COMMUN. MATH. SCI.
Vol. 17, No. 7, pp. 1757–1777

THE FACTORIZATION METHOD FOR PARTIALLY PENETRABLE OBSTACLE IN ELASTIC SCATTERING*

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Abstract. This paper considers the elastic scattering problem of a partially penetrable obstacle. By partially penetrable obstacle, we mean that the elastic incident waves can only transmit from partial boundary into the interior of the obstacle. Firstly, using the boundary integral equation method, the direct scattering problem is discussed in a brief way. Then the inverse scattering problem of reconstructing the shape and location of the obstacle from the knowledge of far field patterns due to the incident plane compressional and shear waves is considered. To this end, we use the well known factorization method to deal with it and establish the theory foundation of this method. Finally, some numerical examples are presented to illustrate the validity and feasibility of the proposed method.

Key words. partially penetrable obstacle; factorization method; inverse elastic scattering.

AMS subject classifications. 35R30; 35Q30.

1. Introduction

Elastic scattering problems have received a lot of attention in recent years. There have been some research achievements for elastic scattering problems by penetrable obstacle, in which case, the elastic wave fields pass through the whole boundary of the obstacle. In certain conditions, however, a penetrable obstacle may touch a rigid crack, then the elastic waves fields can only transmit from part of the boundary, which is the problem under consideration. As far as we know, there are few papers involving such a special transmission problem except for the paper [1], in which the authors use the linear sampling method to reconstruct a combined scatterer consisting of a penetrable obstacle and a hard crack touching with each other. Similar transmission problems also appear in electromagnetic scattering [2, 3].

The inverse transmission problems in acoustic, electromagnetic and elastic scattering have obtained abundant research results. Some of them are stated as follows. The Newton iteration method is used to solve an inverse transmission problem for the Helmholtz equation in [4]. The study of the factorization method for recovering a penetrable obstacle with a general conductive boundary condition is presented in [5], and for determining a cavity bounded by a penetrable anisotropic inhomogeneous medium from internal measurements of the cavity is given by [44]. The identification of a penetrable obstacle with mixed transmission conditions is shown in [6] by the linear sampling method. See [42] for a direct linear sampling approach to imaging scatterers in an acoustic waveguide, and see [43] for multi-frequency reconstruction of sound soft and penetrable obstacles via the linear sampling method. In the inverse electromagnetic

*Received: November 2, 2018; Accepted (in revised form): March 10, 2019. Communicated by Liliana Borcea.
This paper is supported by the Fundamental Research Funds for South-Central University for Nationalities, No.CZT18010 and the National Natural Science Foundation of People's Republic of China No.11801575.

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scattering, the linear sampling method is applied in [7] to determine the shape and surface conductivity of a partially coated dielectric from a knowledge of the far field pattern of the scattered electromagnetic wave. In the paper [8], the author deduces the formulas, which give direct links between the far field data and the unknowns of the inverse problem, to reconstruct complex obstacles by giving the shapes, deciding the nature of the obstacles (penetrable or impenetrable), and localizing the support of the coating. For the inverse scattering of elastic waves, the enclosure method is established to reconstruct penetrable unknown inclusions in a plane elastic body by boundary measurements in [9]. The topological derivative method [41] is presented for elastic-wave imaging of underground cavities. The problem of detecting a penetrable obstacle is solved by using an iterative two-step method in [10] and the linear sampling method in [11, 12]. We note that the uniqueness results are given, for example, in [13–15], respectively, for acoustic, electromagnetic and elastic inverse transmission problems.

The problem we are interested in is to study the elastic scattering by a partially penetrable obstacle, and the purpose of the present study is to extend the factorization method to the reconstruction of the elastic body from the knowledge of the far field pattern of the scattered fields for elastic plane incident waves. This method is applied in [16] to the inverse elastic scattering problem by penetrable isotropic bodies and later is proved to be suitable for acoustic transmission problem [17]. Different from the impenetrable obstacle scattering, the fundamental data-to-pattern operator G (mapping the boundary data to far field pattern) is no longer injective. By a deep investigation of the nullspace of G and giving the exact description of it via the Dirichlet-to-Neumann map, the authors derive an appropriate decomposition of the far field operator and then explore thoroughly the properties of the involved boundary potential operators such that the abstract functional theoretic result [18] can be employed to characterize the penetrable obstacle. Since the boundary of the obstacle in our problem is made up of the penetrable and impenetrable parts, the Dirichlet-to-Neumann map and the boundary potential operators don't maintain some of the properties as those in the case of completely penetrable obstacle. We adopt different technical ideas to overcome this difficulty and obtain the theoretical framework of the proposed method.

Since the first work [19] on using the factorization method to recover a soft or hard obstacle in acoustic scattering, this method has been applied successfully to other various inverse shape scattering problems for Helmholtz equation and Maxwell equations as well as in electrical impedance tomography, such as [20–24, 44]. When it comes to elastic wave scattering, the recorded literature on the factorization method is rare. See [25] for a rigid cavity, [16] for penetrable bodies mentioned above, and [26] for rigid obstacles by using only the knowledge of the transversal or longitudinal far field pattern corresponding to incident plane shear wave or pressure wave. We refer to the works [27–30] for a recent progress on the factorization method.

The outline of this paper is organized as follows. In Section 2, we formulate direct and inverse scattering problems. Based on the conclusions in [1], a brief derivation to the direct scattering is given by using the boundary integral equation approach. The obtained boundary integral system is useful for the numerical experiments. In Section 3, under some suitable assumptions, a rigorous proof of the factorization method for the reconstruction of the partially penetrable obstacle is provided. The properties of the involved decomposition operators for the far field operator are proved to satisfy the range identity theorem [18]. The numerical simulations are presented in Section 4 to justify the validity of our method.

2. The direct and inverse scattering problems

Consider the scattering of time harmonic elastic plane wave $\mathbf{u}^{in}$ by a bounded partially penetrable inclusion $D_i \subset \mathbb{R}^2$. The boundary ∂D_i of the connected domain D_i is assumed to be Lipschitz continuous and has a Lipschitz dissection $\partial D_i = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_1 and Γ_1 are open subsets of ∂D_i and are disjoint, denoting the penetrable and impenetrable portion of the boundary, respectively. The unbounded domain $\mathbb{R}^2 \backslash \overline{D}_i$ is denoted by D_e. Both regions of D_i and D_e are occupied by isotropic and homogeneous elastic medium with constant density ρ_α, Lamé constants μ_α and λ_α satisfying $\mu_\alpha > 0, 2\mu_\alpha + \lambda_\alpha > 0$, for $\alpha = i, e$ and we assume that $\rho_i \geq \rho_e$, $\mu_i \geq \mu_e$, $\lambda_i \geq \lambda_e$. Then the generated wave fields by the obstacle are the scattered one $\mathbf{u}$ defined in D_e and the transmitted field $\mathbf{v}$ in D_i, which are governed by the Navier equations

$$\begin{cases} \mu_e \Delta \mathbf{u} + (\mu_e + \lambda_e) \nabla (\nabla \cdot \mathbf{u}) + \rho_e \omega^2 \mathbf{u} = \mathbf{0} & \text{in } D_e, \\ \mu_i \Delta \mathbf{v} + (\mu_i + \lambda_i) \nabla (\nabla \cdot \mathbf{v}) + \rho_i \omega^2 \mathbf{v} = \mathbf{0} & \text{in } D_i, \end{cases} \tag{2.1}$$

where $\omega > 0$ is the circular frequency. From now on, we denote by Δ_α^* the Lamé operator $\mu_\alpha \Delta + (\mu_\alpha + \lambda_\alpha) \nabla (\nabla \cdot)$ for brevity.

A description of some notations is given as follows. For $\mathbf{x} \in \mathbb{R}^2$, let $\hat{\mathbf{x}}$ be the unit vector $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|$ and $\mathbf{x}^\perp$ be the vector obtained by rotating $\mathbf{x}$ anticlockwise by $\pi/2$. As usual, we use the notations $\mathbf{a} \cdot \mathbf{b}$ to represent the scalar product and $\mathbf{a} \times \mathbf{b}$ to present the vector product for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. For a vector function $\mathbf{u} = [u^1, u^2]^\top$ and a matrix function $W = [\mathbf{w}^1, \mathbf{w}^2]^\top$, the symbols $\nabla \mathbf{u}$ and $\nabla \cdot W$ are denoted respectively by

$$\nabla \mathbf{u} = [\nabla u^1, \nabla u^2]^\top, \quad \nabla \cdot W = [\nabla \cdot \mathbf{w}^1, \nabla \cdot \mathbf{w}^2]^\top.$$

Let $\mathbf{n}$ be the unit outward normal vector of the boundaries ∂D_i. On the penetrable part of the boundary ∂D_i, the following transmission boundary conditions are satisfied

$$\begin{cases} \mathbf{u} + \mathbf{u}^{in} = \mathbf{v} & \text{on } \Gamma_1, \\ T_e \mathbf{u} + T_e \mathbf{u}^{in} = T_i \mathbf{v} + i\lambda \mathbf{v} & \text{on } \Gamma_1. \end{cases} \tag{2.2}$$

Here, $\lambda < 0$ is the constant surface conductivity, T_α is the surface stress operator on Γ_1 which is given by

$$\begin{aligned} T_\alpha \mathbf{w} &= (2\mu_\alpha \mathbf{n} \cdot \nabla + \lambda_\alpha \mathbf{n} \nabla \cdot - \mu_\alpha \mathbf{n}^\perp \nabla^\perp \cdot) \mathbf{w} \\ &= \begin{bmatrix} (\lambda_\alpha + 2\mu_\alpha) \dfrac{\partial w_1}{\partial x_1} + \lambda_\alpha \dfrac{\partial w_2}{\partial x_2} & \mu_\alpha (\dfrac{\partial w_1}{\partial x_2} + \dfrac{\partial w_2}{\partial x_1}) \\ \mu_\alpha (\dfrac{\partial w_1}{\partial x_2} + \dfrac{\partial w_2}{\partial x_1}) & \lambda_\alpha \dfrac{\partial w_1}{\partial x_1} + (\lambda_\alpha + 2\mu_\alpha) \dfrac{\partial w_2}{\partial x_2} \end{bmatrix} \mathbf{n}. \end{aligned}$$

The Dirichlet boundary condition is imposed on both sides of the impenetrable part Γ_2,

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_2, \tag{2.3}$$

$$\mathbf{u} + \mathbf{u}^{in} = \mathbf{0} \quad \text{on } \Gamma_2. \tag{2.4}$$

The incident wave is assumed to be a longitudinal plane wave with the form

$$\mathbf{u}^{in} = \mathbf{u}_p^{in} = \mathbf{d} e^{ik_{p,e} \mathbf{x} \cdot \mathbf{d}}, \quad \mathbf{d} \in \mathbb{S},$$

where $\mathbb{S}$ is the unit circle in $\mathbb{R}^2$ and $\mathbf{d}$ is the incident direction, or a transversal plane wave with the form

$$\mathbf{u}^{in} = \mathbf{u}_s^{in} = \mathbf{q}e^{ik_{s,e}\mathbf{x}\cdot\mathbf{d}}, \quad \mathbf{q}, \mathbf{d} \in \mathbb{S},$$

where $\mathbf{q}$ is the polarization direction such that $\mathbf{q} \perp \mathbf{d}$. The wave numbers of compressional and shear waves $k_{p,e}$ and $k_{s,e}$, respectively are given by

$$k_{p,e} = \omega\sqrt{\frac{\rho_e}{2\mu_e + \lambda_e}} \quad \text{and} \quad k_{s,e} = \omega\sqrt{\frac{\rho_e}{\mu_e}}.$$

The wave numbers $k_{p,i}$ and $k_{s,i}$ can be defined in a similar way.

Due to the Helmholtz decomposition theorem [31], the scattered field $\mathbf{u}$ can be decomposed as

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s, \quad \mathbf{u}_p = -\frac{1}{k_{p,e}^2}\nabla(\nabla\cdot\mathbf{u}), \quad \mathbf{u}_s = -\frac{1}{k_{s,e}^2}\nabla^{\perp}(\nabla^{\perp}\cdot\mathbf{u}),$$

where $\mathbf{u}_p$ denotes the longitudinal wave and $\mathbf{u}_s$ is the transversal wave. Furthermore, each displacement field $\mathbf{u}_a (a = p, s)$ should satisfy the Helmholtz equation

$$\Delta\mathbf{u}_a + k_{a,e}^2\mathbf{u}_a = \mathbf{0},$$

and the Kupradze radiation condition [32]

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\mathbf{u}_p}{\partial r} - ik_{p,e}\mathbf{u}_p\right) = \mathbf{0}, \quad \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\mathbf{u}_s}{\partial r} - ik_{s,e}\mathbf{u}_s\right) = \mathbf{0}, \quad r = |\mathbf{x}| \tag{2.5}$$

uniformly in all directions $\hat{\mathbf{x}} \in \mathbb{S}$. In the sequel, the solution of Navier Equation (2.1) satisfying the Kupradze radiation condition is called the radiating solution. It holds that the radiating solution to the Navier equation has the asymptotic expansions of the forms [33, 34]

$$\mathbf{u}(\mathbf{x}) = \frac{e^{ik_{p,e}|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}u_p^{\infty}(\hat{\mathbf{x}})\hat{\mathbf{x}} + \frac{e^{ik_{s,e}|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}u_s^{\infty}(\hat{\mathbf{x}})\hat{\mathbf{x}}^{\perp} + O(|\mathbf{x}|^{-3/2}), \ |\mathbf{x}| \to \infty \tag{2.6}$$

and

$$T_{e,\hat{\mathbf{x}}}\mathbf{u}(\mathbf{x}) = \frac{i\omega^2}{k_{p,e}}\frac{e^{ik_{p,e}|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}u_p^{\infty}(\hat{\mathbf{x}})\hat{\mathbf{x}} + \frac{i\omega^2}{k_{s,e}}\frac{e^{ik_{s,e}|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}u_s^{\infty}(\hat{\mathbf{x}})\hat{\mathbf{x}}^{\perp} + O(|\mathbf{x}|^{-1}), \ |\mathbf{x}| \to \infty, \tag{2.7}$$

where $u_p^{\infty}(\hat{\mathbf{x}})$ is the compressional far field pattern of $\mathbf{u}$ and $u_s^{\infty}(\hat{\mathbf{x}})$ is the shear far field pattern of $\mathbf{u}$. The far field pattern of the scattered field $\mathbf{u}$ is defined by

$$\mathbf{u}^{\infty}(\hat{\mathbf{x}}) = (u_p^{\infty}(\hat{\mathbf{x}}), u_s^{\infty}(\hat{\mathbf{x}})).$$

We call the direct scattering problem (2.1)–(2.5) as **DP**, and the classical boundary integral equation method can be used to solve it. Here we just give a brief discussion on this issue and obtain an equivalent boundary integral system for the sake of handling the inverse scattering problem. We refer to the papers [33, 35] applying the boundary integral equation method to solve the elastic obstacle scattering problems and to [1] for mixed obstacle scattering problem.

We now deal with the direct scattering problem **DP** and firstly introduce some Sobolev spaces. Let $H^1(D_i)$ and $H^1_{loc}(D_e)$ be the usual Sobolev spaces with $H^{1/2}(\partial D_i)$ being the trace space. We introduce the following trace spaces on $\Gamma_l, l=1,2$.

$$
\begin{aligned}
[H^{1/2}(\Gamma_l)]^2 &= \{\mathbf{u}|_{\Gamma_l}:\ \mathbf{u}\in[H^{1/2}(\partial D_i)]^2\},\\
[\widetilde{H}^{1/2}(\Gamma_l)]^2 &= \{\mathbf{u}\in[H^{1/2}(\partial D_i)]^2:\ \mathrm{supp}\,\mathbf{u}\subseteq\overline{\Gamma_l}\},\\
[H^{-1/2}(\Gamma_l)]^2 &= \left([\widetilde{H}^{1/2}(\Gamma_l)]^2\right)',\ \text{the dual space of } [\widetilde{H}^{1/2}(\Gamma_l)]^2,\\
[\widetilde{H}^{-1/2}(\Gamma_l)]^2 &= \left([H^{1/2}(\Gamma_l)]^2\right)',\ \text{the dual space of } [H^{1/2}(\Gamma_l)]^2.
\end{aligned}
$$

Consider a general problem: let $\mathbf{f}\in[H^{1/2}(\Gamma_1)]^2$, $\mathbf{g}\in[H^{-1/2}(\Gamma_1)]^2$ and $\mathbf{h}\in[H^{-1/2}(\Gamma_2)]^2$ seek a radiating solution $\mathbf{u}\in[H^1_{loc}(D_e)]^2$ and $\mathbf{v}\in[H^1(D_i)]^2$ such that

$$
\begin{cases}
\Delta_e^*\mathbf{u}+\rho_e\omega^2\mathbf{u} &= \mathbf{0} \quad \text{in } D_e,\\
\Delta_i^*\mathbf{v}+\rho_i\omega^2\mathbf{v} &= \mathbf{0} \quad \text{in } D_i,\\
\mathbf{u}-\mathbf{v} &= \mathbf{f} \quad \text{on } \Gamma_1,\\
T_e\mathbf{u}-T_i\mathbf{v}-i\lambda\mathbf{v} &= \mathbf{g} \quad \text{on } \Gamma_1,\\
\mathbf{v} &= \mathbf{0} \quad \text{on } \Gamma_2,\\
\mathbf{u} &= \mathbf{h} \quad \text{on } \Gamma_2.
\end{cases}
\tag{2.8}
$$

The fundamental solution, which is also called Green's tensor of the Navier equation in free space, is given by

$$
\Gamma_\alpha(\mathbf{x},\mathbf{y}) = \frac{i}{4\mu_\alpha}H_0^{(1)}(k_{s,\alpha}|\mathbf{x}-\mathbf{y}|)I+\frac{i}{4\omega^2}\nabla_x^\top\nabla_x(H_0^{(1)}(k_{s,\alpha}|\mathbf{x}-\mathbf{y}|)-H_0^{(1)}(k_{p,\alpha}|\mathbf{x}-\mathbf{y}|))
$$

for $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$ and $\mathbf{x}\neq\mathbf{y}$, where $H_0^{(1)}(\cdot)$ is the Hankel function of the first kind of order zero. The following four boundary integral operators in terms of the fundamental solution will be used

$$
\begin{aligned}
(H^\alpha_{jl}\mathbf{g})(\mathbf{x}) &= \int_{\Gamma_j}\Gamma_\alpha(\mathbf{x},\mathbf{y})\cdot\mathbf{g}(\mathbf{y})ds(\mathbf{y}), && \mathbf{x}\in\Gamma_l,\\
(K^\alpha_{jl}\mathbf{g})(\mathbf{x}) &= \int_{\Gamma_j}[T_{\alpha,\mathbf{y}}\Gamma_\alpha(\mathbf{x},\mathbf{y})]^\top\cdot\mathbf{g}(\mathbf{y})ds(\mathbf{y}), && \mathbf{x}\in\Gamma_l,\\
(K'^\alpha_{jl}\mathbf{g})(\mathbf{x}) &= \int_{\Gamma_j}T_{\alpha,\mathbf{x}}\Gamma_\alpha(\mathbf{x},\mathbf{y})\cdot\mathbf{g}(\mathbf{y})ds(\mathbf{y}), && \mathbf{x}\in\Gamma_l,\\
(L^\alpha_{jl}\mathbf{g})(\mathbf{x}) &= T_{\alpha,\mathbf{x}}\int_{\Gamma_j}[T_{\alpha,\mathbf{y}}\Gamma_\alpha(\mathbf{x},\mathbf{y})]^\top\cdot\mathbf{g}(\mathbf{y})ds(\mathbf{y}), && \mathbf{x}\in\Gamma_l,
\end{aligned}
$$

for $j,l=1,2,3$. See [36] for the following mapping properties

$$
\begin{aligned}
&H^\alpha_{ll}:[\widetilde{H}^{-1/2}(\Gamma_l)]^2\to[H^{1/2}(\Gamma_l)]^2, && K^\alpha_{ll}:[\widetilde{H}^{1/2}(\Gamma_l)]^2\to[H^{1/2}(\Gamma_l)]^2,\\
&K'^\alpha_{ll}:[\widetilde{H}^{-1/2}(\Gamma_l)]^2\to[H^{-1/2}(\Gamma_l)]^2, && L^\alpha_{ll}:[\widetilde{H}^{1/2}(\Gamma_l)]^2\to[H^{-1/2}(\Gamma_l)]^2.
\end{aligned}
$$

Next, the layer potentials will be used to solve problem (2.8) and we suppose the solution pair $(\mathbf{u},\mathbf{v})$ in the form of combined single-and double-layer potentials

$$
\mathbf{u}(\mathbf{x}) = \int_{\Gamma_1}\left\{\Gamma_e(\mathbf{x},\mathbf{y})\cdot\mathbf{b}(\mathbf{y})+[T_{e,\mathbf{y}}\Gamma_e(\mathbf{x},\mathbf{y})]^\top\cdot\mathbf{a}(\mathbf{y})\right\}ds(\mathbf{y})
$$

$$+\int_{\Gamma_2}\Gamma_e(\mathbf{x},\mathbf{y})\cdot\mathbf{c}(\mathbf{y})ds(\mathbf{y}),\ \mathbf{x}\in D_e, \tag{2.9}$$

$$\mathbf{v}(\mathbf{x})=\int_{\Gamma_1}\Big\{\Gamma_i(\mathbf{x},\mathbf{y})\cdot\mathbf{b}(\mathbf{y})+[T_{i,\mathbf{y}}\Gamma_i(\mathbf{x},\mathbf{y})]^\top\cdot\mathbf{a}(\mathbf{y})\Big\}ds(\mathbf{y})$$
$$+\int_{\Gamma_2}\Gamma_i(\mathbf{x},\mathbf{y})\cdot\mathbf{e}(\mathbf{y})ds(\mathbf{y}),\ \mathbf{x}\in D_i \tag{2.10}$$

with $\mathbf{a}\in[\widetilde{H}^{1/2}(\Gamma_1)]^2$, $\mathbf{b}\in[\widetilde{H}^{-1/2}(\Gamma_1)]^2$, $\mathbf{c}\in[\widetilde{H}^{-1/2}(\Gamma_2)]^2$ and $\mathbf{e}\in[\widetilde{H}^{1/2}(\Gamma_2)]^2$ being undetermined densities. Note that such combined single-and double-layer potentials make sure that $(\mathbf{u},\mathbf{v})$ belongs to $[H^1_{loc}(D_e)]^2\times[H^1(D_i)]^2$

By the jump relations of single-and double-layer potentials [36], the boundary conditions in Equation (2.8) yield a boundary integral system

$$\begin{bmatrix} M_{11} & M_{12} & K'^{e}_{21} & -K'^{i}_{21}-i\lambda H^i_{21} \\ K^e_{11}-K^i_{11}+I & H^e_{11}-H^i_{11} & H^e_{21} & -H^i_{21} \\ K^e_{12} & H^e_{12} & H^e_{22} & 0 \\ K^i_{12} & H^i_{12} & 0 & H^i_{22} \end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\\ \mathbf{e}\end{bmatrix}=\begin{bmatrix}\mathbf{g}\\ \mathbf{f}\\ \mathbf{h}\\ \mathbf{0}\end{bmatrix}, \tag{2.11}$$

where $M_{11}=L^e_{11}-L^i_{11}-i\lambda K^i_{11}+\frac{i\lambda}{2}I$, $M_{12}=K'^{e}_{11}-K'^{i}_{11}-i\lambda H^i_{11}-I$. Denote by A the boundary integral operator on the left side of above equation, and define the Sobolev spaces

$$X:=[\widetilde{H}^{1/2}(\Gamma_1)]^2\times[\widetilde{H}^{-1/2}(\Gamma_1)]^2\times[\widetilde{H}^{1/2}(\Gamma_2)]^2\times[\widetilde{H}^{-1/2}(\Gamma_2)]^2,$$

$$X^*:=[H^{-1/2}(\Gamma_1)]^2\times[H^{1/2}(\Gamma_1)]^2\times[H^{-1/2}(\Gamma_2)]^2\times[H^{1/2}(\Gamma_2)]^2,$$

one can observe that $A:X\to X^*$ is a bounded operator.

Furthermore, following the ideas in [1], it can be proved that the operator A has a bounded inverse operator. Thus the boundary integral system (2.11) is solvable, from which we can deduce that problem (2.8) possesses the solution in the form of (2.9) and (2.10). So, we conclude that the direct scattering problem **DP** is well posed.

The inverse scattering problem under consideration is the determination of the partially penetrable obstacle D_i, which is regarded as **IP**. The inversion data is the knowledge of the far field pattern $\mathbf{u}^\infty(\hat{\mathbf{x}},\mathbf{d};\mathbf{t})$ of the scattering field $\mathbf{u}(\mathbf{x},\mathbf{d};\mathbf{t})$ for all observation direction $\hat{\mathbf{x}}\in\mathbb{S}$, incident direction $\mathbf{d}\in\mathbb{S}$ and the polarization $\mathbf{t}=\mathbf{d}$ or $\mathbf{q}$ associated with the incident plane wave $\mathbf{d}e^{ik_{p,e}\mathbf{x}\cdot\mathbf{d}}$ or $\mathbf{q}e^{ik_{s,e}\mathbf{x}\cdot\mathbf{d}}$.

We aim at extending the factorization method to the inverse elastic scattering problems **IP** and now introduce the elastic Herglotz wavefunction with density $\tau=(\tau_p,\tau_s)\in[L^2(\mathbb{S})]^2$ defined by

$$\tilde{\mathbf{v}}_\tau(\mathbf{x})=e^{-i\pi/4}\int_{\mathbb{S}}\Big\{\sqrt{\frac{k_{p,e}}{\omega}}e^{ik_{p,e}\mathbf{d}\cdot\mathbf{x}}\mathbf{d}\tau_p(\mathbf{d})+\sqrt{\frac{k_{s,e}}{\omega}}e^{ik_{s,e}\mathbf{d}\cdot\mathbf{x}}\mathbf{d}^\perp\tau_s(\mathbf{d})\Big\}ds(\mathbf{d}),\quad \mathbf{x}\in\mathbb{R}^2. \tag{2.12}$$

The Hilbert space $[L^2(\mathbb{S})]^2$ throughout this paper is equipped with the inner product

$$\langle\mathbf{g},\mathbf{h}\rangle=\frac{\omega}{k_{p,e}}\int_{\mathbb{S}}g_p\overline{h_p}ds+\frac{\omega}{k_{s,e}}\int_{\mathbb{S}}g_s\overline{h_s}ds,\quad \mathbf{g},\mathbf{h}\in[L^2(\mathbb{S})]^2.$$

The elastic far field operator $F:[L^2(\mathbb{S})]^2\to[L^2(\mathbb{S})]^2$ is defined by

$$(F\tau)(\hat{\mathbf{x}})=e^{-i\pi/4}\int_{\mathbb{S}}\Big\{\sqrt{\frac{k_{p,e}}{\omega}}\mathbf{u}^\infty(\hat{\mathbf{x}},\mathbf{d};\mathbf{d})\tau_p(\mathbf{d})+\sqrt{\frac{k_{s,e}}{\omega}}\mathbf{u}^\infty(\hat{\mathbf{x}},\mathbf{d};\mathbf{d}^\perp)\tau_s(\mathbf{d})\Big\}ds(\mathbf{d}),\quad \hat{\mathbf{x}}\in\mathbb{S}, \tag{2.13}$$

where $\mathbf{u}^\infty$ is the far field pattern of the scattered field $\mathbf{u}$ to the problem (2.1)–(2.5). We know from the superposition principle that it is the far field pattern of the resulting scattered field aroused by the incidence of Herglotz wavefunction.

The factorization method for inverse problem **IP** relies on the far field equation

$$(F\mathbf{g}_z)(\hat{\mathbf{x}})=\Gamma_e^\infty(\hat{\mathbf{x}},\mathbf{z};\mathbf{p})\quad \text{for}\quad \mathbf{g}_z\in[L^2(\mathbb{S})]^2,\hat{\mathbf{x}}\in\mathbb{S}, \tag{2.14}$$

where $\Gamma_e^\infty(\hat{\mathbf{x}},\mathbf{z};\mathbf{p})=(\Gamma_{p,e}^\infty(\hat{\mathbf{x}},\mathbf{z};\mathbf{p}),\Gamma_{s,e}^\infty(\hat{\mathbf{x}},\mathbf{z};\mathbf{p}))$ is the far field pattern of an elastic point source $\Gamma_e(\mathbf{x},\mathbf{z};\mathbf{p})=\Gamma_e(\mathbf{x},\mathbf{z})\cdot\mathbf{p}$ in $\mathbf{z}\in\mathbb{R}^2$ with the polarization direction $\mathbf{p}\in\mathbb{S}$. The longitudinal and transverse parts of $\Gamma_e(\mathbf{x},\mathbf{y};\mathbf{p})$ are respectively given by

$$\Gamma_{p,e}^\infty(\hat{\mathbf{x}},\mathbf{y};\mathbf{p})=\frac{1}{2\mu+\lambda}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{p,e}}}e^{-ik_{p,e}\hat{\mathbf{x}}\cdot\mathbf{y}}\hat{\mathbf{x}}\cdot\mathbf{p} \tag{2.15}$$

and

$$\Gamma_{s,e}^\infty(\hat{\mathbf{x}},\mathbf{y};\mathbf{p})=\frac{1}{\mu}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{s,e}}}e^{-ik_{s,e}\hat{\mathbf{x}}\cdot\mathbf{y}}\hat{\mathbf{x}}^\perp\cdot\mathbf{p}. \tag{2.16}$$

The factorization method is to establish the theoretical basis for the following phenomena. The behavior of the solution $\mathbf{g}_z$ for the varying point $\mathbf{z}$ plays a role as an indicator function, from where we know whether the test point $\mathbf{z}$ is located in the unknown obstacle D_i and thereby obtain the location information on the obstacle.

3. The factorization method

This section concerns with the inverse problem **IP**. We proceed in three steps.

- properly decompose the far field operator F in the form $F=G^*TG$,
- use the test function $\Gamma_e^\infty(\hat{\mathbf{x}},\mathbf{z};\mathbf{p})$ to characterize the obstacle D_i,
- bridge between the test function and the data operator F by the data-to-pattern operator G.

The difficulty is to verify that the operators T and G satisfy the assumptions in the range identity theorem. We made partial use of the idea in [16] to show the coercivity of the operator T.

We begin with the definitions of some operators related to the inverse problem.

In this paper, assume that ω is not an eigenvalue for following interior Dirichlet problem.

$$\begin{cases}\Delta_e^*\widehat{\mathbf{u}}+\rho_e\omega^2\widehat{\mathbf{u}}=\mathbf{0} & \text{in } D_i,\\ \widehat{\mathbf{u}}=\varphi & \text{on } \Gamma_1,\\ \widehat{\mathbf{u}}=\psi & \text{on } \Gamma_2,\end{cases} \tag{3.1}$$

where $\varphi\in[H^{1/2}(\Gamma_1)]^2$ and $\psi\in[H^{1/2}(\Gamma_2)]^2$. Then we can define the Dirichlet-to-Neumann map $\Lambda:[H^{1/2}(\Gamma_1)]^2\times[H^{1/2}(\Gamma_2)]^2\to[H^{-1/2}(\Gamma_1)]^2$ by

$$\Lambda(\varphi,\psi)=T_e\widehat{\mathbf{u}}|_{\Gamma_1}. \tag{3.2}$$

Let φ,ψ be given as in problem (3.1) and consider the following problem

$$\begin{cases} \Delta_e^*\mathbf{u}+\rho_e\omega^2\mathbf{u} &= \mathbf{0} \quad \text{in } D_e,\\ \Delta_i^*\mathbf{v}+\rho_i\omega^2\mathbf{v} &= \mathbf{0} \quad \text{in } D_i,\\ \mathbf{u}-\mathbf{v} &= \varphi \quad \text{on } \Gamma_1,\\ T_e\mathbf{u}-T_i\mathbf{v}-i\lambda\mathbf{v} &= \Lambda(\varphi,\psi) \quad \text{on } \Gamma_1,\\ \mathbf{v} &= \mathbf{0} \quad \text{on } \Gamma_2,\\ \mathbf{u} &= \psi \quad \text{on } \Gamma_2. \end{cases} \tag{3.3}$$

Also, $\mathbf{u}$ is required to satisfy Kupradze radiation condition. The well posedness of this problem defines a data-to-pattern operator $G:[H^{1/2}(\Gamma_1)]^2\times[H^{1/2}(\Gamma_2)]^2\to[L^2(\mathbb{S})]^2$ by

$$G(\varphi,\psi)(\hat{\mathbf{x}})=\mathbf{u}^\infty(\hat{\mathbf{x}}), \tag{3.4}$$

where $\mathbf{u}^\infty(\hat{\mathbf{x}})$ is the far field pattern of the scattered field $\mathbf{u}$.

Let $\tilde{\mathbf{v}}_\tau$ be the Herglotz wavefunction and we define the Herglotz wave operator $H:[L^2(\mathbb{S})]^2\to[H^{1/2}(\Gamma_1)]^2\times[H^{1/2}(\Gamma_2)]^2$ by

$$H\tau=(\tilde{\mathbf{v}}_\tau|_{\Gamma_1},\tilde{\mathbf{v}}_\tau|_{\Gamma_2}). \tag{3.5}$$

Then, it follows that

$$F\tau=-G(H\tau). \tag{3.6}$$

In order to apply the factorization method, a deep study of the relevant operators is necessary.

Obviously, the adjoint operator $H^*:[\widetilde{H}^{-1/2}(\Gamma_1)]^2\times[\widetilde{H}^{-1/2}(\Gamma_2)]^2\to[L^2(\mathbb{S})]^2$ of H is given in the form

$$\begin{aligned}&(H^*(\phi,\theta))(\mathbf{d})\\ &=e^{i\pi/4}\Big(\int_{\Gamma_1}\sqrt{\frac{\omega}{k_{p,e}}}e^{-ik_{p,e}\mathbf{d}\cdot\mathbf{x}}\mathbf{d}\cdot\phi(\mathbf{x})ds(\mathbf{x}),\int_{\Gamma_1}\sqrt{\frac{\omega}{k_{s,e}}}e^{-ik_{s,e}\mathbf{d}\cdot\mathbf{x}}\mathbf{d}^\perp\cdot\phi(\mathbf{x})ds(\mathbf{x})\Big)\\ &\quad+e^{i\pi/4}\Big(\int_{\Gamma_2}\sqrt{\frac{\omega}{k_{p,e}}}e^{-ik_{p,e}\mathbf{d}\cdot\mathbf{x}}\mathbf{d}\cdot\theta(\mathbf{x})ds(\mathbf{x}),\int_{\Gamma_2}\sqrt{\frac{\omega}{k_{s,e}}}e^{-ik_{s,e}\mathbf{d}\cdot\mathbf{x}}\mathbf{d}^\perp\cdot\theta(\mathbf{x})ds(\mathbf{x})\Big)\end{aligned}$$

for $\mathbf{d}\in\mathbb{S}$. We can prove with the help of (2.15), (2.16) that the function $1/(\sqrt{8\pi\omega})H^*(\phi,\theta)$ is just the far field pattern of the following potential

$$\widehat{\mathbf{w}}(\mathbf{x})=\int_{\Gamma_1}\Gamma_e(\mathbf{x},\mathbf{y})\cdot\phi(\mathbf{y})ds(\mathbf{y})+\int_{\Gamma_2}\Gamma_e(\mathbf{x},\mathbf{y})\cdot\theta(\mathbf{y})ds(\mathbf{y}) \tag{3.7}$$

for $(\phi,\theta)\in[\widetilde{H}^{-1/2}(\Gamma_1)]^2\times[\widetilde{H}^{-1/2}(\Gamma_2)]^2$ and $\mathbf{x}\in\mathbb{R}^2\backslash\overline{\partial D_i}$.

Now, consider the interior transmission problem as below,

$$\begin{cases} \Delta_e^*\mathbf{w}+\rho_e\omega^2\mathbf{w} &= \mathbf{0} \quad \text{in } D_i,\\ \Delta_i^*\widehat{\mathbf{v}}+\rho_i\omega^2\widehat{\mathbf{v}} &= \mathbf{0} \quad \text{in } D_i,\\ \mathbf{w}-\widehat{\mathbf{v}} &= \mathbf{0} \quad \text{on } \Gamma_1,\\ T_e\mathbf{w}-T_i\widehat{\mathbf{v}}-i\lambda\widehat{\mathbf{v}} &= \phi \quad \text{on } \Gamma_1,\\ \widehat{\mathbf{v}} &= \mathbf{0} \quad \text{on } \Gamma_2,\\ \mathbf{w} &= \mathbf{0} \quad \text{on } \Gamma_2. \end{cases} \tag{3.8}$$

We refer [37] for a detailed analysis on a similar problem in acoustic scattering. The values of ω are called transmission eigenvalues, if the homogeneous interior transmission problem (3.8) has a non-trivial solution. Throughout this paper, we assume that ω is not a transmission eigenvalue and the problem (3.8) is well posed.

Noting the jump relation of the single layer potential and the boundary conditions in problem (3.8), a simple calculation yields

$$T_e\widehat{\mathbf{w}}_+ - T_i\widehat{\mathbf{v}} - i\lambda\widehat{\mathbf{v}} = T_e\widehat{\mathbf{w}}_+ + \phi - T_e\mathbf{w} = T_e\widehat{\mathbf{w}}_- - T_e\mathbf{w}, \text{ on } \Gamma_1,$$

$$\widehat{\mathbf{w}}_+ - \widehat{\mathbf{v}} = \widehat{\mathbf{w}}_- - \mathbf{w}, \text{ on } \Gamma_1 \text{ and } \widehat{\mathbf{w}}_+ = \widehat{\mathbf{w}}_- = \widehat{\mathbf{w}}_- - \mathbf{w}, \text{ on } \Gamma_2.$$

The notation $\widehat{\mathbf{w}}_\pm$ means $\lim_{h\to 0^+}\widehat{\mathbf{w}}(\mathbf{x}\pm h\mathbf{n})$ for $\mathbf{x}\in\Gamma$ and $T_e\widehat{\mathbf{w}}_\pm$ represents $\lim_{h\to 0^+}(2\mu_i\mathbf{n}\cdot\nabla + \lambda_i\mathbf{n}\nabla\cdot - \mu_i\mathbf{n}^\perp\nabla^\perp\cdot)\widehat{\mathbf{w}}(\mathbf{x}\pm h\mathbf{n})$ for $\mathbf{x}\in\Gamma$. In addition, we can obtain that

$$(T_e\widehat{\mathbf{w}}_- - T_e\mathbf{w})|_{\Gamma_1} = \Lambda((\widehat{\mathbf{w}}_- - \mathbf{w})|_{\Gamma_1}, \widehat{\mathbf{w}}_+|_{\Gamma_2})$$

since $\widehat{\mathbf{w}} - \mathbf{w}$ satisfies problem (3.1) in the domain D_i with $\varphi = \widehat{\mathbf{w}}_- - \mathbf{w}$ on Γ_1, $\psi = \widehat{\mathbf{w}}_- - \mathbf{w}$ on Γ_2.

Therefore, $(\widehat{\mathbf{w}}, \widehat{\mathbf{v}})$ satisfies following problem

$$\begin{cases} \Delta_e^*\widehat{\mathbf{w}} + \rho_e\omega^2\widehat{\mathbf{w}} = \mathbf{0} & \text{in } D_e, \\ \Delta_i^*\widehat{\mathbf{v}} + \rho_i\omega^2\widehat{\mathbf{v}} = \mathbf{0} & \text{in } D_i, \\ \widehat{\mathbf{w}} - \widehat{\mathbf{v}} = \widehat{\mathbf{w}}_- - \mathbf{w} & \text{on } \Gamma_1, \\ T_e\widehat{\mathbf{w}} - T_i\widehat{\mathbf{v}} - i\lambda\widehat{\mathbf{v}} = \Lambda((\widehat{\mathbf{w}}_- - \mathbf{w})|_{\Gamma_1}, \widehat{\mathbf{w}}_+|_{\Gamma_2}) & \text{on } \Gamma_1, \\ \widehat{\mathbf{v}} = \mathbf{0} & \text{on } \Gamma_2, \\ \widehat{\mathbf{w}} = \widehat{\mathbf{w}}_+ & \text{on } \Gamma_2. \end{cases} \tag{3.9}$$

According to the definition of the operator G, we have that

$$G((\widehat{\mathbf{w}}_- - \mathbf{w})|_{\Gamma_1}, \widehat{\mathbf{w}}_+|_{\Gamma_2}) = \widehat{\mathbf{w}}^\infty.$$

Define $T: [\widetilde{H}^{-1/2}(\Gamma_1)]^2 \times [\widetilde{H}^{-1/2}(\Gamma_2)]^2 \to [H^{1/2}(\Gamma_1)]^2 \times [H^{1/2}(\Gamma_2)]^2$ by

$$T(\phi,\theta) = ((\widehat{\mathbf{w}}_- - \mathbf{w})|_{\Gamma_1}, \widehat{\mathbf{w}}_+|_{\Gamma_2}). \tag{3.10}$$

Then we obtain

$$H^*(\phi,\theta) = \sqrt{8\pi\omega}\,G(T(\phi,\theta)),$$

which implies $H\tau = \sqrt{8\pi\omega}T^*G^*\tau$. So the far field operator has the factorized form

$$F = -\sqrt{8\pi\omega}\,GT^*G^*. \tag{3.11}$$

The decomposition technique of the far field operator F is more difficult than and is different from that used in the papers [16,17]. We here adopt a ideological line based on scattering problems, and get the desired decomposition. Next, we turn our attention to the properties of the operator T and then explore the operator G, which are the key for the factorization method.

Before giving the proof of the next lemma, we would like to remind the reader of Betti's first integral formula. Let D be a bounded domain with Lipschitz continuous boundary ∂D, for two vectors $\mathbf{v},\mathbf{w}\in[H^1(D)]^2$ with $\Delta^*\mathbf{v}\in[H^1(D)]^2$, it holds that

$$\int_D E_\alpha(\mathbf{v},\overline{\mathbf{w}})d\mathbf{x}+\int_D \Delta_\alpha^*\mathbf{v}\cdot\overline{\mathbf{w}}d\mathbf{x}=\int_{\partial D} T_\alpha\mathbf{v}\cdot\overline{\mathbf{w}}ds,$$

where the sesquilinear form $E_\alpha(\mathbf{v},\mathbf{w})$ is given as

$$\begin{aligned}E_\alpha(\mathbf{v},\mathbf{w})=&(2\mu_\alpha+\lambda_\alpha)\Big(\frac{\partial v_1}{\partial x_1}\frac{\partial w_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\frac{\partial w_2}{\partial x_2}\Big)+\mu_\alpha\Big(\frac{\partial v_1}{\partial x_2}\frac{\partial w_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\frac{\partial w_2}{\partial x_1}\Big)\\&+\lambda_\alpha\Big(\frac{\partial v_1}{\partial x_1}\frac{\partial w_2}{\partial x_2}+\frac{\partial v_2}{\partial x_2}\frac{\partial w_1}{\partial x_1}\Big)+\mu_\alpha\Big(\frac{\partial v_1}{\partial x_2}\frac{\partial w_2}{\partial x_1}+\frac{\partial v_2}{\partial x_1}\frac{\partial w_1}{\partial x_2}\Big).\end{aligned}$$

LEMMA 3.1. *The imaginary part of the operator T is strictly positive, i.e.*

$$Im\langle T(\phi,\theta),(\phi,\theta)\rangle>0,\ \textit{for all}\ (\phi,\theta)\in[\widetilde{H}^{-1/2}(\Gamma_1)]^2\times[\widetilde{H}^{-1/2}(\Gamma_2)]^2\ \textit{with}\ (\phi,\theta)\neq\mathbf{0}.$$

Proof. Using Betti's first formula and the boundary conditions of problem (3.8), taking note of the following equalities

$$\phi=T_e\mathbf{w}-T_i\widehat{\mathbf{v}}-i\lambda\widehat{\mathbf{v}}=T_e\widehat{\mathbf{w}}_--T_e\widehat{\mathbf{w}}_+,\ \widehat{\mathbf{w}}_+=\widehat{\mathbf{w}}_-\text{ on }\Gamma_1,$$
$$\theta=T_e\widehat{\mathbf{w}}_--T_e\widehat{\mathbf{w}}_+,\ \widehat{\mathbf{w}}_+=\widehat{\mathbf{w}}_-\text{ on }\Gamma_2,$$

we have from the definition of the operator T that

$$\begin{aligned}\langle(\phi,\theta),T(\phi,\theta)\rangle&=\int_{\Gamma_1}\phi\cdot(\overline{\widehat{\mathbf{w}}_-}-\overline{\mathbf{w}})ds+\int_{\Gamma_2}\theta\cdot\overline{\widehat{\mathbf{w}}_+}ds\\
&=\int_{\Gamma_1}\phi\cdot\overline{\widehat{\mathbf{w}}_-}ds-\int_{\Gamma_1}\phi\cdot\overline{\mathbf{w}}ds+\int_{\Gamma_2}\theta\cdot\overline{\widehat{\mathbf{w}}_+}ds\\
&=\int_{\Gamma_1}(T_e\widehat{\mathbf{w}}_--T_e\widehat{\mathbf{w}}_+)\cdot\overline{\widehat{\mathbf{w}}_-}ds-\int_{\Gamma_1}(T_e\mathbf{w}-T_i\widehat{\mathbf{v}}-i\lambda\widehat{\mathbf{v}})\cdot\overline{\mathbf{w}}ds\\
&\quad+\int_{\Gamma_2}(T_e\widehat{\mathbf{w}}_--T_e\widehat{\mathbf{w}}_+)\cdot\overline{\widehat{\mathbf{w}}_+}ds\\
&=(\int_{\Gamma_1}T_e\widehat{\mathbf{w}}_-\cdot\overline{\widehat{\mathbf{w}}_-}ds+\int_{\Gamma_2}T_e\widehat{\mathbf{w}}_-\cdot\overline{\widehat{\mathbf{w}}_-}ds)-(\int_{\Gamma_1}T_e\widehat{\mathbf{w}}_+\cdot\overline{\widehat{\mathbf{w}}_+}ds+\int_{\Gamma_2}T_e\widehat{\mathbf{w}}_+\cdot\overline{\widehat{\mathbf{w}}_+}ds)\\
&\quad-\int_\Gamma T_e\mathbf{w}\cdot\overline{\mathbf{w}}ds+\int_\Gamma T_i\widehat{\mathbf{v}}\cdot\overline{\widehat{\mathbf{v}}}ds+i\lambda\int_{\Gamma_1}|\widehat{\mathbf{v}}|^2ds\\
&=\int_{D_i}E_e(\widehat{\mathbf{w}},\overline{\widehat{\mathbf{w}}})d\mathbf{x}-\rho_e\omega^2\int_{D_i}|\widehat{\mathbf{w}}|^2d\mathbf{x}+\int_{B_r\cap D_e}E_e(\widehat{\mathbf{w}},\overline{\widehat{\mathbf{w}}})d\mathbf{x}-\rho_e\omega^2\int_{B_r\cap D_e}|\widehat{\mathbf{w}}|^2d\mathbf{x}\\
&\quad-\int_{D_i}E_e(\mathbf{w},\overline{\mathbf{w}})d\mathbf{x}+\rho_e\omega^2\int_{D_i}|\mathbf{w}|^2d\mathbf{x}+\int_{D_i}E_i(\widehat{\mathbf{v}},\overline{\widehat{\mathbf{v}}})d\mathbf{x}-\rho_i\omega^2\int_{D_i}|\widehat{\mathbf{v}}|^2d\mathbf{x}\\
&\quad-\int_{\partial B_r}T_e\widehat{\mathbf{w}}\cdot\overline{\widehat{\mathbf{w}}}ds+i\lambda\int_{\Gamma_1}|\widehat{\mathbf{v}}|^2ds,\end{aligned}\tag{3.12}$$

where B_r is a circle centering at the origin with radius r large enough such that $D_i\subset B_r$. Since $\lambda<0$, $E_i(\widehat{\mathbf{v}},\overline{\widehat{\mathbf{v}}})$, $E_e(\mathbf{w},\overline{\mathbf{w}})$ and $E_e(\widehat{\mathbf{w}},\overline{\widehat{\mathbf{w}}})$ are real, we take the imaginary part and have

$$Im\langle T(\phi,\theta),(\phi,\theta)\rangle=-Im\langle(\phi,\theta),T(\phi,\theta)\rangle$$

$$= \frac{\omega^2}{k_{p,e}} \int_{\mathbb{S}} |\widehat{w}_p^\infty|^2 ds + \frac{\omega^2}{k_{s,e}} \int_{\mathbb{S}} |\widehat{w}_s^\infty|^2 ds - \lambda \int_{\Gamma_1} |\widehat{\mathbf{v}}|^2 ds$$
$$\geq 0,$$

where the second identity is obtained by the asymptotic expansions (2.6) and (2.7), and $(\widehat{w}_p^\infty, \widehat{w}_s^\infty)$ is the far field pattern of $\widehat{\mathbf{w}}$.

Let $T(\phi,\theta) = \mathbf{0}$, then $\widehat{\mathbf{w}} - \mathbf{w}$ satisfies problem (3.1) with homogeneous boundary condition. We have $\widehat{\mathbf{w}} - \mathbf{w} = \mathbf{0}$ in the domain D_i due to the fact that ω is not an eigenvalue. Thus, problem (3.9) has homogeneous boundary condition and as a result possesses only zero solution. So we conclude that the potential function $\widehat{\mathbf{w}} = \mathbf{0}$ in $\mathbb{R}^2 \backslash \overline{\partial D_i}$, and the jump relation of the single layer potential indicates $(\phi,\theta) = \mathbf{0}$. ∎

Denote by $\mathcal{T}$ the operator T with $\omega = i$, the corresponding potential $\widehat{\mathbf{w}}$ given by (3.7) and the solution $(\mathbf{w}, \widehat{\mathbf{v}})$ of problem (3.8) are denoted by $\widehat{\mathcal{W}}$ and $(\mathcal{W}, \widehat{\mathcal{V}})$, respectively, then we have the following lemma.

LEMMA 3.2. *The operator T has a decomposition of the form $T = \mathcal{T} + \mathbf{T}$ with (a) $Re\mathcal{T}$ being a coercive operator and (b) $\mathbf{T}$ being a compact one.*

Proof.
(a) Make the same deduction as in (3.12) and notice that $\widehat{\mathcal{W}}$ decays exponentially as $|\mathbf{x}|$ tends to infinity, we arrive at

$$\langle (\phi,\theta), \mathcal{T}(\phi,\theta) \rangle = \int_{\Gamma_1} \phi \cdot (\overline{\widehat{\mathcal{W}}_-} - \overline{\mathcal{W}}) ds + \int_{\Gamma_2} \theta \cdot \overline{\widehat{\mathcal{W}}_+} ds$$
$$= \Big(\int_{D_i} E_e(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) d\mathbf{x} + \rho_e \int_{D_i} |\widehat{\mathcal{W}}|^2 d\mathbf{x} \Big) + \Big(\int_{D_e} E_e(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) d\mathbf{x} + \rho_e \int_{D_e} |\widehat{\mathcal{W}}|^2 d\mathbf{x} \Big)$$
$$- \Big(\int_{D_i} E_e(\mathcal{W}, \overline{\mathcal{W}}) d\mathbf{x} + \rho_e \int_{D_i} |\mathcal{W}|^2 d\mathbf{x} \Big) + \Big(\int_{D_i} E_i(\widehat{\mathcal{V}}, \overline{\widehat{\mathcal{V}}}) d\mathbf{x} + \rho_i \int_{D_i} |\widehat{\mathcal{V}}|^2 d\mathbf{x} \Big)$$
$$+ i\lambda \int_{\Gamma_1} |\widehat{\mathcal{V}}|^2 ds$$
$$:= \Phi_e^-(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) + \Phi_e^+(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) - \Phi_e^-(\mathcal{W}, \overline{\mathcal{W}}) + \Phi_i^-(\widehat{\mathcal{V}}, \overline{\widehat{\mathcal{V}}}) + i\lambda \int_{\Gamma_1} |\widehat{\mathcal{V}}|^2 ds, \tag{3.13}$$

where the subscript α in the notation $\Phi_\alpha^\pm(\cdot,\cdot)$ corresponds to the physical parameters and the superscript $\pm$ represents the interior and exterior of D_i. On the other hand, owing to $\widehat{\mathcal{V}} = \mathcal{W}$ on Γ_1 and $\widehat{\mathcal{V}} = \mathcal{W} = \mathbf{0}$ on Γ_2, the first part of the second term in the third identity of (3.12) can be changed into the following

$$\int_{\Gamma_1} T_e \mathcal{W} \cdot \overline{\mathcal{W}} ds = \int_{\Gamma} T_e \mathcal{W} \cdot \overline{\widehat{\mathcal{V}}} ds.$$

In such a case, formula (3.12) can be rewritten as

$$\langle (\phi,\theta), \mathcal{T}(\phi,\theta) \rangle = \Phi_e^-(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) + \Phi_e^+(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}})$$
$$- \Big(\int_{D_i} E_e(\mathcal{W}, \overline{\widehat{\mathcal{V}}}) d\mathbf{x} + \rho_e \int_{D_i} \mathcal{W} \cdot \overline{\widehat{\mathcal{V}}} d\mathbf{x} \Big) + \Phi_i^-(\widehat{\mathcal{V}}, \overline{\widehat{\mathcal{V}}}) + i\lambda \int_{\Gamma_1} |\widehat{\mathcal{V}}|^2 ds$$
$$:= \Phi_e^-(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) + \Phi_e^+(\widehat{\mathcal{W}}, \overline{\widehat{\mathcal{W}}}) + \Phi_e^-(\widehat{\mathcal{V}}, \overline{\widehat{\mathcal{V}}}) - \Phi_e^-(\mathcal{W}, \overline{\widehat{\mathcal{V}}})$$
$$+ \Phi_i^-(\widehat{\mathcal{V}}, \overline{\widehat{\mathcal{V}}}) - \Phi_e^-(\widehat{\mathcal{V}}, \overline{\widehat{\mathcal{V}}}) + i\lambda \int_{\Gamma_1} |\widehat{\mathcal{V}}|^2 ds$$

$$
\begin{aligned}
&= \Phi_e^-(\widehat{\mathcal{W}},\overline{\widehat{\mathcal{W}}}) + \Phi_e^+(\widehat{\mathcal{W}},\overline{\widehat{\mathcal{W}}}) + \Phi_e^-(\widehat{\mathcal{V}}-\mathcal{W},\overline{\widehat{\mathcal{V}}}) + \Phi_{i-e}^-(\widehat{\mathcal{V}},\overline{\widehat{\mathcal{V}}}) + i\lambda\int_{\Gamma_1}|\widehat{\mathcal{V}}|^2 ds \\
&= \Phi_e^-(\widehat{\mathcal{W}},\overline{\widehat{\mathcal{W}}}) + \Phi_e^+(\widehat{\mathcal{W}},\overline{\widehat{\mathcal{W}}}) + \Phi_e^-(\widehat{\mathcal{V}}-\mathcal{W},\overline{\widehat{\mathcal{V}}}-\overline{\mathcal{W}}) + \Phi_{i-e}^-(\widehat{\mathcal{V}},\overline{\widehat{\mathcal{V}}}) \\
&\quad + \Phi_e^-(\widehat{\mathcal{V}}-\mathcal{W},\overline{\mathcal{W}}) + i\lambda\int_{\Gamma_1}|\widehat{\mathcal{V}}|^2 ds. \qquad (3.14)
\end{aligned}
$$

Since the real sesquilinear form $\Phi_e^-(\mathcal{W},\overline{\mathcal{W}})$ is equal to $\Phi_e^-(\mathcal{W},\overline{\widehat{\mathcal{V}}})$ by observing $\int_\Gamma T_e\mathcal{W}\cdot\overline{\mathcal{W}}ds = \int_\Gamma T_e\mathcal{W}\cdot\overline{\widehat{\mathcal{V}}}ds$, it holds that $\Phi_e^-(\mathcal{W},\overline{\mathcal{W}}) = \Phi_e^-(\widehat{\mathcal{V}},\overline{\mathcal{W}})$, which results in

$$\Phi_e^-(\widehat{\mathcal{V}}-\mathcal{W},\overline{\mathcal{W}}) = 0.$$

In turn according to the strict coercivity property of $\Phi_\alpha^\pm(\cdot,\cdot)$, the continuity of the trace operator and the continuous invertibility of the single-layer potential operator, we can obtain

$$
\begin{aligned}
&Re\langle\mathcal{T}(\phi,\theta),(\phi,\theta)\rangle = Re\langle(\phi,\theta),\mathcal{T}(\phi,\theta)\rangle \\
&\geq \Phi_e^-(\widehat{\mathcal{W}},\overline{\widehat{\mathcal{W}}}) \geq c\|\widehat{\mathcal{W}}\|_{[H^1(D_i)]^2} \\
&\geq c(\|\widehat{\mathcal{W}}|_{\Gamma_1}\|_{[H^{1/2}(\Gamma_1)]^2} + \|\widehat{\mathcal{W}}|_{\Gamma_2}\|_{[H^{1/2}(\Gamma_2)]^2}) \\
&\geq c(\|\phi\|_{[\widetilde{H}^{-1/2}(\Gamma_1)]^2} + \|\theta\|_{[\widetilde{H}^{-1/2}(\Gamma_2)]^2}).
\end{aligned}
$$

An analogous and more detailed argument in Lemma 3.5 of [16] is recommended to the reader. The claim of the continuous invertibility of the single-layer operator is stated as follows.

Let $\mathcal{H}_{jl}^\alpha$ be the single-layer operator corresponding to $\omega = i$, which is defined in Section 2. The single-layer potential function $\widehat{\mathcal{W}}$ defined by (3.7) satisfies interior Dirichlet problem in D_i with boundary data

$$\widehat{\mathcal{W}}|_{\Gamma_1} = \mathcal{H}_{11}^e\phi + \mathcal{H}_{21}^e\theta,\ \widehat{\mathcal{W}}|_{\Gamma_2} = \mathcal{H}_{12}^e\phi + \mathcal{H}_{22}^e\theta,$$

which implies the boundary integral equation

$$\begin{bmatrix}\mathcal{H}_{11}^e & \mathcal{H}_{21}^e\\ \mathcal{H}_{12}^e & \mathcal{H}_{22}^e\end{bmatrix}\begin{bmatrix}\phi\\ \theta\end{bmatrix} := B\begin{bmatrix}\phi\\ \theta\end{bmatrix} = \begin{bmatrix}\widehat{\mathcal{W}}|_{\Gamma_1}\\ \widehat{\mathcal{W}}|_{\Gamma_2}\end{bmatrix}.$$

A standard derivation can show that the operator $B : [\widetilde{H}^{-1/2}(\Gamma_1)]^2 \times [\widetilde{H}^{-1/2}(\Gamma_2)]^2 \to [H^{1/2}(\Gamma_1)]^2 \times [H^{1/2}(\Gamma_2)]^2$ is invertible.

(b) It is easy to verify that $(\mathbf{w}-\mathcal{W}, \widehat{\mathbf{v}}-\widehat{\mathcal{V}}) \in [H^1(D_i)]^2 \times [H^1(D_i)]^2$ satisfies the following source problem

$$
\begin{cases}
\Delta_e^*(\mathbf{w}-\mathcal{W}) - \rho_e(\mathbf{w}-\mathcal{W}) = -\rho_e(1+\omega^2)\mathbf{w} & \text{in } D_i,\\
\Delta_i^*(\widehat{\mathbf{v}}-\widehat{\mathcal{V}}) - \rho_e(\widehat{\mathbf{v}}-\widehat{\mathcal{V}}) = -\rho_e(1+\omega^2)\widehat{\mathbf{v}} & \text{in } D_i,\\
(\mathbf{w}-\mathcal{W}) - (\widehat{\mathbf{v}}-\widehat{\mathcal{V}}) = \mathbf{0} & \text{on } \Gamma_1,\\
T_e(\mathbf{w}-\mathcal{W}) - T_i(\widehat{\mathbf{v}}-\widehat{\mathcal{V}}) - i\lambda(\widehat{\mathbf{v}}-\widehat{\mathcal{V}}) = \mathbf{0} & \text{on } \Gamma_1,\\
\widehat{\mathbf{v}}-\widehat{\mathcal{V}} = \mathbf{0} & \text{on } \Gamma_2,\\
\mathbf{w}-\mathcal{W} = \mathbf{0} & \text{on } \Gamma_2.
\end{cases}
$$

The well posedness of this problem shows that the mapping $(\mathbf{w},\widehat{\mathbf{v}}) \mapsto (\mathbf{w}-\mathcal{W},\widehat{\mathbf{v}}-\widehat{\mathcal{V}})$ from the source to the solution is bounded from $[L^2(D_i)]^2 \times [L^2(D_i)]^2$ into $[H^1(D_i)]^2 \times$

$[H^1(D_i)]^2$. The mapping $\phi \mapsto (\mathbf{w},\widehat{\mathbf{v}})$ from the boundary data ϕ to the solution of problem (3.8) is bounded from $[\widetilde{H}^{-1/2}(\Gamma_1)]^2$ into $[H^1(D_i)]^2 \times [H^1(D_i)]^2$. The compact imbedding from $[H^1(D_i)]^2 \times [H^1(D_i)]^2$ into $[L^2(D_i)]^2 \times [L^2(D_i)]^2$ shows that the mapping $\phi \mapsto (\mathbf{w} - \mathcal{W}, \widehat{\mathbf{v}} - \widehat{\mathcal{V}})$ is compact. Moreover, the difference of the single-layer potential $\widehat{\mathbf{w}} - \widehat{\mathcal{W}}$ has a continuous kernel and thereby is compact from $[\widetilde{H}^{-1/2}(\Gamma_l)]^2$ into $[H^{1/2}(\Gamma_l)]^2$ for $l=1,2$. In conclusion, the mapping $T-\mathcal{T}: [\widetilde{H}^{-1/2}(\Gamma_1)]^2 \times [\widetilde{H}^{-1/2}(\Gamma_2)]^2 \to [H^{1/2}(\Gamma_1)]^2 \times [H^{1/2}(\Gamma_2)]^2$ with

$$(T-\mathcal{T})(\phi,\theta) = \Big((\widehat{\mathbf{w}}_- - \widehat{\mathcal{W}}_-)|_{\Gamma_1} - (\mathbf{w} - \mathcal{W})|_{\Gamma_1}, (\widehat{\mathbf{w}}_+ - \widehat{\mathcal{W}}_+)|_{\Gamma_2} \Big)$$

is compact due to the boundedness of the trace operator. ∎

We next pay attention to the operator G and have the result as below.

Lemma 3.3. *Assume that ω is not a transmission eigenvalue, then the operator G given by (3.4) is injective and has a dense range.*

Proof. Let $G(\varphi,\psi) = \mathbf{0}$, that is the far field pattern of the scattered field $\mathbf{u}$ to problem (3.3) vanishes. Then Rellich's lemma implies that $\mathbf{u} = \mathbf{0}$ in D_e and consequently $\psi = \mathbf{0}$. Recall the solution $\widehat{\mathbf{u}}$ of problem (3.1) and definition of the operator Λ given by (3.2). We thus derive that $(-\widehat{\mathbf{u}},\mathbf{v})$ satisfies interior transmission problem (3.8) with homogenous boundary conditions. The fact that ω is not a transmission eigenvalue yields $(-\widehat{\mathbf{u}},\mathbf{v}) = (\mathbf{0},\mathbf{0})$ in D_i and thereby $\varphi = \mathbf{0}$. So, the operator G is injective.

From the definitions of the far field operator F and G, we observe that $Range(F) \subset Range(G)$. The dense range of F implies the denseness of G and we next show the operator F has a dense range. To this end, we refer to Theorem 2 in [38] (where the scattering problem for a penetrable obstacle is considered) and claim that the following also hold true for the adjoint operator $F^*: [L^2(\mathbb{S})]^2 \to [L^2(\mathbb{S})]^2$ of F

$$F^*\mathbf{g} = \overline{LFL\overline{\mathbf{g}}}, \text{ for } \mathbf{g} \in [L^2(\mathbb{S})]^2, \tag{3.15}$$

where the reflection operator $L: [L^2(\mathbb{S})]^2 \to [L^2(\mathbb{S})]^2$ is defined by

$$(L\mathbf{g})(\mathbf{d}) := \mathbf{g}(-\mathbf{d}), \mathbf{d} \in \mathbb{S}.$$

Indeed, for given Herglotz incident fields $\tilde{\mathbf{v}}_g$ and $\tilde{\mathbf{v}}_h$, let $(\mathbf{u}_g,\mathbf{v}_g)$ and $(\mathbf{u}_h,\mathbf{v}_h)$ be the corresponding solutions to problem (2.1)-(2.5). The author in [25] has shown that

$$\sqrt{8\pi\omega}(F\mathbf{g},\mathbf{h}) = \int_{\partial D_i} (\mathbf{u}_g \cdot T_e \overline{\tilde{\mathbf{v}}_h} - \overline{\tilde{\mathbf{v}}_h} \cdot T_e \mathbf{u}_g) ds.$$

Noting the relation $\overline{\tilde{\mathbf{v}}_h} = \tilde{\mathbf{v}}_{L\overline{h}}$, using the boundary conditions and applying Betti's third formula and radiation conditions, we can obtain

$$\begin{aligned}
&\sqrt{8\pi\omega}(F\mathbf{g},\mathbf{h}) = \int_{\partial D_i} (\mathbf{u}_g \cdot T_e \tilde{\mathbf{v}}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot T_e \mathbf{u}_g) ds \\
&= \int_{\Gamma_1} (\mathbf{u}_g \cdot T_i \mathbf{v}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot T_i \mathbf{v}_g) ds - \int_{\Gamma_1} (\mathbf{u}_g \cdot T_e \mathbf{u}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot T_e \tilde{\mathbf{v}}_g) ds \\
&\quad + i\lambda \int_{\Gamma_1} (\mathbf{u}_g \cdot \mathbf{v}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot \mathbf{v}_g) ds + \int_{\Gamma_2} (\mathbf{u}_g \cdot T_e \tilde{\mathbf{v}}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot T_e \mathbf{u}_g) ds \\
&= \int_{\Gamma_1} (\mathbf{u}_g \cdot T_i \mathbf{v}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot T_i \mathbf{v}_g) ds - \int_{\partial D_i} (\mathbf{u}_g \cdot T_e \mathbf{u}_{L\overline{h}} - \tilde{\mathbf{v}}_{L\overline{h}} \cdot T_e \tilde{\mathbf{v}}_g) ds
\end{aligned}$$

$$
\begin{aligned}
&+i\lambda\int_{\Gamma_1}(\mathbf{u}_g\cdot\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot\mathbf{v}_g)ds+\int_{\Gamma_2}(\mathbf{u}_g\cdot T_e(\tilde{\mathbf{v}}_{L\overline{h}}+\mathbf{u}_{L\overline{h}})-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_e(\mathbf{u}_g+\tilde{\mathbf{v}}_g))ds\\
=&\int_{\partial D_i}(\mathbf{u}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds-\int_{\partial D_i}(\mathbf{u}_{L\overline{h}}\cdot T_e\mathbf{u}_g-\tilde{\mathbf{v}}_g\cdot T_e\tilde{\mathbf{v}}_{L\overline{h}})ds\\
&+i\lambda\int_{\Gamma_1}(\mathbf{u}_g\cdot\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot\mathbf{v}_g)ds+\int_{\Gamma_2}(\mathbf{u}_g\cdot T_e(\tilde{\mathbf{v}}_{L\overline{h}}+\mathbf{u}_{L\overline{h}})-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_e(\mathbf{u}_g+\tilde{\mathbf{v}}_g))ds\\
&-\int_{\Gamma_2}(\mathbf{u}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds. \qquad (3.16)
\end{aligned}
$$

A continuous calculus yields

$$
\begin{aligned}
&\int_{\partial D_i}(\mathbf{u}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds-\int_{\partial D_i}(\mathbf{u}_{L\overline{h}}\cdot T_e\mathbf{u}_g-\tilde{\mathbf{v}}_g\cdot T_e\tilde{\mathbf{v}}_{L\overline{h}})ds\\
=&\int_{\Gamma_1}\Big((\mathbf{v}_g-\tilde{\mathbf{v}}_g)\cdot T_i\mathbf{v}_{L\overline{h}}-(\mathbf{v}_{L\overline{h}}-\mathbf{u}_{L\overline{h}})\cdot T_i\mathbf{v}_g\Big)ds\\
&-\int_{\Gamma_1}\Big(\mathbf{u}_{L\overline{h}}\cdot(T_i\mathbf{v}_g-T_e\tilde{\mathbf{v}}_g)-\tilde{\mathbf{v}}_g\cdot(T_i\mathbf{v}_{L\overline{h}}-T_e\mathbf{u}_{L\overline{h}})\Big)ds\\
&-i\lambda\int_{\Gamma_1}(\mathbf{u}_{L\overline{h}}\cdot\mathbf{v}_g-\tilde{\mathbf{v}}_g\cdot\mathbf{v}_{L\overline{h}})ds+\int_{\Gamma_2}(\mathbf{u}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds\\
&-\int_{\Gamma_2}(\mathbf{u}_{L\overline{h}}\cdot T_e\mathbf{u}_g-\tilde{\mathbf{v}}_g\cdot T_e\tilde{\mathbf{v}}_{L\overline{h}})ds\\
=&\int_{\Gamma_1}(\mathbf{v}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\mathbf{v}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds+\int_{\Gamma_1}(\mathbf{u}_{L\overline{h}}\cdot T_e\tilde{\mathbf{v}}_g-\tilde{\mathbf{v}}_g\cdot T_e\mathbf{u}_{L\overline{h}})ds\\
&-i\lambda\int_{\Gamma_1}(\mathbf{u}_{L\overline{h}}\cdot\mathbf{v}_g-\tilde{\mathbf{v}}_g\cdot\mathbf{v}_{L\overline{h}})ds+\int_{\Gamma_2}(\mathbf{u}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds\\
&-\int_{\Gamma_2}(\mathbf{u}_{L\overline{h}}\cdot T_e\mathbf{u}_g-\tilde{\mathbf{v}}_g\cdot T_e\tilde{\mathbf{v}}_{L\overline{h}})ds\\
=&\int_{\partial D_i}(\mathbf{v}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\mathbf{v}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds+\int_{\partial D_i}(\mathbf{u}_{L\overline{h}}\cdot T_e\tilde{\mathbf{v}}_g-\tilde{\mathbf{v}}_g\cdot T_e\mathbf{u}_{L\overline{h}})ds\\
&-i\lambda\int_{\Gamma_1}(\mathbf{u}_{L\overline{h}}\cdot\mathbf{v}_g-\tilde{\mathbf{v}}_g\cdot\mathbf{v}_{L\overline{h}})ds+\int_{\Gamma_2}(\mathbf{u}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\tilde{\mathbf{v}}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds\\
&-\int_{\Gamma_2}\Big(\mathbf{u}_{L\overline{h}}\cdot(T_e\mathbf{u}_g+T_e\tilde{\mathbf{v}}_g)-\tilde{\mathbf{v}}_g\cdot(T_e\tilde{\mathbf{v}}_{L\overline{h}}+T_e\mathbf{u}_{L\overline{h}})\Big)ds. \qquad (3.17)
\end{aligned}
$$

The sum of the Equations (3.16) and (3.17) shows

$$
\sqrt{8\pi\omega}(F\mathbf{g},\mathbf{h})=\int_{\partial D_i}(\mathbf{v}_g\cdot T_i\mathbf{v}_{L\overline{h}}-\mathbf{v}_{L\overline{h}}\cdot T_i\mathbf{v}_g)ds+\int_{\partial D_i}(\mathbf{u}_{L\overline{h}}\cdot T_e\tilde{\mathbf{v}}_g-\tilde{\mathbf{v}}_g\cdot T_e\mathbf{u}_{L\overline{h}})ds. \qquad (3.18)
$$

Applying Betti's third formula again for $\mathbf{v}_g,\mathbf{v}_{L\overline{h}}$ in the domain D_i, the first integral in (3.18) vanishes and we hence have

$$
\begin{aligned}
\sqrt{8\pi\omega}(F\mathbf{g},\mathbf{h})&=\int_{\partial D_i}(\mathbf{u}_{L\overline{h}}\cdot T_e\tilde{\mathbf{v}}_g-\tilde{\mathbf{v}}_g\cdot T_e\mathbf{u}_{L\overline{h}})ds\\
&=\int_{\partial D_i}(\mathbf{u}_{L\overline{h}}\cdot T_e\overline{\tilde{\mathbf{v}}_{L\overline{g}}}-\overline{\tilde{\mathbf{v}}_{L\overline{g}}}\cdot T_e\mathbf{u}_{L\overline{h}})ds\\
&=\sqrt{8\pi\omega}(FL\overline{\mathbf{h}},L\overline{\mathbf{g}})=\sqrt{8\pi\omega}(\mathbf{g},\overline{LFL\overline{\mathbf{h}}}), \qquad (3.19)
\end{aligned}
$$

it is observed from this that (3.15) holds true.

As a result, the injectivity of the operator F indicates its denseness. Now let $F\mathbf{g}=\mathbf{0}$, that is the far field pattern of the scattered field $\mathbf{u}$ to problem (2.1)-(2.5) with Herglotz incident field $\tilde{\mathbf{v}}_g$ is zero. Rellich's lemma shows that $\mathbf{u}=\mathbf{0}$ in D_e. Thus $(\mathbf{v},\tilde{\mathbf{v}}_g)$ satisfies interior transmission problem (3.8) with homogenous boundary conditions and accordingly Herglotz wavefunction $\tilde{\mathbf{v}}_g$ equals to zero in D_i. Hence $\mathbf{g}=\mathbf{0}$ and the lemma follows. ◻

LEMMA 3.4. *Assume that ω is not a transmission eigenvalue, for the far field pattern $\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})=(\Gamma_{p,e}^\infty(\cdot,\mathbf{z};\mathbf{p}),\Gamma_{s,e}^\infty(\cdot,\mathbf{z};\mathbf{p}))$ of the elastic point source $\Gamma_e(\cdot,\mathbf{z};\mathbf{p})=\Gamma_e(\cdot,\mathbf{z})\cdot\mathbf{p}$ in $\mathbf{z}\in\mathbb{R}^2$ with the polarization direction $\mathbf{p}\in\mathbb{S}$ (see (2.15) and (2.16)), then $\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})$ is in the range of G if and only if $\mathbf{z}\in D_i$.*

Proof. We first consider the case $\mathbf{z}\in D_i$ and let $(\mathbf{w},\mathbf{v})$ be the solution of following interior transmission problem

$$\begin{cases} \Delta_e^*\mathbf{w}+\rho_e\omega^2\mathbf{w} &= \mathbf{0} \quad \text{in } D_i,\\ \Delta_i^*\mathbf{v}+\rho_i\omega^2\mathbf{v} &= \mathbf{0} \quad \text{in } D_i,\\ \mathbf{w}-\mathbf{v} &= -\Gamma_e(\cdot,\mathbf{z};\mathbf{p}) \quad \text{on } \Gamma_1,\\ T_e\mathbf{w}-T_i\mathbf{v}-i\lambda\mathbf{v} &= -T_e\Gamma_e(\cdot,\mathbf{z};\mathbf{p}) \quad \text{on } \Gamma_1,\\ \mathbf{v} &= \mathbf{0} \quad \text{on } \Gamma_2,\\ \mathbf{w} &= -\Gamma_e(\cdot,\mathbf{z};\mathbf{p}) \quad \text{on } \Gamma_2. \end{cases}$$

Then $(\Gamma_e(\cdot,\mathbf{z};\mathbf{p}),\mathbf{v})$ satisfies problem (3.3) with boundary data $\varphi=-\mathbf{w}|_{\Gamma_1}$, $\psi=-\mathbf{w}|_{\Gamma_2}$ and $\Lambda(\varphi,\psi)=-T_e\mathbf{w}|_{\Gamma_1}$. The definition of the operator G shows that $G(-\mathbf{w}|_{\Gamma_1},-\mathbf{w}|_{\Gamma_2})=\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})$, which implies that $\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})$ belongs to $Range(G)$.

Now let $\mathbf{z}\in D_e$ and on the contrary assume that $\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})\in Range(G)$. Then there exists $\varphi\in[H^{1/2}(\Gamma_1)]^2$ and $\psi\in[H^{1/2}(\Gamma_2)]^2$ such that $G(\varphi,\psi)=\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})$. Let $(\mathbf{u},\mathbf{v})$ satisfy problem (3.3) with boundary data φ, $\Lambda(\varphi,\psi)$ and ψ. We then also have $G(\varphi,\psi)=\mathbf{u}^\infty$ and hence $\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})=\mathbf{u}^\infty$ by the injectivity of the operator G. It holds that $\mathbf{u}=\Gamma_e(\cdot,\mathbf{z};\mathbf{p})$ by Rellich's lemma and unique continuation principle, which is a contradiction due to the fact that $\mathbf{u}$ belongs to $[H^1(D_e)]^2$ but $\Gamma_e(\cdot,\mathbf{z};\mathbf{p})$ has a singularity at the point $\mathbf{z}$. The proof is then completed. ◻

Combining with Lemma 3.1-3.4 and Picard's range criterion, and then applying the range identity theorem [18] we arrive at the main result of this paper.

THEOREM 3.5. *Recall the far field equation given by*

$$(F\mathbf{g}_z)(\hat{\mathbf{x}})=\Gamma_e^\infty(\hat{\mathbf{x}},\mathbf{z};\mathbf{p}) \quad \textit{for} \quad \mathbf{g}_z\in[L^2(\mathbb{S})]^2,\hat{\mathbf{x}}\in\mathbb{S},$$

then we have

$$\mathbf{z}\in D_i \Longleftrightarrow \Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p})\in Range((F_\sharp)^{1/2}),$$

and as a result

$$\mathbf{z}\in D_i \Longleftrightarrow \|\mathbf{g}_z\|_{[L^2(\mathbb{S})]^2}=\sum_{j=1}^{\infty}\frac{|\langle\Gamma_e^\infty(\cdot,\mathbf{z};\mathbf{p}),\sigma_j\rangle_{[L^2(\mathbb{S})]^2}|^2}{|\mu_j|}<\infty, \tag{3.20}$$

where $\{\mu_j,\sigma_j\}$ is an eigensystem of the operator $F_\sharp=|Re(F)|+|Im(F)|$. In other words, *the sign of the function*

$$W(\mathbf{z})=\left[\sum_{j=1}^{\infty}\frac{|\langle\Gamma_e^{\infty}(\cdot,\mathbf{z};\mathbf{p}),\sigma_j\rangle_{[L^2(\mathbb{S})]^2}|^2}{|\mu_j|}\right]^{-1}$$

is just the characteristic function of the obstacle D_i.

In the subsequence, we turn to the study of the factorization method from the knowledge of the transversal far field pattern u_s^{∞} for using only the incident plane shear waves, and this inverse problem is called **IPs**. The analysis is along the lines of [26]. We note that the discussion on using the knowledge of the longitudinal far field pattern u_p^{∞} corresponding to the incident plane pressure waves, to determine the obstacle D_i is analogous, and we name this inverse problem **IPp**.

Given the projection operator $P_s:[L^2(\mathbb{S})]^2\to L^2(\mathbb{S})$ as

$$P_s\mathbf{g}=g_s \text{ for } \mathbf{g}=(g_p,g_s)\in[L^2(\mathbb{S})]^2.$$

We denote by $P_s^*:L^2(\mathbb{S})\to[L^2(\mathbb{S})]^2$ the adjoint operator of P_s. Define the operator $F_s:L^2(\mathbb{S})\to L^2(\mathbb{S})$ by $F_s:=P_sFP_s^*$, as a result we have the factorization by (3.11)

$$F_s=-\sqrt{8\pi\omega}(P_sG)T^*(P_sG)^*.$$

It is ease to show that the operator G given by (3.4) is compact, which, combining with the denseness result in Lemma 3.3, leads us to deduce that $P_sG:[H^{1/2}(\Gamma_1)]^2\times[H^{1/2}(\Gamma_2)]^2\to L^2(\mathbb{S})$ is compact and dense. The properties of the operator T is presented in Lemma 3.1 and 3.2. So, using the range identity theorem 2.15 of [18], we obtain that the ranges of P_sG and $F_{s\sharp}^{1/2}$ coincide if ω is not a transmission eigenvalue, where $F_{s\sharp}:=|Re(F_s)|+|Im(F_s)|$.

Based on Lemma 3.4, and making necessary modifications in the light of Lemma 3.6 in [26], we have the following result to characterize the obstacle D_i.

LEMMA 3.6. *Assume that ω is not a transmission eigenvalue, then the far field pattern $P_s(\Gamma_e^{\infty}(\cdot,\mathbf{z};\mathbf{p}))=\Gamma_{s,e}^{\infty}(\cdot,\mathbf{z};\mathbf{p}))$ is in the range of P_sG if and only if $\mathbf{z}\in D_i$.*

In conclusion, we have the following theorem.

THEOREM 3.7. *Assume that ω is not a transmission eigenvalue, then*

$$\mathbf{z}\in D_i\Longleftrightarrow\sum_{j=1}^{\infty}\frac{|\langle\Gamma_{s,e}^{\infty}(\cdot,\mathbf{z};\mathbf{p}),\delta_j\rangle_{L^2(\mathbb{S})}|^2}{|\eta_j|}<\infty, \tag{3.21}$$

where $\{\delta_j,\eta_j\}$ is an eigensystem of the operator $F_{s\sharp}$.

4. The numerical algorithm and results

In this section, we report some numerical experiments to illustrate the validity of the factorization method. In all examples we assume that the host elastic medium has Lamé constants $\lambda_e=1,\mu_e=2$, the included medium has Lamé constants $\lambda_i=1.5,\mu_i=2.5$ and the mass densities take value $\rho_e=\rho_i=1$.

Firstly, we solve the direct scattering problem (2.1)-(2.5) to produce the forward data synthetically. To this end, we use the collocation and quadrature approaches [39,40] to solve the boundary integral Equation (2.11) with $\mathbf{f}=-\mathbf{u}^{in}|_{\Gamma_1}$, $\mathbf{g}=-T_e\mathbf{u}^{in}|_{\Gamma_1}$, $\mathbf{h}=-\mathbf{u}^{in}|_{\Gamma_2}$. Then the scattered field is obtained by the combined potentials (2.9), and

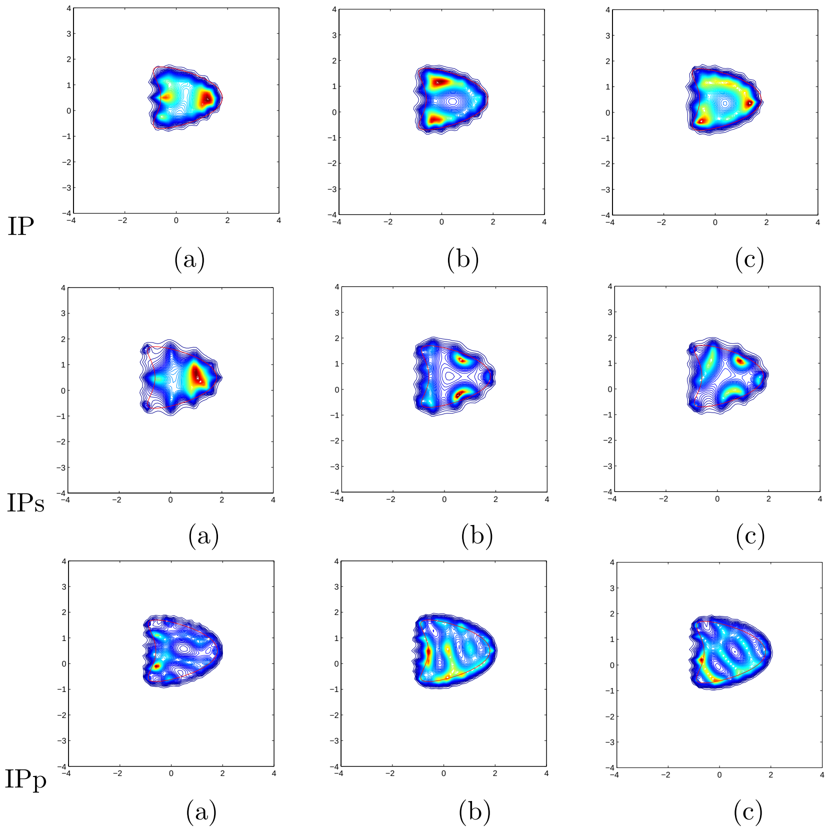

FIG. 4.1. *Reconstruction of the kite for noise level=1%, circular frequencies $\omega=4$, with polarization directions $\mathbf{p}=[0,1]$ in (a), $\mathbf{p}=[-1,0]$ in (b), and $\mathbf{p}=-[\sqrt{2}/2,\sqrt{2}/2]$ in (c). The first, second and third lines correspond to the experiment results for problems **IP**, **IPs** and **IPp**, respectively.*

we deduce the far field data from there, in which the far field patterns of the single-and double-layer potentials are computed by

$$(H^{\infty}_{j,a}\mathbf{g})(\hat{\mathbf{x}})=\beta_a\int_{\Gamma_j}J_a(\hat{\mathbf{x}})\mathbf{g}(\mathbf{y})e^{-ik_{a,e}\hat{\mathbf{x}}\cdot\mathbf{y}}ds(\mathbf{y})$$

and

$$(K^{\infty}_{j,a}\mathbf{g})(\hat{\mathbf{x}})=\gamma_a\int_{\Gamma_j}J_a(\hat{\mathbf{x}})B(\hat{\mathbf{x}},\mathbf{y})\mathbf{g}(\mathbf{y})e^{-ik_{a,e}\hat{\mathbf{x}}\cdot\mathbf{y}}ds(\mathbf{y}),$$

respectively, with the coefficients

$$\beta_p=\frac{1}{2\mu_e+\lambda_e}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{p,e}}},\quad \beta_s=\frac{1}{\mu_e}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{s,e}}},$$

$$\gamma_p=\frac{e^{-i\pi/4}}{2\mu_e+\lambda_e}\sqrt{\frac{k_{p,e}}{8\pi}},\quad \gamma_s=\frac{e^{-i\pi/4}}{\mu_e}\sqrt{\frac{k_{s,e}}{8\pi}},$$

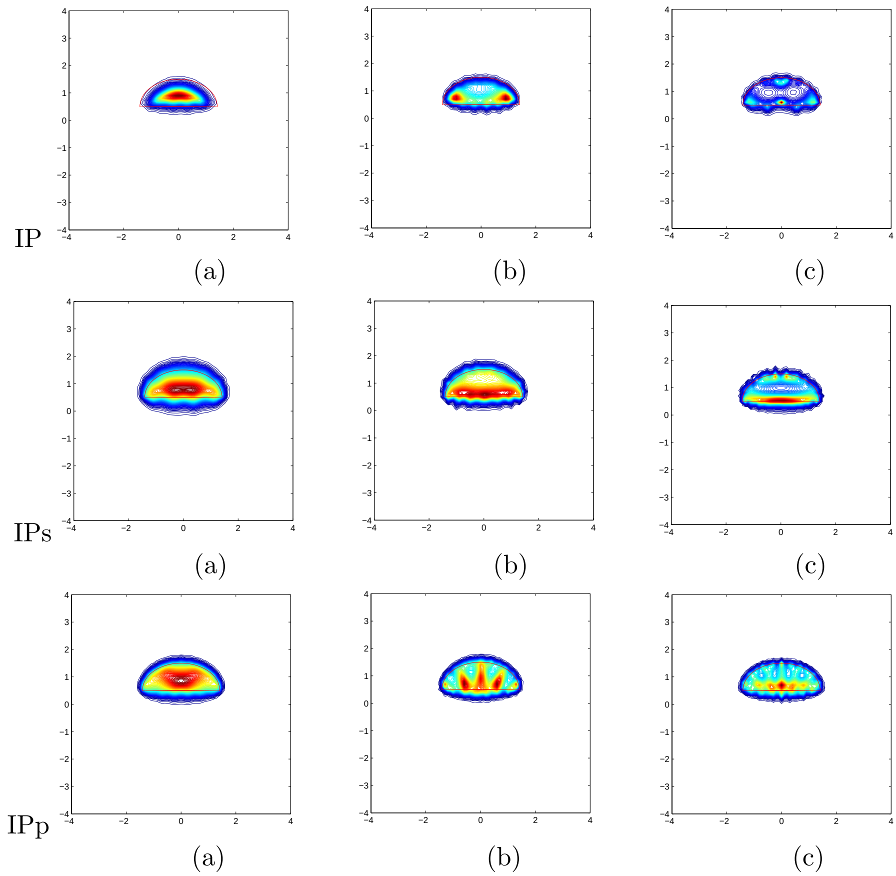

FIG. 4.2. *Reconstruction of the semi-ellipse and bottom line for* $\mathbf{p}=[1,0]^\top$*, noise level=1%, with circular frequencies* $\omega=3$ *in (a),* $\omega=5$ *in (b), and* $\omega=7$ *in (c). The first, second and third lines correspond to the experiment results for problems* ***IP****,* ***IPs*** *and* ***IPp****, respectively.*

and the matrices $J_p=\hat{\mathbf{x}}\hat{\mathbf{x}}^\top/|\hat{\mathbf{x}}|^2$, $J_s=I-J_p$ and

$$B(\hat{\mathbf{x}},\mathbf{y})=\lambda_e\hat{\mathbf{x}}\mathbf{n}(\mathbf{y})^\top+\mu_e\mathbf{n}(\mathbf{y})\hat{\mathbf{x}}^\top+\mu_e\mathbf{n}(\mathbf{y})\cdot\hat{\mathbf{x}}I.$$

Secondly, we treat with the far field Equation (2.14). Given N incident directions $\mathbf{d}_l=(cos(2\pi l/N),sin(2\pi l/N))^\top, l=1,\ldots,N$, and N observation directions $\hat{\mathbf{x}}_m=(cos(2\pi m/N),sin(2\pi m/N))^\top, m=1,\ldots,N$, the limited data of the far field patterns $\mathbf{u}^\infty(\hat{\mathbf{x}},\mathbf{d};\mathbf{d})$ and $\mathbf{u}^\infty(\hat{\mathbf{x}},\mathbf{d};\mathbf{d}^\perp)$ for N plane compressional and shear waves, respectively, are obtained and hence we can discretize approximatively the far field operator F by matrix $F_N\in\mathbb{C}^{2N\times 2N}$

$$F_N=\frac{2\pi}{N}e^{-i\pi/4}\begin{bmatrix}\sqrt{\frac{k_{p,e}}{\omega}}u_p^\infty(\hat{\mathbf{x}}_m,\mathbf{d}_l;\mathbf{d}_l) & \sqrt{\frac{k_{s,e}}{\omega}}u_p^\infty(\hat{\mathbf{x}}_m,\mathbf{d}_l;\mathbf{d}_l^\perp)\\ \sqrt{\frac{k_{p,e}}{\omega}}u_s^\infty(\hat{\mathbf{x}}_m,\mathbf{d}_l;\mathbf{d}_l) & \sqrt{\frac{k_{s,e}}{\omega}}u_s^\infty(\hat{\mathbf{x}}_m,\mathbf{d}_l;\mathbf{d}_l^\perp)\end{bmatrix}.$$

The test function on the right side of (2.14) can be approximated by a column vector

$\Gamma^\infty_{e,N} \in \mathbb{C}^{2N}$ given by

$$\Gamma^\infty_{e,N} = \begin{bmatrix} \frac{1}{2\mu_e+\lambda_e}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{p,e}}}e^{-ik_{p,e}\hat{\mathbf{x}}_m\cdot\mathbf{z}}\hat{\mathbf{x}}_m\cdot\mathbf{p} \\ \frac{1}{\mu_e}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{s,e}}}e^{-ik_{s,e}\hat{\mathbf{x}}_m\cdot\mathbf{z}}\hat{\mathbf{x}}_m^\perp\cdot\mathbf{p} \end{bmatrix}.$$

Finally, solve the discretization form of the far field equation, choose a region containing the expected obstacle and calculate the value of the indicator function $W(\mathbf{z})$ for every sampling point $\mathbf{z}$ lying in this region. Then we plot $W(\mathbf{z})$ with 100 contour lines at fixed polarization $\mathbf{p}$. In the reconstruction, the far field data are given for 40 incident directions and 40 observation directions equally distributed on the unit circle and we use a grid of 81×81 equally spaced sampling points on the rectangle $[-4,4]\times[-4,4]$.

For the problem **IPs**, the incident shear wave is $\mathbf{d}_l^\perp e^{ik_{s,e}\mathbf{x}\cdot\mathbf{d}_l}$, the far field operator F_s is approximated by the $N\times N$ matrix $F_{sN} = (\sqrt{\frac{k_{s,e}}{\omega}}u_s^\infty(\hat{\mathbf{x}}_m,\mathbf{d}_l;\mathbf{d}_l^\perp))_{m,l}$, and the test function $F^\infty_{s,e}$ is given approximately by a N dimensional column vector $F^\infty_{sN,e} = (\frac{1}{\mu_e}\frac{e^{i\pi/4}}{\sqrt{8\pi k_{s,e}}}e^{-ik_{s,e}\hat{\mathbf{x}}_m\cdot\mathbf{z}}\hat{\mathbf{x}}_m^\perp\cdot\mathbf{p})_m$. The description of the numerical experiment for problem **IPp** is omitted.

We give the reconstruction results by the following two examples. In Figure 4.1, a kite

$$\partial D_i := \left\{\Big(1.2\cos(s\pi)+0.6\cos(2s\pi), 1.2\sin(s\pi)+0.5\Big) : 0\le s\le 2\right\}$$

is considered, the top half of the kite is penetrable and the bottom half is impenetrable. In Figure 4.2, the penetrable semi-ellipse

$$\Gamma_1 := \left\{\Big(\sqrt{2}\cos(s\pi), \sin(s\pi)+0.5\Big) : 0\le s\le 1\right\}$$

with the impenetrable bottom line

$$\Gamma_2 := \left\{\Big(-\cot(\frac{s\pi}{6}), 0.5\Big) : 1\le s\le 5\right\}$$

is reconstructed. In all the figures, the red lines represent the original scatterers.

The numerical experiments show the viability of the factorization method for the reconstructions of the partially penetrable obstacles. In addition, we observe that: The polarization direction $\mathbf{p}$ has a certain influence on the experiments, since the characteristic function is associated with $\mathbf{p}$. The reaction to the circular frequency ω is sensitive in the numerical examples and the experimental effect is just relatively good for $\omega = 3,4,5,6$, which shows the factorization method works effectively in the resonance region. The reconstruction by using the complete far field pattern is more reliable than using only the S-part or the P-part of the far field pattern. The possible explanation is given in [26].

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