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MULTIPLE EQUILIBRIA AND TRANSITIONS IN SPHERICAL MHD EQUATIONS*

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Abstract. In this study, we aim to describe the first dynamic transitions of the MHD equations in a thin spherical shell. It is well known that the MHD equations admit a motionless steady state solution with constant vertically aligned magnetic field and linearly conducted temperature. This basic solution is stable for small Rayleigh numbers R and loses its stability at a critical threshold R_c. There are two possible sources for this instability. Either a set of real eigenvalues or a set of non-real eigenvalues cross the imaginary axis at R_c. We restrict ourselves to the study of the first case. In this case, by the center manifold reduction, we reduce the full PDE to a system of $2l_c+1$ ODE's where l_c is a positive integer. We exhibit the most general reduction equation regardless of l_c. Then, it is shown that for $l_c=1,2$, the system either exhibits a continuous transition accompanied by an attractor homeomorphic to $2l_c$ dimensional sphere which contains steady states of the system or a drastic transition accompanied by a repeller bifurcated on $\mathrm{R}<\mathrm{R}_c$. We show that there are parameter regimes where both types of transitions are realized. Besides, several identities involving the triple products of gradients of spherical harmonics are derived, which are useful for the study of related problems.

Keywords. Magnetohydrodynamics convection; dynamical transition theory; spherical harmonics; linear stability; principle of exchange of stabilities.

AMS subject classifications. 76W05; 76E25; 82D10; 37N20; 37G30.

1. Introduction

It is known that the thermal convection led by the buoyancy plays an important role in the heat transfer in a thermal system, such as the general atmospheric circulation, the formation of winds and oceanic currents, movements within the Earth's mantle, and the complex activity in the atmosphere of the sun. Convection is also used in engineering practices of homes, industrial processes, cooling of equipment, etc. Among these thermal systems, the phenomenon of magnetic convection with its importance in the understanding of the dynamo processes [1, 2], industrial processes and astrophysics [3, 4], has attracted in the past as also nowadays the attention of so many researchers [5–10] from the perspective of numerical simulation and the linear stability analysis. In this paper, we tackle the problem of first dynamic transitions associated with the magnetic convection governing the 3D incompressible MHD equations in a spherical shell.

Our main toolbox is the dynamical transition theory recently developed by Ma and Wang, see [11]. This theory aims to search for the full set of transition states usually described by a local attractor. According to this theory, dynamic transitions in dissipative systems can be classified into three distinct types: continuous, catastrophic and random. Roughly speaking, a continuous transition is characterized by the continuous appearance of a local attractor; a catastrophic transition is characterized by an immediate jump from the basic state to another state and a random transition is said to occur when the system exhibits both continuous and catastrophic transitions depending on the initial conditions. Due to the practical feature of the theory, which offers detailed

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dynamic analysis from one state to another, it has been applied to study a lot of phase transition phenomena, such as: the transitions of Rayleigh-Bénard convection [12–14], the formation of Taylor vortex led by the centrifugal instability [15,16], the plasma perturbation systems [17] and the pattern formation in ocean dynamics [18–20]. Recently, the theory has been generalized by Liu et al. [21,22] making it suitable for the stochastic transition problems.

There are several recent papers dealing with dynamic transitions in problems related to the current study. The problem has been studied in the context of dynamical transition theory in 3D rectangular domains in [23]. In that paper, the authors describe the dynamic transitions as well as pattern formations of symmetric patterns such as rolls, rectangles and hexagons. From the perspective of the dynamical transition theory, the convection problem without a magnetic field in a spherical shell is studied in [24]. In that paper, the authors show that the first transition of the pure thermal convection without the magnetic effects has a continuous transition characterized by the attractor bifurcation theory [25]. They also manage to describe the structure of the local attractors when the first critical eigenspace is either 3 or 5 dimensional. Recently, the results in [24] have been extended to the double diffusive case in the context of thermohaline circulation in [26].

As we discuss in this study, the addition of a magnetic field greatly changes the nature of the problem. First, due to the non-selfadjointess of the linear operator, there are two sources for the first transition of the system. Namely the transition is either caused by a finite set of real or complex eigenvalues crossing the imaginary axis as the control parameter Rayleigh number exceeds a critical threshold. In this study we restrict ourselves to the study of the case of critical crossing of real eigenvalues. In this case, the number of first eigenvalues depending on the system parameters must be an odd integer greater than or equal to three. By reducing the system onto the first critical eigenspace, we obtain a system of ODE's with cubic nonlinearities. One of the main achievements of the study is obtaining the exact coefficients of this reduced system by the center manifold reduction. To carry out the center manifold reduction, a basis for the phase space alternative to the eigenbasis of the associated linear operator is constructed. This alternative approach allows us to make all computations related to the nonlinear interactions of the different modes manually and then verify them using a symbolic computation software. For the manual computations, we derive some integral identities involving triple products of spherical harmonics and their gradients over a sphere, which are presented in the Appendix and could be useful in the study of related problems.

In this article, we only focus on the case where the first eigenspace is either 3 or 5 dimensional as in [24, 26] as higher dimensional cases lead to ODE systems with cross cubic nonlinear terms which exhibit more complex behavior. In these two cases, we manage to show that there are two possible transition types. More precisely, the transition is either continuous accompanied by an attractor which is homeomorphic to a sphere with one dimension less than the dimension of the critical eigenspace or it is abrupt, meaning that the system suddenly moves to states away from the basic solution after the transition. We show that there are parameter regimes where these two types of transitions are possible.

The rest of the paper is arranged as follows. In Section 2, we formulate our problem. In Section 3, we introduce our main results which include the linear stability result, the verification of the PES condition, the main transition theorem and the proofs. The discussion involving the relation between the specific transition types and the control parameter regimes and the conclusion are provided in Section 4.

2. Governing equations and the functional setting

2.1. Governing equations. We will consider the incompressible MHD convection in a thin spherical shell. The main motivation for thin shell assumption is the fact that this geometry is the natural setting for geophysical flows. The thin spherical shell can be approximated (see [27,28]) as the Cartesian product $S_a^2 \times (0,h)$ where S_a^2 is the 2D sphere with radius a and h denotes the height of the fluid layer. The Boussinesq approximation yields the following set of equations.

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho_0}(\nabla p + \rho g \hat{\mathbf{e}}_z) + \mu_0(\nabla\times\mathbf{H})\times\mathbf{H} + \nu\Delta\mathbf{u},\\
&\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T = \kappa_T \Delta T,\\
&\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{H} = (\mathbf{H}\cdot\nabla)\mathbf{u} + \kappa_H\Delta\mathbf{H},\\
&\mathrm{div}\mathbf{u} = \mathrm{div}\mathbf{H} = 0,
\end{aligned} \tag{2.1}$$

where $\mathbf{u}$ is the 3D velocity, T is the temperature, $\mathbf{H}$ is the magnetic field, ρ is the density, $\hat{\mathbf{e}}_z$ is the unit vector in the z-direction, ν, κ_T, κ_H, g are all positive constants denoting the kinematic diffusivity, the thermal diffusivity, the magnetic diffusivity and the gravitational constant, respectively.

In the Boussinesq approximation, the density ρ is considered constant except in the body forcing term where it is given by an equation of state. We consider a linear equation of state as

$$\rho = \rho_0[1 - a_T(T - T_0)],$$

where $a_T > 0$ is the thermal expansion coefficient, and T_0, ρ_0 are the temperature and density at the boundary $z=0$, respectively.

The Equations (2.1) possess a basic steady state solution given by

$$\begin{aligned}
&\mathbf{u}_{\mathrm{ss}} = 0,\\
&\mathbf{H}_{\mathrm{ss}} = H_0\hat{\mathbf{e}}_z,\\
&T_{\mathrm{ss}} = T_0 - (T_0 - T_1)\frac{z}{h},\\
&p_{\mathrm{ss}} = p_0 - g\rho_0\left(a_T(T_0-T_1)\frac{z^2}{2h}\right),
\end{aligned} \tag{2.2}$$

where T_1 represents the fixed temperature at the upper boundary $z=h$. Thus the basic solution represents a motionless state with a constant vertically aligned magnetic field and linearly conducted temperature. The nondimensional form of (2.1), obtained as in [23], is given by

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathfrak{p}_1\left(-\nabla p + \mathrm{R}T\hat{\mathbf{e}}_z + \Delta\mathbf{u} + \mathrm{Q}\frac{\partial \mathbf{H}}{\partial z} + \frac{\mathrm{Q}}{\mathfrak{p}_2}(\mathbf{H}\cdot\nabla)\mathbf{H}\right),\\
&\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T = \Delta T + u_3,\\
&\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{H} = (\mathbf{H}\cdot\nabla)\mathbf{u} + \mathfrak{p}_2\left(\frac{\partial \mathbf{u}}{\partial z} + \Delta\mathbf{H}\right),\\
&\mathrm{div}\mathbf{u} = \mathrm{div}\mathbf{H} = 0,
\end{aligned} \tag{2.3}$$

where R is the Rayleigh number, Q is the Chandrasekhar number and $\mathfrak{p}_1$, $\mathfrak{p}_2$ are the Prandtl and magnetic Prandtl numbers, respectively, defined as

$$\mathrm{R}=\frac{a_T g(T_0-T_1)h^3}{\kappa_T\nu}, \qquad \mathrm{Q}=\frac{\mu_0 H_0^2 h^2}{\kappa_H \nu},$$
$$\mathfrak{p}_1=\frac{\nu}{\kappa_T}, \qquad \mathfrak{p}_2=\frac{\kappa_H}{\kappa_T}.$$

All the above dimensionless parameters are positive. In particular, the Rayleigh number $R>0$ which means that $T_0>T_1$, in other words the fluid layer is heated from below. Here we set the nondimensional spatial domain $\Omega=S_r^2\times(0,1)$ with the nondimensional radius $r=\frac{a}{h}$, and the variables in (2.3) are also in nondimensional form. Moreover ∇, ∇_u, div and Δ will denote both the scalar and vectorial differential operators given by

$$\begin{aligned}
&\nabla_u f=u\cdot\nabla f=\frac{1}{r}\left(u_\theta\frac{\partial f}{\partial\theta}+\frac{u_\varphi}{\sin\theta}\frac{\partial f}{\partial\varphi}\right),\\
&\Delta f=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right],\\
&\nabla_u v=\frac{1}{r}\left(u_\theta\frac{\partial v_\theta}{\partial\theta}+\frac{u_\varphi}{\sin\theta}\frac{\partial v_\theta}{\partial\varphi}-u_\varphi v_\varphi\cot\theta\right)\hat{e}_\theta+\frac{1}{r}\left(u_\theta\frac{\partial v_\varphi}{\partial\theta}+\frac{u_\varphi}{\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}+u_\varphi v_\theta\cot\theta\right)\hat{e}_\varphi,\\
&\Delta u=\left(\Delta u_\theta-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}-\frac{u_\theta}{r^2\sin^2\theta}\right)\hat{e}_\theta+\left(\Delta u_\varphi+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}-\frac{u_\varphi}{r^2\sin^2\theta}\right)\hat{e}_\varphi,\\
&\mathrm{div}\ u=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi},
\end{aligned}$$

where f is a scalar function, $u=u_\theta\hat{e}_\theta+u_\varphi\hat{e}_\varphi$ and $v=v_\theta\hat{e}_\theta+v_\varphi\hat{e}_\varphi$ are 2 dimensional vector valued functions.

Let $\mathbf{u}=u+w\hat{\mathbf{e}}_z$ be the 3D velocity field where $u=u_\theta\hat{e}_\theta+u_\varphi\hat{e}_\varphi$ is the horizontal velocity field. Similarly, $\mathbf{H}=H+M\hat{\mathbf{e}}_z$ is the 3D magnetic field vector where $H=H_\theta\hat{e}_\theta+H_\varphi\hat{e}_\varphi$ is the 2D horizontal component of the magnetic field. With this notation, the Equations (2.3) become

$$\begin{aligned}
&u_t+\nabla_u u+wu_z=\mathfrak{p}_1\left(-\nabla p+\Delta u+u_{zz}+\mathrm{Q}H_z+\frac{\mathrm{Q}}{\mathfrak{p}_2}(\nabla_H H+MH_z)\right),\\
&w_t+\nabla_u w+ww_z=\mathfrak{p}_1\left(-p_z+\mathrm{R}T+\Delta w+w_{zz}+\mathrm{Q}M_z+\frac{\mathrm{Q}}{\mathfrak{p}_2}(\nabla_H M+MM_z)\right),\\
&T_t+\nabla_u T+wT_z-w=\Delta T+T_{zz},\\
&H_t+\nabla_u H+wH_z=\nabla_H u+Mu_z+\mathfrak{p}_2(u_z+\Delta H+H_{zz}),\\
&M_t+\nabla_u M+wM_z=\nabla_H w+Mw_z+\mathfrak{p}_2(w_z+\Delta M+M_{zz}),\\
&\mathrm{div}\ u+w_z=0,\\
&\mathrm{div}\ H+M_z=0.
\end{aligned} \tag{2.4}$$

In the present paper the Equations (2.4) are examined with the following boundary conditions

$$w=T=H=\frac{\partial u}{\partial z}=\frac{\partial M}{\partial z}=0, \quad \text{at } z=0,1. \tag{2.5}$$

2.2. Functional settings. We first represent the main Equations (2.4) subject to the boundary condition (2.5) as an abstract ODE in a Banach space. For this purpose, we define the functional spaces as

$$\begin{aligned}\mathcal{H}=&\Big\{(u,w,T,H,M)\in L^2(\Omega)^7\,|\,\text{div }u=\text{div }H=0,\\&\int_\Omega u=\int_\Omega M=0, w=T=H=0 \text{ at } z=0,1\Big\},\\\mathcal{H}_1=&\Big\{(u,w,T,H,M)\in H^2(\Omega)^7\,|\,\text{div }u=\text{div }H=0,\\&\int_\Omega u=\int_\Omega M=0, w=T=\frac{\partial M}{\partial z}=\frac{\partial u}{\partial z}=H=0, \text{ at } \ z=0,1\Big\}.\end{aligned}\tag{2.6}$$

The inner product in $\mathcal{H}$ is defined for the vectors $\Psi_i=(u_i,w_i,T_i,H_i,M_i)$, $i=1,2$ as

$$\langle\Psi_1,\Psi_2\rangle=\int_0^\pi\int_0^{2\pi}\int_0^1(u_1\overline{u_2}+w_1\overline{w_2}+T_1\overline{T_2}+H_1\overline{H_2}+M_1\overline{M_2})r^2\sin\theta\; dzd\varphi d\theta.$$

Now the main Equations (2.4) with the boundary conditions (2.5) become

$$\frac{\partial\Psi}{\partial t}=L\Psi+\mathcal{G}(\Psi,\Psi),\qquad \Psi(0)=\Psi_0,\tag{2.7}$$

where the linear operator $L:\mathcal{H}_1\to\mathcal{H}$ and the bilinear operator $\mathcal{G}:\mathcal{H}_1\times\mathcal{H}_1\to\mathcal{H}$ are defined by

$$L(\Psi)=\mathcal{P}\begin{bmatrix}\mathfrak{p}_1\left((\Delta+\frac{\partial^2}{\partial z^2})u+\mathrm{Q}\frac{\partial H}{\partial z}\right)\\\mathfrak{p}_1\left((\Delta+\frac{\partial^2}{\partial z^2})w+\mathrm{R}T+\mathrm{Q}\frac{\partial M}{\partial z}\right)\\(\Delta+\frac{\partial^2}{\partial z^2})T+w\\\mathfrak{p}_2\left((\Delta+\frac{\partial^2}{\partial z^2})H+\frac{\partial u}{\partial z}\right)\\\mathfrak{p}_2\left((\Delta+\frac{\partial^2}{\partial z^2})M+\frac{\partial w}{\partial z}\right)\end{bmatrix},\tag{2.8}$$

$$\mathcal{G}(\Psi_1,\Psi_2)=-\mathcal{P}\begin{bmatrix}\nabla_{u_1}u_2+w_1\frac{\partial u_2}{\partial z}-\mathrm{Q}\frac{\mathfrak{p}_1}{\mathfrak{p}_2}\left(\nabla_{H_1}H_2+M_1\frac{\partial H_2}{\partial z}\right)\\\nabla_{u_1}w_2+w_1\frac{\partial w_2}{\partial z}-\mathrm{Q}\frac{\mathfrak{p}_1}{\mathfrak{p}_2}\left(\nabla_{H_1}M_2+M_1\frac{\partial M_2}{\partial z}\right)\\\nabla_{u_1}T_2+w_1\frac{\partial T_2}{\partial z}\\\nabla_{u_1}H_2+w_1\frac{\partial H_2}{\partial z}-\nabla_{H_1}u_2-M_1\frac{\partial u_2}{\partial z}\\\nabla_{u_1}M_2+w_1\frac{\partial M_2}{\partial z}-\nabla_{H_1}w_2-M_1\frac{\partial w_2}{\partial z}\end{bmatrix}.\tag{2.9}$$

Here $\mathcal{P}:L^2(\Omega)^7\to\mathcal{H}$ is the usual Leray projection onto divergence-free vector fields. For convenience, we denote

$$G(\Psi)=\mathcal{G}(\Psi,\Psi),$$

which will be used throughout the text.

3. Main results

3.1. Linear stability. The linear stability analysis of the problem is well studied, see for example the excellent classical treatment in [5]. Here we present the linear stability of the problem in a form which is suitable for the current analysis. The eigenvalue problem of the linearized equations with the boundary conditions (2.5) are

$$\mathfrak{p}_1((\Delta+\partial_{zz})u+\mathrm{Q}H_z-\nabla p)=\beta u, \tag{3.1}$$
$$\mathfrak{p}_1((\Delta+\partial_{zz})w+\mathrm{Q}M_z+\mathrm{R}T-p_z)=\beta w, \tag{3.2}$$
$$(\Delta+\partial_{zz})T+w=\beta T, \tag{3.3}$$
$$\mathfrak{p}_2((\Delta+\partial_{zz})H+u_z)=\beta H, \tag{3.4}$$
$$\mathfrak{p}_2((\Delta+\partial_{zz})M+w_z)=\beta M, \tag{3.5}$$
$$\mathrm{div}\ u+w_z=0, \tag{3.6}$$
$$\mathrm{div} H+M_z=0, \tag{3.7}$$

where β denotes an eigenvalue.

Note that due to zero mean conditions for u and M in (2.6), constant u with all the other variables being zero or constant M with all the other variables being zero are not allowed as eigensolutions. Hence there are no fixed zero eigenvalues for all R.

Case I. For an eigensolution independent of z, we must have $w=T=0$ and $H=0$ due to the boundary conditions. In this case, the Equation (3.5) yields

$$\mathfrak{p}_2\Delta M=\beta M, \tag{3.8}$$

which is the eigenvalue problem for the Laplacian operator on the sphere. If we define

$$\alpha_l^2=\frac{l(l+1)}{r^2}, \qquad l\in\mathbb{N},$$

then (3.8) has an eigenfunction for each $l\in\mathbb{N}$ and $m\in\mathbb{Z}$ such that $|m|\le l$ given by the spherical harmonics $M=Y_{lm}(\theta,\varphi)$ with corresponding eigenvalue $\beta=-\mathfrak{p}_2\alpha_l^2$.

Taking the curl of (3.1) and using the incompressibility condition we have

$$\mathfrak{p}_1\Delta\nabla\times u=\beta\nabla\times u,$$

which has a solution for each $l\in\mathbb{N}$ and $m\in\mathbb{Z}$ with $|m|\le l$ given by

$$u=\mathrm{curl} Y_{lm}\hat{\mathbf{e}}_z=\nabla\times Y_{lm}\hat{\mathbf{e}}_z=\frac{1}{r\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}\hat{e}_\theta-\frac{1}{r}\frac{\partial Y_{lm}}{\partial\theta}\hat{e}_\varphi, \tag{3.9}$$

with corresponding eigenvalue $\beta=-\mathfrak{p}_1\alpha_l^2$.

Case II. For the more general case, we shall use the separation of variables on the spherical shell as

$$\begin{aligned} u&=\nabla f(\theta,\varphi)\chi'(z),\\ w&=\alpha^2 f(\theta,\varphi)\chi(z),\\ T&=f(\theta,\varphi)\Theta(z),\\ H&=\nabla f(\theta,\varphi)\Phi'(z),\\ M&=\alpha^2 f(\theta,\varphi)\Phi(z). \end{aligned} \tag{3.10}$$

Substituting (3.10) into the incompressibility equation (3.6) gives a solution for each $l\in\mathbb{N}$ and $m\in\mathbb{Z}$ with $|m|\le l$

$$f=Y_{lm}(\theta,\varphi), \qquad \alpha^2=\alpha_l^2=\frac{l(l+1)}{r^2}.$$

By taking the gradient of (3.2) and subtracting from the z-derivative of (3.1), we have

$$\mathfrak{p}_1\left((D^2-\alpha^2)^2\chi+\mathrm{Q}\ D(D^2-\alpha^2)\Phi-\mathrm{R}\Theta\right)=\beta(D^2-\alpha^2)\chi. \tag{3.11}$$

In addition to (3.11), the Equations (3.3)–(3.6), with the help of the above separation of variables, become

$$(D^2-\alpha^2)\Theta+\alpha^2\chi=\beta\Theta, \tag{3.12}$$

$$\mathfrak{p}_2\left((D^2-\alpha^2)\Phi+D\ \chi\right)=\beta\Phi, \tag{3.13}$$

where the operator D represents the derivative with respect to z. Meanwhile the corresponding boundary conditions (2.5) become

$$\chi=\Theta=\Phi=D^2\chi=D^2\Phi=0, \qquad \text{at } z=0,1. \tag{3.14}$$

By considering the Equations (3.12), (3.13) and (3.14) we may take

$$\begin{aligned}\chi&=\sin n\pi z,\\ \Theta&=\frac{\alpha_l^2}{n^2\pi^2+\alpha_l^2+\beta_{ln}}\sin n\pi z,\\ \Phi&=\frac{\mathfrak{p}_2 n\pi}{\mathfrak{p}_2(n^2\pi^2+\alpha_l^2)+\beta_{ln}}\cos n\pi z.\end{aligned}$$

Upon plugging the above ansatz into (3.11), we obtain the dispersion relation (3.17) for the eigenvalues.

Now we summarize the results. The set of all eigenfunctions and eigenvalues can be indexed by Ψ_{lmn} and β_{ln} where $l,m,n\in\mathbb{Z}$, $l\geq 0$, $|m|\leq l$, $n\geq 0$ and $(l,n)\neq(0,0)$.

(1) When $l=0$, $n\neq 0$, the eigenpairs are

$$\beta_{0n}=-n^2\pi^2, \quad \Psi_{00n}=(0,0,T=\sin n\pi z,0,0). \tag{3.15}$$

(2) When $l\neq 0$ and $n=0$, the two families of eigenpairs are

$$\begin{aligned}\beta_{l0}^1&=-\mathfrak{p}_1\alpha_l^2, &\quad \Psi_{lm0}^1&=(\nabla\times Y_{lm}\hat{\mathbf{e}}_z,0,0,0,0),\\ \beta_{l0}^2&=-\mathfrak{p}_2\alpha_l^2, &\quad \Psi_{lm0}^1&=(0,0,0,0,Y_{lm}).\end{aligned} \tag{3.16}$$

(3) When $l\neq 0$ and $n\neq 0$, there are three families of eigenvalues which we order as $\Re(\beta_{ln}^1)\geq\Re(\beta_{ln}^2)\geq\Re(\beta_{ln}^3)$ corresponding to the solutions of the following dispersion equation

$$\beta^3+b_0\beta^2+b_1\beta+b_2=0, \tag{3.17}$$

where

$$\begin{aligned}b_0&=\gamma_{l,n}^2(\mathfrak{p}_1+\mathfrak{p}_2+1),\\ b_1&=\mathfrak{p}_1\left((\mathfrak{p}_2+\frac{\mathfrak{p}_2}{\mathfrak{p}_1}+1)\gamma_{l,n}^4+\mathfrak{p}_2\mathrm{Q}n^2\pi^2-\alpha^2\mathrm{R}\gamma_{l,n}^{-2}\right),\\ b_2&=\mathfrak{p}_1\mathfrak{p}_2(\gamma_{l,n}^6+\mathrm{Q}n^2\pi^2\gamma_{l,n}^2-\mathrm{R}\alpha_l^2),\end{aligned}$$

and

$$\gamma_{l,n}^2=n^2\pi^2+\alpha_l^2. \tag{3.18}$$

(4) The eigenfunctions related to the above three eigenvalues, β_{ln}^k, $k=1,2,3$ are

$$\Psi_{lmn}^k = \begin{cases} u = n\pi\nabla Y_{lm}\cos n\pi z, \\ w = \alpha_l^2 Y_{lm}\sin n\pi z, \\ T = \dfrac{\alpha_l^2}{\gamma_{l,n}^2+\beta_{ln}^k} Y_{lm}\sin n\pi z, \\ H = -\dfrac{n^2\pi^2}{\gamma_{l,n}^2+\frac{\beta_{ln}^k}{\mathfrak{p}_2}}\nabla Y_{lm}\sin n\pi z, \\ M = \dfrac{\alpha_l^2 n\pi}{\gamma_{l,n}^2+\frac{\beta_{ln}^k}{\mathfrak{p}_2}} Y_{lm}\cos n\pi z. \end{cases} \tag{3.19}$$

3.1.1. Adjoint problem. For the reduction of the full system to a system of ODE's, we need to study the eigenproblem of the adjoint linear operator L^* which is defined as

$$\langle L\Psi_1, \Psi_2\rangle = \langle \Psi_1, L^*\Psi_2\rangle.$$

The eigenproblem for the adjoint linear operator reads

$$\begin{aligned} &\mathfrak{p}_1\left((\Delta+\partial_{zz})u^* - \nabla p^*\right) - \mathfrak{p}_2 H_z^* = \overline{\beta}u^*, \\ &\mathfrak{p}_1\left((\Delta+\partial_{zz})w^* - p_z^*\right) - \mathfrak{p}_2 M_z^* + T^* = \overline{\beta}w^*, \\ &(\Delta+\partial_{zz})T^* + \mathfrak{p}_1 \mathrm{R} w^* = \overline{\beta}T^*, \\ &\mathfrak{p}_2(\Delta+\partial_{zz})H^* - \mathfrak{p}_1 \mathrm{Q} u_z^* = \overline{\beta}H^*, \\ &\mathfrak{p}_2(\Delta+\partial_{zz})M^* - \mathfrak{p}_1 \mathrm{Q} w_z^* = \overline{\beta}M^*, \\ &\mathrm{div} u^* + w_z^* = 0, \\ &\mathrm{div} H^* + M_z = 0. \end{aligned} \tag{3.20}$$

We only need to determine the adjoint eigenvector for the case $l\geq 1$ and $n\geq 1$. Following a similar analysis as for the determination of the eigenvectors for the linear operator, we obtain for $l\geq 1$, $n\geq 1$, $|m|\leq l$, $k=1,2,3$,

$$\Psi_{lmn}^{*k} = \begin{cases} u^* = n\pi\nabla Y_{lm}\cos n\pi z, \\ w^* = \alpha_l^2 Y_{lm}\sin n\pi z, \\ T^* = \dfrac{\mathfrak{p}_1 \mathrm{R}\alpha_l^2}{\gamma_{l,n}^2+\overline{\beta}_{ln}^k} Y_{lm}\sin n\pi z, \\ H^* = \dfrac{\mathfrak{p}_1 n^2\pi^2 \mathrm{Q}}{\mathfrak{p}_2\gamma_{l,n}^2+\overline{\beta_{ln}^k}}\nabla Y_{lm}\sin n\pi z, \\ M^* = -\dfrac{\mathfrak{p}_1 n\pi\alpha_l^2 \mathrm{Q}}{\mathfrak{p}_2\gamma_{l,n}^2+\overline{\beta_{ln}^k}} Y_{lm}\cos n\pi z. \end{cases} \tag{3.21}$$

3.1.2. Principle of exchange of stabilities(PES). Now we present the principle of exchange of stabilities which is a conclusion of the linear stability analysis carried out in the previous section. It is evident that the modes without vertical structure or modes without horizontal structure are always stable by (3.15) and (3.16). By analyzing (3.17), one can easily obtain the critical Rayleigh numbers for the onset of steady and oscillatory transitions. We present the results in a slightly different form and provide a short proof of PES condition for completeness, see also [23].

Now let us define

$$\mathrm{R}(l)=\frac{\gamma_{l,1}^2}{\alpha_l^2}(\gamma_{l,1}^4+\mathrm{Q}\pi^2), \tag{3.22}$$

$$\mathrm{R}_c=\min_{l\in\mathbb{N}}\mathrm{R}(l)=\mathrm{R}(l_c)=\frac{\gamma_{l_c,1}^2}{\alpha_{l_c}^2}(\gamma_{l_c,1}^4+\mathrm{Q}\pi^2), \tag{3.23}$$

$$\widetilde{\mathrm{R}}(l)=\frac{(\mathfrak{p}_2+1)(\mathfrak{p}_1+\mathfrak{p}_2)}{\mathfrak{p}_1}\frac{\gamma_{l,1}^2}{\alpha_l^2}[\gamma_{l,1}^4+\frac{\mathfrak{p}_2\mathfrak{p}_1}{(\mathfrak{p}_2+1)(\mathfrak{p}_1+1)}\mathrm{Q}\pi^2], \tag{3.24}$$

$$\widetilde{\mathrm{R}}_c=\widetilde{\mathrm{R}}(\tilde{l}_c)=\frac{(\mathfrak{p}_2+1)(\mathfrak{p}_1+\mathfrak{p}_2)}{\mathfrak{p}_1}\frac{\gamma_{\tilde{l}_c,1}^2}{\alpha_{\tilde{l}_c}^2}[\gamma_{\tilde{l}_c,1}^4+\frac{\mathfrak{p}_2\mathfrak{p}_1}{(\mathfrak{p}_2+1)(\mathfrak{p}_1+1)}\mathrm{Q}\pi^2], \tag{3.25}$$

$$\mathrm{Q}_0(l)=\gamma_{l,1}^4\frac{\mathfrak{p}_2(\mathfrak{p}_1+1)}{\pi^2\mathfrak{p}_1(1-\mathfrak{p}_2)}. \tag{3.26}$$

Theorem 3.1. *The following assertions hold true.*

a) If $\mathfrak{p}_2\geq 1$ or $Q<Q_0(\tilde{l}_c)$ then R_c is the first critical Rayleigh number. That is, there is a finite set of critical indices minimizing (3.23) *such that the condition*

$$\begin{aligned} &\beta_{l_c1}^1(R)\begin{cases}<0, & R<R_c,\\ =0, & R=R_c,\\ >0, & R>R_c,\end{cases}\\ &\Re\beta_{ln}(R_c)<0, \quad l\neq l_c,\ n\neq 1,\end{aligned} \tag{3.27}$$

holds true generically in the parameter space.

b) If $\mathfrak{p}_2<1$ and $Q>Q_0(l_c)$ then $\widetilde{R}_c$ is the first critical Rayleigh number. That is, there is a finite set of critical indices minimizing (3.25) *such that the condition*

$$\begin{aligned} &\Re\beta_{\tilde{l}_c1}^1(R)=\Re\beta_{\tilde{l}_c1}^2(R)\begin{cases}<0, & R<\widetilde{R}_c,\\ =0, & R=\widetilde{R}_c,\\ >0, & R>\widetilde{R}_c,\end{cases}\\ &\Re\beta_{ln}(\widetilde{R}_c)<0, \quad l\neq\tilde{l}_c,\ n\neq 1,\end{aligned} \tag{3.28}$$

holds true generically in the parameter space.

Proof. Solving $b_2=0$ for R in (3.17) we find that (3.17) has 0 as a root at $\mathrm{R}=\mathrm{R}(l)$. Thus the real eigenvalues indexed by $l\geq 1$, $n=1$ cross the imaginary axis as R crosses $\mathrm{R}(l)$. Similarly, solving $\frac{b_2}{b_0}=b_1$ for R in (3.17) so that (3.17) has $\pm i\rho$, $\rho>0$, as roots, we see that the complex eigenvalues indexed by $l\geq 1$, $n=1$ cross the imaginary axis as R crosses $\widetilde{R}(l)$. By taking the minimum of $\mathrm{R}(l)$ and $\widetilde{\mathrm{R}}(l)$ over $l\geq 1$, the critical Rayleigh numbers R_c and $\widetilde{\mathrm{R}}_c$ are obtained.

When $\mathfrak{p}_2=1$, $\mathrm{R}(l)<\widetilde{\mathrm{R}}(l)$ for all $l\geq 1$, and as a result $\mathrm{R}_c<\widetilde{\mathrm{R}}_c$. For $\mathfrak{p}_2\neq 1$, by comparing $\mathrm{R}(l)$ and $\widetilde{\mathrm{R}}(l)$, a simple algebraic manipulation shows that

$$\mathrm{R}(l)\begin{cases}<\widetilde{\mathrm{R}}(l), & \mathrm{Q}<\mathrm{Q}_0(l),\\ =\widetilde{\mathrm{R}}(l), & \mathrm{Q}=\mathrm{Q}_0(l),\\ >\widetilde{\mathrm{R}}(l), & \mathrm{Q}>\mathrm{Q}_0(l).\end{cases}$$

By (3.23) and (3.25), it is clear that if $\mathfrak{p}_2>1$ then $\mathrm{Q}_0(l)<0$ and $\mathrm{R}(l)\leq\widetilde{\mathrm{R}}(l)$ for all $l\geq 1$ which implies that $\mathrm{R}_c<\widetilde{\mathrm{R}}_c$. When $\mathfrak{p}_2<1$ and $\mathrm{Q}<\mathrm{Q}_0(\tilde{l}_c)$, we have $\mathrm{R}_c\leq\mathrm{R}(\tilde{l}_c)<\widetilde{\mathrm{R}}(\tilde{l}_c)=\widetilde{\mathrm{R}}_c$ and R_c is the first critical Rayleigh number. Similarly, if $\mathrm{Q}>\mathrm{Q}_0(l_c)$, then $\mathrm{R}_c=\mathrm{R}(l_c)>\widetilde{\mathrm{R}}(l_c)\geq\widetilde{R}_c$ and $\widetilde{\mathrm{R}}_c$ is the first critical Rayleigh number. That is

$$\mathrm{Q}>\mathrm{Q}_0(l_c)\Longrightarrow\widetilde{\mathrm{R}}_c<\mathrm{R}_c. \tag{3.29}$$

□

REMARK 3.1. When $\mathfrak{p}_2<1$, there must exist a critical Chandrasekhar number Q such that the two neutral stability curves for R_c and $\widetilde{\mathrm{R}}_c$ intersect giving rise to a transition from a set of eigenvalues some of which are complex and the rest are real. This type of transition is beyond the scope of the current study.

3.2. Transition theorem. By Theorem 3.1, we know that the system must undergo a transition as the Rayleigh number exceeds the critical Rayleigh number given by

$$\min\left\{R_c,\widetilde{R}_c\right\}.$$

For this paper, we restrict ourselves to the study of the transitions under the PES condition (3.27), i.e. when the eigenvalues which first cross the imaginary axis are real, which is equivalent to the condition

$$R_c=\min\left\{R_c,\widetilde{R}_c\right\}. \tag{3.30}$$

Theorem 3.1 gives sufficient conditions for the condition (3.30) to hold. Namely, when $\mathfrak{p}_2\geq 1$ or $Q<Q_0(\tilde{l}_c)$, the condition (3.30) holds.

Second, the condition (3.27) holds only generically in the parameter space when it indeed holds. That is, for parameters in a set of measure zero in the parameter space, it is possible that $\beta^1_{l_c,1}=\beta^1_{l_c+1,1}$ giving rise to a transition from $4l_c+4$ eigenvalues. This nongeneric case is very interesting by its implications for the pattern formations of the system, however the reduction in this case is more difficult. We will only deal with the generic case in this work.

To describe the transition which takes place at $R=R_c$, we need to reduce the full PDE system to a system of ODE's by the center manifold reduction and then analyze this ODE system. The details of the procedure can be found in [11]. For this purpose let us define the $2l_c+1$ dimensional eigenspace

$$E_1=\operatorname{span}\left\{\sum_{m=-l_c}^{l_c}x_m\Psi^1_{l_cm1}\,|\,x_{-m}=(-1)^m\overline{x_m},x_m\in\mathbb{C}\right\}\subset\mathcal{H},$$

spanned by the first critical eigenvectors with corresponding eigenvalues $\beta^1_{l_c,1}$ satisfying (3.27). Note that E_1 consists of real valued functions since by the symmetry $\overline{Y}_{lm}=(-1)^mY_{l-m}$ of the spherical harmonics, it follows that $\overline{\Psi}^1_{l_cm1}=(-1)^m\Psi^1_{l_cm1}$.

We show that the reduction of the main equations onto the critical subspace E_1 is given by the following $2l_c+1$ equations. For each $m\in\mathbb{Z}$, $-l_c\leq m\leq l_c$,

$$\frac{dx_m}{dt}=\beta^1_{l_c1}\,x_m+\omega_{l_c}x_m|x|^2+\sum_{l,m_1,p_1}\xi_{l,l_c}c^{m_1,m_2,m}_{l_c,l,l_c}c^{p_1,p_2,m_2}_{l_c,l_c,l}x_{m_1}x_{p_1}x_{p_2}, \tag{3.31}$$

where $|x|^2=\sum_{j=-l_c}^{l_c}|x_j|^2$, $m_2=m-m_1$, $p_2=m_2-p_1$ and the sum is over $l\in\{2,4,\dots,2l_c\}$, $|m_1|\le l_c$, and $|p_1|\le l_c$. Here the tripling coefficients of the spherical harmonics are defined by

$$c_{l_1,l_2,l}^{m_1,m_2,m}=\frac{1}{r^2}\int_{S_r^2}Y_{l_1m_1}Y_{l_2m_2}\overline{Y_{lm}}dS^2.$$

The coefficients w_{l_c} and ξ_{l,l_c} depend on the system parameters whose explicit expressions are given later by (3.67) and (3.68).

THEOREM 3.2. *Suppose that R_c is the first critical Rayleigh number and the generic condition (3.27) is satisfied. In this case, the stability and transitions of the basic steady state solution (2.2) near $R=R_c$ is governed by a system of $2l_c+1$ ODE's given by (3.31). Particularly, for $l_c=1$ and $l_c=2$ the sum in (3.31) reduces significantly since*

$$\sum_{l,m_1,p_1}\xi_{l,l_c}c_{l_c,l,l_c}^{m_1,m_2,m}c_{l_c,l_c,l}^{p_1,p_2,m_2}x_{m_1}x_{p_1}x_{p_2}=\begin{cases}\dfrac{\xi_{2,1}|x|^2}{5\pi}, & l_c=1,\\ \dfrac{5\xi_{2,2}+9\xi_{4,2}}{49\pi}|x|^2, & l_c=2.\end{cases}$$

Thus when $l_c=1$, the reduced equations read

$$\frac{dx_m}{dt}=\beta_{11}^1\ x_m+q_1x_m|x|^2+o(3),\qquad m=-1,0,1, \tag{3.32}$$

with

$$q_1=\omega_1+\frac{\xi_{2,1}}{5\pi}. \tag{3.33}$$

When $l_c=2$, the reduced equations read

$$\frac{dx_m}{dt}=\beta_{21}^1\ x_m+q_2x_m|x|^2+o(3),\qquad m=-2,-1,0,1,2, \tag{3.34}$$

with

$$q_2=\omega_2+\frac{5\xi_{2,2}+9\xi_{4,2}}{49\pi}. \tag{3.35}$$

Proof. The proof of the theorem is given in Section 3.3. ∎

From Theorem 3.2, we see that in the cases of $l_c=1$ and $l_c=2$, the transition is described by a single dimensionless number, q_1 and q_2. For $l_c\ge 3$, mixed cubic terms appear in (3.31) which complicates the analysis. We plan to address this case elsewhere. The simplicity of the reduced Equations (3.32) and (3.34) allows us to state the following theorem which governs the first transition of the system.

THEOREM 3.3. *Suppose that R_c is the first critical Rayleigh number and the generic condition (3.27) is satisfied. Assume further that $l_c=1$ or $l_c=2$. Then the following statements hold true.*

(1) When $R<R_c$, the basic state (2.2) is locally asymptotically stable.

(2) If $q_{l_c}<0$ then there is a continuous transition at $R=R_c$ and a bifurcated attractor Σ_R, which is homeomorphic to the $2l_c$ dimensional sphere, bifurcates on $R>R_c$. Moreover, Σ_R has the following approximation

$$\Sigma_R=\left\{\sum_{m=-l_c}^{l_c}x_m\Psi_{l_cm1}^1\,|\,x_{-m}=(-1)^m x_m,\ \sum_{m=-l_c}^{l_c}|x_m|^2=-\frac{\beta_{l_c,1}^1}{q_{l_c}}\right\}+o(\beta_{l_c,1}^1).$$

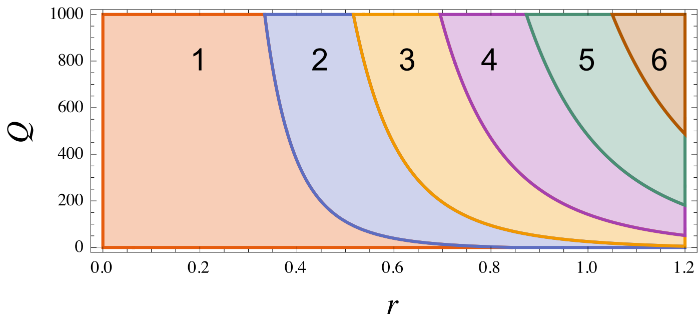

FIG. 3.1. *The regions of the critical wave index l_c as a function of the aspect ratio r and the Chandrasekhar number Q.*

The attractor Σ_R attracts $\mathcal{H}\setminus\Gamma$, where Γ is the stable manifold of $\phi=0$ with codimension $2l_c+1$.

(3) *When $l_c=1$, the S^2-attractor consists precisely of degenerate steady states. When $l_c=2$, the S^4-attractor contains at least a S^2 subset of degenerate steady states.*

(4) *If $q_{l_c}>0$ then there is a drastic transition on $R>R_c$. In particular, a repeller Σ_R bifurcates on $R<R_c$ as given above and there exists an open and dense neighborhood U of $\Psi=0$ in $\mathcal{H}$ such that for any initial condition $\Psi_0\in U$ and for every $R_c<R<R_c+\epsilon$ with some $\epsilon>0$, the solution $\Psi_R(t,\Psi_0)$ of (2.7) satisfies*

$$\limsup_{t\to\infty}\|\Psi_R(t,\Psi_0)\|\geq\delta>0.$$

for some δ which is independent of R.

Proof. The proof is a modification of the one in [24] and is included only for completeness. By the linear analysis in the previous subsection, apparently, the assertion (1) is true. By the attractor bifurcation theorem in [11], to show (2) and (4), we only need to show that $\Psi=0$ is stable (unstable) at $\mathrm{R}=\mathrm{R}_c$. From Theorem 3.2, one can easily see that the stability (instability) of $\Psi=(u,w,T,H,M)=0$ at $\mathrm{R}=\mathrm{R}_c$ is equivalent to $q_{l_c}<0$ ($q_{l_c}>0$). That is, the Assertions (2) and (4) hold true. To show (3), let us denote $L(\Psi)=A(\Psi)+B_{\mathrm{R}}(\Psi)$, where

$$A(\Psi)=\mathcal{P}\begin{bmatrix}\mathfrak{p}_1(\Delta+\frac{\partial^2}{\partial z^2})u\\ \mathfrak{p}_1(\Delta+\frac{\partial^2}{\partial z^2})w\\ (\Delta+\frac{\partial^2}{\partial z^2})T\\ \mathfrak{p}_2(\Delta+\frac{\partial^2}{\partial z^2})H\\ \mathfrak{p}_2(\Delta+\frac{\partial^2}{\partial z^2})M\end{bmatrix},\quad B_{\mathrm{R}}(\Psi)=\mathcal{P}\begin{bmatrix}\mathfrak{p}_1\mathrm{Q}\frac{\partial H}{\partial z}\\ \mathfrak{p}_1\mathrm{R}T+\mathfrak{p}_1\mathrm{Q}\frac{\partial M}{\partial z}\\ w\\ \mathfrak{p}_2\frac{\partial u}{\partial z}\\ \mathfrak{p}_2\frac{\partial w}{\partial z}\end{bmatrix}. \tag{3.36}$$

Apparently, the operator A is invertible. Then, the steady problem

$$L\Psi=-G(\Psi),$$

can be rewritten as

$$\left(id+A^{-1}B_{\mathrm{R}}\right)\Psi=-A^{-1}G(\Psi). \tag{3.37}$$

One can easily check that the operator $id + A^{-1}B_{\mathrm{R}} : \mathcal{H} \to \mathcal{H}$ is a completely continuous field, and the operator $A^{-1}G : \mathcal{H} \to \mathcal{H}$ is compact. Hence by the Krasnoselski bifurcation theorem (see e.g. Theorem 1.10 in [29]), (3.37) has a nontrivial steady state bifurcation at R_c, as the critical eigenvalue has odd multiplicity $2l_c + 1$, which is the dimension of the eigenspace. Due to the spherical symmetry of the governing equations, this steady state will generate a S^2-set of steady states. The degeneracy of this steady state is evident from the reduced Equation (3.32) and (3.34). The proof is complete. ∎

3.3. Proof of the Theorem 3.2.

3.3.1. Approximation of the center manifold. Let Ψ_c be the projection of the solution onto E_1. Hence we have

$$\Psi_c = \sum_{m=-l_c}^{l_c} x_m(t) \Psi^1_{l_c m 1} \in E_1.$$

Since the first eigenvalues which cross the imaginary axis are real, the center manifold function can be approximated by the below equation, see [11].

$$-\mathcal{L}\Phi(\Psi_c) = P_2 G(\Psi_c) + o(2). \tag{3.38}$$

Here

$$P_2 : \mathcal{H} \to E_2 := \{\Psi \in \mathcal{H}_1 \mid \langle \Psi, \Psi^{1*}_{l_c,m,1} \rangle = 0 \text{ for all } |m| \le l_c\},$$

is the canonical projection, $\mathcal{L} = L|_{E_2}$, L is the linear operator and G is the nonlinear operator in (2.7) and

$$o(2) = o\left(|x|^2\right) + O(|\beta^1_{l_c 1}(\mathrm{R})||x|^2), \quad \text{as } \mathrm{R} \to \mathrm{R}_c, |x| \to 0,$$

where $|x|^2 = \sum_{m=-l_c}^{l_c} |x_m|^2$.

In order to resolve (3.38) we will use the natural basis of our phase space to expand the center manifold. This is accomplished by considering the eigenvectors of the following system of linear operator with the same boundary conditions as in (2.5).

$$\begin{aligned}
&(\Delta + \partial_{zz})u - \nabla p = \beta u,\\
&(\Delta + \partial_{zz})w - p_z = \beta w,\\
&(\Delta + \partial_{zz})T = \beta T,\\
&(\Delta + \partial_{zz})H = \beta H,\\
&(\Delta + \partial_{zz})M = \beta M,\\
&\mathrm{div}\ u + w_z = 0,\\
&\mathrm{div} H + M_z = 0.
\end{aligned} \tag{3.39}$$

The eigenvectors of the above equation constitute a basis for the phase space. It is not difficult to see that the eigenvectors of (3.39) are as follows.

$$\begin{aligned}
&e_{00n} = (0, 0, \sin n\pi z, 0, 0),\\
&e^1_{lm0} = (\nabla \times Y_{lm}\hat{\mathbf{e}}_z, 0, 0, 0, 0),\\
&e^2_{lm0} = (0, 0, 0, 0, Y_{lm})\\
&e^1_{lmn} = (n\pi \nabla Y_{lm} \cos n\pi z, \alpha_l^2 Y_{lm} \sin n\pi z, 0, 0, 0),\\
&e^2_{lmn} = (0, 0, Y_{lm} \sin n\pi z, 0, 0),\\
&e^3_{lmn} = (0, 0, 0, -n\pi \nabla Y_{lm} \sin n\pi z, \alpha_l^2 Y_{lm} \cos n\pi z),
\end{aligned} \tag{3.40}$$

where $l\in\mathbb{N}$, $|m|\leq l$, $m\in\mathbb{N}$, $n\in\mathbb{N}$, and $\nabla\times Y_{lm}\hat{\mathbf{e}}_z$ is given in (3.9). The above basis is an alternative to the eigenbasis of the linear operator (2.8) of the original system. The main advantage of using the above basis is that it avoids having a very complicated dependence on the system parameters that the eigenbasis of (3.1)–(3.7) has, since the coefficients in (3.19) depend on the eigenvalues β which are determined as the roots of cubic polynomial Equation (3.17).

Using the above basis and in virtue of (3.38), the center manifold function can be expanded by modes with $n=0,2$ thanks to the orthogonality of the trigonometric functions and $l=2,4,\dots,2l_c$ thanks to the orthogonality of the spherical harmonics, see (5.7). Thus we can write

$$\Phi=y_{002}e_{002}+\sum_{\substack{l,m\\k=1,2}} y^k_{lm0}e^k_{lm0}+\sum_{\substack{l,m\\k=1,2,3}} y^k_{lm2}e^k_{lm2}+o(2), \tag{3.41}$$

where the sums run over $l\in\{2,4,\dots,2l_c\}$ and $|m|\leq l$. The coefficients of the center manifold can be evaluated as

$$\begin{aligned}
y_{002}&=-\frac{\langle G(\Psi_c),e_{002}\rangle}{\langle e_{002},L^*e_{002}\rangle},\\
y^k_{lm0}&=-\frac{\langle G(\Psi_c),e^k_{lm0}\rangle}{\langle e^k_{lm0},L^*e^k_{lm0}\rangle}, \qquad k=1,2,\\
y^k_{lm2}&=-\frac{\langle G(\Psi_c),e^k_{lm2}\rangle}{\langle e^k_{lm2},L^*e^k_{lm2}\rangle}, \qquad k=1,2,3.
\end{aligned} \tag{3.42}$$

Recalling that L^* is the adjoint linear operator which is given by the left-hand side of (3.20), the terms in the denominator are easily found as

$$L^*e_{002}=-4\pi^2e_{002}, \qquad L^*e^1_{lm0}=-\mathfrak{p}_1\alpha_l^2e^1_{lm0} \qquad L^*e^2_{lm0}=-\mathfrak{p}_2\alpha_l^2e^2_{lm0},$$

and

$$\begin{aligned}
L^*e^1_{lmn}&=\left\{-\mathfrak{p}_1\gamma^2_{l,n}n\pi\nabla Y_{lm}\cos n\pi z,-\mathfrak{p}_1\gamma^2_{l,n}\alpha_l^2Y_{lm}\sin n\pi z,0,0,0\right\},\\
L^*e^2_{lmn}&=\left\{0,*,-\gamma^2_{l,n}Y_{lm}\sin n\pi z,0,0\right\},\\
L^*e^3_{lmn}&=\left\{*,*,0,\mathfrak{p}_2\gamma^2_{l,n}n\pi\nabla Y_{lm}\sin n\pi z,-\mathfrak{p}_2\gamma^2_{l,n}\alpha_l^2Y_{lm}\cos n\pi z\right\},
\end{aligned}$$

where $*$ denotes a nonzero term which will not enter into the calculations and hence is omitted.

In what follows, we will use the below notation

$$\int_\Omega f d\Omega=\int_{z=0}^1\int_{S_r^2} f dS^2 dz=\int_{z=0}^1\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} fr^2\sin\theta d\theta d\varphi dz.$$

By (3.19), the critical eigenvectors are

$$\psi^1_{l_cm}=\left\{\hat{u}_c\nabla Y_{l_cm}\cos\pi z,\hat{w}_cY_{lm}\sin\pi z,\hat{T}_cY_{l_cm}\sin\pi z,\hat{H}_c\nabla Y_{l_cm}\sin\pi z,\hat{M}_cY_{lm}\cos\pi z\right\},$$

where

$$\hat{u}_c=\pi, \quad \hat{w}_c=\alpha^2_{l_c}, \quad \hat{T}_c=\alpha^2_{l_c}/\gamma^2_{l_c,1}, \quad \hat{H}_c=-\pi^2/\gamma^2_{l_c,1} \quad \hat{M}_c=\alpha^2_{l_c}\pi/\gamma^2_{l_c,1}. \tag{3.43}$$

3.3.2. Computation of y_{002}. We start with the computation of y_{002}.

$$\langle G(\Psi_c), e_{002}\rangle = -\int_\Omega (u_c\cdot\nabla T_c + w_c\frac{\partial T_c}{\partial z})\sin 2\pi z d\Omega$$
$$= -\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}\hat{u}_c\hat{T}_c\int_0^1 \cos\pi z\sin\pi z\sin 2\pi z dz\int_{S_r^2}\nabla Y_{l_c,m_1}\cdot\nabla Y_{l_c,m_2} dS^2$$
$$-\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}\hat{w}_c\hat{T}_c\pi\int_0^1 \sin\pi z\cos\pi z\sin 2\pi z dz\int_{S_r^2} Y_{l_c,m_1}Y_{l_c,m_2} dS^2. \tag{3.44}$$

Using

$$\sum_{m=-l_c}^{l_c}(-1)^m x_m x_{-m} = \sum_{m=-l_c}^{l_c} x_m\overline{x_m} = |x|^2,$$

and (5.4), (5.5) in (3.44) gives

$$\langle G(\Psi_c), e_{002}\rangle = -\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}(-1)^{m_2}\delta_{m_1,-m_2} r^2\frac{1}{4}\left(\hat{u}_c\hat{T}_c\alpha_{l_c}^2 + \hat{w}_c\hat{T}_c\right)$$
$$= -r^2\frac{1}{4}\left(\hat{u}_c\hat{T}_c\alpha_{l_c}^2 + \pi\hat{w}_c\hat{T}_c\right)|x|^2. \tag{3.45}$$

We also have

$$\langle e_{002}, L^* e_{002}\rangle = -4\pi^2\int_0^1 \sin^2 2\pi z dz\int_{S_r^2} dS^2 = -8r^2\pi^3. \tag{3.46}$$

Combining (3.45) and (3.46), we obtain y_{002} as in (3.52).

3.3.3. Computation of y^1_{lm0}. We will show that the higher frequency 2D horizontal velocity field modes e^1_{lm0} have no affect on the transition. To see this, we will show that

$$\langle G(\Psi_c), e^1_{lm0}\rangle = 0,$$

which in turn implies that $y^1_{lm0} = 0$ by (3.42). For this, we first expand

$$\langle G(\Psi_c), e^1_{lm0}\rangle = -\frac{1}{2}\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}$$
$$\times\left(\left(\hat{u}_c\hat{u}_c - \frac{\mathrm{Q}\mathfrak{p}_1}{\mathfrak{p}_2}\hat{H}_c\hat{H}_c\right)g^{m_1m_2m}_{l_cl_cl} - \pi\left(\hat{w}_c\hat{u}_c + \frac{\mathrm{Q}\mathfrak{p}_1}{\mathfrak{p}_2}\hat{M}_c\hat{H}_c\right)f^{m_1m_2m}_{l_cl_cl}\right),$$

where the tensors $g^{m_1m_2m}_{l_cl_cl}$ and $f^{m_1m_2m}_{l_cl_cl}$ are defined in (5.12). By the anti-symmetry property (5.14), we have

$$\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}f^{m_1m_2m}_{l_cl_cl} = \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}\frac{1}{2}\left(f^{m_1m_2m}_{l_cl_cl} + f^{m_2m_1m}_{l_cl_cl}\right) = 0.$$

Similar argument holds for

$$\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}g^{m_1m_2m}_{l_cl_cl} = 0.$$

3.3.4. Computation of y^2_{lm0}. A straightforward computation shows that

$$
\begin{aligned}
\langle G(\Psi_c), e^2_{lm0}\rangle &= -\int_\Omega \left(\nabla_{u_c} M_c + w_c \frac{\partial M_c}{\partial z} - \nabla_{H_c} w_c - M_c \frac{\partial w_c}{\partial z}\right)\overline{Y}_{lm} \\
&= -\sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} \left(\frac{\hat{u}_c \hat{M}_c - \hat{H}_c \hat{w}_c}{2} b^{m_1,m_2,m}_{l_c,l_c,l} - \pi \hat{w}_c \hat{M}_c c^{m_1,m_2,m}_{l_c,l_c,l}\right) r^2, \qquad (3.47)
\end{aligned}
$$

where $c^{m_1,m_2,m}_{l_c,l_c,l}$ and $b^{m_1,m_2,m}_{l_c,l_c,l}$ are the triple product integral coefficients defined by (5.8) and (5.6).

Also direct computation shows that

$$
\langle e^2_{lm0}, L^* e^2_{lm0}\rangle = -\mathfrak{p}_2 \alpha_l^2 \int_0^1 dz \int_{S^2_r} Y_{lm} \overline{Y}_{lm} dS^2 = -\mathfrak{p}_2 \alpha_l^2 r^2. \qquad (3.48)
$$

Finally combining (3.47), (3.48) and (3.43) we obtain y^2_{lm0} as in (3.54).

3.3.5. Computation of y^1_{lm2}.

$$
\begin{aligned}
&\langle G(\Psi_c), e^1_{lm1}\rangle \\
&= -\int_\Omega \left(u_c \cdot \nabla u_c + w_c \frac{\partial u_c}{\partial z} - \frac{Q\mathfrak{p}_1}{\mathfrak{p}_2}\left(H_c \cdot \nabla H_c + M_c \frac{\partial H_c}{\partial z}\right)\right) 2\pi \nabla \overline{Y}_{lm} \cos 2\pi z \\
&\quad - \int_\Omega \left(u_c \cdot \nabla w_c + w_c \frac{\partial w_c}{\partial z} - \frac{Q\mathfrak{p}_1}{\mathfrak{p}_2}\left(H_c \cdot \nabla M_c + M_c \frac{\partial M_c}{\partial z}\right)\right) \alpha_l^2 \overline{Y}_{lm} \sin 2\pi z \\
&= -\frac{\pi}{2} \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} \left(\hat{u}_c \hat{u}_c e^{m_1 m_2 m}_{l_c l_c l} + \pi \hat{u}_c \hat{w}_c d^{m_1 m_2 m}_{l_c l_c l}\right) r^2 \\
&\quad - \frac{\pi}{2} \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} \frac{Q\mathfrak{p}_1}{\mathfrak{p}_2} \left(\hat{H}_c \hat{H}_c e^{m_1 m_2 m}_{l_c l_c l} - \pi \hat{M}_c \hat{H}_c e^{m_1 m_2 m}_{l_c l_c l}\right) r^2 \\
&\quad - \frac{\alpha_l^2}{4} \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} \left(\hat{u}_c \hat{w}_c b^{m_1 m_2 m}_{l_c l_c l} + \hat{w}_c \hat{w}_c \pi c^{m_1 m_2 m}_{l_c l_c l}\right) r^2 \\
&\quad + \frac{\alpha_l^2}{4} \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} \frac{Q\mathfrak{p}_1}{\mathfrak{p}_2} \left(\hat{H}_c \hat{M}_c b^{m_1 m_2 m}_{l_c l_c l} - \pi \hat{M}_c \hat{M}_c c^{m_1 m_2 m}_{l_c l_c l}\right) r^2 \\
&= \frac{\pi \alpha_l^2 (4\alpha_{l_c}^2 - \alpha_l^2)(\mathfrak{p}_1 \pi^2 Q + \mathfrak{p}_2 \gamma^4_{l_c,1}) r^2}{8 \mathfrak{p}_2 \gamma^2_{l_c,1}} \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} c^{m_1 m_2 m}_{l_c l_c l}, \qquad (3.49)
\end{aligned}
$$

where $b^{m_1 m_2 m}_{l_c l_c l}$, $d^{m_1 m_2 m}_{l_c l_c l}$, $e^{m_1 m_2 m}_{l_c l_c l}$ are as defined in (5.8), (5.9), (5.10).

3.3.6. Computation of y^2_{lm2}. Direct computation yields

$$
\begin{aligned}
\langle G(\Psi_c), e^2_{lm1}\rangle &= -\int_\Omega \left(u_c \cdot \nabla T_c + w_c \frac{\partial T_c}{\partial z}\right) \overline{Y}_{lm} \sin 2\pi z \\
&= -\frac{\pi}{2} \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} \frac{1}{4} \left(\hat{u}_c \hat{T}_c b^{m_1 m_2 m}_{l_c l_c l} + \pi \hat{w}_c \hat{T}_c c^{m_1 m_2 m}_{l_c l_c l}\right) r^2 \\
&= \frac{\pi \alpha^2_{l_c}}{8 \gamma^2_{l_c,1}} (-4\alpha^2_{l_c} + \alpha_l^2) r^2 \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1} x_{m_2} c^{m_1 m_2 m}_{l_c l_c l}, \qquad (3.50)
\end{aligned}
$$

3.3.7. Computation of y^3_{lm2}. We will show that the higher frequency magnetic field modes e^3_{lm2} do not affect the transition.

$$\begin{aligned}
&\langle G(\Psi_c), e^3_{lm2}\rangle\\
&= -\int_\Omega \left(u_c\cdot\nabla H_c + w_c\frac{\partial H_c}{\partial z} - H_c\cdot\nabla u_c - M_c\frac{\partial u_c}{\partial z}\right)(-2\pi)\nabla\overline{Y}_{lm}\sin 2\pi z\\
&\quad -\int_\Omega \left(u_c\cdot\nabla M_c + w_c\frac{\partial M_c}{\partial z} - H_c\cdot\nabla w_c - M_c\frac{\partial w_c}{\partial z}\right)\alpha_l^2\overline{Y}_{lm}\cos 2\pi z\\
&= \sum_{m_1,m_2=-l_c}^{l_c} x_{m_1}x_{m_2}\frac{1}{4}\left(\hat{w}_c\hat{H}_c + \hat{M}_c\hat{u}_c\right)\left(2\pi^2 d^{m_1m_2m}_{l_cl_cl} - \alpha_l^2 b^{m_1m_2m}_{l_cl_cl}\right) = 0, \qquad (3.51)
\end{aligned}$$

since $\hat{w}_c\hat{H}_c + \hat{M}_c\hat{u}_c = 0$ by (3.43).

3.3.8. The summary of the center manifold coefficients. We present below the results of our computations in the previous section.

$$y_{002} = A_{l_c02}|x|^2, \qquad (3.52)$$

$$y^1_{lm0} = 0, \qquad (3.53)$$

$$y^2_{lm0} = A_{l_cl0}\sum_{\substack{m_1,m_2=-l_c,\\ m_1+m_2=m}}^{l_c} c^{m_1,m_2,m}_{l_c,l_c,l}\; x_{m_1}\; x_{m_2}, \qquad (3.54)$$

$$y^k_{lm2} = A^k_{l_cl2}\sum_{\substack{m_1,m_2=-l_c,\\ m_1+m_2=m}}^{l_c} c^{m_1,m_2,m}_{l_c,l_c,l}\; x_{m_1}\; x_{m_2}, \quad k=1,2, \qquad (3.55)$$

$$y^3_{lm2} = 0. \qquad (3.56)$$

Here, the coefficients are obtained as follows:

$$\begin{aligned}
&A_{l_c02} = -\frac{\alpha^4_{l_c}}{16\pi^2\gamma^2_{l_c,1}}, \qquad A_{l_cl0} = \frac{\alpha^2_{l_c}\pi^2}{2\mathfrak{p}_2\gamma^2_{l_c,1}}\\
&A^1_{l_cl2} = \frac{a_{l_cl}b_{l_cl}}{f^1_{l_cl}}, \qquad A^2_{l_cl2} = \frac{a_{l_cl}}{f^2_{l_cl}}, \qquad A^3_{l_cl2} = 0,\\
&a_{l_cl} = \frac{\pi\alpha^2_{l_c}r^2}{4\gamma^2_{l_c,1}}\left(2\alpha^2_{l_c} - \frac{1}{2}\alpha_l^2\right), \quad b_{l_cl} = \frac{\alpha_l^2}{\alpha^2_{l_c}}\left(\frac{\mathfrak{p}_1}{\mathfrak{p}_2}\pi^2\mathrm{Q} + \gamma^4_{l_c,1}\right),\\
&f^1_{l_cl} = \langle e^1_{l_cm2}, L^* e^1_{l_cm2}\rangle = -\frac{\mathfrak{p}_1}{2}\alpha_l^2 r^2\gamma^4_{l,2},\\
&f^2_{l_cl} = \langle e^2_{l_cm2}, L^* e^2_{l_cm2}\rangle = -\frac{1}{2}r^2\gamma^2_{l,2}.
\end{aligned} \qquad (3.57)$$

3.3.9. The Computation of the reduced equations. Now, we let $\Psi = \Psi_c + \Phi$, where $\Psi_c \in E_1$ and Φ is the center manifold function. We plug Ψ in (2.7) and project the resulting equation onto E_1 to obtain

$$\frac{dx_m}{dt} = \beta^1_{l_c1}\; x_m + \frac{\langle G(\Psi_c+\Phi), \Psi^{1*}_{l_cm1}\rangle}{\langle \Psi^1_{l_cm1}, \Psi^{1*}_{l_cm1}\rangle} + o(3). \qquad (3.58)$$

The nonlinear part of the equation represents the mixing of critical modes with the modes spanning the center manifold.

Since the integral of the triple product of any combination of $\cos\pi z$, $\sin\pi z$ over the unit interval is zero, one gets right away that

$$\langle G(\Psi_c,\Psi_c),\Psi^{1*}_{l_cm1}\rangle = 0.$$

This implies the vanishing of any quadratic nonlinearities in (3.58). Also as $\Phi = O(|x|^2)$, one gets $G(\Phi,\Phi) = O(|x|^4)$ and hence

$$\begin{aligned}\langle G(\Psi_c+\Phi),\Psi^{1*}_{l_cm1}\rangle &= \langle G(\Psi_c,\Psi_c),\Psi^{1*}_{l_cm1}\rangle + \langle G(\Psi_c,\Phi),\Psi^{1*}_{l_cm1}\rangle + \langle G(\Phi,\Psi_c),\Psi^{1*}_{l_cm1}\rangle\\ &= \langle G_s(\Phi,\Psi_c),\Psi^{1*}_{l_cm1}\rangle + O(|x|^4),\end{aligned}$$

where $G_s(\Psi_1,\Psi_2) = G(\Psi_1,\Psi_2) + G(\Psi_2,\Psi_1)$. The reduced equations become

$$\frac{dx_m}{dt} = \beta^1_{l_c1}\, x_m + \frac{\langle G_s(\Psi_c,\Phi),\Psi^{1*}_{l_cm1}\rangle}{\langle \Psi^1_{l_cm1},\Psi^{1*}_{l_cm1}\rangle} + o(3). \tag{3.59}$$

The denominator $\langle \Psi^1_{l_cm1},\Psi^{1*}_{l_cm1}\rangle$ of the nonlinear term in (3.59) depends only on l_c and it can be written for arbitrary l_c explicitly as

$$g_{l_c} = \langle \Psi^1_{l_cm1},\Psi^{1*}_{l_cm1}\rangle = \frac{1}{2}\alpha^2_{l_c}r^2\left[\frac{\mathfrak{p}_1 \mathrm{R}\alpha^2_{l_c}}{\gamma^4_{l_c,1}} + \gamma^2_{l_c,1}\left(1-\frac{\mathfrak{p}_1\mathrm{Q}\pi^2}{\mathfrak{p}_2\gamma^4_{l_c,1}}\right)\right], \tag{3.60}$$

where $\gamma^2_{l_c,1} = \pi^2 + \alpha^2_{l_c}$ as we have defined before.

The numerator of the nonlinear term in (3.59) can be dealt with as follows.

$$\begin{aligned}
&\langle G_s(\Psi_c,\Phi),\Psi^{1*}_{l_cm1}\rangle = \sum_{m_1} x_{m_1}\langle G_s(\Psi^1_{l_cm_11},\Phi),\Psi^*_{l_c,m,1}\rangle\\
&= \sum_{m_1} x_{m_1}\left\langle G_s\left(\Psi^1_{l_cm_11}, y_{002}e_{002} + \sum_{m_2} y^2_{lm_20}e^2_{lm_20} + \sum_{m_2,k} y^k_{lm_22}e^k_{lm_22}\right),\Psi^*_{l_c,m,1}\right\rangle\\
&= \sum_{m_1} x_{m_1}y_{002}\langle G_s(\Psi^1_{l_cm_11},e_{002}),\Psi^*_{l_c,m,1}\rangle\\
&\quad + \sum_{m_1,m_2} x_{m_1}y^2_{lm_20}\langle G_s(\Psi^1_{l_cm_11},e^2_{lm_20}),\Psi^*_{l_c,m,1}\rangle\\
&\quad + \sum_{l,m_1,m_2,k} x_{m_1}y^k_{lm_22}\langle G_s(\Psi^1_{l_cm_11},e^k_{l,m_2,2}),\Psi^*_{l_c,m,1}\rangle\\
&:= P_{0m2}(x) + \sum_{l=1}^{2l_c}(P_{lm0}(x)+P_{lm2}(x)), \qquad |m|\le l_c.
\end{aligned}$$

The computations of the above terms are similar to the computations of the center manifold coefficients which were previously carried out. For example, the first term above can be computed as shown below.

$$\begin{aligned}P_{0m2}(x) &= \sum_{m_1} x_{m_1}\langle G_s(\Psi^1_{l_cm_11},e_{002}),\Psi^*_{l_c,m,1}\rangle\\ &= -\sum_{m_1} x_{m_1}\int_\Omega w^1_{l_cm_11}2\pi\cos 2\pi z\overline{T^*_{l_c,m,1}}\end{aligned}$$

$$
\begin{aligned}
&= -2\pi \hat{w}_c \hat{T}_c^* \sum_{m_1} x_{m_1} \int_\Omega Y_{l_c m_1} \sin\pi z \cos 2\pi z \overline{Y_{l_c m}} \sin\pi z \\
&= \frac{\hat{w}_c \hat{T}_c^* \pi}{2} r^2 \sum_{m_1} x_{m_1} \delta_{m_1,m} \\
&= \frac{\hat{w}_c \hat{T}_c^* \pi}{2} x_m r^2, \qquad (3.61)
\end{aligned}
$$

where $\hat{T}_c^* = \frac{\mathfrak{p}_1 \mathrm{R} \alpha_{l_c}^2}{\gamma_{l_c,1}^2}$ as given in (3.21).

With the above notations, the reduced Equations (3.59) become

$$
\frac{dx_m}{dt} = \beta_{l_c 1}^1 \; x_m + \frac{1}{g_{l_c}} \left(P_{0m2}(x) + \sum_{l=1}^{2l_c} P_{lm0}(x) + P_{lm2}(x) \right), \qquad |m| \le l_c, \tag{3.62}
$$

where $P_{0m2}(x)$, $P_{lm0}(x)$, $P_{lm2}(x)$ represent cubic polynomials in x_j and are related to the center manifold coefficients y_{002}, y_{lm0}^2, y_{lm2}^k, $k=1,2$, respectively. We now give their explicit expressions.

$$
P_{0m2}(x) = S_{l_c}^1 x_m y_{002}, \tag{3.63}
$$

$$
P_{lm0}(x) := S_{l_c,l}^2 \sum_{m_1} \sum_{m_2} c_{l_c,l,l_c}^{m_1,m_2,m} x_{m_1} y_{lm_2 0}^2, \tag{3.64}
$$

$$
P_{lm2}(x) := \sum_{k=1}^{2} S_{l_c,l}^{3,k} \sum_{m_1} \sum_{m_2} c_{l_c,l,l_c}^{m_1,m_2,m} x_{m_1} y_{lm_2 2}^k. \tag{3.65}
$$

where the sums run over $|m_1| \le l_c$, $|m_2| \le l_c$, and

$$
\begin{aligned}
S_{l_c}^1 &= \frac{\alpha_{l_c}^4}{2\gamma_{l_c,1}^2} r^2 \mathfrak{p}_1 \pi \mathrm{R} \\
S_{l_c,l}^2 &= \frac{\mathfrak{p}_1}{\mathfrak{p}_2} \pi^2 \mathrm{Q} r^2 \left(\frac{\pi^2}{2} \frac{\alpha_l^2}{\gamma_{l_c,1}^2} - \alpha_{l_c}^2 \right), \\
S_{l_c,l}^{3,1} &= \frac{\alpha_l^2}{4\gamma_{l_c,1}^2} r^2 \pi \left(\frac{1}{2}\alpha_l^2 - 2\alpha_{l_c}^2\right) \left[\frac{\mathfrak{p}_1}{\mathfrak{p}_2} \pi^2 \mathrm{Q} - \gamma_{l_c,1}^4 \right], \\
S_{l_c,l}^{3,2} &= \frac{\alpha_{l_c}^2}{4\gamma_{l_c,1}^2} r^2 \pi \left(2\alpha_{l_c}^2 - \frac{1}{2}\alpha_l^2\right) \mathfrak{p}_1 \mathrm{R}.
\end{aligned} \tag{3.66}
$$

Finally, it is possible to write the equation in the form (3.31) with

$$
\omega_{l_c} = \frac{1}{g_{l_c}} S_{l_c}^1 A_{l_c 02}, \tag{3.67}
$$

and

$$
\xi_{l,l_c} = \frac{1}{g_{l_c}} \left(S_{l_c,l}^2 A_{l_c l 0} + S_{l_c,l}^{3,1} A_{l_c l 2}^1 + S_{l_c,l}^{3,2} A_{l_c l 2}^2 \right), \tag{3.68}
$$

where the A-coefficients are given by (3.57).

4. Conclusion

4.1. The dependence of transition types on the system parameters. In this section, we analyze the transition number q_1 given by (3.33) which dictates the type of transition under the critical crossing condition (3.27) with $l_c=1$. We have seen that similar analysis holds true for $l_c=2$ case which we omit in this study for brevity. According to Theorem 3.3, the system exhibits either a continuous transition to a local attractor containing degenerate steady states if $q_1<0$ or a catastrophic transition to states far away from the basic state if $q_1>0$.

We note that the sign of the denominator term g_{l_c} given by (3.60) is always positive at $R=R_c$ given by (3.23) as it can be written in the form

$$g_{l_c}=\frac{1}{2}\alpha_{l_c}^2 r^2\frac{\mathfrak{p}_1}{\gamma_{l_c,1}^2}\frac{(\mathfrak{p}_2-1)\pi^2}{\mathfrak{p}_2}(Q-Q_0(l_c))$$

where Q_0 is given by (3.26). By our assumption we have either $\mathfrak{p}_2\geq 1$ or if $\mathfrak{p}_2<1$ then we must have $Q<Q_0(l_c)$, see (3.29), and in both cases $g_{l_c}>0$. Thus the contribution of g_{l_c} is irrelevant for the determination of the sign of the transition number.

A symbolic manipulation shows that q_1 given by (3.33) can be written as

$$\begin{aligned}
&q_1=k(a_1Q^2+b_1Q+c_1)\\
&a_1=3\pi^6\alpha_1^2>0\\
&b_1=8\pi^4(3\pi^2-2\gamma_{1,1}^2)\gamma_{2,2}^4+\mathfrak{p}_2^2\left(\pi^2\alpha_1^2\gamma_{1,1}^2\gamma_{2,2}^2(\pi^2-5\gamma_{2,2}^2)\right)\\
&c_1=\mathfrak{p}_2^2\left(\alpha_1^2\gamma_{1,1}^6\gamma_{2,2}^2(\pi^2-5\gamma_{2,2}^2)-\frac{1}{\mathfrak{p}_1^2}3\pi^2\alpha_1^2\gamma_{1,1}^8\right)<0
\end{aligned}\tag{4.1}$$

where k is a positive constant. Equation (4.1) implies that $q_1<0$ if Q is sufficiently small and becomes positive as Q increases. Furthermore q_1 is negative for sufficiently large $\mathfrak{p}_2$ (since $\pi^2<5\gamma_{2,2}^2$). Finally q_1 is negative if $\mathfrak{p}_1$ is sufficiently small and the effect of $\mathfrak{p}_1$ on q_1 diminishes quadratically as $\mathfrak{p}_1$ increases.

Now, we will show that both types of transitions described by Theorem 3.3 are possible. Theorem 3.3 is valid under the following conditions:

- The PES condition (3.27) should hold. For this, either the condition $\mathfrak{p}_2>1$ or the condition $Q<Q_0(\tilde{l}_c)$ given by (3.26) must hold. The determination of the number $Q_0(\tilde{l}_c)$ is non-trivial but one can easily determine a lower estimate $Q_0'<Q_0(\tilde{l}_c)$ given by
$$Q_0'=\pi^2\frac{\mathfrak{p}_2(\mathfrak{p}_1+1)}{\mathfrak{p}_1(1-\mathfrak{p}_2)},$$
by using the fact $\gamma_{l_c,1}^4>\pi^4$. Thus this condition holds if either $Q<Q_0'$ or $\mathfrak{p}_2>1$.
- To have $l_c=1$, Q must be smaller than some Q_1 depending on the radius r which can be obtained from (3.23), see also Figure 3.1.

Now let us fix $r=.7$, $\mathfrak{p}_1=1$ and $\mathfrak{p}_2=0.1$. For these values we find that $Q_0'=2.19$ and $Q_1=13.7$. Hence, for $Q<Q_0'$ and $Q<Q_1$ the assumptions hold true. With these parameters in (4.1), we find that

$$q_1=k(-14.72+34.85Q+0.12Q^2)\begin{cases}<0, & 0<Q<0.42\\ >0, & Q>0.42\end{cases}$$

showing that both types of transition can occur.

The transition is determined by the nonlinear interactions of the higher frequency modes with the critical modes. We have already discussed in the proof that the higher frequency horizontal velocity field modes e^1_{lm0} and the magnetic field modes e^3_{lm2} have no effect on the transition for any l_c. The effect of the heat conduction mode $T=\sin 2z$, is always stabilizing, because

$$S_1^1 A_{102} = \frac{-\mathfrak{p}_1\alpha_1^6(\pi^2\mathrm{Q}+\gamma_{1,1}^4)}{32\pi\gamma_{1,1}^2} < 0.$$

The effect of the vertically homogeneous vertical magnetic field mode $M=Y_{2m}$ is destabilizing (resp. stabilizing) for $r<2/\pi$ (resp. $r>2/\pi$), due to

$$\frac{S_{1,2}^2 A_{202}}{5\pi} = \frac{\mathfrak{p}_1\pi^3\mathrm{Q}\alpha_1^4(3\pi^2-2\gamma_{1,1}^2)}{20\mathfrak{p}_2^2\gamma_{1,1}^4} = \frac{\mathfrak{p}_1\pi^3\mathrm{Q}\alpha_1^4(\pi^2-\frac{4}{r^2})}{20\mathfrak{p}_2^2\gamma_{1,1}^4}.$$

The effect of the velocity mode $(u,w)=(2\pi\nabla Y_{lm}\cos 2\pi z, \alpha_2^2 Y_{2m}\sin 2\pi z)$ is

$$\frac{S_{1,2}^{3,1} A_{122}^1}{5\pi} = \frac{3\pi\alpha_1^6(\mathfrak{p}_1^2\pi^4\mathrm{Q}^2-\mathfrak{p}_2^2\gamma_{1,1}^8)}{160\mathfrak{p}_1\mathfrak{p}_2^2\gamma_{1,1}^4\gamma_{2,2}^4}$$

and its effect depends on the sign of $\mathfrak{p}_1^2\pi^4\mathrm{Q}^2-\mathfrak{p}_2^2\gamma_{1,1}^8$. The effect of the temperature mode $T=Y_{2m}\sin 2\pi z$ is

$$\frac{S_{1,2}^{3,2} A_{122}^2}{5\pi} = \frac{\mathfrak{p}_1\pi\alpha_1^6(\pi^2\mathrm{Q}+\gamma_{1,1}^4)}{160\gamma_{1,1}^2\gamma_{2,2}^2} > 0$$

and is always destabilizing.

4.2. Discussion. In this paper we study the first dynamical transitions from the basic steady profile of the MHD equations as the Rayleigh number is increased. We consider the case where the first $2l_c+1$ critical eigenvalues of the linear operator are real where l_c is a positive integer. To reduce the full PDE to a system of $2l_c+1$ ODE's, our main tool is the center manifold reduction. To carry out this reduction, we construct a suitable basis for the phase space and derive some identities involving the triple product of gradients of spherical harmonics, which we believe can be used in the related problems. The transition is fully described by the stability analysis of this reduced model. We derive the most general reduced system irrespective of the integer l_c. Then by specializing to $l_c=1,2$ cases, we show that the system exhibits either continuous transitions or drastic transitions. In the continuous transition scenario, an attractor which is homeomorphic to $2l_c$ dimensional sphere containing an S^2-set of degenerate steady states of the system bifurcates as the Rayleigh number R crosses the critical Rayleigh number R_c. In the drastic transition scenario, the system has a repeller bifurcated on $\mathrm{R}<\mathrm{R}_c$ and the system moves abruptly from the basic state to a new state which is away from the basic state. We show that for $l_c=1,2$ cases, the first transition is continuous if the Chandrasekhar number Q is sufficiently small, or the Prandtl number $\mathfrak{p}_1$ is sufficiently small or the magnetic Prandtl number $\mathfrak{p}_2$ is sufficiently large. Furthermore, we demonstrate a parameter regime such that by increasing Q, the type of transition changes from continuous to drastic, which is essentially different from the transition in the convection on a spherical shell without a magnetic field.

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Appendix. Spherical harmonics. In this section we recall some properties of spherical harmonics and derive some identities on the integral of triple product of gradients of spherical harmonics.

First of all, the spherical harmonics are the eigenfunctions of the Laplacian on the sphere S_r^2 with radius r, i.e. for $l \in \mathbb{N}$ and $m \in \mathbb{N}$, $|m| \leq l$, they satisfy

$$\Delta Y_{lm} = -\alpha_l^2 Y_{lm}, \qquad \text{in } S_r^2 \tag{5.1}$$

where

$$\alpha_l^2 = \frac{l(l+1)}{r^2}.$$

The spherical harmonics are orthogonal in $L^2(S_r^2)$, and we further assume that they are normalized via

$$\int_{S_r^2} Y_{l_1 m_1} \overline{Y_{l_2 m_2}} dS^2 = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_{l_1 m_1} \overline{Y_{l_2 m_2}} \sin\theta r^2 d\theta d\varphi = r^2 \delta_{l_1,l_2} \delta_{m_1,m_2}. \tag{5.2}$$

Spherical harmonics satisfy the symmetry condition

$$Y_{l,m} = (-1)^m \overline{Y_{l,-m}} \tag{5.3}$$

By (5.2) and (5.3), we obtain

$$\int_{S_1^2} Y_{l_1,m_1} Y_{l_2,m_2} dS^2 = (-1)^{m_2} r^2 \delta_{l_1,l_2} \delta_{m_1,-m_2}. \tag{5.4}$$

Now integrating by parts and using (5.1) and (5.3), we obtain

$$\begin{aligned} \int_{S_r^2} \nabla Y_{l_1,m_1} \cdot \nabla Y_{l_2,m_2} dS^2 &= -\int_{S_r^2} \Delta Y_{l_1,m_1} Y_{l_2,m_2} dS^2 \\ &= \alpha_l^2 r^2 (-1)^{m_2} \delta_{l_1,l_2} \delta_{m_1,-m_2}. \end{aligned} \tag{5.5}$$

Now we deal with triple products of spherical harmonics. First, we define

$$c_{l_1,l_2,l}^{m_1,m_2,m} = \frac{1}{r^2} \int_{S_r^2} Y_{l_1 m_1} Y_{l_2 m_2} \overline{Y_{lm}} dS^2. \tag{5.6}$$

It is well known ([30]) that the tripling coefficients $c_{l_1,l_2,l_3}^{m_1,m_2,m_3}$ vanish unless all of the following conditions hold:

(1) The triangle condition, i.e. $|l_{j_1} - l_{j_2}| \leq l_{j_3}$ for all distinct j_1, j_2, $j_3 \in \{1,2,3\}$,

(2) $m_1 + m_2 = m_3$,

(3) $l_1 + l_2 + l_3$ is an even integer which guarantees that the integrand is an even function. This is a parity conservation law.
Thus

$$c_{l_c,l_c,l}^{m_1,m_2,m} = 0 \qquad \text{unless} \qquad m = m_1 + m_2 \quad \text{and} \quad l \in \{2,4,\ldots,2l_c\}. \tag{5.7}$$

In what follows, we derive some identities which are used in the derivation of the center manifold function.

Proposition 5.1. *The following integrals in terms of (5.6) hold true:*

$$b_{l_1,l_2,l}^{m_1,m_2,m} = \frac{1}{r^2}\int_{S_r^2} \nabla Y_{l_1m_1}\cdot\nabla Y_{l_2m_2}\overline{Y_{lm}}dS^2 = \frac{1}{2}\left(\alpha_{l_1}^2+\alpha_{l_2}^2-\alpha_l^2\right)c_{l_1,l_2,l}^{m_1,m_2,m}, \tag{5.8}$$

$$d_{l_1,l_2,l}^{m_1,m_2,m} = \frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\nabla Y_{l_2m_2}\cdot\nabla\overline{Y_{lm}}dS^2 = \frac{1}{2}(-\alpha_{l_1}^2+\alpha_{l_2}^2+\alpha_l^2)c_{l_1,l_2,l}^{m_1,m_2,m}, \tag{5.9}$$

$$\begin{aligned} e_{l_1,l_2,l}^{m_1,m_2,m} &= \frac{1}{r^2}\int_{S_r^2} \nabla Y_{l_1m_1}\cdot\nabla\nabla Y_{l_2m_2}\cdot\nabla\overline{Y_{lm}}dS^2 \\ &= \frac{1}{2}\left(\alpha_{l_1}^2+\alpha_{l_2}^2-\alpha_l^2\right)d_{l_1,l_2,l}^{m_1,m_2,m}. \end{aligned}\tag{5.10}$$

In particular, we have

$$\begin{aligned} b_{l_c,l_c,l}^{m_1,m_2,m} &= \frac{1}{2}\left(2\alpha_{l_c}^2-\alpha_l^2\right)c_{l_c,l_c,l}^{m_1,m_2,m}, \\ d_{l_c,l_c,l}^{m_1,m_2,m} &= \frac{1}{2}\alpha_l^2 c_{l_c,l_c,l}^{m_1,m_2,m}, \\ e_{l_c,l_c,l}^{m_1,m_2,m} &= \frac{1}{4}\alpha_l^2\left(2\alpha_{l_c}^2-\alpha_l^2\right)c_{l_c,l_c,l}^{m_1,m_2,m}. \end{aligned}\tag{5.11}$$

Proof. The identity (5.8) can be seen by integrating by parts thrice and using (5.1). The calculation goes as follows.

$$\begin{aligned} b_{l_1,l_2,l}^{m_1,m_2,m} &= \alpha_{l_1}^2 c_{l_1,l_2,l}^{m_1,m_2,m} - \frac{1}{r^2}\int_{S_r^2}\nabla Y_{l_1m_1}\cdot Y_{l_2m_2}\nabla\overline{Y_{lm}}dS^2 \\ &= \alpha_{l_1}^2 c_{l_1,l_2,l}^{m_1,m_2,m} + \frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\nabla Y_{l_2m_2}\cdot\nabla\overline{Y_{lm}}dS^2 - \alpha_l^2 c_{l_1,l_2,l}^{m_1,m_2,m} \\ &= (\alpha_{l_1}^2-\alpha_l^2)c_{l_1,l_2,l}^{m_1,m_2,m} - b_{l_1,l_2,l}^{m_1,m_2,m} + \alpha_{l_2}^2 c_{l_1,l_2,l}^{m_1,m_2,m}, \end{aligned}$$

from which the result follows. The identity (5.9) is similar and we omit its proof. For (5.10),

$$\begin{aligned} e_{l_1,l_2,l}^{m_1,m_2,m} &= -\frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\nabla\cdot\left(\nabla\nabla Y_{l_2m_2}\cdot\nabla\overline{Y_{lm}}\right)dS^2 \\ &= -\frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\left(\nabla\cdot\nabla\nabla Y_{l_2m_2}\cdot\nabla\overline{Y_{lm}} + \operatorname{trace}\left(\nabla\nabla Y_{l_2m_2}\nabla\nabla\overline{Y_{lm}}\right)\right)dS^2 \\ &= \alpha_{l_2}^2 d_{l_1,l_2,l}^{m_1,m_2,m} - \mathcal{I} \end{aligned}$$

Using the Einstein summation over indices $i,j=1,2$ and D_1, D_2 denoting the horizontal derivatives, we obtain

$$\begin{aligned} \mathcal{I} &= \frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\operatorname{trace}\left(\nabla\nabla Y_{l_2m_2}\nabla\nabla\overline{Y_{lm}}\right)dS^2 \\ &= \frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}D_{ij}Y_{l_2m_2}D_{ij}\overline{Y_{lm}}dS^2 \\ &= -\frac{1}{r^2}\int_{S_r^2} D_jY_{l_1m_1}D_iY_{l_2m_2}D_{ij}\overline{Y_{lm}}dS^2 + \alpha_l^2 d_{l_1,l_2,l}^{m_1,m_2,m} \\ &= \left(-\alpha_{l_1}^2+\alpha_l^2\right)d_{l_1,l_2,l}^{m_1,m_2,m} + \frac{1}{r^2}\int_{S_r^2} D_jY_{l_1m_1}D_{ij}Y_{l_2m_2}D_i\overline{Y_{lm}}dS^2 \\ &= \left(-\alpha_{l_1}^2+\alpha_{l_2}^2+\alpha_l^2\right)d_{l_1,l_2,l}^{m_1,m_2,m} - \mathcal{I} \end{aligned}$$

Hence the result (5.10) follows. ∎

PROPOSITION 5.2. *Let*

$$f_{l_1l_2l}^{m_1m_2m} = \frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\nabla Y_{l_2m_2}\cdot(\nabla\times\overline{Y_{lm}}\hat{\mathbf{e}}_z)dS^2$$
$$g_{l_1l_2l}^{m_1m_2m} = \frac{1}{r^2}\int_{S_r^2} \nabla Y_{l_1m_1}\cdot(\nabla\nabla Y_{l_2m_2})\cdot(\nabla\times\overline{Y_{lm}}\hat{\mathbf{e}}_z)dS^2 \tag{5.12}$$

Then we have

$$g_{l_1l_2l}^{m_1m_2m} = \frac{1}{2}(\alpha_{l_1}^2+\alpha_{l_2}^2-\alpha_l^2)f_{l_1l_2l}^{m_1m_2m}, \tag{5.13}$$

and the anti-symmetry properties

$$f_{l_1l_2l}^{m_1m_2m} = -f_{l_2l_1l}^{m_2m_1m}, \qquad g_{l_1l_2l}^{m_1m_2m} = -g_{l_2l_1l}^{m_2m_1m}, \tag{5.14}$$

hold true.

Proof. First notice that

$$g_{l_1l_2l}^{m_1m_2m} = -\frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\nabla\cdot\left(\nabla\nabla Y_{l_2m_2}\cdot(\nabla\times\overline{Y_{lm}}\hat{\mathbf{e}}_z)\right)dS^2 = \mathcal{I}+\mathcal{J}$$

where

$$\mathcal{I} = -\frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}(\nabla\cdot\nabla\nabla Y_{l_2m_2})\cdot(\nabla\times\overline{Y_{lm}}\hat{\mathbf{e}}_z)dS^2 = \alpha_{l_2}^2 f_{l_1l_2l}^{m_1m_2m}.$$

Denoting $Z_1 = D_2\overline{Y_{lm}}$ and $Z_2 = -D_1\overline{Y_{lm}}$, we have by three integration by parts

$$\begin{aligned}\mathcal{J} &= -\frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}\mathrm{trace}\left(\nabla\nabla Y_{l_2m_2}\nabla(\nabla\times\overline{Y_{lm}}\hat{\mathbf{e}}_z)\right)dS^2 \\ &= -\frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}D_{ij}Y_{l_2m_2}D_iZ_j dS^2 = \frac{1}{2}(\alpha_{l_1}^2-\alpha_{l_2}^2-\alpha_l^2)f_{l_1l_2l}^{m_1m_2m}\end{aligned}$$

Hence (5.13) is proved and as a consequence we only need to prove the anti-symmetry of $f_{l_1l_2l}^{m_1m_2m}$ in (5.14). We can alternatively express $f_{l_1l_2l}^{m_1m_2m}$ as

$$f_{l_1l_2l}^{m_1m_2m} = \frac{1}{r^2}\int_{S_r^2} Y_{l_1m_1}J(Y_{l_2m_2},\overline{Y_{lm}})dS^2$$

where J is the advective nonlinearity

$$J(f,g) = (D_1fD_2g - D_2fD_1g).$$

The symmetries of the following trilinear form

$$\int_{S_r^2} f_1J(f_2,f_3)dS^2 = \int_{S_r^2} f_2J(f_3,f_1)dS^2 = -\int_{S_r^2} f_2J(f_1,f_3)dS^2$$

for any suitable f_1, f_2, f_3 is well-known and can easily be shown to hold. Thus, (5.14) is proved. ∎

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