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ASYMPTOTIC TRAVELING WAVE FOR A PRICING MODEL WITH MULTIPLE CREDIT RATING MIGRATION RISK*

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Abstract. In this paper, an asymptotic traveling wave of a free boundary problem related to a pricing model for corporate bond with multiple credit rating migration risk is studied. The pricing model is captured by a free boundary problem, whose existence, uniqueness and regularity of the solution are obtained such that the rationality of the model is guaranteed. The existence of a unique traveling wave in the free boundary problem is established with some risk discount rate condition satisfied. The inductive method is applied to overcome the multiplicity of free boundaries. We prove that the solution of the pricing model for corporate bond is convergent to the traveling wave, which shows a clear dynamics of price change for the corporate bond.

Keywords. Traveling wave; Asymptotic behavior; Free boundary problem; Multiple credit rating migration; Pricing model for corporate bond.

AMS subject classifications. 35K10; 60H10; 91G40.

1. Introduction

The rapid development of globalization and complexity of financial markets leads to higher requirement on management of credit risks. Credit risks include default risks and credit rating migration risks. The existing literature has paid much attention on the default risks. Since the 2008 subprime crisis and the 2010 European debit crisis, the credit rating migration risk plays more and more remarkable roles in credit risk analysis. Therefore, researches on credit rating migration risks are attracting more and more attention.

Both structural models and reduced form models are traditional models for default risks. Structural models assume that default occurs when the value of firm falls below some insolvency threshold, see Merton [23], Black and Cox [2], Leland [16], Longstarff and Schwartz [18], Leland and Toft [17], Briys and de Varenne [3], Bessembinder et al. [1] and so forth. For the reduced form models, an exogenous default intensity is applied, see Jarrow and Turnbull [11], Lando [14], Duffie and Singleton [5] and so forth. Both models have been widely adopted in different settings in practice and show their strength and weakness.

In existing literature on credit rating migrations, Markov chain is the mainly employed approach, where a transferring intensity matrix is adapted. Then the reduced form framework is directly developed for dynamic processes of credit rating migrations, see Jarrow et al. [12], Das and Tufano [4], Lando [15], Thomas et al. [25] and so forth. Some authors considered another perspective where the value of the firm is an important factor in the credit rating migrations, see Hu et al. [10], Liang and Zeng [19], Liang et

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al. [20], Liang et al. [21], Liang et al. [22] and so forth. For instance, the 2010 European debit crisis caused several European countries to be downgraded on their credit ratings and resulted in a lot of difficulties. The primary origin of the crisis was the unsustainable levels of sovereign debts in these countries due to their bad economic behaviors accumulated in the former years. In such a setting, Markov chain cannot fully capture the migration of credit ratings. To reply to this problem, Liang and Zeng [19], Liang et al. [20] apply structural model for pricing corporate bonds with credit rating migrations relative to the asset value. They set a predetermined migration threshold to divide the value of the firm into high and low rating levels, where the values follow different stochastic processes. Subsequently, Hu et al. [10] improved the model by introducing the proportion of the debt and the value of the firm as the threshold of the boundary of the credit rating migration. The model is transferred into a free boundary problem of partial differential equations (PDE). The existence, uniqueness and regularity of the free boundary problem are obtained. Liang et al. [22] extended the work of Hu et al. [10] to a pricing model where the volatility of the bond price strongly depends on potential credit rating migration and stochastic change of the interest rate. The existence, uniqueness and regularity of the solution for the model are established. In particular, following the work of Hu et al. [10], Liang et al. [21] carry out the first study associating the traveling wave to the problem on credit rating migrations. They have proved that the solution of the free boundary problem transferred from the pricing model for credit rating migration is convergent to a traveling wave with an explicit form, through the Lyapnov function approach. Traveling waves exist widely in nature, especially in physics, chemistry and ecology etc. The traveling waves have been studied in both theoretical and applied mathematics in different areas, see Feng and Knobel [6], Morita [24], Wang [26] and so forth.

Two credit ratings are considered in their models, the high and low credit ratings respectively in credit rating migrations. In practice, we should consider more credit ratings in credit rating migration problem, when accessing the credit of a corporation. For instance, on the evening of December 16, 2009, the Standard & Poors, an international rating agency, downgraded the long term sovereign credit rating of Greece from A- to BBB+. Wu and Liang [27] considered a pricing model for corporate bond with multiple credit rating migration risk. They discussed numeric scheme, stability of numeric algorithm, convergence order and calibration of parameters. Subsequently, Yin et al. [28] extended the work of Wu and Liang [27] to a model with stochastic interest rate. The model improves the previous existing bond models with only two credit ratings or multiple ratings but with a constant interest rate.

In this paper, due to the practical significance of multiple credit ratings when assessing the credit level of a firm, the authors devote to considering the asymptotic traveling wave in a pricing model of corporate bond with multiple credit rating migrations. Indeed, our work is an extension of Liang et al. [21], where two credit ratings are considered. Thus, the overall strategy of the current paper is the same as that in [21]. Firstly, the pricing model is transferred into an equivalent free boundary problem. To ensure the model is well posed, the existence globally in time, uniqueness and regularity of the solution are proved. The procedure to achieve these results follows similarly the argument of Liang et al. [21], with some necessary modifications to fit the case of multiple credit ratings. Secondly, the existence and uniqueness of traveling wave in the model is established through a delicate application of inductive method. We claim that this is the highlight of the current paper. In the work of Liang et al. [21], since it is assumed that there are only two credit ratings in the model, the explicit formula for the traveling wave is presented, which directly verifies the existence of a unique traveling wave. However, in the case of multiple credit ratings, it fails to obtain the explicit formula of the traveling wave. In the current paper, the authors prove the existence of the traveling wave by showing a semi-explicit formula associated with a nonlinear parameter system with any finite dimension, which is proved solvable. Thirdly, the convergence for the solution of the model to the traveling wave is proved by the construction of a Lyapunov function. The form of the Lyapunov function is similar to that in [21] but the authors show that it is also applicable in the case of multiple credit ratings.

Naturally, one should focus on the differences between the cases of multiple credit ratings and only two credit ratings. In the corresponding free boundary problem, there should be multiple free boundaries rather than only one free boundary in the problem. In the work of Liang et al. [21], it has been shown that the existence and convergence of the traveling wave are subject to some constraint on the risk discount rate factor. That is, the risk discount rate should be between the half squares of the volatilities in the high and low credit ratings. For the case of multiple credit ratings, it is found that as long as the risk discount rate is between the half squares of the volatilities in the highest and lowest credit ratings, the traveling wave exists and the solution of the free boundary problem converges to the traveling wave. Meanwhile, the phenomenon of asymptotic traveling wave does not depend on the relations between the risk discount rate and the volatilities in other credit ratings. In this paper, one can capture some information on the distribution of the free boundaries, especially the order of them. We do not need to consider this event in the case of two credit ratings, since only one free boundary is involved. These free boundaries are shown uniformly bounded and convergent to corresponding limits, where the constraint on the risk discount rate plays an important role. Compared with the case of two credit ratings, as expounded in the last paragraph, the technical highlight in the case of the multiple credit ratings is the existence of a unique traveling wave. In the case of two credit ratings, an explicit formula for the traveling wave is given. However, due to the multiplicity of free boundaries, a nonlinear parameters system is derived and proved solvable by the delicate application of inductive method, which implies the existence of a unique traveling wave in the case of multiple credit ratings.

The paper is organized as follows. In Section 2, the baseline model and the corresponding PDE problem are proposed. The model is transferred into a free boundary problem of PDE in Section 3. In Section 4, an approximation for the free boundary problem is presented and the existence of a unique traveling wave in the free boundary problem is founded in Section 5. In Section 6, some estimates for the free boundaries are given. Then through the approximated problem, the existence and uniqueness of the solution in the free boundary problem are obtained in Section 7. In Section 8, we show that the solution of the free boundary problem is convergent to the traveling wave. Conclusion and discussion are presented in Section 9.

2. Baseline model

2.1. Assumptions. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space. In the current paper, we assume that the firm issues only one corporate bond, which is a contingent claim of its value on this probability space.

ASSUMPTION 2.1 (The firm asset with credit rating migration). *Let S_t, $t\geq 0$, be the value of the firm in the risk neutral situation. It satisfies the following Black-Scholes-type models*

$$dS_t = rS_t dt + \sigma_N S_t dW_t, \text{ in the first highest credit rating region,}$$

$$dS_t = rS_t dt + \sigma_{N-1} S_t dW_t, \text{ in the second highest credit rating region,}$$
$$\cdots$$
$$dS_t = rS_t dt + \sigma_1 S_t dW_t, \text{ in the second lowest credit rating region,}$$
$$dS_t = rS_t dt + \sigma_0 S_t dW_t, \text{ in the first lowest credit rating region,}$$

where r is the risk free interest rate and

$$\sigma_N < \sigma_{N-1} < \cdots < \sigma_1 < \sigma_0 \tag{2.1}$$

represent volatilities of the value of the firm under different credit ratings respectively. They are assumed to be positive constants. Volatilities in high credit rating regions should be smaller than those in low credit rating regions. A firm with low credit rating might be accompanied with high volatility leading to high risk. W_t is the Brownian motion which generates the filtration $\mathscr{F}_t$.

Assumption 2.2 (The corporate bond). *The firm issues only one corporate zero-coupon bond with face value K. We focus on the effect of the firm value with multiple credit rating migration to the bond. Therefore, the discount value of the bond is considered. Denote by ϕ_t the discount value of the bond at time t. Thus, on the maturity time T, an investor can get $\phi_T = \min\{S_T, K\}$.*

Assumption 2.3 (The risk discount[1] rate). *A nonnegative constant δ is introduced to represent the risk discount rate on the proportion of the debt and the firm value from the bond maturity. Financially, the risk discount rate implies that the credit rating migration is more sensitive to the proportion of the debt and the firm value as maturity approaches.*

Assumption 2.4 (The credit rating migration times). *The rating regions are determined by the proportion of the debt and the firm value. The credit rating migration times are the first moments when the credit ratings of the firm are downgraded or upgraded. Select a sequence of constants*

$$0 < \gamma_N < \gamma_{N-1} < \cdots < \gamma_1 < 1$$

as the threshold proportions of the debt and the value. Define the credit rating migration times as follows:

$$\tau_0 = \inf\left\{ t > 0 : \frac{\phi_0}{S_0} e^{\delta T} \in (\gamma_1, \infty),\ \frac{\phi_t}{S_t} e^{\delta(T-t)} \in (-\infty, \gamma_1] \right\},$$

$$\tau_N = \inf\left\{ t > 0 : \frac{\phi_0}{S_0} e^{\delta T} \in (-\infty, \gamma_N),\ \frac{\phi_t}{S_t} e^{\delta(T-t)} \in [\gamma_N, \infty) \right\},$$

and for $n = 1, 2, \cdots, N-1$,

$$\tau_n = \inf\left\{ t > 0 : \frac{\phi_0}{S_0} e^{\delta T} \in (\gamma_{n+1}, \gamma_n),\ \frac{\phi_t}{S_t} e^{\delta(T-t)} \in (-\infty, \gamma_{n+1}] \cup [\gamma_n, \infty) \right\}.$$

[1] A risk discount refers to a situation where an investor is willing to accept a lower expected return in exchange for lower risk or volatility. The degree to which any particular investor, whether individual or firm, is willing to trade risk for return depends on the particular risk tolerance and investment goals of that investor.

2.2. Cash flows. Once the credit rating migrates before the maturity T, a virtual substitute termination happens, namely that the bond is virtually terminated and substituted by a new one with a new credit rating. There is a virtual cash flow of the bond. Denote $\phi_n(t,y)$, $n=0,1,\cdots,N$, the values of the bond in each credit rating. The values $\phi_n(t,y)$ are calculated in a way of conditional expectations as follows:

$$\begin{aligned}\phi_0(t,y)=&\mathbb{E}^{t,y}\Big[e^{-r(T-t)}\min\{S_T,K\}\chi_{\tau_0\geq T}\\&\quad+e^{-r(\tau_0-t)}\phi_1(\tau_0,S_{\tau_0})\chi_{t<\tau_0<T}\Big|S_t=y,\gamma_1 e^{-\delta(T-t)}<\frac{\phi_0(t,y)}{y}\Big],\\ \phi_N(t,y)=&\mathbb{E}^{t,y}\Big[e^{-r(T-t)}\min\{S_T,K\}\chi_{\tau_N\geq T}\\&\quad+e^{-r(\tau_N-t)}\phi_{N-1}(\tau_N,S_{\tau_N})\chi_{t<\tau_N<T}\Big|S_t=y,\gamma_N e^{-\delta(T-t)}>\frac{\phi_N(t,y)}{y}\Big],\end{aligned}$$

for $n=1,2,\cdots,N-1$,

$$\begin{aligned}\phi_n(t,y)=&\mathbb{E}^{t,y}\Big[e^{-r(T-t)}\min\{S_T,K\}\chi_{\tau_n\geq T}\\&+e^{-r(\tau_n-t)}\phi_{n-1}(\tau_n,S_{\tau_n})\chi_{\phi_n(\tau_n,S_{\tau_n})=\gamma_n S_{\tau_n},t<\tau_n<T}\\&+e^{-r(\tau_n-t)}\phi_{n+1}(\tau_n,S_{\tau_n})\chi_{\phi_n(\tau_n,S_{\tau_n})=\gamma_{n+1} S_{\tau_n},t<\tau_n<T}\Big|\\&S_t=y,\gamma_{n+1}e^{-\delta(T-t)}<\frac{\phi_n(t,y)}{y}<\gamma_n e^{-\delta(T-t)}\Big],\end{aligned}$$

where χ is the indicator function. ($\chi_{event}=1$ if "event" happens. Otherwise, $\chi_{event}=0$.)

2.3. The PDE problem. The conditional expectations given in Subsection 2.2 imply that the value of the bond is continuous when it passes the rating threshold value, namely that

$$\phi_n=\phi_{n-1},\ n=1,2,\cdots,N \text{ on the rating migration boundaries.} \tag{2.2}$$

By the Feynman-Kac formula, ϕ_n, $n=0,1,\cdots,N$, are the functions of the firm value S and time t. They satisfy the following PDE in their corresponding regions:

$$\begin{aligned}&\frac{\partial\phi_0}{\partial t}+\frac{\sigma_0^2}{2}S^2\frac{\partial^2\phi_0}{\partial S^2}+rS\frac{\partial\phi_0}{\partial S}-r\phi_0=0,\ \phi_0>\gamma_0 e^{-\delta(T-t)}S,\ t>0,\\&\frac{\partial\phi_N}{\partial t}+\frac{\sigma_N^2}{2}S^2\frac{\partial^2\phi_N}{\partial S^2}+rS\frac{\partial\phi_N}{\partial S}-r\phi_N=0,\ \phi_N<\gamma_N e^{-\delta(T-t)}S,\ t>0,\end{aligned}\tag{2.3}$$

and for $n=1,2,\cdots,N-1$,

$$\frac{\partial\phi_n}{\partial t}+\frac{\sigma_n^2}{2}S^2\frac{\partial^2\phi_n}{\partial S^2}+rS\frac{\partial\phi_n}{\partial S}-r\phi_n=0,\ \gamma_{n+1}e^{-\delta(T-t)}S<\phi_n<\gamma_n e^{-\delta(T-t)}S,\ t>0,$$

with the terminal conditions

$$\phi_n(T,S)=\min\{S,K\},\ n=0,1,\cdots,N.$$

Meanwhile, according to the Black-Scholes theory (see Jiang [13]), it holds as well that

$$\frac{\partial\phi_n}{\partial S}=\frac{\partial\phi_{n-1}}{\partial S},\ n=1,2,\cdots,N \text{ on the rating migration boundaries.} \tag{2.4}$$

3. Free boundary problem

In this section, we transfer the pricing model proposed in the last section to a free boundary problem. Through the standard transformation of variables $x = \log S$ and remaining $T-t$ as t, define

$$\varphi(t,x) = \phi_n(T-t, e^x) \text{ in } n\text{-th rating region}, \; n = 0,1,\cdots,N.$$

Through (2.2) and (2.4), we then derive the equation system (2.3) as

$$\frac{\partial \varphi}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 \varphi}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial \varphi}{\partial x} + r\varphi = 0, \; -\infty < x < \infty, \; t > 0, \tag{3.1}$$

where $\sigma = \sigma(\varphi, x)$ is given as

$$\sigma = \sigma(\varphi,x) = \begin{cases} \sigma_N, \; \varphi < \gamma_N e^{x-\delta t}, \\ \sigma_n, \; \gamma_{n+1} e^{x-\delta t} \le \varphi < \gamma_n e^{x-\delta t}, \; n = 1,2,\cdots,N-1, \\ \sigma_0, \; \varphi \ge \gamma_1 e^{x-\delta t}. \end{cases} \tag{3.2}$$

Without loss of generality, we assume $K = 1$. Then (3.1) is supplemented with the initial condition

$$\varphi(0,x) = \min\{e^x, 1\}, \; -\infty < x < \infty. \tag{3.3}$$

In this paper, we shall prove that the domain could be separated by N free boundaries $x = \lambda_n(t)$, $n = 1,2,\cdots,N$. The boundaries $\lambda_n(t)$, $n = 1,2,\cdots,N$, are a priori unknown since they should be solved through the equations

$$\varphi(t, \lambda_n(t)) = \gamma_n e^{\lambda_n(t) - \delta t}, \; n = 1,2,\cdots,N, \tag{3.4}$$

where the solution φ is also a priori unknown. Since we have assumed that the system (2.3) is valid across the free boundaries, we can derive from (2.2) and (2.4) that

$$\varphi(t, \lambda_n(t)-) = \varphi(t, \lambda_n(t)+) = \gamma_n e^{\lambda_n(t) - \delta t}, \tag{3.5}$$

$$\frac{\partial \varphi}{\partial x}(t, \lambda_n(t)-) = \frac{\partial \varphi}{\partial x}(t, \lambda_n(t)+), \tag{3.6}$$

for $n = 1,2,\cdots,N$.

This is a nonlinear problem with nonlinearity and discontinuity in the coefficient of the highest order term. Liang et al. [21] established a comparison principle to overcome the difficulty of energy estimates. The following comparison principle is a slightly modified version, which can be applied to our model. The proof is similar to that in Liang et al. [21] and we refer the readers to the detailed proof of Theorem 3.1 in [21].

PROPOSITION 3.1 (Comparison principle). *Let*

$$\mathscr{L}_i[\varphi_i] \triangleq \frac{\partial \varphi_i}{\partial t} - \frac{\sigma_i^2}{2}\frac{\partial^2 \varphi_i}{\partial x^2} - \left(r - \frac{\sigma_i^2}{2}\right)\frac{\partial \varphi_i}{\partial x} + r\varphi_i, \; i = 1,2,$$

where $\sigma_i = \sigma_i(t,x)$ satisfy

$$\sigma_1(t,x) \le \sigma_N + \sum_{n=0}^{N-1} (\sigma_n - \sigma_{n+1}) H(\varphi_1 - \gamma_{n+1} e^{x-\delta t}), \tag{3.7}$$

$$\sigma_2(t,x) \geq \sigma_N + \sum_{n=0}^{N-1} (\sigma_n - \sigma_{n+1}) H(\varphi_2 - \gamma_{n+1} e^{x-\delta t}), \tag{3.8}$$

where H is the Heaviside function satisfying $H(x)=1$ for $x \geq 0$ and $H(x)=0$ for $x<0$. Suppose that there exists $B>0$, depending on T, such that

$$\sigma_i(t,x) = \sigma_N,\ x>B,\ 0 \leq t \leq T,\ i=1,2,$$
$$\sigma_i(t,x) = \sigma_0,\ x<-B,\ 0 \leq t \leq T,\ i=1,2.$$

For either $i=1$ or $i=2$,

$$\frac{\partial^2 \varphi_i}{\partial x^2} - \frac{\partial \varphi_i}{\partial x} \leq 0,\ -\infty < x < \infty,\ 0 \leq t \leq T. \tag{3.9}$$

Assume also that $\varphi_i \in W^{1,2}_{\infty,loc}((0,T)\times\mathbb{R}) \cap C([0,T]\times\mathbb{R}) \cap L^\infty((0,T)\times\mathbb{R})$, $i=1,2$, and

$$\mathscr{L}_1[\varphi_1] \geq \mathscr{L}_2[\varphi_2],\ \varphi_1(0,x) \geq \varphi_2(0,x),\ -\infty < x < \infty,\ t>0,$$

then

$$\varphi_1(t,x) \geq \varphi_2(t,x),\ -\infty < x < \infty,\ 0 < t \leq T.$$

REMARK 3.1. The assumption of $W^{1,2}_{\infty,loc}$ can be replaced by $W^{1,2}_{p,loc}$, $p \geq 3$.

As we shall establish the existence of a traveling wave and convergence to it, it will be more convenient for us to work on

$$u(t,\xi) = e^{rt}\varphi(t,x),\ \xi = x+ct,\ \eta_n(t) = \lambda_n(t)+ct,\ n=1,2,\cdots,N,$$

where $c=r-\delta$. Then (3.1)-(3.6) are transformed into

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial \xi^2} - \left(\delta - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial \xi} = 0,\ -\infty < \xi < \infty,\ t>0, \tag{3.10}$$

where σ is defined as

$$\sigma = \sigma(u,\xi) = \begin{cases} \sigma_N,\ u < \gamma_N e^{\xi}, \\ \sigma_n,\ \gamma_{n+1} e^{\xi} \leq u < \gamma_n e^{\xi},\ n=1,2,\cdots,N-1, \\ \sigma_0,\ u \geq \gamma_1 e^{\xi}, \end{cases} \tag{3.11}$$

with initial condition

$$u(0,\xi) = \min\{e^{\xi},1\},\ -\infty < \xi < \infty, \tag{3.12}$$

and free boundary conditions

$$u(t,\eta_n(t)-) = u(t,\eta_n(t)+) = \gamma_n e^{\eta_n(t)}, \tag{3.13}$$
$$\frac{\partial u}{\partial \xi}(t,\eta(t)-) = \frac{\partial u}{\partial \xi}(t,\eta(t)+), \tag{3.14}$$

for $n=1,2,\cdots,N$.

4. Approximation for the problem

In this section, an approximation for solution of the free boundary problem is given as a bridge for the existence and uniqueness of solution. Rewrite (3.11) as

$$\sigma = \sigma_N + \sum_{n=0}^{N-1} (\sigma_n - \sigma_{n+1}) H(u - \gamma_{n+1} e^{\xi}). \tag{4.1}$$

We approximate the Heaviside function H by a C^∞ function H_ϵ such that

$$H_\epsilon(x) = 0 \text{ for } x < -\epsilon, \; H_\epsilon(x) = 1 \text{ for } x > 0, \; H_\epsilon'(x) \geq 0 \text{ for } -\infty < x < \infty.$$

Consider the approximated problem

$$\mathscr{L}^\epsilon[u_\epsilon] \triangleq \frac{\partial u_\epsilon}{\partial t} - \frac{\sigma_\epsilon^2}{2} \frac{\partial^2 u_\epsilon}{\partial \xi^2} - \left(\delta - \frac{\sigma_\epsilon^2}{2}\right) \frac{\partial u_\epsilon}{\partial \xi} = 0, \; -\infty < \xi < \infty, \; t > 0, \tag{4.2}$$

with initial condition

$$u_\epsilon(0,\xi) = \min\{e^\xi, 1\}, \; -\infty < \xi < \infty, \tag{4.3}$$

where

$$\sigma_\epsilon = \sigma_N + \sum_{n=0}^{N-1} (\sigma_n - \sigma_{n+1}) H_\epsilon(u_\epsilon - \gamma_{n+1} e^{\xi}).$$

Through a classical fixed point argument for PDE, it is easy to check that (4.2)-(4.3) admits a unique classical solution u_ϵ. We then proceed to derive some estimates for u_ϵ in the following argument. The following lemmas in this section are analogous to the corresponding results in [21]. Some necessary modifications are needed to fit the case of multiple credit ratings in the model.

REMARK 4.1. The function H_ϵ is a modification of H. We can see that there exists a small buffer when changing from 0 to 1. In practice, the length of this buffer corresponding to upgrading or downgrading is affected by the bond price. Furthermore, we could set H_ϵ monotone with respect to ϵ. People could control the length of this buffer area to make their benefits optimal.

LEMMA 4.1. *Denote u_ϵ as the solution of (4.2) with initial condition (4.3). Then u_ϵ satisfies*

$$0 \leq u_\epsilon(t,\xi) \leq 1 \textit{ for } -\infty < \xi < \infty, \; 0 \leq t < \infty.$$

Proof. It is easy to verify that 0 is a lower solution while 1 is a upper solution. The lemma is the direct result of comparison principle. ∎

LEMMA 4.2. *Denote u_ϵ as the solution of (4.2) with initial condition (4.3). Then u_ϵ satisfies*

$$\frac{\partial^2 u_\epsilon}{\partial \xi^2} - \frac{\partial u_\epsilon}{\partial \xi} < 0, \; -\infty < \xi < \infty, \; t > 0.$$

Proof. Denote

$$w \triangleq \frac{\partial u_\epsilon}{\partial t} - \delta \frac{\partial u_\epsilon}{\partial \xi} = \frac{\sigma_\epsilon^2}{2}\left(\frac{\partial^2 u_\epsilon}{\partial \xi^2} - \frac{\partial u_\epsilon}{\partial \xi}\right).$$

Differentiating (4.2) with respect to t gives

$$\begin{aligned}&\frac{\partial^2 u_\epsilon}{\partial t^2}-\frac{\sigma_\epsilon^2}{2}\frac{\partial^3 u_\epsilon}{\partial t\partial\xi^2}-\left(\delta-\frac{\sigma_\epsilon^2}{2}\right)\frac{\partial^2 u_\epsilon}{\partial t\partial\xi}\\ =&\sigma_\epsilon\left(\frac{\partial^2 u_\epsilon}{\partial\xi^2}-\frac{\partial u_\epsilon}{\partial\xi}\right)\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon'(u_\epsilon-\gamma_{n+1}e^\xi)\frac{\partial u_\epsilon}{\partial t}.\end{aligned}\tag{4.4}$$

Differentiating (4.2) with respect to ξ gives

$$\begin{aligned}&\frac{\partial^2 u_\epsilon}{\partial t\partial\xi}-\frac{\sigma_\epsilon^2}{2}\frac{\partial^3 u_\epsilon}{\partial\xi^3}-\left(\delta-\frac{\sigma_\epsilon^2}{2}\right)\frac{\partial^2 u_\epsilon}{\partial\xi^2}\\ =&\sigma_\epsilon\left(\frac{\partial^2 u_\epsilon}{\partial\xi^2}-\frac{\partial u_\epsilon}{\partial\xi}\right)\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon'(u_\epsilon-\gamma_{n+1}e^\xi)\delta\left(\frac{\partial u_\epsilon}{\partial\xi}-\gamma_{n+1}e^\xi\right).\end{aligned}\tag{4.5}$$

Associating (4.4) with (4.5) gives

$$\mathscr{L}^\epsilon[w]=\frac{2w}{\sigma_\epsilon}\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon'(u_\epsilon-\gamma_{n+1}e^\xi)\left(\frac{\partial u_\epsilon}{\partial t}-\delta\left(\frac{\partial u_\epsilon}{\partial\xi}-\gamma_{n+1}e^\xi\right)\right).$$

At $t=0$, w produces a Dirac measure of intensity -1 at $\xi=0$ and $w(0,\xi)=0$ for both $\xi<0$ and $\xi>0$. By further approximating the initial data with smooth functions if necessary, we derive that $w<0$ by maximum principle (see Theorem 3.7 in Chapter 3 of Hu [9]), which completes the proof of the lemma. ∎

LEMMA 4.3. *Denote u_ϵ as the solution of (4.2) with initial condition (4.3). Then u_ϵ satisfies*

$$-1\le\frac{\partial u_\epsilon}{\partial\xi}-u_\epsilon\le 0,\ \frac{\partial u_\epsilon}{\partial\xi}\ge 0.$$

Proof. Denote $w=\partial u_\epsilon/\partial\xi-u_\epsilon$. Then for the first inequality, w satisfies

$$\frac{\partial w}{\partial t}=\left(\frac{\partial}{\partial\xi}-1\right)\frac{\partial u_\epsilon}{\partial t}=\frac{\sigma_\epsilon^2}{2}\frac{\partial^2 w}{\partial\xi^2}+\left(\delta-\frac{\sigma_\epsilon^2}{2}\right)\frac{\partial w}{\partial\xi}+\sigma_\epsilon\frac{\partial\sigma_\epsilon}{\partial\xi}\frac{\partial w}{\partial\xi}.$$

Together with the fact that

$$\frac{\partial\sigma_\epsilon}{\partial\xi}=\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon'(u_\epsilon-\gamma_{n+1}e^\xi)\left(\frac{\partial u_\epsilon}{\partial\xi}-\gamma_{n+1}e^\xi\right),$$

it holds that

$$\mathscr{L}_1^\epsilon[w]\triangleq\mathscr{L}^\epsilon[w]-\sigma_\epsilon\frac{\partial w}{\partial\xi}\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon'(u_\epsilon-\gamma_{n+1}e^\xi)\left(\frac{\partial u_\epsilon}{\partial\xi}-\gamma_{n+1}e^\xi\right)=0.$$

Hence, w would not reach any positive maximum point in the region. It is clear as well that initially $w(0,\xi)=0$ for $\xi<0$ and $w(0,\xi)=-1$ for $\xi>0$. Then it follows by maximum principle that $w(t,\xi)\le 0$ for $-\infty<\xi<\infty$, $t>0$. It is also clear that $\mathscr{L}_1^\epsilon[-1]=0$. Thus, we have $w(t,\xi)\ge -1$ for $-\infty<\xi<\infty$, $t>0$ by comparison principle.

For the second inequality, differentiating (4.2) with respect to ξ, we get

$$\mathscr{L}^\epsilon\left[\frac{\partial u_\epsilon}{\partial \xi}\right] = \sigma_\epsilon\left(\frac{\partial^2 u_\epsilon}{\partial \xi^2} - \frac{\partial u_\epsilon}{\partial \xi}\right)\sum_{n=0}^{N-1}(\sigma_n - \sigma_{n+1})H_\epsilon'(u_\epsilon - \gamma_{n+1}e^\xi)\left(\frac{\partial u_\epsilon}{\partial \xi} - \gamma_{n+1}e^\xi\right). \quad (4.6)$$

As in the proof of the first inequality, take

$$l(\xi) \triangleq \sigma_\epsilon\left(\frac{\partial^2 u_\epsilon}{\partial \xi^2} - \frac{\partial u_\epsilon}{\partial \xi}\right)\sum_{n=0}^{N-1}(\sigma_n - \sigma_{n+1})H_\epsilon'(u_\epsilon - \gamma_{n+1}e^\xi)\left(\frac{\partial u_\epsilon}{\partial \xi} - \gamma_{n+1}e^\xi\right) \quad (4.7)$$

as a given function. If $-\epsilon \le u_\epsilon - \gamma_n e^\xi \le 0$ holds for some fixed n, then associating with the first inequality, it holds that

$$-1-\epsilon \le \frac{\partial u_\epsilon}{\partial \xi} - \gamma_n e^\xi \le 0,$$

which implies that $l(\xi) \ge 0$ with the result from Lemma 4.2. Thus, we are able to apply the comparison principle by noticing that initially $\partial u_\epsilon(0,\xi)/\partial\xi = e^\xi > 0$ for $\xi < 0$ and $\partial u_\epsilon(0,\xi)/\partial\xi = 0$ for $\xi > 0$. Then it follows that $\partial u_\epsilon/\partial\xi \ge 0$. ∎

LEMMA 4.4. *Denote u_ϵ as the solution of (4.2) with initial condition (4.3). Then there exist constants C_1, C_2 and C_3, independent of ϵ, such that*

$$-C_3 - \frac{C_2}{\sqrt{t}}e^{-C_1\xi^2/t} \le \frac{\partial u_\epsilon}{\partial t} \le \delta, \ -\infty < \xi < \infty, \ 0 < t < \infty.$$

Proof. Through Lemmas 4.1-4.3, it is easy to see that

$$\frac{\partial u_\epsilon}{\partial t} = \frac{\sigma_\epsilon^2}{2}\frac{\partial^2 u_\epsilon}{\partial \xi^2} + \left(\delta - \frac{\sigma_\epsilon^2}{2}\right)\frac{\partial u_\epsilon}{\partial \xi} = \frac{\sigma_\epsilon^2}{2}\left(\frac{\partial^2 u_\epsilon}{\partial \xi^2} - \frac{\partial u_\epsilon}{\partial \xi}\right) + \delta\frac{\partial u_\epsilon}{\partial \xi} \le \delta\frac{\partial u_\epsilon}{\partial \xi} \le \delta.$$

This establishes the second inequality. Next, we establish the first inequality. Since $u_\epsilon(0,0) = 1 > \gamma_1$ and by uniform Hölder continuity of the solution, there exists $\rho > 0$, independent of ϵ, such that

$$u_\epsilon(t,\xi) > \frac{1+\gamma_1}{2} > \gamma_1 e^\xi \text{ for } |\xi| \le \rho, \ 0 \le t \le \rho^2.$$

Thus $\sigma_\epsilon = \sigma_0$ for $|\xi| \le \rho$, $0 \le t \le \rho^2$. It follows from the standard parabolic estimates (see Chapter 4 in Garrori and Menaldi [8]) that

$$\frac{\partial u_\epsilon}{\partial t} \ge -C_2 - \frac{C_2}{\sqrt{t}}e^{-C_1\xi^2/t} \text{ for } |\xi| < \frac{\rho}{2}, \ 0 < t \le \frac{\rho^2}{4}, \quad (4.8)$$

where C_1, $C_2 > 0$ are constants independent of ϵ. In particular, this implies that there exists a constant $C_3 > 0$ independent of ϵ, such that

$$\frac{\partial u_\epsilon}{\partial t} \ge -C_3 \text{ on } \left\{0 < t < \frac{\rho^2}{4}, |\xi| = \frac{\rho}{2}\right\} \cup \left\{t = \frac{\rho^2}{4}, |\xi| < \frac{\rho}{2}\right\}. \quad (4.9)$$

Define

$$\mathscr{L}_2^\epsilon[v] \triangleq \mathscr{L}^\epsilon[v] - \left(\frac{\partial^2 u_\epsilon}{\partial \xi^2} - \frac{\partial u_\epsilon}{\partial \xi}\right)\sigma_\epsilon\sum_{n=0}^{N-1}(\sigma_n - \sigma_{n+1})H_\epsilon'(u_\epsilon - \gamma_{n+1}e^\xi)v.$$

Then it holds that $\mathscr{L}_2^\epsilon[\partial u_\epsilon/\partial t]=0$. Thus, we have

$$\mathscr{L}_2^\epsilon\left[\frac{\partial u_\epsilon}{\partial t}+C_3\right]=-\left(\frac{\partial^2 u_\epsilon}{\partial \xi^2}-\frac{\partial u_\epsilon}{\partial \xi}\right)\sigma_\epsilon\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon'(u_\epsilon-\gamma_{n+1}e^\xi)C_3\geq 0.$$

Consider the region $Q=[0,\infty)\times(-\infty,\infty)\setminus\overline{Q_\rho}$, where $Q_\rho=(0,\rho^2/4)\times(-\rho/2,\rho/2)$. Then by maximum principle, we can conclude that $\partial u_\epsilon/\partial t+C_3\geq 0$ on this region, together with (4.8), which implies the establishment of the first inequality. □

5. Existence of traveling wave

In this paper, we devote to capturing the phenomenon of asymptotic traveling wave in the model, which could give us a profile for the bond price. The precondition is the existence of traveling wave. The following theorem solves this problem by showing the existence of a traveling wave in (3.10). Moreover, such traveling wave is unique. We shall assume that the risk discount rate δ satisfies

$$\frac{\sigma_N^2}{2}<\delta<\frac{\sigma_0^2}{2}. \tag{5.1}$$

This hypothesis on the risk discount rate not only suffices the existence of traveling wave, but also directs the convergence to the traveling wave, which will be verified in Section 8.

THEOREM 5.1. *Suppose that the risk discount rate δ satisfies (5.1). The following problem*

$$\frac{\sigma^2}{2}\psi''(\xi)+\left(\delta-\frac{\sigma^2}{2}\right)\psi'(\xi)=0, \tag{5.2}$$

for $\xi\in(-\infty,\eta_1^*)\cup(\eta_1^*,\eta_2^*)\cup\cdots\cup(\eta_{N-1}^*,\eta_N^*)\cup(\eta_N^*,\infty)$, with conditions

$$\psi(\eta_n^*+)=\psi(\eta_n^*-)=\gamma_n e^{\eta_n^*},\ \psi'(\eta_n^*+)=\psi'(\eta_n^*-), \tag{5.3}$$

for $n=1,2,\cdots,N$, and

$$\psi(+\infty)=1,\ \psi(-\infty)=0, \tag{5.4}$$

where

$$\sigma=\begin{cases}\sigma_0,\ \xi\leq\eta_1^*,\\ \sigma_n,\ \eta_n^*<\xi\leq\eta_{n+1}^*,\ n=1,2,\cdots,N-1,\\ \sigma_N,\ \xi>\eta_N^*,\end{cases}$$

admits a unique solution $(\psi,\eta_n^*,n=1,2,\cdots,N)$ solved as

$$\psi(\xi)=\alpha_0+\beta_0 e^{(1-2\delta/\sigma_0^2)\xi},\ \xi\leq\eta_1^*,$$
$$\psi(\xi)=\alpha_n+\beta_n e^{(1-2\delta/\sigma_n^2)\xi},\ \eta_n^*<\xi\leq\eta_{n+1}^*,$$

for $n=1,2,\cdots,N-1$, and

$$\psi(\xi)=\alpha_N+\beta_N e^{(1-2\delta/\sigma_N^2)\xi},\ \xi>\eta_N^*,$$

where α_n, β_n, $n=0,1,2,\cdots,N$, *satisfy*

$$\alpha_0=0,\ \alpha_N=1, \tag{5.5}$$

$$\alpha_n+\beta_n e^{(1-2\delta/\sigma_n^2)\eta_{n+1}^*}=\gamma_{n+1}e^{\eta_{n+1}^*}, \tag{5.6}$$

$$\alpha_{n+1}+\beta_{n+1}e^{(1-2\delta/\sigma_{n+1}^2)\eta_{n+1}^*}=\gamma_{n+1}e^{\eta_{n+1}^*}, \tag{5.7}$$

$$\sigma_{n+1}^2\beta_n e^{(1-2\delta/\sigma_n^2)\eta_{n+1}^*}(\sigma_n^2-2\delta)=\sigma_n^2\beta_{n+1}e^{(1-2\delta/\sigma_{n+1}^2)\eta_{n+1}^*}(\sigma_{n+1}^2-2\delta), \tag{5.8}$$

for $n=0,1,2,\cdots,N-1$.

Proof. We just need to show that system (5.5)-(5.8) for $n=0,1,2,\cdots,N-1$ is solvable with δ, σ_0, σ_n, γ_n as known parameters. We would like to achieve this result inductively. As $N=1$, system (5.5)-(5.8) is simplified into

$$\alpha_0=0,\ \alpha_1=1,$$

$$\alpha_0+\beta_0 e^{(1-2\delta/\sigma_0^2)\eta_1^*}=\gamma_1 e^{\eta_1^*},$$

$$\alpha_1+\beta_1 e^{(1-2\delta/\sigma_1^2)\eta_1^*}=\gamma_1 e^{\eta_1^*},$$

$$\sigma_1^2\beta_0 e^{(1-2\delta/\sigma_0^2)\eta_1^*}(\sigma_0^2-2\delta)=\sigma_0^2\beta_1 e^{(1-2\delta/\sigma_1^2)\eta_1^*}(\sigma_1^2-2\delta),$$

and δ satisfies $\sigma_1^2<2\delta<\sigma_0^2$. For this case, it is easy to solve that

$$\eta_1^*=\log\sigma_0^2(2\delta-\sigma_1^2)-\log 2\delta\gamma_1(\sigma_0^2-\sigma_1^2),$$

$$\beta_0=\gamma_1 e^{2\delta\eta_1^*/\sigma_0^2},\ \beta_1=(\gamma_1 e^{\eta_1^*}-1)e^{(2\delta/\sigma_1^2-1)\eta_1^*}.$$

Thus, system (5.5)-(5.8) is solvable as $N=1$. Meanwhile, the traveling wave solution obtained for the case $N=1$ is consistent with the one in Lemma 4.7 of [21]. The constraint $\sigma_1^2<2\delta<\sigma_0^2$ not only guarantees the solvability of the system, but also guarantees the fact that the function $\psi(\xi)\to 1$ as $\xi\to\infty$ and $\psi(\xi)\to 0$ as $\xi\to-\infty$. Now we suppose that system (5.5)-(5.8) is solvable as $N=k$ for some $k\in\mathbb{N}$ with δ satisfying $\sigma_k^2<2\delta<\sigma_0^2$. Then we need to show that it is also solvable as $N=k+1$ with δ satisfying $\sigma_{k+1}^2<2\delta<\sigma_0^2$.

If $\sigma_k^2<2\delta<\sigma_0^2$, we set $\alpha_k=1$. Then by the hypothesis, system (5.5)-(5.8) for $n=0,1,2,\cdots,k-1$ is solvable. Since $\alpha_k=1$, $\gamma_k e^{\eta_k^*}\le 1$ and

$$\alpha_k+\beta_k e^{(1-2\delta/\sigma_k^2)\eta_k^*}=\gamma_k e^{\eta_k^*},$$

it is known that $\beta_k\le 0$. Define functions f_1, f_2 as

$$f_1(x)=\gamma_{k+1}e^x-\beta_k e^{(1-2\delta/\sigma_k^2)x},$$

$$f_2(x)=\gamma_k e^x-\beta_k e^{(1-2\delta/\sigma_k^2)x}.$$

Since $\gamma_k>\gamma_{k+1}$, we have $f_2(x)>f_1(x)$ for all $x\in\mathbb{R}$. We know that $f_2(\eta_k^*)=1>f_1(\eta_k^*)$. Meanwhile, it is easy to see that $f_1(x)\to\infty$ as $x\to\pm\infty$ since $2\delta>\sigma_k^2$. Then there exists x^* such that $f_1(x^*)=1$ by mean value theorem. Let $\eta_{k+1}^*=x^*$ and there holds

$$1+\beta_k e^{(1-2\delta/\sigma_{k+1}^2)\eta_{k+1}^*}=\gamma_{k+1}e^{\eta_{k+1}^*}. \tag{5.9}$$

Furthermore, we can solve that

$$\alpha_{k+1}=C_1,\ \beta_{k+1}=\frac{\beta_k\sigma_{k+1}^2(\sigma_k^2-2\delta)}{\sigma_k^2(\sigma_{k+1}^2-2\delta)}e^{(2\delta/\sigma_{k+1}^2-2\delta/\sigma_k^2)\eta_{k+1}^*},$$

where

$$C_1=\gamma_{k+1}e^{\eta_{k+1}^*}-\frac{\beta_k\sigma_{k+1}^2(\sigma_k^2-2\delta)}{\sigma_k^2(\sigma_{k+1}^2-2\delta)}e^{(1-2\delta/\sigma_k^2)\eta_{k+1}^*}.$$

This claims that the following system

$$\begin{aligned}
&\alpha_0=0,\ \alpha_{k+1}=C_1,\\
&\alpha_n+\beta_n e^{(1-2\delta/\sigma_n^2)\eta_{n+1}^*}=\gamma_{n+1}e^{\eta_{n+1}^*},\\
&\alpha_{n+1}+\beta_{n+1}e^{(1-2\delta/\sigma_{n+1}^2)\eta_{n+1}^*}=\gamma_{n+1}e^{\eta_{n+1}^*},\\
&\sigma_{n+1}^2\beta_n e^{(1-2\delta/\sigma_n^2)\eta_{n+1}^*}(\sigma_n^2-2\delta)=\sigma_n^2\beta_{n+1}e^{(1-2\delta/\sigma_{n+1}^2)\eta_{n+1}^*}(\sigma_{n+1}^2-2\delta),
\end{aligned}$$

for $n=0,1,2,\cdots,k$, is solvable. It is equivalent to claiming that the problem

$$\frac{\sigma^2}{2}\psi''(\xi)+\left(\delta-\frac{\sigma^2}{2}\right)\psi'(\xi)=0,$$

for $\xi\in(-\infty,\eta_1^*)\cup(\cup_{n=1}^k(\eta_n^*,\eta_{n+1}^*))\cup(\eta_{k+1}^*,\infty)$, with conditions

$$\psi(\eta_n^*+)=\psi(\eta_n^*-)=\gamma_n e^{\eta_n^*},\psi'(\eta_n^*+)=\psi'(\eta_n^*-),$$

for $n=1,2,\cdots,k+1$, and

$$\psi(+\infty)=C_1,\psi(-\infty)=0,$$

admits a unique solution. Note that $\alpha_{k+1}=C_1>0$. Make a transformation $\theta(y)=\psi(y+\log C_1)/C_1$. Then θ satisfies

$$\frac{\sigma^2}{2}\theta''(y)+\left(\delta-\frac{\sigma^2}{2}\right)\theta'(y)=0,$$

for $y\in(-\infty,\eta_1^*-\log C_1)\cup(\cup_{n=1}^k(\eta_n^*-\log C_1,\eta_{n+1}^*-\log C_1))\cup(\eta_{k+1}^*-\log C_1,\infty)$, with conditions

$$\theta((\eta_n^*-\log C_1)+)=\theta((\eta_n^*-\log C_1)-)=\gamma_n e^{\eta_n^*-\log C_1},$$

$$\theta'((\eta_n^*-\log C_1)+)=\theta'((\eta_n^*-\log C_1)-),$$

for $n=1,2,\cdots,k+1$, and

$$\theta(+\infty)=1,\ \theta(-\infty)=0,$$

where

$$\sigma=\sigma(y)=\begin{cases}\sigma_0,\ y\le\eta_1^*-\log C_1,\\ \sigma_n,\ \eta_n^*-\log C_1<y\le\eta_{n+1}^*-\log C_1,\ n=1,2,\cdots,k,\\ \sigma_{k+1},\ y>\eta_{k+1}^*-\log C_1.\end{cases}$$

This implies that if the condition $\sigma_k^2 < 2\delta < \sigma_0^2$ holds, system (5.5)-(5.8) for $n = 0,1,2,\cdots,k$ is solvable.

Next we suppose that $\sigma_{k+1}^2 < 2\delta < \sigma_1^2$. In this case, we set $\alpha_1 = 0$. Then by the hypothesis, system (5.5)-(5.8) for $n = 1,2,\cdots,k$ is solvable and we can derive that

$$\eta_1^* = \frac{\sigma_1^2}{2\delta}(\log\beta_1 - \log\gamma_1),$$
$$\alpha_0 = C_2,\ \beta_0 = \frac{\beta_1\sigma_0^2(\sigma_1^2 - 2\delta)}{\sigma_1^2(\sigma_0^2 - 2\delta)} e^{(2\delta/\sigma_0^2 - 2\delta/\sigma_1^2)\eta_1^*},$$

where

$$C_2 = \gamma_1 e^{\eta_1^*} - \frac{\beta_1\sigma_0^2(\sigma_1^2 - 2\delta)}{\sigma_1^2(\sigma_0^2 - 2\delta)} e^{(1-2\delta/\sigma_1^2)\eta_1^*}.$$

This claims that system

$$\alpha_0 = C_2,\ \alpha_{k+1} = 1, \tag{5.10}$$
$$\alpha_n + \beta_n e^{(1-2\delta/\sigma_n^2)\eta_{n+1}^*} = \gamma_{n+1} e^{\eta_{n+1}^*}, \tag{5.11}$$
$$\alpha_{n+1} + \beta_{n+1} e^{(1-2\delta/\sigma_{n+1}^2)\eta_{n+1}^*} = \gamma_{n+1} e^{\eta_{n+1}^*}, \tag{5.12}$$
$$\sigma_{n+1}^2 \beta_n e^{(1-2\delta/\sigma_n^2)\eta_{n+1}^*}(\sigma_n^2 - 2\delta) = \sigma_n^2 \beta_{n+1} e^{(1-2\delta/\sigma_{n+1}^2)\eta_{n+1}^*}(\sigma_{n+1}^2 - 2\delta), \tag{5.13}$$

for $n = 0,1,2,\cdots,k$, is solvable. Let $\alpha_0' = \alpha_0 - C_2 = 0$, $\zeta_1^* = \log(\gamma_1 e^{\eta_1^*} - C_2) - \log\gamma_1$. Note that $\gamma_1 e^{\eta_1^*} > C_2$ and ζ_1^* is well defined. Then it holds that

$$\alpha_0' + \beta_0' e^{(1-2\delta/\sigma_0^2)\zeta_1^*} = \gamma_1 e^{\zeta_1^*},$$

where $\beta_0' = \beta_0 e^{(1-2\delta/\sigma_0^2)(\eta_1^* - \zeta_1^*)}$. Denote

$$\beta_1' \triangleq \frac{\beta_0'\sigma_1^2(\sigma_0^2 - 2\delta)}{\sigma_0^2(\sigma_1^2 - 2\delta)} e^{(2\delta/\sigma_1^2 - 2\delta/\sigma_0^2)\zeta_1^*}.$$

Then it holds that

$$\sigma_1^2 \beta_0' e^{(1-2\delta/\sigma_0^2)\zeta_1^*}(\sigma_0^2 - 2\delta) = \sigma_0^2 \beta_1' e^{(1-2\delta/\sigma_1^2)\zeta_1^*}(\sigma_1^2 - 2\delta).$$

We can choose α_1' such that

$$\alpha_1' + \beta_1' e^{(1-2\delta/\sigma_1^2)\zeta_1^*} = \gamma_1 e^{\zeta_1^*}.$$

Define

$$g_1(x) = \gamma_1 e^x - \beta_1' e^{(1-2\delta/\sigma_1^2)x},$$
$$g_2(x) = \gamma_2 e^x - \beta_1' e^{(1-2\delta/\sigma_1^2)x}.$$

Since $\gamma_2 < \gamma_1$, we have $g_2 < g_1$ and $g_2(\zeta_1^*) < g_1(\zeta_1^*) = \alpha_1'$. On the other hand, it holds that $g_2(x) \to \infty$ as $x \to \infty$ since $2\delta < \sigma_1^2$. Then there exists ζ_2^* such that $g_2(\zeta_2^*) = \alpha_1'$, namely that

$$\alpha_1' + \beta_1' e^{(1-2\delta/\sigma_1^2)\zeta_2^*} = \gamma_2 e^{\zeta_2^*}.$$

Denote

$$\beta_2' \triangleq \frac{\beta_1'\sigma_2^2(\sigma_1^2-2\delta)}{\sigma_1^2(\sigma_2^2-2\delta)} e^{(2\delta/\sigma_2^2-2\delta/\sigma_1^2)\zeta_2^*}.$$

Then it holds that

$$\sigma_2^2\beta_1' e^{(1-2\delta/\sigma_1^2)\zeta_2^*}(\sigma_1^2-2\delta)=\sigma_1^2\beta_2' e^{(1-2\delta/\sigma_2^2)\zeta_2^*}(\sigma_2^2-2\delta).$$

We can choose α_2' such that

$$\alpha_2'+\beta_2' e^{(1-2\delta/\sigma_2^2)\zeta_2^*}=\gamma_2 e^{\zeta_2^*}.$$

With the programming going on, we can construct a system from system (5.10)-(5.13) for $n=0,1,2,\cdots,k$ as follows:

$$\alpha_n'+\beta_n' e^{(1-2\delta/\sigma_n^2)\zeta_{n+1}^*}=\gamma_{n+1}e^{\zeta_{n+1}^*},$$
$$\alpha_{n+1}'+\beta_{n+1}' e^{(1-2\delta/\sigma_{n+1}^2)\zeta_{n+1}^*}=\gamma_{n+1}e^{\zeta_{n+1}^*},$$
$$\sigma_{n+1}^2\beta_n' e^{(1-2\delta/\sigma_n^2)\zeta_{n+1}^*}(\sigma_n^2-2\delta)=\sigma_n^2\beta_{n+1}' e^{(1-2\delta/\sigma_{n+1}^2)\zeta_{n+1}^*}(\sigma_{n+1}^2-2\delta),$$

for $n=0,1,2,\cdots,k$. This claims that the problem

$$\frac{\sigma^2}{2}\theta''(\xi)+\left(\delta-\frac{\sigma^2}{2}\right)\theta'(\xi)=0,$$

for $\xi\in(-\infty,\zeta_1^*)\cup(\cup_{n=1}^k(\zeta_n^*,\zeta_{n+1}^*))\cup(\zeta_{k+1}^*,\infty)$, with conditions

$$\theta(\zeta_n^*+)=\theta(\zeta_n^*-)=\gamma_n e^{\zeta_n^*},\ \theta'(\zeta_n^*)=\theta'(\zeta_n^*),$$

for $n=1,2,\cdots,k+1$, and

$$\theta(+\infty)=C_3,\ \theta(-\infty)=0,$$

where

$$\sigma=\begin{cases}\sigma_0,\ \xi\le\zeta_1^*,\\ \sigma_n,\ \zeta_n^*<\xi\le\zeta_{n+1}^*,\ n=1,2,\cdots,k,\\ \sigma_{k+1},\ \xi>\zeta_{k+1}^*,\end{cases}$$

admits a unique solution. Make a transformation $v(y)=\theta(y+\log C_3)/C_3$. Then v satisfies

$$\frac{\sigma^2}{2}v''(y)+\left(\delta-\frac{\sigma^2}{2}\right)v'(y)=0,$$

for $y\in(-\infty,\zeta_1^*-\log C_3)\cup\left(\cup_{n=1}^k(\zeta_n^*-\log C_3,\zeta_{n+1}^*-\log C_3)\right)\cup(\zeta_{k+1}^*-\log C_3,\infty)$, with conditions

$$v((\zeta_n^*-\log C_3)+)=v((\zeta_n^*-\log C_3)-)=\gamma_n e^{\zeta_n^*-\log C_3},$$

$$v'((\zeta_n^*-\log C_3)+)=v'((\zeta_n^*-\log C_3)-),$$

for $n=1,2,\cdots,k+1$, and

$$v(+\infty)=1,\ v(-\infty)=0,$$

where

$$\sigma=\begin{cases}\sigma_0,\ y\leq\zeta_1^*-\log C_3,\\ \sigma_n,\ \zeta_n^*-\log C_3<y\leq\zeta_{n+1}^*-\log C_3, n=1,2,\cdots,k,\\ \sigma_{k+1},\ y>\zeta_{k+1}^*-\log C_3.\end{cases}$$

This equivalently claims that if the condition $\sigma_{k+1}^2<2\delta<\sigma_1^2$ holds, system (5.5)-(5.8) for $n=0,1,2,\cdots,k$ is solvable.

Combining the result under the condition $\sigma_k^2<2\delta<\sigma_0^2$ and the result under the condition $\sigma_{k+1}^2<2\delta<\sigma_1^2$, we conclude that if the condition $\sigma_{k+1}^2<2\delta<\sigma_0^2$ holds, system (5.5)-(5.8) is solvable as $N=k+1$, which completes the proof of the theorem. ◻

6. Estimates for free boundaries

Denote by η_n^ϵ, $n=1,2,\cdots,N$, the approximated free boundaries. They are the implied solutions of the equations

$$u_\epsilon(t,\eta_n^\epsilon(t))=\gamma_n e^{\eta_n^\epsilon(t)},\ n=1,2,\cdots,N. \tag{6.1}$$

In this section, we derive some estimates for the approximated free boundaries.

Lemma 6.1. *Let η_n^ϵ, $n=1,2,\cdots,N$, be the approximated free boundaries defined as (6.1). Then the sequence η_n^ϵ, $n=1,2,\cdots,N$, is strictly increased with respect to n, i.e.*

$$\eta_1^\epsilon<\eta_2^\epsilon<\cdots<\eta_{N-1}^\epsilon<\eta_N^\epsilon.$$

Proof. For fixed n, we claim that $\eta_n^\epsilon(t)\neq\eta_{n+1}^\epsilon(t)$ for any $t\geq 0$. To obtain a contradiction, we suppose that there exists $t_0\geq 0$, such that

$$\eta_n^\epsilon(t_0)=\eta_{n+1}^\epsilon(t_0),$$

which implies that

$$u_\epsilon(t_0,\eta_n^\epsilon(t_0))e^{-\eta_n^\epsilon(t_0)}=u_\epsilon(t_0,\eta_{n+1}^\epsilon(t_0))e^{-\eta_{n+1}^\epsilon(t_0)}$$

Nevertheless, as for any $t\geq 0$, it holds that

$$u_\epsilon(t,\eta_n^\epsilon(t))e^{-\eta_n^\epsilon(t)}=\gamma_n\neq\gamma_{n+1}=u_\epsilon(t,\eta_{n+1}^\epsilon(t))e^{-\eta_{n+1}^\epsilon(t)},$$

which contradicts the hypothesis. To see the order of η_n^ϵ and η_{n+1}^ϵ, we have

$$\frac{\partial}{\partial\xi}(u_\epsilon(t,\xi)e^{-\xi})=\frac{\partial u_\epsilon/\partial\xi-u_\epsilon}{e^\xi}\leq 0$$

by Lemma 4.3. Since

$$u_\epsilon(t,\eta_n^\epsilon(t))e^{-\eta_n^\epsilon(t)}=\gamma_n>\gamma_{n+1}=u_\epsilon(t,\eta_{n+1}^\epsilon(t))e^{-\eta_{n+1}^\epsilon(t)},$$

this implies that $\eta_n^\epsilon(t)<\eta_{n+1}^\epsilon(t)$ for $t\geq 0$. ◻

Remark 6.1. If the approximated free boundaries η_n^ϵ, $n=1,2,\cdots,N$, converge to the free boundaries η_n, $n=1,2,\cdots,N$, then by Lemma 6.1, the domain could be divided into regions as $\{(t,\xi):\xi\le\eta_1(t)\}$, $\{(t,\xi):\xi>\eta_N(t)\}$ and for $n=1,2,\cdots,N-1$, $\{(t,\xi):\eta_n(t)<\xi\le\eta_{n+1}(t)\}$, or equivalently, $\{(t,x):x\le\lambda_1(t)\}$, $\{(t,x):x>\lambda_N(t)\}$ and for $n=1,2,\cdots,N-1$, $\{(t,x):\lambda_n(t)<x\le\lambda_{n+1}(t)\}$. Indeed, this convergence should be held, which will be claimed subsequently.

It is easy to see that the comparison principle in Proposition 3.1 is correct for $u(t,\xi)$ as well under corresponding assumptions, by which, the upper and lower bounds for the approximated free boundaries η_n^ϵ, $n=1,2,\cdots,N$, are obtained.

Lemma 6.2. *Let $\eta_n^\epsilon, n=1,2,\cdots,N$, be the approximated free boundaries defined by (6.1). Then for sufficiently small ϵ, $\eta_n^\epsilon, n=1,2,\cdots,N$, satisfy*

$$\kappa_\epsilon\le\eta_1^\epsilon<\eta_2^\epsilon<\cdots<\eta_{N-1}^\epsilon<\eta_N^\epsilon\le-\log\gamma_N,$$

where κ_ϵ is given as

$$\kappa_\epsilon\equiv\frac{\sigma_0^2}{2\delta}\log\frac{\beta_0}{\gamma_1}+\frac{\sigma_0^2}{2\delta}\log\left(1-\frac{2\delta}{\sigma_0^2}\right)+\left(1-\frac{\sigma_0^2}{2\delta}\right)\max\left\{\log\frac{1+\epsilon}{\gamma_n}-\eta_n^*,\ n=1,2,\cdots,N\right\},$$

β_0, η_n^*, $n=1,2,\cdots,N$, are given in Theorem 5.1.

Proof. Let u_ϵ be the solution of the approximated problem (4.2). Define

$$\xi_\epsilon=\max\left\{\log\frac{1+\epsilon}{\gamma_n}-\eta_n^*,\ n=1,2,\cdots,N\right\},$$

where η_n^*, $n=1,2,\cdots,N$, are the boundaries given in Theorem 5.1. Then the function $v_\epsilon(t,\xi)\triangleq\psi(\xi-\xi_\epsilon)$, where ψ is the solution of the problem (5.2)-(5.4), satisfies the following problem

$$\hat{\mathscr{L}}^\epsilon[v_\epsilon]\triangleq\frac{\partial v_\epsilon}{\partial t}-\frac{\hat\sigma_\epsilon^2}{2}\frac{\partial^2 v_\epsilon}{\partial\xi^2}-\left(\delta-\frac{\hat\sigma_\epsilon^2}{2}\right)\frac{\partial v_\epsilon}{\partial\xi}=0,$$

where

$$\hat\sigma_\epsilon(\xi)=\begin{cases}\sigma_0,\ \xi\le\xi_\epsilon+\eta_1^*,\\ \sigma_n,\ \xi_\epsilon+\eta_n^*<\xi\le\xi_\epsilon+\eta_{n+1}^*,\ n=1,2,\cdots,N-1,\\ \sigma_N,\ \xi_\epsilon+\eta_N^*<\xi.\end{cases}$$

For $\xi\le\xi_\epsilon+\eta_1^*$, it trivially holds that $\hat\sigma_\epsilon\ge\sigma_\epsilon$. Now consider $\xi\in(\xi_\epsilon+\eta_n^*,\xi_\epsilon+\eta_{n+1}^*]$, for which $\hat\sigma_\epsilon(\xi)=\sigma_n$, $n=1,2,\cdots,N$. Note that $\xi>\xi_\epsilon+\eta_n^*\ge\log(1+\epsilon)-\log\gamma_n$, thus it holds that

$$u_\epsilon(t,\xi)+\epsilon\le1+\epsilon=\gamma_n e^{\log(1+\epsilon)-\log\gamma_n}<\gamma_n e^\xi\le\gamma_k e^\xi$$

for all $k\le n$, namely that $u_\epsilon(t,\xi)-\gamma_k e^\xi<-\epsilon$ for all $k\le n$. The definition of σ_ϵ gives

$$\sigma_\epsilon\le\sigma_N+\sum_{k=n}^{N-1}(\sigma_k-\sigma_{k-1})=\sigma_n,$$

which shows that $\hat\sigma_\epsilon(\xi)\ge\sigma_\epsilon(t,\xi)$ for $\xi\in(\xi_\epsilon+\eta_n^*,\xi_\epsilon+\eta_{n+1}^*]$. Thus, it holds that $\hat\sigma_\epsilon\ge\sigma_\epsilon$. Let $w_\epsilon(t,\xi)\triangleq v_\epsilon(t,\xi)-M$, where $M>0$ is determined subsequently. Then w_ϵ satisfies $\hat{\mathscr{L}}^\epsilon[w_\epsilon]=0$. It holds that for $\xi\ge0$,

$$w_\epsilon(0,\xi)=v_\epsilon(0,\xi)-M<v_\epsilon(0,\xi)<1=u_\epsilon(0,\xi).$$

Since $\xi_\epsilon+\eta_1^*\geq 0$, then for $\xi\leq 0\leq \xi_\epsilon+\eta_1^*$, there holds

$$w_\epsilon(0,\xi)=\psi(\xi-\xi_\epsilon)-M=\beta_0 e^{\mu(\xi-\xi_\epsilon)}-M,$$

where β_0 is given in Theorem 5.1 and $\mu=1-2\delta/\sigma_0^2$. We need to choose M such that for $\xi\leq 0$,

$$w_\epsilon(0,\xi)=\beta_0 e^{\mu(\xi-\xi_\epsilon)}-M\leq e^{\xi}=u_\epsilon(0,\xi).$$

Define the function

$$f(\xi)=\beta_0 e^{\mu(\xi-\xi_\epsilon)}-e^{\xi}.$$

Then it satisfies $f(-\infty)=0$ and $f(0)=\beta_0 e^{-\mu\xi_\epsilon}-1$. Since $\beta_0=\gamma_1 e^{2\delta\eta_1^*/\sigma_0^2}$, it holds that

$$f(0)=\beta_0 e^{-\mu\xi_\epsilon}-1=\gamma_1 e^{2\delta\eta_1^*/\sigma_0^2-\mu\xi_\epsilon}-1.$$

By the definition of ξ_ϵ, we have

$$\gamma_1 e^{2\delta\eta_1^*/\sigma_0^2-\mu\xi_\epsilon}\leq \gamma_1 e^{2\delta\eta_1^*/\sigma_0^2+\mu\log\gamma_1+\mu\eta_1^*-\mu\log(1+\epsilon)}=\gamma_1 e^{\eta_1^*+\mu\log\gamma_1-\mu\log(1+\epsilon)}=\frac{\gamma_1^{1+\mu}e^{\eta_1^*}}{(1+\epsilon)^{\mu}}.$$

With the fact that $\gamma_1 e^{\eta_1^*}\leq 1$ and $\gamma_1<1$, it holds that $\gamma_1^{1+\mu}e^{\eta_1^*}(1+\epsilon)^{-\mu}<1$ and $f(0)<0$. Letting

$$f'(\xi)=\beta_0\mu e^{\mu(\xi-\xi_\epsilon)}-e^{\xi}=0,$$

we can solve that

$$\xi^*=\frac{\log\beta_0+\log\mu-\mu\xi_\epsilon}{1-\mu}.$$

Again by $\gamma_1 e^{\eta_1^*}\leq 1$ and $2\delta<\sigma_0^2$, it holds that $\beta_0=\gamma_1 e^{2\delta\eta_1^*/\sigma_0^2}<1$. Meanwhile, it holds that $\mu<1$ and $\xi_\epsilon>0$, which implies that $\xi^*<0$. It is easy to see that if

$$\xi<\hat{\xi}\triangleq\frac{\log\beta_0-\mu\xi_\epsilon}{1-\mu},$$

then $f(\xi)>0$. On the other hand, $f(-\infty)=0$. Thus, since $\xi^*<\hat{\xi}$, $f(\xi)$ attains its positive maximum at ξ^*. We take

$$M\equiv f(\xi^*)=\beta_0^{1/(1-\mu)}(\mu^{\mu/(1-\mu)}-\mu^{1/(1-\mu)})e^{-\mu\xi_\epsilon/(1-\mu)}.$$

Thus, we have shown that $w_\epsilon(0,\xi)\leq u_\epsilon(0,\xi)$. With the fact that $\hat{\sigma}_\epsilon\geq\sigma_\epsilon$ satisfying the conditions (3.7)-(3.8) in the statement of Proposition 3.1, it holds that $w_\epsilon(t,\xi)\leq u_\epsilon(t,\xi)$ for $-\infty<\xi<\infty$, $t>0$ by the comparison principle of Proposition 3.1, the proof of which can be referred to Theorem 3.1 in [21]. In particular, for $\xi\leq\xi_\epsilon+\eta_1^*$, $t>0$,

$$w_\epsilon(t,\xi)=\beta_0 e^{\mu(\xi-\xi_\epsilon)}-M.$$

The inequality $w_\epsilon(t,\xi)>\gamma_1 e^{\xi}$ is equivalent to

$$\beta_0 e^{\mu(\xi-\xi_\epsilon)}-M>\gamma_1 e^{\xi},$$

namely that

$$h(\xi) \triangleq \beta_0 e^{\mu(\xi-\xi_\epsilon)} - M - \gamma_1 e^{\xi} > 0.$$

Letting

$$h'(\xi) = \beta_0 \mu e^{\mu(\xi-\xi_\epsilon)} - \gamma_1 e^{\xi} = 0$$

gives

$$\xi^{**} = \frac{\log\beta_0 + \log\mu - \mu\xi_\epsilon - \log\gamma_1}{1-\mu}.$$

Then

$$h(\xi^{**}) = \beta_0^{1/(1-\mu)}(\mu^{\mu/(1-\mu)} - \mu^{1/(1-\mu)})(\gamma_1^{-\mu/(1-\mu)} - 1)e^{-\mu\xi_\epsilon/(1-\mu)} > 0.$$

Thus, we have

$$u_\epsilon(t,\xi^{**}) \geq w_\epsilon(t,\xi^{**}) > \gamma_1 e^{\xi^{**}},$$

which implies that $\{\xi \leq \xi^{**}\}$ is in the lowest rating region and $\eta_1^\epsilon(t) \geq \xi^{**}$. ∎

REMARK 6.2. If we consider the base case $N=1$ and set $(\gamma,\eta^*) = (\gamma_1,\eta_1^*)$, $\beta = \beta_0 = \gamma e^{2\delta\eta^*/\sigma_0^2}$ and $\alpha_\epsilon = \gamma e^{\eta^*}/(1+\epsilon)$, then with

$$\eta^* = \log\sigma_0^2(2\delta - \sigma_1^2) - \log 2\delta\gamma_1(\sigma_0^2 - \sigma_1^2),$$

it is found that

$$\kappa_\epsilon = \frac{1}{1-\beta}\log\alpha_\epsilon\beta\frac{(1+\epsilon)^{1-\beta}}{\gamma^{1-\beta}},$$

which agrees with the corresponding lower bound in Lemma 4.8 of [21], after correcting the proof there using the same definition of $\hat{\sigma}_\epsilon$ in Lemma 6.3.

LEMMA 6.3. *For any $T>0$, there exists $C_T>0$, independent of ϵ, such that the derivatives of the approximated free boundaries η_n^ϵ, $n=1,2,\cdots,N$, are bounded by*

$$-C_T \leq \frac{d\eta_n^\epsilon}{dt} \leq C_T \ \ for\ 0<t<T,\ n=1,2,\cdots,N.$$

Proof. Clearly, it holds that

$$\frac{d\eta_n^\epsilon}{dt} = \frac{\partial u_\epsilon(t,\eta_n^\epsilon(t))/\partial t}{u_\epsilon(t,\eta_n^\epsilon(t)) - \partial u_\epsilon(t,\eta_n^\epsilon(t))/\partial x},\ n=1,2,\cdots,N.$$

It follows from Lemma 4.4 that

$$-C_0 \leq \frac{\partial u_\epsilon}{\partial t}(t,\eta_n^\epsilon(t)) \leq \delta,\ t\geq 0,\ n=1,2,\cdots,N,$$

where $C_0>0$ is a constant independent of ϵ. To finish the proof, it is sufficient to establish a result

$$u_\epsilon(t,\eta_n^\epsilon(t)) - \frac{\partial u_\epsilon}{\partial x}(t,\eta_n^\epsilon(t)) \geq C_*$$

for some constant $C_*>0$ independent of ϵ. Let $\mathscr{L}_1^\epsilon$ be the operator defined in Lemma 4.3. As shown in Lemma 4.3, $w\equiv u_\epsilon-\partial u_\epsilon/\partial\xi$ satisfies $\mathscr{L}_1^\epsilon[w]=0$ and $w(0,\xi)=1$ for $\xi>0$, $w(0,\xi)=0$ for $\xi<0$. From Lemma 6.2, there exists $R>0$, independent of ϵ, such that

$$-R+1\le\eta_n^\epsilon(t)\le R-1,\ 0<t\le T,\ n=1,2,\cdots,N.$$

As in the proof of Lemma 4.4, there exists $\rho>0$, independent of ϵ, such that

$$u_\epsilon(t,\xi)>\frac{1+\gamma_1}{2}\ge\gamma_1e^\xi,\ |\xi|\le\rho,\ 0<t\le\rho^2,$$

which implies that $\{|\xi|\le\rho\}$ is in the lowest rating region. Then for $0\le t\le\rho^2$, we have

$$\eta_N^\epsilon(t)>\eta_{N-1}^\epsilon(t)>\cdots>\eta_2^\epsilon(t)>\eta_1^\epsilon(t)\ge\rho.$$

Consider the region

$$\Gamma=\left\{\frac{\rho}{2}<\xi<R,0<t<\rho^2\right\}\cup\{-R\le\xi\le R,\rho^2\le t\le T\}.$$

The parabolic boundary of the region Γ consists of five line segments. On the initial segment $\{(0,\xi):\rho/2\le\xi\le R\}$, it holds that $w(0,\xi)=1$. The segment $\{(t,R):0\le t\le T\}$ is completely in the highest rating region and the other segments $\{(t,\rho/2):0\le t\le\rho^2\}$, $\{(\rho^2,\xi):-R\le\xi\le\rho/2\}$ and $\{(t,-R):\rho^2\le t\le T\}$ are in the lowest rating region. Note that they are independent of ϵ. Thus, by compactness and strong maximum principle, on these four boundaries, $w\ge C>0$ for some constant independent of ϵ. It follows that $w\ge\min\{1,C\}\equiv C_*$ on the region Γ and completes the proof of the lemma. ∎

7. Existence and uniqueness for the problem

Lemmas 4.2-4.4 provide estimates for the approximated solution u_ϵ. By taking the limit as $\epsilon\to0$ along a subsequence if necessary, we derive the existence of solution for problem (3.10)-(3.14). Lemmas 6.1-6.3 show that there are uniform estimates in space $C^1([0,T])$ for approximated free boundaries η_n^ϵ, $n=1,2,\cdots,N$. Therefore, the limits of η_n^ϵ, $n=1,2,\cdots,N$, as $\epsilon\to0$ exist, which are denoted by η_n, $n=1,2,\cdots,N$. The η_n, $n=1,2,\cdots,N$, are the free boundaries of the problem (3.10)-(3.14). Recalling that $u(t,\xi)=e^{rt}\varphi(t,\xi)$, $\xi=x+ct$, $\eta_n(t)=\lambda_n(t)+ct$, $n=1,2,\cdots,N$, where $c=r-\delta$, we have the following theorem for existence immediately.

Theorem 7.1. *The free boundary problem (3.1)-(3.6) admits a solution $(\varphi,\lambda_n,n=1,2,\cdots,N)$ with $\varphi\in W_\infty^{1,2}([0,T]\times(-\infty,\infty)\setminus\overline{Q_\rho})\cap W_\infty^{0,1}([0,T]\times(-\infty,\infty))$ for any $\rho>0$, where $Q_\rho=(0,\rho^2)\times(-\rho,\rho)$ and $\lambda_n\in W^1([0,T])$. Furthermore, the solution satisfies*

$$\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial x}\le0,\ -\infty<x<\infty,\ 0<t\le T.$$

Meanwhile, the comparison principle in Proposition 3.1 implies the following theorem for uniqueness.

Theorem 7.2. *The solution $(\varphi,\lambda_n,n=1,2,\cdots,N)$ of the boundary problem (3.1)-(3.6) with $\varphi\in\{\cap_{\rho>0}W_\infty^{1,2}([0,T]\times(-\infty,\infty)\setminus\overline{Q_\rho})\}\cap W_\infty^{0,1}([0,T]\times(-\infty,\infty))$, $\lambda_n\in C([0,T])$ satisfying*

$$\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial\varphi}{\partial x}\le0,\ -\infty<x<\infty,\ 0<t\le T,$$

is unique.

8. Convergence to the traveling wave

8.1. Estimates for large space. Theorem 5.1 has shown the existence of a unique traveling wave ψ for the boundary problem (3.10)-(3.14). In this section, we devote to proving that when the risk discount rate δ satisfies (5.1), the solution of this boundary problem converges to this traveling wave as time tends to infinity. For the convergence, as $\psi(+\infty)=1$, $\psi(-\infty)=0$, we give the estimates in the infinity for u through u_ϵ by upper and lower solutions.

LEMMA 8.1. *Denote u_ϵ as the solution of (4.2) with initial condition (4.3). There hold*

$$0\le u_\epsilon(t,\xi)\le e^{(1-2\delta/\sigma_0^2)(\xi-\kappa_\epsilon)},\ \xi<\kappa_\epsilon,\ t>0,$$

and

$$1-e^{(1-2\delta/\sigma_N^2)(\xi-\log((1+\epsilon)/\gamma_N))}\le u_\epsilon(t,\xi)\le 1,\ \xi>\log\frac{1+\epsilon}{\gamma_N},\ t>0,$$

where κ_ϵ is given in Lemma 6.2.

Proof. Define the function f_u as

$$f_u(\xi)=e^{(1-2\delta/\sigma_0^2)(\xi-\kappa_\epsilon)}$$

on the region $(t,\xi)\in[0,\infty)\times(-\infty,\kappa_\epsilon)$. As $\eta_1^\epsilon(t)\ge\kappa_\epsilon$, this region is in the lowest rating region $u_\epsilon(t,\xi)\ge\gamma_1 e^\xi$ and thus $\sigma_\epsilon=\sigma_0$. Meanwhile, it holds that

$$\mathscr{L}^\epsilon[f_u](\xi)=-\frac{\sigma_0^2}{2}f_u''(\xi)-\left(\delta-\frac{\sigma_0^2}{2}\right)f_u'(\xi)=0,\ -\infty<\xi<\infty.$$

Recalling that $\delta<\sigma_0^2/2$ and $\kappa_\epsilon<0$, we have initially $f_u(\xi)>e^\xi=u_\epsilon(0,\xi)$ for $-\infty<\xi<\kappa_\epsilon$. When $\xi=\kappa_\epsilon$, $f_u(\kappa_\epsilon)=1>u_\epsilon(t,\kappa_\epsilon)$. By comparison principle, it holds that $f_u(\xi)\ge u_\epsilon(t,\xi)$ for $t>0$, $\xi<\kappa_\epsilon$. If $\xi>\log((1+\epsilon)/\gamma_N)$, then $u_\epsilon(t,\xi)<\gamma_N e^\xi-\epsilon$ and thus $\sigma_\epsilon=\sigma_N$. Define

$$f_l(\xi)=1-e^{(1-2\delta/\sigma_N^2)(\xi-\log((1+\epsilon)/\gamma_N))}$$

on the region $(t,\xi)\in[0,\infty)\times(\log((1+\epsilon)/\gamma_N),\infty)$. It holds that

$$\mathscr{L}^\epsilon[f_l]=-\frac{\sigma_N^2}{2}f_l''(\xi)-\left(\delta-\frac{\sigma_N^2}{2}\right)f_l'(\xi)=0,\ -\infty<\xi<\infty.$$

Similarly, we can derive that $f_l(\xi)\le u_\epsilon(t,\xi)$ for $\xi>\log((1+\epsilon)/\gamma_N)$, $t>0$. ∎

8.2. Estimates for large time. In this subsection, we consider the large-time behavior for the solution of the free boundary problem and establish the convergence to the traveling wave through constructing a Lyapunov function (see Zelenyak [29], Galaktionov [7]). Firstly we show the formal construction, namely that we ignore the integrability of any integral appearing in the construction, and then we verify the integrability of those integrals in the formal construction through the approximated solution. The formal construction of Lyapunov function is similar to the one by Liang et al. [21]. When associating with the solution of the model and considering the integrability of the integrals in the Lyapunov function, we need some corresponding modifications to fit the case of multiple credit ratings as well. We can see that the integrability depends on the highest and lowest volatilities corresponding to the lowest and highest credit ratings, but independent of volatilities in other credit ratings. This also extends the corresponding results for the case of two credit ratings by Liang et al. [21].

8.2.1. Formal construction of Lyapunov function. Let $\mathscr{G}(\xi,u,q)$ be a function to be determined and let

$$E[u](t)=\int_{-\infty}^{\infty}\mathscr{G}(\xi,u(t,\xi),u_\xi(t,\xi))d\xi. \tag{8.1}$$

Formally, assuming the integrability, we have

$$\begin{aligned}\frac{d}{dt}E[u](t)&=\int_{-\infty}^{\infty}(\mathscr{G}_u u_t+\mathscr{G}_q u_{\xi t})d\xi\\&=\int_{-\infty}^{\infty}u_t(\mathscr{G}_u-\mathscr{G}_{q\xi}-\mathscr{G}_{qu}u_\xi-\mathscr{G}_{qq}u_{\xi\xi})d\xi\\&=\int_{-\infty}^{\infty}u_t\left(\mathscr{G}_u-\mathscr{G}_{q\xi}-\mathscr{G}_{qu}u_\xi-\mathscr{G}_{qq}\left(\frac{2}{\sigma^2}u_t+\frac{2}{\sigma^2}\left(\frac{\sigma^2}{2}-\delta\right)u_\xi\right)\right)d\xi\\&=-\int_{-\infty}^{\infty}\frac{2}{\sigma^2}\mathscr{G}_{qq}u_t^2d\xi+\int_{-\infty}^{\infty}u_t\left(\mathscr{G}_u-\mathscr{G}_{q\xi}-\mathscr{G}_{qu}u_\xi-\mathscr{G}_{qq}\left(1-\frac{2\delta}{\sigma^2}\right)u_\xi\right)d\xi\\&=-\int_{-\infty}^{\infty}\frac{2}{\sigma^2}\mathscr{G}_{qq}u_t^2d\xi,\end{aligned}$$

provided we take $\mathscr{G}$ such that for all $-\infty<\xi<\infty$, $0\le u\le 1$ and $0\le q\le 1$,

$$\mathscr{G}_u(\xi,u,q)-\mathscr{G}_{q\xi}(\xi,u,q)-q\mathscr{G}_{qu}(\xi,u,q)-q\mathscr{G}_{qq}(\xi,u,q)\left(1-\frac{2\delta}{\sigma^2(u,\xi)}\right)=0. \tag{8.2}$$

We set $\rho(\xi,u,q)=\mathscr{G}_{qq}(\xi,u,q)$. Assuming $\mathscr{G}(\xi,u,0)=\mathscr{G}_q(\xi,u,0)=0$, we have

$$\int_0^q(q-m)\rho(\xi,u,m)dm=\int_0^q(q-m)d\mathscr{G}_q(\xi,u,m)=\int_0^q\mathscr{G}_q(\xi,u,m)dm=\mathscr{G}(\xi,u,q). \tag{8.3}$$

Thus, it holds that

$$\begin{aligned}\mathscr{G}_u&=\int_0^q(q-m)\rho_u(\xi,u,m)dm,\\\mathscr{G}_q&=\int_0^q\rho(\xi,u,m)dm,\\\mathscr{G}_{q\xi}&=\int_0^q\rho_\xi(\xi,u,m)dm,\\\mathscr{G}_{qu}&=\int_0^q\rho_u(\xi,u,m)dm,\end{aligned}$$

and

$$q\mathscr{G}_{qq}=q\rho(\xi,u,q)=\int_0^q\frac{d}{dm}(\rho(\xi,u,m)m)dm=\int_0^q(\rho(\xi,u,m)+\rho_q(\xi,u,m))dm.$$

Then (8.2) can be written as

$$\begin{aligned}&\int_0^q(q\rho_u(\xi,u,m)-m\rho_u(\xi,u,m)-\rho_\xi(\xi,u,m)-q\rho_u(\xi,u,m))dm\\&\quad-\int_0^q\left(1-\frac{2\delta}{\sigma^2(u,\xi)}\right)(\rho(\xi,u,m)+\rho_q(\xi,u.m)m)dm=0.\end{aligned}$$

Thus (8.2) is satisfied if for $-\infty<\xi<\infty$, $0\le u\le 1$ and $0\le m\le 1$, it holds that

$$m\rho_u(\xi,u,m)+\rho_\xi(\xi,u,m)+\left(1-\frac{2\delta}{\sigma^2(u,\xi)}\right)(\rho(\xi,u,m)+\rho_q(\xi,u,m)m)=0. \tag{8.4}$$

Formally, let $v(\xi;\xi_0,u_0,q_0)$ be a solution of the equation

$$-v_{\xi\xi}+\left(1-\frac{2\delta}{\sigma^2(v,\xi)}\right)v_\xi=0 \tag{8.5}$$

with conditions

$$v(\xi;\xi_0,u_0,q_0)|_{\xi=\xi_0}=u_0,\ v_\xi(\xi;\xi_0,u_0,q_0)|_{\xi=\xi_0}=q_0.$$

Then it holds that

$$\begin{aligned}
&\frac{d}{d\xi}\rho(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))\\
=&\rho_\xi(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))\\
&+\rho_u(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))v_\xi(\xi;\xi_0,u_0,q_0)\\
&+\rho_q(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))v_{\xi\xi}(\xi;\xi_0,u_0,q_0)\\
=&\rho_\xi(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))\\
&+\rho_u(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))v_\xi(\xi;\xi_0,u_0,q_0)\\
&+\rho_q(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0))v_\xi(\xi;\xi_0,u_0,q_0)\left(1-\frac{2\delta}{\sigma^2(v,\xi)}\right)\\
=&-\left(1-\frac{2\delta}{\sigma^2(v,\xi)}\right)\rho(\xi,v(\xi;\xi_0,u_0,q_0),v_\xi(\xi;\xi_0,u_0,q_0)),
\end{aligned}$$

where the last equality is due to (8.4). Thus, it holds that

$$\rho(\xi_0,u_0,q_0)=C(v(0;\xi_0,u_0,q_0),v_\xi(0;\xi_0,u_0,q_0))e^{-\int_0^{\xi_0}(1-2\delta/\sigma^2(v(\zeta;\xi_0,u_0,q_0),\zeta))d\zeta},$$

where $C(u,q)$ is an arbitrary function. Take $C(u,q)\equiv 1$. Replacing ξ_0 by ξ, u_0 by u and q_0 by q, we have

$$\rho(\xi,u,q)=e^{-\int_0^{\xi}(1-2\delta/\sigma^2(v(\zeta;\xi,u,q),\zeta))d\zeta}. \tag{8.6}$$

Integrating the Lyapunov function and assuming $E(t)\ge 0$, we obtain

$$\int_{t_0}^{T}\int_{-\infty}^{\infty}\frac{2}{\sigma^2}\rho u_t^2 d\xi dt=E(t_0)-E(t)\le E(t_0).$$

8.2.2. Lyapunov function through approximated solution. The formal process to use a Lyapunov function is shown in the last subsection. As explained in Liang et al. [21], there are two problems with this formal construction. Firstly, function ρ grows exponentially as $\xi\to\pm\infty$. Thus, it should be shown that the derivative of the solution with respect to time u_t converges exponentially to zero at a faster rate as $\xi\to\pm\infty$, such that the integration is valid. Secondly, the coefficient in (8.5) is discontinuous and the theory of ordinary differential equations (ODE) cannot be applied directly. To address these two issues, we should work on constructing a Lyapunov function for approximated solution u_ϵ with all estimates independent of ϵ. We begin this process by defining

$$E_\epsilon^R[u_\epsilon] = \int_{-R}^{R} \mathscr{G}_\epsilon\left(\xi, u_\epsilon(t,\xi), \frac{\partial u_\epsilon}{\partial \xi}(t,\xi)\right) d\xi,$$

for $R > 0$, where $\mathscr{G}_\epsilon$ is defined by

$$\mathscr{G}_\epsilon(\xi,u,q) = \int_0^q (q-m)\rho_\epsilon(\xi,u,m)dm,$$

satisfying $\mathscr{G}_\epsilon(\xi,u,0) = \partial\mathscr{G}_\epsilon/\partial q(\xi,u,0) = 0$, ρ_ϵ is defined by (8.6) with $\sigma(v(\zeta;\xi,u,q),\zeta) = \sigma_N + \sum_{n=0}^{N-1}(\sigma_n - \sigma_{n+1})H(v(\zeta;\xi,u,q) - \gamma_{n+1}e^\zeta)$ replaced by $\sigma_\epsilon(v_\epsilon(\zeta;\xi,u,q),\zeta) = \sigma_N + \sum_{n=0}^{N-1}(\sigma_n - \sigma_{n+1})H_\epsilon(v_\epsilon(\zeta;\xi,u,q) - \gamma_{n+1}e^\zeta)$, and $v_\epsilon(\zeta;\xi,u,q)$ is the solution of the following equation

$$-\frac{d^2 v_\epsilon}{d\zeta^2} + \left(1 - \frac{2\delta}{\sigma_\epsilon^2(v_\epsilon,\zeta)}\right)\frac{dv_\epsilon}{d\zeta} = 0, \tag{8.7}$$

with conditions

$$v_\epsilon(\zeta;\xi,u,q)|_{\zeta=\xi} = u, \quad \frac{dv_\epsilon}{d\zeta}(\zeta;\xi,u,q)|_{\zeta=\xi} = q.$$

Since

$$1 - \frac{2\delta}{\sigma_N^2} \le 1 - \frac{2\delta}{\sigma_\epsilon^2(v_\epsilon,\zeta)} \le 1 - \frac{2\delta}{\sigma_0^2},$$

it is easy to see from the theory of ODE that (8.7) can be solved for any given u, q, ξ on the real line $\zeta \in \mathbb{R}$. Thus $\mathscr{G}_\epsilon$ is well defined. According to the definition of $\mathscr{G}_\epsilon$ and following the process (8.2)-(8.6), it holds that

$$\frac{\partial\mathscr{G}_\epsilon}{\partial u} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q\partial\xi} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q\partial u}\frac{\partial u_\epsilon}{\partial\xi} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q^2}\frac{\partial u_\epsilon}{\partial\xi}\left(1 - \frac{2\delta}{\sigma_\epsilon^2(u_\epsilon,\xi)}\right) = 0.$$

Therefore, we have

$$\begin{aligned}
\frac{d}{dt}E_\epsilon^R[u_\epsilon](t) &= \int_{-R}^{R}\left(\frac{\partial\mathscr{G}_\epsilon}{\partial u}\frac{\partial u_\epsilon}{\partial t} + \frac{\partial\mathscr{G}_\epsilon}{\partial q}\frac{\partial^2 u_\epsilon}{\partial\xi\partial t}\right)d\xi \\
&= \frac{\partial\mathscr{G}_\epsilon}{\partial q}\frac{\partial u_\epsilon}{\partial t}\Big|_{-R}^{R} + \int_{-R}^{R}\frac{\partial u_\epsilon}{\partial t}\left(\frac{\partial\mathscr{G}_\epsilon}{\partial u} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q\partial\xi} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q\partial u}\frac{\partial u_\epsilon}{\partial\xi} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q^2}\frac{\partial^2 u_\epsilon}{\partial\xi^2}\right)d\xi \\
&= \frac{\partial\mathscr{G}_\epsilon}{\partial q}\frac{\partial u_\epsilon}{\partial t}\Big|_{-R}^{R} - \int_{-R}^{R}\frac{2}{\sigma_\epsilon^2}\frac{\partial^2\mathscr{G}_\epsilon}{\partial q^2}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 d\xi \\
&\quad + \int_{-R}^{R}\frac{\partial u_\epsilon}{\partial t}\left(\frac{\partial\mathscr{G}_\epsilon}{\partial u} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q\partial\xi} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q\partial u}\frac{\partial u_\epsilon}{\partial\xi} - \frac{\partial^2\mathscr{G}_\epsilon}{\partial q^2}\left(1 - \frac{2\delta}{\sigma_\epsilon^2}\right)\frac{\partial u_\epsilon}{\partial\xi}\right)\right)d\xi \\
&= \frac{\partial\mathscr{G}_\epsilon}{\partial q}\frac{\partial u_\epsilon}{\partial t}\Big|_{-R}^{R} - \int_{-R}^{R}\frac{2}{\sigma_\epsilon^2}\frac{\partial^2\mathscr{G}_\epsilon}{\partial q^2}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 d\xi \\
&= \frac{\partial\mathscr{G}_\epsilon}{\partial q}\frac{\partial u_\epsilon}{\partial t}\Big|_{-R}^{R} - \int_{-R}^{R}\frac{2}{\sigma_\epsilon^2}\rho_\epsilon\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 d\xi.
\end{aligned}$$

LEMMA 8.2. *Denote u_ϵ as the solution of (4.2) with initial condition (4.3). Then for any $K_1>0$, there exist K_2, $C_2>0$, independent of ϵ, such that*

$$\left|\frac{\partial u_\epsilon}{\partial \xi}\right|+\left|\frac{\partial u_\epsilon}{\partial t}\right|\le C_2 e^{K_2 t-K_1\xi},\ (t,\xi)\in(0,\infty)\times\left(\log\frac{1+\epsilon}{\gamma_N},\infty\right). \tag{8.8}$$

There exist K_3, $C_3>0$, independent of ϵ, such that

$$\left|\frac{\partial u_\epsilon}{\partial \xi}\right|+\left|\frac{\partial u_\epsilon}{\partial t}\right|\le C_3 e^{K_3 t+\xi},\ (t,\xi)\in(0,\infty)\times(-\infty,\kappa_\epsilon), \tag{8.9}$$

where κ_ϵ is given in Lemma 6.2.

Proof. Define the operators as follows

$$\begin{aligned}\mathscr{G}^\epsilon_{\sigma_N}[\cdot]&\triangleq\frac{\partial}{\partial t}-\frac{\sigma_N^2}{2}\frac{\partial^2}{\partial\xi^2}+\left(\frac{\sigma_N^2}{2}-\delta\right)\frac{\partial}{\partial\xi},\\ \mathscr{G}^\epsilon_{\sigma_0}[\cdot]&\triangleq\frac{\partial}{\partial t}-\frac{\sigma_0^2}{2}\frac{\partial^2}{\partial\xi^2}+\left(\frac{\sigma_0^2}{2}-\delta\right)\frac{\partial}{\partial\xi}.\end{aligned}$$

Then

$$\begin{aligned}&\mathscr{G}^\epsilon_{\sigma_N}[u_\epsilon]=\mathscr{G}^\epsilon_{\sigma_N}\left[\frac{\partial u_\epsilon}{\partial\xi}\right]=\mathscr{G}^\epsilon_{\sigma_N}\left[\frac{\partial u_\epsilon}{\partial t}\right]=0,\ (t,\xi)\in(0,\infty)\times\left(\log\frac{1+\epsilon}{\gamma_N},\infty\right),\\ &\mathscr{G}^\epsilon_{\sigma_0}[u_\epsilon]=\mathscr{G}^\epsilon_{\sigma_0}\left[\frac{\partial u_\epsilon}{\partial\xi}\right]=\mathscr{G}^\epsilon_{\sigma_0}\left[\frac{\partial u_\epsilon}{\partial t}\right]=0,\ (t,\xi)\in(0,\infty)\times(-\infty,\kappa_\epsilon).\end{aligned}$$

In Lemmas 4.3 and 4.4, we have already established

$$\sup_{0<t<\infty}\left(\left|\frac{\partial u_\epsilon}{\partial\xi}\right|+\left|\frac{\partial u_\epsilon}{\partial t}\right|\right)\Bigg|_{\xi=\log((1+\epsilon)/\gamma_N)}+\sup_{0<t<\infty}\left(\left|\frac{\partial u_\epsilon}{\partial\xi}\right|+\left|\frac{\partial u_\epsilon}{\partial t}\right|\right)\Bigg|_{\xi=\kappa_\epsilon}\le C,$$

where $C>0$ is a constant. On the other hand, we have

$$\begin{aligned}&\frac{\partial u_\epsilon}{\partial\xi}(0,\xi)=\frac{\partial u_\epsilon}{\partial t}(0,\xi)=0,\ \xi\in\left(\log\frac{1+\epsilon}{\gamma_N},\infty\right),\\ &\frac{\partial u_\epsilon}{\partial\xi}(0,\xi)=e^\xi,\ \frac{\partial u_\epsilon}{\partial t}(0,\xi)=\delta e^\xi,\ \xi\in(-\infty,\kappa_\epsilon).\end{aligned}$$

Then for any given K_1, there exist K_2, $C_2>0$ such that $\mathscr{G}^\epsilon_{\sigma_N}[C_2e^{K_2t-K_1\xi}]\ge 0$ for $(t,\xi)\in(0,\infty)\times(\log((1+\epsilon)/\gamma_N),\infty)$, which implies that

$$-C_2e^{K_2t-K_1\xi}\le\frac{\partial u_\epsilon}{\partial t}+\frac{\partial u_\epsilon}{\partial\xi}\le C_2e^{K_2t-K_1\xi},$$

and then (8.8). A similar application of the comparison principle claims that there exist K_3, $C_3>0$ such that (8.9) holds. ∎

LEMMA 8.3. *There exist C_1, C_2, C_3, C_4, C_5, $C_6>0$, independent of ϵ, such that ρ_ϵ satisfies*

$$C_1\le\rho_\epsilon(\xi,u,q)\le C_2,\ \kappa_\epsilon-\epsilon\le\xi\le\log\frac{1+\epsilon}{\gamma_N}+\epsilon,$$

$$C_3 e^{(2\delta/\sigma_N^2-1)\xi} \le \rho_\epsilon(\xi,u,q) \le C_4 e^{(2\delta/\sigma_N^2-1)\xi},\ \xi > \log\frac{1+\epsilon}{\gamma_N}+\epsilon,$$
$$C_5 e^{(2\delta/\sigma_0^2-1)\xi} \le \rho_\epsilon(\xi,u,q) \le C_6 e^{(2\delta/\sigma_0^2-1)\xi},\ \xi < \kappa_\epsilon - \epsilon,$$

$\mathscr{G}_\epsilon$ satisfies

$$\frac{C_1}{2}q^2 \le \mathscr{G}_\epsilon(\xi,u,q) \le \frac{C_2}{2}q^2,\ \kappa_\epsilon-\epsilon \le \xi \le \log\frac{1+\epsilon}{\gamma_N}+\epsilon,$$
$$\frac{C_3}{2}q^2 e^{(2\delta/\sigma_N^2-1)\xi} \le \mathscr{G}_\epsilon(\xi,u,q) \le \frac{C_4}{2}q^2 e^{(2\delta/\sigma_N^2-1)\xi},\ \xi > \log\frac{1+\epsilon}{\gamma_N}+\epsilon,$$
$$\frac{C_5}{2}q^2 e^{(2\delta/\sigma_0^2-1)\xi} \le \mathscr{G}_\epsilon(\xi,u,q) \le \frac{C_6}{2}q^2 e^{(2\delta/\sigma_0^2-1)\xi},\ \xi < \kappa_\epsilon - \epsilon,$$

$\partial\mathscr{G}_\epsilon/\partial q$ satisfies

$$C_1 q \le \frac{\partial \mathscr{G}_\epsilon}{\partial q}(\xi,u,q) \le C_2 q,\ \kappa_\epsilon-\epsilon \le \xi \le \log\frac{1+\epsilon}{\gamma_N}+\epsilon,$$
$$C_3 q e^{(2\delta/\sigma_N^2-1)\xi} \le \frac{\partial \mathscr{G}_\epsilon}{\partial q}(\xi,u,q) \le C_4 q e^{(2\delta/\sigma_N^2-1)\xi},\ \xi > \log\frac{1+\epsilon}{\gamma_N}+\epsilon,$$
$$C_5 q e^{(2\delta/\sigma_0^2-1)\xi} \le \frac{\partial \mathscr{G}_\epsilon}{\partial q}(\xi,u,q) \le C_6 q e^{(2\delta/\sigma_0^2-1)\xi},\ \xi < \kappa_\epsilon - \epsilon,$$

for $q \ge 0$, where κ_ϵ is given in Lemma 6.2.

Proof. We know from (8.6) that

$$\rho_\epsilon(\xi,u,q) = e^{-\int_0^\xi (1-2\delta/\sigma_\epsilon^2(v_\epsilon(\zeta;\xi,u,q),\zeta))d\zeta}.$$

For $\kappa_\epsilon - \epsilon \le \xi \le \log((1+\epsilon)/\gamma_N)+\epsilon$, it is easy to derive that there exist C_1, $C_2 > 0$ such that

$$C_1 \le \rho_\epsilon(\xi,u,q) \le C_2,\ \kappa_\epsilon-\epsilon \le \xi \le \log\frac{1+\epsilon}{\gamma_N}+\epsilon.$$

For $\xi > \log((1+\epsilon)/\gamma_N)+\epsilon$,

$$\begin{aligned}\rho_\epsilon(\xi,u,q) &= e^{(\int_0^{\log((1+\epsilon)/\gamma_N)+\epsilon} + \int_{\log((1+\epsilon)/\gamma_N)+\epsilon}^{\xi})(2\delta/\sigma_\epsilon^2(v_\epsilon(\zeta;\xi,u,q),\zeta)-1)d\zeta} \\ &= e^{\int_0^{\log((1+\epsilon)/\gamma_N)+\epsilon}(2\delta/\sigma_\epsilon^2(v_\epsilon(\zeta;\xi,u,q),\zeta)-1)d\zeta} e^{(2\delta/\sigma_N^2-1)(\xi-\log((1+\epsilon)/\gamma_N)-\epsilon)}.\end{aligned}$$

Thus there exist C_3, $C_4 > 0$ such that

$$C_3 e^{2\delta/\sigma_N^2-1} \le \rho_\epsilon(\xi,u,q) \le C_4 e^{2\delta/\sigma_N^2-1}$$

for $\xi > \log((1+\epsilon)/\gamma_N)+\epsilon$. Similarly, for $\xi < \kappa_\epsilon - \epsilon$, we have

$$\rho_\epsilon(\xi,u,q) = e^{\int_{\kappa_\epsilon-\epsilon}^{0}(1-2\delta/\sigma_\epsilon^2(v_\epsilon(\zeta;\xi,u,q),\zeta))d\zeta} e^{(2\delta/\sigma_0^2-1)(\xi-\kappa_\epsilon+\epsilon)},$$

which implies that there exist C_5, $C_6 > 0$, such that

$$C_5 e^{2\delta/\sigma_0^2-1} \le \rho_\epsilon(\xi,u,q) \le C_6 e^{2\delta/\sigma_0^2-1}$$

for $\xi<\kappa_\epsilon-\epsilon$. Moreover, since

$$\mathscr{G}_\epsilon(\xi,u,q)=\int_0^q(q-m)\rho_\epsilon(\xi,u,m)dm,\ \frac{\partial\mathscr{G}_\epsilon}{\partial q}=\int_0^q\rho_\epsilon(\xi,u,m)dm,$$

then applying the estimates for ρ_ϵ, we can derive the estimates for $\mathscr{G}_\epsilon$ and $\partial\mathscr{G}_\epsilon/\partial q$. □

Through the Lemmas 8.2 and 8.3, we have

$$C'e^{2K_2t-2K_1\xi+(2\delta/\sigma_N^2-1)\xi}\le\mathscr{G}_\epsilon\left(\xi,u_\epsilon(t,\xi),\frac{\partial u_\epsilon}{\partial\xi}(t,\xi)\right)\le C''e^{2K_2t-2K_1\xi+(2\delta/\sigma_N^2-1)\xi},$$

for $\xi>\log((1+\epsilon)/\gamma_N)+\epsilon$, and

$$C'e^{2K_3t+\xi+2\delta/\sigma_0^2\xi}\le\mathscr{G}_\epsilon\left(\xi,u_\epsilon(t,\xi),\frac{\partial u_\epsilon}{\partial\xi}(t,\xi)\right)\le C''e^{2K_3t+\xi+2\delta/\sigma_0^2\xi},$$

for $\xi<\kappa_\epsilon-\epsilon$, where C', C'' are fixed constants. This implies that

$$\lim_{R\to\infty}\int_{-R}^R\mathscr{G}_\epsilon\left(\xi,u_\epsilon(t,\xi),\frac{\partial u_\epsilon}{\partial\xi}(t,\xi)\right)d\xi=\int_{-\infty}^\infty\mathscr{G}_\epsilon\left(\xi,u_\epsilon(t,\xi),\frac{\partial u_\epsilon}{\partial\xi}(t,\xi)\right)d\xi.$$

Similar applications of Lemmas 8.2 and 8.3 lead to

$$\lim_{R\to\infty}\frac{\partial\mathscr{G}_\epsilon}{\partial q}(t,\pm R)\frac{\partial u_\epsilon}{\partial t}(t,\pm R)=0,$$
$$\lim_{R\to\infty}\int_{-R}^R\frac{2\rho_\epsilon}{\sigma_\epsilon^2}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2d\xi=\int_{-\infty}^\infty\frac{2\rho_\epsilon}{\sigma_\epsilon^2}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2d\xi.$$

Following the formal procedure with these results, we derive

$$\int_{t_0}^T\int_{-\infty}^\infty\frac{2\rho_\epsilon}{\sigma_\epsilon^2}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2d\xi dt\le E_\epsilon^\infty[u_\epsilon](t_0)\le C,$$

where C is independent of ϵ. By the estimates for ρ_ϵ in Lemma 8.3, we have

$$\frac{2\rho_\epsilon}{\sigma_\epsilon^2}=\frac{2\rho_\epsilon}{(\sigma_N+\sum_{n=0}^{N-1}(\sigma_n-\sigma_{n+1})H_\epsilon(u_\epsilon-\gamma_{n+1}e^\xi))^2}\ge C_0>0,$$

where C_0 is independent of ϵ. Then it holds that

$$\int_{t_0}^\infty\int_{-\infty}^\infty u_t^2(t,\xi)d\xi dt<\infty. \tag{8.10}$$

8.3. Convergence. Let $u^n(t,\xi)=u(t+n,\xi)$ and consider u^n as a sequence of functions on $[0,1]\times\mathbb{R}$. Since $u^n(t,\xi)$ is a bounded sequence in $W_\infty^{1,2}([0,1]\times\mathbb{R})$, we derive by embedding theorem that there exists a subsequence n_j of n and a function $\widetilde{\psi}(t,\xi)$ such that as $n_j\to\infty$,

$$u^{n_j}\to\widetilde{\psi}\text{ in }C^{(1+\alpha)/2,1+\alpha}([0,1]\times[-R,R]),\ 0<\alpha<1, \tag{8.11}$$

for any $R>1$. Furthermore, by taking a further subsequence if necessary, it holds that

$$u_t^{n_j}\xrightarrow{w*}\widetilde{\psi}_t,\ u_{\xi\xi}^{n_j}\xrightarrow{w*}\widetilde{\psi}_{\xi\xi}\text{ in }L^\infty([0,1]\times\mathbb{R}),$$

and thus

$$\|\widetilde{\psi}_t\|_{L^\infty}\le\liminf_{n\to\infty}\|u_t^{n_j}\|_{L^\infty}\le C,\ \ \|\widetilde{\psi}_{\xi\xi}\|_{L^\infty}\le\liminf_{n\to\infty}\|u_{\xi\xi}^{n_j}\|_{L^\infty}\le C.$$

Since (8.10) implies that

$$\int_0^1\int_{-\infty}^{\infty}(u_t^n)^2(t,\xi)d\xi dt=\int_n^{n+1}\int_{-\infty}^{\infty}u_t^2(t,\xi)d\xi dt\to 0 \text{ as } n=n_j\to\infty,$$

we have $\int_0^1\int_{-\infty}^{\infty}\widetilde{\psi}_t^2 d\xi dt=0$. It follows that $\widetilde{\psi}_t\equiv 0$, which implies that $\widetilde{\psi}(t,\xi)$ is independent of t and depends only on ξ. The following estimates on u_ϵ are then passed to u and then $\widetilde{\psi}$,

$$0\le\widetilde{\psi}(\xi)\le 1,\ \widetilde{\psi}_\xi(\xi)\ge 0,\ \widetilde{\psi}_\xi(\xi)-\widetilde{\psi}(\xi)\le 0,\ \widetilde{\psi}_{\xi\xi}(\xi)-\widetilde{\psi}_\xi(\xi)\le 0.$$

Now suppose that

$$\liminf_{n_j\to\infty}\min_{0\le t\le 1}\eta_i(t+n_j)=\underline{\eta}_i^*\le\overline{\eta}_i^*=\limsup_{n_j\to\infty}\max_{0\le t\le 1}\eta_i(t+n_j),\ i=1,2,\cdots,N.$$

We choose $\underline{t}_{i,j}$, $\overline{t}_{i,j}\in[0,1]$ such that $\min_{0\le t\le 1}\eta_i(t+n_j)=\eta_i(\underline{t}_{i,j}+n_j)$ and $\max_{0\le t\le 1}\eta_i(t+n_j)=\eta_i(\overline{t}_{i,j}+n_j)$. Taking subsequences along which liminf and limsup are achieved, together with the free boundary condition $u^n(\eta_i(t+n),t)=\gamma_i e^{\eta_i(t+n)}$ and (8.11), it is deduced that

$$\widetilde{\psi}(\underline{\eta}_i^*)=\gamma_i e^{\underline{\eta}_i^*},\ \widetilde{\psi}(\overline{\eta}_i^*)=\gamma_i e^{\overline{\eta}_i^*},\ i=1,2,\cdots,N.$$

We claim that $\underline{\eta}_i^*=\overline{\eta}_i^*$ for all $i=1,2,\cdots,N$. If this is not the case, then the following results

$$\frac{d}{d\xi}(e^{-\xi}\widetilde{\psi}(\xi))\le 0,\ e^{-\underline{\eta}_i^*}\widetilde{\psi}(\underline{\eta}_i^*)=e^{-\overline{\eta}_i^*}\widetilde{\psi}(\overline{\eta}_i^*)=\gamma_i,\ i=1,2,\cdots,N,$$

imply

$$\widetilde{\psi}(\xi)\equiv\gamma_i e^{\xi},\ \underline{\eta}_i^*<\xi<\overline{\eta}_i^*,\ i=1,2,\cdots,N.$$

Fixing some i, it is easy to check that u^n satisfies the following equation

$$u_{\xi\xi}^n-u_\xi^n=\frac{2(u_t^n-\delta u_\xi^n)}{(\sigma_N+\sum_{i=0}^{N-1}(\sigma_i-\sigma_{i+1})H(u^n-\gamma_{i+1}e^{\xi}))^2},\ \underline{\eta}_i^*<\xi<\overline{\eta}_i^*,\ 0\le t\le 1. \quad (8.12)$$

It is clear that $2u_t^n/(\sigma_N+\sum_{i=0}^{N-1}(\sigma_i-\sigma_{i+1})H(u^n-\gamma_{i+1}e^{\xi}))^2$ converges in L^2 to zero. By (8.11), u_ξ^n converges uniformly to $\widetilde{\psi}_\xi(\xi)=\gamma_i e^{\xi}$ for $\underline{\eta}_i^*<\xi<\overline{\eta}_i^*$ and hence for $n\gg 1$,

$$\frac{-2\delta u_\xi^n}{(\sigma_N+\sum_{i=0}^{N-1}(\sigma_i-\sigma_{i+1})H(u^n-\gamma_{i+1}e^{\xi}))^2}\le -\frac{\delta\gamma_i}{\sigma_0^2}e^{\xi},\ \underline{\eta}_i^*<\xi<\overline{\eta}_i^*,\ 0\le t\le 1.$$

The left-hand side of (8.12) converges weak $*$ in L^∞ to $\widetilde{\psi}_{\xi\xi}-\widetilde{\psi}_\xi$, which equals to zero for $\underline{\eta}_i^*<\xi<\overline{\eta}_i^*$. Then by taking a limit in (8.12) as $n=n_j\to\infty$, we obtain $0\le-\delta\gamma_i e^{\xi}/\sigma_0^2$, which leads to a contradiction. Hence, we have proved $\underline{\eta}_i^*=\overline{\eta}_i^*$ for all $i=1,2,\cdots,N$. It is clear $\widetilde{\psi}$ satisfies (5.2) and (5.3). By Lemma 8.1, $\widetilde{\psi}$ satisfies (5.4) as well. By the result of uniqueness, it holds that $\psi \equiv \widetilde{\psi}$, where ψ is the traveling wave in Theorem 5.1. Moreover, the uniqueness implies that all subsequence limit should be uniform and thus the full sequence must converge as $n \to \infty$.

THEOREM 8.1 (Convergence to traveling wave). *Let φ be the solution of the free boundary problem (3.1)-(3.6). Then $e^{rt}\varphi(t)$ converges uniformly to the traveling wave ψ, where ψ is the solution of (5.2)-(5.4). The wave speed is given as $c = \delta - r$, where δ satisfies $\sigma_N^2 < 2\delta < \sigma_0^2$.*

9. Conclusion and discussion

In this paper, we have studied the phenomenon of asymptotic traveling wave in the pricing model for corporate bond with multiple credit rating migration risk. The results in this paper extend the work of Liang et al. [21], where two credit ratings are considered. In mathematics, the existence, uniqueness and regularity of solution in the model are obtained, which verifies the rationality of the model. The traveling wave solution is established through the delicate application of inductive method. The form of the traveling wave is semi-explicit, since it is related to a nonlinear system of parameters. This is different from the corresponding result of Liang et al. [21], where the traveling wave solution is explicit. Such difference is caused by the multiplicity of credit ratings. The solvability of the nonlinear system of parameters implies the existence of the traveling wave solution. Then by constructing a Lyapunov function, it is shown that the solution of the model converges to the traveling wave solution. Interestingly, the existence and convergence condition of the traveling wave is that the risk discount rate is between the half squares of the highest and lowest volatilities, regardless of the volatilities in other credit ratings.

The problem shows not only its own interests in mathematics, such as a traveling wave with multiple free boundaries, but also shows some interpretations in finance. For instance, firstly, one can understand that the pricing solution will keep closer to a pattern along a certain direction at certain speed. Secondly, one can approximate the pricing solution by the traveling wave, which has a semi-explicit analyzing form and can be unfolded through some effective numeric solvers, when the time is far away from the maturity. Thirdly, one can learn that the speed of the traveling wave is the difference between the risk discount rate and the risk-free interest rate. Conclusively, the pattern of traveling wave in the pricing model could help realize more precise prediction for the price of the corporate bond under multiple credit rating migration.

In [27, 28], the authors consider models with stochastic change of interest rate. In particular, the model in [28] not only captures multiple credit rating migration but also involves stochastic interest rate. Therefore, there rises a question on traveling wave in a model with both multiple credit rating migration and stochastic interest rate, which needs to be addressed. We would address this problem in our following work.

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