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EXISTENCE OF WEAK SOLUTIONS TO THE STEADY TWO-PHASE FLOW*

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Abstract. In this paper, we prove the existence of weak solutions to the steady two-phase flow. The result holds in three dimensions on the condition that the adiabatic constants $\gamma,\theta>1$ and $\gamma>\frac{7}{3}$, $\theta=1$. By constructing a special example, we show that the weak solutions are non-unique. It turns out that the uniform approximation scheme restricts the type of weak solutions, which leads to some open problems.

Keywords. two-phase model; weak solutions; non-uniqueness.

AMS subject classifications. 35D30; 35Q30; 76T10.

1. Introduction

In this paper, we are concerned about the steady problem of two-phase flow in a bounded domain $\Omega\in\mathbb{R}^3$, which is described by the Navier-Stokes equation with a pressure law in two variables

$$\operatorname{div}(\rho u)=0, \tag{1.1}$$
$$\operatorname{div}(nu)=0, \tag{1.2}$$
$$\operatorname{div}[(\rho+n)u\otimes u]+\nabla p(\rho,n)-\mu\Delta u-(\mu+2\lambda)\nabla\operatorname{div}u=(\rho+n)f, \tag{1.3}$$

with the boundary condition

$$u|_{\partial\Omega}=0, \tag{1.4}$$

and the mass conservations

$$\int_\Omega \rho\ dx=M>0,\quad \int_\Omega n\ dx=N>0. \tag{1.5}$$

Here $p(\rho,n)=\rho^\gamma+n^\theta$ is the pressure with the adiabatic constants $\gamma\geq1$, $\theta\geq1$, ρ and n denote the densities of the two phases, u is the velocity of the fluid, f stands for the external volume force; μ and λ are fixed constant viscosity coefficients, which satisfy thermodynamic constraints $\mu>0$, $2\mu+3\lambda\geq0$.

If we only consider a single density, the model becomes the well-known steady compressible Navier-Stokes equations:

$$\operatorname{div}(\rho u)=0, \tag{1.6}$$
$$\operatorname{div}(\rho u\otimes u)+\nabla\rho^\gamma-\mu\Delta u-(\mu+2\lambda)\nabla\operatorname{div}u=\rho f. \tag{1.7}$$

For the general large data, Lions [12] used the method of weak convergence to obtain the existence of the global renormalized weak solutions for all $\gamma>\frac{5}{3}$ in 3D. He established

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the delicate approximation scheme

$$\alpha(\rho-h)+\operatorname{div}(\rho u)=\epsilon\Delta\rho, \tag{1.8}$$
$$\alpha(\rho+h)u+\frac{1}{2}\operatorname{div}(\rho u\otimes u)+\frac{1}{2}\rho u\cdot\nabla u+\nabla\rho^\gamma+\delta\nabla\rho^\beta-\mu\Delta u-(\mu+2\lambda)\nabla\operatorname{div}u=\rho f, \tag{1.9}$$

which not only keeps the mass conservation, but also helped us to obtain energy inequality. Adapting the concept of oscillation defect measure developed in [6], Novo and Novotný [15] were able to show the existence result for $\gamma>\frac{3}{2}$ if f is a potential. The following works focused on how to improve the integrability of the density and reduced the condition to the physical limit $\gamma\geq 1$. The breakthrough came from the work [7] and [18], in which they both obtained L^∞-estimate of the inverse Laplacian of the pressure to improve the estimate of the density. The methods were developed by Březina and Novotný [3], which enabled them to show the existence of weak solutions to the spatially periodic case in 3D for $\gamma>\frac{3+\sqrt{41}}{8}$ if f is a potential, or $\gamma>\frac{1+\sqrt{13}}{3}$ if $f\in L^\infty(\Omega)$. In papers [8] and [9], Frehse, Steinhauer, and Weigant extended their local weighted estimate for the pressure to a global estimate to treat the case $\gamma>\frac{4}{3}$ in 3D and $\gamma\geq 1$ in 2D. The first result for $\gamma>1$ in three dimensions was due to the work of Jiang and Zhou [11], in which they established a new coupled estimate for both kinetic energy and pressure. The method was generalized to solve the case of slip boundary condition by Jessle and Novotny [10] and finally Plotnikova and Weigant [19] extended it to the case of Dirichlet boundary condition for all $\gamma>1$. To our best knowledge, the problem $\gamma=1$ in three dimensions is still left open.

Recently, there came out a lot of results about the weak solutions for the two-phase model in three dimension. Most of the models can be seen as a special case of the following system:

$$\alpha^+ +\alpha^- =1, \tag{1.10}$$
$$\partial_t(\alpha^\pm\rho^\pm)+\operatorname{div}(\alpha^\pm\rho^\pm u^\pm)=0, \tag{1.11}$$
$$\partial_t(\alpha^\pm\rho^\pm u^\pm)+\operatorname{div}(\alpha^\pm\rho^\pm u^\pm\otimes u^\pm)+\alpha^\pm\nabla p(\rho^\pm)=\operatorname{div}(\alpha^\pm\tau^\pm), \tag{1.12}$$

where $\alpha^+\geq 0$, $\alpha^-\geq 0$ are the volume fractions of the fluid $+$ and the fluid $-$, respectively. Moreover, $\rho^\pm$, $u^\pm$, p and $\tau^\pm$ stand for the densities, the velocity of each fluid, the pressure, the strain, respectively. For more details about the model, we refer to [1].

By taking $u^+=u^-$, $\rho=\alpha^+\rho^+$ and $n=\alpha^-\rho^-$, we obtain the system (1.1)-(1.3) with time development

$$\rho_t+\operatorname{div}(\rho u)=0, \tag{1.13}$$
$$n_t+\operatorname{div}(nu)=0, \tag{1.14}$$
$$[(\rho+n)u]_t+\operatorname{div}[(\rho+n)u\otimes u]+\nabla p(\rho,n)-\mu\Delta u-(\mu+2\lambda)\nabla\operatorname{div}u=0. \tag{1.15}$$

Vasseur, Wen, and Yu [20] prove the existence of weak solutions to the above system for $\gamma>\frac{9}{5}$, $\theta>\frac{9}{5}$. Moreover, if the initial data satisfies the equivalence condition $\frac{1}{c_0}\rho\leq n\leq c_0\rho$, $c_0\geq 1$, the result can be improved to $\gamma>\frac{9}{5}$, $\theta\geq 1$. Considering another type of the pressure, D. Bresch, P. Mucha, and E. Zatorska in [2] study the two-fluid Stokes system

$$\rho_t+\operatorname{div}(\rho u)=0, \tag{1.16}$$
$$n_t+\operatorname{div}(nu)=0, \tag{1.17}$$
$$\nabla p(\rho,n)=\mu\Delta u+(\mu+2\lambda)\nabla\operatorname{div}u. \tag{1.18}$$

Here the pressure $p(\rho,n)$ is given by

$$A_-\rho_-^\theta = A_+\rho_+^\gamma, \tag{1.19}$$

$$\rho\rho_- + n\rho_+ = \rho_-\rho_+, \tag{1.20}$$

A_- and A_+ are positive constants. They obtain the global weak solutions for $\gamma,\theta>1$, without the equivalence condition. Very recently, using the equivalence condition, Novotný, and Pokorný [16] improve the result of Vasseur, Wen, and Yu [20] to general pressure and allow $\gamma\geq\frac{9}{5}$, $\theta>0$. The pressure includes $p(\rho,n)=\rho^\gamma+n^\theta$ and even the non-monotone functions.

There are also some papers considering the related problem of the two-phase flow. For some special initial data, Dong, Zhu, and Xue [4] show the blow up of smooth solutions to the Cauchy problem of (1.13)-(1.15). Following the work of [2], Li, Sun, and Zatorska in [14] study a special case of (1.10)-(1.12) with a common velocity and algebraic pressure closure. They prove the existence, uniqueness and stability of global weak solutions in dimension one with arbitrarily large initial data. Moreover, Wen, and Zhu [21] consider a two-phase system with magnetic field and show global existence and uniqueness of strong solution as well as the time decay estimates. However, the pressure is $p(\rho,n)=\rho^\gamma+n$. Li, and Sun [13] derive from a two-dimensional compressible MHD model to obtain a (1.13)-(1.15)-type system with the pressure term $p=\rho^\gamma+n^2$. They prove the global weak solution for $\gamma>1$ with the initial density away from vacuum and the initial magnetic field bounded.

The goal of this paper is to show the existence and non-uniqueness of weak solutions for the two-phase model. Using an approximation scheme similar to (1.8)-(1.9), we establish the existence of weak solution and prove that the densities are equivalent. In order to obtain $\gamma>1$, $\theta>1$, we adopt the weighted estimates of the energy and pressure developed by [18] and cut-off function developed by [15]. The restriction on the adiabatic constants $\gamma>\frac{7}{3}$, $\theta=1$ is due to the fact that we use Bogovskii operator to improve the integration of the density, see Lemma 3.3 for the details. An example of non-uniqueness is constructed, which describes two independent bubbles at vacuum. Solutions of this type are also called rest states or equilibria since $u=0$ everywhere. The above analysis inspires us to find a new scheme to approximate the constructed solution. We list it as some open problems in the final section. Throughout this paper, we denote the sense of distributions by $\mathcal{D}'$, C_0^∞ by $\mathcal{D}$ and positive constant (possibly different) by C.

2. Main result

We begin our discussion with the following definition.

DEFINITION 2.1. *We shall call (ρ,n,u) a renormalized weak solution of (1.1)-(1.5) if*

(1) $\rho\geq 0$, $n\geq 0$, $(\rho,n,u)\in L^\gamma(\Omega)\times L^\theta(\Omega)\times W_0^{1,2}(\Omega)$;

(2) Equations (1.1)-(1.3) hold in $\mathcal{D}'(\Omega)$, (1.1) and (1.2) hold in $\mathcal{D}'(\mathbb{R}^3)$, provided (ρ,n,u) is prolonged to zero in $\mathbb{R}^3/\Omega$;

(3) Equations (1.1), (1.2) are satisfied in the sense of renormalized solutions, i.e.

$$\operatorname{div}(b(f)u)+[b'(f)f-b(f)]\operatorname{div}u=0 \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{2.1}$$

for any

$$b\in C^0[0,\infty)\cap C^1(0,\infty), \tag{2.2}$$

$$\sup_{t\in(0,1)}|t^{k_1}b'(t)|<\infty, \quad \text{for} \quad \text{some} \quad k_1\in(0,1), \tag{2.3}$$

$$\sup_{t\in(1,\infty)} |t^{-k_2}b'(t)|<\infty, \quad \text{for} \quad \text{some} \quad k_2\le\frac{\gamma}{2}-1, \tag{2.4}$$

where $f=\rho,n$.

Note that the definition is quite similar to the Navier-Stokes system, we may construct an approximation scheme similar to (1.8)-(1.9), i.e.

$$\alpha(\rho-h)+\operatorname{div}(\rho u)=\epsilon\Delta\rho, \tag{2.5}$$

$$\alpha(n-k)+\operatorname{div}(nu)=\epsilon\Delta n, \tag{2.6}$$

$$\begin{aligned}\alpha(\rho+n+h+k)u+\frac{1}{2}\operatorname{div}[(\rho+n)u\otimes u]+\frac{1}{2}(\rho+n)u\cdot\nabla u\\ +\nabla\rho^\gamma+\nabla n^\theta+\delta\nabla(\rho+n)^\beta-(\rho+n)f=\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u,\end{aligned} \tag{2.7}$$

which is completed by the boundary conditions

$$\left(\epsilon\frac{\partial\rho}{\partial\nu},\ \epsilon\frac{\partial n}{\partial\nu},\ u\right)\bigg|_{\partial\Omega}=0. \tag{2.8}$$

Here $h=M/|\Omega|$, $k=N/|\Omega|$ and the additional term $\delta\nabla(\rho+n)^\beta$ is used to improve the adiabatic constant γ and θ. Further investigation shows that the uniform approximation scheme only gives a solution with equivalent densities, which leads to our main result.

THEOREM 2.1. *Let Ω be a bounded domain with C^2 boundary and $f\in L^\infty(\Omega)$. Then for $\gamma,\theta>1$ or $\gamma>\frac{7}{3}$, $\theta=1$, approximation system (2.5)-(2.8) only allows a renomalized weak solution such that*

$$n=C_0\rho \quad a.e. \quad \text{in } \Omega,$$

where $C_0=N/M$.

REMARK 2.1. We will only give the proof for the case $\gamma>\frac{7}{3}$, $\theta=1$. For $\gamma,\theta>1$, we can use the equivalence of the densities and follow the work [15], [18] to obtain the weak solution.

The problem seems to be solved. However, it is easy to construct a solution with independent densities.

EXAMPLE 2.1. *Let $u\equiv0$ and $f=\nabla\phi$, we deduce from (1.3) that*

$$\nabla\rho^\gamma+\nabla n^\theta=(\rho+n)\nabla\phi. \tag{2.9}$$

Here $\phi\in W^{1,\infty}$ be a function satisfying:

(1) There are $t,T\in\mathbb{R}$, $t<T$, such that for any $s\in(t,T)$ the level sets $\{x\in\Omega;\phi(x>s)\}=A_1^{(s)}\cup A_2^{(s)}$, where $A_i^{(s)}$ $(i=1,2)$ are nonempty disjoint domains;

(2) There always holds that $A_i^{(T)}=\emptyset$;

(3) There always holds that $A_i^{(s_2)}\subset A_i^{(s_1)}$ if $s_1,s_2\in(t,T)$, $s_1<s_2$.

Then we have the following weak solution to the problem (2.9):

$$\rho^{k_1}(x)=\left[\frac{\gamma-1}{\gamma}(\phi(x)-k_1)^+\right]^{\frac{1}{\gamma-1}} I_{A_1^{(k_1)}}, \tag{2.10}$$

$$n^{k_2}(x)=\left[\frac{\theta-1}{\theta}(\phi(x)-k_2)^+\right]^{\frac{1}{\theta-1}} I_{A_2^{(k_2)}}. \tag{2.11}$$

Here the unknown constant k_1, k_2 are determined by M, N respectively. It is easy to show that the mass functions $m_1(k)=\int_\Omega \rho^k$, $m_2(k)=\int_\Omega n^k$ are continuous decreasing functions on (t,T) with $m_1(T)=m_2(T)=0$, Thus, for any $M\in(0,m_c)$, $N\in(0,m_c)$, $m_c=\min\{m_1(t),m_2(t)\}$, we can find $k_1\in(t,T)$, $k_2\in(t,T)$ such that

$$\int_\Omega \rho^{k_1}\ dx=M,\quad \int_\Omega n^{k_2}\ dx=N. \tag{2.12}$$

The above analysis shows that the weak solutions of the steady two-phase flow are non-unique. Then we wonder what kind of system can approximate the solution with densities independent. We will return to this topic in Section 4 and give two new schemes for this problem.

3. The proof of Theorem 2.1

We will first prove the existence of the system (2.5)-(2.8) and then show equivalence of the two densities. The method for the existence theorem can also be employed in the approximation scheme (4.1)-(4.3) with a little modification. The condition of equivalence will reduce the problem into Navier-Stokes equations with a pressure $\rho^\gamma+C_0\rho+\delta\rho^\beta$. The following work is to pass to the limit $\epsilon\to 0^+$, $\alpha\to 0^+$, $\delta\to 0^+$. And we will adopt the concept of renormalized solution and the smooth property of the effective viscous flux to establish the strong convergence of the density.

3.1. The existence of the approximation system. The existence of the approximation system (2.5)-(2.8) can be proved by using the Schaefer's fixed point theorem.

THEOREM 3.1. *[5] Let X be a Banach space, and $\varGamma$ is a continuous and compact mapping $\varGamma\colon\ X\longmapsto X$, such that the set*

$$\{u\in X|u=t\varGamma u\quad \text{for}\quad \text{some}\quad 0\le t\le 1\}$$

is bounded in X. Then $\varGamma$ has a fixed point.

Given $\varepsilon>0,\alpha>0,\delta>0$, we define the space $W_0^{1,\infty}=\{g\in W^{1,\infty}(\Omega),\ g=0 \text{ on } \partial\Omega\}$ and operator

$$\varGamma:(W_0^{1,\infty})^3\to(W_0^{1,\infty})^3,$$

with $u=\varGamma U$ being a solution of the following system

$$\nu\Delta u+(\nu+\lambda)\nabla\mathrm{div}u=-F(\rho,n,U)\quad \text{in}\quad \Omega, \tag{3.1}$$

where

$$\begin{aligned}F(\rho,n,U)=&\alpha(\rho+h)U+\frac{1}{2}\mathrm{div}(\rho U\otimes U)+\frac{1}{2}\rho U\cdot\nabla U\\&+\nabla\rho^\gamma+\nabla n+\delta\nabla(\rho+n)^\beta-(\rho+n)f,\end{aligned} \tag{3.2}$$

and $\rho=\mathcal{S}_1U$, $n=\mathcal{S}_2U$ are given by the following proposition.

PROPOSITION 3.1. *[17] Let $\alpha,\varepsilon>0$, Ω be a bounded domain of class C^2. Then there exist mappings*

$$\mathcal{S}_i:(W_0^{1,\infty})^3\to W^{2,p},\quad 1<p<\infty,\quad i=1,2$$

such that $\rho=\mathcal{S}_1U$, $n=\mathcal{S}_2U$ *and for any* $\eta\in C^\infty(\Omega)$,

$$\epsilon\int_\Omega \nabla\rho\cdot\nabla\eta\ dx+\alpha\int_\Omega(\rho-h)\eta\ dx-\int_\Omega\rho U\cdot\nabla\eta\ dx=0, \tag{3.3}$$

$$\epsilon\int_\Omega \nabla n\cdot\nabla\eta\ dx+\alpha\int_\Omega(n-k)\eta\ dx-\int_\Omega nU\cdot\nabla\eta\ dx=0. \tag{3.4}$$

Moreover,

$$\int_\Omega\rho\ dx=\int_\Omega h\ dx,\quad \int_\Omega n\ dx=\int_\Omega k\ dx,\quad \rho,n\geq 0, \tag{3.5}$$

$$\|\rho\|_{W^{2,p}(\Omega)}+\|n\|_{W^{2,p}(\Omega)}\leq C(1+\|U\|_{W^{1,\infty}(\Omega)}). \tag{3.6}$$

Similar to the case in Navier-Stokes system [17], it is easy to show that Γ is a continuous and compact mapping from $(W_0^{1,\infty})^3$ to $(W_0^{1,\infty})^3$. And the approximation system can be solved by means of the Schaefer's fixed point theorem as soon as the following lemma holds.

LEMMA 3.1. *Assume that* $t\in[0,1]$ *and* $u\in(W_0^{1,\infty})^3$ *satisfies* $u=t\Gamma(u)$. *Then*

$$\|u\|_{W^{1,\infty}(\Omega)}\leq C,\quad t\in[0,1],$$

where C *is independent of* t.

Proof. Testing the momentum equation by u and integrating over Ω, we have

$$\begin{aligned}
&\alpha t\int_\Omega(\rho+n+h+k)u^2\ dx+\frac{\alpha\gamma t}{\gamma-1}\int_\Omega(\rho-h)(\rho^{\gamma-1}-h^{\gamma-1})dx\\
&+\frac{\alpha\delta\beta t}{\beta-1}\int_\Omega(\rho+n-h-k)\left((\rho+n)^{\beta-1}-(h+k)^{\beta-1}\right)dx\\
&+\beta\delta\epsilon t\int_\Omega(\rho+n)^{\beta-2}|\nabla(\rho+n)|^2\ dx+\mu\int_\Omega|\nabla u|^2\ dx+(\mu+2\lambda)\int_\Omega|\mathrm{div}u|^2\ dx\\
\leq&\int_\Omega(\rho+n)fu\ dx+\frac{\alpha\gamma}{\gamma-1}\int_\Omega(h-\rho)h^{\gamma-1}\ dx+\frac{\alpha\delta\beta}{\beta-1}\int_\Omega(h-\rho)h^{\beta-1}+\int_\Omega n\mathrm{div}u\ dx.
\end{aligned} \tag{3.7}$$

By the virtue of the imbeddings $W^{1,2}\hookrightarrow L^6$, $L^{3\beta}\hookrightarrow L^{\frac{6}{5}}$, $L^{3\beta}\hookrightarrow L^2$, one gets

$$\|\nabla u\|^2_{L^2(\Omega)}+\|\nabla(\rho+n)^{\frac{\beta}{2}}\|^2_{L^2(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k)\left(1+\|\rho+n\|^2_{L^{3\beta}(\Omega)}\right). \tag{3.8}$$

On the other hand, using the Poincaré inequality, we obtain

$$\|\rho+n\|^{\frac{\beta}{2}}_{L^\beta(\Omega)}\leq C\left(\|\nabla(\rho+n)^{\frac{\beta}{2}}\|_{L^2(\Omega)}+\|\rho+n\|^{\frac{\beta}{2}}_{L^{\frac{\beta}{2}}(\Omega)}\right). \tag{3.9}$$

Note that $\int_\Omega(\rho+n)\ dx=M+N$, by interpolation inequality and Sobolev imbedding $W^{1,2}\hookrightarrow L^6$, we arrive at

$$\|\rho+n\|_{L^{3\beta}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k), \tag{3.10}$$

which, combined with (3.8) and boundary condition, implies

$$\|u\|_{W^{1,2}(\Omega)}+\|\rho\|_{L^{3\beta}(\Omega)}+\|n\|_{L^{3\beta}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k). \tag{3.11}$$

Next, we consider the following elliptic equation

$$\epsilon\Delta\rho=\alpha(\rho-h)+\operatorname{div}(\rho u), \quad \text{in} \quad \Omega, \tag{3.12}$$

$$\nabla\rho\cdot\nu=0, \quad \text{in} \quad \partial\Omega. \tag{3.13}$$

Applying the elliptic regularity theorem, we have

$$\|\rho\|_{W^{1,p}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k)(\|\rho u\|_{L^p(\Omega)}+\|u\|_{L^p(\Omega)}), \tag{3.14}$$

where $p=6\beta/(\beta+2)$. Then by the imbedding $W^{1,2}\hookrightarrow L^6$ and (3.11), one gets

$$\|\rho\|_{W^{2,2}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k). \tag{3.15}$$

Similarly, we have

$$\|n\|_{W^{2,2}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k). \tag{3.16}$$

Taking the advantage of the bootstrapping via

$$\begin{aligned} F(\rho,n,u)=&\alpha(\rho+n+h+k)u+\frac{1}{2}\operatorname{div}[(\rho+n)u\otimes u]+\frac{1}{2}(\rho+n)u\cdot\nabla u\\ &+\gamma\rho^{\gamma-1}\nabla\rho+\nabla n+\delta\beta(\rho+n)^{\beta-1}\nabla(\rho+n)-(\rho+n)f, \end{aligned} \tag{3.17}$$

we finally obtain

$$\|\rho\|_{W^{2,q}(\Omega)}+\|n\|_{W^{2,q}(\Omega)}+\|u\|_{W^{2,q}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k), \quad 1<q<\infty. \tag{3.18}$$

□

Now, we summarize what we have proved and give the following proposition on the weak solutions of the approximation (2.5)-(2.8).

PROPOSITION 3.2. *Suppose $\beta>\max\{3,\gamma\}$, $\alpha,\epsilon,\delta>0$. Then there exists a weak solution (ρ,n,u) with the following properties:*

(1)

$$\|\rho\|_{W^{2,q}(\Omega)}+\|n\|_{W^{2,q}(\Omega)}+\|u\|_{W^{2,q}(\Omega)}\leq C(\epsilon,\alpha,\delta,\Omega,f,h,k), \quad 1<q<\infty; \tag{3.19}$$

(2)

$$\int_\Omega \rho\ dx=M, \quad \int_\Omega n\ dx=N, \quad \rho,n\geq 0. \tag{3.20}$$

3.2. The equivalence of the two densities. We observe that the two continuity equations have the same velocity, boundary condition and approximation parameters ϵ, α. Moreover, the solution of the continuity equation is nonnegative (if $h,k\geq 0$), which enables us to show the equivalence of the density.

LEMMA 3.2. *Assume that (ρ,n) is the solution stated in Proposition 3.2, then*

$$n=C_0\rho \quad \text{a.e. in } \Omega, \tag{3.21}$$

where $C_0 = k/h = N/M$.

Proof. Multiplying the Equation (2.5) by C_0 and then subtracting (2.6) from the resultant equation, we have

$$\alpha(C_0\rho - n) + \mathrm{div}[(C_0\rho - n)u] = \epsilon\Delta(C_0\rho - n). \tag{3.22}$$

By Proposition 3.1, one obtains

$$C_0\rho - n = 0 \quad \text{a.e. in } \Omega. \tag{3.23}$$

□

Now the approximation system reduces to steady Navier-Stokes equations with an additional pressure term $C_0\rho$, namely

$$\alpha(\rho - h) + \mathrm{div}(\rho u) = \epsilon\Delta\rho, \tag{3.24}$$

$$\begin{aligned}\alpha(1+C_0)(\rho+h)u + \frac{1+C_0}{2}\mathrm{div}(\rho u \otimes u) + \frac{1+C_0}{2}\rho u \cdot \nabla u \\ +\nabla\rho^\gamma + C_0\nabla\rho + \delta\nabla\rho^\beta = \mu\Delta u + (\mu+2\lambda)\nabla\mathrm{div}u + (1+C_0)\rho f,\end{aligned} \tag{3.25}$$

with the boundary conditions

$$\left(\epsilon\frac{\partial\rho}{\partial\nu},\ u\right)\bigg|_{\partial\Omega} = 0. \tag{3.26}$$

Following the work of [17], we are able to pass to the limit $\epsilon \to 0^+$, $\alpha \to 0^+$ to obtain the existence of weak solutions to the following system:

$$\mathrm{div}(\rho_\delta u_\delta) = 0, \tag{3.27}$$

$$(1+C_0)\mathrm{div}(\rho_\delta u_\delta \otimes u_\delta) + \nabla P_\delta(\rho_\delta) = \mu\Delta u_\delta + (\mu+2\lambda)\nabla\mathrm{div}u_\delta + (1+C_0)\rho_\delta f, \tag{3.28}$$

with the boundary condition

$$u_\delta\big|_{\partial\Omega} = 0, \tag{3.29}$$

where $P_\delta(\rho_\delta) = \nabla\rho_\delta^\gamma + C_0\nabla\rho_\delta + \delta\nabla\rho_\delta^\beta$. In summary, we have

Proposition 3.3. *Let $\delta \in (0,1]$. Then the approximation system (3.27)-(3.29) has at least a weak renormalized solution (ρ_δ, u_δ) satisfying the following properties:*

(i) $\rho_\delta \geq 0$, $\rho_\delta \in L^{2\beta}(\Omega)$, $u_\delta \in W_0^{1,2}(\Omega)$, $\int_\Omega \rho_\delta = M > 0$;

(ii) Equation (3.27) holds in the renormalized sense, i.e.

$$\int_\Omega \{\psi(\rho_\delta)u_\delta \cdot \nabla\xi + \xi[\rho_\delta\psi'(\rho_\delta) - \psi(\rho_\delta)]\mathrm{div}u_\delta\}\,dx = 0, \tag{3.30}$$

for any $\xi \in C^\infty(\Omega)$ and $\psi \in C^1[0,\infty)$ with

$$|\psi(t)| + |t\psi'(t)| \leq C(1+|t|^\beta), \quad t \in [0,\infty); \tag{3.31}$$

(iii) Equation (3.28) holds in the sense of $\mathcal{D}'(\Omega)$, i.e.

$$\begin{aligned}\int_\Omega [(1+C_0)\rho_\delta u_\delta \otimes u_\delta : \nabla\phi + P_\delta(\rho_\delta)\mathrm{div}\phi - \mu\nabla u_\delta : \nabla\phi \\ -(2\mu+\lambda)\mathrm{div}u_\delta\mathrm{div}\phi + (1+C_0)\rho_\delta f \cdot \phi]dx = 0,\end{aligned} \tag{3.32}$$

for any $\phi \in \mathcal{D}(\Omega)$ and in particular, for any test function $\phi \in W_0^{1,2}(\Omega)$.

(iv) Moreover, we have

$$\int_\Omega (|\nabla u_\delta|^2 + |\mathrm{div} u_\delta|^2)\ dx \le C\left(\int_\Omega \rho_\delta f\cdot u_\delta dx + \int_\Omega \rho_\delta \mathrm{div} u_\delta dx\right), \tag{3.33}$$

where C is independent of δ.

3.3. Limit passage. In this subsection, we will pass to the limit $\delta\to 0^+$ and complete the proof of Theorem 2.1. In order to obtain the limit function in $L^1(\Omega)$ for the pressure term, it still needs a better estimate for the density. By the energy inequality (3.33), we have

$$\|u_\delta\|_{W^{1,2}(\Omega)} \le C\left(1+\|\rho_\delta\|_{L^2(\Omega)}\right). \tag{3.34}$$

Then we introduce the *Bogovskii operator*, which is a linear integral operator satisfying

$$\mathcal{B}:\left\{\varphi\in L^p(\Omega): \int_\Omega \varphi = 0\right\} \to W_0^{1,p}, \tag{3.35}$$

$$\mathrm{div}\mathcal{B}[\varphi]=\varphi, \qquad \mathcal{B}[\varphi]|_{\partial\Omega}=0, \tag{3.36}$$

$$\|\mathcal{B}[\varphi]\|_{W^{1,p}(\Omega)} \le c(p)\|\varphi\|_{L^p(\Omega)} \quad \text{for} \quad \text{any} \quad 1<p<\infty. \tag{3.37}$$

By denoting

$$\phi = \mathcal{B}\left[\rho_\varepsilon^\sigma - \fint \rho_\varepsilon^\sigma\right], \qquad \fint \rho_\varepsilon^\sigma = \frac{1}{|\Omega|}\int_\Omega \rho_\varepsilon^\sigma\ dx, \tag{3.38}$$

we have $\phi\in W_0^{1,2}(\Omega)$, which enables ϕ to be a test function for the momentum Equation (3.28). It leads to the following result.

LEMMA 3.3. *Assume $\gamma > \frac{7}{3}$, (ρ_δ, u_δ) be a solution given by Proposition 3.3, then*

$$\|\rho_\delta\|_{L^{\gamma+\sigma}(\Omega)} \le C, \tag{3.39}$$

$$\|u_\delta\|_{W^{1,2}(\Omega)} \le C, \tag{3.40}$$

where C is independent of δ and $\sigma = 2\gamma - 3$.

Proof. Taking the test function ϕ in the momentum Equation (3.28), we have the following identity

$$\begin{aligned}
&\int_\Omega \rho_\delta^{\gamma+\sigma} + C_0\rho_\delta^{1+\sigma} + \delta\int_\Omega \rho_\delta^{\beta+\sigma} \\
&= \int_\Omega \rho_\delta^\gamma \fint_\Omega \rho_\delta^\sigma + \int_\Omega \rho_\delta \fint_\Omega \rho_\delta^\sigma + \delta\int_\Omega \rho_\delta^\beta \fint_\Omega \rho_\delta^\sigma \\
&\quad + \nu\int_\Omega \nabla u_\delta\cdot\nabla\mathcal{B}\left[\rho_\delta^\sigma - \fint \rho_\delta^\sigma\right] + (\nu+\lambda)\int_\Omega \mathrm{div} u_\delta\left(\rho_\delta^\sigma - \fint \rho_\delta^\sigma\right) \\
&\quad - \int_\Omega \rho_\delta u_\delta\otimes u_\delta : \nabla\mathcal{B}\left[\rho_\delta^\sigma - \fint \rho_\delta^\sigma\right] - (1+C_0)\int_\Omega \rho_\delta f\cdot\mathcal{B}\left[\rho_\delta^\sigma - \fint \rho_\delta^\sigma\right] = \sum_{i=1}^7 J_i.
\end{aligned} \tag{3.41}$$

(i) As for J_1, we have

$$|J_1| \leq C\|\rho_\delta\|^{\sigma}_{L^{\gamma+\sigma}(\Omega)}\|\rho_\delta\|^{\gamma a(\gamma)}_{L^{\gamma+\sigma}(\Omega)}, \tag{3.42}$$

where we have used Hölder's inequality and the interpolation $\|\rho_\delta\|_{L^s(\Omega)} \leq C\|\rho_\delta\|^{1-a(s)}_{L^1(\Omega)}\|\rho_\delta\|^{a(s)}_{L^{\gamma+\sigma}(\Omega)}$,

$$0 < a(s) = \frac{(\gamma+\sigma)(s-1)}{s(\gamma+\sigma-1)} < 1. \tag{3.43}$$

(ii) Analogously, one gets

$$\|J_2\| \leq CM\|\rho_\delta\|^{\sigma}_{L^{\gamma+\sigma}(\Omega)}, \tag{3.44}$$

$$\|J_3\| \leq C\delta\|\rho_\delta\|^{\sigma}_{L^{\beta+\sigma}(\Omega)}\|\rho_\delta\|^{\frac{(\beta+\sigma)(\beta-1)}{\beta+\sigma-1}}_{L^{\beta+\sigma}(\Omega)}. \tag{3.45}$$

(iii) In view of (3.34) and $L^2 \hookrightarrow L^{\gamma+\sigma}$, together with Hölder's inequality,

$$\begin{aligned} |J_4| + |J_5| &\leq C\|u_\delta\|_{W^{1,2}(\Omega)}\|\rho_\delta^{\sigma}\|_{L^2(\Omega)} \\ &\leq C\|\rho_\delta\|^{1+\sigma}_{L^{\gamma+\sigma}(\Omega)}, \end{aligned} \tag{3.46}$$

provided $\gamma+\sigma \geq 2$, $\gamma > 1$.

(iv) Similarly, by the interpolations

$$\|\rho_\delta\|_{L^{\frac{3(\gamma+\sigma)}{2\gamma-\sigma}}(\Omega)} \leq C\|\rho_\delta\|^{\frac{2\gamma-\sigma-3}{3(\gamma+\sigma-1)}}_{L^1(\Omega)}\|\rho_\delta\|^{\frac{\gamma+4\sigma}{3(\gamma+\sigma-1)}}_{L^{\gamma+\sigma}(\Omega)}, \tag{3.47}$$

$$\|\rho_\delta\|_{L^2(\Omega)} \leq C\|\rho_\delta\|^{\frac{\gamma+\sigma-2}{2(\gamma+\sigma-1)}}_{L^1(\Omega)}\|\rho_\delta\|^{\frac{\gamma+\sigma}{2(\gamma+\sigma-1)}}_{L^{\gamma+\sigma}(\Omega)}, \tag{3.48}$$

we have

$$\begin{aligned} |J_6| &\leq C\|\rho_\delta\|_{L^{\frac{3(\gamma+\sigma)}{2\gamma-\sigma}}(\Omega)}\|u_\delta^2\|_{L^3(\Omega)}\left\|\nabla\mathcal{B}\left[\rho_\delta^{\sigma} - \fint \rho_\delta^{\sigma}\right]\right\|_{L^{\frac{\gamma+\sigma}{\sigma}}(\Omega)} \\ &\leq C\|\rho_\delta\|^{\sigma+\frac{\gamma+4\sigma}{3(\gamma+\sigma-1)}}_{L^{\gamma+\sigma}(\Omega)}\left(1+\|\rho_\delta\|^{\frac{\gamma+\sigma}{2(\gamma+\sigma-1)}}_{L^{\gamma+\sigma}(\Omega)}\right)^2, \end{aligned} \tag{3.49}$$

provided $\sigma \leq 2\gamma - 3$ and $\gamma > \frac{7}{3}$.

(v) Furthermore, the imbedding $L^{\frac{\gamma+\sigma}{\sigma}}(\Omega) \hookrightarrow L^{\frac{\gamma+\sigma}{\gamma+\sigma-1}}(\Omega)$ implies

$$|J_7| \leq C\|\rho_\delta\|^{\sigma}_{L^{\gamma+\sigma}(\Omega)}\left(1+\|\rho_\delta\|_{L^{\gamma+\sigma}(\Omega)}\right). \tag{3.50}$$

$\square$

By Lemma 3.3 and the compact imbedding $W^{1,2} \hookrightarrow L^{p_1}$, $p_1 \in [1,6)$, one has the following limits:

$$\delta\rho_\delta^{\beta} \to 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{3.51}$$

$$u_\delta \to u \quad \text{weakly} \quad \text{in} \quad W_0^{1,2}(\Omega), \tag{3.52}$$

$$u_\delta \to u \quad \text{strongly} \quad \text{in} \quad L^{p_1}(\Omega), \tag{3.53}$$

$$\rho_\delta \to \rho \quad \text{weakly} \quad \text{in} \quad L^{\gamma+\sigma}(\Omega), \tag{3.54}$$

$$\rho_\delta^\gamma \to \overline{\rho^\gamma} \quad \text{weakly} \quad \text{in} \quad L^{\frac{\gamma+\sigma}{\gamma}}(\Omega), \tag{3.55}$$

$$\rho_\delta u_\delta \to \rho u \quad \text{weakly} \quad \text{in} \quad L^{\frac{6\gamma}{6+\gamma}}(\Omega), \tag{3.56}$$

$$\rho_\delta u_\delta \otimes u_\delta \to \rho u \otimes u \quad \text{weakly} \quad \text{in} \quad L^{\frac{3\gamma}{6+\gamma}}(\Omega). \tag{3.57}$$

In summary, the limit of (ρ_δ, u_δ) satisfies the system:

$$\operatorname{div}(\rho u) = 0, \tag{3.58}$$

$$(1+C_0)\operatorname{div}(\rho u \otimes u) + \nabla \overline{\rho^\gamma} + C_0 \nabla \rho = \mu \Delta u + (\mu + 2\lambda)\nabla \operatorname{div} u + (1+C_0)\rho f. \tag{3.59}$$

And what is left to prove is that

$$\overline{\rho^\gamma} = \rho^\gamma \quad \text{a.e.} \quad \text{on} \quad \Omega, \tag{3.60}$$

which needs the strong convergence of the density. Then we consider a series of cut-off functions

$$T_k(z) = kT\left(\frac{z}{k}\right) \quad \text{for} \quad z \in \mathbb{R}, \quad k = 1,2,3,\cdots, \tag{3.61}$$

where $T \in C^\infty(\mathbb{R})$ is a concave function satisfying

$$T(z) = \begin{cases} z, & z \le 1, \\ 2, & z \ge 3. \end{cases} \tag{3.62}$$

It follows that

$$\overline{T_k(\rho)} \to \rho \quad \text{in} \quad L^p(\Omega) \quad \text{for} \quad \text{any} \quad 1 \le p < \gamma, \quad \text{as} \quad k \to \infty, \tag{3.63}$$

since

$$\|\overline{T_k(\rho)} - \rho\|_{L^p(\Omega)} \le \liminf_{\delta \to 0} \|T_k(\rho_\delta) - \rho_\delta\|_{L^p(\Omega)}, \tag{3.64}$$

and

$$\|T_k(\rho_\delta) - \rho_\delta\|_{L^p(\Omega)}^p \le 2^p k^{p-\gamma} \|\rho_\delta\|_{L^\gamma(\Omega)}^\gamma \le C 2^p k^{p-\gamma}. \tag{3.65}$$

Similarly,

$$T_k(\rho) \to \rho \quad \text{in} \quad L^p(\Omega), \quad 1 \le p < \gamma \quad \text{as} \quad k \to \infty. \tag{3.66}$$

Next, by denoting the effective viscous flux

$$H_\delta := \rho_\delta^\gamma + C_0 \rho_\delta - (2\mu + \lambda)\operatorname{div} u_\delta,$$

$$\overline{H} := \overline{\rho^\gamma + C_0 \rho} - (2\mu + \lambda)\operatorname{div} u,$$

we have the following lemma on H_δ and $\overline{H}$.

Lemma 3.4. *For any $\phi \in \mathcal{D}(\Omega)$, we have*

$$\lim_{\delta \to 0^+} \int_\Omega \phi H_\delta T_k(\rho_\delta) dx = \int_\Omega \phi \overline{H}\ \overline{T_k(\rho)} dx. \tag{3.67}$$

Proof. The proof is very similar to the work of [15]. The details are omitted. ∎

Finally, let $L_k \in C^1(\mathbb{R})$ satisfy

$$L_k(z) = \begin{cases} z\log z, & 0 \le z < k; \\ z\log k + z\displaystyle\int_k^z \frac{T_k(s)}{s^2}ds, & z \ge k. \end{cases} \tag{3.68}$$

Note that

$$L_k = \beta_k z - 2k, \quad z \ge 3k, \tag{3.69}$$

$$\beta_k = \log k + \int_k^{3k} \frac{T_k(s)}{s^2}ds + \frac{2}{3}, \tag{3.70}$$

and $tL_k'(t) - L_k(t) = T_k(t)$, we use the concept of renormalized solution to obtain

$$\operatorname{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta)\operatorname{div}u_\delta = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3). \tag{3.71}$$

Integrating this identity over Ω and letting $\delta \to 0^+$, one derives

$$\int_\Omega \overline{T_k(\rho)\operatorname{div}u}\; dx = 0. \tag{3.72}$$

Similarly, we have

$$\int_\Omega T_k(\rho)\operatorname{div}u\; dx = 0. \tag{3.73}$$

As $t \longmapsto t^\gamma$ is convex, T_k is concave on $[0,\infty]$, we can make use of (3.72), (3.73) and Lemma 3.4 to obtain

$$\begin{aligned}
&\limsup_{\delta\to 0}\int_\Omega |T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1} + C_0|T_k(\rho_\delta) - T_k(\rho)|^2\; dx \\
\le &\lim_{\delta\to 0}\int_\Omega (\rho_\delta^\gamma - \rho^\gamma)(T_k(\rho_\delta) - T_k(\rho)) + C_0(\rho_\delta - \rho)(T_k(\rho_\delta) - T_k(\rho))\; dx \\
\le &\lim_{\delta\to 0}\int_\Omega (\rho_\delta^\gamma - \rho^\gamma)(T_k(\rho_\delta) - T_k(\rho)) + C_0(\rho_\delta - \rho)(T_k(\rho_\delta) - T_k(\rho))\; dx \\
&\quad + \int_\Omega (\overline{\rho^\gamma} - \rho^\gamma)(T_k(\rho) - \overline{T_k(\rho)})\; dx \\
\le &\lim_{\delta\to 0}\int_\Omega (\rho_\delta^\gamma + C_0\rho_\delta)T_k(\rho_\delta) - \overline{\rho^\gamma + C_0\rho}\;\overline{T_k(\rho)}\; dx \\
= &\int_\Omega \overline{T_k(\rho)}\operatorname{div}u - \overline{T_k(\rho)\operatorname{div}u}\; dx \\
= &\int_\Omega \left(\overline{T_k(\rho)} - T_k(\rho)\right)\operatorname{div}u\; dx \\
\le &\|\overline{T_k(\rho)} - T_k(\rho)\|_{L^2(\Omega)}\|\operatorname{div}u\|_{L^2(\Omega)},
\end{aligned} \tag{3.74}$$

which, in accordance with (3.63), (3.66) yields

$$\lim_{k\to\infty}\limsup_{\delta\to 0}\int_\Omega \left(|T_k(\rho_\delta) - T_k(\rho)|^{\gamma+1} + C_0|T_k(\rho_\delta) - T_k(\rho)|^2\right)dx = 0. \tag{3.75}$$

Notice that

$$\|\rho_\delta-\rho\|_{L^1(\Omega)}\leq\|\rho_\delta-T_k(\rho_\delta)\|_{L^1(\Omega)}+\|T_k(\rho_\delta)-T_k(\rho)\|_{L^1(\Omega)}+\|T_k(\rho)-\rho\|_{L^1(\Omega)}, \quad (3.76)$$

in accordance with (3.63), (3.66), we obtain the strong convergence of the density in $L^1(\Omega)$.

Therefore we complete the proof of Theorem 2.1.

4. Open problem

(1) Approximation scheme with different parameters
The uniform approximate parameters require the two densities behave similarly and may only fit some weak solutions. A better treatment is to use different parameters, i.e.

$$\alpha_1(\rho-h)+\operatorname{div}(\rho u)=\epsilon_1\Delta\rho, \quad (4.1)$$

$$\alpha_2(n-k)+\operatorname{div}(nu)=\epsilon_2\Delta n, \quad (4.2)$$

$$\begin{aligned}&\alpha_1(\rho+h)u+\alpha_2(n+k)u+\frac{1}{2}\operatorname{div}[(\rho+n)u\otimes u]+\frac{1}{2}(\rho+n)u\cdot\nabla u\\&+\nabla\rho^\gamma+\nabla n^\theta+\delta\nabla(\rho+n)^\beta-(\rho+n)f=\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u. \quad (4.3)\end{aligned}$$

But it will meet a great challenge when solving the pressure in two variables. More specifically, compared to the Navier-Stokes equations, it will be a question of how to solve the relationship between a uniform effective viscous flux and two independent renormalized mass equations.

(2) Equilibria hypothesis
To obtain some solutions with independent densities, we can propose an approximation system as follows:

$$\alpha(\rho-h)+\operatorname{div}(\rho u_1)=\epsilon\Delta\rho, \quad (4.4)$$

$$\alpha(\rho+h)u_1+\frac{1}{2}\operatorname{div}(\rho u_1\otimes u_1)+\frac{1}{2}\rho u_1\cdot\nabla u_1+\nabla\rho^\gamma-\rho f=qLu_1, \quad (4.5)$$

$$\alpha(n-k)+\operatorname{div}(nu_2)=\epsilon\Delta n, \quad (4.6)$$

$$\alpha(n+k)u_2+\frac{1}{2}\operatorname{div}(nu_2\otimes u_2)+\frac{1}{2}nu_2\cdot\nabla u_2+\nabla n^\theta-nf=(1-q)Lu_2, \quad (4.7)$$

where $Lu_i=\mu\Delta u_i+(\mu+2\lambda)\nabla\operatorname{div}u_i$, $i=1,2$, $q(x)\in C^1(\Omega)$, $s\leq q(x)\leq 1-s$, $s\in(0,1)$. Now the main obstacle is whether $u_1=u_2$ almost everywhere. In particularly, we have the following hypothesis: *for arbitrary $f\in L^\infty(\Omega)$, the scheme* (1.8)-(1.9) *always approximates a weak solution for Navier-Stokes equation with $u=0$ almost everywhere.*

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