial_{v_i}f_tG^{-1})\|^2_{L^2(m_1G)}+bt^3\|\nabla_v(\partial_{x_i}f_tG^{-1})\|^2_{L^2(m_1G)}.$$

by our choice on a,b,c. So by taking A large, $12b\le c$, and $0<\eta$ small ($t\in(0,\eta]$), as a consequence

$$\frac{d}{dt}\mathcal{F}(t,f_t)\le -L(\|\nabla_vf_t\|^2_{L^2(m_1)}+t^2\|\nabla_xf_t\|^2_{L^2(m_1)})+C\|f_t\|^2_{L^2(m_1)},$$

for some $L,C>0$, and that ends the proof. ◻

REMARK 4.1. For the case $F=\mathcal{F}^*$, the only difference in the proof is to change (4.6) and (4.7) into

$$\partial_{x_i}\mathcal{L}^*f=\mathcal{L}^*\partial_{x_i}f-\sum_{j=1}^d\partial^2_{x_ix_j}V\partial_{v_j}f,$$

and

$$\partial_{v_i}\mathcal{L}^*f=\mathcal{L}^*\partial_{v_i}f+\partial_{x_i}f+\partial_{v_i}f.$$

The following proofs of this section are true for both cases.

LEMMA 4.3. *Denote* $m_1=G^{-\frac{1}{2}(1+\delta)}$, *then for any* $0<\delta<1$ *we have*

$$\int|\nabla_{x,v}(fm_1)|^2\le\int|\nabla_{x,v}f|^2m_1^2+C\int f^2m_1^2,$$

Proof. We have

$$\begin{aligned}\int|\nabla(fm_1)|^2&=\int|\nabla fm_1+\nabla m_1f|^2\\&=\int|\nabla f|^2m_1^2+\int|\nabla m_1|^2f^2+\int 2fm_1\nabla f\cdot\nabla m_1\\&=\int|\nabla f|^2m_1^2+\int\left(|\nabla m_1|^2-\frac{1}{2}\Delta(m_1^2)\right)f^2,\\&=\int|\nabla f|^2m_1^2-\int\frac{\Delta m_1}{m_1}f^2m_1^2,\end{aligned}$$

since

$$\frac{\Delta m_1}{m_1}=\frac{(1+\delta)^2}{4}(|v|^2+|\nabla_xV(x)|^2)+\frac{1+\delta}{2}(\Delta_xV(x)+d)\ge -C,$$

for some $C>0$, we are done. ∎

LEMMA 4.4. *Nash's inequality: for any $f\in L^1(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$, there exists a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}}\le C_d\|f\|_{L^1}^{\frac{2}{d}}\|\nabla_v f\|_{L^2},$$

For the proof of Nash's inequality, we refer to [12, Section 8.13], for instance.

LEMMA 4.5. *Denote $m_1=G^{-\frac{1}{2}(1+\delta)}$, then for any $0<\delta<1$ we have*

$$\frac{d}{dt}\|f\|_{L^1(m_1)}\le d\|f\|_{L^1(m_1)} \tag{4.8}$$

which implies

$$\|f_t\|_{L^1(m_1)}\le e^{dt}\|f_0\|_{L^1(m_1)}$$

In particular we have

$$\|f_t\|_{L^1(m_1)}\le C\|f_0\|_{L^1(m_1)},\quad \forall t\in(0,\eta], \tag{4.9}$$

for some constant $C>0$.

Proof. By Lemma 5.1 in the next section, letting $p=1$, we have

$$\begin{aligned}\frac{d}{dt}\int|f|m_1=&\int|f|(\Delta_v m_1-v\cdot\nabla_v m_1\\&+v\cdot\nabla_x m_1-\nabla_x V(x)\cdot\nabla_v m_1-K\chi_R m_1)\\ \le&\int|f|\left(\frac{1+\delta}{2}d-\frac{(1+\delta)(1-\delta)}{4}|v|^2\right)m_1\le d\int|f|m_1.\end{aligned}$$

so (4.8) is proved. As $\mathcal{T}m_1=0$, the result is still true when $F=\mathcal{F}^*$. ∎

Now we come to the proof of Theorem 4.1.

Proof. **(Proof of Theorem 4.1.)** We define

$$\mathcal{G}(t,f_t)=B\|f_t\|_{L^1(m_1)}^2+t^Z\mathcal{F}(t,f_t),$$

with $B,Z>0$ to be fixed and $\mathcal{F}$ is defined in Lemma 4.2. We choose $t\in(0,\eta]$, η small such that $(a+b+c)Z\eta^{Z+1}\le\frac{1}{2}L\eta^Z$ (a,b,c,L are also defined Lemma 4.2), by (4.8) and Lemma 4.2 we have

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t,f_t)\le& dB\|f_t\|_{L^1(m_1)}^2+Zt^{Z-1}\mathcal{F}(t,f_t)\\&-Lt^Z(\|\nabla_v f_t\|_{L^2(m_1)}^2+t^2\|\nabla_x f_t\|_{L^2(m_1)}^2)+Ct^Z\|f_t\|_{L^2(m_1)}^2\\ \le& dB\|f_t\|_{L^1(m_1)}^2+Ct^{Z-1}\|f_t\|_{L^2(m_1)}^2\\&-\frac{L}{2}t^Z(\|\nabla_v f_t\|_{L^2(m_1)}^2+t^2\|\nabla_x f_t\|_{L^2(m_1)}^2).\end{aligned}$$

Nash's inequality and Lemma 4.2 implies

$$\|f_tm_1\|_{L^2}\le C\|f_tm_1\|_{L^1}^{\frac{2}{d+2}}\|\nabla_{x,v}(f_tm_1)\|_{L^2}^{\frac{d}{d+2}}$$

$$\le C\|f_t m_1\|_{L^1}^{\frac{2}{d+2}} (\|\nabla_{x,v} f_t m_1\|_{L^2} + C\|f_t m_1\|_{L^2})^{\frac{d}{d+2}}.$$

Using Young's inequality, we have

$$\|f_t\|_{L^2(m_1)}^2 \le C_\epsilon t^{-\frac{3}{2}d} \|f\|_{L^1(m_1)}^2 + \epsilon t^3 (\|\nabla_{x,v} f_t\|_{L^2(m_1)}^2 + C\|f_t\|_{L^2(m_1)}^2).$$

Taking ϵ small such that $C\epsilon\eta^3 \le \frac{1}{2}$, we deduce

$$\|f_t\|_{L^2(m_1)}^2 \le 2C_\epsilon t^{-\frac{3}{2}d} \|f\|_{L^1(m_1)}^2 + 2\epsilon t^3 \|\nabla_{x,v} f_t\|_{L^2(m_1)}^2.$$

Taking ϵ small we have

$$\frac{d}{dt}\mathcal{G}(t,f_t) \le dB\|f_t\|_{L^1(m_1)}^2 + C_1 t^{Z-1-\frac{3}{2}d} \|f_t\|_{L^1(m_1)}^2,$$

for some $C_1 > 0$. Choosing $Z = 1 + \frac{3}{2}d$, and using (4.9), we deduce

$$\forall t \in (0,\eta], \quad \mathcal{G}(t,f_t) \le \mathcal{G}(0,f_0) + C_2\|f_0\|_{L^1(m_1)}^2 \le C_3\|f_0\|_{L^1(m_1)}^2,$$

which ends the proof. ∎

5. $S_\mathcal{B}$ decay in larger spaces

The aim of this section is to prove the following decay estimate for the semigroup $S_\mathcal{B}$ which will be useful in the last section where we will prove Theorem 1.1 in full generality.

THEOREM 5.1. *Let $H = 1 + |x|^2 + 2v\cdot x + 3|v|^2$, for any $\theta \in (0,1)$ and for any $l > 0$, we have*

$$\|S_\mathcal{B}(t)\|_{L^1(H^l)\to L^1(H^{l\theta})} \lesssim (1+t)^{-a},$$

where

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}}.$$

We start with an elementary identity.

LEMMA 5.1. *For the kinetic Fokker Planck operator $\mathcal{L}$, let m be a weight function, for any $p \in [1,\infty]$ we have*

$$\int |f|^{p-1} \operatorname{sign} f (\mathcal{L}f) m^p = -(p-1)\int |\nabla_v (mf)|^2 (m|f|)^{p-2} + \int |f|^p m^p \phi,$$

with

$$\phi = \frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

In particular when $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

Proof. We split the integral as

$$\int \operatorname{sign} f(\mathcal{L}f)|f|^{p-1}m^p = \int \operatorname{sign} f|f|^{p-1}(\mathcal{S}f)m^p + \int \operatorname{sign} f|f|^{p-1}(\mathcal{T}f)m^p.$$

First compute the contribution of the term with operator $\mathcal{T}$

$$\int \operatorname{sign} f|f|^{p-1}(\mathcal{T}f)m^p = \frac{1}{p}\int \mathcal{T}(|f|^p)m^p = -\int |f|^p m^p \frac{\mathcal{T}m}{m}.$$

Concerning the term with operator $\mathcal{S}$, we split it also into two parts

$$\int (\mathcal{S}f)\operatorname{sign} f|f|^{p-1}m^p = \int \operatorname{sign} f|f|^{p-1}m^p(\Delta_v f + \operatorname{div}_v(vf)) := C_1 + C_2.$$

We first compute the C_2 term, to get

$$\begin{aligned} C_2 &= \int \operatorname{sign} f|f|^{p-1}m^p(df + v\cdot\nabla_v f) \\ &= \int d|f|^p m^p - \frac{1}{p}\int |f|^p \operatorname{div}_v(vm^p) \\ &= \int |f|^p\left[\left(1-\frac{1}{p}\right)d - v\cdot\frac{\nabla_v m}{m}\right]m^p. \end{aligned}$$

Then turning to the C_1 term, we have

$$\begin{aligned} C_1 &= \int \operatorname{sign} f|f|^{p-1}m^p\Delta_v f = -\int \nabla_v(\operatorname{sign} f|f|^{p-1}m^p)\cdot\nabla_v f \\ &= \int -(p-1)|\nabla_v f|^2|f|^{p-2}m^p - \frac{1}{p}\int \nabla_v|f|^p\cdot\nabla_v(m^p). \end{aligned}$$

Using $\nabla_v(mf) = m\nabla_v f + f\nabla_v m$, we deduce

$$\begin{aligned} C_1 = &-(p-1)\int |\nabla_v(mf)|^2|f|^{p-2}m^{p-2} + (p-1)\int |\nabla_v m|^2|f|^p m^{p-2} \\ &+\frac{2(p-1)}{p^2}\int \nabla_v(|f|^p)\cdot\nabla_v(m^p) - \frac{1}{p}\int \nabla_v(|f|^p)\cdot\nabla_v(m^p) \\ = &-(p-1)\int |\nabla_v(mf)|^2|f|^{p-2}m^p + (p-1)\int |\nabla_v m|^2|f|^p m^{p-2} \\ &-\frac{p-2}{p^2}\int |f|^p\Delta_v m^p. \end{aligned}$$

Using that $\Delta_v m^p = p\Delta_v m\ m^{p-1} + p(p-1)|\nabla_v m|^2 m^{p-2}$, we obtain

$$C_1 = -(p-1)\int |\nabla_v(mf)|^2|f|^{p-2}m^{p-2} + \int |f|^p m^p\left[\left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v m|^2}{m^2}\right].$$

We conclude by combining the above equalities. ∎

Proof. **(Proof of Theorem 5.1.)** From Lemma 5.1, we have

$$\int \operatorname{sign} f(\mathcal{B}f)|f|^{p-1}m^p = \int \operatorname{sign} f((\mathcal{L} - K\chi_R)f)|f|^{p-1}m^p$$

$$= -(p-1)\int |\nabla_v(mf)|^2(m|f|)^{p-2} + \int |f|^p m^p \phi, \tag{5.1}$$

with

$$\phi = \frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + (\frac{2}{p}-1)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - K\chi_R.$$

When $p=1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - K\chi_R.$$

Let $m = H^k$. We have

$$\frac{\nabla_v m}{m} = k\frac{\nabla_v H}{H}, \quad \frac{\nabla_x m}{m} = k\frac{\nabla_x H}{H},$$

and

$$\frac{\Delta_v m}{m} = \frac{k\Delta_v H}{H} + \frac{k(k-1)|\nabla_v H|^2}{H^2}.$$

Summing up, we have for ϕ

$$\frac{\phi H}{k} = \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} - v\cdot\nabla_v H + v\cdot\nabla_x H - \nabla_x V(x)\cdot\nabla_v H - K\chi_R,$$

From the very definition of H, we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

We then compute

$$\begin{aligned}
&\Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} + v\cdot\nabla_x H - v\cdot\nabla_v H - \nabla_x V(x)\cdot\nabla_v H \\
&= 6 + (k-1)\frac{|6v+2x|^2}{H} + 2|v|^2 + 2x\cdot v - 6|v|^2 \\
&\quad -2x\cdot v - 6v\cdot\nabla_x V(x) - 2x\cdot\nabla_x V(x) \\
&\le (2|v|^2 + C|v| - 6|v|^2) - 2x\cdot\nabla_x V(x) + C \\
&\le -C_1|v|^2 - C_2 x\cdot\nabla_x V(x) + C_3 \\
&\le -C_4 H^{\frac{\gamma}{2}} + K_1\chi_{R_1},
\end{aligned}$$

for some $C_i > 0$. Taking K and R large enough, we have $\phi \le -CH^{\frac{\gamma}{2}-1}$, using this inequality in Equation (5.1), we deduce

$$\begin{aligned}
\frac{d}{dt}Y_4(t) := \frac{d}{dt}\int |f_{\mathcal{B}}(t)|H^k &= \int \operatorname{sign}(f_{\mathcal{B}}(t))(\mathcal{B}f_{\mathcal{B}}(t))H^k \\
&\le -C\int |f_B(t)|H^{k-1+\frac{\gamma}{2}},
\end{aligned} \tag{5.2}$$

for any $k > 1$. In particular for any $l \ge 1$, we can find K and R large enough such that

$$\frac{d}{dt}\int |f_{\mathcal{B}}(t)|H^l \le 0,$$

which readily implies

$$\int |f_{\mathcal{B}}(t)|H^l \leq \int |f_0|H^l := Y_5.$$

Take $k \leq l$, denoting

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in (0,1),$$

the Hölder's inequality

$$\int |f_B(t)|H^k \leq \left(\int |f_B(t)|H^{k-1+\frac{\gamma}{2}}\right)^{\alpha} \left(\int |f_B(t)|H^l\right)^{1-\alpha},$$

implies

$$\left(\int |f_B(t)|H^k\right)^{\frac{1}{\alpha}} \left(\int |f_B(t)|H^l\right)^{\frac{\alpha-1}{\alpha}} \leq \int |f_B(t)|H^{k-1+\frac{\gamma}{2}},$$

From this inequality and (5.2), we get

$$\frac{d}{dt}Y_4(t) \leq -C(Y_4(t))^{\frac{1}{\alpha}} Y_5^{\frac{\alpha-1}{\alpha}}.$$

Using $Y_4(0) \leq Y_5$, after an integration, we deduce

$$Y_4(t) \leq C_\alpha \frac{1}{(1+t)^{\frac{\alpha}{1-\alpha}}} Y_5,$$

which is nothing but the polynomial decay on $S_{\mathcal{B}}$

$$\|S_{\mathcal{B}}(t)\|_{L^p(H^l) \to L^p(H^k)} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l-k}{1-\frac{\gamma}{2}}, \quad \forall 0<k<l, \quad 1 \leq l.$$

We conclude Theorem 5.1 by writing $k = l\theta$, $0<\theta<1$. ∎

6. L^p convergence for the KFP model

Before going to the proof of our main theorem, we need two last deduced results.

LEMMA 6.1. *For any $\epsilon > 0$ small enough, we have*

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)}) \to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

and

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \to L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

for some $a > 0$. Similarly for any $0<b<\frac{\gamma}{2-\gamma}$ and for any $\epsilon>0$ small enough, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)}) \to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} e^{-at^b}, \quad \forall t \geq 0,$$

for $\alpha=\frac{3d+2}{4}$ *and some* $a>0$.

Proof. The first two inequalities are obtained obviously by Lemma 3.1 and the property $\mathcal{A}=K\chi_R$. For the third inequality we split it into two parts, $t\in(0,\eta]$ and $t>\eta$, where η is defined in Theorem 4.1. When $t\in(0,\eta]$, we have $e^{-at^{\frac{\gamma}{2-\gamma}}}\geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$, by Theorem 4.1, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim t^{-\alpha}\lesssim t^{-\alpha}e^{-at^{\frac{\gamma}{2-\gamma}}},\quad \forall t\in(0,\eta],$$

for some $a>0$. When $t\geq\eta$, by Theorem 4.1, we have

$$\|S_{\mathcal{B}}(\eta)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim\eta^{\alpha}\lesssim 1,$$

and by Lemma 3.1

$$\|S_{\mathcal{B}}(t-\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})}\lesssim e^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

gathering the two inequalities, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}\lesssim t^{-\alpha}e^{-at^b},\quad \forall t>\eta,$$

for any $0<b<\frac{\gamma}{2-\gamma}$, the proof is ended by combining the two cases above. $\square$

LEMMA 6.2. *Similarly as Lemma 6.1, for any* $p\in(2,\infty)$, *we have*

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^2(G^{-\frac{1}{2}})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},\quad \forall t\geq 0.$$

and

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^p(G^{-\frac{1}{2}})\to L^p(G^{-\frac{1}{2}})}\lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},\quad \forall t\geq 0.$$

for some $a>0$. *And for any* $0<b<\frac{\gamma}{2-\gamma}$ *we have*

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^p(G^{-\frac{1}{2}})}\lesssim t^{-\beta}e^{-at^b},\quad \forall t\geq 0.$$

for some $\beta>0$ *and some* $a>0$.

The proof of Lemma 6.2 is similar to the proof of Lemma 6.1 and is thus skipped.

LEMMA 6.3. *Let* X,Y *be two Banach spaces,* $S(t)$ *a semigroup such that for all* $t\geq 0$ *and some* $0<a$, $0<b<1$ *we have*

$$\|S(t)\|_{X\to X}\leq C_X e^{-at^b},\quad \|S(t)\|_{Y\to Y}\leq C_Y e^{-at^b},$$

and for some $0<\alpha$, *we have*

$$\|S(t)\|_{X\to Y}\leq C_{X,Y}t^{-\alpha}e^{-at^b}.$$

Then we can have that for all integers $n>0$

$$\|S^{(*n)}(t)\|_{X\to X}\leq C_{X,n}t^{n-1}e^{-at^b},$$

similarly

$$\|S^{(*n)}(t)\|_{Y\to Y} \le C_{Y,n} t^{n-1} e^{-at^b},$$

and

$$\|S^{(*n)}(t)\|_{X\to Y} \le C_{X,Y,n} t^{n-\alpha-1} e^{-at^b}.$$

In particular for $\alpha+1<n$, and for any $b^*<b$

$$\|S^{(*n)}(t)\|_{X\to Y} \le C_{X,Y,n} e^{-at^{b^*}}.$$

The proof of Lemma 6.3 is the same as Lemma 2.5 in [14], plus the fact $t^b \le s^b + (t-s)^b$ for any $0\le s\le t, 0<b<1$.

Then we come to the final proof.

Proof. **(Proof of Theorem 1.1.)** We only prove the case when $m = G^{\frac{p-1}{p}(1+\epsilon)}$, $p\in[1,2]$, for the proof of the other cases, one need only replace the use of Lemma 6.1 in the following proof by Lemma 6.2 and Theorem 4.1. We will prove $p=1$ first, this time we need to prove

$$\|S_{\mathcal{L}}(I-\Pi)(t)\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^b},$$

for any $0<b<\frac{\gamma}{2-\gamma}$, where I is the identity operator and Π is a projection operator defined by

$$\Pi(f) = \mathcal{M}(f)G.$$

First, iterating the Duhamel's formula, we split it into 3 terms

$$S_{\mathcal{L}}(I-\Pi) = (I-\Pi)\{S_{\mathcal{B}} + \sum_{l=1}^{n-1} (S_{\mathcal{B}}\mathcal{A})^{(*l)} * (S_{\mathcal{B}})\} + \{(I-\Pi)S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}}(t))^{(*n)},$$

and we will estimate them separately. By Lemma 3.1, we have

$$\|S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \tag{6.1}$$

the first term is thus estimated. For the second term, still using Lemma 3.1, we get

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^1\to L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

by Lemma 6.3, we have

$$\|(S_{\mathcal{B}}(t)\mathcal{A})^{(*l)}\|_{L^1\to L^1} \lesssim t^{l-1} e^{-at^{\frac{\gamma}{2-\gamma}}},$$

together with (6.1) the second term is estimated. For the last term by Lemma 3.1

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon})\to L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

By Lemma 6.1 and 6.3, for any $0<b<\frac{\gamma}{2-\gamma}$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*(n-1))}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{n-\alpha-2} e^{-at^b},$$

finally by Theorem 3.1, we have

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Taking $n>\alpha+2$ the third term is estimated, thus the proof of case $p=1$ is concluded by gathering the inequalities above. As the case $p=2$ is already proved in Theorem 3.1, the case $p\in(1,2)$ follows by interpolation. ∎

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