COMMUN. MATH. SCI.
Vol. 17, No. 6, pp. 1737–1755

FAST COMMUNICATION

ON UNIFORM SECOND ORDER NONLOCAL APPROXIMATIONS TO LINEAR TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract. In this paper, nonlocal approximations are considered for linear two-point boundary value problems (BVPs) with Dirichlet and mixed boundary conditions, respectively. These nonlocal formulations are constructed from nonlocal variational problems that are analogous to local problems. The well-posedness and regularity of the resulting nonlocal problems are established, along with the convergence to local problem as the nonlocal horizon parameter δ tends to 0. Uniform second order accuracy with respect to δ of the nonlocal approximation to the local solution, spatially in the pointwise sense, can be achieved under suitable conditions. Numerical simulations are carried out to examine the order of convergence rate, which also motivate further refined asymptotic estimates.

Keywords. nonlocal two-point boundary value problems; nonlocal operator and maximum principle; nonlocal Dirichlet and Neumann-type problems with volume-constraints; local limit; the weak regularity of nonlocal solutions.

AMS subject classifications. 82C21; 65R20; 65M60; 46N20; 45A05.

1. Introduction

Nonlocal models are becoming a rich research field, with connections to subjects such as the peridynamical theory of continuum mechanics, nonlocal wave propagation, and nonlocal diffusion process, see [3, 7, 17, 24, 25]. By utilizing the alternative integral-type nonlocal operators to evade the explicit application of spatial derivatives, the effectiveness of nonlocal models has been demonstrated for complex processes involving singular solutions and anomalous behavior. For example, peridynamics has been successfully used in simulating material singularities [1, 15, 18, 20], such as the crack nucleation and growth, fracture and failure of composites, polycrystals and nanofiber networks.

Different from local PDEs, nonlocal models associated with nonlocal diffusion and nonlocal peridynamics studied in [7] often involve a horizon parameter δ to characterize the range of nonlocal interaction. While nonlocal models are often used not as approximations of local PDEs, it is still natural to ask, at least in simple benchmark settings where local PDEs remain valid, whether or not the solutions of nonlocal analog are consistent with solutions of classical PDEs when nonlocal effects vanish [6]. More precisely, as the horizon parameter $\delta \to 0$, whether the solutions of the nonlocal models converge to that of the corresponding local PDE under suitable assumptions on the kernel functions and the given data [7]. Such consistency is quite useful not only for the modeling, but also the validation/verification of numerical simulations, which offer confidence in capturing the underlying physics of nonlocal problems and demonstrating

*Received: December 10, 2018; Accepted (in revised form): September 15, 2019. Communicated by Jianfeng Lu.

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their mathematical meaning. Moreover, in cases that nonlocal models converge to a corresponding limit, it is interesting to investigate what are the proper assumptions on the data and solutions to ensure the optimal order of convergence. This is the main task of this work.

On the continuum level, several techniques have been developed to investigate the local limit, including Talyor expansions with sufficiently smooth solutions [17, 18] and functional analytical means without extra regularity assumptions [13, 14]. In the discrete level, a theory of asymptotically compatible schemes was proposed in [22, 23] to guarantee the consistency of numerical schemes with the local limit as both mesh size h and nonlocal parameter δ tend to zero. Meanwhile, the issue of consistency between nonlocal models and local PDEs also arises in various studies of PDEs that utilize nonlocal relaxations. A good example for the latter is the SPH method originally proposed in [12] and [11] to simulate local continuum models. While a variety of studies on the convergence of nonlocal continuum models to local limits have been carried out for bounded domain problems [4,7,8,10,23,25], some issues still remain open. In particular, it is desirable to see if there are some systematic relaxations of well-posed nonlocal models that can be high-order approximations, with respect to the horizon parameter δ, to the local limit under minimal regularity assumptions and for various types of boundary conditions or volume constraints.

Generically, nonlocal approximations to many familiar local differential operators, in the free space, can be of second-order rate with respect to the horizon parameter δ that measures the range of nonlocal interactions [9, 25]. This property becomes more complicated to establish for problems defined on a bounded domain. The aim of this paper is to present some formulations of linear nonlocal models such that their solutions converge to those of limiting local BVPs, subject to various boundary conditions, in a second-order rate with respect to parameter δ. The convergence order is measured spatially in a pointwise sense, so that the rate remains uniform across the spatial domain. To this end, we first introduce nonlocal variational problems with nonlocal Dirichlet and Neumann-type constraints that are analogous to classical elliptic equations. After establishing the well-posedness of these nonlocal problems, we investigate the regularity of nonlocal solutions.

Specifically, under suitable assumptions that the source term and boundary data are given as piecewise continuous and bounded functions, we show that nonlocal solutions are generically only continuous and bounded. This weak regularity of nonlocal solutions brings new challenge to the convergence order analysis of nonlocal models to their local limit. To avoid extra regularity assumptions on the nonlocal solutions, we present, for Dirichlet-type problems, the construction of the nonlocal boundary data using the information at the boundary and the first-order derivatives of the limiting solution to local BVPs. Such a construction is reminiscent of the auxiliary function approach used in [21] but with new derivations.

For nonlocal Neumann-type data, if we also apply the nonlocal maximum principle directly, the initial five terms of Taylor expansion of the solution to local BVPs are required to theoretically guarantee a pointwise second-order approximation. A computational finding from numerical experiments shows that the second-order convergence rate is achieved even if only the initial two terms of Taylor expansion are used. To clarify this last point, we give a detailed analysis to find out the underneath mathematical theory. This interesting observation is again consistent to a similar claim stated in [21] where a different justification is made based on the construction of a barrier function. The formulation and approach presented in this work provide more direct and intuitive understanding.

The organization of the paper is as follows. In Section 2, the definition of nonlocal operator is introduced associated with a prescribed primitive kernel function, and the maximum principle of nonlocal operator is demonstrated as an analog of the local second-order elliptic operator. In Section 3, the formulations of nonlocal models with Dirichlet-type and Neumann-type nonlocal constraints are presented. In Section 4, we show that the solutions of the resulting nonlocal problems converge to that of local BVPs in a pointwise second-order rate as the horizon parameter δ tends to zero. In Section 5, we present a refined asymptotic estimate that provides the mathematical clarification to the experimental observations (described in Section 6) that not all information used for imposing nonlocal Neumann boundary data in the Section 4 are required. This paper ends with a conclusion.

2. Nonlocal operators and maximum principle

To streamline the notation, we first present the definition of some useful kernels.

DEFINITION 2.1. *A (one-dimensional) parent kernel function γ_1 is defined as a nonnegative function which satisfies the following properties:*

- $\gamma_1(-s)=\gamma_1(s)$ *for all* $s\in\mathbb{R}$;
- γ_1 *is piecewise smooth on* $\mathbb{R}$;
- $\gamma_1(s)>0$ *for almost all* $s\in\mathbb{R}$ *with* $|s|<1$;
- $\gamma_1(s)=0$ *for all* $s\in\mathbb{R}$ *with* $|s|>1$.

A (rescaled) kernel function γ_δ (with horizon $\delta>0$) is determined by a rescaling of a parent kernel γ_1 through

$$\gamma_\delta(s)=\frac{1}{\delta^3}\gamma_1\left(\frac{s}{\delta}\right).$$

In the above definition, by *piecewise smooth*, we mean that the function and its derivatives are bounded except at a finite number of points.

Given an interval $\Omega=(a,b)\subset\mathbb{R}$ (not necessarily bounded), for all $\delta<|\Omega|/2$, we introduce the following notations:

$$\Omega_\delta^- =(a+\delta,b-\delta),\quad \Omega_\delta^{-,c}=(a,a+\delta)\cup(b-\delta,b),$$
$$\Omega_\delta^+ =(a-\delta,b+\delta),\quad \Omega_\delta^{+,c}=(a-\delta,a)\cup(b,b+\delta).$$

We here list the definitions of some notations often used in the following.

- The space of bounded, continuous functions is denoted by

$$C_b(\Omega):=\{u\in C(\Omega): u \text{ is bounded}\}.$$

- For any non-negative integer m, we define

$$C_b^m(\Omega):=\{u\in C_b(\Omega):\forall \text{ nonegative integer } k\le m, u^{(k)}\in C_b(\Omega)\}.$$

- A function u measurable on Ω is said to be essentially bounded on Ω if there is a constant K such that $|u(x)|\le K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω, and is denoted by $\operatorname{esssup}_{x\in\Omega}|u(x)|$, i.e.

$$\operatorname*{esssup}_{x\in\Omega}|u(x)|:=\inf_{\mu(E)=0}\{\sup_{x\in\Omega\setminus E}|u(x)|\},$$

where $\mu(E)$ is the Lebesgue measure of set E.

Definition 2.2. *Given an interval $\Omega=(a,b)$, a nonlocal diffusion operator with horizon $\delta<|\Omega|/2$ associated with a prescribed parent kernel function γ_1 is defined as the following linear operator*

$$\mathcal{L}_{\Omega,\delta}u(x)=\int_\Omega [u(x)-u(y)]\gamma_\delta(x-y)dy, \quad \forall x\in\Omega. \tag{2.1}$$

By Young's inequality for integral operators and the assumption on the kernel that $\gamma_1\in L^1$, we have the following result.

Lemma 2.1. *The linear mapping $\mathcal{L}_{\Omega,\delta}$ is bounded from $L^2(\Omega)$ to $L^2(\Omega)$.*

The nonlocal operator considered previously has a close connection with local differential operators [6]. Actually, given a function $u\in C_b^4(\Omega_\delta^+)$, for any $x\in\Omega$ with $\delta<|\Omega|/2$, it holds that

$$\begin{aligned}\mathcal{L}_{\Omega_\delta^+,\delta}u(x)&=\frac{1}{\delta^3}\int_{\Omega_\delta^+}[u(x)-u(y)]\gamma_1\left(\frac{x-y}{\delta}\right)dy\\ &=\frac{1}{2\delta^2}\int_{-1}^{1}[2u(x)-u(x+s\delta)-u(x-s\delta)]\gamma_1(s)\,ds\\ &=\frac{1}{2}\int_{-1}^{1}\frac{2u(x)-u(x+s\delta)-u(x-s\delta)}{s^2\delta^2}s^2\gamma_1(s)\,ds\\ &=\mathcal{L}_{\Omega,0}u(x)+\mathcal{O}(\delta^2),\end{aligned}$$

where we have defined

$$\sigma=\frac{1}{2}\int_{-1}^{1}s^2\gamma_1(s)ds>0, \quad \mathcal{L}_{\Omega,0}u(x)=-\sigma\frac{d^2u(x)}{dx^2}. \tag{2.2}$$

This implies that acting on the smooth functions suitably away from the boundary, the nonlocal operator converges to the local differential operator with a second order rate by taking the horizon δ as a small asymptotic parameter.

Analogous to local elliptic differential operators, nonlocal operators may also admit the maximum principle. For example, we have

Theorem 2.1. *Let $\Omega=(a,b)$ be a bounded interval, and u a piecewise smooth function satisfying $u|_\Omega\in C_b(\Omega)$ and $u|_{\Omega_\delta^{+,c}}\in C_b(\Omega_\delta^{+,c})$ for some $\delta<|\Omega|/2$. Then the following statements are valid:*

(1) $\mathcal{L}_{\Omega_\delta^+,\delta}u|_\Omega\in C_b(\Omega)$ and $\mathcal{L}_{\Omega_\delta^+,\delta}u|_{\Omega_\delta^{+,c}}\in C_b(\Omega_\delta^{+,c})$;

(2) if $\mathcal{L}_{\Omega_\delta^+,\delta}u|_\Omega\le 0$, then $\operatorname{ess\,sup}_\Omega u\le \operatorname{ess\,sup}_{\Omega_\delta^{+,c}}u$;

(3) if $\mathcal{L}_{\Omega_\delta^+,\delta}u|_{\Omega\cup(b,b+\delta)}\le 0$, then $\operatorname{ess\,sup}_{\Omega\cup(b,b+\delta)}u\le\operatorname{ess\,sup}_{(a-\delta,a)}u$.

Proof. The first statement is obvious according to the definition of the primitive kernel function γ_1 and the definition of the nonlocal operator $\mathcal{L}_{\Omega_\delta^+,\delta}$, see (2.1). In the sequel we merely prove the third statement, since the proof of the second statement is actually analogous but simpler to that of the third statement. Let us consider the following case first:

$$\mathcal{L}_{\Omega_\delta^+,\delta}u|_{\Omega\cup(b,b+\delta)}\le -\sigma_*<0, \tag{2.3}$$

where σ_* is an arbitrary positive number. If

$$\operatorname*{ess\,sup}_{\Omega\cup(b,b+\delta)}u>\operatorname*{ess\,sup}_{(a-\delta,a)}u,$$

then for all $\epsilon > 0$, there exists a continuous point $x_* = x_*(\epsilon) \in \Omega \cup (b, b+\delta)$, such that

$$u(x_*) - u(y) \geq -\epsilon, \ a.e. \ \forall y \in \Omega_\delta^+.$$

Therefore, it holds that

$$\mathcal{L}_{\Omega_\delta^+,\delta} u(x_*) = \int_{\Omega_\delta^+} [u(x_*) - u(y)] \gamma_\delta(x-y) dy \geq -2\epsilon \int_0^\delta \gamma_\delta(s) ds.$$

Due to (2.3), this is impossible if we set

$$\epsilon = \frac{\sigma_*}{2} \left(\int_0^\delta \gamma_\delta(s) ds \right)^{-1}.$$

In the general case, let us put

$$\phi_\delta(x) = (x - b - \delta)^2. \tag{2.4}$$

Since for all $x \in \Omega \cup (b, b+\delta)$, it holds that

$$\begin{aligned}
\mathcal{L}_{\Omega_\delta^+,\delta} \phi_\delta(x) &= \int_{-\delta}^{\min(\delta, b+\delta-x)} [\phi_\delta(x) - \phi_\delta(x+s)] \gamma_\delta(s) ds \\
&= - \int_{-\delta}^{\min(\delta, b+\delta-x)} [s^2 + 2s(x-b-\delta)] \gamma_\delta(s) ds \\
&\leq -\sigma - (x-b-\delta) \int_{-\delta}^{\min(\delta, b+\delta-x)} s \gamma_\delta(s) ds \leq -\sigma,
\end{aligned}$$

with an arbitrary $\epsilon > 0$, we have

$$\mathcal{L}_{\Omega_\delta^+,\delta} (u + \epsilon \phi_\delta) \leq -\epsilon \sigma < 0.$$

By the previous argument, we have

$$\operatorname*{ess\,sup}_{\Omega \cup (b, b+\delta)} (u + \epsilon \phi_\delta) \leq \operatorname*{ess\,sup}_{(a-\delta, a)} (u + \epsilon \phi_\delta).$$

Sending ϵ to zero, we finish the proof. □

We note that although maximum principles are known to be valid and have been used for various scalar nonlocal equations [21, 22], the weak version shown in the above requires only piecewise continuity of the solution which could be very fitting to situations where nonlocal models might be applicable.

Next, we state the pointwise a priori estimates in a precise form to distinguish them from the local counterpart.

THEOREM 2.2. *Let $\Omega = (a, b)$ be a bounded interval, and u a piecewise smooth function satisfying $u|_\Omega \in C_b(\Omega)$ and $u|_{\Omega_\delta^{+,c}} \in C_b(\Omega_\delta^{+,c})$ for some $\delta < |\Omega|/2$. Then there exists a constant $c > 0$, independent of u but depending on $|\Omega|$, such that*

$$\|u\|_{\infty,\Omega} \leq \|u\|_{\infty,\Omega_\delta^{+,c}} + c \|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega}, \tag{2.5}$$

$$\|u\|_{\infty,\Omega\cup(b,b+\delta)} \leq \|u\|_{\infty,(a-\delta,a)} + c \|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(b,b+\delta)}, \tag{2.6}$$

$$\|u\|_{\infty,\Omega\cup(a-\delta,a)} \le \|u\|_{\infty,(b,b+\delta)} + c\|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(a-\delta,a)}. \tag{2.7}$$

Proof. We only prove (2.6). Set

$$v^{\pm} = \pm u + \frac{1}{\sigma}\|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(b,b+\delta)}\phi_\delta,$$

where σ is given in (2.2) and ϕ_δ is defined as in (2.4), then we have

$$\mathcal{L}_{\Omega_\delta^+,\delta} v^{\pm} = \pm\mathcal{L}_{\Omega_\delta^+,\delta} u + \frac{1}{\sigma}\|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(b,b+\delta)}\mathcal{L}_{\Omega_\delta^+,\delta}\phi_\delta \le 0.$$

Applying Theorem 2.1, we derive

$$\operatorname*{ess\,sup}_{\Omega\cup(b,b+\delta)} v^{\pm} \le \|u\|_{\infty,(a-\delta,a)} + \frac{1}{\sigma}\|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(b,b+\delta)} \max_{(a-\delta,a)} \phi_\delta.$$

Therefore, we have

$$\begin{aligned}\|u\|_{\infty,\Omega\cup(b,b+\delta)} &\le \max v^{\pm} + \frac{1}{\sigma}\|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(b,b+\delta)} \max_{\Omega_\delta^+}\phi_\delta \\ &\le \|u\|_{\infty,(a-\delta,a)} + \frac{2}{\sigma}\|\mathcal{L}_{\Omega_\delta^+,\delta} u\|_{\infty,\Omega\cup(b,b+\delta)} \max_{\Omega_\delta^+}\phi_\delta.\end{aligned}$$

The inequality (2.7) can be derived similarly. This ends the proof. ∎

We note that the contribution of the nonlocal boundary terms is over regions of nonzero volume to the bounds on the right-hand sides of (2.5) and (2.6). This is a particular feature of nonlocal problems.

3. Nonlocal problems on bounded domain

In this section, we formulate variational problems associated with the nonlocal operator $\mathcal{L}_{\Omega_\delta^+,\delta}$. To achieve this, we recall the boundary value problems of local operators first, then present the nonlocal analog. The formulation of nonlocal Dirichlet-type problems is similar to that in [13], while the one involving Neumann-type data is similar to that in [14, 21]. In comparison, we propose more careful treatment of the boundary data to achieve higher order consistency to the local limit. Additional references and more detailed discussions on nonlocal problems can be found in [5–7].

3.1. Local BVPs. Let $\Omega=(a,b)$ be a bounded interval. Given a source function $f\in L^2(\Omega)$ and two boundary values u_0 and u_1 at the boundary points a and b, the classical Dirichlet principle reads as:

$$\mathcal{J}_0^{\mathcal{D}}(u) = \inf_{\{v\in H^1(\Omega),\ v(a)=u_0,\ v(b)=u_1\}} \mathcal{J}_0^{\mathcal{D}}(v), \tag{3.1}$$

where

$$\mathcal{J}_0^{\mathcal{D}}(v) = \frac{1}{2}(\sigma\nabla v,\nabla v)_\Omega - (f,v)_\Omega,\ \forall v\in H^1(\Omega). \tag{3.2}$$

The weak form of (3.1) is the following:

$$\begin{aligned}&\textit{Find } u\in H^1(\Omega) \textit{ with } u(a)=u_0,\ u(b)=u_1,\ \textit{s.t.}\\ &(\sigma\nabla u,\nabla v)_\Omega = (f,v)_\Omega,\ \forall v\in H_0^1(\Omega).\end{aligned} \tag{3.3}$$

The strong form of (3.1) is to find $u \in H^2(\Omega)$ such that

$$\begin{aligned} \mathcal{L}_{\Omega,0} u &= f \quad \text{in} \quad \Omega, \\ u(a) &= u_0, \quad u(b) = u_1, \end{aligned} \tag{3.4}$$

where the local operator $\mathcal{L}_{\Omega,0}$ is defined as in (2.2). It is known that the problem (3.4) admits a unique solution for $f \in L^2(\Omega)$, and its regularity can be raised to $C_b^{m+2}(\Omega)$ if $f \in C_b^m(\Omega)$ for some $m \geq 0$.

On the other hand, given a source function $f \in L^2(\Omega)$ and two boundary data u_0 and $\tilde{u}_1$ at the two boundary points a and b, the variational form of the mixed BVP reads as:

$$\mathcal{J}_0^{\mathcal{N}}(u) = \inf_{\{v \in H^1(\Omega),\ v(a) = u_0\}} \mathcal{J}_0^{\mathcal{N}}(v), \tag{3.5}$$

where

$$\mathcal{J}_0^{\mathcal{N}}(v) = \frac{1}{2}(\sigma \nabla v, \nabla v)_\Omega - (f, v)_\Omega - \tilde{u}_1 v(b),\ \forall v \in H^1(\Omega). \tag{3.6}$$

The weak form of (3.5) is the following:

$$\begin{aligned} &\textit{Find } u \in H^1(\Omega) \text{ with } u(a) = u_0,\ \textit{s.t.} \\ &(\sigma \nabla u, \nabla v)_\Omega = (f, v)_\Omega + \tilde{u}_1 v(b),\ \forall v \in H^1(\Omega) \text{ with } v(a) = 0. \end{aligned} \tag{3.7}$$

The strong form of (3.5) is to find $u \in H^2(\Omega)$, such that

$$\begin{aligned} \mathcal{L}_{\Omega,0} u &= f \quad \text{in} \quad \Omega, \\ u(a) &= u_0, \quad \sigma u'(b) = \tilde{u}_1. \end{aligned} \tag{3.8}$$

Again, it is known that the mixed BVP (3.8) admits a unique solution for all $f \in L^2(\Omega)$, and its regularity can be raised to $C_b^{m+2}(\Omega)$ if $f \in C_b^m(\Omega)$.

3.2. Nonlocal constrained value problems. Now let us consider the formulations of nonlocal problems with nonlocal constraints as discussed in [7]. Given two bounded domains Ω_1 and Ω_2, let us introduce

$$A^\delta_{\Omega_1,\Omega_2}(u,v) = \frac{1}{2}\int_{x\in\Omega_1}\int_{y\in\Omega_2} [u(x)-u(y)][v(x)-v(y)]\gamma_\delta(x-y)dydx. \tag{3.9}$$

This is a symmetric nonnegative definite bilinear form on $L^2(\Omega_1) \times L^2(\Omega_2)$. Besides, it holds that

$$A^\delta_{\Omega_1,\Omega_2} = A^\delta_{\Omega_2,\Omega_1}. \tag{3.10}$$

Now, given a source function $f \in L^2(\Omega)$ and a nonlocal boundary data $g \in L^2(\Omega_\delta^{+,c})$, by replacing the quadratic term in (3.2) with $A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(v,v)/2$, we formulate the variational form of nonlocal Dirichlet constrained value problems (CVP) as:

$$\mathcal{J}_\delta^{\mathcal{D}}(u) = \inf_{\{v \in L^2(\Omega_\delta^+),\ v = g \text{ on } \Omega_\delta^{+,c}\}} \mathcal{J}_\delta^{\mathcal{D}}(v), \tag{3.11}$$

where

$$\mathcal{J}_\delta^{\mathcal{D}}(v) = \frac{1}{2}A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(v,v) - (f,v)_\Omega,\ \forall v \in L^2(\Omega_\delta^+). \tag{3.12}$$

The weak form of (3.11) is the following:

$$\begin{aligned}&\textit{Find } u\in L^2(\Omega_\delta^+) \textit{ with } u=g \text{ on } \Omega_\delta^{+,c},\ \textit{s.t.}\\ &A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(u,v)=(f,v)_\Omega,\ \forall v\in L^2_c(\Omega_\delta^{+,c}),\end{aligned}\tag{3.13}$$

where we have set the constrained test space as

$$L^2_c(\Omega_\delta^{+,c})=\{v\in L^2(\Omega_\delta^+): v=0 \text{ in } \Omega_\delta^{+,c}\}.\tag{3.14}$$

The strong form of (3.11) is to find $u\in L^2(\Omega_\delta^{+,c})$, such that

$$\begin{aligned}&\mathcal{L}_{\Omega_\delta^+,\delta}u=f \quad \text{in} \quad \Omega,\\ &u=g \quad \text{on} \quad \Omega_\delta^{+,c}.\end{aligned}\tag{3.15}$$

The need to impose constraint of u on a region $\Omega_\delta^{+,c}$ (the so-called interaction domain and often denoted as Ω_I) of nonzero measure is a character of the nonlocal model, see [5–7] for more discussions.

On the other hand, given a source function $f\in L^2(\Omega)$ defined on Ω, nonlocal boundary data functions $g\in L^2(a-\delta,a)$ and $w\in L^2(b,b+\delta)$ defined in regions of nonzero volume, by replacing the quadratic term in (3.6) with $A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(v,v)/2$, and the term $\tilde{u}_1 v(b)$ with $(w,v)_{(b,b+\delta)}$, we formulate the variational form of nonlocal mixed CVP as:

$$\mathcal{J}_\delta^{\mathcal{N}}(u)=\inf_{\{v\in L^2(\Omega_\delta^+),\ v=g \text{ in } (a-\delta,a)\}}\mathcal{J}_\delta^{\mathcal{N}}(v),\tag{3.16}$$

where

$$\mathcal{J}_\delta^{\mathcal{N}}(v)=\frac{1}{2}A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(v,v)-(f,v)_\Omega-(w,v)_{(b,b+\delta)},\ \forall v\in L^2(\Omega_\delta^+).\tag{3.17}$$

By defining a constrained test space accordingly,

$$L^2_n(\Omega_\delta^{+,c})=\{v\in L^2(\Omega_\delta^+): v=0 \text{ in } (a-\delta,a)\},\tag{3.18}$$

the weak form of (3.16) can be stated as follows:

$$\begin{aligned}&\textit{Find } u\in L^2(\Omega_\delta^+) \text{ with } u=g \text{ in } (a-\delta,a),\ \textit{s.t.}\\ &A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(u,v)=(f,v)_\Omega+(w,v)_{(b,b+\delta)},\ \forall v\in L^2_n(\Omega_\delta^+).\end{aligned}\tag{3.19}$$

The strong form of (3.16) is to find $u\in L^2(\Omega_\delta^+)$, such that

$$\begin{aligned}&\mathcal{L}_{\Omega_\delta^+,\delta}u=f \quad \text{in} \quad \Omega,\\ &\mathcal{L}_{\Omega_\delta^+,\delta}u=w \quad \text{in} \quad (b,b+\delta),\\ &u=g \quad \text{in} \quad (a-\delta,a),\end{aligned}\tag{3.20}$$

For more general discussions of formulations of nonlocal models on the bounded domain like the ones above, we refer to [7, 14].

3.3. Well-posedness of nonlocal CVPs.

THEOREM 3.1. *If $f\in C_b(\Omega)$ and $g\in C_b(\Omega_\delta^{+,c})$, then the nonlocal Dirichlet CVP (3.13) admits a unique solution u satisfying $u|_\Omega\in C_b(\Omega)$.*

Proof. The problem (3.13) is equivalent to finding $u\in L^2(\Omega_\delta^+)$ such that $u|_{\Omega_\delta^{+,c}}=g$ and it holds that

$$c_\delta u(x)-\int_\Omega u(y)\gamma_\delta(x-y)dy=\int_{\Omega_\delta^{+,c}} g(y)\gamma_\delta(x-y)dy+f(x), \tag{3.21}$$

where

$$c_\delta=\int_{-\delta}^{\delta}\gamma_\delta(s)ds=\delta^{-2}\int_{-1}^{1}\gamma_1(s)ds.$$

The left-hand side of (3.21) determines a Fredholm operator of the second kind in $L^2(\Omega)$, see [16]. Therefore, to show the well-posedness of (3.13), it suffices to verify that the kernel space is trivial with $f=0$. To show the latter, let $v\in L^2(\Omega_\delta^+)$ lie in the kernel space, then $v|_{\Omega_\delta^{+,c}}=0$, and it holds that

$$A^\delta_{\Omega_\delta^+,\Omega_\delta^+}(v,v)=0.$$

This leads to $v=0$ [25]. Note that for $x\in\Omega$, it holds that

$$c_\delta u-\int_\Omega u(y)\gamma_\delta(x-y)dy=\int_{\Omega_\delta^{+,c}} g(y)\gamma_\delta(x-y)dy. \tag{3.22}$$

In the case that $g\in C_b(\Omega_\delta^{+,c})$, the right-hand side of (3.22) determines a function in $C_b(\Omega)$. Since the solution u belongs to $L^2(\Omega_\delta^+)$, we know that the second term on the left-hand side of (3.22) belongs to $C_b(\Omega)$. The proof is thus complete, considering that c_δ is a positive constant. ∎

THEOREM 3.2. *If $f\in C_b(\Omega)$, $w\in C_b(b,b+\delta)$, and $g\in C_b(a-\delta,a)$, then the nonlocal mixed CVP (3.19) admits a unique solution u which satisfies $u|_\Omega\in C_b(\Omega)$ and $u|_{(b,b+\delta)}\in C_b(b,b+\delta)$.*

The proof of Theorem 3.2 is analogous to Theorem 3.1, and we omit it here. We remark that, for points that are in $(b,b+\delta)$ in case of Neumann-type data, the first term is no longer $c_\delta u(x)$ in (3.21) but another nonzero function of x times $u(x)$.

In Section 5, we need to consider the following nonlocal Neumann CVP in a semi-infinite interval: find $u\in L^2(-\infty,\delta)$, such that

$$\begin{aligned}\mathcal{L}_{(-\infty,\delta),\delta}u&=0 \quad \text{in} \quad (-\infty,0),\\ \mathcal{L}_{(-\infty,\delta),\delta}u&=w \quad \text{in} \quad (0,\delta).\end{aligned} \tag{3.23}$$

THEOREM 3.3. *If $w\in C_b(0,\delta)$ satisfies $(w,1)_{(0,\delta)}=0$, then the nonlocal CVP (3.23) admits a unique solution $u\in L^2(-\infty,\delta)$ which satisfies $u|_{(-\infty,0)}\in C_b(-\infty,0)$ and $u|_{(0,\delta)}\in C_b(0,\delta)$.*

Proof. The weak form of nonlocal CVP (3.23) is to find $u\in L^2(-\infty,\delta)$ such that

$$A^\delta_{(-\infty,\delta),(-\infty,\delta)}(u,v)=(w,v)_{(0,\delta)},\quad \forall v\in L^2(-\infty,\delta). \tag{3.24}$$

The left-hand side determines a Fredholm operator of the second kind [16] in $L^2(-\infty,\delta)$. Considering (3.24), we know that the kernel space is trivial. By the Fredholm theory, the problem (3.24) admits a unique solution in $L^2(-\infty,\delta)$ if w satisfies the compatibility condition $(w,1)_{(0,\delta)}=0$.

Next we prove the regularity. Confined to $(0,\delta)$, we have

$$u(x)\int_{x-\delta}^{\delta}\gamma_\delta(x-y)dy-\int_{x-\delta}^{\delta}u(y)\gamma_\delta(x-y)dy=w(x).$$

Since γ_δ is piecewise smooth, the right-hand side belongs to $C_b(0,\delta)$. The second term on the left-hand side belongs to $C_b(0,\delta)$ since $u|_{(0,\delta)}\in L^2(0,\delta)$. Therefore, we have $u|_{(0,\delta)}\in C_b(0,\delta)$. Confined to $(-\infty,0)$, we have

$$u(x)\int_{-\delta}^{\delta}\gamma_\delta(s)ds=\int_{x-\delta}^{x+\delta}u(y)\gamma_\delta(x-y)dy.$$

The right-hand side belongs to $C_b(-\infty,0)$ since $u\in L^2(-\infty,\delta)$. Therefore, we have $u|_{(-\infty,0)}\in C_b(-\infty,0)$. ∎

4. Order of approximations of nonlocal CVPs to local BVPs

For a prescribed smooth function u, as shown earlier, the function $\mathcal{L}_\delta u$ converges to $\mathcal{L}_0 u$ with a second order rate as $\delta\to 0$, away from the boundary. In this section, we investigate the asymptotic error of nonlocal CVPs. More precisely, we want to figure out for what kind of nonlocal boundary data, solutions of nonlocal CVPs converge to those of local BVPs with a second order asymptotic rate in δ.

THEOREM 4.1. *Let $\Omega=(a,b)$ be a bounded interval. Suppose $u_0\in C_b^4(\Omega)$ is the unique solution to the following local Dirichlet boundary value problem:*

$$\mathcal{L}_{\Omega,0}u_0(x)=f(x),\quad x\in\Omega, \tag{4.1}$$
$$u_0(a)=g_0,\quad u_0(b)=g_1, \tag{4.2}$$

where $f\in C_b^2(\Omega)$. For all $\delta<|\Omega|/2$, let u_δ be the unique solution to the following nonlocal Dirichlet boundary value problem:

$$\mathcal{L}_{\Omega_\delta^+,\delta}u_\delta(x)=f(x),\quad x\in\Omega, \tag{4.3}$$
$$u_\delta(x)=g_0+(x-a)\frac{du_0(a)}{dx},\quad x\in(a-\delta,a), \tag{4.4}$$
$$u_\delta(x)=g_1+(x-b)\frac{du_0(b)}{dx},\quad x\in(b,b+\delta). \tag{4.5}$$

Then it holds that

$$\|u_\delta-u_0\|_{\infty,\Omega}=\mathcal{O}(\delta^2).$$

Proof. Let us define

$$\tilde u_0(x)=\begin{cases}\displaystyle\sum_{m=0}^{4}\frac{(x-a)^m}{m!}\frac{d^m u_0(a)}{dx^m}, & x\in(a-\delta,a),\\ u_0(x), & x\in\Omega,\\ \displaystyle\sum_{m=0}^{4}\frac{(x-b)^m}{m!}\frac{d^m u_0(b)}{dx^m}, & x\in(b,b+\delta).\end{cases}$$

Since $\tilde{u}_0 \in C_b^4(\Omega_\delta^+)$, we have

$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{u}_0 = \mathcal{L}_{\Omega,0}u_0 + \mathcal{O}(\delta^2) = f + \mathcal{O}(\delta^2), \quad \forall x \in \Omega.$$

Therefore, it holds for $x \in \Omega$ that

$$\mathcal{L}_{\Omega_\delta^+,\delta}(u_\delta - \tilde{u}_0) = \mathcal{O}(\delta^2).$$

Applying Theorem 2.2, we derive

$$\|u_\delta - u_0\|_{\infty,\Omega} \le \|u_\delta - \tilde{u}_0\|_{\infty,\Omega_\delta^{+,c}} + \mathcal{O}(\delta^2) = \mathcal{O}(\delta^2).$$

This ends the proof. □

THEOREM 4.2. *Let $\Omega = (a,b)$ be a bounded interval. Suppose $u_0 \in C_b^4(\Omega)$ is the unique solution to the following local mixed boundary value problem:*

$$\mathcal{L}_{\Omega,0}u_0(x) = f(x), \quad x \in \Omega, \tag{4.6}$$

$$u_0(a) = g_0, \quad \sigma u_0'(b) = \tilde{g}_1, \tag{4.7}$$

where $f \in C_b^2(\Omega)$. For all $\delta < |\Omega|/2$, let u_δ be the unique solution to the following nonlocal mixed boundary value problem:

$$\mathcal{L}_{\Omega_\delta^+,\delta}u_\delta(x) = f(x), \quad x \in \Omega, \tag{4.8}$$

$$u_\delta(x) = u_0(a) + (x-a)\frac{du_0(a)}{dx}, \quad x \in (a-\delta, a), \tag{4.9}$$

$$\mathcal{L}_{\Omega_\delta^+,\delta}u_\delta(x) = \mathcal{L}_{\Omega_\delta^+,\delta}\tilde{u}_0(x), \quad x \in (b, b+\delta), \tag{4.10}$$

where

$$\tilde{u}_0(x) = \sum_{m=0}^{4} \frac{(x-b)^m}{m!}\frac{d^m u_0(b)}{dx^m}. \tag{4.11}$$

Then it holds that

$$\|u_\delta - u_0\|_{\infty,\Omega} = \mathcal{O}(\delta^2).$$

Proof. Let us define

$$\hat{u}_0(x) = \begin{cases} \displaystyle\sum_{m=0}^{4} \frac{(x-a)^m}{m!}\frac{d^m u_0(a)}{dx^m}, & x \in (a-\delta, a), \\ u_0(x), & x \in \Omega, \\ \displaystyle\sum_{m=0}^{4} \frac{(x-b)^m}{m!}\frac{d^m u_0(b)}{dx^m}, & x \in (b, b+\delta). \end{cases}$$

Then $\hat{u}_0 \in C_b^4(\Omega_\delta^+)$. Therefore, it holds that

$$\mathcal{L}_{\Omega_\delta^+,\delta}\hat{u}_0(x) = \mathcal{L}_{\Omega,0}u_0(x) + \mathcal{O}(\delta^2) = f(x) + \mathcal{O}(\delta^2), \quad \forall x \in \Omega.$$

If $x \in (b, b+\delta)$, then

$$\mathcal{L}_{\Omega_\delta^+,\delta}\hat{u}_0(x) = \mathcal{L}_{\Omega_\delta^+,\delta}\tilde{u}_0(x) + \mathcal{O}(\delta^2).$$

Therefore, for $x\in\Omega\cup(b,b+\delta)$ we have

$$\mathcal{L}_{\Omega_\delta^+,\delta}(u_\delta-\hat{u}_0)=\mathcal{O}(\delta^2).$$

Applying Theorem 2.2, we derive

$$\|u_\delta-u_0\|_{\infty,\Omega}=\|u_\delta-\hat{u}_0\|_{\infty,\Omega}\le\|u_\delta-\hat{u}_0\|_{\infty,(a-\delta,a)}+\mathcal{O}(\delta^2)=\mathcal{O}(\delta^2).$$

This ends the proof. □

The result of the previous theorem, derived with a high order Taylor expansion of the local solution, is not optimal. It turns out that sharper truncation error analysis can be derived, see the construction of special barrier function in [21] and the refined estimate of the residual terms in the next section for details.

5. Refined asymptotic estimate of Theorem 4.2

Numerical evidences presented in the next section show that to maintain the second-order asymptotic convergence rate for the Neumann part of the boundary, only the terms up to $m=1$ in (4.11) are necessary. This is consistent with similar observations made in [21]. Analysis given in [21] relies on careful examination of the appropriate barrier function. In this section, we intend to clarify this point through similar calculations but in more direct and straightforward fashion.

Firstly, we prove the following theorem. Before the next computation, we here consider a special case that $\Omega_\delta^+=(a-\delta,b+\delta)$ with $b=0$, and the results can be straightforwardly extended to the more general case $\Omega_\delta^+=(a-\delta,b+\delta)$.

THEOREM 5.1. *Let $\Omega=(a,0)$ be a bounded interval. For all $\delta<|\Omega|/2$, let $u_{m,\delta}$ be the unique solution to the following nonlocal mixed boundary value problem:*

$$\mathcal{L}_{\Omega_\delta^+,\delta}u_{m,\delta}(x)=0,\quad x\in\Omega,\tag{5.1}$$

$$u_{m,\delta}(x)=0,\quad x\in(a-\delta,a),\tag{5.2}$$

$$\mathcal{L}_{\Omega_\delta^+,\delta}u_{m,\delta}(x)=\mathcal{L}_{\Omega_\delta^+,\delta}x^m,\quad x\in(0,\delta),\tag{5.3}$$

where m is any integer not less than 2. Then it holds that

$$\|u_{m,\delta}\|_{\infty,\Omega}=\mathcal{O}(\delta^2).$$

At the first glance, it does not seem possible to expect the conclusion in the Theorem 5.1, since for any $x\in(0,\delta)$, we have

$$\begin{aligned}\mathcal{L}_{\Omega_\delta^+,\delta}x^m&=\int_{x-\delta}^{\delta}[x^m-y^m]\gamma_\delta(x-y)dy\\&=\delta^{m-2}\int_{\hat{x}-1}^{1}[\hat{x}^m-\hat{y}^m]\gamma_1(\hat{x}-\hat{y})d\hat{y}\\&=\delta^{m-2}\mathcal{L}_{(-1,1),1}\hat{x}^m=\mathcal{O}(\delta^{m-2}),\quad\hat{x}=\frac{x}{\delta}.\end{aligned}$$

Therefore, if we were to apply Theorem 2.2 directly, then we would only expect

$$\|u_{m,\delta}\|_{\infty,\Omega}=\mathcal{O}(\delta^{m-2}).\tag{5.4}$$

In particular, it would appear that $\|u_{2,\delta}\|_{\infty,\Omega}=\mathcal{O}(1)$ at best.

Before presenting the proof of Theorem 5.1, it is useful to derive the following results.

LEMMA 5.1. *For any smooth function $\varphi(x)$ that is also even, it holds that*

$$\int_0^\delta \mathcal{L}_{\Omega_\delta^+,\delta}\varphi(x)dx = 0.$$

Besides, for any strictly increasing function $\varphi(x)$, it holds that

$$\int_0^\delta \mathcal{L}_{\Omega_\delta^+,\delta}\varphi(x)dx > 0.$$

Proof. A direct computation shows that

$$\begin{aligned}
&\int_0^\delta \mathcal{L}_{\Omega_\delta^+,\delta}\varphi(x)dx \\
&= \int_0^\delta \int_{x-\delta}^\delta [\varphi(x)-\varphi(y)]\gamma_\delta(x-y)dydx \\
&= \int_0^\delta \int_0^\delta [\varphi(x)-\varphi(y)]\gamma_\delta(x-y)dydx + \int_0^\delta \int_{x-\delta}^0 [\varphi(x)-\varphi(y)]\gamma_\delta(x-y)dydx \\
&\equiv \mathcal{T}_1 + \mathcal{T}_2.
\end{aligned}$$

Obviously, by the antisymmetry of the integral in $\mathcal{T}_1$, we have $\mathcal{T}_1 = 0$. If φ is a strictly increasing function, then $\mathcal{T}_2 > 0$. On the other hand, if φ is an even function, it holds that

$$\begin{aligned}
\mathcal{T}_2 &= \int_{-\delta}^0 \int_0^{\delta+y} [\varphi(x)-\varphi(y)]\gamma_\delta(x-y)dxdy \\
&= \int_0^\delta \int_0^{\delta-y} [\varphi(x)-\varphi(-y)]\gamma_\delta(x+y)dxdy \\
&= \int_0^\delta \int_{y-\delta}^0 [\varphi(-x)-\varphi(-y)]\gamma_\delta(y-x)dxdy \\
&= \int_0^\delta \int_{y-\delta}^0 [\varphi(x)-\varphi(y)]\gamma_\delta(y-x)dxdy = -\mathcal{T}_2,
\end{aligned}$$

which leads to $\mathcal{T}_2 = 0$. The proof is thus complete. ∎

In the case that $\varphi(x) = x^m$, with m being some positive odd integer, and recalling the proof of the above lemma, we have

$$\begin{aligned}
\int_0^\delta \mathcal{L}_{\Omega_\delta^+,\delta}x^m dx &= \int_0^\delta \int_{x-\delta}^0 [x^m - y^m]\gamma_\delta(x-y)dydx \\
&= \delta^{m-1}\int_0^1 \int_{x-1}^0 [x^m - y^m]\gamma_1(x-y)dydx \equiv \delta^{m-1}c_m.
\end{aligned}$$

Therefore, if we introduce

$$\phi_{m,\delta}(x) = \begin{cases} \mathcal{L}_{(-\infty,\delta),\delta}x^m, \ m \geq 2 \text{ even}, \\ \mathcal{L}_{(-\infty,\delta),\delta}x, \ m = 1, \\ \mathcal{L}_{(-\infty,\delta),\delta}x^m - c_m c_{m-2}^{-1}\delta^2\mathcal{L}_{(-\infty,\delta),\delta}x^{m-2}, \ m \geq 3 \text{ odd}, \end{cases} \tag{5.5}$$

we have

$$\int_0^\delta \phi_{m,\delta}(x)dx = 0, \ \forall m \geq 2. \tag{5.6}$$

Note that the function $\phi_{m,\delta}(x)$ admits the scaling property, i.e.,

$$\phi_{m,\delta}(x) = \delta^{m-2}\phi_{m,1}(x/\delta), \quad \forall m \geq 2. \tag{5.7}$$

Instead of considering the problem (5.1)-(5.3), we consider ($m \geq 1$):

$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{u}_{m,\delta}(x) = 0, \quad x \in \Omega, \tag{5.8}$$
$$\tilde{u}_{m,\delta}(x) = 0, \quad x \in (a-\delta, a), \tag{5.9}$$
$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{u}_{m,\delta}(x) = \phi_{m,\delta}(x), \quad x \in (0,\delta). \tag{5.10}$$

In the case that $m = 1$, the function $\tilde{v}_{1,\delta}(x) = \tilde{u}_{1,\delta}(x) - (x-a)$ solves

$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{v}_{1,\delta}(x) = 0, \quad x \in \Omega, \tag{5.11}$$
$$\tilde{v}_{1,\delta}(x) = a - x, \quad x \in (a-\delta, a), \tag{5.12}$$
$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{v}_{1,\delta}(x) = 0, \quad x \in (0,\delta). \tag{5.13}$$

Applying Theorem 2.2, we have

$$\|\tilde{v}_{1,\delta}\|_{\infty,\Omega} = \mathcal{O}(\delta),$$

which leads to

$$\|\tilde{u}_{1,\delta}\|_{\infty,\Omega} = \mathcal{O}(1). \tag{5.14}$$

For $m \geq 2$, let us consider the following semi-infinite domain problem:

$$\mathcal{L}_{(-\infty,1),1}\psi_m(x) = 0, \quad x \in (-\infty, 0), \tag{5.15}$$
$$\mathcal{L}_{(-\infty,1),1}\psi_m(x) = \phi_{m,1}(x), \quad x \in (0,1). \tag{5.16}$$

Thanks to Theroem 3.3, there exists a unique solution $\psi_m \in L^2(-\infty,1)$ independent of δ, which satisfies $\psi_m|_{(-\infty,0)} \in C_b(-\infty,0)$. Then the function $\tilde{v}_{m,\delta}(x) = \delta^m\psi_m(x/\delta) - \tilde{u}_{m,\delta}(x)$ solves:

$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{v}_{m,\delta}(x) = 0, \quad x \in \Omega, \tag{5.17}$$
$$\tilde{v}_{m,\delta}(x) = \delta^m\psi_m(x/\delta), \quad x \in (a-\delta, a), \tag{5.18}$$
$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{v}_{m,\delta}(x) = 0, \quad x \in (0,\delta). \tag{5.19}$$

Applying Theorem 2.2, we have

$$\|\tilde{v}_{m,\delta}\|_{\infty,\Omega} = \mathcal{O}(\delta^m), \quad \forall m \geq 2.$$

This leads to

$$\|\tilde{u}_{m,\delta}\|_{\infty,\Omega} = \mathcal{O}(\delta^m), \quad \forall m \geq 2.$$

Now we are ready to prove the theorem.

Proof. **(Proof of Theorem 5.1.)** If $m \geq 2$ is an even integer, the theorem holds since $u_{m,\delta} = \tilde{u}_{m,\delta}$. If $m \geq 3$ is an odd integer, it suffices to prove the following

$$\|u_{m,\delta}\|_{\infty,\Omega} = \mathcal{O}(\delta^{m-1}). \tag{5.20}$$

Recalling the Definition (5.5) of $\phi_{m,d}$, thanks to (5.14), we have

$$u_{3,\delta} = \tilde{u}_{3,\delta} + c_3 c_1^{-1} \delta^2 \tilde{u}_{1,\delta} = \mathcal{O}(\delta^2).$$

The estimate (5.20) holds for $m = 3$. For each odd integer $m > 3$, we have

$$\begin{aligned} \|u_{m,\delta}\|_{\infty,\Omega} &= \|\tilde{u}_{m,\delta} + c_m c_{m-2}^{-1} \delta^2 u_{m-2,\delta}\|_{\infty,\Omega} \\ &\leq \|\tilde{u}_{m,\delta}\|_{\infty,\Omega} + \|c_m c_{m-2}^{-1} \delta^2 u_{m-2,\delta}\|_{\infty,\Omega} \\ &= \mathcal{O}(\delta^m) + \delta^2 \mathcal{O}(\delta^{m-3}) = \mathcal{O}(\delta^{m-1}). \end{aligned}$$

Therefore, the estimate (5.20) holds by induction. ◻

Based on the conclusion of Theorem 5.1, we immediately arrive at an improved asymptotic estimate of Theorem 4.2 as follows.

THEOREM 5.2. *Let $\Omega = (a,b)$ be a bounded interval. Suppose $u_0 \in C_b^4(\Omega)$ is the unique solution to the following local mixed boundary value problem (4.6). For all $\delta < |\Omega|/2$, let u_δ be the unique solution to the following nonlocal mixed boundary value problem (4.11) with*

$$\tilde{u}_0(x) = u_0(b) + \frac{du_0}{dx}(b)(x-b). \tag{5.21}$$

Then it holds that

$$\|u_\delta - u_0\|_{\infty,\Omega} = \mathcal{O}(\delta^2).$$

6. Numerical experiments

Above, we theoretically proved the uniform second order convergence, as $\delta \to 0$, of nonlocal CVPs to the local limit with our proposed nonlocal boundary data.

We now provide numerical examples to demonstrate the sharpness of our theoretical analysis. We consider the kernel function γ_δ with $\delta \in (0,1)$ given by

$$\gamma_\delta(x,y) = 3\delta^{-3} \chi_{[-\delta,\delta]}(x-y).$$

In all calculations, we use the linear finite element method by taking a sufficiently small mesh size h, and consider the affect in the reduction of δ. In the situation of $h \ll \delta$, we can investigate the convergence properties of the nonlocal problem to the local problem by taking $\delta \to 0$, see [2, 19, 23]. In this situation, the theory developed in [22, 23] ensures that an asymptotically compatible scheme can numerically solve the nonlocal problem such that its numerical solutions can correctly converge to those of the corresponding local problem.

As what we will see later, the convergence has an optimal second order accuracy with respect to the variable δ. This implies that the analysis of nonlocal boundary value problems proposed here are valid. In the following, we respectively consider the nonlocal CVPs with Dirichlet and mixed boundary conditions.

6.1. Nonlocal Dirichlet CVP. Theorem 4.1 shows that the nonlocal problem (4.3)-(4.5) converges to the corresponding local problem (4.1)-(4.2) as $\delta \to 0$ in second order. Two examples are provided to verify the theoretical analysis.

EXAMPLE 6.1. *To investigate the quantitative accuracy of our analysis in Theorem 4.1, we construct an exact solution* $u(x) = x^2 + x^4 + x^5 + x^7$ *to the local problem* (4.1)-(4.2) *in the computational domain* $(a,b) = (0,1)$. *For the local problem, we have* $f(x) = -(42x^5 + 20x^3 + 12x^2 + 2)$, $u_0(a) = 0$ *and* $u_0(b) = 4$. *For the nonlocal problem, we consider two kind of boundary conditions:*

(1) The first kind of Dirichlet boundary condition (DBC1):

$$u_\delta(x) = u_0(a), \quad x \in (a-\delta, a), \quad u_\delta(x) = u_0(b), \quad x \in (b, b+\delta).$$

(2) The second kind of Dirichlet boundary condition (DBC2):

$$u_\delta(x) = u_0(a) + (x-a)\frac{du_0(a)}{dx}, \quad x \in (a-\delta, a),$$
$$u_\delta(x) = u_0(b) + (x-b)\frac{du_0(b)}{dx}, \quad x \in (b, b+\delta).$$

The first kind of boundary condition here means the boundary condition of nonlocal problems over a layer with the width δ is extended by directly using the information of boundary values of local problem. The second kind of boundary condition here means the boundary condition of nonlocal problems over a layer with the width δ is extended by using the information of both boundary values and derivative values of local problem, this is the first order Taylor expansion. As what we can see in Theorem 4.1, the constructions of boundary conditions above for nonlocal problems will lead to the first and second order convergences with respect to horizon parameter δ. To demonstrate the analysis, in the simulations we refine $\delta = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$ and use a spatial mesh size $h = 2^{-13}$ that can provide sufficient resolution. Table 6.1 shows the first and second order convergences by using $DBC1$ and $DBC2$, respectively.

δ	$DBC1$	order	$DBC2$	order
2^{-5}	2.366×10^{-1}	$--$	9.081×10^{-3}	$--$
2^{-6}	1.166×10^{-1}	1.02	2.264×10^{-3}	2.00
2^{-7}	5.784×10^{-2}	1.01	5.651×10^{-4}	2.00
2^{-8}	2.881×10^{-2}	1.01	1.412×10^{-4}	2.00
2^{-9}	1.437×10^{-2}	1.00	3.532×10^{-5}	2.00

TABLE 6.1. *(Example 6.1) L^∞-errors and convergence orders between nonlocal solutions and local solutions by refining δ and fixing the mesh size h^{-13}.*

EXAMPLE 6.2. *We construct the exact solution of the local problem as* $u_0(x) = \cos(\pi x) + \sin(\pi x)$ *over the computational domain* $[a,b] = [0,1]$. *For the local problem, we have* $f(x) = -\pi^2(\cos(\pi x) + \sin(\pi x))$, $u_0(a) = 1$ *and* $u_0(b) = -1$. *For the nonlocal problem, we investigate two kinds of boundary conditions $DBC1$ and $DBC2$ as shown in Example 6.1. In the simulations we refine $\delta = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}$ and fix the spatial mesh size $h = 2^{-13}$ as in the previous example. Again, Table 6.2 shows the first and second order convergences with $DBC1$ and $DBC2$ respectively.*

From Tables 6.1 and 6.2, we see that the boundary data play important roles in the asymptotic convergence rate. That is, it is necessary to have suitable nonlocal constraints (data) for nonlocal problems in order to lead to a second-order asymptotic convergence.

δ	$DBC1$	order	$DBC2$	order
2^{-5}	4.124×10^{-2}	$--$	1.175×10^{-3}	$--$
2^{-6}	2.033×10^{-2}	1.02	2.935×10^{-4}	2.00
2^{-7}	1.001×10^{-2}	1.01	7.332×10^{-5}	2.00
2^{-8}	5.028×10^{-3}	1.01	1.833×10^{-5}	2.00
2^{-9}	2.509×10^{-3}	1.00	4.587×10^{-6}	2.00

TABLE 6.2. *(Example 6.2) L^∞-errors and convergence orders between nonlocal solutions and local solutions by refining δ and fixing the mesh size h^{-13}.*

6.2. Nonlocal mixed CVP. Directly using maximum principle, Theorem 4.2 shows that the nonlocal problem (4.8)-(4.10) converges to the local problem (4.6)-(4.7) as $\delta\to0$ in the second-order rate, but it requires the fourth-order Taylor expansion for the Neumann boundary data. In the following simulations, we implement the left Dirichlet boundary condition by $DBC2$, and two kinds of Neumann boundary conditions, namely, taking

$$\mathcal{L}_{\Omega_\delta^+,\delta}\tilde{u}_0(x)=\int_{x-\delta}^{b+\delta}(\tilde{u}_0(x)-\tilde{u}_0(y))\gamma(x-y)dy,\quad x\in(b,b+\delta)$$

with

(1) the first kind of Neumann boundary condition (NBC1): $\tilde{u}_0(x)$ is given by (5.21).
(2) the second kind of Neumann boundary condition (NBC2):

$$\tilde{u}_0(x)=\sum_{m=0}^{2}\frac{(x-b)^m}{m!}\frac{d^m u_0(b)}{dx^m}.$$

As we can see in the following numerical results, both *NBC1 and* NBC2 result in the second-order asymptotic convergence. This is explained in the refined asymptotic estimate in Theorem 5.2. Two numerical examples are given to substantiate the analytical studies.

δ	$NBC1$	order	$NBC2$	order
2^{-5}	4.710×10^{-3}	$--$	1.165×10^{-3}	$--$
2^{-6}	1.159×10^{-3}	2.02	2.921×10^{-4}	2.00
2^{-7}	2.283×10^{-4}	2.01	7.316×10^{-5}	2.00
2^{-8}	7.152×10^{-5}	2.01	1.831×10^{-5}	2.00
2^{-9}	1.785×10^{-5}	2.00	4.584×10^{-6}	2.00

TABLE 6.3. *(Example 6.3) L^∞-errors and convergence orders between nonlocal solutions and local solutions by refining δ and fixing the mesh size h^{-13}.*

EXAMPLE 6.3. *To investigate the quantitative accuracy of the asymptotic compatibility, we construct the solution $u_0(x)=\cos(\pi x)+\sin(\pi x)$ to local mixed CVPs over the computational domain $(a,b)=(0,1)$. For the local problem, we have $f(x)=-\pi^2(\cos(\pi x)+\sin(\pi x))$, $u_0(a)=1$ and $u_0'(b)=-\pi$.*

In the simulations, we refine $\delta=2^{-2},2^{-3},2^{-4},2^{-5},2^{-6}$ and fix the spatial mesh size $h=2^{-13}$. Table 6.3 shows the second-order convergence in the L^∞-norm by using $NBC1$ and $NBC2$, respectively

δ	$NBC1$	order	$NBC2$	order
2^{-5}	2.190×10^{-2}	$--$	3.810×10^{-3}	$--$
2^{-6}	5.700×10^{-3}	1.94	9.520×10^{-4}	2.00
2^{-7}	1.452×10^{-3}	1.97	2.380×10^{-4}	2.00
2^{-8}	3.664×10^{-4}	1.99	5.958×10^{-5}	2.00
2^{-9}	9.204×10^{-5}	1.99	1.497×10^{-5}	1.99

TABLE 6.4. *(Example 6.4) L^∞-errors and convergence orders between nonlocal solutions and local solutions by refining δ and fixing the mesh size h^{-13}.*

EXAMPLE 6.4. *We construct another exact solution of the local problem as $u_0(x)=\exp(2x)(\cos(\pi x)+\sin(\pi x))$ over the computational domain $[a,b]=[0,1]$. For the local problem, we have $f(x)=-u_0''(x)$, $u_0(a)=1$ and $u_0(b)=-(2+\pi)\exp(2)$. Again, in the simulations, we refine $\delta=2^{-2},2^{-3},2^{-4},2^{-5},2^{-6}$ and fix the spatial mesh size $h=2^{-13}$. Again, Table 6.4 shows the second-order convergence with $NBC1$ and $NBC2$ respectively.*

From Tables 6.3 and 6.4, one can see that only the first order Taylor expansion for the Neumann boundary data is required to produce the second-order convergence rate. The detailed theoretical analysis is established in Section 5.

7. Conclusion

In this work, we carefully investigate the formulations of nonlocal constrained value problems (CVPs) and their asymptotic order of convergence to the local limit. The regularity and well-posedness of the resulting nonlocal CVPs are also studied for given source terms and boundary data. Without any extra regularity assumption on nonlocal solutions, we construct suitable boundary data so that nonlocal solutions converge to the local solution in the second-order rate with respect to the horizon parameter δ. For nonlocal Neumann boundary data, a refined asymptotic estimate is given to understand the underlying mechanism. Numerical examples are reported to further verify the theoretical predictions. The findings and techniques presented in this work may be useful in not only benchmark studies of nonlocal models and their local limits but also constructions of more accurate numerical discretizations to local PDEs based on nonlocal integral relaxations to differential operators [8].

Acknowledgements. Qiang Du is supported in part by the U.S. NSF grants DMS-1719699, AFOSR MURI Center for material failure prediction through peridynamics, and the ARO MURI W911NF-15-1-0562 on Fractional PDEs for Conservation Laws and Beyond: Theory, Numerics and Applications. Jiwei Zhang is partially supported by NSFC under Nos. 11771035, and the Natural Science Foundation of Hubei Province No. 2019CFA007, and Xiangtan University 2018ICIP01. Chunxiong Zheng is supported by Natural Science Foundation of Xinjiang Autonomous Region under No. 2019D01C026, and National Natural Science Foundation under No. 11771248.

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