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GENERALIZED KELVIN-VOIGT EQUATIONS FOR NONHOMOGENEOUS AND INCOMPRESSIBLE FLUIDS*

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Abstract. In this work, we consider the Kelvin-Voigt equations for non-homogeneous and incompressible fluid flows with the diffusion and relaxation terms described by two distinct power-laws. Moreover, we assume that the momentum equation is perturbed by an extra term, which, depending on whether its signal is positive or negative, may account for the presence of a source or a sink within the system. For the associated initial-boundary value problem, we study the existence of weak solutions as well as the large-time behavior of the solutions. In the case the extra term is a sink, we prove the global existence of weak solutions and we establish the conditions for the polynomial time decay and for the exponential time decay of these solutions. If the extra term is a source, we show how the exponents of nonlinearity must interact to ensure the local existence of weak solutions.

Keywords. Kelvin-Voigt equations; nonhomogeneous and incompressible fluids; power-laws; existence; large-time behavior.

AMS subject classifications. 35Q35; 76D05; 35Q30; 76D03; 35D30.

1. Introduction

Viscoelastic materials, as for instance polymers, are ones which can exhibit all intermediate range of properties between an elastic solid and a viscous fluid. These materials have some memory in the sense that they can come back to some previous state when the shear stress is removed. Pure viscous fluids have no memory, while elastic solids have perfect memory as long as it is within the yield stress conditions. The parameter that estimates the memory of materials is the dimensionless Deborah number, which expresses the ratio between the stress relaxation time and the time scale of observation. Low Deborah numbers always indicate fluid-like behaviour, whereas high Deborah numbers means solid-like response, the limits of the interval being zero (pure viscous fluid) and infinity (pure elastic solid). Simple constitutive relations describing the behaviour of linear viscoelastic materials are usually obtained by combining Hook's law of linear elasticity with Newton's law of viscosity. In these equations, stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\varepsilon}$ are related via a linear differential equation whose most general expression can be written in the form

$$a_0\boldsymbol{\sigma}+a_1\frac{d\boldsymbol{\sigma}}{dt}+\cdots+a_n\frac{d^n\boldsymbol{\sigma}}{dt^n}=b_0\boldsymbol{\varepsilon}+b_1\frac{d\boldsymbol{\varepsilon}}{dt}+\cdots+b_n\frac{d^n\boldsymbol{\varepsilon}}{dt^n}, \tag{1.1}$$

where a_0, a_1, $\dots$, a_n and b_0, b_1, $\dots$, b_n are nonnegative coefficients. This differential equation can be justified by using the Boltzmann superposition principle, or by considering mechanical models constructed using elastic springs and viscous dashpots. For instance, if the only nonzero coefficients in the differential Equation (1.1), are $a_0=1$,

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$a_1=\kappa$ and $b_1=2\mu$, where κ denotes the relaxation time and μ is the fluid viscosity, we obtain the following stress-strain relation, empirically suggested by Maxwell [19] in 1868 for the characterization of viscous fluids with elastic properties,

$$\boldsymbol{\sigma}+\kappa\frac{d\boldsymbol{\sigma}}{dt}=2\mu\frac{d\boldsymbol{\varepsilon}}{dt}.$$

If, on the other hand, the only nonzero coefficients in the differential Equation (1.1) are $a_0=1$, $b_0=E$ and $b_1=2\mu$, where E is the elastic modulus, then we get the following stress-strain relation, proposed by Kelvin [27] in 1865 for the description of elastic solids with viscous properties,

$$\boldsymbol{\sigma}=E\boldsymbol{\varepsilon}+2\mu\frac{d\boldsymbol{\varepsilon}}{dt}. \tag{1.2}$$

Departing from the Kelvin stress-strain relation (1.2), Voigt [29] derived in 1892 a system of equations describing the behavior of elastic solids with viscous properties, which is known today by the names of Voigt equations, Kelvin equations, or Kelvin-Voigt equations. Similar to the Kelvin-Voigt equations for viscoelastic bodies, Oskolkov [22] derived in 1973 the following system of equations for homogeneous and incompressible fluids with elastic properties,

$$\operatorname{div}\mathbf{v}=0, \tag{1.3}$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right)=\rho\mathbf{f}-\boldsymbol{\nabla}\pi+\mu\triangle\mathbf{v}+\kappa\triangle\mathbf{v}_t, \tag{1.4}$$

where $\mathbf{v}$ denotes the velocity field, π is the pressure, ρ is the constant density, and $\mathbf{f}$ is the external force field. The momentum Equation (1.4) can be justified by considering $a_0=1$, $b_1=2\mu$ and $b_2=2\kappa$ as the only nonzero coefficients in the general stress-strain relation (1.1), and taking

$$\frac{d\boldsymbol{\varepsilon}}{dt}=\mathbf{D}(\mathbf{v}):=\frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{\nabla}\mathbf{v}^T\right).$$

With this notation, the stress-strain relation from which Equations (1.3)-(1.4) were derived, can be written in the form

$$\boldsymbol{\sigma}=2\mu\mathbf{D}(\mathbf{v})+2\kappa\frac{\partial\mathbf{D}(\mathbf{v})}{\partial t}. \tag{1.5}$$

Oskolkov denoted system (1.3)-(1.4) as the Kelvin-Voigt equations for fluid flows with elastic properties, but neither Kelvin nor Voigt have suggested any stress-strain relation or system of governing equations for viscoelastic fluids. The name of Kelvin-Voigt fluids or just Kelvin fluids, or Voigt fuids, has appeared directly or indirectly in a vast literature of the last 50 years, in some cases even before the work of Oskolkov. For instance, in 1966, Ladyzhenskaya [13] has suggested these equations as a regularization to the 3-dimensional Navier-Stokes equations to ensure the existence of unique global solutions. In rheology, the first work using a stress-strain relation similar to (1.5), seems to have been due to Pavlovsky [21] who in 1971 has used this relation to model what he called weakly concentrated water-polymer mixtures. Motivated by the stress-strain relation (1.5), we shall consider, in the present work, the natural generalization of (1.5) for power-law fluids,

$$\boldsymbol{\sigma}=\mu|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v})+\kappa|\mathbf{D}(\mathbf{v})|^{q-2}\frac{\partial\mathbf{D}(\mathbf{v})}{\partial t}, \tag{1.6}$$

where p and q are constants ranging in the interval $(1,\infty)$. It is well-known that in the particular case of $\kappa=0$, (1.6) reduces to the stress-strain relation for shear-thinning fluids if $1<p<2$, Newtonian fluids if $p=2$ and shear-thickening fluids if $p>2$. If $\kappa\neq 0$, $q=2$ and p is assumed to range in the interval $(1,\infty)$, then (1.6) can be used to model generalized Newtonian fluids with elastic properties. Considering a nonhomogeneous and incompressible fluid flow for which stresses and strains are related by (1.6), then the corresponding system of governing equations is the following,

$$\begin{aligned}&\frac{\partial(\rho\mathbf{v})}{\partial t}+\operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v})\\ &=\rho\mathbf{f}-\boldsymbol{\nabla}\pi+\operatorname{div}\left(\mu|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v})+\varkappa|\mathbf{D}(\mathbf{v})|^{q-2}\frac{\partial\mathbf{D}(\mathbf{v})}{\partial t}\right)+\gamma|\mathbf{v}|^{m-2}\mathbf{v}.\end{aligned}\tag{1.7}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0,\tag{1.8}$$

$$\operatorname{div}\mathbf{v}=0.\tag{1.9}$$

In the momentum Equation (1.7), the extra term $\gamma|\mathbf{v}|^{m-2}\mathbf{v}$, with $\gamma\in(-\infty,\infty)$ and $m\in(1,\infty)$, accounts for a sink or a source within the system. If $\gamma<0$, this term describes a sink, whereas if $\gamma>0$ it depicts a source. The case $\gamma=0$, reduces the momentum Equation (1.7) to the usual case, *i.e.* without the presence of any source or sink. The presence of the source/sink term in the momentum equation can be justified in different applications of porous media flows and in continuous electromagnetic media. In porous media flows, the extra term $\gamma|\mathbf{v}|^{m-2}\mathbf{v}$ is known as the (generalized) Forchheimer term and it is important to characterize the resistance made by the rigid matrix of the porous medium to the flow, in particular when the pore Reynolds number exceeds 10 (see *e.g.* the handbook edited by Vafai [28]). For some quasi-stationary processes in crystalline semiconductors, the extra term $\gamma|\mathbf{v}|^{m-2}\mathbf{v}$ is of fundamental importance to model the density of sources or sinks of free electrons in the semiconductor lattice (see *e.g.* Al'shin *et al.* [1]). Being aware that neither Kelvin nor Voigt have suggested a model for fluid flows with elastic properties, we shall keep the name Kelvin-Voigt equations for the system (1.7)-(1.9) to maintain the uniformity of the nomenclature introduced by Oskolkov for this type of problems. In this work, we consider the Equations (1.7)-(1.8) satisfied in a general domain

$$Q_T:=\Omega\times(0,T),\qquad\text{with}\quad\Gamma_T:=\partial\Omega\times(0,T),$$

where $\Omega\subset\mathbb{R}^d$, with $d\geq 2$, is a bounded domain with its boundary denoted by $\partial\Omega$. It should be noted that, by using (1.8) and (1.9), Equations (1.7) and (1.8) can be written in the following nonconservative form,

$$\begin{aligned}&\rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right)\\ &=\rho\mathbf{f}-\boldsymbol{\nabla}\pi+\mathbf{div}\left(\mu|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v})+\varkappa|\mathbf{D}(\mathbf{v})|^{q-2}\frac{\partial\mathbf{D}(\mathbf{v})}{\partial t}\right)+\gamma|\mathbf{v}|^{m-2}\mathbf{v},\end{aligned}$$

$$\frac{\partial\rho}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\rho=0.$$

We supplement the system (1.7)-(1.8) with the following initial and boundary conditions

$$\rho\mathbf{v}=\rho_0\mathbf{v}_0\quad\text{and}\quad\rho=\rho_0\qquad\text{in }\Omega,\quad\text{when }t=0,\tag{1.10}$$

$$\mathbf{v}=\mathbf{0} \qquad \text{on } \Gamma_T. \tag{1.11}$$

The problem we study here is thus the following: given $\mathbf{v}_0$, ρ_0 and $\mathbf{f}$, to find $\mathbf{v}$, π and ρ satisfying to (1.7)-(1.11). This problem is very general and therefore encompasses many other situations of fluids modelling, some of them already mentioned above. For instance, if $\gamma=0$ and $p=q=2$ in the momentum Equation (1.7), we recover the Kelvin-Voigt model for nonhomogeneous and incompressible fluids, whose particular case $\rho=Constant$ was studied by Oskolkov [22], Ladyzhenskaya [13] and Pavlovsky [21]. On the other hand, Equations (1.7)-(1.11), in the case of $\varkappa=0$, $p=2$ and $\gamma=0$ in (1.7), have been used since the 1960s to describe flows of nonhomogeneous viscous incompressible fluids. First results of the modern mathematical theory for these fluids apparently were obtained by Antontsev, Kazhikov and Monakhov [2–4] and then developed by many authors, among whom were Ladyzhenskaya and Solonnikov [14], Lemoine [15], Lions [17] and Simon [25]. The equations for power-law fluids, *i.e.* the case of $\varkappa=0$ and $\gamma=0$, but with the possibility of $p\neq 2$ in (1.7), have been studied by Zhikov and Pastukhova [30,31]. Mathematical questions involving Kelvin-Voigt's equations for homogeneous incompressible viscous fluids, *i.e.* the case of $\varkappa\neq 0$, $\gamma=0$ and $p=q=2$ in (1.7), and constant ρ, were considered by Oskolkov [22] (see also [23,24]) and by Zvyagin and his collaborators (see [32] and the references cited therein). Still within the scope of these Kelvin-Voigt's equations (for homogeneous incompressible viscous fluids), we should mention the work by Korpusov and Sveshnikov [12] (see also Korpusov [11]), who apparently were the first to consider an extra source term in the momentum equation. For their model, which corresponds to taking $\rho=1$, $p=q=2$, $\mu=\varkappa=1$, $\mathbf{f}=\mathbf{0}$, $\gamma=1$, and $m=4$ in (1.7)-(1.11), the authors have proven the local existence of weak solutions and found sufficient conditions in the initial data for the global-in-time existence. They have also estimated, from above and below, the finite time blow-up of the solutions, whenever it occurs. A wide class of equations of Sobolev type with nonlinear sink or source terms were considered by the same authors in subsequent works (see Al'shin *et al.* [1] and the references cited therein). More recently, Antontsev and Khompysh [5,6] have addressed some mathematical issues for the Kelvin-Voigt equations in the case of homogeneous fluids and for general p. The unique solvability of the homogeneous Kelvin-Voigt equations with anisotropic diffusion, relaxation and damping is considered in [7]. To the authors' best knowledge, Kelvin-Voigt equations for nonhomogeneous incompressible fluids have not yet been considered in previous works.

The present paper is organized as follows. In Section 2 we introduce the function spaces and the notion of weak solutions we shall work with in this work. Section 3 is devoted to establish the global existence of weak solutions to our problem in the case of $q=2$ and $\gamma\leq 0$, whereas the case of $q=2$ and $\gamma>0$ is studied in Section 4. To perform this study, we construct the Galerkin approximations $\mathbf{v}_n$ and ρ_n and derive their first and second a priori estimates. Next, using compactness arguments together with the monotonicity method, we realize a passage to the limit as $n\to\infty$. In Section 5, we prove a local existence theorem for the case of $q=2$ and $\gamma>0$ and under conditions not covered by the result of Section 4. Section 6 is devoted to study the large-time behavior of the solutions to our problem. Here, we establish polynomial and exponential time-decay properties for the weak solutions that also hold for the case $q\neq 2$, assuming of course that such solutions exist.

2. Auxiliary results

In this section we introduce the necessary definitions and preliminary results to state the main results of this paper. For the definitions and notations of the function spaces used throughout the paper and for their properties, we address the reader to *e.g.*

the monographs [2, 4, 16–18, 26] cited in this work. We just fix the following notations for the functions spaces of mathematical fluid mechanics,

$$\begin{aligned}&\mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div} \mathbf{v} = 0\},\\ &\mathbf{H} := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{L}^2(\Omega),\\ &\mathbf{V}_p := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{W}^{1,p}(\Omega).\end{aligned}$$

If $p = 2$, we denote $\mathbf{V}_p$ simply by $\mathbf{V}$.

When establishing relations between the velocity gradient and its symmetric part, Korn's inequality plays an important role.

Lemma 2.1 (Korn's inequality). *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$. For any $p \in (1, \infty)$ and any function $\mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega)$ there exists a positive constant $C = C(p, \Omega)$ such that*

$$\frac{1}{C_K} \|\nabla \mathbf{v}\|_{p,\Omega} \leq \|\mathbf{D}(\mathbf{v})\|_{p,\Omega} \leq C_K \|\nabla \mathbf{v}\|_{p,\Omega}. \tag{2.1}$$

Proof. See *e.g.* [8, 10, 20]. □

The following nonlinear version of Grönwall's inequality will be used to establish important estimates when proving the local existence result in Section 5.

Lemma 2.2. *If $y : \mathbb{R}^+ \longrightarrow [0, \infty)$ is a continuous function such that*

$$y(t) \leq C_1 \int_0^t y^\mu(s) ds + C_2, \quad t \in \mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 and C_2, then

$$y(t) \leq C_2 \left(1 - (\mu - 1) C_1 C_2^{\mu-1} t\right)^{-\frac{1}{\mu-1}} \quad \text{for} \ \ 0 \leq t < t_{\max} := \frac{1}{(\mu - 1) C_1 C_2^{\mu-1}}.$$

Proof. See *e.g.* [2]. □

Next, we recall the following lemma which we shall use, in Section 5, to study the asymptotic behavior of the solutions to our problem.

Lemma 2.3. *Suppose that a positive, differentiable function $\Phi(t)$ satisfies for any $t > 0$ the inequality*

$$\Phi'(t) + C \Phi^\alpha(t) \leq 0,$$

where α and C are positive constants. Then the following estimates are valid:

(1) If $\alpha \in (0, 1)$, then

$$0 \leq \Phi(t) \leq \left(\Phi(0)^{1-\alpha} - (1 - \alpha) C t\right)_+^{\frac{1}{1-\alpha}} \quad \forall\ t \geq 0, \quad \text{where} \ \ f_+ = \max\{0, f\}.$$

(2) If $\alpha = 1$, then

$$\Phi(t) \leq \Phi(0) e^{-Ct} \quad \forall\ t \geq 0;$$

(3) If $\alpha > 1$, then

$$\Phi(t) \leq \Phi(0) \left(1 + t C (\alpha - 1) \Phi(0)^{\alpha-1}\right)^{-\frac{1}{\alpha-1}} \quad \forall\ t \geq 0.$$

Proof. See *e.g.* [2]. ∎

We now define the notion of weak solutions to the problem (1.7)-(1.11) we shall work with in this work.

DEFINITION 2.1. *Let $d\geq 2$, $1<q,\ p,\ m<\infty$ and assume that $\mathbf{f}\in\mathbf{L}^2(Q_T)$. A pair of functions $(\mathbf{v},\rho)$ is a weak solution to the problem (1.7)-(1.11), if:*

(1) $\mathbf{v}\in L^\infty(0,T;\mathbf{H}\cap\mathbf{V}_q)\cap L^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T)$;

(2) $\rho>0$ a.e. in Q_T, $\rho\in C([0,T];L^\lambda(\Omega))$ for all $\lambda\in[1,\infty)$ and $\rho|\mathbf{v}|^2\in L^\infty(0,T;L^1(\Omega))$;

(3) $\mathbf{v}(0)=\mathbf{v}_0$ and $\rho(0)=\rho_0$, with $\rho_0\geq 0$ a.e. in Ω;

(4) For every $\boldsymbol{\varphi}\in\mathcal{V}$ there holds for a.a. $t\in[0,T]$

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega \rho(t)\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x}+\frac{\varkappa}{q-1}\int_\Omega |\mathbf{D}(\mathbf{v}(t))|^{q-2}\mathbf{D}(\mathbf{v}(t)):\mathbf{D}(\boldsymbol{\varphi})\,d\mathbf{x}\right)\\ &\qquad+\mu\int_\Omega |\mathbf{D}(\mathbf{v}(t))|^{p-2}\mathbf{D}(\mathbf{v}(t)):\mathbf{D}(\boldsymbol{\varphi})\,d\mathbf{x}+\int_\Omega(\rho(t)\mathbf{v}(t)\cdot\boldsymbol{\nabla})\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x}\\ &=\int_\Omega\rho(t)\mathbf{f}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x}+\gamma\int_\Omega|\mathbf{v}(t)|^{m-2}\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,d\mathbf{x}.\end{aligned}\tag{2.2}$$

(5) For every $\phi\in\mathrm{C}_0^\infty(\Omega)$ there holds for a.a. $t\in[0,T]$

$$\frac{d}{dt}\int_\Omega\rho(t)\phi\,d\mathbf{x}+\int_\Omega\rho(t)\mathbf{v}(t)\cdot\boldsymbol{\nabla}\phi\,d\mathbf{x}=0.\tag{2.3}$$

REMARK 2.1. As usual, conditions $\mathbf{v}(0)=\mathbf{v}_0$ and $\rho(0)=\rho_0$ are interpreted in the following sense,

$$\lim_{t\to 0^+}\int_\Omega\mathbf{v}(t)\boldsymbol{\varphi}\,d\mathbf{x}=\int_\Omega\mathbf{v}_0\boldsymbol{\varphi}\,d\mathbf{x}\quad\forall\ \boldsymbol{\varphi}\in\mathcal{V},\qquad \lim_{t\to 0^+}\int_\Omega\rho(t)\phi\,d\mathbf{x}=\int_\Omega\rho_0\phi\,d\mathbf{x}\quad\forall\ \phi\in\mathrm{C}_0^\infty(\Omega).$$

3. Global existence: the case $\gamma\leq 0$

In this section, we shall only consider the case of

$$\gamma\leq 0\tag{3.1}$$

and

$$q=2.\tag{3.2}$$

The case $\gamma>0$ will be considered in Section 3. With respect to Assumption (3.2), it should be mentioned that existence results for generalized Kelvin-Voigt equations (with $q\neq 2$) are completely open. Despite the fact that existence of solutions is proved only in the case of $q=2$, many estimates and integral relations are proved for the case $q\neq 2$, since they are used in Section 6. For the results we aim to establish here, let us define the quantity

$$s:=\max\{q,p\},\tag{3.3}$$

which will resume to $s:=\max\{2,p\}$ in the case that (3.2) holds.

THEOREM 3.1. *Let M_1 and M_2, with $M_1\leq M_2$, be two positive constants such that*

$$0<M_1:=\inf_{\mathbf{x}\in\Omega}\rho_0(\mathbf{x})\leq\rho_0(\mathbf{x})\leq\sup_{\mathbf{x}\in\Omega}\rho_0(\mathbf{x})=:M_2<\infty\qquad\forall\ \mathbf{x}\in\overline{\Omega},\tag{3.4}$$

and let

$$\mathbf{v}_0 \in \mathbf{V} \cap \mathbf{V}_p \cap \mathbf{L}^m(\Omega), \tag{3.5}$$

$$\mathbf{f} \in \mathbf{L}^2(Q_T). \tag{3.6}$$

Assume, in addition to (3.1), (3.2), (3.4) and (3.5)-(3.6), that one of the following alternatives is fulfilled,

$$2 \le d \le 4 \qquad \text{and} \qquad p > 1, \tag{3.7}$$

$$d \ge 3 \qquad \text{and} \qquad p \ge \frac{d}{2}, \tag{3.8}$$

$$d \le m \qquad \text{and} \qquad \gamma \ne 0, \tag{3.9}$$

If

$$s > \frac{4d}{d+4} \tag{3.10}$$

then the problem (1.7)-(1.11) has, at least, a weak solution $(\mathbf{v}, \rho)$ *in the sense of Definition 2.1 in the cylinder* Q_T*.*

Moreover, the weak solutions to the problem (1.7)-(1.11) satisfy the following estimates,

$$0 < M_1 \le \rho(\mathbf{x},t) \le M_2 < \infty \qquad \forall\ (\mathbf{x},t) \in Q_T, \tag{3.11}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\mathbf{v}(t)\|_{2,\Omega}^2 + \|\nabla \mathbf{v}(t)\|_{2,\Omega}^2\right) + \|\nabla \mathbf{v}\|_{p,Q_T}^p + |\gamma| \|\mathbf{v}\|_{m,Q_T}^m \\ &\quad \le C_1 \left(\|\mathbf{v}_0\|_{2,\Omega}^2 + \|\nabla \mathbf{v}_0\|_{2,\Omega}^2 + \|\mathbf{f}\|_{2,Q_T}^2\right), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\nabla \mathbf{v}(t)\|_{p,\Omega}^p + |\gamma| \|\mathbf{v}(t)\|_{m,\Omega}^m\right) + \|\mathbf{v}_t\|_{2,Q_T}^2 + \|\nabla \mathbf{v}_t\|_{2,Q_T}^2 \\ &\quad \le C_2 \left(\|\nabla \mathbf{v}_0\|_{p,\Omega}^p + |\gamma| \|\mathbf{v}_0\|_{m,\Omega}^m + \|\mathbf{f}\|_{2,Q_T}^2 + 1\right), \end{aligned} \tag{3.13}$$

where C_1 *and* C_2 *are positive constants whose dependence is stated below at (3.33) and (3.51), respectively.*

Remark 3.1. The alternative conditions (3.7)-(3.9) play an important role to derive useful estimates of the time-dependent convective integral term written in (3.38) in Subsection 3.3.

Remark 3.2. In view of (3.2)-(3.3) and of (3.1), Assumption (3.10) assures that one of the following conditions holds,

$$p > \frac{4d}{d+4}, \tag{3.14}$$

$$2 \le d < 4. \tag{3.15}$$

Note that condition (3.14) is equivalent to $4 < p^*$, whereas (3.15) implies the equivalence $d < 4 \Leftrightarrow 4 < 2^*$, where by s^* we denote, as usual, the Sobolev conjugate exponent of s, i.e. $s^* = \frac{ds}{d-s}$ if $1 \le s < d$, s^* is any real number in the interval $[1, \infty)$ if $s = d$, and

$s^*=\infty$ if $s>d$. Conditions (3.14) and (3.15) are of the utmost importance to assure the convergence to zero of the integral terms X_n^{31} and X_n^{32} defined in Subsection 3.5.

REMARK 3.3. Using the information of (3.14)-(3.15), we can see that the alternative conditions (3.7)-(3.8) could be more simply written as follows,

$$2\leq d<4 \qquad \text{and} \qquad 1<p\leq 2,$$
$$d\geq 4 \qquad \text{and} \qquad p>2.$$

However there is no simplest way to write the case when (3.9) holds together with (3.10).

Proof. For the sake of comprehension, we shall split the proof of Theorem 3.1 into several steps.

3.1. Galerkin's approximations. We construct a solution to the problem (1.7)-(1.10) as a limit of the Galerkin approximations. Let $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ be an orthonormal family in $\mathbf{L}^2(\Omega)$ formed by functions of $\mathcal{V}$ whose linear combinations are dense in $\mathbf{V}\cap\mathbf{V}_p\cap\mathbf{L}^m(\Omega)$. It can be proved (see e.g. [16,18]), that if $\mathbf{v}_0\in\mathbf{V}\cap\mathbf{V}_p\cap\mathbf{L}^m(\Omega)$ (see Assumption (3.5)), then there exists a sequence $\{\mathbf{v}_{n,0}\}_{n\in\mathbb{N}}$ such that, for each $n\in\mathbb{N}$,

$$\mathbf{v}_{n,0}(\mathbf{x})=\sum_{k=1}^{n}c_{k,0}^n\boldsymbol{\psi}_k(\mathbf{x}), \qquad c_{k,0}^n\in\mathbb{R}, \tag{3.16}$$

and

$$\mathbf{v}_{n,0}\longrightarrow\mathbf{v}_0 \quad \text{strongly in } \mathbf{V}\cap\mathbf{V}_p\cap\mathbf{L}^m(\Omega), \quad \text{as } n\to\infty. \tag{3.17}$$

Thus, from (3.17), one easily has

$$\|\mathbf{v}_{n,0}\|_{2,\Omega}\leq\|\mathbf{v}_0\|_{2,\Omega}, \qquad \|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}\leq\|\boldsymbol{\nabla}\mathbf{v}_0\|_{2,\Omega}, \tag{3.18}$$
$$\|\mathbf{v}_{n,0}\|_{m,\Omega}\leq\|\mathbf{v}_0\|_{m,\Omega}, \qquad \|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{p,\Omega}\leq\|\boldsymbol{\nabla}\mathbf{v}_0\|_{p,\Omega}. \tag{3.19}$$

Given $n\in\mathbb{N}$, let us consider the corresponding n-dimensional space, say $\mathbf{V}^n$, spanned by $\boldsymbol{\psi}_1,\ \ldots,\ \boldsymbol{\psi}_n$. For each $n\in\mathbb{N}$, we search for approximate solutions

$$\mathbf{v}_n(\mathbf{x},t)=\sum_{k=1}^{n}c_k^n(t)\boldsymbol{\psi}_k(\mathbf{x}), \quad \boldsymbol{\psi}_k\in\mathbf{V}^n, \qquad \text{and} \quad \rho_n(\mathbf{x},t), \tag{3.20}$$

where the functions $c_1^n(t),\ldots,c_n^n(t)$ are obtained from the following system of ordinary differential equations derived from (2.2),

$$\begin{aligned}&\int_\Omega\rho_n(t)\left[\mathbf{v}_{n,t}(t)+(\mathbf{v}_n(t)\cdot\boldsymbol{\nabla})\mathbf{v}_n(t)\right]\cdot\boldsymbol{\psi}_j\,d\mathbf{x}\\&\quad+\varkappa\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{q-2}\mathbf{D}(\mathbf{v}_{n,t}(t)):\boldsymbol{\nabla}\boldsymbol{\psi}_j\,d\mathbf{x}+\mu\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{p-2}\mathbf{D}(\mathbf{v}_n(t)):\boldsymbol{\nabla}\boldsymbol{\psi}_j\,d\mathbf{x}\\&=\int_\Omega\rho_n\mathbf{f}(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}+\gamma\int_\Omega|\mathbf{v}_n(t)|^{m-2}\mathbf{v}_n(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}=0, \qquad \text{where} \quad \mathbf{v}_{n,t}:=\frac{\partial\mathbf{v}_n}{\partial t},\end{aligned} \tag{3.21}$$

for $k=1,\ 2,\ \ldots,\ n$, and where ρ_n satisfies the following equation in the classical sense,

$$\rho_{nt}+\mathbf{v}_n\cdot\boldsymbol{\nabla}\rho_n=0, \qquad \text{where} \quad \rho_{nt}:=\frac{\partial\rho_n}{\partial t}. \tag{3.22}$$

System (3.21)-(3.22) is supplemented with the following Cauchy data

$$c_k^n(0)=c_{k,0}^n \quad \text{and} \quad \rho_n(0)=\rho_0 \quad \text{in } \Omega, \tag{3.23}$$

where, for each $k\in\{1,\dots,n\}$, $c_{k,0}^n$ is given in (3.16).

For simplicity, we assume that $\rho_0\in \mathrm{C}^1(\overline{\Omega})$. Then we have

$$\rho_n(\mathbf{x},t)=\rho_0(y_n(\tau,\mathbf{x},t)\big|_{\tau=0}) \tag{3.24}$$

where y_n is the solution to the Cauchy problem

$$\frac{dy_n}{d\tau}=\mathbf{v}_n(y_n,\tau), \qquad y_n\big|_{\tau=t}=\mathbf{x}.$$

This problem has a unique solution y_n for $\mathbf{v}_n$ given by (3.20) with $c_k^n\in C([0,T])$. Moreover, according to (3.4) and (3.24), one has

$$0<M_1=\inf_{\mathbf{x}\in\overline{\Omega}}\rho_0(\mathbf{x})\le\rho_n(\mathbf{x},t)\le\sup_{\mathbf{x}\in\overline{\Omega}}\rho_0(\mathbf{x})=M_2<\infty \qquad \forall\ (\mathbf{x},t)\in Q_T. \tag{3.25}$$

See [3,14] for questions regarding the unique solvability of the problem formed by (3.22) and $(3.23)_2$.

On the other hand, system (3.21) can be written in the form

$$\mathbf{A}\frac{d\mathbf{c}}{dt}=\mathbf{b}, \quad \mathbf{c}(0)=\mathbf{c}_0, \tag{3.26}$$

where $\mathbf{A}=\{A_{jk}^n\}_{j,k=1}^n$, $\mathbf{c}=\{c_k^n\}_{k=1}^n$, $\mathbf{b}=\{b_j^n\}_{j=1}^n$, with

$$A_{jk}^n(t):=\int_\Omega \rho_n\boldsymbol{\psi}_j\cdot\boldsymbol{\psi}_k d\mathbf{x}+\varkappa\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{q-2}\mathbf{D}(\boldsymbol{\psi}_k):\boldsymbol{\nabla}\boldsymbol{\psi}_j\,d\mathbf{x},$$

$$\begin{aligned}b_j^n(t):=&\int_\Omega\rho_n\mathbf{f}(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}+\gamma\int_\Omega|\mathbf{v}_n(t)|^{m-2}\mathbf{v}_n(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}-\int_\Omega\rho_n(t)(\mathbf{v}_n(t)\cdot\boldsymbol{\nabla})\mathbf{v}_n(t)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}\\ &-\mu\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{p-2}\mathbf{D}(\mathbf{v}_n(t)):\boldsymbol{\nabla}\boldsymbol{\psi}_j\,d\mathbf{x},\end{aligned}$$

$$\mathbf{c}=(c_1^n,\ \dots,c_n^n) \qquad \text{and} \qquad \mathbf{c}_0=(c_1^n(0),\dots,c_n^n(0)).$$

Taking into account that the family $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ is linearly independent, then, in view of (3.25), $(\mathbf{A}\boldsymbol{\xi},\boldsymbol{\xi})>0$ for all $\boldsymbol{\xi}\in\mathbb{R}^n\setminus\{\mathbf{0}\}$. Thus, we can write (3.26) in the form

$$\frac{d\mathbf{c}}{dt}=\mathbf{A}^{-1}\mathbf{b}, \quad \mathbf{c}(0)=\mathbf{c}_0. \tag{3.27}$$

By Carathodory's theorem, the Cauchy problem (3.27) has, at least, a solution $\mathbf{c}$ in a neighborhood, say $(0,T_0)$, with $T_0>0$, of the initial condition.

3.2. First a priori estimate. Next, we need to derive an *a priori* estimate independent of n and valid for all $t\in(0,T)$. Multiplying (3.21) by $c_k^n(t)$ and summing with respect to k, from 1 to n, we obtain the relations

$$\int_\Omega\rho_n(t)\left[\mathbf{v}_{n,t}(t)+(\mathbf{v}_n(t)\cdot\boldsymbol{\nabla})\mathbf{v}_n(t)\right]\cdot\mathbf{v}_n(t)\,d\mathbf{x}$$
$$+\varkappa\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{q-2}\mathbf{D}(\mathbf{v}_{n,t}(t)):\mathbf{D}(\mathbf{v}_n(t))\,d\mathbf{x}+\mu\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^p\,d\mathbf{x}$$

$$=\int_\Omega \rho_n(t)\mathbf{f}(t)\cdot\mathbf{v}_n(t)\,d\mathbf{x}+\gamma\int_\Omega |\mathbf{v}_n(t)|^m\,d\mathbf{x}. \tag{3.28}$$

Using Equation (3.22) and the fact that $\operatorname{div}\mathbf{v}_n=0$, we derive from (3.28)

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho_n(t)|\mathbf{v}_n(t)|^2+\frac{\varkappa}{q}|\mathbf{D}(\mathbf{v}_n(t))|^q\right)d\mathbf{x}+\int_\Omega\left(\mu|\mathbf{D}(\mathbf{v}_n(t))|^p-\gamma|\mathbf{v}_n(t)|^m\right)d\mathbf{x}$$
$$=\int_\Omega \rho_n(t)\mathbf{f}(t)\cdot\mathbf{v}_n(t)\,d\mathbf{x}. \tag{3.29}$$

Integrating (3.29) between 0 and $t\in(0,T_0)$ and using (3.23) together with Korn's inequality (2.1) and with (3.25), we obtain

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|^2_{2,\Omega}+\frac{\varkappa}{qC_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|^q_{q,\Omega}+\int_0^t\left(\frac{\mu}{C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(s)\|^p_{p,\Omega}-\gamma\|\mathbf{v}_n(s)\|^m_{m,\Omega}\right)ds$$
$$\leq\frac{M_2}{2}\|\mathbf{v}_{n,0}\|^2_{2,\Omega}+\frac{\varkappa C_K}{q}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|^q_{q,\Omega}+\int_0^t\int_\Omega\rho_n(s)\mathbf{f}(s)\cdot\mathbf{v}_n(s)\,d\mathbf{x}ds. \tag{3.30}$$

where C_K is the Korn's inequality constant. Then, using Schwarz and Cauchy inequalities together with (3.25) on the last term of (3.30), we obtain, under the validity of Assumptions (3.1) and (3.2),

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|^2_{2,\Omega}+\frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|^2_{2,\Omega}+\int_0^t\left(\frac{\mu}{C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(s)\|^p_{p,\Omega}+|\gamma|\,\|\mathbf{v}_n(s)\|^m_{m,\Omega}\right)ds$$
$$\leq\frac{M_2}{2}\|\mathbf{v}_{n,0}\|^2_{2,\Omega}+\frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|^2_{2,\Omega}+\frac{M_2}{2}\left(\int_0^t\|\mathbf{f}(s)\|^2_{2,\Omega}ds+\int_0^t\|\mathbf{v}_n(s)\|^2_{2,\Omega}ds\right). \tag{3.31}$$

From (3.31), we can use Grönwall's inequality to show that

$$\|\mathbf{v}^n(t)\|^2_{2,\Omega}\leq C\left(\|\mathbf{v}_{n,0}\|^2_{2,\Omega}+\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|^2_{2,\Omega}+\|\mathbf{f}\|^2_{2,Q_T}\right),\quad C=C(M_1,M_2,\varkappa,C_K,t). \tag{3.32}$$

Thus, using (3.32) together with (3.18), we obtain from (3.31)

$$\sup_{t\in[0,T]}\left(\|\mathbf{v}_n(t)\|^2_{2,\Omega}+\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|^2_{2,\Omega}\right)+\|\boldsymbol{\nabla}\mathbf{v}_n\|^p_{p,Q_T}+|\gamma|\,\|\mathbf{v}_n\|^m_{m,Q_T}$$
$$\leq C\left(\|\mathbf{v}_0\|^2_{2,\Omega}+\|\boldsymbol{\nabla}\mathbf{v}_0\|^2_{2,\Omega}+\|\mathbf{f}\|^2_{2,Q_T}\right):=K_0,\qquad C=C(M_1,M_2,\varkappa,\mu,C_K,T). \tag{3.33}$$

Due to Assumptions (3.5)-(3.6), the right-hand side of (3.33) is finite and hence the solution to the Cauchy problem (3.27) found in the previous step can be continued to the entire interval $[0,T]$.

In particular, from (3.33), one has

$$\sup_{t\in[0,T]}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|^2_{2,\Omega}\leq K_0. \tag{3.34}$$

3.3. Second a priori estimate. Let us now multiply (3.21) by $\frac{dc^n_k(t)}{dt}$ and summing with respect to k from 1 to n, we obtain

$$\int_\Omega\rho_n(t)|\mathbf{v}_{n,t}(t)|^2d\mathbf{x}+\int_\Omega\rho_n(t)(\mathbf{v}_n(t)\cdot\boldsymbol{\nabla})\mathbf{v}_n(t)\cdot\mathbf{v}_{n,t}(t)d\mathbf{x}$$
$$+\varkappa\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{q-2}|\mathbf{D}(\mathbf{v}_{n,t}(t))|^2\,d\mathbf{x}+\mu\int_\Omega|\mathbf{D}(\mathbf{v}_n(t))|^{p-2}\mathbf{D}(\mathbf{v}_n(t)):\mathbf{D}(\mathbf{v}_{n,t}(t))\,d\mathbf{x}$$

$$=\int_\Omega \rho_n(t)\mathbf{f}(t)\cdot\mathbf{v}_{n,t}(t)\,d\mathbf{x}+\gamma\int_\Omega |\mathbf{v}_n(t)|^{m-2}\mathbf{v}_n(t)\cdot\mathbf{v}_{n,t}(t)\,d\mathbf{x}. \tag{3.35}$$

After some calculations on (3.35) and using (3.25), we arrive, in the case that (3.1) and (3.2) hold, at the relation

$$\frac{d}{dt}\left(\frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|_{p,\Omega}^p+\frac{|\gamma|}{m}\|\mathbf{v}_n(t)\|_{m,\Omega}^m\right)+M_1\|\mathbf{v}_{n,t}(t)\|_{2,\Omega}^2+\varkappa\|\mathbf{D}(\mathbf{v}_{n,t}(t))\|_{2,\Omega}^2$$
$$\leq I(t)+\int_\Omega \rho_n(t)\mathbf{f}(t)\cdot\mathbf{v}_{n,t}(t)\,d\mathbf{x}, \tag{3.36}$$

where

$$I(t):=-\int_\Omega \rho_n(t)\left(\mathbf{v}_n(t)\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n(t)\cdot\mathbf{v}_{n,t}(t)\,d\mathbf{x}.$$

To estimate the last term of (3.36), we use Schwarz and Cauchy inequalities together with (3.25) as follows

$$\left|\int_\Omega \rho_n(t)\mathbf{f}(t)\cdot\mathbf{v}_{n,t}(t)\,d\mathbf{x}\right|\leq\frac{\varepsilon M_2}{2}\|\mathbf{v}_{n,t}(t)\|_{2,\Omega}^2+\frac{M_2}{2\varepsilon}\|\mathbf{f}(t)\|_{2,\Omega}^2. \tag{3.37}$$

Let us estimate $I(t)$ in the case of $d>2$ (the case $d=2$ is easier). Using Hölder's inequality together with (3.25), one has

$$|I(t)|\leq M_2\|\mathbf{v}_{n,t}(t)\|_{2^*,\Omega}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}\|\mathbf{v}_n(t)\|_{d,\Omega},\qquad \frac{1}{2^*}+\frac{1}{2}+\frac{1}{d}=1. \tag{3.38}$$

If $d\leq 4$ (Assumption (3.7)), we can use Sobolev and Korn inequalities so that

$$|I(t)|\leq C_1\|\mathbf{D}(\mathbf{v}_{n,t}(t))\|_{2,\Omega}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2, \tag{3.39}$$

where $C_1=C(M_2,C_K,d,\Omega)$ is a positive constant. Integrating (3.39) between 0 and $t\in(0,T)$, we have by virtue of (3.34) and Cauchy's inequality,

$$\int_0^t |I(s)|\,ds\leq C_2\int_0^t \|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}\,ds\leq\frac{\varkappa}{2}\int_0^t\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}^2\,ds+C_3, \tag{3.40}$$

where $C_2=C(C_1,K_0)$ and $C_3=C(C_2,\varkappa,t)$ are positive constants. Integrating (3.36) between 0 and $t\in(0,T)$, plugging (3.37) and (3.39) into the resulting inequality, choosing $\varepsilon=\frac{M_1}{M_2}$, and using (3.40) together with (3.19), we arrive at the estimate

$$\frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|_{p,\Omega}^p+\frac{|\gamma|}{m}\|\mathbf{v}_n(t)\|_{m,\Omega}^m+\int_0^t\left(\frac{M_1}{2}\|\mathbf{v}_{n,t}(s)\|_{2,\Omega}^2+\frac{\varkappa}{2}\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}^2\right)ds$$
$$\leq\frac{\mu}{p}\|\boldsymbol{\nabla}\mathbf{v}_0\|_{p,\Omega}^p+\frac{|\gamma|}{m}\|\mathbf{v}_0\|_{m,\Omega}^m+\frac{M_2^2}{2M_1}\int_0^t\|\mathbf{f}(s)\|_{2,\Omega}^2ds+C_3. \tag{3.41}$$

If $\frac{d}{2}\leq p<d$, we use Sobolev and Korn inequalities in (3.38) in such a way that

$$|I(t)|\leq C_1\|\mathbf{D}(\mathbf{v}_{n,t}(t))\|_{2,\Omega}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{p,\Omega}, \tag{3.42}$$

where $C_1=C(M_2,C_K,d,\Omega)$ is a positive constant.

If $p=d$, we have by Sobolev's inequality that

$$\|\mathbf{v}_n(t)\|_{s,\Omega} \leq C\|\nabla \mathbf{v}_n(t)\|_{p,\Omega} \qquad \text{for arbitrary} \quad s\in[1,\infty) \tag{3.43}$$

and for some positive constant $C=C(s,p,d,\Omega)$. In particular, we may take $s=d$ in (3.43) so that, by virtue of (3.38), we obtain again the same type of inequality (3.42).

If $p>d$, we use Morrey's inequality before the Hölder, Sobolev and Korn inequalities to obtain the following alternative of (3.38) and (3.42),

$$\begin{aligned} |I(t)| \leq & M_2\|\mathbf{v}_n(t)\|_{C^{0,1-\frac{d}{p}}(\Omega)} \int_\Omega |\mathbf{v}_{n,t}(t)||\nabla \mathbf{v}_n(t)|\,d\mathbf{x} \\ \leq & C_1\|\nabla \mathbf{v}_n(t)\|_{p,\Omega}\|\mathbf{D}(\mathbf{v}_{n,t}(t))\|_{2,\Omega}\|\nabla \mathbf{v}_n(t)\|_{2,\Omega} \end{aligned} \tag{3.44}$$

for some positive constant $C_1=C(M_2,C_K,d,p,\Omega)$.

If, in addition, $p\leq 2$, then from (3.42) we also get (3.39) for some positive constant $C_1=C(M_2,p,C_K,d,\Omega)$ and the proof follows as in the case of $d\leq 4$. If otherwise $p>2$, we argue as we did for (3.40), in particular using (3.34), so that, by virtue of (3.42) or (3.44), one has

$$\begin{aligned} \int_0^t |I(s)|\,ds \leq & C_2\int_0^t \|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}\|\mathbf{D}(\mathbf{v}_n(s))\|_{p,\Omega}\,ds \\ \leq & \frac{\varkappa}{2}\int_0^t \|\mathbf{D}(\mathbf{v}_{n,t}(s))\|^2_{2,\Omega}\,ds + C_3\int_0^t \|\mathbf{D}(\mathbf{v}_n(s))\|^p_{p,\Omega}\,ds + C_4, \end{aligned} \tag{3.45}$$

where $C_2=C(C_1,K_0)$, $C_3=C(C_2,\varkappa)$ and $C_4=C(C_2,\varkappa,T)$ are positive constants. Proceeding as we did for (3.41), but now using (3.45), we derive

$$\begin{aligned} &\frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|^p_{p,\Omega} + \frac{|\gamma|}{m}\|\mathbf{v}_n(t)\|^m_{m,\Omega} + \int_0^t \left(\frac{M_1}{2}\|\mathbf{v}_{n,t}(s)\|^2_{2,\Omega} + \frac{\varkappa}{2}\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|^2_{2,\Omega}\right)ds \\ \leq & \frac{\mu}{p}\|\nabla \mathbf{v}_0\|^p_{p,\Omega} + \frac{|\gamma|}{m}\|\mathbf{v}_0\|^m_{m,\Omega} + \frac{M_2^2}{2M_1}\int_0^t \|\mathbf{f}(s)\|^2_{2,\Omega}ds + C_3\int_0^t \|\mathbf{D}(\mathbf{v}_n(s))\|^p_{p,\Omega}\,ds + C_4. \end{aligned} \tag{3.46}$$

Thus, combining (3.46) with Grönwall's inequality for the function

$$y(t) = \frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|^p_{p,\Omega} + \frac{|\gamma|}{m}\|\mathbf{v}_n(t)\|^m_{m,\Omega},$$

one has

$$\begin{aligned} &\|\mathbf{D}(\mathbf{v}_n(t))\|^p_{p,\Omega} + \frac{|\gamma|}{m}\|\mathbf{v}_n(t)\|^m_{m,\Omega} + \int_0^t \left(\frac{M_1}{2}\|\mathbf{v}_{n,t}(s)\|^2_{2,\Omega} + \frac{\varkappa}{2}\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|^2_{2,\Omega}\right)ds \\ \leq & C\left(\frac{\mu}{p}\|\nabla \mathbf{v}_0\|^p_{p,\Omega} + \frac{|\gamma|}{m}\|\mathbf{v}_0\|^m_{m,\Omega} + \frac{M_2^2}{2M_1}\int_0^t \|\mathbf{f}(s)\|^2_{2,\Omega}ds + 1\right) \end{aligned} \tag{3.47}$$

for some positive constant $C=C(C_2,\varkappa,T)$.

Assume now that $\gamma\neq 0$. In this case, if $d\leq m$ (Assumption (3.9)), we use the Sobolev, Korn and Hölder inequalities in (3.38) so that

$$|I(t)| \leq C_1\|\mathbf{D}(\mathbf{v}_{n,t}(t))\|_{2,\Omega}\|\nabla \mathbf{v}_n(t)\|_{2,\Omega}\|\mathbf{v}_n(t)\|_{m,\Omega}, \tag{3.48}$$

where $C_1=C(M_2,m,C_K,d,\Omega)$ is a positive constant. Using (3.48) and arguing as we did to derive (3.40) and (3.45), in particular using (3.34), one has

$$\begin{aligned}\int_0^t |I(s)|\,ds \le & C_2\int_0^t \|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}\|\mathbf{v}_n(s)\|_{m,\Omega}\,ds\\ &\le \frac{\varkappa}{2}\int_0^t \|\mathbf{D}(\mathbf{v}_{n,t}(s))\|^2_{2,\Omega}\,ds + C_3\int_0^t \|\mathbf{v}_n(s)\|^m_{m,\Omega}\,ds + C_4,\end{aligned} \tag{3.49}$$

where $C_2=C(C_1,K_0)$, $C_3=C(C_2,\varkappa)$ and $C_4=C(C_2,\varkappa,T)$ are positive constants. Note that if $m\ge d$, then for sure $m\ge 2$. Using now (3.49), we obtain the counterpart of (3.46),

$$\begin{aligned}&\frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|^p_{p,\Omega}+\frac{|\gamma|}{m}\|\mathbf{v}_n(t)\|^m_{m,\Omega}+\int_0^t\left(\frac{M_1}{2}\|\mathbf{v}_{n,t}(s)\|^2_{2,\Omega}+\frac{\varkappa}{2}\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|^2_{2,\Omega}\right)ds\\ \le&\frac{\mu}{p}\|\boldsymbol{\nabla}\mathbf{v}_0\|^p_{p,\Omega}+\frac{|\gamma|}{m}\|\mathbf{v}_0\|^m_{m,\Omega}+\frac{M_2^2}{2M_1}\int_0^t\|\mathbf{f}(s)\|^2_{2,\Omega}ds+C_3\int_0^t\|\mathbf{v}_n(s)\|^m_{m,\Omega}ds+C_4.\end{aligned} \tag{3.50}$$

Arguing as we did for (3.47), we can show that (3.50) implies the same type inequality of (3.47), where now the positive constant C depends also on m instead of p.

In any case, taking the supremum in $[0,T]$ of (3.41), or of (3.47), and using Korn's inequality (2.1), we obtain

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|^p_{p,\Omega}+|\gamma|\|\mathbf{v}_n(t)\|^m_{m,\Omega}\right)+\|\mathbf{v}_{n,t}\|^2_{2,Q_T}+\|\boldsymbol{\nabla}\mathbf{v}_{n,t}\|^2_{2,Q_T}\\ &\le C\left(\|\boldsymbol{\nabla}\mathbf{v}_0\|^p_{p,\Omega}+|\gamma|\|\mathbf{v}_0\|^m_{m,\Omega}+\|\mathbf{f}\|^2_{2,Q_T}+1\right):=K_1,\end{aligned} \tag{3.51}$$

where $C=C(M_1,M_2,C_K,\varkappa,\mu,m,p,K_0,d,\Omega,T)$ is a positive constant. Again, due to the Assumptions (3.5)-(3.6), the right-hand side of (3.51) is finite.

3.4. Passage to the limit as $n\to\infty$

By means of reflexivity and up to some subsequences, estimate (3.33) implies that

$$\boldsymbol{\nabla}\mathbf{v}_n \longrightarrow \boldsymbol{\nabla}\mathbf{v} \quad \text{weakly in } \mathbf{L}^p(Q_T)\cap\mathbf{L}^2(Q_T), \quad \text{as } n\to\infty, \tag{3.52}$$

$$\mathbf{v}_n \longrightarrow \mathbf{v} \quad \text{weakly in } \mathbf{L}^m(Q_T)\cap\mathbf{L}^2(Q_T), \quad \text{as } n\to\infty \tag{3.53}$$

and estimate (3.51) assures that

$$\mathbf{v}_{n,t}\longrightarrow\mathbf{v}_t \quad \text{weakly in } \mathbf{L}^2(0,T;\mathbf{H}\cap\mathbf{V}), \quad \text{as } n\to\infty. \tag{3.54}$$

In particular, (3.33) and (3.51) allow us to use Korn's inequality (2.1) so that we can infer, for some subsequences,

$$|\mathbf{D}(\mathbf{v}_n)|^{p-2}\mathbf{D}(\mathbf{v}_n)\longrightarrow\mathbf{S} \quad \text{weakly in } \mathbf{L}^{p'}(Q_T), \quad \text{as } n\to\infty, \tag{3.55}$$

$$\mathbf{D}(\mathbf{v}_{n,t})\longrightarrow\mathbf{D}(\mathbf{v}_t) \quad \text{weakly in } \mathbf{L}^2(Q_T), \quad \text{as } n\to\infty. \tag{3.56}$$

Estimate (3.33) also implies that, for some subsequence, $|\mathbf{v}_n|^{m-2}\mathbf{v}_n\longrightarrow\mathbf{w}$ weakly in $\mathbf{L}^{m'}(Q_T)$, as $n\to\infty$. On the other hand, from the estimates (3.33) and (3.51), we have

$$\begin{aligned}&\mathbf{v}_n \ \text{ is uniformly bounded in } \ L^s(0,T;\mathbf{W}^{1,s}_0(\Omega)),\\ &\mathbf{v}_{n,t} \ \text{ is uniformly bounded in } \ \mathbf{L}^2(Q_T),\end{aligned}$$

for the exponent s defined in (3.3). Then, due to the following compact and continuous imbedding,

$$\mathbf{W}_0^{1,s}(\Omega)\hookrightarrow\hookrightarrow\mathbf{L}^r(\Omega)\hookrightarrow\mathbf{L}^2(\Omega), \qquad \forall\, r:2\leq r<s^*,$$

we can apply Aubin-Lions compactness lemma so that, for some subsequence,

$$\mathbf{v}_n\longrightarrow\mathbf{v} \quad \text{strongly in } L^s(0,T;\mathbf{L}^r(\Omega)), \text{ with } 1\leq r<s^*, \quad \text{as } n\to\infty. \tag{3.57}$$

As a consequence of (3.57) and Riesz-Fischer's theorem, we have up to some subsequence,

$$\mathbf{v}_n\longrightarrow\mathbf{v} \quad \text{a.e. in } Q_T, \quad \text{as } n\to\infty. \tag{3.58}$$

From (3.58), it follows that $|\mathbf{v}_n|^{m-2}\mathbf{v}_n\longrightarrow|\mathbf{v}|^{m-2}\mathbf{v}$ a.e. in Ω, and, due to (3.33), $|\mathbf{v}_n|^{m-2}\mathbf{v}_n$ is uniformly bounded in $\mathbf{L}^{m'}(Q_T)$. Thus, we see that, in view of the uniqueness of the limit, it must be

$$|\mathbf{v}_n|^{m-2}\mathbf{v}_n\longrightarrow|\mathbf{v}|^{m-2}\mathbf{v} \quad \text{weakly in } \mathbf{L}^{m'}(Q_T), \quad \text{as } n\to\infty, \tag{3.59}$$

Next, we note that from (3.25)

$$\rho_n \quad \text{is uniformly bounded in } L^\lambda(Q_T) \quad \forall\,\lambda:\ 1\leq\lambda\leq\infty, \tag{3.60}$$

and, by means of reflexivity, there exists a subsequence (still denoted by) ρ_n such that

$$\rho_n\longrightarrow\rho \quad \text{weakly in } \ L^\lambda(Q_T), \quad \text{as } \ n\to\infty, \quad \forall\,\lambda:\ 1<\lambda<\infty, \tag{3.61}$$

where ρ satisfies (1.8) and (3.25).

Moreover applying (3.33), (3.51) and the Sobolev continuous imbedding $\mathbf{W}_0^{1,s}(\Omega)\hookrightarrow\mathbf{L}^r(\Omega)$, valid for $1\leq\ r\leq s^*$, where s is defined in (3.3) and s^* is its Sobolev conjugate exponent, we obtain for all $t\in[0,T]$

$$\|\mathbf{v}_n(t)\|_{r,\Omega}\leq C_1\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{s,\Omega}\leq C_2, \tag{3.62}$$

where $C_1=C(d,s,r,\Omega)$ and $C_2=C(C_1,K_0 \text{ or } K_1)$ are positive constants. In particular, for $r=4$, we have, for all $t\in[0,T]$, one of the following alternatives,

$$\|\mathbf{v}_n(t)\|_{4,\Omega}\leq C_1\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{p,\Omega}\leq C_2 \quad \text{if} \quad p\geq\frac{4d}{4+d}, \tag{3.63}$$

$$\|\mathbf{v}_n(t)\|_{4,\Omega}\leq C_1\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}\leq C_2 \quad \text{if} \quad d\leq 4. \tag{3.64}$$

Now, using the Equation (3.22) together with (3.62)-(3.64), it can be proved that

$$\frac{\partial\rho_n}{\partial t} \quad \text{is uniformly bounded in } L^s(0,T;W^{-1,s^*}(\Omega)). \tag{3.65}$$

Moreover the following compact and continuous imbeddings hold for all λ such that $s^*\leq\lambda<\infty$,

$$L^\lambda(\Omega)\hookrightarrow\hookrightarrow W^{-1,\lambda}(\Omega)\hookrightarrow W^{-1,s^*}(\Omega). \tag{3.66}$$

Then, (3.60), (3.65) and (3.66) allow us to use Aubin-Lions compactness lemma so that, for some subsequence,

$$\rho_n\longrightarrow\rho \quad \text{strongly in } C([0,T];W^{-1,\lambda}(\Omega)), \quad \text{as } n\to\infty, \tag{3.67}$$

for $s^* \le \lambda < \infty$.

On the other hand, it follows from (1.10)$_2$ and (3.23)$_2$ together with (3.24) (the last not only for ρ_n but also for ρ) that

$$\|\rho_n(t)\|_{2,\Omega}^2 = \|\rho_0\|_{2,\Omega}^2 \quad \text{and} \quad \|\rho(t)\|_{2,\Omega}^2 = \|\rho_0\|_{2,\Omega}^2 \quad \forall\, t \in [0,T]. \tag{3.68}$$

Thus, applying (3.61) and (3.68) together with (3.67), we get for all $t \in [0,T]$

$$\|\rho_n(t) - \rho(t)\|_{2,\Omega}^2 = \|\rho_n(t)\|_{2,\Omega}^2 - \|\rho(t)\|_{2,\Omega}^2 + 2\int_\Omega (\rho(t) - \rho_n(t))\rho(t)\, d\mathbf{x} \longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{3.69}$$

Then, as a consequence of (3.25) and (3.69), we have

$$\|\rho_n(t) - \rho(t)\|_{\lambda,\Omega}^\lambda \longrightarrow 0, \quad \text{as} \quad n \to \infty, \quad \forall\, \lambda : 1 \le \lambda < \infty. \tag{3.70}$$

Hence, (3.67) and (3.70) together with the argument used in [17, pp. 42, 45], assure that

$$\rho_n \longrightarrow \rho \quad \text{strongly in } C([0,T]; L^\lambda(\Omega)), \quad \text{as } n \to \infty, \quad \forall\, \lambda : 1 \le \lambda < \infty. \tag{3.71}$$

As a consequence of the application of (3.56) and (3.57) together with (3.71), we have

$$\rho_n \mathbf{v}_{n,t} \longrightarrow \rho \mathbf{v}_t \quad \text{weakly in } \mathbf{L}^2(Q_T), \quad \text{as } n \to \infty, \tag{3.72}$$

$$\rho_n \mathbf{v}_n \longrightarrow \rho \mathbf{v} \quad \text{strongly in } \mathbf{L}^r(Q_T), \text{ with } 1 \le r < s^*, \quad \text{as } n \to \infty. \tag{3.73}$$

In fact, to prove (3.73) we use the Minkowski and Sobolev inequalities together with (3.25), (3.61) and (3.62), to write

$$\begin{aligned}
&\|\rho_n \mathbf{v}_n - \rho \mathbf{v}\|_{r,Q_T}^r \\
\le\, & C_1 \left(\sup_{t\in[0,T]} \|\rho_n(t) - \rho(t)\|_{r,\Omega}^r \int_0^T \|\boldsymbol{\nabla} \mathbf{v}_n(t)\|_{s,\Omega}^r dt + \int_0^T \|\rho(t)\|_{r,\Omega}^r \|\mathbf{v}_n(t) - \mathbf{v}(t)\|_{r,\Omega}^r dt \right) \\
\le\, & C_2 \left(\sup_{t\in[0,T]} \|\rho_n(t) - \rho(t)\|_{r,\Omega}^r + \|\mathbf{v}_n - \mathbf{v}\|_{r,Q_T}^r \right),
\end{aligned}$$

for some positive constants $C_1 = C(r,s,d,\Omega)$ and $C_2 = C(C_1, M_2)$. Then the convergence of the right-hand side to zero follows from (3.57) and (3.71). To prove (3.72), we write

$$\int_{Q_T} (\rho_n \mathbf{v}_{n,t} - \rho \mathbf{v}_t) \cdot \mathbf{w}\, d\mathbf{x}dt = \int_{Q_T} (\rho_n - \rho) \mathbf{v}_{n,t} \cdot \mathbf{w}\, d\mathbf{x}dt + \int_{Q_T} \rho(\mathbf{v}_{n,t} - \mathbf{v}_t) \cdot \mathbf{w}\, d\mathbf{x}dt$$

and observe that the convergence to zero of the second right-hand side integral is due to (3.55) and to the fact that, in view of (3.25), $\rho\mathbf{w} \in \mathbf{L}^2(\Omega)$ for any $\mathbf{w} \in \mathbf{L}^2(\Omega)$. The convergence of the first right-hand side integral follows from the application of the Hölder and Sobolev inequalities together with (3.51), so that

$$\begin{aligned}
\int_{Q_T} (\rho_n - \rho) \mathbf{v}_{n,t} \cdot \mathbf{w}\, d\mathbf{x}dt &\le \int_0^T \|\rho_n(t) - \rho(t)\|_{d,\Omega} \|\mathbf{v}_{n,t}(t)\|_{2^*,\Omega} \|\mathbf{w}(t)\|_{2,\Omega} dt \\
&\le C \sup_{t\in[0,T]} \|\rho_n(t) - \rho(t)\|_{d,\Omega} \|\mathbf{w}\|_{2,Q_T}
\end{aligned}$$

for some positive constant $C=C(d,\Omega,K_1)$. The convergence to zero of the last term is due to (3.71).

Then, gathering the information of (3.33), (3.51), (3.62)-(3.64) and (3.71), we can prove that

$$\rho_n(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n\longrightarrow\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\quad\text{strongly in}\quad\mathbf{L}^1(Q_T),\quad\text{as } n\to\infty. \tag{3.74}$$

In fact, writing the corresponding integral as

$$\begin{aligned}&\int_{Q_T}[\rho_n(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n-\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]\,d\mathbf{x}dt\\&=\int_{Q_T}(\rho_n-\rho)(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n\,d\mathbf{x}dt+\int_{Q_T}\rho[(\mathbf{v}_n-\mathbf{v})\cdot\boldsymbol{\nabla}]\mathbf{v}_n\,d\mathbf{x}dt+\int_{Q_T}\rho(\mathbf{v}\cdot\boldsymbol{\nabla})(\mathbf{v}_n-\mathbf{v}_n)\,d\mathbf{x}dt,\end{aligned}$$

we see that the first right-hand side integral converges to zero by the application of the Hölder and Sobolev inequalities together with (3.62)-(3.64) and (3.71),

$$\begin{aligned}\int_{Q_T}(\rho_n-\rho)(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n\,d\mathbf{x}dt\leq&\sup_{t\in[0,T]}\|\rho_n(t)-\rho(t)\|_{\lambda,\Omega}\int_0^T\|\mathbf{v}_n(t)\|_{s^*,\Omega}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{s,\Omega}dt\\\leq&C\sup_{t\in[0,T]}\|\rho_n(t)-\rho(t)\|_{\lambda,\Omega}\longrightarrow0,\quad\text{as } n\to\infty,\end{aligned}$$

for any $\lambda\in(1,\infty)$ such that $\frac{1}{\lambda}+\frac{1}{s^*}+\frac{1}{s}\leq1$. The convergence to zero of the second integral follows from the application of Hölder's inequality together with (3.25), (3.33) and (3.57),

$$\begin{aligned}\int_{Q_T}\rho[(\mathbf{v}_n-\mathbf{v})\cdot\boldsymbol{\nabla}]\mathbf{v}_n\,d\mathbf{x}dt\leq&C_1\|\mathbf{v}_n-\mathbf{v}\|_{2,Q_T}\|\boldsymbol{\nabla}\mathbf{v}_n\|_{2,Q_T}\\\leq&C_2\|\mathbf{v}_n-\mathbf{v}\|_{2,Q_T}\longrightarrow0,\quad\text{as } n\to\infty.\end{aligned}$$

The third integral converges to zero, due to (3.52) and because $\rho\mathbf{v}\in\mathbf{L}^2(Q_T)$ in view of (3.25).

Let now ζ be a continuously differentiable function on $[0,T]$ such that $\zeta(T)=0$. Multiplying (3.21), in the case of $q=2$, by ζ and integrating by parts in $[0,T]$, we obtain

$$\begin{aligned}&-\int_{Q_T}\rho_n\mathbf{v}_n\cdot\boldsymbol{\psi}_j\zeta'\,d\mathbf{x}dt+\int_{Q_T}\rho_n(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n\cdot\boldsymbol{\psi}_j\zeta\,d\mathbf{x}dt+\varkappa\int_{Q_T}\mathbf{D}(\mathbf{v}_{n,t}):\mathbf{D}(\boldsymbol{\psi}_j)\zeta\,d\mathbf{x}dt\\&\quad+\mu\int_{Q_T}|\mathbf{D}(\mathbf{v}_n)|^{p-2}\mathbf{D}(\mathbf{v}_n):\mathbf{D}(\boldsymbol{\psi}_j)\zeta\,d\mathbf{x}dt-\gamma\int_{Q_T}|\mathbf{v}_n|^{m-2}\mathbf{v}_n\cdot\boldsymbol{\psi}_j\zeta\,d\mathbf{x}dt\\&=\zeta(0)\int_\Omega\rho_n(0)\mathbf{v}_n(0)\cdot\boldsymbol{\psi}_j\,d\mathbf{x}+\int_{Q_T}\rho_n\mathbf{f}\cdot\boldsymbol{\psi}_j\zeta\,d\mathbf{x}dt.\end{aligned}\tag{3.75}$$

Multiplying (3.22) by $\phi=\eta\zeta$, with $\eta\in C^1(\Omega)$, $\zeta\in C^1(0,T)$ and $\phi(T)=0$, and integrating by parts into Q_T, we arrive at

$$\int_{Q_T}\rho_n(\eta\zeta'+\mathbf{v}_n\cdot\boldsymbol{\nabla}\eta\zeta)\,d\mathbf{x}dt=\zeta(0)\int_\Omega\rho_n(0)\eta\,d\mathbf{x}. \tag{3.76}$$

Then, we use, for each corresponding term of (3.75), the convergence results (3.73), (3.74), (3.56), (3.55), (3.59), $(3.23)_2$ together with $(3.18)_1$, and (3.71), respectively, to pass the Equation (3.75) to the limit $n\to\infty$. To pass the Equation (3.76) to the limit $n\to\infty$, we use (3.71) and (3.73) together with $(3.23)_2$. After all, we obtain

$$\begin{aligned}
&-\int_{Q_T}\rho\mathbf{v}\cdot\boldsymbol{\psi}_j\zeta'\,d\mathbf{x}dt+\int_{Q_T}\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\cdot\boldsymbol{\psi}_j\zeta\,d\mathbf{x}dt+\varkappa\int_{Q_T}\mathbf{D}(\mathbf{v}_t):\mathbf{D}(\boldsymbol{\psi}_j)\zeta\,d\mathbf{x}dt\\
&\quad+\mu\int_{Q_T}\mathbf{S}:\mathbf{D}(\boldsymbol{\psi}_j)\zeta\,d\mathbf{x}dt-\gamma\int_{Q_T}|\mathbf{v}|^{m-2}\mathbf{v}\cdot\boldsymbol{\psi}_j\zeta\,d\mathbf{x}dt\\
&=\zeta(0)\int_\Omega\rho_0\mathbf{v}_0\cdot\boldsymbol{\psi}_j\,d\mathbf{x}+\int_{Q_T}\mathbf{f}\cdot\boldsymbol{\psi}_j\zeta\,d\mathbf{x}dt
\end{aligned}\tag{3.77}$$

for all $j\in\{1,\dots,n\}$, and

$$\int_{Q_T}\rho(\eta\zeta'+\mathbf{v}\cdot\boldsymbol{\nabla}\eta\zeta)\,d\mathbf{x}dt=\zeta(0)\int_\Omega\rho_0\eta\,d\mathbf{x}.\tag{3.78}$$

By linearity, Equation (3.77) holds for any finite linear combination of $\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_n$, and, by a continuity argument, it is still true for any $\boldsymbol{\psi}\in\mathcal{V}$. Now writing (3.77) with $\zeta\in C_0^\infty(0,T)$ and assuming, for now, that

$$\mathbf{S}=|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v}),\tag{3.79}$$

we see that $\mathbf{v}$ satisfies (2.2) in the sense of distributions on $[0,T]$. By the same reasoning, we can infer, from (3.78), that ρ satisfies (2.3) in the sense of distributions on $[0,T]$.

3.5. Use of the monotonicity. Let us now show the validity of (3.79) by the application of the Minty trick.

Firstly, we observe that since the set

$$\mathbf{Z}:=\{\mathbf{z}=\boldsymbol{\psi}\zeta:\boldsymbol{\psi}\in\mathcal{V},\ \zeta\in C_0^\infty(0,T)\}$$

is dense in $\mathbf{L}^\infty(0,T;\mathbf{V})\cap\mathbf{L}^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T)$, we can take $\mathbf{z}=\mathbf{v}_n$ and $\mathbf{z}=\mathbf{v}$ as test functions in (3.75) and (3.77), respectively, so that

$$\begin{aligned}
&\int_{Q_T}\rho_n\mathbf{v}_{n,t}\cdot\mathbf{v}_n\,d\mathbf{x}dt+\int_{Q_T}\rho_n(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n\cdot\mathbf{v}_n\,d\mathbf{x}dt+\varkappa\int_{Q_T}\mathbf{D}(\mathbf{v}_{n,t}):\mathbf{D}(\mathbf{v}_n)\,d\mathbf{x}dt\\
&\quad+\mu\int_{Q_T}|\mathbf{D}(\mathbf{v}_n)|^p\,d\mathbf{x}dt=\int_{Q_T}\rho_n\mathbf{f}\cdot\mathbf{v}_n\,d\mathbf{x}dt+\gamma\int_{Q_T}|\mathbf{v}_n|^m\,d\mathbf{x}dt.
\end{aligned}\tag{3.80}$$

and

$$\begin{aligned}
&\int_{Q_T}\rho\mathbf{v}_t\cdot\mathbf{v}\,d\mathbf{x}dt+\int_{Q_T}\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x}dt+\varkappa\int_{Q_T}\mathbf{D}(\mathbf{v}_t):\mathbf{D}(\mathbf{v})\,d\mathbf{x}dt\\
&\quad+\mu\int_{Q_T}|\mathbf{D}(\mathbf{v})|^p\,d\mathbf{x}dt=\int_{Q_T}\rho\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}dt+\gamma\int_{Q_T}|\mathbf{v}|^m\,d\mathbf{x}dt.
\end{aligned}\tag{3.81}$$

Next, using Assumption (3.1) together with the monotonicity property of the operator $\mathbf{F}(\boldsymbol{\xi})=|\boldsymbol{\xi}|^{r-2}\boldsymbol{\xi}$ for either $r=p$ or $r=m$ (see e.g. [18]), we have

$$\begin{aligned}
X_n:=&\mu\int_{Q_T}\left(|\mathbf{D}(\mathbf{v}_n)|^{p-2}\mathbf{D}(\mathbf{v}_n)-|\mathbf{D}(\mathbf{z})|^{p-2}\mathbf{D}(\mathbf{z})\right):(\mathbf{D}(\mathbf{v}_n)-\mathbf{D}(\mathbf{z}))\,d\mathbf{x}dt\\
&+|\gamma|\int_{Q_T}\left(|\mathbf{v}_n|^{m-2}\mathbf{v}_n-|\mathbf{z}|^{m-2}\mathbf{z}\right)\cdot(\mathbf{v}_n-\mathbf{z})\,d\mathbf{x}dt\geq 0\qquad\forall\ \mathbf{z}\in\mathbf{Z}.
\end{aligned}\tag{3.82}$$

Expanding (3.82) and using the identity (3.80),

$$
\begin{aligned}
0 \le X_n = &\int_{Q_T} \rho_n \mathbf{f}\cdot\mathbf{v}_n\, d\mathbf{x}dt - \int_{Q_T} \rho_n \mathbf{v}_{n,t}\cdot\mathbf{v}_n\, d\mathbf{x}dt - \int_{Q_T} \rho_n(\mathbf{v}_n\cdot\boldsymbol{\nabla})\mathbf{v}_n\cdot\mathbf{v}_n\, d\mathbf{x}dt \\
&-|\gamma|\int_{Q_T} |\mathbf{v}_n|^{m-2}\mathbf{v}_n\cdot\mathbf{z}\, d\mathbf{x}dt - |\gamma|\int_{Q_T} |\mathbf{z}|^{m-2}\mathbf{z}\cdot\mathbf{v}_n\, d\mathbf{x}dt - \varkappa\int_{Q_T} \mathbf{D}(\mathbf{v}_{n,t}):\mathbf{D}(\mathbf{v}_n)\, d\mathbf{x}dt \\
&-\mu\int_{Q_T} |\mathbf{D}(\mathbf{v}_n)|^{p-2}\mathbf{D}(\mathbf{v}_n):\mathbf{D}(\mathbf{z})\, d\mathbf{x}dt - \mu\int_{Q_T} |\mathbf{D}(\mathbf{z})|^{p-2}\mathbf{D}(\mathbf{z}):\mathbf{D}(\mathbf{v}_n)\, d\mathbf{x}dt \\
&+\mu\int_{Q_T} |\mathbf{D}(\mathbf{z})|^{p}\, d\mathbf{x}dt + |\gamma|\int_{Q_T} |\mathbf{z}|^{m}\, d\mathbf{x}dt \\
:= &X_n^1 - X_n^2 - X_n^3 - |\gamma|X_n^4 - |\gamma|X_n^5 - \varkappa X_n^6 - \mu X_n^7 - \mu X_n^8 + \mu X_n^9 + |\gamma| X_n^{10}.
\end{aligned}
\tag{3.83}
$$

Taking the limsup of (3.83) and using the properties of limsup and liminf, we have

$$
\begin{aligned}
\limsup_{n\to\infty} X_n \le &\limsup_{n\to\infty} X_n^1 - \liminf_{n\to\infty} X_n^2 - \liminf_{n\to\infty} X_n^3 - |\gamma|\liminf_{n\to\infty} X_n^4 - |\gamma|\liminf_{n\to\infty} X_n^5 \\
&- \varkappa\liminf_{n\to\infty} X_n^6 - \mu\liminf_{n\to\infty} X_n^7 - \mu\liminf_{n\to\infty} X_n^8 + \mu\limsup_{n\to\infty} X_n^9 + |\gamma|\limsup_{n\to\infty} X_n^{10}
\end{aligned}
\tag{3.84}
$$

Now, we shall evaluate all the limits appearing in the right-hand side of (3.84).

For the term X_n^1, we use the Assumption (3.6) together with (3.73) to show that

$$X_n^1 \longrightarrow \int_{Q_T} \rho\mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}dt, \quad \text{as } n\to\infty.$$

The convergence

$$X_n^2 \longrightarrow \int_{Q_T} \rho\mathbf{v}_t\cdot\mathbf{v}\, d\mathbf{x}dt := X^2, \quad \text{as } n\to\infty.$$

is justified, because we can write

$$X_n^2 = \int_{Q_T} \rho_n\mathbf{v}_{n,t}\cdot(\mathbf{v}_n-\mathbf{v})\, d\mathbf{x}dt + \int_{Q_T} \rho_n\mathbf{v}_{n,t}\cdot\mathbf{v}\, d\mathbf{x}dt := X_n^{21} + X_n^{22},$$

and, since $X_n^{21} \longrightarrow 0$, as $n\to\infty$, due to (3.25) and (3.51) together with (3.57), and once that $X_n^{22} \longrightarrow X^2$, as $n\to\infty$, by virtue of (3.72).

With respect to the convergence of the term X_n^3, we can prove that

$$
\begin{aligned}
X_n^3 = &\int_{Q_T} [(\rho_n\mathbf{v}_n - \rho\mathbf{v})\cdot\boldsymbol{\nabla}]\mathbf{v}_n\cdot\mathbf{v}_n\, d\mathbf{x}dt + \int_{Q_T} (\rho\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}_n\cdot(\mathbf{v}_n-\mathbf{v})\, d\mathbf{x}dt \\
&+\int_{Q_T} (\rho\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}_n\cdot\mathbf{v}\, d\mathbf{x}dt := X_n^{31} + X_n^{32} + X_n^{33} \longrightarrow \int_{Q_T} (\rho\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\cdot\mathbf{v}\, d\mathbf{x}dt := X^{33}, \text{ as } n\to\infty.
\end{aligned}
$$

In fact, by the Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
|X_n^{31}| &\le \|\rho_n\mathbf{v}_n - \rho\mathbf{v}\|_{4,Q_T}\|\boldsymbol{\nabla}\mathbf{v}_n\|_{2,Q_T}\|\mathbf{v}_n\|_{4,Q_T} \\
&\le C\|\rho_n\mathbf{v}_n - \rho\mathbf{v}\|_{4,Q_T} \longrightarrow 0, \quad \text{as } n\to\infty,
\end{aligned}
$$

due to (3.51), (3.63)-(3.64) and (3.73). Note that the application of (3.73) is possible, since, by Assumption (3.10), $s > \frac{4d}{d+4} \Leftrightarrow 4 < s^*$. By a similar argument, we have

$$
\begin{aligned}
|X_n^{32}| &\le M_2\|\mathbf{v}_n - \mathbf{v}\|_{4,Q_T}\|\boldsymbol{\nabla}\mathbf{v}_n\|_{2,Q_T}\|\mathbf{v}\|_{4,Q_T}, \\
&\le C\|\mathbf{v}_n - \mathbf{v}\|_{4,Q_T} \longrightarrow 0, \quad \text{as } n\to\infty,
\end{aligned}
$$

due to (3.25), (3.51) and (3.57). Moreover, $X_n^{33} \longrightarrow X^{33}$, as $n\to\infty$, by virtue of (3.52) and because, due to (3.25), (3.33) and (3.51), $\rho|\mathbf{v}|^2 \in \mathbf{L}^{s'}(Q_T)$ for $s \geq \frac{3d}{d+2}$.

The convergence of the terms X_n^4 and X_n^5 follow from (3.59) and (3.53), respectively,

$$X_n^4 \longrightarrow \int_{Q_T} |\mathbf{v}|^{m-2}\mathbf{v}\cdot\mathbf{z}\, d\mathbf{x}dt, \quad \text{as } n\to\infty,$$

$$X_n^5 \longrightarrow \int_{Q_T} |\mathbf{z}|^{m-2}\mathbf{z}\cdot\mathbf{v}\, d\mathbf{x}dt, \quad \text{as } n\to\infty.$$

Regarding the convergence of the term X_n^6, we first note that, as we shall prove later on,

$$\mathbf{v} \in C_w([0,T];\mathbf{H}\cap\mathbf{V}), \tag{3.85}$$

where $C_w([0,T];\mathbf{H}\cap\mathbf{V})$ denotes the subspace of $L^\infty([0,T];\mathbf{H}\cap\mathbf{V})$ formed by weakly continuous functions from $[0,T]$ onto $\mathbf{H}\cap\mathbf{V}$. Hence, (3.85) implies the quantities $\mathbf{v}(0)$, $\mathbf{v}(T)$, $\mathbf{D}(\mathbf{v}(0))$ and $\mathbf{D}(\mathbf{v}(T)))$ are meaningful. In particular,

$$\mathbf{v}(0) = \mathbf{v}_0 \qquad \text{and} \qquad \boldsymbol{\nabla}\mathbf{v}(0) = \boldsymbol{\nabla}\mathbf{v}_0.$$

Thus, we can use integration by parts to write X_n^6 in the form

$$\begin{aligned} X_n^6 =& \frac{1}{2}\int_0^T \frac{d}{dt}\|\mathbf{D}(\mathbf{v}_n(t))\|_{2,\Omega}^2 dt = \frac{1}{2}\left(\|\mathbf{D}(\mathbf{v}_n(T))\|_{2,\Omega}^2 - \|\mathbf{D}(\mathbf{v}_n(0))\|_{2,\Omega}^2\right) \\ =& \frac{1}{2}\|\mathbf{D}(\mathbf{v}_n(T)) - \mathbf{D}(\mathbf{v}(T))\|_{2,\Omega}^2 - \frac{1}{2}\|\mathbf{D}(\mathbf{v}_n(0)) - \mathbf{D}(\mathbf{v}(0))\|_{2,\Omega}^2 \\ &+ \int_{Q_T} \mathbf{D}(\mathbf{v}_t):(\mathbf{D}(\mathbf{v}_n) - \mathbf{D}(\mathbf{v}))\, d\mathbf{x}dt + \int_{Q_T} \mathbf{D}(\mathbf{v}_{n,t}):\mathbf{D}(\mathbf{v})\, d\mathbf{x}dt \\ :=& \frac{1}{2}X_n^{61} - \frac{1}{2}X_n^{62} + X_n^{63} + X_n^{64}. \end{aligned}$$

Using again the properties of lim sup and lim inf, we have

$$\liminf_{n\to\infty} X_n^6 \geq \frac{1}{2}\liminf_{n\to\infty} X_n^{61} - \frac{1}{2}\limsup_{n\to\infty} X_n^{62} + \liminf_{n\to\infty} X_n^{63} + \liminf_{n\to\infty} X_n^{64}.$$

From the definition of X_n^{61}, one easily sees that $X_n^{61} \geq 0$, and, by virtue of (3.17), we get

$$X_n^{62} \longrightarrow 0, \quad \text{as } n\to\infty.$$

On the other hand, by the convergence results (3.52) and (3.56), one has

$$X_n^{63} \longrightarrow 0 \quad \text{and} \quad X_n^{64} \longrightarrow \int_{Q_T} \mathbf{D}(\mathbf{v}_t):\mathbf{D}(\mathbf{v})\, d\mathbf{x}dt, \quad \text{as } n\to\infty.$$

In view of this, we have

$$\liminf_{n\to\infty} X_n^6 \geq \int_{Q_T} \mathbf{D}(\mathbf{v}_t):\mathbf{D}(\mathbf{v})\, d\mathbf{x}dt.$$

Finally, since the convergence of X_n^9 and X_n^{10} is trivial, we just have to justify the convergence of X_n^7 and X_n^8, which follows from (3.55) and (3.52), respectively,

$$X_n^7 \longrightarrow \int_{Q_T} \mathbf{S}:\mathbf{D}(\mathbf{z})\, d\mathbf{x}dt, \quad \text{as } n\to\infty,$$

$$X_n^8 \longrightarrow \int_{Q_T} |\mathbf{D}(\mathbf{z})|^{p-2}\mathbf{D}(\mathbf{z}):\mathbf{D}(\mathbf{v})\,d\mathbf{x}dt, \quad \text{as } n\to\infty.$$

Gathering the information of (3.83) and (3.84) together with the convergence results for X_n^1, ..., X_n^8 depicted above, we obtain

$$\begin{aligned}0\leq&\int_{Q_T}\rho\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}dt-\int_{Q_T}\rho\mathbf{v}_t\cdot\mathbf{v}\,d\mathbf{x}dt-\int_{Q_T}\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x}dt\\&-|\gamma|\int_{Q_T}|\mathbf{v}|^{m-2}\mathbf{v}\cdot\mathbf{z}\,d\mathbf{x}dt-|\gamma|\int_{Q_T}|\mathbf{z}|^{m-2}\mathbf{z}\cdot\mathbf{v}\,d\mathbf{x}dt-\varkappa\int_{Q_T}\mathbf{D}(\mathbf{v}_t):\mathbf{D}(\mathbf{v})\,d\mathbf{x}dt\\&-\mu\int_{Q_T}\mathbf{S}:\mathbf{D}(\mathbf{z})\,d\mathbf{x}dt-\mu\int_{Q_T}\mathbf{S}:\mathbf{D}(\mathbf{v})\,d\mathbf{x}dt+\mu\int_{Q_T}|\mathbf{D}(\mathbf{z})|^p\,d\mathbf{x}dt+|\gamma|\int_{Q_T}|\mathbf{z}|^m\,d\mathbf{x}dt.\end{aligned}\tag{3.86}$$

Thus, by (3.81) and (3.86), we achieve

$$\begin{aligned}\mu&\int_{Q_T}\left(\mathbf{S}-|\mathbf{D}(\mathbf{z})|^{p-2}\mathbf{D}(\mathbf{z})\right)\cdot\left(\mathbf{D}(\mathbf{v})-\mathbf{D}(\mathbf{z})\right)d\mathbf{x}dt\\&+|\gamma|\int_{Q_T}\left(|\mathbf{v}|^{m-2}\mathbf{v}-|\mathbf{z}|^{m-2}\mathbf{z}\right)\cdot(\mathbf{v}-\mathbf{z})\,d\mathbf{x}dt\geq0\qquad\forall\ \mathbf{z}\in\mathbf{Z}.\end{aligned}\tag{3.87}$$

By means of density, (3.87) still holds for all $\mathbf{z}\in\mathbf{L}^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T)$. Thus, taking $\mathbf{z}=\mathbf{v}\mp\delta\mathbf{w}$ for an arbitrary $\mathbf{w}\in\mathbf{L}^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T)$ and $\delta>0$, it follows from (3.87)

$$\begin{aligned}\pm\Bigg(\mu&\int_{Q_T}\left(\mathbf{S}-|\mathbf{D}(\mathbf{v})\mp\delta\mathbf{D}(\mathbf{w})|^{p-2}\left(\mathbf{D}(\mathbf{v})\mp\delta\mathbf{D}(\mathbf{w})\right)\right)\cdot\mathbf{D}(\mathbf{w})\,d\mathbf{x}dt\\&+|\gamma|\int_{Q_T}\left(|\mathbf{v}|^{m-2}\mathbf{v}-|\mathbf{v}\mp\delta\mathbf{w}|^{m-2}(\mathbf{v}+\delta\mathbf{w})\right)\cdot\mathbf{w}\,d\mathbf{x}dt\Bigg)\geq0.\end{aligned}\tag{3.88}$$

Letting $\delta\to0$ in (3.88), we obtain

$$\pm\int_{Q_T}\left(\mathbf{S}-|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v})\right)\cdot\mathbf{D}(\mathbf{w})\,d\mathbf{x}dt\geq0.$$

Due to the arbitrariness of $\mathbf{w}$, we see that it must be $\mathbf{S}=|\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v})$ which proves (3.79).

3.6. Proof that $\mathbf{v}\in C_w([0,T];\mathbf{H}\cap\mathbf{V})$. We first note that, from (3.33) and (3.51), respectively, we also have for some subsequences

$$\mathbf{v}_n\longrightarrow\mathbf{v}\quad\text{weakly}-*\text{ in }\mathbf{L}^\infty(0,T;\mathbf{H}\cap\mathbf{V}),\quad\text{as } n\to\infty,\tag{3.89}$$

$$\mathbf{v}_n\longrightarrow\mathbf{v}\quad\text{weakly}-*\text{ in }\mathbf{L}^\infty(0,T;\mathbf{L}^m(\Omega)\cap\mathbf{V}_p),\quad\text{as } n\to\infty.\tag{3.90}$$

Thus, we have from (3.52)-(3.53) and (3.89) that

$$\mathbf{v}\in L^\infty(0,T;\mathbf{H}\cap\mathbf{V})\cap L^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T),\tag{3.91}$$

and from (3.90),

$$\mathbf{v}\in\mathbf{L}^\infty(0,T;\mathbf{L}^m(\Omega)\cap\mathbf{V}_p).\tag{3.92}$$

On the other hand, from (3.54),

$$\mathbf{v}_t \in L^2(0,T;\mathbf{H}\cap\mathbf{V}). \tag{3.93}$$

Let us set now

$$\mathbf{Y} := \mathbf{L}^2(\Omega)\cap\mathbf{L}^m(\Omega)\cap\mathbf{V}\cap\mathbf{V}_p \quad \text{and} \quad j := \max\{2,p,m\}.$$

Due to (3.91) and (3.92), one can easily see that

$$\mathbf{v}\in L^j(0,T;\mathbf{Y}). \tag{3.94}$$

From (3.77), we can write

$$\rho\mathbf{v}_t = \kappa\,\mathbf{div}(\mathbf{D}(\mathbf{v}_t)) + \mu\,\mathbf{div}\mathbf{S} - \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \rho\mathbf{f} + \gamma|\mathbf{v}|^{m-2}\mathbf{v}$$

in the sense of distributions on $\Omega\times(0,T)$. Observe that when we have used (3.93) we have not yet proved (3.79). Now, we can show that

$$\int_0^T \|\kappa\,\mathbf{div}(\mathbf{D}(\mathbf{v}_t(t)))\|_{\mathbf{V}'}^{j'}\,dt \le \kappa^{j'} C_K \int_0^T \|\boldsymbol{\nabla}\mathbf{v}_t(t)\|_{2,\Omega}^{j'} dt \le C_2\|\boldsymbol{\nabla}\mathbf{v}_t\|_{2,Q_T}^{j'} < \infty,$$

due to (3.93) and $j\ge 2$,

$$\int_0^T \|\mu\,\mathbf{div}\mathbf{S}(t)\|_{\mathbf{V}_p'}^{j'}\,dt \le \mu^{j'}\int_0^T \|\mathbf{S}(t)\|_{p',\Omega}^{j'} dt \le C_2\|\mathbf{S}\|_{p',Q_T}^{j'} < \infty,$$

due to (3.55) and $j\ge p$,

$$\int_0^T \|\rho(t)\,(\mathbf{v}(t)\cdot\nabla)\mathbf{v}(t)\|_{2,\Omega}^{j'}\,dt \le C_3\left(\operatorname{ess\,sup}_{t\in[0,T]}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{2,\Omega}\right)^{2j'} < \infty,$$

due to (3.25) and (3.91),

$$\int_0^T \|\rho(t)\mathbf{f}(t)\|_{2,\Omega}^{j'}\,dt \le M_2^{j'}\int_0^T \|\mathbf{f}(t)\|_{2,\Omega}^{j'}\,dt \le C_4\|f\|_{2,Q_T}^{j'} < \infty,$$

due to (3.25) and (3.91),

$$\int_0^T \|\rho(t)\mathbf{f}(t)\|_{2,\Omega}^{j'}\,dt \le M_2^{j'}\int_0^T \|\mathbf{f}(t)\|_{2,\Omega}^{j'}\,dt \le C_4\|f\|_{2,Q_T}^{j'} < \infty,$$

due to (3.6), (3.25) and $j\ge 2$,

$$\int_0^T \left\|\gamma|\mathbf{v}(t)|^{m-2}\mathbf{v}(t)\right\|_{m',\Omega} dt \le |\gamma|^{j'}\int_0^T \|\mathbf{v}(t)\|_{m,\Omega}^{j'(m-1)} dt \le C_5\|\mathbf{v}\|_{m,Q_T}^{j'(m-1)} < \infty,$$

due to (3.53) and $j\ge m$,

where $C_1=C(j,\varkappa,C_K,T)$, $C_2=C(j,p,T)$, $C_3=C(j,M_2,\Omega)$, $C_4=C(j,M_2,T)$ and $C_5=C(j,M_2,\Omega)$ are positive constants. This yields $\rho\mathbf{v}_t\in L^{j'}(0,T;\mathbf{Y}')$ and consequently, due to (3.25),

$$\mathbf{v}_t\in L^{j'}(0,T;\mathbf{Y}'). \tag{3.95}$$

Then, from (3.94) and (3.95), we can use *e.g.* Lemma 1.2 of [26, Chapter III] to show the validity of (3.85).

3.7. Proof of the estimates (3.11)-(3.13). Since the norm is weakly lower semicontinuous, we can use the convergence results (3.52)-(3.53) and (3.89) to derive (3.12). The estimate (3.13) follows by using the same reasoning and the convergence results (3.54) and (3.90). With respect to the estimate (3.11), it follows from (3.25) and (3.67).

The proof of Theorem 3.1 is now completed. □

4. Global existence: the case $\gamma > 0$

In this section we are interested in establishing an existence result to the problem (1.7)-(1.11) in the case of

$$\gamma > 0. \tag{4.1}$$

Here again we shall only assume that $q = 2$.

THEOREM 4.1. *Assume that (3.2), (3.4), (3.5)-(3.6) and (4.1) hold, and that one of the alternatives written in (3.7)-(3.9) is fulfilled. In addition, assume that (3.10) holds as well. If one of the following conditions holds,*

$$m \leq 2, \tag{4.2}$$

$$2 < m < p, \tag{4.3}$$

and if, in the case of (4.3), we additionally have

$$2(m-1) \leq p^*, \tag{4.4}$$

then the problem (1.7)-(1.11) has, at least, a weak solution $(\mathbf{v}, \rho)$ in the sense of Definition 2.1 in the cylinder Q_T.

Moreover, this weak solution satisfies (3.11) and the following estimates

$$\begin{aligned} &\sup_{t \in [0,T]} \left(\|\mathbf{v}(t)\|_{2,\Omega}^2 + \|\boldsymbol{\nabla}\mathbf{v}(t)\|_{2,\Omega}^2 \right) + \|\boldsymbol{\nabla}\mathbf{v}\|_{p,Q_T}^p + \|\mathbf{v}\|_{m,Q_T}^m \\ &\leq C_1 \left(\|\mathbf{v}_0\|_{2,\Omega}^2 + \|\boldsymbol{\nabla}\mathbf{v}_0\|_{2,\Omega}^2 + \|\mathbf{f}\|_{2,Q_T}^2 \right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\sup_{t \in [0,T]} \left(\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{p,\Omega}^p + \|\mathbf{v}(t)\|_{m,\Omega}^m \right) + \|\mathbf{v}_t\|_{2,Q_T}^2 + \|\boldsymbol{\nabla}\mathbf{v}_t\|_{2,Q_T}^2 \\ &\leq C_2 \left(\|\boldsymbol{\nabla}\mathbf{v}_0\|_{p,\Omega}^p + \|\mathbf{v}_0\|_{m,\Omega}^m + \|\mathbf{f}\|_{2,Q_T}^2 + 1 \right), \end{aligned} \tag{4.6}$$

where C_1 and C_2 are the positive constants from (4.14) and (4.19) written below.

Proof. The proof of Theorem 4.1 follows very closely the proof of Theorem 3.1. Thus, we shall just point out the differences that we have here due to the new Assumption (4.1).

4.1. Galerkin's approximations. The existence of the Galerkin approximations proved in Step 3.1 of the proof of Theorem 3.1 holds in this case as well.

4.2. First a priori estimate. Assuming here that (4.1) holds, we proceed as we did for (3.31) to obtain

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\mu}{C_K}\int_0^t \|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p ds$$

$$\le \frac{M_2}{2}\|\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{M_2}{2}\left(\int_0^t \|\mathbf{f}(s)\|_{2,\Omega}^2 ds + \int_0^t \|\mathbf{v}_n(s)\|_{2,\Omega}^2 ds\right) + \gamma\int_0^t \|\mathbf{v}_n(s)\|_{m,\Omega}^m ds. \tag{4.7}$$

If $m \le 2$ (Assumption (4.2)), then we can use Young's inequality to show that

$$\|\mathbf{v}_n(t)\|_{m,\Omega}^m \le C\left(\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + 1\right), \tag{4.8}$$

for some positive constant $C = C(m,\Omega)$. Thus, plugging (4.8) into (4.7), we get

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\mu}{C_K}\int_0^t \|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p ds$$
$$\le \frac{M_2}{2}\|\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{M_2}{2}\int_0^t \|\mathbf{f}(s)\|_{2,\Omega}^2 ds + C_1\int_0^t \|\mathbf{v}_n(s)\|_{2,\Omega}^2 ds + C_2 \tag{4.9}$$

for some positive constants $C_1 = C(M_2,m,\gamma,\Omega)$ and $C_2 = C(m,\gamma,\Omega,T)$. Combining (4.9) with Grönwall's inequality applied to the function

$$y(t) = \frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2,$$

one has

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\mu}{C_K}\int_0^t \|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p ds$$
$$\le C\left(\frac{M_2}{2}\|\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{M_2}{2}\int_0^t \|\mathbf{f}(s)\|_{2,\Omega}^2 ds + 1\right), \tag{4.10}$$

for some positive constant $C = C(M_2,m,\gamma,\Omega,T)$.

If now $m < p$ (Assumption (4.3)), we can use Korn's inequality to derive

$$\|\mathbf{v}_n(t)\|_{m,\Omega}^m \le \varepsilon\|\boldsymbol{\nabla}(\mathbf{v}_n(t))\|_{p,\Omega}^p + C(\varepsilon) \tag{4.11}$$

for $\varepsilon > 0$ and for some positive constant $C = C(\varepsilon,m,p)$. Plugging (4.11) into (4.7), choosing $\varepsilon = \frac{\mu}{2C_K|\gamma|}$ and proceeding as we did for (4.10), we obtain

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\mu}{2C_K}\int_0^t \|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p ds$$
$$\le \frac{M_2}{2}\|\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{M_2}{2}\int_0^t \|\mathbf{f}(s)\|_{2,\Omega}^2 ds + C, \tag{4.12}$$

for some positive constant $C = C(m,p,\mu,\gamma,\Omega,T)$. Adding the absorption/diffusion integral term to both sides of (4.10), or of (4.12), and using (4.8), or (4.11), just in the integral added to the right-hand side, we obtain by the same argument

$$\frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2 + \frac{\mu}{2C_K}\int_0^t \|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p ds + \gamma\int_0^t \|\mathbf{v}_n(s)\|_{m,\Omega}^m ds$$
$$\le C\left(\frac{M_2}{2}\|\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}^2 + \frac{M_2}{2}\int_0^t \|\mathbf{f}(s)\|_{2,\Omega}^2 ds + 1\right). \tag{4.13}$$

Taking the supremum of (4.13) in $[0,T]$ and using (3.18), we achieve

$$\sup_{t\in[0,T]}\left(\|\mathbf{v}_n(t)\|_{2,\Omega}^2+\|\nabla\mathbf{v}_n(t)\|_{2,\Omega}^2\right)+\|\nabla\mathbf{v}_n\|_{p,Q_T}^p+\gamma\|\mathbf{v}_n\|_{m,Q_T}^m \\ \leq C\left(\|\mathbf{v}_0\|_{2,\Omega}^2+\|\nabla\mathbf{v}_0\|_{2,\Omega}^2+\|\mathbf{f}\|_{2,Q_T}^2+1\right):=\widetilde{K}_0, \tag{4.14}$$

where $C=C(M_1,M_2,C_K,\varkappa,\mu,m,\gamma,T)$ is a positive constant. When we take the supremum of (4.12) this constant depends also on p, whereas in the case of (4.10) it depends also on Ω. Then, due to the Assumptions (3.5)-(3.6), the right-hand side of (4.14) is finite.

4.3. Second a priori estimate Using the main Assumption (4.1) of this case and considering that one of the two alternatives, (3.7) or (3.8), holds, we obtain, by proceeding as we did for (3.41) and (3.47), the following type inequality

$$\frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|_{p,\Omega}^p+\int_0^t\left(\frac{M_1}{2}\|\mathbf{v}_{n,t}(s)\|_{2,\Omega}^2+\frac{\varkappa}{2}\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}^2\right)ds \\ \leq C\left(\frac{\mu}{p}\|\nabla\mathbf{v}_0\|_{p,\Omega}^p+\frac{M_2^2}{2M_1}\int_0^t\|\mathbf{f}(s)\|_{2,\Omega}^2ds+\frac{\gamma}{m}\|\mathbf{v}_n(t)\|_{m,\Omega}^m-\frac{\gamma}{m}\|\mathbf{v}_0\|_{m,\Omega}^m+1\right), \tag{4.15}$$

where now the positive constant C depends on $\widetilde{K}_0$ defined above at (4.14).

If $m\leq 2$, then we can use (4.8) together with (4.14) to obtain the estimate

$$\|\mathbf{v}_n(t)\|_{m,\Omega}^m\leq C_1\left(\sup_{t\in[0,T]}\|\mathbf{v}_n(t)\|_{2,\Omega}^2+1\right)\leq C_2, \tag{4.16}$$

where C_1 is the constant from (4.8) and $C_2=C(C_1,\widetilde{K}_0)$ is another positive constant.

If $m<p$ (which holds due to (4.3)), we can use the Sobolev, Korn and Young inequalities so that

$$\|\mathbf{v}_n(t)\|_{m,\Omega}^m\leq C_1\|\mathbf{D}(\mathbf{v}_n(t))\|_{p,\Omega}^m\leq\frac{m\mu}{2p\gamma C}\|\mathbf{D}(\mathbf{v}_n(t))\|_{p,\Omega}^p+C_2 \tag{4.17}$$

for some positive constants $C_1=C(\gamma,m,p,d,C_K,\Omega)$ and $C_2=C(C_1,\mu)$.

In any case, plugging (4.16) or (4.17) into (4.15), then adding the absorption/diffusion term to both sides of the resulting inequality and using again (4.16) or (4.17) just in the term added to the right-hand side, one has

$$\frac{\mu}{p}\|\mathbf{D}(\mathbf{v}_n(t))\|_{p,\Omega}^p+\frac{\gamma}{m}\|\mathbf{v}_n(t)\|_{m,\Omega}^m+\int_0^t\left(\frac{M_1}{2}\|\mathbf{v}_{n,t}(s)\|_{2,\Omega}^2+\frac{\varkappa}{2}\|\mathbf{D}(\mathbf{v}_{n,t}(s))\|_{2,\Omega}^2\right)ds \\ \leq C\left(\frac{\mu}{p}\|\nabla\mathbf{v}_0\|_{p,\Omega}^p+\frac{M_2^2}{2M_1}\int_0^t\|\mathbf{f}(s)\|_{2,\Omega}^2ds+\frac{\gamma}{m}\|\mathbf{v}_0\|_{m,\Omega}^m+1\right) \tag{4.18}$$

for some positive constant $C=C(M_2,C_K,m,p,\gamma,\varkappa,d,\Omega,t,\widetilde{K}_0)$. Then, taking the supreme in $[0,T]$ of (4.18) and using Korn's inequality (2.1), we obtain the counterpart inequality of (3.51)

$$\sup_{t\in[0,T]}\left(\|\nabla\mathbf{v}_n(t)\|_{p,\Omega}^p+\gamma\|\mathbf{v}_n(t)\|_{m,\Omega}^m\right)+\|\mathbf{v}_{n,t}\|_{2,Q_T}^2+\|\nabla\mathbf{v}_{n,t}\|_{2,Q_T}^2 \\ \leq C\left(\|\nabla\mathbf{v}_0\|_{p,\Omega}^p+\gamma\|\mathbf{v}_0\|_{m,\Omega}^m+\|\mathbf{f}\|_{2,Q_T}^2+1\right):=\widetilde{K}_1, \tag{4.19}$$

where $C=C(M_1,M_2,C_K,\varkappa,\mu,m,p,\widetilde{K}_0,d,\Omega,T)$ is a positive constant.

4.4. Passage to the limit as $n\to\infty$ The convergence results proved in Step 3.4 of the proof of Theorem 3.1 hold in this case as well. In view of this, the rest of the proof of Theorem 4.1 is straightforward. □

5. Local existence

In this section, we prove a local-in-time existence theorem to the problem (1.7)-(1.11) with (4.1) being satisifed, *i.e.* when $\gamma>0$. Once again we shall only consider the case of $q=2$, and we recall the notation (3.3) in this case, $s=\max\{2,p\}$.

THEOREM 5.1. *Assume that (3.2), (3.4), (3.5)-(3.6) and (4.1) hold. If one of the following conditions hold,*

$$2<m\leq 2^{*}, \tag{5.1}$$

$$2<p\leq m<p\left(1+\frac{2}{d}\right), \tag{5.2}$$

and if in the case of (5.2) there additionally holds (4.4), then there exists $T_{\max}\in(0,T)$ defined at (5.6), or (5.11), below such that the problem (1.7)-(1.11) has, at least, a weak solution $(\mathbf{v},\rho)$ in the sense of Definition 2.1 in the cylinder $Q_{T_{\max}}$.

Proof. To prove Theorem 5.1, one just needs to derive estimates (4.5) and (4.6) for a small interval $(0,T_{\max})$, with $T_{\max}\in(0,T)$ suitably chosen. Let us assume that

$$s<m\leq s^{*},$$

where we again recall that $s=\max\{2,p\}$.

First we note that, since by Assumptions (5.1)-(5.2), $2<m$, we can use the Hölder and Young inequalities in the fourth term of the right-hand side of (4.7) such that

$$\begin{aligned}&\frac{M_1}{2}\|\mathbf{v}_n(t)\|_{2,\Omega}^2+\frac{\varkappa}{2C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2+\frac{\mu}{C_K}\int_0^t\|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p ds\\ \leq&\frac{M_2}{2}\|\mathbf{v}_{n,0}\|_{2,\Omega}^2+\frac{\varkappa C_K}{2}\|\boldsymbol{\nabla}\mathbf{v}_{n,0}\|_{2,\Omega}^2+\frac{M_2}{2}\int_0^t\|\mathbf{f}(s)\|_{2,\Omega}^2ds+2\gamma\int_0^t\|\mathbf{v}_n(s)\|_{m,\Omega}^m ds+C\end{aligned} \tag{5.3}$$

for some positive constant $C=C(M_2,\gamma,m,\Omega,T)$.

If $s=2$, we use Sobolev's inequality so that

$$\|\mathbf{v}_n(s)\|_{m,\Omega}^m\leq C\left(\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2\right)^{\frac{m}{2}} \tag{5.4}$$

for some positive constant $C=C(m,d,\Omega)$. Plugging (5.4) into (5.3), we obtain the following integral inequality

$$Y(t)\leq C_1\int_0^t Y(s)^{\frac{m}{2}}ds+C_2\qquad\text{for}\ \ Y(t):=\|\mathbf{v}_n(t)\|_{2,\Omega}^2+\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2, \tag{5.5}$$

and where $C_1=C(\gamma,m,d,\Omega)$ and $C_2=C(M_1,\varkappa,C_K,\widetilde{K}_0)$ are positive constants, being $\widetilde{K}_0$ the constant from the estimate (4.14). Since, by Assumption (5.1), $m>2$, we can use Lemma 2.2 in (5.5) to show that

$$Y(t)\leq C_2\left(1-\frac{m-2}{2}C_1C_2^{\frac{m-2}{2}}t\right)^{-\frac{2}{m-2}}\quad\text{for}\ \ 0\leq t\leq T_{\max}<\frac{2}{(m-2)C_1C_2^{\frac{m-2}{2}}}. \tag{5.6}$$

Thus, proceeding as we did for (4.14), we get from (5.5)-(5.6) the following local estimate,

$$\sup_{t\in[0,T_{\max}]}\left(\|\mathbf{v}_n(t)\|_{2,\Omega}^2+\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{2,\Omega}^2\right)+\|\boldsymbol{\nabla}\mathbf{v}_n\|_{p,Q_{T_{\max}}}^p+\gamma\|\mathbf{v}_n\|_{m,Q_{T_{\max}}}^m \\ \leq C\left(\|\mathbf{v}_0\|_{2,\Omega}^2+\|\boldsymbol{\nabla}\mathbf{v}_0\|_{2,\Omega}^2+\|\mathbf{f}\|_{2,Q_T}^2+1\right):=\widetilde{K}_2<\infty, \tag{5.7}$$

where $C=C(M_1,M_2,C_K,\Omega,\varkappa,\mu,\gamma,m,p,T_{\max})$ is a positive constant.

If $s=p$, we use interpolation together with Sobolev's inequality so that

$$\|\mathbf{v}_n(t)\|_{m,\Omega}^m\leq C\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{p,\Omega}^{\theta m}\|\mathbf{v}_n(t)\|_{2,\Omega}^{(1-\theta)m} \qquad \text{for} \;\; \theta:=\frac{m-2}{m}\,\frac{dp}{dp-2(d-p)}\in(0,1), \tag{5.8}$$

and where $C=C(m,p,d,\Omega)$ is a positive constant. Since, by Assumption (5.2),

$$\frac{\theta m}{p}=\frac{d(m-2)}{dp-2(d-p)}<1\Leftrightarrow 2<m<p\left(1+\frac{2}{d}\right),$$

we can use Young's inequality in (5.8) so that

$$\|\mathbf{v}_n(t)\|_{m,\Omega}^m\leq\varepsilon\|\boldsymbol{\nabla}\mathbf{v}_n(t)\|_{p,\Omega}^p+C\left(\|\mathbf{v}_n(t)\|_{2,\Omega}^2\right)^\mu \qquad \text{for} \;\; \mu:=\frac{(1-\theta)m}{2}\,\frac{p}{p-\theta m} \tag{5.9}$$

and for some positive constant $C=C(m,p,d,\Omega,\varepsilon)$. Plugging (5.9) into (4.7) and choosing $\varepsilon=\frac{\mu}{2C_K}$ in the resulting inequality, we achieve the following integro-differential inequality

$$Y(t)\leq C_1\int_0^t Y(s)^\mu ds+C_2 \qquad \text{for} \;\; Y(t):=\|\mathbf{v}_n(t)\|_{2,\Omega}^2, \tag{5.10}$$

and where $C_1=C(\gamma,\mu,m,p,d,C_K,\Omega)$ and $C_2=C(M_1,\widetilde{K}_0)$ are positive constants. Since $p>0$, one has $\mu>1$ and therefore we can use again Lemma 2.2, now in (5.10), to show that

$$Y(t)\leq C_2\left(1-(\mu-1)C_1C_2^{\mu-1}t\right)^{-\frac{1}{\mu-1}} \qquad \text{for} \;\; 0\leq t\leq T_{\max}<\frac{1}{(\mu-1)C_1C_2^{\mu-1}}. \tag{5.11}$$

Proceeding again as we did for (4.14), we get now from (5.10)-(5.11) the same type of local estimate as in (5.7).

In any case, the corresponding local estimate to (4.5) follows as in the proof of Theorem 4.1.

The proof of the corresponding local estimate to (4.6) is entirely analogous to the one we have performed when we proved Theorem 4.1. ◻

6. Large-time behavior

In this section, we study the large-time behavior properties of the weak solutions to the initial and boundary-value problem (1.7)-(1.11). For the analysis we shall perform, we assume that problem (1.7)-(1.11) has, at least, a weak solution, including for the open case $q\neq 2$. Throughout this section, we shall assume that (3.1) holds, *i.e.* that $\gamma\leq 0$, and that

$$\mathbf{v}_0\in\mathbf{H}\cap\mathbf{V}_q. \tag{6.1}$$

Alternatively to (3.6), we assume that one of the following conditions on the force field holds,

$$\mathbf{f}\in\mathbf{L}^{p'}(Q_T)\cap\mathbf{L}^2(\Omega), \tag{6.2}$$

$$\mathbf{f}\in\mathbf{L}^{m'}(Q_T)\cap\mathbf{L}^2(\Omega), \tag{6.3}$$

where p' and m' denote the Hölder conjugates of p and m.

Let us define the functions

$$\Phi(t):=\frac{1}{2}\|\sqrt{\rho(t)}\mathbf{v}(t)\|^2_{2,\Omega}+\frac{\varkappa}{q}\|\nabla\mathbf{v}(t)\|^q_{q,\Omega}, \tag{6.4}$$

$$\Psi(t):=\mu\|\nabla\mathbf{v}(t)\|^p_{p,\Omega}+|\gamma|\|\mathbf{v}(t)\|^m_{m,\Omega}. \tag{6.5}$$

Due to the definition of (6.5), it is important to distinguish the case when γ might be 0 from the case

$$\gamma<0. \tag{6.6}$$

In the following result, we establish energy inequalities satisfied by the weak solutions to the problem (1.7)-(1.11) when one of the conditions, (3.1) or (6.6), holds.

Theorem 6.1 (Energy inequality). *Let $(\mathbf{v},\rho)$ be a weak solution to the problem (1.7)-(1.11) in the sense of Definition 2.1 and assume that (6.1) holds. Then the weak solutions to the problem (1.7)-(1.11) satisfy the following energy inequality,*

$$\Phi'(t)+\Psi(t)\leq C\int_\Omega\rho(t)\mathbf{f}(t)\cdot\mathbf{v}(t)d\mathbf{x}\qquad\forall\ t\in[0,T], \tag{6.7}$$

for some positive constant $C=C(p,q,\Omega)$. In particular, the following assertions hold:

1. *If (3.1) holds together with (6.2), then there exists a positive constant $C=C(M_2,p,q,\mu,C_K,d,\Omega)$ such that*

$$\Phi'(t)+\Psi(t)\leq C\|\mathbf{f}(t)\|^{p'}_{p',\Omega}\qquad\forall\ t\in[0,T]. \tag{6.8}$$

2. *If (6.6) holds together with (6.3), then there exists a positive constant $C=C(M_2,p,q,m,|\gamma|)$ such that*

$$\Phi'(t)+\Psi(t)\leq C\|\mathbf{f}(t)\|^{m'}_{m',\Omega}\qquad\forall\ t\in[0,T]. \tag{6.9}$$

Proof. To prove (6.7), we first note that by proceeding as we did to prove (3.33), but without using Assumption (3.2), we can prove, in the case that (6.6) holds, that

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\|\mathbf{v}_n(t)\|^2_{2,\Omega}+\|\nabla\mathbf{v}_n(t)\|^q_{q,\Omega}\right)+\|\nabla\mathbf{v}_n\|^p_{p,Q_T}+|\gamma|\,\|\mathbf{v}_n\|^m_{m,Q_T}\\&\leq C\left(\|\mathbf{v}_0\|^2_{2,\Omega}+\|\nabla\mathbf{v}_0\|^q_{q,\Omega}+\|\mathbf{f}\|^2_{2,Q_T}\right),\end{aligned} \tag{6.10}$$

where now the positive constant C also depends on q. Note also that, in this case, the Galerkin approximations, whose existence is established in Section 3.1, have to be chosen in such a way that instead of $(3.18)_2$, one has $\|\nabla\mathbf{v}_{n,0}\|_{q,\Omega}\leq\|\nabla\mathbf{v}_0\|_{q,\Omega}$.

From Assumptions (3.6), (6.6) and (6.1), it is assured that the right-hand side of (6.10) is finite and thus, by means of reflexivity, there exists a subsequence (still denoted by) $\mathbf{v}_n$ such that

$$\mathbf{v}_n\longrightarrow\mathbf{v}\quad\text{weakly-}*\text{ in }L^\infty(0,T;\mathbf{H}\cap\mathbf{V}_q),\quad\text{as }n\to\infty \tag{6.11}$$

$$\mathbf{v}_n \longrightarrow \mathbf{v} \quad \text{weakly in } L^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T), \quad \text{as } n\to\infty \tag{6.12}$$

We now use Assumption (6.6) and integrate Equation (3.29) between t_0 and t_1, with $t_0<t_1$ and $t_0,t_1\in[0,T)$. Then, using Korn's inequality (2.1) in the resulting equation, we obtain

$$\begin{aligned}&\left[\frac{1}{2}\|\sqrt{\rho_n(s)}\mathbf{v}_n(s)\|_{2,\Omega}^2+\frac{\varkappa}{qC_K}\|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{q,\Omega}^q\right]\Bigg|_{s=t_0}^{s=t_1}\\&+\int_{t_0}^{t_1}\left(\frac{\mu}{C_K}\|\boldsymbol{\nabla}\mathbf{v}_n(s)\|_{p,\Omega}^p+|\gamma|\|\mathbf{v}_n(s)\|_{m,\Omega}^m\right)ds\leq\int_{t_0}^{t_1}\int_\Omega\rho_n\mathbf{f}\cdot\mathbf{v}_n\,d\mathbf{x}ds.\end{aligned} \tag{6.13}$$

Using the convergence results (6.11)-(6.12) together with a classical property of weak limits, and doing some elementary transformations, we obtain, when taking the limit inf of (6.13), as $n\to\infty$, the following relation,

$$\begin{aligned}&\left[\frac{1}{2}\|\sqrt{\rho(s)}\mathbf{v}(s)\|_{2,\Omega}^2+\frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}(s)\|_{q,\Omega}^q\right]\Bigg|_{s=t_0}^{s=t_1}+\int_{t_0}^{t_1}\left(\mu\|\boldsymbol{\nabla}\mathbf{v}(s)\|_{p,\Omega}^p+|\gamma|\|\mathbf{v}(s)\|_{m,\Omega}^m\right)ds\\&\leq C_0\int_{t_0}^{t_1}\int_\Omega\rho\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}ds,\end{aligned}$$

where $C_0=\left(\min\left\{1,\frac{1}{C_K}\right\}\right)^{-1}$. For the convergence of the first and last terms of (6.13), we also use the convergence results (3.71) and (3.73) together with (3.11).

Thus, we can write for every t, $t+\Delta t\in[0,T]$, with $\Delta t>0$,

$$\begin{aligned}&\frac{1}{|\Delta t|}\left[\frac{1}{2}\|\sqrt{\rho(s)}\mathbf{v}(s)\|_{2,\Omega}^2+\frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}(s)\|_{q,\Omega}^q\right]\Bigg|_{s=t}^{s=t+\Delta t}\\&\leq-\frac{1}{|\Delta t|}\int_t^{t+\Delta t}\left(\mu\|\boldsymbol{\nabla}\mathbf{v}(s)\|_{p,\Omega}^p+|\gamma|\|\mathbf{v}(s)\|_{m,\Omega}^m\right)ds+\frac{C_0}{|\Delta t|}\int_t^{t+\Delta t}\int_\Omega\rho\mathbf{f}\cdot\mathbf{v}d\mathbf{x}ds.\end{aligned} \tag{6.14}$$

Since $\mathbf{v}\in\mathbf{L}^p(0,T;\mathbf{V}_p)\cap\mathbf{L}^m(Q_T)$ and $\int_\Omega\rho\mathbf{f}\cdot\mathbf{v}d\mathbf{x}\in L^1[0,T]$, due to (3.11), (3.53) and Assumption (3.6), then every term on the right-hand side of (6.14) has a limit, for all $t\in[0,T]$ and as $\Delta t\to 0$. This in turn yields the existence of a limit of the left-hand side of this inequality, for all $t\in[0,T]$ and as $\Delta t\to 0$. Whence, we can write

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\sqrt{\rho(t)}\mathbf{v}(t)\|_{2,\Omega}^2+\frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^q\right)+\mu\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{p,\Omega}^p+|\gamma|\|\mathbf{v}(t)\|_{m,\Omega}^m\\&\leq C_0\int_\Omega\rho(t)\mathbf{f}(t)\cdot\mathbf{v}(t)d\mathbf{x}\end{aligned} \tag{6.15}$$

for all $t\in[0,T]$. From (6.15), we easily derive (6.7).

Let us assume that (6.2) holds. Then, using the Hölder, Sobolev and Young inequalities together with (3.11), we can show that

$$\left|\int_\Omega\rho(t)\mathbf{f}(t)\cdot\mathbf{v}(t)d\mathbf{x}\right|\leq\frac{\mu}{2}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{p,\Omega}^p+C\|\mathbf{f}(t)\|_{p',\Omega}^{p'}, \tag{6.16}$$

for some positive constant $C=C(M_2,p,\mu,C_K,d,\Omega)$.

Alternatively, if we assume that now (6.3) holds, then we use the Hölder and Young inequalities, again together with (3.11), so that

$$\left|\int_\Omega\rho(t)\mathbf{f}(t)\cdot\mathbf{v}(t)d\mathbf{x}\right|\leq\frac{|\gamma|}{2}\|\mathbf{v}(t)\|_{m,\Omega}^m+C\|\mathbf{f}(t)\|_{m',\Omega}^{m'}, \tag{6.17}$$

for other positive constant $C=C(M_2,m,|\gamma|)$.

The energy inequalities (6.8) and (6.9) now follow from plugging (6.16) and (6.17), respectively, into (6.15). ∎

REMARK 6.1. We observe that, in the case of $p\leq 2$ or $m\leq 2$, conditions (6.2) or (6.3) can be replaced just by $\mathbf{f}\in\mathbf{L}^{p'}(Q_T)$ or $\mathbf{f}\in\mathbf{L}^{m'}(Q_T)$, respectively. In these cases, note that the existence of weak solutions to the problem (1.7)-(1.11) is assured by the case (3.7) of Theorem 3.1.

Let us now consider a weak solution $(\mathbf{v},\rho)$ to the problem (1.7)-(1.11) in the sense of Definition 2.1 and let the function $\Phi(t)$, defined by (6.4), be the energy function associated to our problem. We are interested in solutions to the problem (1.7)-(1.11) with a finite energy $\Phi(t)$. Therefore, we may assume that exists a positive constant K such that

$$\Phi(t)\leq K \quad \forall\, t\in[0,T]. \tag{6.18}$$

Normalizing (6.18), we obtain

$$E(t)<1 \quad \forall\, t\in[0,T], \qquad E(t):=\frac{\Phi(t)}{K}. \tag{6.19}$$

From (6.4) and (6.19) it follows that

$$\frac{\varkappa}{Kq}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^{q}<E(t)<1 \quad \forall\, t\in[0,T]. \tag{6.20}$$

In the next result, we establish the conditions for the solutions to the problem (1.7)-(1.11) to decay in time according to the following power

$$\alpha:=\frac{p}{\min\{2,q\}}.$$

THEOREM 6.1 (Power decay). *Let $(\mathbf{v},\rho)$ be a weak solution to the problem (1.7)-(1.11) in the sense of Definition 2.1 and assume that condition (6.1) holds. In addition, assume that*

$$\frac{2d}{d+2}\leq q\leq p \qquad \text{and} \qquad \alpha>1. \tag{6.21}$$

If $\mathbf{f}=0$ a.e. in Q_T and (3.1) holds, then there exists a positive constant C, independent of t, such that

$$\Phi(t)\leq C(1+t)^{-\frac{1}{\alpha-1}} \quad \forall\, t\geq 0. \tag{6.22}$$

If $\mathbf{f}\neq 0$, but there exist positive constants $C_{\mathbf{f}}$ and σ, with $\sigma\geq\alpha'$, such that

$$\|\mathbf{f}(t)\|_{s,\Omega}^{s}\leq C_{\mathbf{f}}(1+t)^{-\sigma} \quad \forall\, t\in[0,T], \tag{6.23}$$

for $s=p'$ when (3.1) holds, or $s=m'$ when (6.6) holds, then there exists a positive constant C, independent of t, such that

$$\Phi(t)\leq C(1+t)^{-\frac{\alpha}{\alpha-1}} \quad \forall\, t\geq 0, \tag{6.24}$$

where α', p' and m' denote the Hölder conjugates of α, p and m.

Proof. In this proof, we shall use constants C indexed in the natural numbers to distinguish them from one inequality to another.

If $q \geq \frac{2d}{d+2}$, we can use Sobolev's inequality together with (3.11) so that

$$\|\sqrt{\rho(t)}\mathbf{v}(t)\|_{2,\Omega} \leq C_1 \|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega} \tag{6.25}$$

for some positive constant $C_1 = C(M_2, q, d, \Omega)$. Using (6.25) together with (6.4)-(6.5), (6.20), Hölder's inequality and assuming that $q \leq p$, we get the following chain of inequalities

$$\begin{aligned}
&\Phi(t) \leq C_1 \|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^2 + \frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^q \\
&\leq C_1 \left(\frac{Kq}{\varkappa}\right)^{\frac{2}{q}} \left(\frac{\varkappa}{Kq}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^q\right)^{\frac{2}{q}} + K\left(\frac{\varkappa}{Kq}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^q\right) \\
&\leq C_2 \left(\frac{\varkappa}{Kq}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{q,\Omega}^q\right)^{\min\left\{1,\frac{2}{q}\right\}}, \quad C_2 = C_1\left(\frac{Kq}{\varkappa}\right)^{\frac{2}{q}} + K \\
&\leq C_3 \left(\mu\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{p,\Omega}^p\right)^{\beta} \leq C_3 \Psi^{\beta}(t), C_3 = C_2\left(\frac{\varkappa}{Kq}|\Omega|^{\frac{p-q}{p}}\right)^{\min\left\{1,\frac{2}{q}\right\}}\left(\frac{1}{\mu}\right)^{\beta}, \beta = \frac{\min\{2,q\}}{p}.
\end{aligned} \tag{6.26}$$

Plugging (6.26) into (6.8) or (6.9), we obtain the following nonlinear differential inequality

$$\Phi'(t) + C_4\Phi^{\alpha}(t) \leq C_5\|\mathbf{f}(t)\|_{s,\Omega}^s \quad \forall\, t \in [0,T], \qquad \alpha = \frac{1}{\beta} = \frac{p}{\min\{2,q\}}, \tag{6.27}$$

for either $s = p'$ or $s = m'$, and where $C_4 = C_3^{-\alpha}$ and C_5 stands for the constant C given in (6.8) or (6.9).

If $\mathbf{f} = \mathbf{0}$ a.e. in Q_T, and once that, by Assumption $(6.21)_2$, $\alpha > 1$, we can immediately apply Lemma 2.3 to (6.27) so that

$$\Phi(t) \leq \left(\Phi(0)^{1-\alpha} + C_4(\alpha-1)t\right)^{-\frac{1}{\alpha-1}} \quad \forall\, t \in [0,T], \tag{6.28}$$

where C_4 is the corresponding constant from (6.27). From (6.28), one easily derives the inequality (6.22) with

$$C = \left[\max\left\{\Phi(0)^{1-\alpha}, C_4(\alpha-1)\right\}\right]^{-\frac{1}{\alpha-1}},$$

where

$$\Phi(0) = \frac{1}{2}\|\sqrt{\rho_0}\mathbf{v}_0\|_{2,\Omega}^2 + \frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}_0\|_{q,\Omega}^q \leq \frac{M_2}{2}\|\mathbf{v}_0\|_{2,\Omega}^2 + \frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}_0\|_{q,\Omega}^q.$$

If $\mathbf{f} \neq 0$ and satisfies (6.23), we plug this information into (6.27), for $s = p'$ when (3.1) holds, or for $s = m'$ when (6.6) holds, to obtain

$$\Phi'(t) + C_4\Phi^{\alpha}(t) \leq C_6(1+t)^{-\frac{\alpha}{\alpha-1}} \quad \forall\;\, t \in [0,T], \tag{6.29}$$

where $C_6 = C_5 C_{\mathbf{F}}$, and $C_{\mathbf{F}}$ is given in (6.23).

Let us look now for solutions of the form $y(t)=C(1+t)^{-\frac{1}{\alpha-1}}$ to the ordinary differential equation

$$y'(t)+C_4y^{\alpha}(t)=C_*(1+t)^{-\frac{\alpha}{\alpha-1}}, \tag{6.30}$$

where C_4 is the same constant from (6.29) and C_* is a constant to be found below, whereas C is a positive solution to the equation

$$f(C)=0,\ f(C):=-\frac{1}{\alpha-1}C+C_4C^{\alpha}-C_*.$$

Since $f(0)=-C_*$, $C_*>0$ and $\lim_{C\to+\infty}f(C)=+\infty$, by Bolzano's theorem the equation $f(C)=0$ has a root $\overline{C}>0$. On the other hand, $\overline{C}$ can be estimated from below as follows

$$\overline{C}=\left[\frac{1}{C_4}\left(C_*+\frac{1}{\alpha-1}\overline{C}\right)\right]^{\frac{1}{\alpha}}>\left(\frac{\overline{C}}{(\alpha-1)C_4}\right)^{\frac{1}{\alpha}}\Rightarrow\overline{C}>\left(\frac{1}{(\alpha-1)C_4}\right)^{\frac{1}{\alpha-1}}, \tag{6.31}$$

for the constant C_4 given in (6.30).

Next, we introduce the new function

$$G(t):=\Phi(t)-y(t)\equiv\Phi(t)-\overline{C}(1+t)^{-\frac{\alpha}{\alpha-1}},$$

which in turn satisfies the following linear differential inequality

$$G'(t)+C_7G(t)\le C_8(1+t)^{-\frac{\alpha}{\alpha-1}}, \tag{6.32}$$

where

$$C_7:=C_4\alpha\int_0^1[\lambda\Phi(t)+(1-\lambda)y(t)]^{\alpha-1}\,d\lambda\ge0 \text{ and } C_8:=C_{\mathbf{F}}-C_*.$$

Now, we choose the constant C_* introduced at (6.30) in such a way that $C_8\le0$, i.e. $C_*\ge C_{\mathbf{F}}$. Using this information, the linear differential inequality (6.32) becomes homogeneous. Solving the resulting homogeneous differential inequality, we obtain

$$G(t)\le G(0)e^{-\int_0^t C_3(\tau)d\tau}\le G(0)=\Phi(0)-\overline{C}. \tag{6.33}$$

Imposing also that

$$G(0)\le0\Leftrightarrow\overline{C}\ge\Phi(0)=\frac{1}{2}\|\sqrt{\rho_0}\mathbf{v}_0\|^2_{2,\Omega}+\frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}_0\|^q_{q,\Omega}\ge\frac{M_1}{2}\|\mathbf{v}_0\|^2_{2,\Omega}+\frac{\varkappa}{q}\|\boldsymbol{\nabla}\mathbf{v}_0\|^q_{q,\Omega},$$

it follows, from (6.33), that $G(t)\le0$ for all $t\in[0,T]$. As a consequence, by the estimate (6.31), we can see that a choice of $C_*=\max\{C_{\mathbf{F}},C_4\Phi^{\alpha}(0)\}$ proves that

$$0\le\Phi(t)\le\overline{C}(1+t)^{-\frac{\alpha}{\alpha-1}},$$

and, finally, the decay property (6.24) follows by taking $C_{\mathbf{f}}=\overline{C}$. ☐

In the next result we study the limit case of $\alpha=1$ in the nonlinear differential inequality (6.27).

THEOREM 6.2 (Exponential decay). *Let $(\mathbf{v},\rho)$ be a weak solution to the problem (1.7)-(1.11) in the sense of Definition 2.1 and assume that condition (6.1) holds. In addition, assume that one of the following conditions hold,*

$$\frac{2d}{d+2}\leq q=p\leq 2 \qquad \text{such that (3.1) holds,}$$
$$q=p \quad \text{and} \quad m=2 \qquad \text{such that (6.6) holds.}$$

If $\mathbf{f}=0$ a.e. in Q_T and (3.1) holds, then there exists a positive constant C, independent of t, such that

$$\Phi(t)\leq \Phi(0)e^{-Ct} \quad \forall\ t\geq 0. \tag{6.34}$$

If $\mathbf{f}\neq 0$ and

$$\int_0^\infty \|\mathbf{f}(\tau)\|_{s,\Omega}^s d\tau<\infty, \tag{6.35}$$

where $s=p'$ if (3.1) holds together with (6.2), or $s=m'$ if (6.6) holds together with (6.3), then there exist positive constants C_1 and C_2 such that

$$\Phi(t)\leq e^{-C_1 t}\left(\Phi(0)+C_2\int_0^t e^{C_1\tau}\|\mathbf{f}(\tau)\|_{s,\Omega}^s d\tau\right) \quad \forall\ t\geq 0. \tag{6.36}$$

Proof. Let us assume first that $q=p\leq 2$. In this case, we have $\alpha=1$ in (6.27) and $\beta=1$ in (6.26). As a consequence, we obtain from (6.27) and in the case of $q\geq\frac{2d}{d+2}$, that

$$\Phi'(t)+C_1\Phi(t)\leq C_2\|\mathbf{f}(t)\|_{s,\Omega}^s \quad \forall\ t\in[0,T], \tag{6.37}$$

where C_1 and C_2 stand here for the constants C_4 and C_5 given in (6.27), in the particular case of $\alpha=1$.

If $\mathbf{f}=0$ a.e. in Q_T, we can immediately apply Lemma 2.3 to (6.37) to get the exponential decay (6.34) with $C=C_1$.

If $\mathbf{f}\neq 0$, but (6.35) holds, then we apply Grönwall's inequality to (6.37) so that (6.36) holds true.

Let us now assume that $q=p$ and $m=2$. In this case, proceeding as we did for (6.26), we obtain from (6.4)-(6.5) and such that (6.6) holds,

$$\Phi(t)\leq\left(\frac{M_2}{2}\|\mathbf{v}(t)\|_{m,\Omega}^m+\frac{\varkappa C_K}{q}\|\boldsymbol{\nabla}\mathbf{v}(t)\|_{p,\Omega}^p\right)\leq C\Psi(t), \quad C=\max\left\{\frac{\varkappa C_K}{\mu q},\frac{M_2}{2|\gamma|}\right\}. \tag{6.38}$$

Plugging (6.38) into (6.8) or (6.9), we obtain a linear differential inequality of the same type as (6.37). Repeating the arguments of the previous case, we can prove the exponential decays (6.34) and (6.36) in this case as well. ∎

REMARK 6.2. After the existence of $\mathbf{v}$ and ρ have been shown, we can recover the pressure π, from the weak formulation (2.2), by using a version of de Rham's lemma due to Bogovskiĭ and Pileckas (see *e.g.* [9, Theorems III.3.1 and III.5.2]). Important issues regarding, not only the recovery of the pressure from the weak formulation of Kelvin-Voigt type problems, but also the study of its regularity, will be the aim of a forthcoming work.

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REFERENCES

[1] A.B. Al'shin, M.O. Korpusov, and A.G. Sveshnikov, *Blow-up in Nonlinear Sobolev Type Equations*, De Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter Co., Berlin, 15, 2011. 1, 1

[2] S.N. Antontsev, J.I. Diaz, and S. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, 48, 2002. 1, 2, 2, 2

[3] S.N. Antontsev and A.V. Kazhikhov, *Mathematical study of flows of nonhomogeneous fluids (in Russian)*, Lectures at the Novosibirsk State University, 1973. 1, 3.1

[4] S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids* (Translation from the original Russian edition, Nauka, Novosibirsk, 1983), North-Holland, Amsterdam, 1990. 1, 2

[5] S.N. Antontsev and K. Khompysh, *Generalized Kelvin-Voigt equations with p-Laplacian and source/absorption terms*, J. Math. Anal. Appl., 456(1):99–116, 2017. 1

[6] S.N. Antontsev and K. Khompysh, *Kelvin-Voight equation with p-Laplacian and damping term: Existence, uniqueness and blow-up*, J. Math. Anal. Appl., 446(2):1255–1273, 2017. 1

[7] S.N. Antontsev, H.B. de Oliveira, and K. Khompysh, *Kelvin-Voigt equations perturbed by anisotropic relaxation, diffusion and damping*, J. Math. Anal. Appl., 473:1122–1154, 2019. 1

[8] M. Fuchs, *On stationary incompressible Norton fluids and some extensions of Korn's inequality*, Z. Anal. Anwendungen, 13(2):191–197, 1994. 2

[9] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-state Problems*, Springer, New York, 2011. 6.2

[10] V.A. Kondratiev and O.A. Oleinik, *On Korn's inequalities*, C. R. Acad. Sci. Paris - Serie I, 308:483–487, 1989. 2

[11] M.O. Korpusov, *Blow-up of solutions of a class of strongly non-linear equations of Sobolev type*, Izv. Math., 68(4):783–832, 2004. 1, 6

[12] M.O. Korpusov, A.G. Sveshnikov, *Blow-up of Oskolkov's system of equations*, Sb. Math., 200(4): 549–572, 2009. 1, 6

[13] O.A. Ladyzenskaya, *On certain nonlinear problems of the theory of continuous media*, In Internat. Congress of Mathematicians, Moscow, Abstracts of Reports, 149, 1966. 1, 1

[14] O.A. Ladyzenskaya and V.A. Solonnikov, *Unique solvability of an initial-and boundary value problem for viscous incompressible nonhomogeneous fluids*, Translated from the Russian in J. Soviet Math., 9(5):697–749, 1978. 1, 3.1

[15] J. Lemoine, *On non-homogeneous viscous incompressible fluids. Existence of regular solutions*, Comment. Math. Univ. Carolin., 38(4):697–715, 1997. 1

[16] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969. 2, 3.1

[17] P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Volume 1: Incompressible Models*, Oxford Lecture Series in Mathematics and its Applications, Clarendon Press, Oxford, 3, 1996. 1, 2, 3.4

[18] J. Malek, J. Nečas, M. Rokyta, and M.Růžička, *Weak and Measure-Valued Solutions to Evolutionary PDEs*. Applied Mathematics and Mathematical Computation, Chapman & Hall, London, 13, 1996. 2, 3.1, 3.5

[19] J.C. Maxwell, *On the dynamical theory of gases*, Phil. Mag., 35(235):129–145, 1868. 1

[20] P.P. Mosolov and V.P. Mjasnikov, *A proof of Korn's inequality (Russian)*, Dokl. Akad. Nauk SSSR, 201:36–39, 1971. 2

[21] V.A. Pavlovsky, *On the theoretical description of weak water solutions of polymers*, Dokl. Akad. Nauk SSSR, 200(4):809–812, 1971. 1, 1

[22] A.P. Oskolkov, *The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers. (Russian)*, Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 38:98–136, 1973. 1, 1

[23] A.P. Oskolkov, *On the theory of unsteady flows of Kelvin-Voigt fluids*, Translated from the Russian in J. Math. Sci., 28(5):751–758, 1985. 1

[24] A.P. Oskolkov, *Nonlocal problems of motion for equations of the KelvinVoight fluids*, Translated from the Russian in J. Math. Sci., 75(6):2058–2078, 1995. 1

[25] J. Simon, *Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure*, SIAM J. Math. Anal., 21(5):1093–1117, 1990. 1

[26] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979. 2, 3.6

[27] W. Thompson, *On the elasticity and viscosity of metals*, Proc. Roy. Soc. London A, 14:289–297, 1865. 1

[28] K. Vafai (ed.), *Handbook of Porous Media*, CRC Press, Boca Raton, 2015. 1

[29] W. Voigt, *Ueber innere Reibung fester Korper, insbesondere der Metalle*, Annalen der Physik, 283(2):671–693, 1892. 1

[30] V.V. Zhikov and S.E. Pastukhova, *On the solvability of the Navier-Stokes system for a compressible non-Newtonian fluid*, Translated from the Russian in Dokl. Math., 80(1):511–515, 2009. 1

[31] V.V. Zhikov and S.E. Pastukhova, *On the solvability of the Navier-Stokes system for a nonhomogeneous non-Newtonian fluid*, Translated from the Russian in Dokl. Math., 79(3):403–407, 2009. 1

[32] V.G. Zvyagin and M.V. Turbin, *Investigation of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids*, Translated from the Russian in J. Math. Sci., 168(2):157–308, 2010. 1