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ANALYSIS OF TRANSIENT ACOUSTIC SCATTERING BY AN ELASTIC OBSTACLE*

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Abstract. Consider the scattering of an acoustic plane wave by a bounded elastic obstacle which is immersed in a homogeneous medium. This paper concerns the transient analysis of such a coupled acoustic-elastic wave propagation problem. A compressed coordinate transformation is proposed to reduce equivalently the scattering problem into an initial-boundary value problem in a bounded domain over a finite time interval. The reduced problem is shown to have a unique weak solution by using the Galerkin method. The stability estimate and a priori estimates with explicit time dependence are obtained for the weak solution. The proposed method and the reduced model problem are useful for numerical simulations.

Keywords. Time domain; acoustic wave equation; elastic wave equation; well-posedness and stability; a priori estimate.

AMS subject classifications. 78A45; 35A15.

1. Introduction

Consider the scattering of a time-domain acoustic plane wave by a bounded penetrable obstacle which is immersed in a free space occupied by a homogeneous acoustic medium. The obstacle is assumed to be made of a homogeneous and isotropic elastic medium. When the incident wave hits on the surface of the obstacle, the scattered acoustic wave will be generated in the open space. Meanwhile, an elastic wave is induced inside the obstacle. This scattering phenomenon leads to an acoustic-elastic interaction problem. The surface divides the whole space into the interior and exterior of the obstacle where the wave propagation is governed by the elastic wave equation and the acoustic wave equation, respectively. The acoustic and elastic wave equations are coupled on the surface through two continuity conditions: the kinematic interface condition and the dynamic condition. The dynamic interaction between an elastic structure and surrounding air or fluid medium is encountered in many areas of engineering and industrial design and identification [15, 17, 35, 37], such as detection of submerged objects, vibration analysis for aircraft and automobiles, and ultrasound vibro-acoustography.

The acoustic-elastic interaction problems have continuously attracted much attention by many researchers. There are a lot of available mathematical and numerical results, especially for the time-harmonic wave equations [8, 11, 16, 26, 28–30, 33, 39, 40, 43]. The time-domain problems have received considerable attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity [4, 38, 42]. Many approaches are attempted to solve numerically the time-domain problems such as coupling of boundary element and finite element with different time quadratures [12, 14, 18, 27, 34, 40]. Comparing with the time-harmonic scattering problems, the time-domain problems are less studied due to the additional challenge of the temporal dependence. The analysis can be found in [6, 19, 20, 32] for the time-domain acoustic and electromagnetic scattering problems. We refer to [21] and [22] for the

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mathematical analysis of the time-dependent elastic scattering problems in unbounded structures.

The wave scattering usually involves exterior boundary value problems such as the acoustic-elastic interaction problem which is discussed in this paper. The unbounded domain needs to be truncated into a bounded one. Therefore, appropriate boundary conditions are required on the boundaries of the truncated domains so that no artificial wave reflection occurs. Such boundary conditions are called transparent boundary conditions (TBCs) or non-reflecting boundary conditions [36]. They are the subject matter of much ongoing research [1,23–25]. The research on the perfectly matched layer (PML) technique has undergone a tremendous development since Berenger proposed a PML for solving the Maxwell equations [3, 41]. The basic idea of the PML technique is to surround the domain of interest by a layer of finite thickness fictitious material which absorbs all the waves coming from inside the computational domain. When the waves reach the outer boundary of the PML region, their values are so small that the homogeneous Dirichlet boundary conditions can be imposed. Comparing with the PML method for the time-harmonic scattering problems, the rigorous mathematical analysis is much more sophisticated for the time-domain PML method due to the challenge of the dependence of the absorbing medium on all frequencies [5, 7, 9, 10, 31].

Recently, Bao et al. have done some mathematical analysis for the time-domain acoustic-elastic interaction problem in two dimensions [2]. The problem was reformulated into an initial-boundary value problem in a bounded domain by employing a time-domain TBC. Using the Laplace transform and energy method, they showed that the reduced variational problem has a unique weak solution in the frequency domain and obtain the stability estimate for the solution in the time-domain. A priori estimates with explicit time dependence were achieved for the solution of the time-domain variational problem. In addition, the PML method was discussed and a first order symmetric hyperbolic system was considered for the truncated PML problem. It was shown that the system has a unique strong solution and the stability is also obtained for the solution.

In this paper, we carry out the mathematical analysis for the two- and three-dimensional acoustic-elastic interaction problems by using a different method. It is known that waves have finite speed of propagation in the time-domain, which differs from the infinite speed of propagation for time-harmonic waves. We make use of this fact and propose a compressed coordinate transformation to reduce the problem equivalently into an initial-boundary value problem in a bounded domain. Given any time T, we consider the problem in the time interval $(0,T]$. The method begins with constructing an annulus to surround the obstacle. The inner sphere can be chosen as close as possible to the obstacle, but the radius of the outer sphere should be chosen sufficiently large so that the scattered acoustic wave cannot reach it at time $t=T$. Hence the homogeneous Dirichlet boundary condition can be imposed on the outer sphere. Then we apply the change of variables and compress the annulus into a much smaller annulus by mapping the outer sphere into a sphere which is slightly larger than the inner sphere while keeping the inner sphere unchanged. The reduced problem can be formulated in a compact domain where the obstacle is only enclosed by a thin annulus. Based on the Galerkin method and energy estimates, we prove the existence and uniqueness of the weak solution for the corresponding variational problem. Furthermore, we obtain a priori estimates with explicit dependence on the time for the pressure of the acoustic wave and the displacement of the elastic wave. The method does not introduce any approximation or truncation error. It avoids the complicated error or convergence analysis which needs to be carefully done for the TBC or PML method. Therefore, the reduced model problem is potentially suitable for numerical simulations due to its simplicity and small computational domain.

The paper is organized as follows. In Section 2, we introduce the model equations for the acoustic-elastic interaction problem and propose the compressed coordinate transformation to reduce the problem into an initial-boundary value problem. Section 3 is devoted to the analysis of the reduced problem, where the well-posedness and stability are addressed, and a priori estimates with explicit time dependence are obtained for the time-domain variational problem. The paper is concluded with some general remarks in Section 4. To avoid distraction from the main results, we present in the Appendices the details of the change of variables for the compressed coordinate transformation.

2. Problem formulation

In this section, we introduce the problem geometry and model equations, and propose a compressed coordinate transformation to reduce the acoustic-elastic scattering problem into an initial boundary value problem in a bounded domain over a finite time interval.

2.1. Problem geometry. Consider a bounded elastic obstacle which may be described by the bounded domain $D \subset \mathbb{R}^d$ with a Lipschitz continuous boundary ∂D, where $d = 2$ or 3. We assume that D is occupied by an isotropic linearly elastic medium which is characterized by a constant mass density $\rho_2 > 0$ and Lamé parameters λ, μ satisfying $\mu > 0, \lambda + \mu > 0$. The obstacle's surface divides the whole space $\mathbb{R}^d$ into the interior domain D and the exterior domain $\mathbb{R}^d \setminus \bar{D}$. The elastic wave and the acoustic wave propagates inside D and $\mathbb{R}^d \setminus \bar{D}$, respectively. The exterior domain $\mathbb{R}^d \setminus \bar{D}$ is assumed to be connected and filled with a homogeneous, compressible, and inviscid air or fluid with a constant density $\rho_1 > 0$. It is known that the acoustic wave has a finite speed of propagation in the time-domain. Hence, for any given time $T > 0$, we may always pick a sufficiently large $R > 0$ such that the acoustic wave cannot reach the surface $\partial B_R = \{x \in \mathbb{R}^d : |x| = R\}$. Denote the ball $B_a = \{x \in \mathbb{R}^d : |x| < a\}$ with the boundary $\partial B_a = \{x \in \mathbb{R}^d : |x| < a\}$, where $a > 0$ is a constant such that $\bar{D} \subset B_a$. Usually we have $a \ll R$. Let b be an appropriate constant satisfying $a < b \ll R$. Define $B_b = \{x \in \mathbb{R}^d : |x| < b\}$ and $\partial B_b = \{x \in \mathbb{R}^d : |x| = b\}$. We shall consider a compressed coordinate transformation which compresses the annulus $\{x \in \mathbb{R}^d : a < |x| < R\}$ into the much smaller annulus $\{x \in \mathbb{R}^d : a < |x| < b\}$ by mapping ∂B_R into ∂B_b while keeping ∂B_a unchanged. Then the acoustic-elastic interaction problem will be formulated in the bounded domain B_b. The problem geometry is shown in Figure 2.1.

2.2. The model equations. Let the obstacle be illuminated by an acoustic plane wave $p^{\rm inc}(x,t) = \vartheta(ct - x \cdot d)$, where ϑ is a smooth function with a compact support and $d \in \mathbb{S}^{d-1}$ is a unit propagation direction vector. The acoustic wave field in $\mathbb{R}^d \setminus \bar{D}$ is governed by the conservation and the dynamics equations in the time-domain:

$$\nabla p(x,t) = -\rho_1 \partial_t \boldsymbol{v}(x,t), \quad c^2 \rho_1 \nabla \cdot \boldsymbol{v}(x,t) = -\partial_t p(x,t), \quad x \in \mathbb{R}^d \setminus \bar{D},\ t > 0, \tag{2.1}$$

where p is the pressure, $\boldsymbol{v}$ is the velocity, $\rho_1 > 0$ and $c > 0$ are the density and wave speed, respectively. Eliminating the velocity $\boldsymbol{v}$ from (2.1), we may easily verify that the pressure p satisfies the acoustic wave equation

$$\frac{1}{c^2} \partial_t^2 p(x,t) - \Delta p(x,t) = 0, \quad x \in \mathbb{R}^d \setminus \bar{D},\ t > 0. \tag{2.2}$$

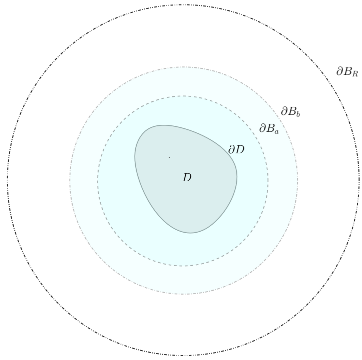

FIG. 2.1. *Problem geometry of the acoustic scattering by a bounded elastic obstacle.*

The scattered field $p^{\mathrm{sc}}=p-p^{\mathrm{inc}}$ is excited due to the interaction between the incident field and the obstacle. It follows from (2.2) and the expression of the plane incident wave p^{inc} that the scattered field p^{sc} also satisfies the acoustic wave equation

$$\frac{1}{c^2}\partial_t^2 p^{\mathrm{sc}}(x,t)-\Delta p^{\mathrm{sc}}(x,t)=0,\quad x\in\mathbb{R}^d\setminus\bar{D},\ t>0. \tag{2.3}$$

By assuming that the incident field vanishes for $t\le 0$, i.e., the system is assumed to be quiescent at the beginning, we may impose the homogeneous initial conditions for the scattered field

$$p^{\mathrm{sc}}|_{t=0}=\partial_t p^{\mathrm{sc}}|_{t=0}=0\quad \text{in } \mathbb{R}^d\setminus\bar{D}.$$

Since the acoustic wave in (2.3) has a finite speed of propagation, for any given time $T>0$, we may always pick a sufficiently large $R>0$ such that the scattered field p^{sc} cannot reach the surface ∂B_R, i.e., the homogeneous Dirichlet boundary condition can be imposed

$$p^{\mathrm{sc}}=0\quad \text{on } \partial B_R\times(0,T].$$

Recall that the domain D is occupied by a linear and isotropic elastic body. Under the hypothesis of small amplitude oscillations in the obstacle, the elastic wave satisfies the linear elasticity equation

$$\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}(x,t))-\rho_2\partial_t^2(\boldsymbol{u}(x,t))=0,\quad x\in D,\ t>0. \tag{2.4}$$

where $\boldsymbol{u}=(u_1,\ldots,u_d)^\top$ is the displacement vector, $\rho_2>0$ is the density, and the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by the generalized Hooke's law:

$$\boldsymbol{\sigma}(\boldsymbol{u})=2\mu\boldsymbol{\epsilon}(\boldsymbol{u})+\lambda\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u}))I,\quad \boldsymbol{\epsilon}(\boldsymbol{u})=\frac{1}{2}\big(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top\big). \tag{2.5}$$

Here the Lamé constants μ,λ satisfy $\mu>0,\lambda+\mu>0$, I is the identity matrix, $\boldsymbol{\epsilon}(\boldsymbol{u})$ is known as the strain tensor, and $\nabla\boldsymbol{u}$ is the displacement gradient tensor defined by

$$\nabla\boldsymbol{u}=\begin{bmatrix}\partial_{x_1}u_1 & \cdots & \partial_{x_d}u_1\\ \vdots & \ddots & \vdots\\ \partial_{x_1}u_d & \cdots & \partial_{x_d}u_d\end{bmatrix}.$$

Substituting (2.5) into (2.4), we obtain the time-domain Navier equation

$$\mu\Delta \boldsymbol{u}(x,t)+(\lambda+\mu)\nabla\nabla\cdot \boldsymbol{u}(x,t)-\rho_2\partial_t^2\boldsymbol{u}(x,t)=0,\quad x\in D,\ t>0.$$

Since the system is assumed to be quiescent, the displacement vector is constrained by the homogeneous initial conditions:

$$\boldsymbol{u}(x,t)|_{t=0}=\partial_t\boldsymbol{u}(x,t)|_{t=0}=0,\quad x\in D.$$

To describe the coupling of acoustic and elastic waves at the interface, the kinematic interface condition is imposed to ensure the continuity of the normal component of the velocity on ∂D:

$$\boldsymbol{n}_D\cdot \boldsymbol{v}(x,t)=\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u}(x,t),\quad x\in\partial D,\ t>0,$$

where $\boldsymbol{n}_D$ is the unit normal vector on ∂D pointing towards $\mathbb{R}^2\setminus\bar{D}$. Noting $-\rho_1\partial_t\boldsymbol{v}(x,t)=\nabla p(x,t)$, we have

$$\partial_{\boldsymbol{n}_D}p(x,t)=\boldsymbol{n}_D\cdot\nabla p(x,t)=-\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u}(x,t),\quad x\in\partial D,\ t>0.$$

In addition, the following dynamic interface condition is required

$$-p(x,t)\boldsymbol{n}_D=\mu\partial_{\boldsymbol{n}_D}\boldsymbol{u}(x,t)+(\lambda+\mu)(\nabla\cdot\boldsymbol{u}(x,t))\boldsymbol{n}_D,\quad x\in\partial D,\ t>0.$$

To summarize, the acoustic scattering by an elastic obstacle can be formulated as an initial boundary value problem in the bounded domain B_R over the finite time interval $(0,T]$:

$$\begin{cases}\frac{1}{c^2}\partial_t^2p-\Delta p=0, & \text{in } B_R\setminus\bar{D}\times(0,T],\\ p=p^{\mathrm{inc}}, & \text{on } \partial B_R\times(0,T],\\ p|_{t=0}=\partial_tp|_{t=0}=0, & \text{in } B_R\setminus\bar{D},\\ \mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}-\rho_2\partial_t^2\boldsymbol{u}=0, & \text{in } D\times(0,T],\\ \boldsymbol{u}|_{t=0}=\partial_t\boldsymbol{u}|_{t=0}=0, & \text{in } D,\\ \partial_{\boldsymbol{n}_D}p=-\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u} & \text{on } \partial D\times(0,T],\\ -p\boldsymbol{n}_D=\mu\partial_{\boldsymbol{n}_D}\boldsymbol{u}+(\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{n}_D, & \text{on } \partial D\times(0,T].\end{cases}\tag{2.6}$$

Now we introduce some useful notation. The scalar, vector, and matrix real-valued L^2 inner products are defined by

$$(a,b)_D:=\int_D ab\,\mathrm{d}x,\quad (\boldsymbol{a},\boldsymbol{b})_D:=\int_D\boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}x,\quad (\boldsymbol{A},\boldsymbol{B})_D:=\int_D\boldsymbol{A}:\boldsymbol{B}\,\mathrm{d}x,$$

where the colon denotes the Frobenius inner product of square matrices, i.e., $\boldsymbol{A}:\boldsymbol{B}=\mathrm{tr}(AB^\top)$. When using complex-valued functions, the complex conjugate will be used as needed. Let Ω be a bounded open domain with Lipschitz boundary $\partial\Omega$. Denote by $L^2(\Omega)$ the space of square integrable functions in Ω equipped with the norm $\|\cdot\|_{L^2(\Omega)}$. Let $H^s(\Omega), s\in\mathbb{R}$ be the standard Sobolev space equipped with the norm $\|\cdot\|_{H^s(\Omega)}$. Denote $\boldsymbol{L}^2(D)=L^2(D)^d$, $\boldsymbol{H}^1(D)=H^1(D)^d$, and $\boldsymbol{L}^2(\partial D)=L^2(\partial D)^d$, which have norms characterized by

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(D)}=\sum_{j=1}^d\|u_j\|^2_{L^2(D)},\quad \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(D)}=\sum_{j=1}^d\|u_j\|^2_{H^1(D)},\quad \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\partial D)}=\sum_{j=1}^d\|u_j\|^2_{L^2(\partial D)}.$$

The 2-norm of the gradient tensor is defined by

$$\|\nabla\boldsymbol{u}\|^2_{L^2(D)^{d\times d}}=\sum_{j=1}^d\int_D|\nabla u_j|^2\mathrm{d}x.$$

2.3. The reduced problem. In this section, we propose the compressed coordinate transformation to reduce equivalently the acoustic-elastic interaction problem (2.6) into an initial boundary value problem in a much smaller domain B_b. Although R is chosen to be large enough so that the scattered wave cannot reach ∂B_R, b does not have to be large as long as $b > a$. The width of the annulus $b-a$ can be small and the annulus $B_b \setminus \bar{B}_a$ can be put as close as possible to enclose the obstacle D, which makes it particularly attractive for the numerical simulation.

Consider the change of variables

$$\rho = \zeta(r) = \begin{cases} r, & r \in [0,a), \\ \eta(r), & r \in [a,b], \end{cases}$$

where

$$\eta(r) = \frac{\xi(r)}{(b-r)(R-b)+(b-a)^2}, \quad \xi(r) = a^2(R-b) + r(a^2 + (b-2a)R).$$

A simple calculation yields

$$\eta'(r) = \frac{(R-a)^2(b-a)^2}{((b-r)(R-b)+(b-a)^2)^2}.$$

It is clear to note that

$$\eta(a) = a, \quad \eta(b) = R, \quad \eta'(a) = 1,$$

which imply that the function $\zeta \in C^1[0,b]$ is positive and monotonically increasing, i.e., $\zeta > 0$ and $\zeta' > 0$. Hence, the transform ζ keeps the ball B_a to itself while compresses the annulus $B_R \setminus \bar{B}_a$ into the annulus $B_b \setminus \bar{B}_a$. Define $\Omega = B_b \setminus \bar{D}$ and its boundary $\partial\Omega = \partial D \cup \partial B_b$.

Let v be the transformed scattered field of p under the change of variables. It follows from the Appendices that v satisfies

$$\frac{\beta}{c^2}\partial_t^2 v - \nabla \cdot (M \nabla v) = 0 \quad \text{in } \Omega \times (0,T],$$

where the variable coefficients

$$\beta = \frac{\zeta\zeta'}{r}, \quad M = Q \begin{bmatrix} \frac{\zeta}{r\zeta'} & 0 \\ 0 & \frac{r\zeta'}{\zeta} \end{bmatrix} Q^\top, \quad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{for } d=2$$

and

$$\beta = \frac{\zeta^2}{r^2}, \quad M = Q \begin{bmatrix} \frac{\zeta^2}{r^2\zeta'} & 0 & 0 \\ 0 & \zeta' & 0 \\ 0 & 0 & \zeta' \end{bmatrix} Q^\top, \quad Q = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \quad \text{for } d=3.$$

Here (r,θ) and (r,θ,φ) are the polar and spherical coordinates in the two- and three-dimensions, respectively. It is easy to note that β is a continuous positive function, Q is an orthonormal matrix, and M is a symmetric positive definite matrix with continuous matrix entries. The details are given in the Appendices.

For the given p^{inc}, there exists a smooth lifting v_0 which has a compact support contained in $\Omega\times[0,T]$ and satisfies the boundary conditions $v_0=p^{\text{inc}}$ on ∂B_b. Hence we may equivalently consider the following initial boundary value problem

$$\begin{cases} \dfrac{\beta}{c^2}\partial_t^2 p-\nabla\cdot(M\nabla p)=f & \text{in } \Omega\times(0,T], & (2.7a)\\ p=0 & \text{on } \partial B_b\times(0,T], & (2.7b)\\ p|_{t=0}=g, \quad \partial_t p|_{t=0}=h & \text{in } \Omega, & (2.7c)\\ \mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}-\rho_2\partial_t^2\boldsymbol{u}=0 & \text{in } D\times(0,T], & (2.7d)\\ \boldsymbol{u}|_{t=0}=\partial_t\boldsymbol{u}|_{t=0}=0 & \text{in } D, & (2.7e)\\ \partial_{\boldsymbol{n}_D}p=-\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u}, & \text{on } \partial D\times(0,T], & (2.7f)\\ -p\boldsymbol{n}_D=\mu\partial_{\boldsymbol{n}_D}\boldsymbol{u}+(\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{n}_D & \text{on } \partial D\times(0,T]. & (2.7g) \end{cases}$$

where $f\in L^2(\Omega), g\in\widetilde{H}_0^1(\Omega):=\{u\in H^1(\Omega): u=0 \text{ on } \partial B_b\}, h\in L^2(\Omega)$.

3. Well-posedness

In this section, we examine the well-posedness of the reduced initial-boundary value problem (2.7) and present a priori estimates for the solution.

3.1. Existence and uniqueness. Taking the inner products in (2.7a) and (2.7d) with the test functions $q\in\widetilde{H}_0^1(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{H}^1(D)$, respectively, we arrive at the variational problem: to find $(p,\boldsymbol{u})\in\widetilde{H}_0^1(\Omega)\times\boldsymbol{H}^1(D)$ for all $t>0$ such that

$$\frac{\beta}{c^2}\left(\partial_t^2 p,q\right)_\Omega-\left(\nabla\cdot(M\nabla p),q\right)_\Omega=(f,q)_\Omega, \quad \forall q\in\widetilde{H}_0^1(\Omega),$$

$$\rho_2\left(\partial_t^2\boldsymbol{u},\boldsymbol{v}\right)_D-\left(\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u},\boldsymbol{v}\right)_D=0, \quad \forall\boldsymbol{v}\in\boldsymbol{H}^1(D).$$

Using the integration by parts and initial and boundary conditions (2.7c), (2.7e), (2.7b), and (2.7f)–(2.7g), we have

$$\left(\frac{\beta}{c^2}\partial_t^2 p,q\right)_\Omega+a_0[p,q;t]-\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u})q\mathrm{d}s=(f,q)_\Omega, \quad \forall q\in\widetilde{H}_0^1(\Omega),$$

$$\left(\rho_2\partial_t^2\boldsymbol{u},\boldsymbol{v}\right)_D+a_1[\boldsymbol{u},\boldsymbol{v};t]+\int_{\partial D}p\boldsymbol{n}_D\cdot\boldsymbol{v}\mathrm{d}s=0, \quad \forall\boldsymbol{v}\in\boldsymbol{H}^1(D),$$

where the bilinear forms

$$a_0[p,q;t]=\int_\Omega(M^{1/2}\nabla p)\cdot(M^{1/2}\nabla q)\mathrm{d}x,$$

$$a_1[\boldsymbol{u},\boldsymbol{v};t]=\mu\int_D(\nabla\boldsymbol{u}):(\nabla\boldsymbol{v})\mathrm{d}x+(\lambda+\mu)\int_D(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{v})\mathrm{d}x.$$

Suppose that $(p(x,t),\boldsymbol{u}(x,t))$ are smooth solutions of (2.7) and define the associated mappings $\mathrm{p}:[0,T]\to\widetilde{H}_0^1(\Omega)$ and $\mathbf{u}:[0,T]\to\boldsymbol{H}^1(D)$ by

$$[\mathrm{p}(t)](x):=p(x,t), \quad x\in\Omega,\ t\in[0,T],$$
$$[\mathbf{u}(t)](x):=\boldsymbol{u}(x,t), \quad x\in D,\ t\in[0,T].$$

Introduce the function $\mathrm{f}:[0,T]\to L^2(\Omega)$ by

$$[\mathrm{f}(t)](x):=f(x,t),\quad x\in\Omega,\ t\in[0,T].$$

We seek a weak solution $(\mathrm{p},\mathbf{u})$ satisfying $(\mathrm{p}'',\mathbf{u}'')\in H^{-1}(\Omega)\times \boldsymbol{H}^{-1}(D)$ for a.e. $t\in[0,T]$. Hence the inner product $(\cdot,\cdot)$ can also be interpreted as the pairing $\langle\cdot,\cdot\rangle$ which is defined between the dual spaces of H^{-1} and H^1.

DEFINITION 3.1. *We say that the function* $(\mathrm{p},\mathbf{u})\in L^2(0,T;\widetilde{H}_0^1(\Omega))\times L^2(0,T;\boldsymbol{H}^1(D))$ *with* $(\mathrm{p}',\mathbf{u}')\in L^2(0,T;L^2(\Omega))\times L^2(0,T;\boldsymbol{L}^2(D))$ *and* $(\mathrm{p}'',\mathbf{u}'')\in L^2(0,T;H^{-1}(\Omega))\times L^2(0,T;\boldsymbol{H}^{-1}(D))$ *is a weak solution of the initial boundary value problem* (2.7) *if it satisfies*

(1) $\forall \mathrm{q}\in\widetilde{H}_0^1(\Omega)$, $\mathbf{v}\in\boldsymbol{H}^1(D)$ *and a.e.* $t\in[0,T]$,

$$\begin{aligned}\Big(\frac{\beta}{c^2}\mathrm{p}'',\mathrm{q}\Big)_\Omega+\big(\rho_1\rho_2\mathbf{u}'',\mathbf{v}\big)_{\mathrm{D}}+\mathrm{a}_0[\mathrm{p},\mathrm{q};\mathrm{t}]&\\ +\rho_1\big(a_1[\mathbf{u},\mathbf{v};t]+a_2[\mathrm{p},\mathbf{v};t]+a_3[\mathbf{u},\mathrm{q};\mathrm{t}]\big)&=(\mathrm{f},\mathrm{q})_\Omega,\end{aligned}$$

where

$$\begin{aligned}a_2[\mathrm{p},\mathbf{v};t]&=\int_{\partial D}\mathrm{p}\boldsymbol{n}_D\cdot\mathbf{v}\mathrm{d}s,\\ a_3[\mathbf{u},\mathrm{q};t]&=\int_{\partial D}-(\boldsymbol{n}_D\cdot\mathbf{u}'')\mathrm{q}\mathrm{d}s.\end{aligned}$$

(2) $\mathrm{p}(0)=g,\ \mathrm{p}'(0)=h.$

We adopt the Galerkin method to construct the weak solution of the initial boundary value problem (2.7) by solving a finite dimensional approximation. We refer to [13] for the method to construct the weak solutions of the general second order parabolic and hyperbolic equations. The method begins with selecting orthogonal basis functions: select functions $w_k:=(w_k^i(x),w_k^e(x))^\top,k\in\mathbb{N}$ by requiring that the smooth functions $\{w_k^i\}_{k=1}^\infty$, $\{w_k^e\}_{k=1}^\infty$ is the standard orthogonal basis of $L^2(\Omega)$ and $L^2(\partial\Omega)$ respectively, and $\{w_k^i\}_{k=1}^\infty$ is also the orthogonal basis of $H_0^1(\Omega)$; select functions $W_k:=(W_k^i(x),W_k^e(x))^\top,k\in\mathbb{N}$ by requiring that the smooth functions $\{W_k^i\}_{k=1}^\infty$, $\{W_k^e\}_{k=1}^\infty$ is the standard orthogonal basis of $\boldsymbol{L}^2(D)$ and $\boldsymbol{L}^2(\partial D)$ respectively, and $\{W_k^i\}_{k=1}^\infty$ is also the orthogonal basis of $\boldsymbol{H}_0^1(D)$.

For positive integers s,l, $N:=s+l$, $w_k^e=0\,(k=1,\ldots,s)$, $w_k^i=0\,(k=s+1,\ldots,N)$, let

$$\mathrm{p}_N(t):=\sum_{j=1}^N p_{Nj}(t)w_j=\sum_{j=1}^s p_{Nj}^i(t)w_j^i+\sum_{j=s+1}^N p_{Nj}^e(t)w_j^e. \tag{3.1}$$

For positive integers m,n, $M:=m+n$, $W_k^i=0\,(k=1,\cdots,m)$, $W_k^e=0\,(k=m+1,\cdots,M)$, let

$$\mathbf{u}_M(t):=\sum_{j=1}^M u_{Mj}(t)W_j=\sum_{j=1}^m u_{Mj}^i(t)W_j^i+\sum_{j=m+1}^M u_{Mj}^e(t)W_j^e. \tag{3.2}$$

The coefficients $p_{Nj}(t),u_{Mj}(t)$ satisfy the initial conditions

$$p_{Nj}(0)=(g,w_j),\quad p_{Nj}^{'}(0)=(h,w_j),\quad u_{Mj}(0)=0,\quad u_{Mj}^{'}(0)=0, \tag{3.3}$$

and $\mathrm{p}_N(t), \mathbf{u}_M(t)$ satisfy the equation

$$(\mathrm{f}, w_k)_\Omega = \Big(\frac{\beta}{c^2}\mathrm{p}_N'', w_k\Big)_\Omega + \big(\rho_1\rho_2\mathbf{u}_M'', W_j\big)_D + a_0[\mathrm{p}_N, w_k; t] \\ + \rho_1\big(a_1[\mathbf{u}_M, W_j; t] + a_2[\mathrm{p}_N, W_j; t] + a_3[\mathbf{u}_M, w_k; t]\big), \tag{3.4}$$

for $k = 1, \ldots, N, j = 1, \ldots, M, t \in [0, T]$.

THEOREM 3.1. *For each $M, N \in \mathbb{N}$, there exist unique functions $\mathrm{p}_N, \mathbf{u}_M$, which are given in the form of (3.1)–(3.2) and satisfy (3.3)–(3.4).*

Proof. Since $\{w_k^i\}_{k=1}^\infty$ and $\{w_k^e\}_{k=1}^\infty$ are the orthogonal bases of $L^2(\Omega)$ and $L^2(\partial\Omega)$; $\{W_k^i\}_{k=1}^\infty$ and $\{W_k^e\}_{k=1}^\infty$ are the orthogonal bases of $\boldsymbol{L}^2(D)$ and $\boldsymbol{L}^2(\partial D)$, we have from (3.1)–(3.2) that

$$(\mathrm{p}_N''(t), w_k)_\Omega = p_{Nk}''(t), \quad k = 1, \ldots, N, \tag{3.5}$$

$$(\mathbf{u}_M''(t), W_j)_D = u_{Mj}''(t), \quad j = 1, \ldots, M. \tag{3.6}$$

It follows from (3.4) that

$$a_0[\mathrm{p}_N, w_k; t] = \sum_{j=1}^N d_k^j(t) p_{Nj}(t), \tag{3.7}$$

$$a_1[\mathbf{u}_M, W_k; t] = \sum_{j=1}^M c_k^j(t) u_{Mj}(t), \tag{3.8}$$

where $d_k^j(t) = a_0[w_j, w_k; t]$, $j, k = 1, \ldots, N$ and $c_k^j(t) = a_1[W_j, W_k; t]$, $j, k = 1, \ldots, M$. Define $D = [d_k^j]_{N\times N}$ and $C = [c_k^j]_{M\times M}$.

Recall that the matrix M is symmetric positive definite. It follows from the definition of $a_0[p, q; t]$ and $a_1[\boldsymbol{u}, \boldsymbol{v}; t]$ that there exist positive constants $C_j, j = 1, \ldots, 4$ such that

$$C_1\|p\|_{H^1(\Omega)}^2 \le |a_0[p, p; t]| \le C_2\|p\|_{H^1(\Omega)}^2, \quad \forall p \in H_0^1(\Omega)$$

and

$$C_3\|\boldsymbol{u}\|_{\boldsymbol{H}^1(D)}^2 \le |a_1[\boldsymbol{u}, \boldsymbol{u}; t]| \le C_4\|\boldsymbol{u}\|_{\boldsymbol{H}^1(D)}^2, \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(D),$$

which imply that the bilinear forms a_0 and a_1 are coercive in $H_0^1(\Omega)$ and $\boldsymbol{H}_0^1(D)$, respectively, i.e., there exists a positive constant C such that

$$a_0[p, p; t] \ge C\|p\|_{H^1(\Omega)}^2, \quad \forall p \in H_0^1(\Omega),$$

$$a_1[\boldsymbol{u}, \boldsymbol{u}; t] \ge C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(D)}^2, \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(D).$$

Similarly, we have from (3.4) that

$$a_2[\mathrm{p}_N, W_k; t] = \rho_1 \sum_{j=1}^N e_k^j(t) p_{Nj}(t), \tag{3.9}$$

$$a_3[\mathbf{u}_M, w_k; t] = \rho_1 \sum_{j=1}^M l_k^j(t) u_{Mj}'', \tag{3.10}$$

where $e_k^j(t)=(w_j\boldsymbol{n}_D,W_k)_{\partial D}$, $j=1,\dots,N,k=1,\dots,M$ and $l_k^j(t)=(\boldsymbol{n}_D\cdot W_j,w_k)_{\partial D}$, $j=1,\dots,M,k=1,\dots,N$. Define $E=[e_k^j]_{n\times l}$ and $L=[l_k^j]_{l\times n}$.

Let

$$f^k(t)=(\mathrm{f}(t),w_k)_\Omega,\quad k=1,\dots,N.$$

Substituting (3.5)–(3.10) into (3.4), we obtain a linear system of second order equations

$$A\mathbf{U}_{\overline{M}}''+B\mathbf{U}_{\overline{M}}=F, \tag{3.11}$$

subject to the initial conditions (3.3), where $\overline{M}=M+N$,

$$A=\underbrace{\left(\begin{array}{c:c:c:c} \rho_1\rho_2 I_{m\times m} & & & \\ \hdashline & \rho_1\rho_2 I_{n\times n} & & \\ \hdashline & & \frac{\beta}{c^2} I_{s\times s} & \\ \hdashline & L_{l\times n} & & \frac{\beta}{c^2} I_{l\times l} \end{array}\right)}_{M\qquad\qquad\qquad N}\begin{array}{l}\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}M\\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}N\end{array},$$

$$B=\underbrace{\left(\begin{array}{c:c:c:c} C_{m\times m} & & & \\ \hdashline & C_{n\times n} & & E_{n\times l} \\ \hdashline & & D_{s\times s} & \\ \hdashline & & & D_{l\times l} \end{array}\right)}_{M\qquad\qquad\qquad N}\begin{array}{l}\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}M\\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}N\end{array},$$

$$\mathbf{U}_{\overline{M}}=(\underbrace{u^i_{M1},\cdots,u^i_{Mm}}_{m};\underbrace{u^e_{M(m+1)},\cdots,u^e_{MM}}_{n};\underbrace{p^i_{N1},\cdots,p^i_{Ns}}_{s};\underbrace{p^e_{N(s+1)},\cdots,p^e_{NN}}_{l})^\top,$$

$$F=(\underbrace{0,\cdots,0}_{m};\underbrace{0,\cdots,0}_{n};\underbrace{f^1,\cdots,f^s}_{s};\underbrace{0,\cdots,0}_{l})^\top,$$

Since A is invertible, it follows from the standard theory of ordinary differential equations that there exists a unique C^2 function $\mathbf{U}_{\overline{M}}(t)$ consisting of p_N and $\mathbf{u}_M$ which satisfy (3.3)–(3.4) for $t\in[0,T]$. $\square$

Define two product spaces

$$\mathcal{H}^1:=\boldsymbol{H}^1(D)\times\boldsymbol{L}^2(\partial D)\times H^1(\Omega)\times L^2(\partial\Omega),$$
$$\mathcal{L}^2:=\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(\partial D)\times L^2(\Omega)\times L^2(\partial\Omega).$$

Let $U=(\mathbf{u}^i;\mathbf{u}^e;\mathrm{p}^i;\mathrm{p}^e)^\top$. The norms of U in $\mathcal{H}^1$ and $\mathcal{L}^2$ are defined by

$$\|U\|^2_{\mathcal{H}^1}=\|\mathbf{u}^i\|^2_{\boldsymbol{H}^1(D)}+\|\mathbf{u}^e\|^2_{\boldsymbol{L}^2(\partial D)}+\|\mathrm{p}^i\|^2_{H^1(\Omega)}+\|\mathrm{p}^e\|^2_{L^2(\partial\Omega)},$$

$$\|U\|_{\mathcal{L}^2}^2 = \|\mathbf{u}^i\|_{\boldsymbol{L}^2(D)}^2 + \|\mathbf{u}^e\|_{\boldsymbol{L}^2(\partial D)}^2 + \|\mathrm{p}^i\|_{L^2(\Omega)}^2 + \|\mathrm{p}^e\|_{L^2(\partial\Omega)}^2.$$

THEOREM 3.2. *There exists a positive constant C depending only on Ω, D, T, and the coefficients of the acoustic-elastic interaction problem (2.6) such that*

$$\max_{t\in[0,T]} \left(\|\mathbf{u}'_{\overline{M}}(t)\|_{\mathcal{L}^2}^2 + \|\mathbf{u}_{\overline{M}}(t)\|_{\mathcal{H}^1}^2\right) + \|\mathbf{u}''_{\overline{M}}(t)\|_{L^2(0,T;\mathcal{H}^{-1})}^2$$
$$\leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|g\|_{H^1(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2\right), \quad \overline{M} = 1,2,\ldots,$$

where $\mathbf{u}_{\overline{M}} = (\mathbf{u}_M, \mathrm{p}_N)^\top$.

Proof. It is easy to see that the lower triangular matrix A has a bounded inverse A^{-1}. We have from (3.11) that

$$\mathbf{U}''_{\overline{M}} + A^{-1}B\mathbf{U}_{\overline{M}} = A^{-1}F. \tag{3.12}$$

Taking the inner product with $\mathbf{U}'_{\overline{M}}$ on both sides of (3.12) yields

$$(\mathbf{U}''_{\overline{M}}, \mathbf{U}'_{\overline{M}}) + (A^{-1}B\mathbf{U}_{\overline{M}}, \mathbf{U}'_{\overline{M}}) = (A^{-1}F, \mathbf{U}'_{\overline{M}}) \quad \text{for a.e. } t\in[0,T]. \tag{3.13}$$

Observe that

$$(\mathbf{U}''_{\overline{M}}, \mathbf{U}'_{\overline{M}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathbf{U}'_{\overline{M}}\|^2\right). \tag{3.14}$$

Combining (3.13)–(3.14) and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{U}'_{\overline{M}}\|^2 &\leq |(A^{-1}B\mathbf{U}_{\overline{M}}, \mathbf{U}'_{\overline{M}})| + |(A^{-1}F, \mathbf{U}'_{\overline{M}})|\\ &\leq \frac{1}{2}\left(\|A^{-1}B\mathbf{U}_{\overline{M}}\|^2 + \|\mathbf{U}'_{\overline{M}}\|^2 + \|A^{-1}F\|^2 + \|\mathbf{U}'_{\overline{M}}\|^2\right)\\ &\leq \frac{1}{2}\|A^{-1}B\|^2\|\mathbf{U}_{\overline{M}}\|^2 + \|\mathbf{U}'_{\overline{M}}\|^2 + \frac{1}{2}\|A^{-1}F\|^2.\end{aligned} \tag{3.15}$$

It is clear to note that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{U}_{\overline{M}}\|^2 \leq |(\mathbf{U}_{\overline{M}}, \mathbf{U}'_{\overline{M}})| \leq \frac{1}{2}\left(\|\mathbf{U}'_{\overline{M}}\|^2 + \|\mathbf{U}_{\overline{M}}\|^2\right). \tag{3.16}$$

Using (3.15) and (3.16), we may consider the inequality

$$\alpha'(t) \leq C_1\alpha(t) + \delta(t), \quad t\in[0,T],$$

where $\alpha(t) = \|\mathbf{U}'_{\overline{M}}\|^2 + \|\mathbf{U}_{\overline{M}}\|^2$, $C_1 = \max\left\{(1+\|A^{-1}B\|_{op}^2), 3\right\}$, $\delta(t) = \|A^{-1}F\|$. Here $\|\cdot\|_{op}$ denotes the operator norm. It follows from the Grönwall inequality that

$$\alpha(t) \leq e^{C_1 t}\left(\alpha(0) + \int_0^t \delta(s)\mathrm{d}s\right), \quad t\in[0,T].$$

A simple calculation yields

$$\alpha(t) \leq e^{CT}\left(\alpha(0) + \|f\|_{L^2[0,T;L^2(\Omega)]}\right), \tag{3.17}$$

where

$$\alpha(0) = \|\mathbf{U}'_{\overline{M}}(0)\|^2 + \|\mathbf{U}_{\overline{M}}(0)\|^2 \leq \left(\|h\|_{L^2(\Omega)}^2 + \|g\|_{H^1(\Omega)}^2\right).$$

By Parseval's equality, we have

$$\|F\|^2=\sum_{k=1}^{m}|(f,w_k)|^2\le\|f\|^2_{L^2(\Omega)}$$

and

$$\|\mathbf{U}'_{\overline{M}}(0)\|^2+\|\mathbf{U}_{\overline{M}}(0)\|^2=\sum_{k=1}^{m}|(h,w_k)|^2+\sum_{k=1}^{m}|(g,w_k)|^2\le\|h\|^2_{L^2(\Omega)}+\|g\|^2_{H^1(\Omega)}.$$

In fact, we may have from straightforward calculations that

$$\begin{aligned}
&\|\mathbf{u}_{\overline{M}}(t)\|^2_{\mathcal{H}^1}\\
&=\int_D\left(\sum_{j=1}^{m}u^i_{Mj}(t)W^i_j\cdot\sum_{k=1}^{m}u^i_{Mk}(t)W^i_k\right)+\left(\sum_{j=1}^{m}u^i_{Mj}(t)\nabla W^i_j:\sum_{k=1}^{m}u^i_{Mk}(t)\nabla W^i_k\right)\mathrm{d}x\\
&+\int_\Omega\left(\sum_{j=M+1}^{M+s}p^i_{Nj}(t)w^i_j\sum_{k=M+1}^{M+s}p^i_{Nk}(t)w^i_k\right)+\left(\sum_{j=M+1}^{M+s}p^i_{Nj}(t)\nabla w^i_j\cdot\sum_{k=M+1}^{M+s}p^i_{Nk}(t)\nabla w^i_k\right)\mathrm{d}x\\
&+\sum_{j=m+1}^{M}|u^e_{Mj}(t)|^2\|W^e_j\|^2_{\mathbf{L}^2(\partial D)}+\sum_{j=M+s+1}^{\overline{M}}|p^e_{Nj}(t)|^2\|w^e_j\|^2_{L^2(\partial\Omega)}\\
&=\sum_{k=1}^{\overline{M}}|\mathbf{U}^k_{\overline{M}}|^2=\|\mathbf{U}_{\overline{M}}\|^2.
\end{aligned}\tag{3.18}$$

Similarly,

$$\|\mathbf{u}'_{\overline{M}}(t)\|^2_{\mathcal{L}^2}=\|\mathbf{U}'_{\overline{M}}\|^2.\tag{3.19}$$

Combining (3.17)–(3.19) leads to

$$\|\mathbf{u}'_{\overline{M}}\|^2_{\mathcal{L}^2}+\|\mathbf{u}_{\overline{M}}\|^2_{\mathcal{H}^1}\le C\left(\|g\|^2_{H^1(\Omega)}+\|h\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(0,T;L^2(\Omega))}\right).$$

Noting that $t\in[0,T]$ is arbitrary, it follows that

$$\max_{t\in[0,T]}\left(\|\mathbf{u}'_{\overline{M}}\|^2_{\mathcal{L}^2}+\|\mathbf{u}_{\overline{M}}\|^2_{\mathcal{H}^1}\right)\le C\left(\|g\|^2_{H^1(\Omega)}+\|h\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(0,T;L^2(\Omega))}\right).\tag{3.20}$$

For any $\mathbf{v}\in\mathcal{H}^1,\|\mathbf{v}\|_{\mathcal{H}^1}\le1$, let $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$, where $\mathbf{v}_1\in\operatorname{span}\{\widetilde{W}_1,\ldots,\widetilde{W}_{\overline{M}}\}$ and $(\mathbf{v}_2,\widetilde{W}_k)=0,k=1,\ldots,\overline{M}$, where $\widetilde{W}_k:=(W_k(x),w_k(x))^\top$, $w_k(x)=0$ $(k=1,\cdots,M)$, $W_k(x)=0$ $(k=M+1,\cdots,\overline{M})$. Note that

$$\mathbf{v}_1(t)=\sum_{j=1}^{\overline{M}}v_{\overline{M}j}(t)\widetilde{W}_j,\quad\|\mathbf{v}_1\|_{\mathcal{H}^1}\le1.$$

Let $\mathbf{V}_{\overline{M}}=(v_{\overline{M}1},\cdots v_{\overline{M}M})$. By the definition of the operator norm

$$\|\mathbf{u}''_{\overline{M}}\|_{\mathcal{H}^{-1}}=\sup_{\|\mathbf{v}\|=1}\langle\mathbf{u}''_{\overline{M}},\mathbf{v}\rangle=\sup_{\|\mathbf{v}\|=1}\langle\mathbf{u}''_{\overline{M}},\mathbf{v}_1\rangle.\tag{3.21}$$

It follows from (3.1), (3.2), and (3.12) that

$$\langle\mathbf{u}''_{\overline{M}},\mathbf{v}\rangle=(\mathbf{u}''_{\overline{M}},\mathbf{v})_{\mathcal{H}^1}=(\mathbf{u}''_{\overline{M}},\mathbf{v}_1)_{\mathcal{H}^1}$$

$$=(\mathbf{U}''_{\overline{M}},\mathbf{V}_{\overline{M}})\le C\left(\|f\|_{L^2(\Omega)}+\|\mathbf{U}_{\overline{M}}\|\right)\quad\text{for a.e. } t\in[0,T].$$

In fact, we may easily verify that

$$\begin{aligned}(\mathbf{u}''_{\overline{M}},\mathbf{v}_1)_{\mathcal{H}^1}&=\left(\sum_{l=1}^{\overline{M}}u_{\overline{M}l}(t)\widetilde{W}_l,\sum_{j=1}^{\overline{M}}v_{\overline{M}j}(t)\widetilde{W}_j\right)_{\mathcal{H}^1}=(\mathbf{U}''_{\overline{M}},\mathbf{V}_{\overline{M}})\\&\le\|\mathbf{U}''_{\overline{M}}\|\|\mathbf{V}_{\overline{M}}\|\le\|A^{-1}F-A^{-1}B\mathbf{U}_{\overline{M}}\|\le C\left(\|f\|_{L^2(\Omega)}+\|\mathbf{U}_{\overline{M}}\|\right).\end{aligned}\tag{3.22}$$

Following from (3.18), (3.21), and (3.22) gives

$$\|\mathbf{u}''_{\overline{M}}\|_{\mathcal{H}^{-1}}\le C\left(\|f\|_{L^2(\Omega)}+\|\mathbf{u}_{\overline{M}}\|_{\mathcal{H}^1}\right).$$

Hence,

$$\begin{aligned}\int_0^T\|\mathbf{u}''_{\overline{M}}\|^2_{\mathcal{H}^{-1}}\mathrm{d}t&\le C\int_0^T\left(\|f\|_{L^2(\Omega)}+\|\mathbf{u}_{\overline{M}}\|_{\mathcal{H}^1}\right)^2\mathrm{d}t\\&\le C\left(\|g\|^2_{H^1(\Omega)}+\|h\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned}\tag{3.23}$$

The proof is completed after combining (3.20) and (3.23). ∎

Now we pass to the limits in the Galerkin approximations to obtain the existence of a weak solution.

THEOREM 3.3. *There exists a weak solution of the initial boundary value problem* (2.7).

Proof. It follows from the energy estimate in Theorem 3.2 that

$$\begin{aligned}&\{\mathbf{u}_{\overline{M}}\}_{M,N=1}^{\infty}\text{ is bounded in } L^2(0,T;\mathcal{H}^1),\\&\{\mathbf{u}'_{\overline{M}}\}_{M,N=1}^{\infty}\text{ is bounded in } L^2(0,T;\mathcal{L}^2),\\&\{\mathbf{u}''_{\overline{M}}\}_{M,N=1}^{\infty}\text{ is bounded in } L^2(0,T;\mathcal{H}^{-1}).\end{aligned}$$

Therefore, there exists a subsequence still denoted as $\{\mathbf{u}_{\overline{M}}\}_{M,N=1}^{\infty}$ and $\mathbf{u}\in L^2(0,T;\mathcal{H}^1)$ with $\mathbf{u}'\in L^2(0,T;\mathcal{L}^2)$ and $\mathbf{u}''\in L^2(0,T;\mathcal{H}^{-1})$ such that

$$\begin{cases}\mathbf{u}_{\overline{M}}\rightharpoonup\mathbf{u} & \text{weakly in } L^2(0,T;\mathcal{H}^1),\\ \mathbf{u}'_{\overline{M}}\rightharpoonup\mathbf{u}' & \text{weakly in } L^2(0,T;\mathcal{L}^2),\\ \mathbf{u}''_{\overline{M}}\rightharpoonup\mathbf{u}'' & \text{weakly in } L^2(0,T;\mathcal{H}^{-1}),\end{cases}\tag{3.24}$$

which imply

$$\begin{cases}(\mathbf{u}_M,\mathrm{p}_N)^\top\rightharpoonup(\mathbf{u},\mathrm{p})^\top,\\ (\mathbf{u}'_M,\mathrm{p}'_N)^\top\rightharpoonup(\mathbf{u}',\mathrm{p}')^\top,\\ (\mathbf{u}''_M,\mathrm{p}''_N)^\top\rightharpoonup(\mathbf{u}'',\mathrm{p}'')^\top.\end{cases}$$

Next we fix integers N_1,N_2 and choose functions $\mathrm{q}\in C^1([0,T];H_0^1(\Omega)\times L^2(\partial\Omega))$ and $\mathbf{v}\in C^1([0,T];\boldsymbol{H}_0^1(D)\times\boldsymbol{L}^2(\partial D))$ of the form

$$\mathrm{q}(t)=\sum_{k=1}^{N_1}q_{N_1k}(t)w_k,\quad \mathbf{v}(t)=\sum_{j=1}^{N_2}v_{N_2j}(t)W_j,\tag{3.25}$$

where $q_{N_1 k}, v_{N_2 j}, k=1,\dots,N_1, j=1,\dots,N_2$ are smooth functions. Letting $m\geq \max\{N_1,N_2\}$ where $m=\min\{M,N\}$, we have from (3.4) that

$$\int_0^T\Big(\frac{\beta^2}{c}\langle \mathrm{p}_m^{''},\mathrm{q}\rangle+\rho_1\rho_2\langle \mathbf{u}_m^{''},\mathbf{v}\rangle+a_0[\mathrm{p}_m,\mathrm{q};t]+\rho_1 a_1[\mathbf{u}_m,\mathbf{v};t]$$
$$+\rho_1\int_{\partial D}(\mathrm{p}_m n_D\cdot\mathbf{v}-(n_D\cdot\mathbf{u}_m^{''})\mathrm{q})\mathrm{d}s\Big)\mathrm{d}t=\int_0^T(\mathrm{f},\mathrm{q})\mathrm{d}t. \tag{3.26}$$

Using (3.24) and taking the limits $m\to\infty$ in (3.26) yields

$$\int_0^T\Big(\frac{\beta^2}{c}\langle \mathrm{p}^{''},\mathrm{q}\rangle+\rho_1\rho_2\langle \mathbf{u}^{''},\mathbf{v}\rangle+a_0[\mathrm{p},\mathrm{q};t]+\rho_1 a_1[\mathbf{u},\mathbf{v};t]$$
$$+\rho_1\int_{\partial D}(\mathrm{p} n_D\cdot\mathbf{v}-(n_D\cdot\mathbf{u}^{''})\mathrm{q})\mathrm{d}s\Big)\mathrm{d}t=\int_0^T(\mathrm{f},\mathrm{q})\mathrm{d}t, \tag{3.27}$$

which holds for any function $\mathrm{q}\in L^2([0,T];\widetilde{H}_0^1(\Omega))$ and $\mathbf{v}\in L^2([0,T];\boldsymbol{H}^1(D))$ since functions of the form (3.25) are dense in the space. Moreover, we have from (3.27) that for any $\widetilde{\mathrm{q}}\in H_0^1(\Omega),\widetilde{\mathbf{v}}\in\boldsymbol{H}^1(D)$ and $t\in[0,T]$

$$\frac{\beta^2}{c}\langle \mathrm{p}^{''},\widetilde{\mathrm{q}}\rangle+\rho_1\rho_2\langle \mathbf{u}^{''},\widetilde{\mathbf{v}}\rangle+a_0[\mathrm{p},\widetilde{\mathrm{q}};t]+\rho_1 a_1[\mathbf{u},\widetilde{\mathbf{v}};t]$$
$$+\rho_1\int_{\partial D}(\mathrm{p} n_D\cdot\widetilde{\mathbf{v}}-(n_D\cdot\mathbf{u}^{''})\widetilde{\mathrm{q}})\mathrm{d}s=(\mathrm{f},\widetilde{\mathrm{q}})$$

and

$$\mathrm{p}\in C(0,T;L^2(\Omega)),\quad \mathrm{p}^{'}\in C(0,T;H^{-1}(\Omega)),$$
$$\mathbf{u}\in C(0,T;\boldsymbol{L}^2(D)),\quad \mathbf{u}^{'}\in C(0,T;\boldsymbol{H}^{-1}(D)).$$

Next is to verify

$$\mathrm{p}|_{t=0}=g,\quad \mathrm{p}'|_{t=0}=h. \tag{3.28}$$

Choose any function $\mathrm{q}\in C^2([0,T];\widetilde{H}_0^1(\Omega))$ with $\mathrm{q}(T)=\mathrm{q}^{'}(T)=0$ and $\mathbf{v}\in C^2([0,T];\boldsymbol{H}^1(D))$ with $\mathbf{v}(T)=\mathbf{v}^{'}(T)=0=\mathbf{v}(0)=\mathbf{v}^{'}(0)$. Using the integration by parts twice with respect to t in (3.27) gives

$$\int_0^T\Big(\frac{\beta^2}{c}(\mathrm{q}^{''},\mathrm{p})+\rho_1\rho_2(\mathbf{v}^{''},\mathbf{u})+a_0[\mathrm{p},\mathrm{q};t]+\rho_1 a_1[\mathbf{u},\mathbf{v};t]$$
$$+\rho_1\int_{\partial D}(\mathrm{p} n_D\cdot\mathbf{v}-(n_D\cdot\mathbf{u})\mathrm{q}^{''})\mathrm{d}s\Big)\mathrm{d}t=\int_0^T(\mathrm{f},\mathrm{q})\mathrm{d}t-(\mathrm{p}(0),\mathrm{q}^{'}(0))+\langle \mathrm{p}^{'}(0),\mathrm{q}(0)\rangle. \tag{3.29}$$

Similarly, we have from (3.26) that

$$\int_0^T\Big(\frac{\beta^2}{c}(\mathrm{q}^{''},\mathrm{p}_m)+\rho_1\rho_2(\mathbf{v}^{''},\mathbf{u}_m)+a_0[\mathrm{p}_m,\mathrm{q};t]+\rho_1 a_1[\mathbf{u}_m,\mathbf{v};t]$$
$$+\rho_1\int_{\partial D}(\mathrm{p}_m n_D\cdot\mathbf{v}-(n_D\cdot\mathbf{u}_m)\mathrm{q}^{''})\mathrm{d}s\Big)\mathrm{d}t$$
$$=\int_0^T(\mathrm{f},\mathrm{q})\mathrm{d}t-(\mathrm{p}_m(0),\mathrm{q}^{'}(0))+\langle \mathrm{p}_m^{'}(0),\mathrm{q}(0)\rangle. \tag{3.30}$$

Taking the limits $m\to\infty$ in (3.30), using (3.3) and (3.24), we get

$$\begin{aligned}&\int_0^T \Big(\frac{\beta^2}{c}(\mathrm{q}^{''},\mathrm{p})+\rho_1\rho_2(\mathbf{v}^{''},\mathbf{u})+a_0[\mathrm{p},\mathrm{q};t]+\rho_1 a_1[\mathbf{u},\mathbf{v};t]\\&\qquad+\rho_1\int_{\partial D}(\mathrm{p}n_D\cdot\mathbf{v}-(n_D\cdot\mathbf{u})\mathrm{q}^{''})\mathrm{d}s\Big)\mathrm{d}t\\&=\int_0^T(\mathrm{f},\mathrm{q})\mathrm{d}t-(g,\mathrm{q}^{'}(0))+\langle h,\mathrm{q}(0)\rangle.\end{aligned}\tag{3.31}$$

Comparing (3.29) and (3.31), we conclude (3.28) since $\mathrm{q}(0)$ and $\mathrm{q}^{'}(0)$ are arbitrary. Hence $(\mathrm{p},\mathbf{u})$ is a weak solution of the initial boundary value problem (2.7). ◻

Taking the partial derivatives of (2.7d), (2.7e), and the second term of (2.7f) with respect to t, we consider

$$\begin{cases}\frac{\beta}{c^2}\partial_t^2 p-\nabla\cdot(M\nabla p)=f & \text{in } \Omega\times(0,T], \quad (3.32a)\\ p=0 & \text{on } \partial B_b\times(0,T], (3.32b)\\ p=g,\quad \partial_t p=h & \text{in } \Omega\times\{t=0\} \quad (3.32c)\\ \mu\Delta(\partial_t\boldsymbol{u})+(\lambda+\mu)\nabla\nabla\cdot(\partial_t\boldsymbol{u})-\rho_2\partial_t^2(\partial_t\boldsymbol{u})=0 & \text{in } D\times(0,T], \quad (3.32d)\\ (\partial_t\boldsymbol{u})=0,\quad \partial_t^2\boldsymbol{u}=\rho_2^{-1}(\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u})|_{t=0}=0 & \text{in } D\times\{t=0\}, \quad (3.32e)\\ \partial_{\boldsymbol{n}_D}p=-\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u} & \text{in } \partial D\times(0,T], \quad (3.32f)\\ -(\partial_t p)\boldsymbol{n}_D=\mu\partial_{\boldsymbol{n}_D}(\partial_t\boldsymbol{u})+(\lambda+\mu)(\nabla\cdot(\partial_t\boldsymbol{u}))\boldsymbol{n}_D & \text{in } \partial D\times(0,T]. \quad (3.32g)\end{cases}$$

THEOREM 3.4. *The initial boundary value problem (3.32) has a unique weak solution.*

Proof. It suffices to show that $p=0,\boldsymbol{u}=0$ if $f=g=h=0$. Fix $0\le t\le T$ and let

$$E(t):=E_1(t)+E_2(t),$$

where

$$\begin{aligned}E_1(t)&=\|\frac{\sqrt{\beta}}{c}\partial_t p\|^2_{L^2(\Omega)}+\|M^{\frac12}\nabla p\|^2_{L^2(\Omega)},\\ E_2(t)&=\|\sqrt{\rho_1\rho_2}\ \partial_t^2\boldsymbol{u}\|^2_{\boldsymbol{L}^2(D)}+\|\sqrt{\rho_1(\lambda+\mu)}\ \nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(D)}+\|\sqrt{\rho_1\mu}\ \nabla(\partial_t\boldsymbol{u})\|^2_{L^2(D)^{d\times d}}.\end{aligned}$$

Then for each $t\in[0,T]$, we have

$$E(t)-E(0)=\int_0^t E^{'}(\tau)\mathrm{d}\tau=\int_0^t E_1^{'}(\tau)\mathrm{d}\tau+\int_0^t E_2^{'}(\tau)\mathrm{d}\tau.\tag{3.33}$$

Following from (3.32) and the integration by parts, we obtain

$$\begin{aligned}\int_0^t E_1^{'}(\tau)\mathrm{d}\tau&=2\int_0^t\int_\Omega\Big(\frac{\beta}{c^2}(\partial_\tau^2 p)(\partial_\tau p)+(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)\Big)\mathrm{d}x\mathrm{d}\tau\\&=2\int_0^t\int_\Omega\Big((\partial_\tau p)(\nabla\cdot(M\nabla p))+(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)+(\partial_\tau p)f\Big)\mathrm{d}x\mathrm{d}\tau\\&=2\int_0^t\int_\Omega\Big(-(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)+(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)+(\partial_\tau p)f\Big)\mathrm{d}x\mathrm{d}\tau\\&\quad+2\int_0^t\int_{\partial B_b}0\mathrm{d}s\mathrm{d}\tau-2\int_0^t\int_{\partial D}(\partial_{\boldsymbol{n}_D}p)(\partial_\tau p)\mathrm{d}s\mathrm{d}\tau\end{aligned}$$

$$= 2\int_0^t \rho_1 \int_{\partial D} (\boldsymbol{n}_D \cdot \partial_\tau^2 \boldsymbol{u})(\partial_\tau p)\mathrm{d}s\mathrm{d}\tau \tag{3.34}$$

and

$$\begin{aligned}
\int_0^t E_2'(\tau)\mathrm{d}\tau &= 2\int_0^t \int_D \left(\rho_1\rho_2(\partial_\tau^3 \boldsymbol{u})\cdot(\partial_\tau^2\boldsymbol{u}) + \rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})]\right)\mathrm{d}x\mathrm{d}\tau \\
&\quad + 2\int_0^t \int_D \left(\rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})] : [\nabla(\partial_\tau\boldsymbol{u})]\right)\mathrm{d}x\mathrm{d}\tau \\
&= 2\int_0^t \int_D \left(\rho_1\mu(\Delta(\partial_\tau\boldsymbol{u}))\cdot(\partial_\tau^2\boldsymbol{u}) + \rho_1(\lambda+\mu)(\nabla\nabla(\partial_\tau\boldsymbol{u}))\cdot(\partial_\tau^2\boldsymbol{u})\right)\mathrm{d}x\mathrm{d}\tau \\
&\quad + 2\int_0^t \int_D \left(\rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})] + \rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})] : [\nabla(\partial_\tau\boldsymbol{u})]\right)\mathrm{d}x\mathrm{d}\tau \\
&= 2\int_0^t \int_D \left(-\rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})] : [\nabla(\partial_\tau\boldsymbol{u})] - \rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})]\right)\mathrm{d}x\mathrm{d}\tau \\
&\quad + 2\int_0^t \int_D \left(\rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})] + \rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})] : [\nabla(\partial_\tau\boldsymbol{u})]\right)\mathrm{d}x\mathrm{d}\tau \\
&\quad + 2\int_0^t \int_{\partial D} \rho_1\left(\mu\partial_{\boldsymbol{n}_D}(\partial_\tau\boldsymbol{u})\cdot(\partial_\tau^2\boldsymbol{u}) + (\lambda+\mu)(\nabla\cdot(\partial_\tau\boldsymbol{u})\boldsymbol{n}_D)\cdot(\partial_\tau^2\boldsymbol{u})\right)\mathrm{d}x\mathrm{d}\tau \\
&= -2\int_0^t \int_{\partial D} \rho_1(\partial_\tau p)(\boldsymbol{n}_D\cdot\partial_\tau^2\boldsymbol{u})\mathrm{d}s\mathrm{d}\tau. \qquad (3.35)
\end{aligned}$$

It is easy to note that if $f = g = h = 0$, we have

$$E(0) = 0.$$

Thus, combining (3.33)–(3.35), we obtain

$$E(t) = E_1(t) + E_2(t) = 0,$$

which implies that

$$\partial_t p = \nabla p = \partial_t^2 \boldsymbol{u} = \nabla\cdot(\partial_t\boldsymbol{u}) = \nabla(\partial_t\boldsymbol{u}) = 0.$$

Thus we obtain from initial conditions in (3.32) that $p = 0, \boldsymbol{u} = 0$ if $f = g = h = 0$, which completes the proof. ∎

3.2. Stability. In this section we discuss the stability estimate for the unique weak solution of the initial boundary value problem (3.32).

THEOREM 3.5. *Let $(p, \boldsymbol{u})$ be the unique weak solution of the initial boundary value problem (3.32). Given $f \in L^2(\Omega), g \in \widetilde{H}_0^1(\Omega), h \in L^2(\Omega)$, there exists a positive constant C such that*

$$\begin{aligned}
&\max_{t\in[0,T]} \Big\{ \|\partial_t p(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla p(\cdot,t)\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\qquad + \|\partial_t^2\boldsymbol{u}(\cdot,t)\|^2_{\boldsymbol{L}^2(D)} + \|\nabla\cdot(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(D)} + \|\nabla(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(D)^{d\times d}} \Big\} \\
&\le C\left(\|f\|^2_{L^1(0,T;L^2(\Omega))} + \|g\|^2_{H^1(\Omega)} + \|h\|^2_{L^2(\Omega)} \right).
\end{aligned}$$

Proof. It follows from the discussion in the previous section that the initial boundary value problem (2.7) has a unique weak solution $(p,\boldsymbol{u})$ satisfying

$$p\in L^2(0,T;\widetilde{H}_0^1(\Omega))\cap H^1(0,T;L^2(\Omega)),$$
$$\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^1(D))\cap H^1(0,T;\boldsymbol{L}^2(D)).$$

For any $t\in[0,T]$, consider the energy function

$$E(t):=E_1(t)+E_2(t),$$

where

$$E_1(t)=\|\frac{\sqrt{\beta}}{c}\partial_t p\|^2_{L^2(\Omega)}+\|M^{\frac12}\nabla p\|^2_{L^2(\Omega)},$$
$$E_2(t)=\|\sqrt{\rho_1\rho_2}\ \partial_t^2\boldsymbol{u}\|^2_{\boldsymbol{L}^2(D)}+\|\sqrt{\rho_1(\lambda+\mu)}\ \nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(D)}+\|\sqrt{\rho_1\mu}\ \nabla(\partial_t\boldsymbol{u})\|^2_{L^2(D)^{d\times d}}.$$

Then for each $t\in[0,T]$, we have

$$E(t)-E(0)=\int_0^t E'(\tau)\mathrm{d}\tau=\int_0^t E_1'(\tau)\mathrm{d}\tau+\int_0^t E_2'(\tau)\mathrm{d}\tau. \tag{3.36}$$

By (3.32) and the integration by parts, we obtain

$$\begin{aligned}\int_0^t E_1'(\tau)\mathrm{d}\tau&=2\int_0^t\int_\Omega\Big(\frac{\beta}{c^2}(\partial_\tau^2 p)(\partial_\tau p)+(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)\Big)\mathrm{d}x\mathrm{d}\tau\\
&=2\int_0^t\int_\Omega\Big((\partial_\tau p)(\nabla\cdot(M\nabla p))+(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)+(\partial_\tau p)f\Big)\mathrm{d}x\mathrm{d}\tau\\
&=2\int_0^t\int_\Omega\Big(-(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)+(M^{\frac12}\nabla(\partial_\tau p))\cdot(M^{\frac12}\nabla p)+(\partial_\tau p)f\Big)\mathrm{d}x\mathrm{d}\tau\\
&\quad+2\int_0^t\int_{\partial B_b}0\mathrm{d}s\mathrm{d}\tau-2\int_0^t\int_{\partial D}(\partial_{\boldsymbol{n}_D}p)(\partial_\tau p)\mathrm{d}s\mathrm{d}\tau\\
&=2\int_0^t\rho_1\int_{\partial D}(\boldsymbol{n}_D\cdot\partial_\tau^2\boldsymbol{u})(\partial_\tau p)\mathrm{d}s\mathrm{d}\tau+2\int_0^t\int_\Omega(\partial_\tau p)f\mathrm{d}x\mathrm{d}\tau,\end{aligned}\tag{3.37}$$

and

$$\begin{aligned}\int_0^t E_2'(\tau)\mathrm{d}\tau&=2\int_0^t\int_D\big(\rho_1\rho_2(\partial_\tau^3\boldsymbol{u})\cdot(\partial_\tau^2\boldsymbol{u})+\rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})]\big)\mathrm{d}x\mathrm{d}\tau\\
&\quad+2\int_0^t\int_D\big(\rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})]:[\nabla(\partial_\tau\boldsymbol{u})]\big)\mathrm{d}x\mathrm{d}\tau\\
&=2\int_0^t\int_D\big(\rho_1\mu(\Delta(\partial_\tau\boldsymbol{u}))\cdot(\partial_\tau^2\boldsymbol{u})+\rho_1(\lambda+\mu)(\nabla\nabla(\partial_\tau\boldsymbol{u}))\cdot(\partial_\tau^2\boldsymbol{u})\big)\mathrm{d}x\mathrm{d}\tau\\
&\quad+2\int_0^t\int_D\big(\rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})]+\rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})]:[\nabla(\partial_\tau\boldsymbol{u})]\big)\mathrm{d}x\mathrm{d}\tau\\
&=2\int_0^t\int_D\big(-\rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})]:[\nabla(\partial_\tau\boldsymbol{u})]-\rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})]\big)\mathrm{d}x\mathrm{d}\tau\\
&\quad+2\int_0^t\int_D\big(\rho_1(\lambda+\mu)[\nabla\cdot(\partial_\tau^2\boldsymbol{u})][\nabla\cdot(\partial_\tau\boldsymbol{u})]+\rho_1\mu[\nabla(\partial_\tau^2\boldsymbol{u})]:[\nabla(\partial_\tau\boldsymbol{u})]\big)\mathrm{d}x\mathrm{d}\tau\\
&\quad+2\int_0^t\int_{\partial D}\rho_1\big(\mu\partial_{\boldsymbol{n}_D}(\partial_\tau\boldsymbol{u})\cdot\partial_\tau^2\boldsymbol{u}+(\lambda+\mu)(\nabla\cdot(\partial_\tau\boldsymbol{u})\boldsymbol{n}_D)\cdot(\partial_\tau^2\boldsymbol{u})\big)\mathrm{d}x\mathrm{d}\tau\end{aligned}$$

$$
= -2\int_0^t \int_{\partial D} \rho_1 (\partial_\tau p)(\boldsymbol{n}_D \cdot \partial_\tau^2 \boldsymbol{u}) \mathrm{d}s\mathrm{d}\tau. \tag{3.38}
$$

It is easy to note that

$$
\begin{aligned}
E(0) &= \| \frac{\sqrt{\beta}}{c} \partial_t p|_{t=0} \|^2_{L^2(\Omega)} + \| M^{\frac{1}{2}} \nabla p|_{t=0} \|^2_{\boldsymbol{L}^2(\Omega)} \\
&= \| \frac{\sqrt{\beta}}{c} h \|^2_{L^2(\Omega)} + \| M^{\frac{1}{2}} \nabla g \|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}
$$

Combining (3.36)–(3.38) leads to

$$
\begin{aligned}
& \| \frac{\sqrt{\beta}}{c} \partial_t p \|^2_{L^2(\Omega)} + \| M^{\frac{1}{2}} \nabla p \|^2_{\boldsymbol{L}^2(\Omega)}, \\
& \quad + \| \sqrt{\rho_1 \rho_2}\ \partial_t^2 \boldsymbol{u} \|^2_{\boldsymbol{L}^2(D)} + \| \sqrt{\rho_1(\lambda+\mu)}\ \nabla \cdot (\partial_t \boldsymbol{u}) \|^2_{L^2(D)} + \| \sqrt{\rho_1 \mu}\ \nabla (\partial_t \boldsymbol{u}) \|^2_{L^2(D)^{d\times d}} \\
=& 2\int_0^t \int_\Omega (\partial_\tau p) f \mathrm{d}x\mathrm{d}\tau + \| \frac{\sqrt{\beta}}{c} h \|^2_{L^2(\Omega)} + \| M^{\frac{1}{2}} \nabla g \|^2_{\boldsymbol{L}^2(\Omega)} \\
\le& 2 \max_{t\in[0,T]} \{ \| \partial_t p(\cdot,t) \|_{L^2(\Omega)} \} \| f \|_{L^1(0,T;L^2(\Omega))} + \frac{\beta}{c^2} \| h \|^2_{L^2(\Omega)} + \| M^{\frac{1}{2}} \nabla g \|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}
$$

Using the Young inequality, we obtain

$$
\begin{aligned}
& \max_{t\in[0,T]} \{ \| \partial_t p(\cdot,t) \|^2_{L^2(\Omega)} + \| \nabla p(\cdot,t) \|^2_{\boldsymbol{L}^2(\Omega)} \\
& \qquad + \| \partial_t^2 \boldsymbol{u}(\cdot,t) \|^2_{\boldsymbol{L}^2(D)} + \| \nabla \cdot (\partial_t \boldsymbol{u}(\cdot,t)) \|^2_{L^2(D)} + \| \nabla (\partial_t \boldsymbol{u}(\cdot,t)) \|^2_{L^2(D)^{d\times d}} \} \\
\le& C \left(\| f \|^2_{L^1(0,T;L^2(\Omega))} + \frac{\beta}{c^2} \| h \|^2_{L^2(\Omega)} + \| M^{\frac{1}{2}} \nabla g \|^2_{\boldsymbol{L}^2(\Omega)} \right) \\
\le& C \left(\| f \|^2_{L^1(0,T;L^2(\Omega))} + \| g \|^2_{H^1(\Omega)} + \| h \|^2_{L^2(\Omega)} \right).
\end{aligned}
$$

which completes the proof. ∎

3.3. A priori estimates. In this section we derive an a priori stability estimate for the wave field with an explicit dependence on the time.

The variational problem of (3.32) is to find $(p,\boldsymbol{u}) \in \widetilde{H}_0^1(\Omega) \times \boldsymbol{H}^1(D)$ for $t \in [0,T]$ such that

$$
\begin{aligned}
\int_\Omega \frac{\beta}{c^2} (\partial_t^2 p) q \mathrm{d}x =& -\int_\Omega (M^{\frac{1}{2}} \nabla p) \cdot (M^{\frac{1}{2}} \nabla q) \mathrm{d}x + \int_{\partial D} \rho_1 (\boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u}) q \mathrm{d}s \\
& + \int_\Omega f q \mathrm{d}x, \quad \forall q \in \widetilde{H}_0^1(\Omega),
\end{aligned} \tag{3.39}
$$

and

$$
\begin{aligned}
\int_D \rho_2 \partial_t^2 (\partial_t \boldsymbol{u}) \cdot \boldsymbol{v} \mathrm{d}x =& -\int_D [(\mu \nabla \partial_t \boldsymbol{u}) : (\nabla \boldsymbol{v}) + (\lambda+\mu)((\nabla \cdot \partial_t \boldsymbol{u})(\nabla \cdot \boldsymbol{v}))] \mathrm{d}x \\
& - \int_{\partial D} (\partial_t \rho)(\boldsymbol{n}_D \cdot \boldsymbol{v}) \mathrm{d}s, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(D).
\end{aligned} \tag{3.40}
$$

THEOREM 3.6. *Let $\boldsymbol{u}$ be the unique weak solution of the initial boundary value problem (2.7). Given $f \in L^1[0,T;L^2(\Omega)]$, $g,h \in L^2(\Omega)$, there exist positive constants C_1, C_2 such that*

$$
\| p \|^2_{L^\infty(0,T;L^2(\Omega))} + \| \nabla p \|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}
$$

$$
\begin{aligned}
&+\|\partial_t \boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(D))}+\|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D)^{d\times d})}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D))}\\
\le& C_1\left(\|g\|^2_{L^2(\Omega)}+T^2\|f\|^2_{L^1(0,T;L^2(\Omega))}+T^2\|h\|^2_{L^2(\Omega)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|p\|^2_{L^2(0,T;L^2(\Omega))}+\|\nabla p\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\
&\quad+\|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{L}^2(D))}+\|\nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^{d\times d})}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^2(0,T;L^2(D))}\\
\le& C_2\left(T\|g\|^2_{L^2(\Omega)}+T^3\|f\|^2_{L^1(0,T;L^2(\Omega))}+T^3\|h\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Proof. Let $0<s<T$ and define an auxiliary function

$$
\begin{aligned}
\Psi_1(x,t)&=\int_t^s p(x,\tau)\mathrm{d}\tau, \quad x\in\Omega, \quad 0\le t\le s.\\
\boldsymbol{\Psi}_2(x,t)&=\int_t^s \partial_\tau \boldsymbol{u}(x,\tau)\mathrm{d}\tau, \quad x\in D, \quad 0\le t\le s.
\end{aligned}
$$

It is clear to note that

$$
\Psi_1(x,s)=0, \qquad \partial_t\Psi_1(x,t)=-p(x,t), \tag{3.41}
$$

and

$$
\boldsymbol{\Psi}_2(x,s)=0, \qquad \partial_t\boldsymbol{\Psi}_2(x,t)=-\partial_t\boldsymbol{u}(x,t). \tag{3.42}
$$

For any $\phi(x,t)\in L^2(0,s;L^2(\Omega))$, using integration by parts and (3.41), we have

$$
\begin{aligned}
\int_0^s \phi(x,t)\Psi_1(x,t)\mathrm{d}t&=\int_0^s\left(\phi(x,t)\int_t^s p(x,\tau)\mathrm{d}\tau\right)\mathrm{d}t\\
&=\int_0^s\left[\left(\int_0^t\phi(x,\tau)\mathrm{d}\tau\right)'\left(\int_t^s p(x,\tau)\mathrm{d}\tau\right)\right]\mathrm{d}t\\
&=\left[\left(\int_0^t\phi(x,\tau)\mathrm{d}\tau\right)\left(\int_t^s p(x,\tau)\mathrm{d}\tau\right)\right]\Bigg|_0^s\\
&\quad-\int_0^s\left[\left(\int_0^t\phi(x,\tau)\mathrm{d}\tau\right)\left(\int_t^s p(x,\tau)\mathrm{d}\tau\right)'\right]\mathrm{d}t\\
&=-\int_0^s\left[\left(\int_0^t\phi(x,\tau)\mathrm{d}\tau\right)\left(\int_t^s p(x,\tau)\mathrm{d}\tau\right)'\right]\mathrm{d}t\\
&=-\int_0^s\left[\left(\int_0^t\phi(x,\tau)\mathrm{d}\tau\right)(-p(x,t))\right]\mathrm{d}t\\
&=\int_0^s\left(\int_0^t\phi(x,\tau)\mathrm{d}\tau\right)p(x,t)\mathrm{d}t.
\end{aligned}\tag{3.43}
$$

Taking the test function $q=\Psi_1$ in (3.39) and integrating from $t=0$ to $t=s$ yields

$$
\begin{aligned}
\int_0^s\left(\int_\Omega\frac{\beta}{c^2}(\partial_t^2 p)\Psi_1\mathrm{d}x\right)\mathrm{d}t=&-\int_0^s\left(\int_\Omega(M^{\frac12}\nabla p)\cdot(M^{\frac12}\nabla\Psi_1)\mathrm{d}x\right)\mathrm{d}t\\
&+\int_0^s\left(\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u})\Psi_1\mathrm{d}s\right)\mathrm{d}t+\int_0^s\left(\int_\Omega f\Psi_1\mathrm{d}x\right)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^s \left(\int_\Omega (M^{\frac{1}{2}} \nabla p) \cdot (M^{\frac{1}{2}} \nabla \Psi_1) \mathrm{d}x \right) \mathrm{d}t \\
&\quad + \int_0^s \left(\int_{\partial D} \rho_1 (\boldsymbol{n}_D \cdot \partial_t \boldsymbol{u}) p \mathrm{d}s \right) \mathrm{d}t + \int_0^s \left(\int_\Omega f \Psi_1 \mathrm{d}x \right) \mathrm{d}t.
\end{aligned} \tag{3.44}
$$

It follows from (3.41) that

$$
\begin{aligned}
&\int_0^s \left(\int_\Omega \frac{\beta}{c^2} (\partial_t^2 p) \Psi_1 \mathrm{d}x \right) \mathrm{d}t = \int_\Omega \int_0^s \frac{\beta}{c^2} (\partial_t (\partial_t p \Psi_1) + p \partial_t p) \mathrm{d}t \mathrm{d}x \\
&= \int_\Omega \frac{\beta}{c^2} \left(\partial_t p \Psi_1 \Big|_0^s + \frac{1}{2} |p|^2 \Big|_0^s \right) \mathrm{d}x \\
&= \frac{1}{2} \| \sqrt{\frac{\beta}{c^2}} p(\cdot, s) \|_{L^2(\Omega)}^2 - \frac{1}{2} \| \sqrt{\frac{\beta}{c^2}} g \|_{L^2(\Omega)}^2 - c^{-2} \beta \int_\Omega h(x) \Psi_1(x, 0) \mathrm{d}x.
\end{aligned} \tag{3.45}
$$

It follows from (3.44) and (3.45) that

$$
\begin{aligned}
&\frac{1}{2} \| \sqrt{\frac{\beta}{c^2}} p(\cdot, s) \|_{L^2(\Omega)}^2 + \int_0^s \left(\int_\Omega (M^{\frac{1}{2}} \nabla p) \cdot (M^{\frac{1}{2}} \nabla \Psi_1) \mathrm{d}x \right) \mathrm{d}t \\
&= \int_0^s \left(\int_{\partial D} \rho_1 (\boldsymbol{n}_D \cdot \partial_t \boldsymbol{u}) p \mathrm{d}s \right) \mathrm{d}t + \int_0^s \left(\int_\Omega f(x,t) \Psi_1(x,t) \mathrm{d}x \right) \mathrm{d}t \\
&\quad + \frac{1}{2} \| \sqrt{\frac{\beta}{c^2}} g \|_{L^2(\Omega)}^2 + c^{-2} \beta \int_\Omega h(x) \Psi_1(x, 0) \mathrm{d}x \\
&= \frac{1}{2} \| \sqrt{\frac{\beta}{c^2}} p(\cdot, s) \|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega \left| \int_0^s M^{\frac{1}{2}} \nabla p(x,t) \mathrm{d}t \right|^2 \mathrm{d}x.
\end{aligned} \tag{3.46}
$$

Similarly, for any $\boldsymbol{\Phi}(x,t) \in L^2(0, s; L^2(\Omega)^d)$, using integration by parts and (3.42), we have

$$
\int_0^s \boldsymbol{\Phi}(x,t) \cdot \boldsymbol{\Psi}_2(x,t) \mathrm{d}t = \int_0^s \left(\int_0^t \boldsymbol{\Phi}(x, \tau) \mathrm{d}\tau \right) \cdot \partial_t \boldsymbol{u}(x,t) \mathrm{d}t.
$$

We get from taking the test function $\boldsymbol{v} = \boldsymbol{\Psi}_2$ in (3.40) and integrating from $t=0$ to $t=s$ that

$$
\begin{aligned}
&\int_0^s \left(\int_D \rho_2 \partial_t^2 (\partial_t \boldsymbol{u}) \cdot \boldsymbol{\Psi}_2 \mathrm{d}x \right) \mathrm{d}t \\
&= -\int_0^s \left(\int_D [(\mu \nabla \partial_t \boldsymbol{u}) : (\nabla \boldsymbol{\Psi}_2) + (\lambda + \mu)((\nabla \cdot \partial_t \boldsymbol{u})(\nabla \cdot \boldsymbol{\Psi}_2))] \mathrm{d}x \right) \mathrm{d}t \\
&\quad - \int_0^s \left(\int_{\partial D} (\partial_t p)(\boldsymbol{n}_D \cdot \boldsymbol{\Psi}_2) \mathrm{d}s \right) \mathrm{d}t.
\end{aligned} \tag{3.47}
$$

Using (3.42) and initial condition (3.32e), we deduce

$$
\begin{aligned}
\int_0^s \left(\int_D \rho_2 \partial_t^2 (\partial_t \boldsymbol{u}) \cdot \boldsymbol{\Psi}_2 \mathrm{d}x \right) \mathrm{d}t &= \int_D \int_0^s \rho_2 \left(\partial_t (\partial_t^2 \boldsymbol{u} \cdot \boldsymbol{\Psi}_2 + \partial_t^2 \boldsymbol{u} \cdot \partial_t \boldsymbol{u}) \right) \mathrm{d}t \mathrm{d}x \\
&= \int_D \rho_2 \left((\partial_t^2 \boldsymbol{u} \cdot \boldsymbol{\Psi}_2)|_0^s + \frac{1}{2} |\partial_t \boldsymbol{u}|^2 |_0^s \right) \mathrm{d}x
\end{aligned}
$$

$$= \frac{\rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot,s)\|^2_{\boldsymbol{L}^2(D)} \tag{3.48}$$

and

$$\begin{aligned}\int_0^s \left(\int_{\partial D} (\partial_t p)(\boldsymbol{n}_D \cdot \boldsymbol{\Psi}_2) \mathrm{d}s \right) \mathrm{d}t &= \int_{\partial D} \int_0^s [\partial_t (p(\boldsymbol{n}_D \cdot \boldsymbol{\Psi}_2)) + p(\boldsymbol{n}_D \cdot \partial_t \boldsymbol{u})] \mathrm{d}t \mathrm{d}x \\ &= \int_{\partial D} (p(\boldsymbol{n}_D \cdot \boldsymbol{\Psi}_2))|_0^s \mathrm{d}s + \int_0^s \int_{\partial D} p(\boldsymbol{n}_D \cdot \partial_t \boldsymbol{u}) \mathrm{d}s \mathrm{d}t \\ &= \int_0^s \int_{\partial D} p(\boldsymbol{n}_D \cdot \partial_t \boldsymbol{u}) \mathrm{d}s \mathrm{d}t. \end{aligned} \tag{3.49}$$

Using (3.47), (3.48) and (3.49) yields

$$\begin{aligned}&\frac{\rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot,s)\|^2_{\boldsymbol{L}^2(D)} + \int_0^s \left(\int_D [(\mu \nabla \partial_t \boldsymbol{u}) : (\nabla \boldsymbol{\Psi}_2) + (\lambda+\mu)(\nabla \cdot \partial_t \boldsymbol{u})(\nabla \cdot \boldsymbol{\Psi}_2)] \mathrm{d}x \right) \mathrm{d}t \\ &= \frac{\rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot,s)\|^2_{\boldsymbol{L}^2(D)} \\ &\quad + \frac{1}{2} \left(\mu \left\| \int_0^s \nabla (\partial_t \boldsymbol{u}(\cdot,t)) \mathrm{d}t \right\|^2_{L^2(D)^{d\times d}} + (\lambda+\mu) \int_D \left| \int_0^s \nabla \cdot (\partial_t \boldsymbol{u}(\cdot,t)) \mathrm{d}t \right|^2 \mathrm{d}x \right) \\ &= -\int_0^s \int_{\partial D} p(\boldsymbol{n}_D \cdot \partial_t \boldsymbol{u}) \mathrm{d}s \mathrm{d}t. \end{aligned} \tag{3.50}$$

Multiplying (3.46) by ρ_1 and then adding it to (3.50), we obtain

$$\begin{aligned}&\frac{1}{2} \|\sqrt{\frac{\beta}{c^2}} p(\cdot,s)\|^2_{L^2(\Omega)} + \frac{1}{2} \int_\Omega \left| \int_0^s M^{\frac{1}{2}} \nabla p(\boldsymbol{\cdot},t) \mathrm{d}t \right|^2 \mathrm{d}x + \frac{\rho_1 \rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot,s)\|^2_{\boldsymbol{L}^2(D)} \\ &\quad + \frac{\rho_1}{2} \left(\mu \left\| \int_0^s \nabla (\partial_t \boldsymbol{u}(\cdot,t)) \mathrm{d}t \right\|^2_{L^2(D)^{d\times d}} + (\lambda+\mu) \int_D \left| \int_0^s \nabla \cdot (\partial_t \boldsymbol{u}(\cdot,t)) \mathrm{d}t \right|^2 \mathrm{d}x \right) \\ &= \int_0^s \left(\int_\Omega f(x,t) \Psi_1(x,t) \mathrm{d}x \right) \mathrm{d}t + \frac{1}{2} \|\sqrt{\frac{\beta}{c^2}} g\|^2_{L^2(\Omega)} + c^{-2}\beta \int_\Omega h(x) \Psi_1(x,0) \mathrm{d}x. \end{aligned} \tag{3.51}$$

Next, we estimate the two terms on the left-hand side of (3.51) separately. It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}c^{-2}\beta \int_\Omega h(x) \Psi_1(x,0) \mathrm{d}x &= c^{-2}\beta \int_\Omega h(x) \left(\int_0^s p(x,t) \mathrm{d}t \right) \mathrm{d}x \\ &= c^{-2}\beta \int_0^s \left(\int_\Omega h(x) p(x,t) \mathrm{d}x \right) \mathrm{d}t \\ &\le C \left(\|h\|_{L^2(\Omega)} \right) \int_0^s \|p(\cdot,t)\|_{L^2(\Omega)} \mathrm{d}t. \end{aligned} \tag{3.52}$$

For $0 \le t \le s \le T$, we have from (3.43) that

$$\begin{aligned}\int_0^s \left(\int_\Omega f(x,t) \Psi_1(x,t) \mathrm{d}x \right) \mathrm{d}t &= \int_\Omega \left(\int_0^s \left(\int_0^t f(x,\tau) \mathrm{d}\tau \right) p(x,t) \mathrm{d}t \right) \mathrm{d}x \\ &\le \int_0^s \int_0^t \|f(\cdot,\tau)\|_{L^2(\Omega)} \|p(\cdot,t)\|_{L^2(\Omega)} \mathrm{d}\tau \mathrm{d}t\end{aligned}$$

$$\le \left(\int_0^s \|f(\cdot,t)\|_{L^2(\Omega)}\mathrm{d}t\right)\left(\int_0^s \|p(\cdot,t)\|_{L^2(\Omega)}\mathrm{d}t\right). \tag{3.53}$$

Substituting (3.52)-(3.53) into (3.51), we have for any $s\in[0,T]$ that

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\frac{\beta}{c^2}}p(\cdot,s)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\left|\int_0^s M^{\frac{1}{2}}\nabla p(\cdot,t)\mathrm{d}t\right|^2\mathrm{d}x+\frac{\rho_1\rho_2}{2}\|\partial_t\boldsymbol{u}(\cdot,s)\|^2_{L^2(D)^2}\\&\quad+\frac{\rho_1}{2}\left(\mu\left\|\int_0^s\nabla(\partial_t\boldsymbol{u}(\cdot,t))\mathrm{d}t\right\|^2_{L^2(D)^{d\times d}}+(\lambda+\mu)\int_D\left|\int_0^s\nabla\cdot(\partial_t\boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2\mathrm{d}x\right)\\&\le\frac{\beta}{2c^2}\|g\|^2_{L^2(\Omega)}+\left(\int_0^s\|f(\cdot,t)\|_{L^2(\Omega)}\mathrm{d}t+C\|h\|_{L^2(\Omega)}\right)\int_0^s\|p(\cdot,t)\|_{L^2(\Omega)}\mathrm{d}t.\end{aligned} \tag{3.54}$$

Taking the L^∞- norm with respect to s on both sides of (3.54) yields

$$\begin{aligned}&\|p\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\nabla p\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\&\quad+\|\partial_t\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(D))}+\|\nabla\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D)^{d\times d})}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D))}\\&\le C_1\|g\|^2_{L^2(\Omega)}+C_2T\left(\|f\|_{L^1(0,T;L^2(\Omega))}+\|h\|_{L^2(\Omega)}\right)\|p\|_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

Applying the Young inequality yields

$$\begin{aligned}&\|p\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\nabla p\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\\&\quad+\|\partial_t\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(D))}+\|\nabla\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D)^{d\times d})}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(D))}\\&\le C_1\left(\|g\|^2_{L^2(\Omega)}+T^2\|f\|^2_{L^1(0,T;L^2(\Omega))}+T^2\|h\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Integrating (3.54) with respect to s from 0 to T and using the Cauchy-Schwarz inequality and the Young inequality, we can get

$$\begin{aligned}&\|p\|^2_{L^2(0,T;L^2(\Omega))}+\|\nabla p\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\&\quad+\|\partial_t\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{L}^2(D))}+\|\nabla\boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^{d\times d})}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^2(0,T;L^2(D))}\\&\le C_2\left(T\|g\|^2_{L^2(\Omega)}+T^3\|f\|^2_{L^1(0,T;L^2(\Omega))}+T^3\|h\|^2_{L^2(\Omega)}\right),\end{aligned}$$

which completes the proof. ∎

4. Conclusion

In this paper, we have studied the two- and three-dimensional acoustic-elastic wave scattering problem on a finite time interval. The acoustic and elastic wave equations are coupled on the surface of the elastic obstacle. We propose the compressed coordinate transformation to reduce equivalently the scattering problem into an initial-boundary value problem in a bounded domain. The reduced problem is proved to have a unique weak solution by using the Galerkin method. A priori estimates with explicit time dependence are also established for the acoustic pressure and elastic displacement of the time-domain variational problem. We believe that the method of compressed coordinate transformation can be applied to many other time-domain scattering problems imposed in open domains. The model problem is suitable for numerical simulations. We hope to report the work on the numerical analysis and computation elsewhere in the future.

Appendix A. Change of variables in two dimensions. Let $\boldsymbol{x}=(x,y)\in\mathbb{R}^2$ and $\rho=|\boldsymbol{x}|$. The polar coordinates (ρ,θ) are related to the Cartesian coordinates (x,y) by $x=\rho\cos\theta, y=\rho\sin\theta$. The local orthonormal basis is

$$\boldsymbol{e}_\rho=(\cos\theta,\sin\theta)^\top,\quad \boldsymbol{e}_\theta=(-\sin\theta,\cos\theta)^\top.$$

Denote by ∇_ρ and $\nabla_\rho\cdot$ the gradient operator and the divergence operator in the old coordinates (ρ,θ), respectively. We study the two-dimensional acoustic wave equation:

$$\frac{1}{c^2}\partial_t^2 u(\rho,\theta,t)-\Delta_\rho u(\rho,\theta,t)=0 \quad \text{in } \mathbb{R}^2,\ t>0, \tag{A.1}$$

where Δ_ρ is the Laplace operator and $c>0$ is the wave speed.

Consider the change of variables $\rho=\zeta(r)$, where ζ is a smooth and invertible function. Denote by ∇_r and $\nabla_r\cdot$ the gradient operator and the divergence operator in the new coordinates (r,θ), respectively.

LEMMA A.1. *Let $v(r,\theta,t)=u(\rho,\theta,t)|_{\rho=\zeta(r)}$ be a differentiable scalar function, then*

$$\nabla_\rho u(\rho,\theta,t)|_{\rho=\zeta(r)}=Q\begin{bmatrix}\frac{1}{\zeta'(r)} & 0\\ 0 & \frac{r}{\zeta(r)}\end{bmatrix}Q^\top\nabla_r v(r,\theta,t),$$

where R is an orthonormal matrix given by

$$Q(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

Proof. It follows from the straightforward calculations that

$$\begin{aligned}\nabla_\rho u|_{\rho=\zeta(r)}&=\partial_\rho u|_{\rho=\zeta(r)}\boldsymbol{e}_\rho+\frac{1}{\rho}\partial_\theta u|_{\rho=\zeta(r)}\boldsymbol{e}_\theta\\&=\frac{1}{\zeta'}\partial_r v\boldsymbol{e}_r+\frac{1}{\zeta}\partial_\theta v\boldsymbol{e}_\theta\\&=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\frac{1}{\zeta'} & 0\\ 0 & \frac{r}{\zeta}\end{bmatrix}\begin{bmatrix}\partial_r v\\ \frac{1}{r}\partial_\theta v\end{bmatrix}\\&=Q\begin{bmatrix}\frac{1}{\zeta'} & 0\\ 0 & \frac{r}{\zeta}\end{bmatrix}Q^\top Q\begin{bmatrix}\partial_r v\\ \frac{1}{r}\partial_\theta v\end{bmatrix}\\&=Q\begin{bmatrix}\frac{1}{\zeta'} & 0\\ 0 & \frac{r}{\zeta}\end{bmatrix}Q^\top\nabla_r v,\end{aligned}$$

which completes the proof. ∎

LEMMA A.2. *Let $\boldsymbol{v}(r,\theta,t)=\boldsymbol{u}(\rho,\theta,t)|_{\rho=\zeta(r)}$ be a differentiable vector function, then*

$$\nabla_\rho\cdot\boldsymbol{u}(\rho,\theta,t)|_{\rho=\zeta(r)}=\beta^{-1}(r)\nabla_r\cdot(K(r,\theta)\boldsymbol{v}(r,\theta,t)),$$

where

$$\beta(r)=\frac{\zeta(r)\zeta'(r)}{r},\quad K(r,\theta)=Q\begin{bmatrix}\frac{\zeta(r)}{r} & 0\\ 0 & \zeta'(r)\end{bmatrix}Q^\top.$$

Proof. Let $\boldsymbol{u}=u_\rho\boldsymbol{e}_\rho+u_s\boldsymbol{e}_\theta$ and $\boldsymbol{v}=v_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta$. A simple calculation yields that

$$\begin{aligned}\nabla_\rho\cdot\boldsymbol{u}(\rho,\theta,t)|_{\rho=\zeta(r)}&=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho u_\rho)+\frac{1}{\rho}\partial_\theta(u_\theta)\\&=\frac{1}{\zeta\zeta'}\partial_r(\zeta v_r)+\frac{1}{\zeta}\partial_\theta(v_\theta)\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{\zeta\zeta'}\left[\frac{1}{r}\partial_r\left(r\frac{\zeta}{r}v_r\right)+\frac{1}{r}\partial_\theta\left(\zeta' v_\theta\right)\right]\\
&= \beta^{-1}\nabla_r\cdot\left(\frac{\zeta}{r}v_r\boldsymbol{e}_r+\zeta' v_\theta\boldsymbol{e}_\theta\right)\\
&= \beta^{-1}\nabla_r\cdot\left(Q\begin{bmatrix}\frac{\zeta}{r} & 0\\ 0 & \zeta'\end{bmatrix}Q^\top Q\begin{bmatrix}v_r\\ v_\theta\end{bmatrix}\right)\\
&= \beta^{-1}\nabla_r\cdot(K\boldsymbol{v}),
\end{aligned}
$$

which completes the proof. ∎

LEMMA A.3. *Let $v(r,\theta,t)=u(\rho,\theta,t)|_{\rho=\zeta(r)}$ be a differentiable function, then*

$$
\Delta_\rho u(\rho,\theta,t)|_{\rho=\zeta(r)}=\beta^{-1}(r)\nabla_r\cdot(M(r,\theta)\nabla_r v(r,\theta,t)),
$$

where

$$
M(r,\theta)=Q\begin{bmatrix}\frac{\zeta(r)}{r\zeta'(r)} & 0\\ 0 & \frac{r\zeta'(r)}{\zeta(r)}\end{bmatrix}Q^\top.
$$

Proof. It is easy to note that

$$
\Delta_\rho u|_{\rho=\zeta(r)}=\nabla_\rho\cdot(\nabla_\rho u)|_{\rho=\zeta(r)}.
$$

Using similar steps of the change of variables in the proofs for Lemmas A.1–A.2, we have

$$
\begin{aligned}
\nabla_\rho\cdot(\nabla_\rho u)|_{\rho=\zeta(r)} &= \beta^{-1}\nabla_r\cdot\left(KQ\begin{bmatrix}\frac{1}{\zeta'} & 0\\ 0 & \frac{r}{\zeta}\end{bmatrix}Q^\top\nabla_r v\right)\\
&= \beta^{-1}\nabla_r\cdot\left(Q\begin{bmatrix}\frac{\zeta}{r\zeta'} & 0\\ 0 & \frac{r\zeta'}{\zeta}\end{bmatrix}Q^\top\nabla_r v\right)\\
&= \beta^{-1}\nabla_r\cdot(M\nabla_r v),
\end{aligned}
$$

which completes the proof. ∎

THEOREM A.1. *In the new coordinates (r,θ), the acoustic wave equation (A.1) becomes*

$$
\frac{\beta(r)}{c^2}\partial_t^2 v(r,\theta,t)-\nabla_r\cdot(M(r,\theta)\nabla_r v(r,\theta,t))=0.
$$

Proof. Using Lemma A.2–Lemma A.3, we have from (A.1) that

$$
0=\left(\frac{1}{c^2}\partial_t^2 u-\Delta u\right)\Big|_{\rho=\zeta(r)}=\frac{1}{c^2}\partial_t^2 v-\beta^{-1}\nabla_r\cdot(M\nabla_r v).
$$

The proof is completed by multiplying the above equation by β. ∎

Appendix B. Change of variables in three dimensions. Let $\boldsymbol{x}=(x,y,z)\in\mathbb{R}^3$ and $\rho=|\boldsymbol{x}|$. The spherical coordinates (ρ,θ,φ) are related to the Cartesian coordinates (x,y,z) by $x=\rho\sin\theta\cos\varphi, y=\rho\sin\theta\sin\varphi, z=\rho\cos\theta$. The local orthonormal basis is

$$
\boldsymbol{e}_\rho=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)^\top,
$$

$$\boldsymbol{e}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)^\top,$$
$$\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)^\top.$$

Again, denote by ∇_ρ and $\nabla_\rho\cdot$ the gradient operator and the divergence operator in the old coordinates (ρ,θ,φ), respectively. In this section, we present parallel results for the three-dimensional acoustic wave equation:

$$\frac{1}{c^2}\partial_t^2 u(\rho,\theta,\varphi,t) - \Delta_\rho u(\rho,\theta,\varphi,t) = 0 \quad \text{in } \mathbb{R}^3,\ t>0, \tag{B.1}$$

where Δ_ρ is the Laplace operator and $c>0$ is the wave speed.

Consider the change of variables $\rho=\zeta(r)$, where ζ is a smooth and invertible function. Denote by ∇_r and $\nabla_r\cdot$ the gradient operator and the divergence operator in the new coordinates (r,θ,φ), respectively.

LEMMA B.1. *Let $v(r,\theta,\varphi,t) = u(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)}$ be a differentiable scalar function, then*

$$\nabla_\rho u(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)} = Q\begin{bmatrix} \frac{1}{\zeta'(r)} & 0 & 0 \\ 0 & \frac{r}{\zeta(r)} & 0 \\ 0 & 0 & \frac{r}{\zeta(r)} \end{bmatrix} Q^\top \nabla_r v(r,\theta,\varphi,t),$$

where R is an orthonormal matrix given by

$$Q(\theta,\varphi) = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}.$$

Proof. It follows from the straightforward calculations that

$$\begin{aligned}
\nabla_\rho u|_{\rho=\zeta(r)} &= \partial_\rho u|_{\rho=\zeta(r)}\boldsymbol{e}_\rho + \frac{1}{\rho}\partial_\theta u|_{\rho=\zeta(r)}\boldsymbol{e}_\theta + \frac{1}{\rho\sin\theta}\partial_\varphi u|_{\rho=\zeta(r)}\boldsymbol{e}_\varphi \\
&= \frac{1}{\zeta'}\partial_r v\boldsymbol{e}_r + \frac{1}{\zeta}\partial_\theta v\boldsymbol{e}_\theta + \frac{1}{\zeta\sin\theta}\partial_\varphi v\boldsymbol{e}_\varphi \\
&= \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\zeta'} & 0 & 0 \\ 0 & \frac{r}{\zeta} & 0 \\ 0 & 0 & \frac{r}{\zeta} \end{bmatrix}\begin{bmatrix} \partial_r v \\ \frac{1}{r}\partial_\theta v \\ \frac{1}{r\sin\theta}\partial_\varphi v \end{bmatrix} \\
&= Q\begin{bmatrix} \frac{1}{\zeta'} & 0 & 0 \\ 0 & \frac{r}{\zeta} & 0 \\ 0 & 0 & \frac{r}{\zeta} \end{bmatrix} Q^\top Q\begin{bmatrix} \partial_r v \\ \frac{1}{r}\partial_\theta v \\ \frac{1}{r\sin\theta}\partial_\varphi v \end{bmatrix} \\
&= Q\begin{bmatrix} \frac{1}{\zeta'} & 0 & 0 \\ 0 & \frac{r}{\zeta} & 0 \\ 0 & 0 & \frac{r}{\zeta} \end{bmatrix} Q^\top \nabla_r v,
\end{aligned}$$

which completes the proof. ∎

LEMMA B.2. *Let $\boldsymbol{v}(r,\theta,\varphi,t) = \boldsymbol{u}(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)}$ be a differentiable vector function, then*

$$\nabla_\rho\cdot\boldsymbol{u}(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)} = \beta^{-1}(r)\nabla_r\cdot\left(K(r,\theta,\varphi)\boldsymbol{v}(r,\theta,\varphi,t)\right),$$

where

$$\beta(r)=\frac{\zeta^2(r)\zeta'(r)}{r^2},\quad K(r,\theta,\varphi)=Q\begin{bmatrix}\frac{\zeta^2(r)}{r^2} & 0 & 0\\ 0 & \frac{\zeta(r)\zeta'(r)}{r} & 0\\ 0 & 0 & \frac{\zeta(r)\zeta'(r)}{r}\end{bmatrix}Q^\top.$$

Proof. Let $\boldsymbol{u}=u_\rho\boldsymbol{e}_\rho+u_\theta\boldsymbol{e}_\theta+u_\varphi\boldsymbol{e}_\varphi$ and $\boldsymbol{v}=v_r\boldsymbol{e}_r+v_\theta\boldsymbol{e}_\theta+v_\varphi\boldsymbol{e}_\varphi$. A simple calculation yields that

$$\begin{aligned}&\nabla_\rho\cdot\boldsymbol{u}(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)}\\ =&\left(\frac{1}{\rho^2}\partial_\rho\left(\rho^2u_\rho\right)+\frac{1}{\rho\sin\theta}\partial_\theta(\sin\theta u_\theta)+\frac{1}{\rho\sin\theta}\partial_\varphi(u_\varphi)\right)\bigg|_{\rho=\zeta(r)}\\ =&\frac{1}{\zeta^2\zeta'}\partial_r\left(\zeta^2v_r\right)+\frac{1}{\zeta\sin\theta}\partial_\theta(\sin\theta v_\theta)+\frac{1}{\zeta\sin\theta}\partial_\varphi(v_\varphi)\\ =&\frac{r^2}{\zeta^2\zeta'}\left[\frac{1}{r^2}\partial_r\left(r^2\frac{\zeta^2}{r^2}v_r\right)+\frac{1}{r\sin\theta}\partial_\theta\left(\frac{\zeta\zeta'}{r}\sin\theta v_\theta\right)+\frac{1}{r\sin\theta}\partial_\varphi\left(\frac{\zeta\zeta'}{r}v_\varphi\right)\right]\\ =&\beta^{-1}\nabla_r\cdot\left(\frac{\zeta^2}{r^2}v_r\boldsymbol{e}_r+\frac{\zeta\zeta'}{r}v_\theta\boldsymbol{e}_\theta+\frac{\zeta\zeta'}{r}v_\varphi\boldsymbol{e}_\varphi\right)\\ =&\beta^{-1}\nabla_r\cdot\left(Q\begin{bmatrix}\frac{\zeta^2}{r^2} & 0 & 0\\ 0 & \frac{\zeta\zeta'}{r} & 0\\ 0 & 0 & \frac{\zeta\zeta'}{r}\end{bmatrix}Q^\top Q\begin{bmatrix}v_r\\ v_\theta\\ v_\varphi\end{bmatrix}\right)\\ =&\beta^{-1}\nabla_r\cdot(K\boldsymbol{v}),\end{aligned}$$

which completes the proof. ∎

LEMMA B.3. *Let $v(r,\theta,\varphi,t)=u(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)}$ be a differentiable function, then*

$$\Delta_\rho u(\rho,\theta,\varphi,t)|_{\rho=\zeta(r)}=\beta^{-1}(r)\nabla_r\cdot(M(r,\theta,\varphi)\nabla_r v(r,\theta,\varphi,t)),$$

where

$$M(r,\theta,\varphi)=Q\begin{bmatrix}\frac{\zeta^2(r)}{r^2\zeta'(r)} & 0 & 0\\ 0 & \zeta'(r) & 0)\\ 0 & 0 & \zeta'(r)\end{bmatrix}Q^\top.$$

Proof. It is easy to note that

$$\Delta_\rho u|_{\rho=\zeta(r)}=\nabla_\rho\cdot(\nabla_\rho u)|_{\rho=\zeta(r)}.$$

Using similar steps of the change of variables for the proofs of Lemmas B.1–B.2, we have

$$\begin{aligned}\nabla_\rho\cdot(\nabla_\rho u)|_{\rho=\zeta(r)}&=\beta^{-1}\nabla_r\cdot\left(KQ\begin{bmatrix}\frac{1}{\zeta'} & 0 & 0\\ 0 & \frac{r}{\zeta} & 0\\ 0 & 0 & \frac{r}{\zeta}\end{bmatrix}Q^\top\nabla_r v\right)\\ &=\beta^{-1}\nabla_r\cdot\left(Q\begin{bmatrix}\frac{\zeta^2}{r^2\zeta'} & 0 & 0\\ 0 & \zeta' & 0\\ 0 & 0 & \zeta'\end{bmatrix}Q^\top\nabla_r v\right)\\ &=\beta^{-1}\nabla_r\cdot(M\nabla_r v),\end{aligned}$$

which completes the proof. ∎

THEOREM B.1. *In the new coordinates* (r,θ,φ)*, the acoustic wave Equation* (B.1) *becomes*

$$\frac{\beta(r)}{c^2}\partial_t^2 v(r,\theta,\varphi,t)-\nabla_r\cdot(M(r,\theta,\varphi)\nabla_r v(r,\theta,\varphi,t))=0.$$

Proof. Using Lemma B.2–Lemma B.3, we have from (B.1) that

$$0=\left(\frac{1}{c^2}\partial_t^2 u-\Delta_\rho u\right)\Big|_{\rho=\zeta(r)}=\frac{1}{c^2}\partial_t^2 v-\beta^{-1}\nabla_r\cdot(M\nabla_r v).$$

The proof is completed by multiplying the above equation by β. ∎

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