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A NONLINEAR FUNCTIONAL APPROACH FOR MONO-CLUSTER FLOCKING TO THE DISCRETE CUCKER-SMALE MODEL*

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Abstract. We present a nonlinear functional approach for the discrete Cucker-Smale (C-S) model with a general communication weight, and using the monotonicity of this functional, we provide a simple sufficient condition for the emergence of a mono-cluster flocking in terms of initial data, communication weight function and system parameters. Our proposed nonlinear functionals are the discrete analogues of the continuous nonlinear functionals introduced in [S.-Y. Ha and J.-G. Liu, Commun. Math. Sci., 7:297–325, 2009].

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1. Introduction

After Vicsek's seminal work [35] on the modeling of collective dynamics, research on the collective dynamics has received lots of attention from diverse disciplines such as applied math, control theory, nonlinear dynamics and statistical physics, etc. [14–16, 21, 26, 27, 32, 33, 35] at the beginning of this century. In this paper, we are mainly interested in a mechanical model by two mathematicians Cucker and Smale. In [13], the authors proposed sufficient conditions for the mono-cluster flocking in terms of initial data, decay rate of communication weight and system parameters, and they also showed a phase-transition-like phenomenon depending on the long-ranged or short-ranged nature of the communication weight function. Compared to the previous literature [27, 34] on the flocking modeling, Cucker and Smale provided the first analytical framework for the emergence of mono-cluster flocking using the self-bounding method, and their study was further generalized from different perspectives in [1–4, 6, 7, 9–12, 17, 18, 20, 22–24, 30]. As an alternative methodology for flocking, Ha and Liu introduced nonlinear functional approach for the *continuous* C-S model with a general communication weight, and provided a simple sufficient condition leading to the emergence of mono-cluster flocking (see Section 2.2). Their sufficient condition was expressed as a simple inequality (2.9) involving the initial data, communication weight and coupling strength. As far as the authors know, this nonlinear functional approach has never been extended to the discrete C-S model. Recently, Ha and Zhang [19] succeeded in providing a flocking theorem for the discrete C-S model with a general communication weight by combining a nonlinear functional approach for the continuous model and approximation relationship between the continuous C-S model and discrete C-S model. In fact, their key idea is to use the flocking information for the continuous model to derive analogous results for the discrete model. In this work, we provide a nonlinear functional approach for the discrete model directly without resorting to the information from the continuous model. Now,

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we describe our setting. Consider a flock of N interacting particles (agents) moving in the Euclidean space $\mathbb{R}^d$, and let $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ be the phase-space coordinate of the i-th C-S particle in $\mathbb{R}^{2d}$. Then, the continuous dynamics of phase-space coordinate $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ is governed by the first-order ODE system:

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{v}_i(t), \quad t > 0, \quad i = 1, \cdots, N, \\ \dot{\boldsymbol{v}}_i(t) = \dfrac{\kappa}{N} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N} \psi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|)(\boldsymbol{v}_j(t) - \boldsymbol{v}_i(t)), \\ (\boldsymbol{x}_i, \boldsymbol{v}_i)(0) = (\boldsymbol{x}_i^0, \boldsymbol{v}_i^0), \end{cases} \tag{1.1}$$

where $\kappa > 0$ and $\|\cdot\|$ are the coupling strength and standard ℓ_2-norm in $\mathbb{R}^d$, respectively, and $\psi : (0, \infty) \to (0, \infty)$ is the communication weight function which is locally Lipschitz continuous and satisfies positivity and monotonicity:

$$\psi(r) > 0, \quad \forall\, r > 0, \qquad (\psi(r_1) - \psi(r_2))(r_1 - r_2) \leq 0, \ r_1, r_2 > 0. \tag{1.2}$$

Note that the functions

$$\psi(r) = r^{-\alpha} \quad \text{or} \quad \psi(r) = (1 + r^2)^{-\frac{\alpha}{2}} \quad \text{with } \alpha > 0$$

satisfy the above assumptions (1.2). On the other hand, for the sake of numerical simulations, we need to discretize the continuous system (1.1) by employing suitable discretization schemes. In this paper, we use the first-order forward Euler discretization scheme with time-step $h > 0$. First, we set

$$\boldsymbol{x}_i[n] := \boldsymbol{x}_i(nh), \quad \boldsymbol{v}_i[n] := \boldsymbol{v}_i(nh) \quad \text{for all } n \in \mathbb{N}.$$

Then, the state $(\boldsymbol{x}_i[n], \boldsymbol{v}_i[n])$ is governed by the difference system:

$$\begin{cases} \boldsymbol{x}_i[n+1] = \boldsymbol{x}_i[n] + h\boldsymbol{v}_i[n] \quad n \in \mathbb{N}, \quad i = 1, \cdots, N, \\ \boldsymbol{v}_i[n+1] = \boldsymbol{v}_i[n] + \dfrac{h\kappa}{N} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N} \psi(\|\boldsymbol{x}_i[n] - \boldsymbol{x}_j[n]\|)(\boldsymbol{v}_j[n] - \boldsymbol{v}_i[n]), \\ (\boldsymbol{x}_i, \boldsymbol{v}_i)[0] = (\boldsymbol{x}_i^0, \boldsymbol{v}_i^0). \end{cases} \tag{1.3}$$

The continuous model (1.1) -(1.2) with a regular and bounded ψ has been extensively studied in literature (see a recent survey paper [11] and references therein). In this case, although collisions between particles are possible and the set of all such configurations has measure zero in state space, due to the boundedness of ψ, collisions do not cause any trouble in the well-posedness and flocking estimate. In contrast, for the singular and unbounded ψ, it takes the value of infinity at the moment when particles collide. Thus, the standard Cauchy-Lipschitz theory cannot be applied to yield a global solution. Recently, Peszek and his collaborators in a series of papers [5,25,28,29] have investigated the possibility of collision avoidance in terms of the degree of singularity, when ψ is singular at the origin and short-ranged. In this paper, we instead use the collision avoidance framework from [2], in which some admissible set for the initial configurations was proposed to avoid collisions and existence of a positive lower bound for a positive minimal distance between particles was obtained. In this framework, global smooth solutions exist using the standard ODE theory. Similar issues can also occur for the discrete model (1.3) as well. Motivated by the work [2], we will study the discrete model (1.3) for some class of restricted initial data in which a positive minimal distance between particles can be obtained.

The main result of this paper is to propose a nonlinear functional approach for flocking to the discrete C-S model (1.3) with a monotonically decaying communication weight (1.2), which has been sought for last ten years after Ha-Liu's work [17]. Our proposed nonlinear functionals for the discrete model have the same form as nonlinear functionals for the continuous C-S model except that the integral part is replaced by the discrete sum. More precisely, for $n\geq 1$, the nonlinear functionals $\mathcal{L}_{\pm}[n]$ have the form:

$$\mathcal{L}_{\pm}[n] := \|V[n]\| \pm \kappa \sum_{i=1}^{n} \psi\left(\sqrt{2}\|X[i-1]\|\right)(\|X[i]\| - \|X[i-1]\|).$$

Then, by the matrix formulation of the difference system (1.3) (see Section 2.1), we can show that $\mathcal{L}_{\pm}[n]$ satisfy the monotonicity (Proposition 3.1):

$$\mathcal{L}_{\pm}[n+1] \leq \mathcal{L}_{\pm}[n], \quad n\geq 0.$$

Once the above monotonicity is obtained, then we can use a similar argument as the continuous model to derive the emergence of mono-cluster flocking for the discrete model (1.3). As a direct corollary of flocking estimate, we can also provide a uniform-in-time transition from the discrete model to the continuous model which is valid for whole time interval as the time-step tends to zero (see Corollary 2.1).

The rest of the paper is organized as follows. In Section 2, we provide matrix formulations of (1.1) and (1.3), and review a nonlinear functional approach for the continuous model, and discuss our main results. In Section 3, we present a proof of Theorem 2.2 on the emergence of mono-cluster flocking. In Section 4, we study the uniform-in-time transition from discrete model to continuous model, as the size of time-step h tends to zero. Finally, Section 5 is devoted to a brief summary of our main results and discussion on the possible direction for a future work.

Notation: Throughout the paper, we regard a vector $\boldsymbol{z}=(z^1,\cdots,z^d)\in\mathbb{R}^d$ as a $d\times 1$ matrix. For N column vectors $\boldsymbol{z}_1,\cdots,\boldsymbol{z}_N$ in $\mathbb{R}^d$, $Z=(\boldsymbol{z}_1,\cdots,\boldsymbol{z}_N)^T$ is an $N\times d$ matrix whose rows are given by $\boldsymbol{z}_i$. For two vectors $\boldsymbol{z},\boldsymbol{w}$ in $\mathbb{R}^d$, $\boldsymbol{z}\cdot\boldsymbol{w}$ and $\langle\boldsymbol{z},\boldsymbol{w}\rangle$ denote the same standard inner product in $\mathbb{R}^d$:

$$\langle\boldsymbol{z},\boldsymbol{w}\rangle = \boldsymbol{z}\cdot\boldsymbol{w} = \sum_{i=1}^{d} z^i\cdot w^i.$$

2. Preliminaries and main results

In this section, we present matrix representation of the models (1.1) and (1.3), and briefly review the nonlinear functional approach for the continuous model in [17]. We also briefly describe our main results whose proofs will be presented in the following two sections.

2.1. Matrix-valued system formulation.

Below, we present a decomposition of the N-body velocity space $\mathbb{R}^{N\times d}$ as a direct sum of d-dimensional flocking manifold and its orthogonal complement. More precisely, we set $\triangle$ to be a d-dimensional flocking submanifold which corresponds to the diagonal of $\mathbb{R}^{N\times d}$:

$$\triangle := \left\{(u,\ldots,u)^{\top} \,|\, u\in\mathbb{R}^d\right\},$$

and set $\triangle^\perp$ to be the orthogonal complement of $\triangle$ in $\mathbb{R}^{N\times d}$. Thus, every point $V = (\boldsymbol{v}_1,\dots,\boldsymbol{v}_N)^\top \in \mathbb{R}^{N\times d}$ can be decomposed in a unique way:

$$V = V_\triangle + V_\perp, \quad V_\triangle \in \triangle \quad \text{and} \quad V_\perp \in \triangle^\perp.$$

In fact, this decomposition has a simple explicit form:

$$\bar{\boldsymbol{v}} := \frac{1}{N}\sum_{i=1}^N \boldsymbol{v}_i, \quad V_\triangle = (\bar{\boldsymbol{v}},\dots,\bar{\boldsymbol{v}})^\top \quad \text{and} \quad V_\perp = (\boldsymbol{v}_1 - \bar{\boldsymbol{v}},\dots,\boldsymbol{v}_k - \bar{\boldsymbol{v}})^\top. \tag{2.1}$$

Then, it is easy to see that

$$\langle V_\triangle, V_\perp\rangle := \sum_{i=1}^N \bar{\boldsymbol{v}}\cdot(\boldsymbol{v}_i - \bar{\boldsymbol{v}}) = \bar{\boldsymbol{v}}\cdot\left(\sum_{i=1}^N \boldsymbol{v}_i - N\bar{\boldsymbol{v}}\right) = 0,$$

where $\cdot$ is the standard inner product in $\mathbb{R}^d$.

Next, we rewrite system (1.1) in matrix form. For this, we first note that the right-hand side of the velocity Equation $(1.1)_2$ can be rewritten as

$$\frac{\kappa}{N}\sum_{\substack{j=1\\ j\neq i}}^N \psi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)(\boldsymbol{v}_j - \boldsymbol{v}_i) = -\frac{\kappa}{N}\Big[\Big(\sum_{\substack{j=1\\ j\neq i}}^N \psi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)\Big)\boldsymbol{v}_i - \sum_{\substack{j=1\\ j\neq i}}^N \psi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)\boldsymbol{v}_j\Big]. \tag{2.2}$$

To exprress the R.H.S. of (2.2) as a matrix form, we set

$$\begin{aligned}
&a_{ij}(t) := \psi(\|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|), \quad d_i(t) := \sum_{j=1}^N a_{ij}(t), \quad 1\le i\le N,\\
&A(t) := (a_{ij}(t)), \quad D(t) = \mathrm{diag}(d_1(t),\cdots,d_N(t)), \quad L(t) := D(t) - A(t),\\
&X(t) := (\boldsymbol{x}_1(t),\cdots,\boldsymbol{x}_N(t))^\top \quad \text{and} \quad V(t) := (\boldsymbol{v}_1(t),\cdots,\boldsymbol{v}_N(t))^\top.
\end{aligned} \tag{2.3}$$

Then, the matrix L is a Laplacian matrix (see [8] for definition of Laplacian matrix and its spectral properties). Then, it follows from (1.1), (2.2) and (2.3) that (X,V) satisfies the matrix-valued ODE system:

$$\begin{cases} \dot{X}(t) = V(t), \quad t>0,\\ \dot{V}(t) = -\dfrac{\kappa}{N}L(t)V(t),\\ (X,V)(0) = (X^0,V^0). \end{cases} \tag{2.4}$$

Similarly, the discrete C-S model (1.3) can also be written as

$$\begin{cases} X[n+1] = X[n] + hV[n], \quad n\in\mathbb{N},\\ V[n+1] = \left(I - h\dfrac{\kappa}{N}L[n]\right)V[n],\\ (X,V)[0] = (X^0,V^0). \end{cases} \tag{2.5}$$

Now, we study conservation laws associated with (1.1) and (1.3). For this, similar to (2.1), we introduce the average position $\bar{\boldsymbol{x}}$:

$$\bar{\boldsymbol{x}} := \frac{1}{N}\sum_{i=1}^N \boldsymbol{x}_i = \frac{1}{N}([1,\cdots,1]X)^\top.$$

Note that our C-S models are translation invariant, so linear momenta are conserved as can be seen in the following lemma.

LEMMA 2.1. *Let $(X(t),V(t))$ and $(X[n],V[n])$ be solutions to (1.1) and (1.3), respectively. Then, we have*

$$(i)\ \bar{\boldsymbol{v}}(t)=\bar{\boldsymbol{v}}^0, \quad \bar{\boldsymbol{x}}(t)=\bar{\boldsymbol{x}}^0+t\bar{\boldsymbol{v}}^0, \quad t>0.$$

$$(ii)\ \bar{\boldsymbol{v}}[n]=\bar{\boldsymbol{v}}^0, \quad \bar{\boldsymbol{x}}[n]=\bar{\boldsymbol{x}}^0+nh\bar{\boldsymbol{v}}^0, \quad n\in\mathbb{N}.$$

Proof. The proofs follow from (1.1) and (1.3) by straightforward calculations. ◻

Thanks to Lemma 2.1 and translation invariance of (1.1) and (1.3), from now on, we may assume that

$$\sum_{i=1}^N \boldsymbol{x}_i(t)=\sum_{i=1}^N \boldsymbol{v}_i(t)=0, \text{ for all } t\geq 0, \qquad \sum_{i=1}^N \boldsymbol{x}_i[n]=\sum_{i=1}^N \boldsymbol{v}_i[n]=0, \text{ for all } n\in\mathbb{N}.$$

In other words, we have

$$[1,\cdots,1]X=[1,\cdots,1]V=[0,\cdots,0].$$

This means that we are considering systems (2.4) and (2.5) in the quotient space $\mathbb{E}:=\triangle^\perp$, instead of $\mathbb{R}^{Nd}$.

2.2. The continuous C-S model. In this subsection, we briefly review the nonlinear functional approach for the continuous model (1.1)-(1.2) with a bounded communication weight, and explain how it can be used to derive mono-cluster flocking estimates. In order to reduce the large system (1.1) into a smaller system, we introduce two scalar quantities $\|X\|$ and $\|V\|$ which correspond to the Frobenius norms for state matrix (X,V):

$$\|X\|^2:=\sum_{i=1}^N\|\boldsymbol{x}_i\|^2, \qquad \|V\|^2:=\sum_{i=1}^N\|\boldsymbol{v}_i\|^2,$$

Then, one can easily derive a system of dissipative differential equalities for $\|X(t)\|$ and $\|V(t)\|$ along (2.4) (see [17] for details):

$$\left|\frac{d\|X(t)\|}{dt}\right|\leq\|V(t)\|, \qquad \frac{d\|V(t)\|}{dt}\leq-\kappa\psi(\sqrt{2}\|X(t)\|)\|V(t)\|, \qquad \text{for a.e. } t>0. \tag{2.6}$$

Note that once we can derive a uniform bound for $\|X\|$, we can derive an exponential decay of $\|V\|$. For this purpose, we introduce nonlinear functionals $\mathcal{E}_\pm$ which can be viewed as ℓ^2-energy-type functionals for (2.4):

$$\mathcal{E}_\pm(t):=\|V(t)\|\pm\kappa\int_{\|X^0\|}^{\|X(t)\|}\psi(\sqrt{2}s)ds. \tag{2.7}$$

We differentiate the above functionals and use (2.6) to derive

$$\frac{d}{dt}\mathcal{E}_\pm(t)\leq-\kappa\psi(\sqrt{2}\|X(t)\|)\underbrace{\left(\|V(t)\|\mp\frac{d\|X(t)\|}{dt}\right)}_{\geq 0 \text{ due to } (2.6)_1}\leq 0.$$

We again integrate the above relation to see that $\mathcal{E}_{\pm}(t)$ is non-increasing with respect to t:

$$\|V(t)\| + \kappa \int_{\|X^0\|}^{\|X(t)\|} \psi(\sqrt{2}s)ds \le \|V^0\|, \quad \text{and}$$

$$\|V(t)\| - \kappa \int_{\|X^0\|}^{\|X(t)\|} \psi(\sqrt{2}s)ds \le \|V^0\| \quad \text{for all} \ \ t \ge 0.$$

These two relations can be combined as a stability estimate:

$$\|V(t)\| + \kappa \left| \int_{\|X^0\|}^{\|X(t)\|} \psi(\sqrt{2}s)ds \right| \le \|V^0\| \quad \text{for all} \ \ t \ge 0. \tag{2.8}$$

As an application of the above stability estimate, we obtain the following flocking theorem.

THEOREM 2.1 ([17, Theorem 3.2]). *Suppose that the initial data, communication weight and coupling strength satisfy*

$$\|V^0\| < \kappa \int_{\|X^0\|}^{\infty} \psi(\sqrt{2}s)ds, \tag{2.9}$$

and let (X,V) *be a solution to* (2.4)*. Then, there is an* $x_M > 0$ *such that*

$$\sup_{0 \le t < \infty} \|X(t)\| \le x_M \quad \textit{and} \quad \|V(t)\| \le \|V^0\| e^{-\kappa \psi(\sqrt{2}x_M)t}, \quad t \ge 0.$$

Proof. Detailed proof can be found in [17], but for the comparison with the discrete model, we briefly present the part of the proof regarding the derivation of the uniform bound x_M for $\|X\|$ as follows. We choose x_M to satisfy

$$\|V^0\| = \kappa \int_{\|X^0\|}^{x_M} \psi(\sqrt{2}s)ds.$$

Then, we claim:

$$\|X(t)\| \le x_M, \quad \forall\, t \ge 0.$$

For this, we fix $\varepsilon > 0$ and define a set $\mathcal{S}_\varepsilon$ as follows.

$$\mathcal{S}_\varepsilon := \{T > 0 : \|X(t)\| < x_M + \varepsilon, \ \forall\ 0 \le t \le T\}.$$

We see that $0 \in \mathcal{S}_\varepsilon$ and so the set is nonempty. We need to show

$$\sup \mathcal{S}_\varepsilon = \infty, \qquad \text{for each } \varepsilon > 0.$$

Suppose not, i.e. $\sup \mathcal{S}_\varepsilon = T^* < \infty$. Then, we have

$$\|X(t)\| < x_M + \varepsilon, \quad 0 \le t < T^* \qquad \text{and} \quad \|X(T^*)\| = x_M + \varepsilon.$$

On the other hand, by (2.8) we have

$$\|V^0\| \ge \kappa \int_{\|X^0\|}^{\|X(T^*)\|} \psi(\sqrt{2}s)ds = \kappa \int_{\|X^0\|}^{x_M+\varepsilon} \psi(\sqrt{2}s)ds > \|V^0\|,$$

which is contradictory. Hence, x_M is a uniform bound for $\|X\|$. ∎

REMARK 2.1.

(1) The result of Theorem 2.1 holds for a general non-increasing communication weight ψ.

(2) Note that the condition (2.9) is not only sufficient, but also necessary for the mono-cluster flocking to occur for the two-particle system on the real line $\mathbb{R}$ (see Proposition 3.1 in [10]).

2.3. Description of main results. In this subsection, we briefly discuss our main results for the discrete model (2.5) which are parallel to Theorem 2.1. As mentioned in Introduction, a natural question one may ask is whether the nonlinear functional approach used in the continuous model can be generalized to the discrete model or not. In the sequel, we show that this is the case.

We are now in a position to state our first main result on the emergence of flocking for discrete system (2.5). For this, we consider regular and singular communication weights at the origin:

$$\lim_{r\to 0^+} \psi(r) < +\infty \quad \text{and} \quad \lim_{r\to 0^+} \psi(r) = +\infty.$$

The singular case is more subtle and should be treated more carefully (see Theorem 2.3 below). The first result concerns the exponential flocking for a regular communication weight.

THEOREM 2.2 (Regular communication weight). *Suppose that the initial data, communication weight, coupling strength and time-step satisfy*

$$\lim_{r\to 0^+} \psi(r) < +\infty, \qquad \kappa > 0, \qquad \|V^0\| < \kappa \int_{\|X^0\|}^{\infty} \psi(\sqrt{2}s)ds, \qquad 0 < h \le \frac{1}{\kappa\psi(0^+)}, \quad (2.10)$$

and let (X,V) be a solution to (2.5). Then, there is an $M>0$ such that

$$\sup_{n\in\mathbb{N}} \|X[n]\| \le M \quad \textit{and} \quad \|V[n]\| \le \|V^0\| e^{-h\kappa\psi(\sqrt{2}M)n}, \quad n \ge 0.$$

Proof. The detailed proof will be presented in Section 3. ∎

REMARK 2.2. In [19], Ha and Zhang used rather restricted conditions compared to (2.10): assume $\psi(0^+) \le 1$, and for given initial data (X^0, V^0), let $h>0$ be sufficiently small so that there is an $M>0$ satisfying

$$M > \max\{||X^0||, ||V^0||\}, \quad ||V^0|| < \kappa \int_{||X^0||}^{M} \psi(\sqrt{2}s)ds,$$
$$0 < h \ll \min\left\{1, \frac{\psi(\sqrt{2}M)}{10\kappa}, \frac{M - ||X^0||}{2||V^0||}, \frac{1}{4\kappa}\right\}.$$

The second result deals with singular communication weight.

THEOREM 2.3 (Singular communication weight). *Suppose that the initial data, communication weight, coupling strength and time-step satisfy: there exist $M>0$ and $\rho>0$ such that*

$$\begin{aligned} &\lim_{r\to 0^+} \psi(r) = +\infty, \qquad \kappa > 0, \qquad \|V^0\| < \kappa \int_{\|X^0\|}^{M} \psi(\sqrt{2}s)ds, \\ &\min_{i\neq j} \|\boldsymbol{x}_i^0 - \boldsymbol{x}_j^0\| \ge \rho + \frac{\sqrt{2}\|V^0\|}{\kappa\psi(\sqrt{2}M)}, \quad 0 < h \le \frac{1}{\kappa\psi(\rho)}, \end{aligned} \quad (2.11)$$

and let $(X[n],V[n])$ *be a solution to* (2.5). *Then, we have*

$$\inf_{0\le n<\infty}\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|\ge\rho,\quad \sup_{n\in\mathbb{N}}\|X[n]\|\le M,$$
$$\|V[n]\|\le\|V^0\|e^{-h\kappa\psi(\sqrt{2}M)n}.$$

Proof. The detailed proof will be presented in Section 3. ∎

REMARK 2.3. For the regular case, the flocking result in Theorem 2.2 for the discrete model (2.5) is exactly parallel to that in Theorem 2.1 for the continuous model (2.4). However, for the singular case, the additional condition:

$$\min_{i\neq j}\|\boldsymbol{x}_i^0-\boldsymbol{x}_j^0\|\ge\rho+\frac{\sqrt{2}\|V^0\|}{\kappa\psi(\sqrt{2}M)}$$

is used to guarantee an existence of a positive minimal distance between particles, so that the particles will never accidentally jump into a collision, causing the ill-posedness of (2.5) at that instant.

As a direct corollary on the exponential flocking in Theorem 2.2 and Theorem 2.3, we obtain the uniform-in-time convergence from discrete system (2.5) to the continuous system (2.4), as h tends to zero in the whole time interval $[0,\infty)$.

COROLLARY 2.1. *Suppose that condition* (2.10) *(in regular case) or condition* (2.11) *(in singular case) holds, and let* $(X(t),V(t))$ *and* $(X[n],V[n])$ *be the solutions of* (2.4) *and* (2.5) *with the same initial data, respectively. Then we have*

$$\lim_{h\to 0}\sup_{n\in\mathbb{N}}(\|X[n]-X(nh)\|+\|V[n]-V(nh)\|)=0.$$

Proof. We leave its proof to Section 4. ∎

3. Emergence of mono-cluster flocking

In this section, we provide a nonlinear functional approach for the discrete C-S model, and present proofs of Theorems 2.2 and 2.3.

3.1. A nonlinear functional approach. We begin with several technical lemmas to be used later, and then introduce a discrete analogue of the nonlinear functionals (2.7) for the continuous model.

LEMMA 3.1. *Suppose that the weight function* ψ *satisfies* (1.2), *and initial data* (X^0,V^0) *satisfy zero sum conditions:*

$$\sum_{i=1}^N\boldsymbol{x}_i^0=\sum_{i=1}^N\boldsymbol{v}_i^0=0,$$

and let $(X[n],V[n])$ *be a solution to the discrete system* (2.5). *Then, we have the following inequality for* $\forall\ \boldsymbol{v}\in span\{[1,\cdots,1]\}^\perp\subset\mathbb{R}^N$*:*

$$N\psi\left(\sqrt{2}\|X[n]\|\right)\|\boldsymbol{v}\|^2\le\langle L[n]\boldsymbol{v},\boldsymbol{v}\rangle\le N\psi\left(\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|\right)\|\boldsymbol{v}\|^2.$$

Proof. • (Upper bound estimate): Let $\boldsymbol{v}=(v^1,\cdots,v^N)$. It is easy to see that

$$\langle L[n]\boldsymbol{v},\boldsymbol{v}\rangle=\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N\psi(\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|)(v^i-v^j)v^i$$

$$= \frac{1}{2}\sum_{i\neq j}\psi(\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|)|v^i-v^j|^2. \tag{3.1}$$

We use (1.2) and (3.1) to obtain

$$\begin{aligned}\langle L[n]\boldsymbol{v},\boldsymbol{v}\rangle &\le \frac{1}{2}\psi\left(\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|\right)\sum_{i\neq j}|v^i-v^j|^2\\ &= N\psi\left(\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|\right)\|\boldsymbol{v}\|^2. \end{aligned}\tag{3.2}$$

• (Lower bound estimate): In (3.1), we use

$$\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\| \le \sqrt{2}\|X[n]\|$$

to get

$$\langle L[n]\boldsymbol{v},\boldsymbol{v}\rangle \ge N\psi\left(\sqrt{2}\|X[n]\|\right)\|\boldsymbol{v}\|^2. \tag{3.3}$$

Finally, we combine (3.2) and (3.3) to obtain the desired estimates. □

Next, consider the velocity equation:

$$V[n+1] = \left(I - h\frac{\kappa}{N}L[n]\right)V[n] =: C[n]V[n], \quad n\in\mathbb{N}. \tag{3.4}$$

LEMMA 3.2. *For $n\in\mathbb{N}$, let λ be an eigenvalue of the coefficient matrix $C[n]$ corresponding to an eigenvector in span$\{[1,\cdots,1]\}^{\perp}$. Then, we have*

$$1-h\kappa\psi\left(\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|\right) \le \lambda \le 1-h\kappa\psi\left(\sqrt{2}\|X[n]\|\right).$$

Proof. Let λ be an eigenvalue of $C[n]$ and $\boldsymbol{v}$ be a corresponding eigenvector:

$$C[n]\boldsymbol{v} = \lambda\boldsymbol{v}. \tag{3.5}$$

• Step A (λ is real): Since the matrices A and D in (2.3) are symmetric, the Laplacian matrix L is also symmetric. This implies that the coefficient matrix $C[n]$ in (3.4) is also symmetric. Therefore, all eigenvalues of $C[n]$ are real.
• Step B (Estimate on the range of λ): We use the relation (3.5) to see

$$\lambda\|\boldsymbol{v}\|^2 = \langle C[n]\boldsymbol{v},\boldsymbol{v}\rangle = \left\langle (I-h\frac{\kappa}{N}L[n])\boldsymbol{v},\boldsymbol{v}\right\rangle = \|\boldsymbol{v}\|^2 - \frac{h\kappa}{N}\langle L[n]\boldsymbol{v},\boldsymbol{v}\rangle.$$

Now, we use Lemma 3.1 to obtain the desired estimates. □

Based on Lemma 3.2, we have the decay of $\|V[n]\|$ as follows.

LEMMA 3.3. *For given $n\in\mathbb{N}$, suppose that the time-step h satisfies*

$$0 < h \le \frac{1}{\kappa\psi(\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|)}. \tag{3.6}$$

Then, we have

$$\|V[n+1]\| \le \left(1-h\kappa\psi\left(\sqrt{2}\|X[n]\|\right)\right)\|V[n]\|.$$

Proof. We denote $V^k[n]$ by the k-th column of $V[n]$. Then, since $V^k[n]$ is in the orthogonal complement of $[1,\cdots,1]$, we can decompose $V^k[n]$ into a sum of $N-1$ orthogonal vectors from the $N-1$ remaining orthogonal eigenspaces of the real symmetric matrix $I-h\frac{\kappa}{N}L[n]$, say

$$V^k[n]=\mathbf{u}_1+\cdots+\mathbf{u}_{N-1},\quad \langle \mathbf{u}_i,\mathbf{u}_j\rangle=0,\quad 1\le i\ne j\le N-1.$$

Then, it follows from $(2.5)_2$ that

$$\begin{aligned}\|V^k[n+1]\|^2&=\left\|\left(I-h\frac{\kappa}{N}L[n]\right)V^k[n]\right\|^2=\|\lambda_1\mathbf{u}_1+\cdots+\lambda_{N-1}\mathbf{u}_{N-1}\|^2\\&=\sum_{i=1}^{N-1}\|\lambda_i\mathbf{u}_i\|^2\le\Big(\max_{1\le i\le N-1}\lambda_i^2\Big)\sum_{i=1}^{N-1}\|\mathbf{u}_i\|^2\\&=\Big(\max_{1\le i\le N-1}\lambda_i^2\Big)\left\|\sum_{i=1}^{N-1}\mathbf{u}_i\right\|^2=\Big(\max_{1\le i\le N-1}\lambda_i^2\Big)\left\|V^k[n]\right\|^2.\end{aligned}\tag{3.7}$$

On the other hand, it follows from Lemma 3.2 and (3.6) that we can see that all eigenvalues of $I-h\frac{\kappa}{N}L[n]$ are nonnegative and its largest eigenvalue (excluding the eigenvalue 1 corresponding to $\text{span}\{[1,\cdots,1]\}$) is bounded from above by $1-h\kappa\psi\left(\sqrt{2}\|X[n]\|\right)$. Finally, we sum (3.7) over k to derive the desired estimate. ∎

REMARK 3.1. In [19], under the assumption $\psi(0)=1$, Ha and Zhang derived a rather crude estimate:

$$\begin{aligned}\|V[n+1]\|^2&=\left\|V[n]-h\frac{\kappa}{N}L[n]V[n]\right\|^2=\sum_{k=1}^{d}\left\|V^k[n]-h\frac{\kappa}{N}L[n]V^k[n]\right\|^2\\&=\|V[n]\|^2-2h\frac{\kappa}{N}\sum_{k=1}^{d}\langle V^k[n],L[n]V^k[n]\rangle+\left\|-h\frac{\kappa}{N}L[n]V[n]\right\|^2\\&\le\|V[n]\|^2-2\kappa h\sum_{k=1}^{d}\psi\left(\sqrt{2}\|X[n]\|\right)\|V^k[n]\|^2+2\kappa h\|V[n]\|^2\\&=\left[1+2\kappa h\left(\kappa h-\psi(\sqrt{2}\|X[n]\|)\right)\right]\|V[n]\|^2,\end{aligned}$$

where they used Lemma 3.1 and the relation

$$\begin{aligned}\|-L[n]V[n]\|^2&=\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}\psi(\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|)(\boldsymbol{v}_j[n]-\boldsymbol{v}_i[n])\right\|^2\\&\le\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\|\boldsymbol{v}_j[n]-\boldsymbol{v}_i[n]\|\right)^2\le N\sum_{i,j}\|\boldsymbol{v}_j[n]-\boldsymbol{v}_i[n]\|^2\\&=2N^2\|V[n]\|^2.\end{aligned}\tag{3.8}$$

In (3.8), Lemma 3.1 was not used in the estimation of $\|-L[n]V[n]\|^2$. In other words, the estimates of $\langle V[n],L[n]V[n]\rangle$ were not used when estimating $\|L[n]V[n]\|^2$. However, in this paper, we saw $\boldsymbol{v}_i[n]$ componentwise and devised a way to use the estimates of $\langle \boldsymbol{v},C[n]\boldsymbol{v}\rangle$ for arbitrary $\boldsymbol{v}\in\text{span}\{[1,\cdots,1]\}^\perp$ to estimate $\|C[n]V[n]\|^2$. Here, the idea of decomposing $V^k=[v_1^k,\cdots,v_N^k]^\top$ into a sum of orthogonal eigenvectors was important.

Motivated by the nonlinear functionals in (2.7), we can define the two functionals $\mathcal{L}_+[n]$ and $\mathcal{L}_-[n]$ as follows:

$$\begin{aligned}&\mathcal{L}_\pm[0] := \|V[0]\| \quad \text{and}\\ &\mathcal{L}_\pm[n] := \|V[n]\| \pm \kappa \sum_{i=1}^{n} \psi\left(\sqrt{2}\|X[i-1]\|\right)(\|X[i]\| - \|X[i-1]\|) \quad \text{for} \quad n \geq 1.\end{aligned} \tag{3.9}$$

Note that the R.H.S. of $(3.9)_2$ will be well-defined as long as the term $\psi\left(\sqrt{2}\|X[i-1]\|\right)$ is finite which is always the case for a regular ψ, and $\mathcal{L}_\pm[n]$ has a similar form as $\mathcal{E}_\pm$ for the continuous case.

Proposition 3.1. *For given $n \in \mathbb{N}$, suppose that the time-step h satisfies (3.6), and let $(X[n], V[n])$ be a solution to (2.5). Then, we have*

$$\mathcal{L}_\pm[n+1] \leq \mathcal{L}_\pm[n].$$

Proof. It follows from Lemma 3.3 and $(2.5)_1$ that we have

$$\begin{aligned}&\mathcal{L}_\pm[n+1] - \mathcal{L}_\pm[n]\\ &= \|V[n+1]\| - \|V[n]\| \pm \kappa\psi\left(\sqrt{2}\|X[n]\|\right)(\|X[n+1]\| - \|X[n]\|)\\ &\leq -h\kappa\psi\left(\sqrt{2}\|X[n]\|\right)\|V[n]\| \pm \kappa\psi\left(\sqrt{2}\|X[n]\|\right)(\|X[n+1]\| - \|X[n]\|)\\ &= -\kappa\psi\left(\sqrt{2}\|X[n]\|\right)\|X[n+1] - X[n]\| \pm \kappa\psi\left(\sqrt{2}\|X[n]\|\right)(\|X[n+1]\| - \|X[n]\|)\\ &\leq -\kappa\psi\left(\sqrt{2}\|X[n]\|\right)\Big|\|X[n+1]\| - \|X[n]\|\Big|\\ &\qquad \pm \kappa\psi\left(\sqrt{2}\|X[n]\|\right)(\|X[n+1]\| - \|X[n]\|)\\ &\leq 0.\end{aligned} \tag{3.10}$$

∎

Next, we introduce a useful inequality involving the functional $\mathcal{L}_+$ to be used later.

Proposition 3.2. *For each $n \geq 0$, the functional $\mathcal{L}_+$ satisfies the inequality*

$$\mathcal{L}_+[n] \geq \|V[n]\| + \kappa \int_{\|X[0]\|}^{\|X[n]\|} \psi(\sqrt{2}s)ds.$$

Proof. Due to (3.9), it suffices to prove the following statement: for each $1 \leq i \leq n$, we have

$$\int_{\|X[i-1]\|}^{\|X[i]\|} \psi(\sqrt{2}s)ds \leq \psi\left(\sqrt{2}\|X[i-1]\|\right)(\|X[i]\| - \|X[i-1]\|).$$

To show this, we consider two cases:

$$\text{Either } \|X[i]\| \geq \|X[i-1]\| \quad \text{or} \quad \|X[i]\| < \|X[i-1]\|.$$

- Case A ($\|X[i]\| \geq \|X[i-1]\|$): Since $\psi(\cdot)$ is non-increasing, we have

$$0 \leq \int_{\|X[i-1]\|}^{\|X[i]\|} \psi(\sqrt{2}s)ds \leq \psi\left(\sqrt{2}\|X[i-1]\|\right)(\|X[i]\| - \|X[i-1]\|).$$

- Case B ($\|X[i]\| < \|X[i-1]\|$): Since $\psi(\cdot)$ is non-increasing, we have

$$0 \le \psi\left(\sqrt{2}\|X[i-1]\|\right)(\|X[i-1]\| - \|X[i]\|) \le \int_{\|X[i]\|}^{\|X[i-1]\|} \psi(\sqrt{2}s)ds.$$

Hence we have

$$\int_{\|X[i-1]\|}^{\|X[i]\|} \psi(\sqrt{2}s)ds \le \psi\left(\sqrt{2}\|X[i-1]\|\right)(\|X[i]\| - \|X[i-1]\|) \le 0.$$

Finally, we combine estimates in Case A and Case B to get the desired result. ☐

REMARK 3.2. We will see below that the lower bound for $\mathcal{L}_+[n]$ in Proposition 3.2 is very crucial in establishing mono-cluster flocking estimates. Therefore, $\mathcal{L}_+[n]$ plays the same role as $\mathcal{E}_+(t)$ for the continuous system. However, we cannot derive a similar lower bound for $\mathcal{L}_-[n]$. This is a marked difference between the continuous system and the discrete system.

3.2. A regular communication weight. In this subsection, we provide a proof of Theorem 2.2. The proof is exactly parallel to that of Theorem 2.1. First, we choose $M \ge ||X^0||$ to satisfy

$$\|V^0\| = \kappa \int_{\|X^0\|}^{M} \psi(\sqrt{2}s)ds.$$

Then, we claim:

$$\|X[n]\| \le M, \quad \forall\ n \ge 0.$$

For this, we define a set $\mathcal{S}$ as follows.

$$\mathcal{S} := \{T \in \mathbb{N} : \|X[n]\| \le M,\ n \in \mathbb{N},\ 0 \le n \le T\}.$$

We see that $0 \in \mathcal{S}$ and so the set is nonempty. To prove the claim, we need to show

$$\sup \mathcal{S} = \infty.$$

Suppose not, i.e. $\sup \mathcal{S} = T^* < \infty$. Then, we have

$$\|X[n]\| \le M, \quad 0 \le n \le T^*, \qquad \text{and} \qquad \|X[T^*+1]\| > M.$$

On the other hand, Propositions 3.1 and 3.2 imply

$$\|V^0\| = \mathcal{L}_+[0] \ge \mathcal{L}_+[T^*+1] \ge \kappa \int_{\|X^0\|}^{\|X[T^*+1]\|} \psi(\sqrt{2}s)ds > \kappa \int_{\|X^0\|}^{M} \psi(\sqrt{2}s)ds = \|V^0\|,$$

which is contradictory. Hence, M is a uniform bound for $\|X\|$. For the estimate of velocity variations, we use Lemma 3.3 and inequality:

$$1 - x \le e^{-x}, \quad x > 0$$

to see that for all $n \in \mathbb{N}$,

$$\|V[n]\| \le \left(1 - h\kappa\psi\left(\sqrt{2}M\right)\right)^n \|V^0\| \le \|V^0\| e^{-h\kappa\psi(\sqrt{2}M)n}.$$

This completes the proof of Theorem 2.2.

3.3. A singular communication weight. In this subsection, we provide a proof of Theorem 2.3. Suppose that initial data and system parameters satisfy

$$\begin{aligned} &\kappa > 0, \quad \|V^0\| < \kappa \int_{\|X^0\|}^{M} \psi(\sqrt{2}s)ds, \\ &\min_{i\neq j} \|\boldsymbol{x}_i^0 - \boldsymbol{x}_j^0\| \geq \rho + \frac{\sqrt{2}\|V^0\|}{\kappa\psi(\sqrt{2}M)}, \quad 0 < h \leq \frac{1}{\kappa\psi(\rho)}. \end{aligned} \tag{3.11}$$

• Step A (Derivation on positive minimal distance): In this step, we will derive a positive minimal relative distance for the well-prepared initial data (3.11). More precisely, we claim that for all $1 \leq i \neq j \leq N$ and $n \in \mathbb{N}$,

$$\|\boldsymbol{x}_i[n] - \boldsymbol{x}_j[n]\| \geq \rho. \tag{3.12}$$

For this, we define a set $\mathcal{S}$ as follows.

$$\mathcal{S} := \{T \in \mathbb{N} : \|\boldsymbol{x}_i[n] - \boldsymbol{x}_j[n]\| \geq \rho, \ \forall i \neq j, \ n \in \mathbb{N}, 0 \leq n \leq T\}.$$

Since $0 \in \mathcal{S}$, the set $\mathcal{S}$ is nonempty. To prove (3.12), we need to show

$$\sup \mathcal{S} = \infty.$$

Suppose not, i.e., $\sup \mathcal{S} = T^* < \infty$. Then, we have for all $0 \leq n \leq T^*$,

$$\|\boldsymbol{x}_i[n] - \boldsymbol{x}_j[n]\| \geq \rho, \quad \forall i \neq j, \tag{3.13}$$

and there exist indices $i^* \neq j^*$ such that

$$\|\boldsymbol{x}_{i^*}[T^*+1] - \boldsymbol{x}_{j^*}[T^*+1]\| < \rho. \tag{3.14}$$

On the other hand, by (3.11) and (3.13), the relation (3.6) is satisfied for all $0 \leq n \leq T^*$. Therefore, we use Proposition 3.1 to derive

$$\mathcal{L}_+[n+1] \leq \mathcal{L}_+[n], \qquad \text{for } 0 \leq n \leq T^*.$$

We inductively apply the above relation in n and use the relations (3.11) to see that for all $0 \leq n \leq T^*$,

$$\mathcal{L}_+[n+1] \leq \mathcal{L}_+[0] = \|V[0]\| < \kappa \int_{\|X^0\|}^{M} \psi(\sqrt{2}s)ds.$$

Proposition 3.2 implies

$$\|V[n]\| + \kappa \int_{\|X^0\|}^{\|X[n]\|} \psi(\sqrt{2}s)ds \leq \mathcal{L}_+[n] < \kappa \int_{\|X^0\|}^{M} \psi(\sqrt{2}s)ds, \quad 0 \leq n \leq T^*+1.$$

Clearly, this yields

$$\|X[n]\| \leq M, \qquad \text{for } 0 \leq n \leq T^*+1. \tag{3.15}$$

Now, we use Lemma 3.3 and (3.15) to find that for all $0 \leq n \leq T^*$,

$$\|V[n+1]\| \leq \left(1 - h\kappa\psi\left(\sqrt{2}\|X[n]\|\right)\right)\|V[n]\| \leq \left(1 - h\kappa\psi\left(\sqrt{2}M\right)\right)\|V[n]\|. \tag{3.16}$$

Then, we iterate the relation (3.16) inductively to get

$$\|V[n]\| \le \left(1-h\kappa\psi\left(\sqrt{2}M\right)\right)^n \|V^0\|, \quad 0\le n\le T^*+1. \tag{3.17}$$

On the other hand, it follows from $(2.5)_1$ that

$$\begin{aligned}
&\|\boldsymbol{x}_{i^*}[T^*+1]-\boldsymbol{x}_{j^*}[T^*+1]\| \\
&= \left\|\boldsymbol{x}_{i^*}[0]-\boldsymbol{x}_{j^*}[0]+\sum_{n=0}^{T^*}(\boldsymbol{x}_{i^*}[n+1]-\boldsymbol{x}_{j^*}[n+1]-(\boldsymbol{x}_{i^*}[n]-\boldsymbol{x}_{j^*}[n]))\right\| \\
&= \left\|\boldsymbol{x}_{i^*}[0]-\boldsymbol{x}_{j^*}[0]+h\sum_{n=0}^{T^*}(\boldsymbol{v}_{i^*}[n]-\boldsymbol{v}_{j^*}[n])\right\| \\
&\ge \|\boldsymbol{x}_{i^*}^0-\boldsymbol{x}_{j^*}^0\| - h\sum_{n=0}^{T^*}\|\boldsymbol{v}_{i^*}[n]-\boldsymbol{v}_{j^*}[n]\|.
\end{aligned} \tag{3.18}$$

We combine (3.17) and (3.18) to find

$$\begin{aligned}
&\|\boldsymbol{x}_{i^*}[T^*+1]-\boldsymbol{x}_{j^*}[T^*+1]\| \\
&\ge \|\boldsymbol{x}_{i^*}^0-\boldsymbol{x}_{j^*}^0\| - \sqrt{2}h\sum_{n=0}^{T^*}\left(1-h\kappa\psi\left(\sqrt{2}M\right)\right)^n\|V^0\| \\
&\ge \min_{i\neq j}\|\boldsymbol{x}_i^0-\boldsymbol{x}_j^0\| - \frac{\sqrt{2}\|V^0\|}{\kappa\psi(\sqrt{2}M)} \ge \rho.
\end{aligned}$$

This is contradictory to (3.14). Thus, (3.12) holds for all $n\in\mathbb{N}$:

$$\inf_{n\in\mathbb{N}}\min_{i\neq j}\|\boldsymbol{x}_i[n]-\boldsymbol{x}_j[n]\|\ge\rho.$$

We use Proposition 3.1 to see

$$\mathcal{L}_+[n]\le\mathcal{L}_+[0] \quad \text{for all } n\in\mathbb{N}.$$

Then, the above relation, (3.11) and Proposition 3.2 yield

$$\|X[n]\|\le M \quad \text{for all } n\in\mathbb{N}.$$

• Step B (Estimate on velocity variations): The analysis is the same as in the proof of Theorem 2.2. We use Lemma 3.3 to see that for all $n\in\mathbb{N}$,

$$\|V[n]\|\le\|V^0\|e^{-h\kappa\psi(\sqrt{2}M)n}.$$

This completes the proof of Theorem 2.3.

4. From discrete dynamics to continuous dynamics

In this section, we present a uniform-in-time transition from the discrete C-S model to the continuous C-S model by combining the corresponding result in any finite-time interval and exponential flocking estimate for the discrete model.

Recall that our goal is to derive the estimate:

$$\lim_{h\to 0}\sup_{n\in\mathbb{N}}\left(\|X[n]-X(nh)\|+\|V[n]-V(nh)\|\right)=0.$$

For the derivation of the above estimate, we estimate the velocity variations and spatial variations separately using the same argument.

• Step A (velocity variations): First, we claim:

$$\limsup_{h\to 0}\sup_{n\in\mathbb{N}}\|V[n]-V(nh)\|=0. \tag{4.1}$$

Proof of claim: We will show (4.1) by a contradiction argument. Suppose not, i.e. there exists a constant $\delta>0$ such that

$$\limsup_{h\to 0}\sup_{n\in\mathbb{N}}\|V[n]-V(nh)\|=\delta. \tag{4.2}$$

It follows from Theorem 2.2 or Theorem 2.3 that for all $n\in\mathbb{N}$,

$$\|V[n]\|\le\|V^0\|e^{-h\kappa\psi(\sqrt{2}M)n}. \tag{4.3}$$

On the other hand, it follows from Theorem 2.1 that for all $n\in\mathbb{N}$,

$$\|V(nh)\|\le\|V^0\|e^{-\kappa\psi(\sqrt{2}x_M)hn}. \tag{4.4}$$

By (4.3) and (4.4), there exists a time T_0 such that

$$\|V[n]\|<\frac{\delta}{4},\quad \|V(nh)\|<\frac{\delta}{4},\quad n\ge\lfloor T_0/h\rfloor+1.$$

This yields that for all $n\ge\lfloor T_0/h\rfloor+1$,

$$\|V[n]-V(nh)\|\le\|V[n]\|+\|V(nh)\|<\frac{\delta}{2}. \tag{4.5}$$

We combine (4.2) and (4.5) to see

$$\limsup_{h\to 0}\sup_{n\in\mathbb{N},0\le n\le\lfloor T_0/h\rfloor}\|V[n]-V(nh)\|=\delta$$

which contradicts the classical finite-time interval convergence result in the finite time interval $[0,T_0]$ (see [31]). Hence, we have the uniform-in-time convergence in velocity part.

• Step B (spatial variations): Similar to (4.1), we claim:

$$\limsup_{h\to 0}\sup_{n\in\mathbb{N}}\|X[n]-X(nh)\|=0. \tag{4.6}$$

Suppose not, i.e. there exists a constant $\tilde{\delta}>0$ such that

$$\limsup_{h\to 0}\sup_{n\in\mathbb{N}}\|X[n]-X(nh)\|=\tilde{\delta}. \tag{4.7}$$

For a time $\tilde{T}$ to be determined later, we have for all $n\ge\lfloor\tilde{T}/h\rfloor+1$,

$$X[n]=X[\lfloor\frac{\tilde{T}}{h}\rfloor]+h\sum_{i=\lfloor\frac{\tilde{T}}{h}\rfloor}^{n-1}V[i]\quad\text{and}\quad X(nh)=X(\lfloor\frac{\tilde{T}}{h}\rfloor h)+\int_{\lfloor\frac{\tilde{T}}{h}\rfloor h}^{nh}V(s)ds.$$

Therefore, we have

$$\begin{aligned}
&\|X[n]-X(nh)\| \\
&\le \left\| X[\lfloor \frac{\tilde{T}}{h} \rfloor] - X(\lfloor \frac{\tilde{T}}{h} \rfloor h) \right\| + \left\| \sum_{i=\lfloor \frac{\tilde{T}}{h} \rfloor}^{n-1} \int_{ih}^{(i+1)h} V[i]ds - \sum_{i=\lfloor \frac{\tilde{T}}{h} \rfloor}^{n-1} \int_{ih}^{(i+1)h} V(s)ds \right\| \\
&\le \left\| X[\lfloor \frac{\tilde{T}}{h} \rfloor] - X(\lfloor \frac{\tilde{T}}{h} \rfloor h) \right\| + \sum_{i=\lfloor \frac{\tilde{T}}{h} \rfloor}^{n-1} \int_{ih}^{(i+1)h} \|V[i]-V(s)\| ds \\
&=: \left\| X[\lfloor \frac{\tilde{T}}{h} \rfloor] - X(\lfloor \frac{\tilde{T}}{h} \rfloor h) \right\| + \mathcal{I}[n].
\end{aligned}$$

Let $\mathcal{M} := \max\{M, x_M\}$. For the estimation of $\mathcal{I}[n]$, we use (4.3) and (4.4) to derive

$$\begin{aligned}
\mathcal{I}[n] &\le \sum_{i=\lfloor \frac{\tilde{T}}{h} \rfloor}^{n-1} 2h\|V^0\| e^{-h\kappa\psi(\sqrt{2}\mathcal{M})i} \le 2h\|V^0\| \sum_{i=\lfloor \frac{\tilde{T}}{h} \rfloor}^{\infty} e^{-h\kappa\psi(\sqrt{2}\mathcal{M})i} \\
&= 2h\|V^0\| \frac{e^{-\kappa\psi(\sqrt{2}\mathcal{M})\lfloor \frac{\tilde{T}}{h} \rfloor h}}{1-e^{-h\kappa\psi(\sqrt{2}\mathcal{M})}} \le 2h\|V^0\| \frac{e^{-\kappa\psi(\sqrt{2}\mathcal{M})(\tilde{T}-h)}}{1-e^{-h\kappa\psi(\sqrt{2}\mathcal{M})}}.
\end{aligned}$$

Next, we can take $\tilde{T}$ sufficiently large such that

$$\limsup_{h\to 0} \sup_{n\ge \lfloor \tilde{T}/h \rfloor +1} \mathcal{I}[n] < \tilde{\delta}/2.$$

Then, we have

$$\begin{aligned}
&\limsup_{h\to 0} \sup_{n\in\mathbb{N}} \|X[n]-X(nh)\| \\
&\le \limsup_{h\to 0} \sup_{n\in\mathbb{N}, 0\le n\le \lfloor \tilde{T}/h \rfloor} \|X[n]-X(nh)\| + \limsup_{h\to 0} \sup_{n\in\mathbb{N}, n\ge \lfloor \tilde{T}/h \rfloor +1} \|X[n]-X(nh)\| \\
&\le 2\limsup_{h\to 0} \sup_{n\in\mathbb{N}, 0\le n\le \lfloor \tilde{T}/h \rfloor} \|X[n]-X(nh)\| + \frac{\tilde{\delta}}{2} = \frac{\tilde{\delta}}{2}, \qquad (4.8)
\end{aligned}$$

where the last equality follows from the classical result on the convergence in a finite-time interval [31]. Note that (4.7) and (4.8) are contradictory. Hence, we prove (4.6) and complete the proof.

5. Conclusion

In this paper, we have presented a nonlinear functional approach for the flocking estimate to the discrete C-S model. For the continuous C-S model, Ha and Liu introduced a nonlinear functional approach with a general communication weight. In literature, the flocking estimates for the discrete C-S model have been studied for bounded and algebraically decreasing communication weights using the self-bound argument in [13]. However, as far as the authors know, the nonlinear functional approach for the discrete C-S model has not been done in the last ten years. In a recent work by Ha and Zhang, they derived a flocking estimate for the discrete C-S model with a general communication weight by using the nonlinear functional approach for the continuous C-S model and some approximation argument between the continuous model and discrete model.

Thus, the main novelty of this work is to introduce a discrete analogue of nonlinear functional approach directly. Our proposed nonlinear functional approach is a simple and natural generalization of the continuous counterpart in the discrete regime. As a direct corollary of our nonlinear functional approach, we also show the uniform-in-time convergence from the discrete C-S model to the continuous C-S model under a relaxed framework compared to [19]. Of course, there are still lots of open questions even for the simple discrete C-S model with general communication weight. For the continuous model, there has been some relevant work [10] on the emergence of local flocking. However, similar issues were never addressed in the discrete regime. Thus, it would be very interesting to investigate the emergence of local flocking for the discrete C-S model as well. This will be addressed in a future work.

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