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ASYMPTOTIC-PRESERVING SCHEMES FOR TWO-SPECIES BINARY COLLISIONAL KINETIC SYSTEM WITH DISPARATE MASSES I: TIME DISCRETIZATION AND ASYMPTOTIC ANALYSIS*

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In memory of Professor David Shenou Cai

Abstract. We develop efficient asymptotic-preserving time discretization schemes to solve the disparate mass kinetic system of a binary gas or plasma in the "relaxation time scale" relevant to the epochal relaxation phenomenon. Since the resulting model is associated to a parameter given by the square of the mass ratio between the light and heavy particles, we develop an asymptotic-preserving scheme in a novel decomposition strategy using the penalization method for multiscale collisional kinetic equations. Both the Boltzmann and Fokker-Planck-Landau (FPL) binary collision operators will be considered. Other than utilizing several AP strategies for single-species binary kinetic equations, we also introduce a novel splitting and a carefully designed explicit-implicit approximation, which are guided by the asymptotic analysis of the system. We also conduct asymptotic-preserving analysis for the time discretization, for both space homogenous and inhomogeneous systems.

Keywords. two-species kinetic system; disparate mass; epochal relaxation; asymptotic-preserving method.

AMS subject classifications. 35Q20; 82D10; 65M99.

1. Introduction

We are interested in the numerical approximation of a disparate mass binary gas or plasma system, consisting of the mixture of light particles and the heavy ones. Depending on different scalings, such a mixture exhibits various different and interesting asymptotic behavior which poses tremendous numerical challenges due to both the strongly coupled collisional mechanism, described by the nonlinear and nonlocal Boltzmann or Fokker-Planck-Landau (FPL) collision operators, and multiple time and space scales. In the case of plasma, a mixture of electrons and ions, the equalization of electron and ion temperatures is one of the oldest problems in plasma physics and was initially considered by Landau [24]. See [2,3,12,14,22,23] for more physical description of gas mixtures. By introducing the small scaling parameter, which is the square root of the ratio between the masses of the two kinds of particles, one can obtain various interesting asymptotic limits by different time scalings of the equations, see [3,7,8] for both the Boltzmann and FPL collisions. In particular, under the so-called "relaxation time scale", both particle distribution functions are thermalized and the temperatures evolve toward each other via a relaxation equation. This is the epochal relaxation phenomenon first pointed out by Grad [13], and is the asymptotic regime we are interested in here. For recent numerical studies of the disparate mass problems, see [17,31].

One of the main computational challenges for multiscale kinetic equations for binary interactions is the necessity to resolve the small, microscopic scales numerically

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which are often computationally prohibitive. In this regard, the Asymptotic-Preserving (AP) schemes [15] have been very popular in the kinetic and hyperbolic communities in the last two decades. Such schemes allow one to use *small-scale independent* computational parameters in regimes where one cannot afford to resolve the small physical scalings numerically. Such schemes are designed such that they mimic the asymptotic transition from one scale to another at the discrete level, and also use specially designed explicit-implicit time discretizations so as to reduce the algebraic complexity when implicit discretizations are needed. See review articles [6, 16]. For single species particles, in order to overcome the stiffness of the collision operators, one could penalize the collision operators by simple ones that are easier to invert, see [10, 21], or use exponential Runge-Kutta methods [9, 26], or via the micro-macro decomposition [1, 25]. See also [28]. However, for binary interactions in multispecies models, one encounters extra difficulties due to the coupling of collision terms between different species. The Cauchy problem for the full non-linear homogeneous Boltzmann system describing multi-component monatomic gas mixtures has been studied recently in [11]. For relatively simpler scalings which lead to hydrodynamic limits, multispecies AP schemes were developed in for examples [17, 20, 27]. See also [29], where a spectral-Lagrangian Boltzmann solver for a multi-energy level gas was developed. However, none of the previous works dealt with the disparate mass systems under the long-time scale studied in this paper.

The main challenges to develop efficient AP schemes for the problems under study include: 1) the strong coupling of the binary collision terms between different species; 2) the disparate mass scalings; so different species evolve with different time scales thus different species needed to be treated differently and 3) the long-time scale. In fact, other than utilizing several existing AP techniques for single species problems, we also introduce two *new* ideas: *a novel splitting* of the system, guided by the asymptotic analysis introduced in [7], which is a natural formulation for the design of AP schemes, and identifying less stiff terms from the stiff ones, again taking advantage of the asymptotic behavior of the collision operators. We will handle both the Boltzmann and FPL collision terms, thanks to their bilinear structure, and in the end the algebraic complexity, judged by the kind of algebraic systems to be inverted, somehow similar to the single species counterparts as in [10] and [21].

Due to the complexity of the systems under study, we split our results in several papers. In the current paper we focus on the time discretization, which is the most difficult part for the design of AP schemes for such a system. We will conduct an AP analysis for a simplified version of the time discretization, as was done for their single-species counterpart in [10]. Given the length of the paper, we will leave the numerical experiments in a forthcoming paper.

This paper is organized as follows. In Section 2, we present the physical equations and outline their basic properties and the scalings. We also review the asymptotic analysis in [7] for the space homogenous case, under the relaxation time scaling. In Section 3, an AP time discretization for the space homogeneous equations will be presented, with an asymptotic analysis of its AP property. Section 4 extends the scheme and analysis to the space inhomogeneous case, by combining with the idea of diffusive relaxation schemes in [18, 19] to handle the (also stiff) convection terms. Conclusions and future work will be given in Section 5.

2. An overview

In this section we present the physical equations which include both Boltzmann and FPL collisions, their scalings and fundamental properties, and the asymptotic limit conducted in [7].

2.1. The equations and scalings. Let $f^L(t,x,v)$ and $f^H(t,x,v)$ be the probability density distributions of the light and heavy particles at time t, position x with velocity v. The rescaled, space inhomogeneous equations are given by

$$\frac{\partial f^L}{\partial t} + v^L \cdot \nabla_x f^L + F^L \cdot \nabla_{v^L} f^L = \mathcal{Q}^{LL}(f^L, f^L) + \mathcal{Q}_\varepsilon^{LH}(f^L, f^H), \tag{2.1}$$

$$\frac{\partial f^H}{\partial t} + \varepsilon \left(v^H \cdot \nabla_x f^H + F^H \cdot \nabla_{v^H} f^H \right) = \varepsilon \left[\mathcal{Q}^{HH}(f^H, f^H) + \mathcal{Q}_\varepsilon^{HL}(f^H, f^L) \right], \tag{2.2}$$

where F^L, F^H stand for the force fields. The definitions of collision operators $\mathcal{Q}^{LL}$, $\mathcal{Q}^{HH}$, $\mathcal{Q}_\varepsilon^{LH}$ and $\mathcal{Q}_\varepsilon^{HL}$ representing the binary collisions between light ('L') and heavy ('H') particles, are given in the Appendix, since only some of their properties, not their specific forms, will be used in this paper. Moreover, we assume these are binary interaction operators with transition probability rates presenting the natural symmetries that give rise to the classical conservation laws for mixtures. ε is the square root of the mass ratio between the light and heavy particles.

Define n, u and T as the density, bulk velocity, and temperature

$$n = \int_{\mathbb{R}^3} f(v)\, dv, \qquad u = \frac{1}{n} \int_{\mathbb{R}^3} f(v) v\, dv, \qquad T = \frac{1}{3\rho} \int_{\mathbb{R}^3} f(v) |v-u|^2\, dv, \tag{2.3}$$

and denote $M_{u,T}$ the normalized Maxwellian

$$M_{u,T}(v) = \frac{1}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-u|^2}{2T} \right). \tag{2.4}$$

In [7], three different time scales were introduced which lead to different hydrodynamic limits. We are interested in the third time scale, namely the "relaxation time scale" studied in [7]. The macroscopic limit under this scaling, as well as the design of AP schemes, are the most challenging. The AP schemes that preserve the other two asymptotic limits are easy to design by classical AP strategies so will not be discussed here.

The collision time for the light and heavy species are denoted by t_0^L and t_0^H, respectively. We define $t_0 = t_0^L$ as the basic time scale. Introduce the long-time scaling $t_0' = t_0/\varepsilon^2$ and change of variables $t' = \varepsilon^2 t$, $x' = \varepsilon x$, $F' = F/\varepsilon$, at which both distribution functions will be thermalized and temperatures influence each other via a relaxation equation. Then the evolution equations are given by

$$\frac{\partial f^L}{\partial t} + \frac{1}{\varepsilon} \left(v^L \cdot \nabla_x f^L + F^L \cdot \nabla_{v^L} f^L \right) = \frac{1}{\varepsilon^2} \left[\mathcal{Q}^{LL}(f^L, f^L) + \mathcal{Q}_\varepsilon^{LH}(f^L, f^H) \right], \tag{2.5}$$

$$\frac{\partial f^H}{\partial t} + \left(v^H \cdot \nabla_x f^H + F^H \cdot \nabla_{v^H} f^H \right) = \frac{1}{\varepsilon} \left[\mathcal{Q}^{HH}(f^H, f^H) + \mathcal{Q}_\varepsilon^{HL}(f^H, f^L) \right]. \tag{2.6}$$

Inserting the ansatz

$$\mathcal{Q}_\varepsilon^{LH} = \mathcal{Q}_0^{LH} + \varepsilon \mathcal{Q}_1^{LH} + O(\varepsilon^2), \qquad \mathcal{Q}_\varepsilon^{HL} = \mathcal{Q}_0^{HL} + \varepsilon \mathcal{Q}_1^{HL} + O(\varepsilon^2)$$

into (2.5)–(2.6), one has

$$\frac{\partial f^L}{\partial t} + \varepsilon^{-1} \left(v^L \cdot \nabla_x f^L + F^L \cdot \nabla_{v^L} f^L \right)$$

$$=\varepsilon^{-2}\left(\mathcal{Q}^{LL}(f_\varepsilon^L,f_\varepsilon^L)+\mathcal{Q}_0^{LH}(f_\varepsilon^L,f_\varepsilon^H)\right)+\varepsilon^{-1}\mathcal{Q}_1^{LH}(f_\varepsilon^L,f_\varepsilon^H)+\mathcal{Q}_2^{LH}(f_\varepsilon^L,f_\varepsilon^H)+O(\varepsilon), \quad (2.7)$$

$$\frac{\partial f^H}{\partial t}+v^H\cdot\nabla_x f^H+F^H\cdot\nabla_{v^H}f^H$$
$$=\varepsilon^{-1}\left(\mathcal{Q}^{HH}(f_\varepsilon^H,f_\varepsilon^H)+\mathcal{Q}_0^{HL}(f_\varepsilon^H,f_\varepsilon^L)\right)+\mathcal{Q}_1^{HL}(f_\varepsilon^H,f_\varepsilon^L)+O(\varepsilon). \quad (2.8)$$

Clearly the dynamics of (2.7)–(2.8) have stiff terms associated to the electron-ion mass ratio that naturally enables the development of asymptotic-preserving schemes.

We first give a summary of the propositions and lemmas on the properties of the collision operators given in [5, 30] and summarized in [7] that will be useful in our paper. We call "inter-particle collisions" and "intra-particle collisions" to distinguish binary collisions between different species and like particles in the sequel.

THEOREM 2.1.

(1) For the FPL collision operator,

$$\mathcal{Q}_0^{LH}(f^L,f^H)=n^H q_0(f^L), \qquad q_0(f^L)=\nabla_{v^L}\cdot\left[B(v^L)S(v^L)\nabla_{v^L}f^L(v^L)\right], \quad (2.9)$$
$$\mathcal{Q}_0^{HL}(f^H,f^L)=-2\nabla_{v^H}f^H(v^H)\cdot\int_{\mathbb{R}^3}\frac{B(v^L)}{|v^L|^2}v^L f^L(v^L)\,dv^L.$$

For the Boltzmann collision operator,

$$\mathcal{Q}_0^{LH}(f^L,f^H)=n^H q_0(f^L), q_0(f^L)=\int_{\mathbb{S}^2}B(v^L,\Omega)\left(f^L(v^L-2(v^L,\Omega)\Omega)-f^L(v^L)\right)d\Omega, \quad (2.10)$$

$$\mathcal{Q}_0^{HL}(f^H,f^L)=-2\nabla_{v^H}f^H\cdot\int_{\mathbb{R}^3\times\mathbb{S}^2}B(v^L,\Omega)\,\frac{(v^L,\Omega)^2}{|v^L|^2}v^L f^L(v^L)dv^L d\Omega.$$

(2) For any function f^H,

(i) if f^L is a function of $|v^L|$, then $\mathcal{Q}_0^{LH}(f^L,f^H)=0$,

(ii) if f^L is an even function, then $\mathcal{Q}_0^{HL}(f^H,f^L)=0$.

(3) The conservation properties of the inter-particle collision operators are given by

$$\int_{\mathbb{R}^3}\mathcal{Q}_\varepsilon^{LH}\,dv^L=\int_{\mathbb{R}^3}\mathcal{Q}_\varepsilon^{HL}\,dv^H=0, \quad (2.11)$$
$$\int_{\mathbb{R}^3}\mathcal{Q}_\varepsilon^{LH}\begin{pmatrix}v^L\\|v^L|^2\end{pmatrix}dv^L+\varepsilon\int_{\mathbb{R}^3}\mathcal{Q}_\varepsilon^{HL}\begin{pmatrix}v^H\\|v^H|^2\end{pmatrix}dv^H=0,$$
$$\int_{\mathbb{R}^3}\mathcal{Q}_i^{LH}\,dv^L=\int_{\mathbb{R}^3}\mathcal{Q}_i^{HL}\,dv^H=0, \qquad \forall i\in\mathbb{N}, \quad (2.12)$$
$$\int_{\mathbb{R}^3}\mathcal{Q}_i^{LH}\,v^L\,dv^L+\int_{\mathbb{R}^3}\mathcal{Q}_i^{HL}\,v^H\,dv^H=0, \qquad \forall i\in\mathbb{N},$$
$$\int_{\mathbb{R}^3}\mathcal{Q}_i^{LH}\,|v^L|^2\,dv^L+\int_{\mathbb{R}^3}\mathcal{Q}_{i-1}^{HL}\,|v^H|^2\,dv^H=0, \qquad \forall i\in\mathbb{N}, i\geq 1,$$
$$\int_{\mathbb{R}^3}\mathcal{Q}_0^{LH}\,|v^L|^2\,dv^L=0. \quad (2.13)$$

(4) Introduce the operator

$$\mathcal{Q}_0^L(f^L):=\mathcal{Q}^{LL}(f^L,f^L)+n^H q_0(f^L).$$

For all sufficiently regular f,

$$\int_{\mathbb{R}^3} \mathcal{Q}_0^L(f^L)\ln f\,dv \le 0,$$

and

$$\mathcal{Q}_0^L(f)=0 \Leftrightarrow \exists (n,T)\in[0,\infty)^2 \text{ such that } f=nM_{0,T}, \tag{2.14}$$

where $M_{0,T}$ *is the normalized Maxwellian defined in* (2.4) *with* $u=0$.

(5) *Define* $M_0^L := n_0^L(t)\,M_{0,T_0^L(t)}$. Γ_0^L *is a non-positive self-adjoint operator associated with the inner product*

$$\langle\phi,\psi\rangle=\int_{\mathbb{R}^3}\phi\psi\,M_0^L\,dv$$

on the space $\chi=\{\phi(v),\langle\phi,\phi\rangle<\infty\}$, *and is such that*

$$\textit{ker}\ \Gamma_0^L=\{\phi(v^L) \text{ such that } \exists(a,b)\in\mathbb{R}^2, \phi(v^L)=a+b|v^L|^2\}.$$

For $\psi\in\chi$, *the equation* $\Gamma_0^L\phi=\psi$ *is solvable if and only if*

$$\int_{\mathbb{R}^3}\psi\begin{pmatrix}1\\|v^L|^2\end{pmatrix}M_0^L\,dv^L=0.$$

Then the solution ϕ *is unique in* $\left(\textit{ker}\ \Gamma_0^L\right)^{\perp}$.

2.2. The macroscopic approximation. For clarity of the presentation, we first consider the space homogeneous case of (2.7)–(2.8), so the spatial and velocity gradients on the left-hand side of the equations are omitted. Inserting the Hilbert expansions

$$f_\varepsilon^L=f_0^L+\varepsilon f_1^L+\varepsilon^2 f_2^L+\cdots, \qquad f_\varepsilon^H=f_0^H+\varepsilon f_1^H+\varepsilon^2 f_2^H+\cdots$$

and equating terms of ε leads to: order ε^{-2}:

$$\mathcal{Q}^{LL}(f_0^L,f_0^L)+\mathcal{Q}_0^{LH}(f_0^L,f_0^H)=0; \tag{2.15}$$

order ε^{-1}:

$$0=2\mathcal{Q}^{LL}(f_0^L,f_1^L)+\mathcal{Q}_0^{LH}(f_0^L,f_1^H)+\mathcal{Q}_0^{LH}(f_1^L,f_0^H)+\mathcal{Q}_1^{LH}(f_0^L,f_0^H), \tag{2.16}$$

$$0=\mathcal{Q}^{HH}(f_0^H,f_0^H)+\mathcal{Q}_0^{HL}(f_0^H,f_0^L); \tag{2.17}$$

order ε^0:

$$\begin{aligned}\frac{\partial f_0^L}{\partial t}&=2\mathcal{Q}^{LL}(f_0^L,f_2^L)+\mathcal{Q}^{LL}(f_1^L,f_1^L)+\mathcal{Q}_0^{LH}(f_0^L,f_2^H)+\mathcal{Q}_0^{LH}(f_1^L,f_1^H)+\mathcal{Q}_0^{LH}(f_2^L,f_0^H)\\&\quad+\mathcal{Q}_1^{LH}(f_0^L,f_1^H)+\mathcal{Q}_1^{LH}(f_1^L,f_0^H)+\mathcal{Q}_2^{LH}(f_0^L,f_0^H),\end{aligned} \tag{2.18}$$

$$\frac{\partial f_0^H}{\partial t}=2\mathcal{Q}^{HH}(f_0^H,f_1^H)+\mathcal{Q}_0^{HL}(f_0^H,f_1^L)+\mathcal{Q}_0^{HL}(f_1^H,f_0^L)+\mathcal{Q}_1^{HL}(f_0^H,f_0^L). \tag{2.19}$$

First consider the equation for the heavy particles. By (2.9) and (2.10), we know

$$\mathcal{Q}_0^{LH}(f^L,f^H)=n_0^H\,q_0(f^L),$$

with different $q_0(f^L)$ definitions for the Boltzmann and FPL equations respectively. Using (2.14), Equation (2.15) gives $f_0^L = M_0^L$. By statement (2)(ii) in Theorem 2.1, since f_0^L is an even function, thus

$$\mathcal{Q}_0^{HL}(f_0^H, f_0^L) = 0,$$

and (2.17) reduces to

$$\mathcal{Q}^{HH}(f_0^H, f_0^H) = 0.$$

Using the classical theory of the Boltzmann equation [4], $\exists (n_0^H(t), T_0^H(t)) \in [0,\infty)^2$, $u_0^H(t) \in \mathbb{R}^3$, such that

$$f_0^H = n_0^H(t) M_{u_0^H(t), T_0^H(t)} := M_0^H.$$

By statement (2)(i) in Theorem 2.1,

$$\mathcal{Q}_0^{LH}(f_0^L, f_1^H) = 0,$$

since $f_0^L = M_0^L$ is a function of $|v^L|$. Then (2.16) is an equation for f_1^L, which can be solved by setting

$$\phi_1^L = f_1^L (M_0^L)^{-1}$$

and

$$\Gamma_0^L \phi_1^L = -(M_0^L)^{-1} \mathcal{Q}_1^{LH}(M_0^L, f_0^H), \tag{2.20}$$

where Γ_0^L is an operator defined by

$$\Gamma_0^L \phi = (M_0^L)^{-1} \left[2\mathcal{Q}^{LL}(M_0^L, M_0^L \phi) + n_0^H q_0(M_0^L \phi) \right]. \tag{2.21}$$

According to statement (5) in Theorem 2.1, $\Gamma_0^L \phi = \psi$ is solvable if and only if

$$\int_{\mathbb{R}^3} \psi \begin{pmatrix} 1 \\ |v^L|^2 \end{pmatrix} M_0^L \, dv^L = 0. \tag{2.22}$$

Therefore, we have

$$\psi = -(M_0^L)^{-1} \mathcal{Q}_1^{LH}(M_0^L, f_0^H)$$

in (2.20), and (2.22) is satisfied thanks to statement (5) in Theorem 2.1, thus (2.20) is solvable and its unique solution in $(\ker \Gamma_0^L)^\perp$ is given by

$$f_1^L(v^L) = \frac{1}{T_0^L} M_0^L(v^L) u_0^H \cdot v^L.$$

Since again $\mathcal{Q}_0^{HL}(f_1^H, f_0^L) = 0$, (2.19) is an equation for f_1^H, which can be written in terms of $\phi^H = f_1^H (M_0^H)^{-1}$ with

$$\Gamma_0^H \phi^H = (M_0^H)^{-1} \left[\frac{\partial M_0^H}{\partial t} - \mathcal{Q}_0^{HL}(M_0^H, f_1^L) - \mathcal{Q}_1^{HL}(M_0^H, M_0^L) \right], \tag{2.23}$$

where Γ_0^H is the linearization of $\mathcal{Q}^{HH}$ around a Maxwellian M_0^H:

$$\Gamma_0^H \phi = 2(M_0^H)^{-1} \mathcal{Q}^{HH}(M_0^H, M_0^H \phi).$$

The necessary and sufficient condition of solvability of Equation (2.23) is given by

$$\int_{\mathbb{R}^3}\left[\frac{\partial M_0^H}{\partial t}-\mathcal{Q}_0^{HL}(M_0^H,f_1^L)-\mathcal{Q}_1^{HL}(M_0^H,M_0^L)\right]\begin{pmatrix}1\\ v^H\\ |v^H|^2\end{pmatrix}dv^H=0. \tag{2.24}$$

The calculation in [7] gives

$$\int_{\mathbb{R}^3}\left[\mathcal{Q}_0^{HL}(M_0^H,f_1^L)+\mathcal{Q}_1^{HL}(M_0^H,M_0^L)\right]\begin{pmatrix}1\\ v^H\\ \frac{|v^H|^2}{2}\end{pmatrix}dv^H=\begin{pmatrix}0\\ 0\\ -3\frac{\lambda(T_0^L)}{T_0^L}n_0^L n_0^H(T_0^H-T_0^L)\end{pmatrix}.$$

Inserting it into (2.24), one finally has

$$\frac{d}{dt}\begin{pmatrix}n_0^H\\ n_0^H u_0^H\\ n_0^H(\frac{1}{2}|u_0^H|^2+\frac{3}{2}T_0^H)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ -3\frac{\lambda(T_0^L)}{T_0^L}n_0^L n_0^H(T_0^H-T_0^L)\end{pmatrix}.$$

Therefore the macroscopic limit of the heavy particles, as $\varepsilon\to 0$, is

$$\begin{aligned}&\frac{d}{dt}n_0^H=0,\qquad \frac{d}{dt}n_0^H u_0^H=0,\\ &\frac{d}{dt}\left(\frac{3n_0^H T_0^H}{2}\right)=-3\frac{\lambda(T_0^L)}{T_0^L}n_0^L n_0^H(T_0^H-T_0^L).\end{aligned}$$

Now we consider the light particles. Equation (2.19) is an equation of f_2^L which can be written in terms of $\phi_2^L=f_2^L(M_0^L)^{-1}$ with

$$\Gamma_0^L\phi_2^L=(M_0^L)^{-1}S^L,$$

where Γ_0^L is defined by (2.21) and

$$\begin{aligned}S^L=\frac{\partial M_0^L}{\partial t}&-\mathcal{Q}^{LL}(f_1^L,f_1^L)-\mathcal{Q}_0^{LH}(f_1^L,f_1^H)\\ &-\mathcal{Q}_1^{LH}(M_0^L,f_1^H)-\mathcal{Q}_1^{LH}(f_1^L,M_0^H)-\mathcal{Q}_2^{LH}(M_0^L,M_0^H).\end{aligned} \tag{2.25}$$

According to statement (5) in Theorem 2.1, the necessary and sufficient condition for the existence of f_2^L should be

$$\int_{\mathbb{R}^3}S^L(v^L)\begin{pmatrix}1\\ |v^L|^2\end{pmatrix}dv^L=0. \tag{2.26}$$

The first equation leads to $dn_0^L/dt=0$. By statement (3) in Theorem 2.1,

$$\begin{aligned}&\int \mathcal{Q}_0^{LH}(f_1^L,f_1^H)|v^L|^2\,dv^L=0,\\ &\int \mathcal{Q}_1^{LH}(M_0^L,f_1^H)|v^L|^2\,dv^L=\int \mathcal{Q}_0^{HL}(f_1^H,M_0^L)|v^H|^2\,dv^H=0.\end{aligned}$$

The remaining terms on the right-hand side of (2.25) give

$$\int\left[\mathcal{Q}_1^{LH}(f_1^L,M_0^H)+\mathcal{Q}_2^{LH}(M_0^L,M_0^H)\right]|v^L|^2\,dv^L$$

$$
\begin{aligned}
&= -\int \left[\mathcal{Q}_0^{HL}(M_0^H, f_1^L) + \mathcal{Q}_1^{HL}(M_0^H, M_0^L)\right] |v^H|^2 \, dv^H \\
&= 6\frac{\lambda(T_0^L)}{T_0^L} n_0^L n_0^H (T_0^H - T_0^L).
\end{aligned}
$$

Inserting into (2.26), one obtains the evolution equation for T_0^L:

$$
\frac{d}{dt}\left(\frac{3n_0^L T_0^L}{2}\right) = -3\frac{\lambda(T_0^L)}{T_0^L} n_0^L n_0^H (T_0^L - T_0^H). \tag{2.27}
$$

We now summarize the macroscopic equations for the whole system, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\frac{d}{dt} n_0^L = 0, \\
&\frac{d}{dt}\left(\frac{3n_0^L T_0^L}{2}\right) = -3\frac{\lambda(T_0^L)}{T_0^L} n_0^L n_0^H (T_0^L - T_0^H),
\end{aligned} \tag{2.28}
$$

and

$$
\begin{aligned}
&\frac{d}{dt} n_0^H = 0, \\
&\frac{d}{dt}(n_0^H u_0^H) = 0, \\
&\frac{d}{dt}\left(\frac{3n_0^H T_0^H}{2}\right) = -3\frac{\lambda(T_0^L)}{T_0^L} n_0^L n_0^H (T_0^H - T_0^L).
\end{aligned} \tag{2.29}
$$

3. An asymptotic-preserving time discretization

An AP scheme requires that the discrete version of (2.5)–(2.6) asymptotically approaches the macroscopic equations (2.28)–(2.29) as $\varepsilon \to 0$, when numerical parameters are held fixed. A necessary requirement for such a scheme is some implicit time discretization for the numerical stiff terms, which can be easily inverted [16]. In this section, we design such a time discretization for the space homogeneous equations.

The space homogeneous version of Equations (2.5)–(2.6) is given by

$$
\frac{\partial f^L}{\partial t} = \frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f^L, f^L) + \mathcal{Q}_\varepsilon^{LH}(f^L, f^H)\right], \tag{3.1}
$$

$$
\frac{\partial f^H}{\partial t} = \frac{1}{\varepsilon}\left[\mathcal{Q}^{HH}(f^H, f^H) + \mathcal{Q}_\varepsilon^{HL}(f^H, f^L)\right]. \tag{3.2}
$$

3.1. A splitting of the equation.

We first decompose f into f_0 and f_1,

$$
f^L = f_0^L + \varepsilon f_1^L, \qquad f^H = f_0^H + \varepsilon f_1^H, \tag{3.3}
$$

and insert into the system (3.1)–(3.2), then

$$
\begin{aligned}
\frac{\partial}{\partial t}(f_0^L + \varepsilon f_1^L) &= \frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f_0^L + \varepsilon f_1^L, f_0^L + \varepsilon f_1^L) + \mathcal{Q}_\varepsilon^{LH}(f_0^L + \varepsilon f_1^L, f_0^H + \varepsilon f_1^H)\right] \\
&= \frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_0^L, f_0^L) + 2\varepsilon \mathcal{Q}^{LL}(f_0^L, f_1^L) + \varepsilon^2 \mathcal{Q}^{LL}(f_1^L, f_1^L) \\
&\quad + \mathcal{Q}_\varepsilon^{LH}(f_0^L, f_0^H) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_0^L, f_1^H) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_0^H) + \varepsilon^2 \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_1^H)\Big],
\end{aligned} \tag{3.4}
$$

and

$$\frac{\partial}{\partial t}(f_0^H+\varepsilon f_1^H)=\frac{1}{\varepsilon}\left[\mathcal{Q}^{HH}(f_0^H+\varepsilon f_1^H,f_0^H+\varepsilon f_1^H)+\mathcal{Q}_\varepsilon^{HL}(f_0^H+\varepsilon f_1^H,f_0^L+\varepsilon f_1^L)\right]$$
$$=\frac{1}{\varepsilon}\Bigg[\mathcal{Q}^{HH}(f_0^H,f_0^H)+2\varepsilon\mathcal{Q}^{HH}(f_0^H,f_1^H)+\varepsilon^2\mathcal{Q}^{HH}(f_1^H,f_1^H)$$
$$+\mathcal{Q}_\varepsilon^{HL}(f_0^H,f_0^L)+\varepsilon\mathcal{Q}_\varepsilon^{HL}(f_0^H,f_1^L)+\varepsilon\mathcal{Q}_\varepsilon^{HL}(f_1^H,f_0^L)+\varepsilon^2\mathcal{Q}_\varepsilon^{HL}(f_1^H,f_1^L)\Bigg]. \tag{3.5}$$

Our first key idea is to split (3.4) into two equations for f_0^L, f_1^L respectively,

$$\frac{\partial}{\partial t}f_0^L=\frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_0^L,f_0^L)+\mathcal{Q}_0^{LH}(f_0^L,f_0^H)\Big],$$
$$\frac{\partial}{\partial t}f_1^L=\frac{1}{\varepsilon^2}\Bigg[\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{LH}(f_0^L,f_0^H)-\mathcal{Q}_0^{LH}(f_0^L,f_0^H)\right)+2\mathcal{Q}^{LL}(f_0^L,f_1^L)+\varepsilon\mathcal{Q}^{LL}(f_1^L,f_1^L)$$
$$+\mathcal{Q}_\varepsilon^{LH}(f_0^L,f_1^H)+\mathcal{Q}_\varepsilon^{LH}(f_1^L,f_0^H)+\varepsilon\mathcal{Q}_\varepsilon^{LH}(f_1^L,f_1^H)\Bigg], \tag{3.6}$$

and split (3.5) into two equations for f_0^H, f_1^H respectively:

$$\frac{\partial}{\partial t}f_0^H=\frac{1}{\varepsilon}\Big[\mathcal{Q}^{H}(f_0^H,f_0^H)+\mathcal{Q}_0^{HL}(f_0^H,f_0^L)\Big],$$
$$\frac{\partial}{\partial t}f_1^H=\frac{1}{\varepsilon}\Bigg[\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{HL}(f_0^H,f_0^L)-\mathcal{Q}_0^{HL}(f_0^H,f_0^L)\right)+2\mathcal{Q}^{HH}(f_0^H,f_1^H)+\varepsilon\mathcal{Q}^{HH}(f_1^H,f_1^H)$$
$$+\mathcal{Q}_\varepsilon^{HL}(f_0^H,f_1^L)+\mathcal{Q}_\varepsilon^{HL}(f_1^H,f_0^L)+\varepsilon\mathcal{Q}_\varepsilon^{HL}(f_1^H,f_1^L)\Bigg]. \tag{3.7}$$

This splitting is motivated by the asymptotic analysis presented in Subsection 2.2, and plays the central role in the AP time discretization, which will be introduced in the next subsection.

3.2. Time discretization. First, to have a scheme uniformly stable with respect to ε, it is natural to use the implicit discretizations for all the stiff collision terms, namely, those that appear to be of $O(1)$ inside the brackets on the right-hand side of (3.6)–(3.7). We use the notations $f_{L,0}^n$, $f_{L,1}^n$, $f_{H,0}^n$, $f_{H,1}^n$ to denote the numerical solutions of f_0^L, f_1^L, f_0^H and f_1^H at time step t^n. Consider the light particles. A naive discretization for $f_{L,0}$, $f_{L,1}$ in (3.6) is

$$\frac{f_{L,0}^{n+1}-f_{L,0}^n}{\Delta t}=\frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\Big], \tag{3.8}$$

$$\frac{f_{L,1}^{n+1}-f_{L,1}^n}{\Delta t}=\frac{1}{\varepsilon^2}\Bigg[\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})-\mathcal{Q}_0^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\right)$$
$$+2\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})+\varepsilon\mathcal{Q}^{LL}(f_{L,1}^n,f_{L,1}^n)$$
$$+\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n+1},f_{H,1}^{n+1})+\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})+\varepsilon\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^n,f_{H,1}^n)\Bigg]. \tag{3.9}$$

Consider the time evolution for $f_{H,0}$, $f_{H,1}$. A naive implicit scheme for (3.7) would be:

$$\frac{f_{H,0}^{n+1}-f_{H,0}^{n}}{\Delta t}=\frac{1}{\varepsilon}\left[\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_0^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})\right], \tag{3.10}$$

$$\begin{aligned}\frac{f_{H,1}^{n+1}-f_{H,1}^{n}}{\Delta t}=\frac{1}{\varepsilon}\Bigg[&\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})-\mathcal{Q}_0^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})\right)\\&+2\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,1}^{n+1})+\varepsilon\mathcal{Q}^{HH}(f_{H,1}^{n},f_{H,1}^{n})\\&+\mathcal{Q}_\varepsilon^{HL}(f_{H,0}^{n+1},f_{L,1}^{n+1})+\mathcal{Q}_\varepsilon^{HL}(f_{H,1}^{n+1},f_{L,0}^{n+1})+\varepsilon\mathcal{Q}_\varepsilon^{HL}(f_{H,1}^{n},f_{L,1}^{n})\Bigg],\end{aligned} \tag{3.11}$$

in which the right-hand side is fully implicit, except the terms that are relatively less stiff due to an extra factor of ε. Inverting the above system is algebraically complex due to the nonlinearity, nonlocal nature of the collision operators and the coupling between the two types of particles. Our next key idea is to use the asymptotic behavior of the operators to identify those terms that are not stiff.

3.2.1. Identifying the less stiff terms. First, as $\varepsilon\to 0$,

$$f_{L,0}^{n+1}\to n_0^L M_{0,T_0^L}. \tag{3.12}$$

Since M_{0,T_0^L} is a function of $|v^L|$, according to (2)(i) in Theorem 2.1,

$$\mathcal{Q}_0^{LH}(n_0^L M_{0,T_0^L},f_{H,0}^{n+1})=0,$$

thus

$$\mathcal{Q}_0^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})=O(\varepsilon),$$

which is less stiff and can be implemented explicitly.

Secondly, as $\varepsilon\to 0$, similarly

$$\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n+1},f_{H,1}^{n+1})\to\mathcal{Q}_0^{LH}(f_{L,0}^{n+1},f_{H,1}^{n+1})=O(\varepsilon),$$

so the corresponding term is less stiff and can also be discretized explicitly.

For the less stiff terms $Q_0^{LH}(f_{L,0},f_{H,0})$ and $\mathcal{Q}_\varepsilon^{LH}(f_{L,0},f_{H,1})$ we treat them explicitly, thus our time discretizations for $f_{L,0}$, $f_{L,1}$ are given by

$$\frac{f_{L,0}^{n+1}-f_{L,0}^{n}}{\Delta t}=\frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^{n},f_{H,0}^{n})\right], \tag{3.13}$$

$$\begin{aligned}\frac{f_{L,1}^{n+1}-f_{L,1}^{n}}{\Delta t}=\frac{1}{\varepsilon^2}\Bigg[&\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})-\mathcal{Q}_0^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\right)\\&+2\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})+\varepsilon\mathcal{Q}^{LL}(f_{L,1}^{n},f_{L,1}^{n})\\&+\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n},f_{H,1}^{n})+\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})+\varepsilon\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^{n},f_{H,1}^{n})\Bigg].\end{aligned} \tag{3.14}$$

Similarly for $f_{H,0}$, $f_{H,1}$, we introduce the following time discretizations for $f_{H,0}, f_{H,1}$ by taking advantages of some terms that are actually not stiff:

$$\frac{f_{H,0}^{n+1}-f_{H,0}^{n}}{\Delta t}=\frac{1}{\varepsilon}\left[\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_0^{HL}(f_{H,0}^{n},f_{L,0}^{n})\right], \tag{3.15}$$

$$\frac{f_{H,1}^{n+1}-f_{H,1}^{n}}{\Delta t}=\frac{1}{\varepsilon}\Big[\frac{1}{\varepsilon}\Big(\mathcal{Q}_{\varepsilon}^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})-\mathcal{Q}_{0}^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})\Big) \\ +2\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,1}^{n+1})+\varepsilon\mathcal{Q}^{HH}(f_{H,1}^{n},f_{H,1}^{n}) \\ +\mathcal{Q}_{\varepsilon}^{HL}(f_{H,0}^{n+1},f_{L,1}^{n+1})+\mathcal{Q}_{\varepsilon}^{HL}(f_{H,1}^{n},f_{L,0}^{n})+\varepsilon\mathcal{Q}_{\varepsilon}^{HL}(f_{H,1}^{n},f_{L,1}^{n})\Big], \quad (3.16)$$

where the argument (2)(ii) of Theorem 2.1 is used, that is, since $f_{L,0}^{n+1}$ is asymptotically an even function due to (3.12), one has

$$\mathcal{Q}_{0}^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})=O(\varepsilon),$$

thus the second term on the right-hand side of (3.10) is not stiff. In addition, as $\varepsilon\to 0$,

$$\mathcal{Q}_{\varepsilon}^{HL}(f_{H,1}^{n+1},f_{L,0}^{n+1})\to\mathcal{Q}_{0}^{HL}(f_{H,1}^{n+1},f_{L,0}^{n+1})=O(\varepsilon).$$

Thus the term $\mathcal{Q}_{\varepsilon}^{HL}(f_{H,1}^{n+1},f_{L,0}^{n+1})$ in (3.11) is less stiff and can be approximated explicitly.

3.2.2. Handling of the stiff terms. First, we point out the terms $\mathcal{Q}_{\varepsilon}^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})$, $\mathcal{Q}_{0}^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})$ and $\mathcal{Q}_{\varepsilon}^{HL}(f_{H,0}^{n+1},f_{L,1}^{n+1})$ in (3.16), although implicit, can be obtained *explicitly* since $f_{L,0}^{n+1}$, $f_{H,0}^{n+1}$ and $f_{L,1}^{n+1}$ are already computed from (3.13), (3.14) and (3.15).

Now we take care of the truly stiff and implicit collision terms in schemes (3.13)–(3.14) and (3.15)–(3.16). They will be penalized by an operator that can either be inverted analytically (for the case of the Boltzmann collision [10]) or by a Poisson-type solver (for the case of FPL collision [21]).

(i) For the stiff and nonlinear term $\mathcal{Q}_{\varepsilon}^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})$ in (3.14), motivated by [10, 21], we use $\mathcal{Q}_{0}^{LH}(f_{L,1},f_{H,0})$ which is the leading order asymptotically for ε small, as the penalty operator. The rationale for this is that $\mathcal{Q}_{0}^{LH}(f_{L,1},f_{H,0})$ is much easier to be inverted than $\mathcal{Q}_{\varepsilon}^{LH}(f_{L,1},f_{H,0})$, as will be shown below. We substitute $\mathcal{Q}_{\varepsilon}^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})$ in (3.14) by

$$\underbrace{\mathcal{Q}_{\varepsilon}^{LH}(f_{L,1}^{n},f_{H,0}^{n})-\mathcal{Q}_{0}^{LH}(f_{L,1}^{n},f_{H,0}^{n})}_{\text{less stiff}}+\underbrace{\mathcal{Q}_{0}^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})}_{\text{stiff}}.$$

Integrating both sides of (3.15) in v^H, we get that n_0^H does not change from t^n to t^{n+1}, so we will drop its dependence on n. Thus

$$\mathcal{Q}_{0}^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})=n_0^H\, q_0(f_{L,1}^{n+1}),$$

with q_0 defined in (2.9) and (2.10) for the Boltzmann and FPL equations respectively. For the FPL case,

$$\mathcal{Q}_{0}^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})=n_0^H\,\nabla_{v^L}\cdot\Big[B(v^L)S(v^L)\nabla_{v^L}f_{L,1}^{n+1}(v^L)\Big], \quad (3.17)$$

thus one only needs to invert a linear FP operator. See [21]. For the Boltzmann case,

$$\mathcal{Q}_{0}^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})=n_0^H\int_{\mathbb{S}^2}B(v^L,\Omega)\Big(f_{L,1}^{n+1}(v^L-2(v^L,\Omega)\Omega)-f_{L,1}^{n+1}(v^L)\Big)\,d\Omega$$

$$=n_0^H\int_{\mathbb{S}^2}B(v^L,\Omega)\,f_{L,1}^{n+1}(v^L-2(v^L,\Omega)\Omega)\,d\Omega-n_0^H\,f_{L,1}^{n+1}(v^L)\int_{\mathbb{S}^2}B(v^L,\Omega)\,d\Omega,$$

which is still a nonlocal operator. We use the linear penalty method [25] to remove the stiffness here, that is, substitute the above term by

$$n_0^H \int_{\mathbb{S}^2} B(v^L,\Omega)\left(f_{L,1}^n(v^L-2(v^L,\Omega)\Omega)-f_{L,1}^n(v^L)\right)d\Omega - n_0^H\mu f_{L,1}^n(v^L)+n_0^H\mu f_{L,1}^{n+1}(v^L),$$

where

$$\mu = \max_{v^L}\int_{\mathbb{S}^2} B(v^L,\Omega)\,d\Omega.$$

(See discussions in Remark 3.1 for the use of linear penalty here instead of the BGK penalty of Filbet-Jin [10].)

(ii) To deal with the stiff terms $\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})$ and $\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,0}^{n+1})$ in (3.13) and (3.15) respectively, the BGK penalty is used for the Boltzmann collision operators [10], while a linear Fokker-Planck operator will be used to penalize for the FPL collision case, as done in [21]. Take the term $\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})/\varepsilon^2$ and the Boltzmann equation as an example. The idea is to split it into the summation of a stiff, dissipative part and a non-(or less) stiff, non-dissipative part:

$$\frac{\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})}{\varepsilon^2} = \underbrace{\frac{\mathcal{Q}^{LL}(f_{L,0}^{n},f_{L,0}^{n})-\mathcal{P}(f_{L,0}^n)}{\varepsilon^2}}_{\text{less stiff}} + \underbrace{\frac{\mathcal{P}(f_{L,0}^{n+1})}{\varepsilon^2}}_{\text{stiff}},$$

with $\mathcal{P}(f_{L,0})$ a well-balanced relaxation approximation of $\mathcal{Q}^{LL}(f_{L,0},f_{L,0})$ and defined by

$$\mathcal{P}(f_{L,0}) := \beta_1(M_{\{n,u,T\}}-f_{L,0}), \qquad \beta_1 = \sup_v \left|\frac{\mathcal{Q}^{LL}(f_{L,0},f_{L,0})}{f_{L,0}-M_{\{n,u,T\}}}\right|,$$

and the *local* Maxwellian distribution function is

$$M_{\{n,u,T\}} = \frac{n}{(2\pi T)^{3/2}}\exp\left(-\frac{|v-u|^2}{2T}\right), \tag{3.18}$$

and n, u, T are defined in (2.3) with $f=f_{L,0}$. How to obtain n, u, T from the moment systems of $f_{L,0}$ and $f_{H,0}$ will be discussed below. See the Appendix for more details of the penalization for both the Boltzmann and FPL cases.

(iii) To deal with the nonlinear collision operators $\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})$ in (3.9), since $f_{L,0}^{n+1}$ is already computed from (3.13), this is essentially a linear operator and we use the classical formula [4]

$$\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1}) = \frac{1}{4}\left[\mathcal{Q}^{LL}(f_{L,0}^{n+1}+f_{L,1}^{n+1},f_{L,0}^{n+1}+f_{L,1}^{n+1})-\mathcal{Q}^{LL}(f_{L,0}^{n+1}-f_{L,1}^{n+1},f_{L,0}^{n+1}-f_{L,1}^{n+1})\right]. \tag{3.19}$$

For each collision term on the right-hand side of (3.19) that has the same argument, we adopt the linear penalty method as mentioned in [25] to serve the purpose of removing the stiffness. The reason why the BGK-type penalty method of Filbet-Jin does not work well here will be explained in Remark 3.1 below. The strategy is to substitute $\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})$ by

$$\frac{1}{4}\Big[\mathcal{Q}^{LL}(f_{L,0}^{n}+f_{L,1}^{n},f_{L,0}^{n}+f_{L,1}^{n})+\mu(f_{L,0}^{n}+f_{L,1}^{n})-\mu(f_{L,0}^{n+1}+f_{L,1}^{n+1})$$

$$
-\Big(\mathcal{Q}^{LL}(f^n_{L,0}-f^n_{L,1},f^n_{L,0}-f^n_{L,1})+\mu(f^n_{L,0}-f^n_{L,1})-\mu(f^{n+1}_{L,0}-f^{n+1}_{L,1})\Big)\Big], \qquad (3.20)
$$

where μ is chosen sufficiently large. For the FPL equation, let

$$
\mu > \frac{1}{2}\max_v \lambda(D(g)),
$$

where $g = f_{L,0} \pm f_{L,1}$ and $\lambda(D(g))$ is the spectral radius of D defined by

$$
D(g) = \int_{\mathbb{R}^3} B^L(v^L - v^L_1) S(v^L - v^L_1) g^L_1 \, dv^L_1.
$$

For the Boltzmann equation, let $\mu > \mathcal{Q}^{LL,-}$, where we split the operator $\mathcal{Q}^{LL}$ in (3.20) as

$$
\mathcal{Q}^{LL}(g) = \mathcal{Q}^{LL,+}(g) - g\,\mathcal{Q}^{LL,-}(g),
$$

with the definitions $g = f_{L,0} \pm f_{L,1}$ and

$$
\begin{aligned}
\mathcal{Q}^{LL,+}(g) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^L(v^L - v^L_1, \Omega) g'^{,L} g'^{,L}_1 \, d\Omega dv^L_1,\\
\mathcal{Q}^{LL,-}(g) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^L(v^L - v^L_1, \Omega) g^L_1 \, d\Omega dv^L_1.
\end{aligned}
$$

The collision term $\mathcal{Q}^{HH}(f^{n+1}_{H,0}, f^{n+1}_{H,1})$ in (3.16) is dealt in a similar way.

Now with the penalties plugged into (3.13)–(3.14) and (3.15)–(3.16), our scheme becomes

$$
\frac{f^{n+1}_{L,0}-f^n_{L,0}}{\Delta t} = \frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f^n_{L,0},f^n_{L,0}) - \mathcal{P}(f^n_{L,0}) + \mathcal{P}(f^{n+1}_{L,0}) + \mathcal{Q}^{LH}_0(f^n_{L,0},f^n_{H,0})\Big], \qquad (3.21)
$$

$$
\begin{aligned}
\frac{f^{n+1}_{L,1}-f^n_{L,1}}{\Delta t} &= \frac{1}{\varepsilon^2}\Big[\frac{1}{\varepsilon}\Big(\mathcal{Q}^{LH}_\varepsilon(f^{n+1}_{L,0},f^{n+1}_{H,0}) - \mathcal{Q}^{LH}_0(f^{n+1}_{L,0},f^{n+1}_{H,0})\Big)\\
&+\frac{1}{2}\Big[\mathcal{Q}^{LL}(f^n_{L,0}+f^n_{L,1},f^n_{L,0}+f^n_{L,1})+\mu(f^n_{L,0}+f^n_{L,1})-\mu(f^{n+1}_{L,0}+f^{n+1}_{L,1})\\
&-\Big(\mathcal{Q}^{LL}(f^n_{L,0}-f^n_{L,1},f^n_{L,0}-f^n_{L,1})+\mu(f^n_{L,0}-f^n_{L,1})-\mu(f^{n+1}_{L,0}-f^{n+1}_{L,1})\Big)\Big]\\
&+\varepsilon\mathcal{Q}^{LL}(f^n_{L,1},f^n_{L,1})+\mathcal{Q}^{LH}_\varepsilon(f^n_{L,0},f^n_{H,1})+\big(\mathcal{Q}^{LH}_\varepsilon(f^n_{L,1},f^n_{H,0})-\mathcal{Q}^{LH}_0(f^n_{L,1},f^n_{H,0})\big)\\
&+\mathcal{Q}^{LH}_0(f^{n+1}_{L,1},f^{n+1}_{H,0})+\varepsilon\mathcal{Q}^{LH}_\varepsilon(f^n_{L,1},f^n_{H,1})\Big]; \qquad (3.22)
\end{aligned}
$$

$$
\frac{f^{n+1}_{H,0}-f^n_{H,0}}{\Delta t} = \frac{1}{\varepsilon}\Big[\mathcal{Q}^{HH}(f^{n+1}_{H,0},f^{n+1}_{H,0}) - \mathcal{P}(f^n_{H,0}) + \mathcal{P}(f^{n+1}_{H,0}) + \mathcal{Q}^{HL}_0(f^n_{H,0},f^n_{L,0})\Big], \qquad (3.23)
$$

$$
\begin{aligned}
\frac{f^{n+1}_{H,1}-f^n_{H,1}}{\Delta t} &= \frac{1}{\varepsilon}\Big[\frac{1}{\varepsilon}\Big(\mathcal{Q}^{HL}_\varepsilon(f^{n+1}_{H,0},f^{n+1}_{L,0}) - \mathcal{Q}^{HL}_0(f^{n+1}_{H,0},f^{n+1}_{L,0})\Big)\\
&+\frac{1}{2}\Big[\mathcal{Q}^{HH}(f^n_{H,0}+f^n_{H,1},f^n_{H,0}+f^n_{H,1})+\mu(f^n_{H,0}+f^n_{H,1})-\mu(f^{n+1}_{H,0}+f^{n+1}_{H,1})
\end{aligned}
$$

$$
-\Big(\mathcal{Q}^{HH}(f^n_{H,0}-f^n_{H,1},f^n_{H,0}-f^n_{H,1})+\mu(f^n_{H,0}-f^n_{H,1})-\mu(f^{n+1}_{H,0}-f^{n+1}_{H,1})\Big)\Big]
$$

$$
+\varepsilon\mathcal{Q}^{HH}(f^n_{H,1},f^n_{H,1})+\mathcal{Q}^{HL}_{\varepsilon}(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_{\varepsilon}(f^n_{H,1},f^n_{L,0})+\varepsilon\mathcal{Q}^{HL}_{\varepsilon}(f^n_{H,1},f^n_{L,1})\Big]. \tag{3.24}
$$

REMARK 3.1. In this remark, we will explain why the BGK- or Fokker-Planck-type penalties do not work well and so the linear penalties are used in (3.22) and (3.24). One needs to compute the moment systems in order to define the local Maxwellian $M_{\{n,u,T\}}$ in the penalty operators. Define the vectors

$$
\phi(v^L)=(1,v^L,\frac{|v^L|^2}{2}),\qquad \phi(v^H)=(1,v^H,\frac{|v^H|^2}{2}),
$$

and denote

$$
\phi^L_1=v^L,\qquad \phi^L_2=\frac{|v^L|^2}{2},\qquad \phi^H_1=v^H,\qquad \phi^H_2=\frac{|v^H|^2}{2}. \tag{3.25}
$$

Denote the moments by

$$
n=\int_{\mathbb{R}^3}f(v)dv:=P_0,\qquad nu=\int_{\mathbb{R}^3}vf(v)dv:=P_1,\qquad \int_{\mathbb{R}^3}\frac{1}{2}|v|^2f(v)dv:=P_2. \tag{3.26}
$$

Multiplying (3.21)–(3.22) by $\phi(v^L)$, we obtain the moment systems for $f_{L,0}$, $f_{L,1}$:

$$
\begin{aligned}
&(P_0)^{n+1}_{L,0}=(P_0)^n_{L,0},\\
&(P_1)^{n+1}_{L,0}=(P_1)^n_{L,0}+\frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3}v^L\,\mathcal{Q}^{LH}_0(f^n_{L,0},f^n_{H,0})(v^L)\,dv^L,\\
&(P_2)^{n+1}_{L,0}=(P_2)^n_{L,0},\\
&(P_0)^{n+1}_{L,1}=(P_0)^n_{L,1}+\frac{\mu\Delta t}{\varepsilon^2}\Big((P_0)^n_{L,1}-(P_0)^{n+1}_{L,1}\Big),
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
(P_1)^{n+1}_{L,1}=(P_1)^n_{L,1}+\frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3}\Big[&\frac{1}{\varepsilon}\Big(\mathcal{Q}^{LH}_{\varepsilon}(f^{n+1}_{L,0},f^{n+1}_{H,0})(v^L)-\mathcal{Q}^{LH}_0(f^{n+1}_{L,0},f^{n+1}_{H,0})(v^L)\Big)\\
&+\big(\mathcal{Q}^{LH}_{\varepsilon}(f^n_{L,0},f^n_{H,1})(v^L)+\mathcal{Q}^{LH}_{\varepsilon}(f^n_{L,1},f^n_{H,0})(v^L)-\mathcal{Q}^{LH}_0(f^n_{L,1},f^n_{H,0})(v^L)\big)\\
&+\Big(\mathcal{Q}^{LH}_0(f^{n+1}_{L,1},f^{n+1}_{H,0})(v^L)+\varepsilon\mathcal{Q}^{LH}_{\varepsilon}(f^n_{L,1},f^n_{H,1})(v^L)\Big)\Big]\phi^L_1\,dv^L\\
&+\frac{\mu\Delta t}{\varepsilon^2}\Big((P_1)^n_{L,1}-(P_1)^{n+1}_{L,1}\Big),
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
(P_2)^{n+1}_{L,1}=(P_2)^n_{L,1}+\frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3}\Big[&\frac{1}{\varepsilon}\,\mathcal{Q}^{LH}_{\varepsilon}(f^{n+1}_{L,0},f^{n+1}_{H,0})(v^L)+\mathcal{Q}^{LH}_{\varepsilon}(f^n_{L,0},f^n_{H,1})(v^L)\\
&+\mathcal{Q}^{LH}_{\varepsilon}(f^n_{L,1},f^n_{H,0})(v^L)+\varepsilon\mathcal{Q}^{LH}_{\varepsilon}(f^n_{L,1},f^n_{H,1})(v^L)\Big]\phi^L_2\,dv^L\\
&+\frac{\mu\Delta t}{\varepsilon^2}\Big((P_2)^n_{L,1}-(P_2)^{n+1}_{L,1}\Big),
\end{aligned} \tag{3.29}
$$

The reason why the BGK- or Fokker-Planck-type penalties do not work well for $f_{L,1}$ is due to the complexity of the moment Equation (3.28), in which the term

$\mathcal{Q}_0^{LH}(f_{L,1}^{n+1}, f_{H,0}^{n+1})(v^L)$ is implicit since $f_{L,1}^{n+1}$ is unknown. We find it difficult to invert this term, since both the moment Equation (3.28) and the Equation (3.14) for $f_{L,1}$ involve the same term $f_{L,1}^{n+1}$, thus the entire coupled system (3.13)–(3.14) needs to be inverted all together. Thus it is hard to get the Maxwellian associated with $f_{L,0}+f_{L,1}$ in the BGK- or Fokker-Planck-type penalty operators. Investigating a better approach than the currently used linear penalty method in (3.20) is deferred to a future work.

For the second collision term $\mathcal{Q}^{LL}(f_{L,0}^n - f_{L,1}^n, f_{L,0}^n - f_{L,1}^n)$ in (3.19), the reason we adopt the linear penalty is to avoid negative values of the temperature difference computed from the moment equations of $f_{L,0}$ and $f_{L,1}$ (hence unable to define the Maxwellian in the penalty operators). The difference between the Filbet-Jin (or Jin-Yan) penalty and the linear penalty is that the latter owns an error of $O(\Delta t)$ compared to $O(\varepsilon)$ as in the former, in the AP analysis. See [10]. Another disadvantage of the linear penalty method is that the linear operator does not preserve exactly the mass, momentum and energy as the BGK-type operator does, as mentioned in [10]. Nevertheless, the conservation issues (conservation of mass for each species, and conservation of total momentum and energy for the two species) will be addressed in our follow-up work.

3.2.3. The final numerical scheme. To summarize, the schemes for $f_{L,0}$, $f_{L,1}$ are given by

$$\frac{f_{L,0}^{n+1}-f_{L,0}^n}{\Delta t} = \frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_{L,0}^n, f_{L,0}^n) - \mathcal{P}(f_{L,0}^n) + \mathcal{P}(f_{L,0}^{n+1}) + \mathcal{Q}_0^{LH}(f_{L,0}^n, f_{H,0}^n)\Big], \tag{3.30}$$

$$\begin{aligned}\frac{f_{L,1}^{n+1}-f_{L,1}^n}{\Delta t} = \frac{1}{\varepsilon^2}\Bigg[&\frac{1}{\varepsilon}\Big(\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1}) - \mathcal{Q}_0^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1})\Big)\\
&+\frac{1}{2}\Big[\mathcal{Q}^{LL}(f_{L,0}^n+f_{L,1}^n, f_{L,0}^n+f_{L,1}^n) + \mu(f_{L,0}^n+f_{L,1}^n) - \mu(f_{L,0}^{n+1}+f_{L,1}^{n+1})\\
&-\Big(\mathcal{Q}^{LL}(f_{L,0}^n-f_{L,1}^n, f_{L,0}^n-f_{L,1}^n) + \mu(f_{L,0}^n-f_{L,1}^n) - \mu(f_{L,0}^{n+1}-f_{L,1}^{n+1})\Big)\Big]\\
&+\varepsilon\mathcal{Q}^{LL}(f_{L,1}^n, f_{L,1}^n) + \mathcal{Q}_\varepsilon^{LH}(f_{L,0}^n, f_{H,1}^n) + \big(\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^n, f_{H,0}^n) - \mathcal{Q}_0^{LH}(f_{L,1}^n, f_{H,0}^n)\big)\\
&+\mathcal{Q}_0^{LH}(f_{L,1}^{n+1}, f_{H,0}^{n+1}) + \varepsilon\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^n, f_{H,1}^n)\Bigg].\end{aligned} \tag{3.31}$$

The schemes for $f_{H,0}$, $f_{H,1}$ are given by

$$\frac{f_{H,0}^{n+1}-f_{H,0}^n}{\Delta t} = \frac{1}{\varepsilon}\Big[\mathcal{Q}^{HH}(f_{H,0}^{n+1}, f_{H,0}^{n+1}) - \mathcal{P}(f_{H,0}^n) + \mathcal{P}(f_{H,0}^{n+1}) + \mathcal{Q}_0^{HL}(f_{H,0}^n, f_{L,0}^n)\Big], \tag{3.32}$$

$$\begin{aligned}\frac{f_{H,1}^{n+1}-f_{H,1}^n}{\Delta t} = \frac{1}{\varepsilon}\Bigg[&\frac{1}{\varepsilon}\Big(\mathcal{Q}_\varepsilon^{HL}(f_{H,0}^{n+1}, f_{L,0}^{n+1}) - \mathcal{Q}_0^{HL}(f_{H,0}^{n+1}, f_{L,0}^{n+1})\Big)\\
&+\frac{1}{2}\Big[\mathcal{Q}^{HH}(f_{H,0}^n+f_{H,1}^n, f_{H,0}^n+f_{H,1}^n) + \mu(f_{H,0}^n+f_{H,1}^n) - \mu(f_{H,0}^{n+1}+f_{H,1}^{n+1})\\
&-\Big(\mathcal{Q}^{HH}(f_{H,0}^n-f_{H,1}^n, f_{H,0}^n-f_{H,1}^n) + \mu(f_{H,0}^n-f_{H,1}^n) - \mu(f_{H,0}^{n+1}-f_{H,1}^{n+1})\Big)\Big]\\
&+\varepsilon\mathcal{Q}^{HH}(f_{H,1}^n, f_{H,1}^n) + \mathcal{Q}_\varepsilon^{HL}(f_{H,0}^{n+1}, f_{L,1}^{n+1}) + \mathcal{Q}_\varepsilon^{HL}(f_{H,1}^n, f_{L,0}^n) + \varepsilon\mathcal{Q}_\varepsilon^{HL}(f_{H,1}^n, f_{L,1}^n)\Bigg].\end{aligned} \tag{3.33}$$

We couple with the following equations for moments of $f_{L,0}$ and $f_{H,0}$ (recall (3.26) for the definition):

$$(P_0)_{L,0}^{n+1} = (P_0)_{L,0}^n, \tag{3.34}$$

$$(P_1)_{L,0}^{n+1} = (P_1)_{L,0}^n + \frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3} v^L\, \mathcal{Q}_0^{LH}(f_{L,0}^n, f_{H,0}^n)(v^L)\, dv^L, \tag{3.35}$$

$$(P_2)_{L,0}^{n+1} = (P_2)_{L,0}^n, \tag{3.36}$$

$$(P_0)_{H,0}^{n+1} = (P_0)_{H,0}^n, \tag{3.37}$$

$$(P_1)_{H,0}^{n+1} = (P_1)_{H,0}^n + \frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3} v^H\, \mathcal{Q}_0^{HL}(f_{H,0}^n, f_{L,0}^n)(v^H)\, dv^H, \tag{3.38}$$

$$(P_2)_{H,0}^{n+1} = (P_2)_{H,0}^n + \frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3} \frac{|v^H|^2}{2}\, \mathcal{Q}_0^{HL}(f_{H,0}^n, f_{L,0}^n)(v^H)\, dv^H. \tag{3.39}$$

From the moment system, one computes u from $u = \frac{P_1}{P_0}$ and solves for T by using the formula

$$P_2 = \frac{1}{2}P_0|u|^2 + \frac{3}{2}P_0 T,$$

then obtain the local Maxwellian by the definition

$$M_{n,u,T}(v) = \frac{n}{(2\pi T)^{3/2}}\exp\left(-\frac{|v-u|^2}{2T}\right).$$

$M_{L,0}^{n+1}$ (or $M_{H,0}^{n+1}$) is obtained by n, u, T got from the moment equations of $f_{L,0}$ (or $f_{H,0}$), namely (3.34)–(3.36) (or (3.37)–(3.39)).

The following shows the detailed steps for the implementation of our proposed numerical scheme:

(a) get $M_{L,0}^{n+1}$ from (3.34)–(3.36), then update $f_{L,0}^{n+1}$ from (3.30);

(b) get $M_{H,0}^{n+1}$ from (3.37)–(3.38), then update $f_{H,0}^{n+1}$ from (3.32);

(c) update $f_{L,1}^{n+1}$ from (3.31);

(d) update $f_{H,1}^{n+1}$ from (3.33).

Our scheme, although contains some implicit terms, can be implemented *explicitly* for the case of Boltzmann collision operator, or just needs a linear elliptic solver in the case of FPL operator, as in the case of single species counterpart in [10] and [21]. We would like to mention that higher order time approximation can be extended.

3.3. The AP property. Our goal of this subsection is to prove the AP property of the discretized scheme (3.13)–(3.14) and (3.15)–(3.16).

First, for the light particles, inserting the expansion

$$\mathcal{Q}_\varepsilon^{LH} = \mathcal{Q}_0^{LH} + \varepsilon\mathcal{Q}_1^{LH} + \varepsilon^2\mathcal{Q}_2^{LH} + O(\varepsilon^3)$$

into (3.14), one has

$$\frac{f_{L,1}^{n+1} - f_{L,1}^n}{\Delta t}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^2}\Big[2\mathcal{Q}^{LL}(f_{L,0}^{n+1}, f_{L,1}^{n+1}) + \mathcal{Q}_0^{LH}(f_{L,0}^{n}, f_{H,1}^{n}) + \mathcal{Q}_0^{LH}(f_{L,1}^{n+1}, f_{H,0}^{n+1}) + \mathcal{Q}_1^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1})\Big] \\
&\quad + \frac{1}{\varepsilon}\Big[\mathcal{Q}^{LL}(f_{L,1}^{n}, f_{L,1}^{n}) + \mathcal{Q}_0^{LH}(f_{L,1}^{n}, f_{H,1}^{n}) + \mathcal{Q}_1^{LH}(f_{L,0}^{n}, f_{H,1}^{n}) \\
&\qquad + \mathcal{Q}_1^{LH}(f_{L,1}^{n}, f_{H,0}^{n}) + \mathcal{Q}_2^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1})\Big] \\
&\quad + \mathcal{Q}_1^{LH}(f_{L,1}^{n}, f_{H,1}^{n}) + \mathcal{Q}_2^{LH}(f_{L,0}^{n}, f_{H,1}^{n}) + \mathcal{Q}_2^{LH}(f_{L,1}^{n}, f_{H,0}^{n}).
\end{aligned} \tag{3.40}
$$

First, (3.13) gives

$$
\mathcal{Q}^{LL}(f_{L,0}^{n+1}, f_{L,0}^{n+1}) + \mathcal{Q}_0^{LH}(f_{L,0}^{n}, f_{H,0}^{n}) = O(\varepsilon^2),
$$

thus

$$
f_{L,0}^{n+1} = n_{L,0}^{n+1} M_{0,T_{L,0}^{n+1}} + O(\varepsilon^2 + \Delta t) := M_{L,0}^{n+1} + O(\varepsilon^2 + \Delta t). \tag{3.41}
$$

As for the heavy particles, by (3.15),

$$
\mathcal{Q}^{HH}(f_{H,0}^{n+1}, f_{H,0}^{n+1}) + \underbrace{\mathcal{Q}_0^{HL}(f_{H,0}^{n}, f_{L,0}^{n})}_{=O(\varepsilon^2+\Delta t)} = O(\varepsilon),
$$

which gives

$$
f_{H,0}^{n+1} = n_{H,0}^{n+1} M_{u_{H,0}^{n+1}, T_{H,0}^{n+1}} + O(\varepsilon + \Delta t) := M_{H,0}^{n+1} + O(\varepsilon + \Delta t). \tag{3.42}
$$

According to (3.40),

$$
2\mathcal{Q}^{LL}(f_{L,0}^{n+1}, f_{L,1}^{n+1}) + \underbrace{\mathcal{Q}_0^{LH}(f_{L,0}^{n}, f_{H,1}^{n})}_{=O(\varepsilon^2+\Delta t)} + \mathcal{Q}_0^{LH}(f_{L,1}^{n+1}, f_{H,0}^{n+1}) + \mathcal{Q}_1^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1}) = O(\varepsilon^2), \tag{3.43}
$$

which is an equation for $f_{L,1}^{n+1}$, and can be equivalently written in the form

$$
\phi_L^{n+1} = f_{L,1}^{n+1}(M_{L,0}^{n+1})^{-1}
$$

with

$$
\Gamma_{L,0}\phi_L^{n+1} = -(M_{L,0}^{n+1})^{-1}\mathcal{Q}_1^{LH}(M_{L,0}^{n+1}, M_{H,0}^{n+1}) + O(\varepsilon + \Delta t),
$$

where $\Gamma_{L,0}$ is the linearized operator

$$
\Gamma_{L,0}\phi_L^{n+1} = (M_{L,0}^{n+1})^{-1}\Big[2\mathcal{Q}^{LL}(M_{L,0}^{n+1}, M_{L,0}^{n+1}\phi_L^{n+1}) + \mathcal{Q}_0^{LH}(M_{L,0}^{n+1}\phi_L^{n+1}, M_{H,0}^{n+1})\Big].
$$

Analogous to the continuous case proved in [7], the unique solution in $(\ker(\Gamma_{L,0}))^{\perp}$ is given by

$$
f_{L,1}^{n+1}(v^L) = \underbrace{\frac{M_{L,0}^{n+1}}{T_{L,0}^{n+1}(v^L)} u_{H,0}^{n+1}\cdot v^L}_{:=f_{L,1}^{*,n+1}} + O(\varepsilon + \Delta t), \tag{3.44}
$$

where $f_{L,1}^{*,n+1}$ is used to denote the leading order of $f_{L,1}^{n+1}$.

Multiply (3.40) by ε and add up with (3.13), then

$$\begin{aligned}
&\frac{f_{L,0}^{n+1}-f_{L,0}^{n}}{\Delta t}+\varepsilon\frac{f_{L,1}^{n+1}-f_{L,1}^{n}}{\Delta t} \\
&=\frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^{n},f_{H,0}^{n})\right] \\
&\quad+\frac{1}{\varepsilon}\left[2\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^{n},f_{H,1}^{n})+\mathcal{Q}_0^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_1^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\right] \\
&\quad+\mathcal{Q}^{LL}(f_{L,1}^{n},f_{L,1}^{n})+\mathcal{Q}_0^{LH}(f_{L,1}^{n},f_{H,1}^{n})+\mathcal{Q}_1^{LH}(f_{L,0}^{n},f_{H,1}^{n}) \\
&\quad+\mathcal{Q}_1^{LH}(f_{L,1}^{n},f_{H,0}^{n})+\mathcal{Q}_2^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1}) \\
&\quad+\varepsilon\left[\mathcal{Q}_1^{LH}(f_{L,1}^{n},f_{H,1}^{n})+\mathcal{Q}_2^{LH}(f_{L,0}^{n},f_{H,1}^{n})+\mathcal{Q}_2^{LH}(f_{L,1}^{n},f_{H,0}^{n})\right]. \qquad (3.45)
\end{aligned}$$

Plugging in the leading order of (3.41), (3.44) and comparing the $O(1)$ terms on both sides, one gets

$$\begin{aligned}
\frac{M_{L,0}^{n+1}-M_{L,0}^{n}}{\Delta t}&=\mathcal{Q}^{LL}(f_{L,1}^{*,n},f_{L,1}^{*,n})+\mathcal{Q}_0^{LH}(f_{L,1}^{*,n},f_{H,1}^{*,n})+\mathcal{Q}_1^{LH}(M_{L,0}^{n},f_{H,1}^{*,n}) \\
&\quad+\mathcal{Q}_1^{LH}(f_{L,1}^{*,n},M_{H,0}^{n})+\mathcal{Q}_2^{LH}(M_{L,0}^{n+1},M_{H,0}^{n+1})+O(\Delta t). \qquad (3.46)
\end{aligned}$$

Integrate both sides of (3.46) against $|v^L|^2$ on v^L, then

$$\int \mathcal{Q}^{LL}(f_{L,1}^{*,n},f_{L,1}^{*,n})|v^L|^2\,dv^L=\int \mathcal{Q}_0^{LH}(f_{L,1}^{*,n},f_{H,1}^{*,n})|v^L|^2\,dv^L=0,$$
$$\int \mathcal{Q}_1^{LH}(M_{L,0}^{n},f_{H,1}^{*,n})|v^L|^2\,dv^L=\int \mathcal{Q}_0^{HL}(f_{H,1}^{*,n},M_{L,0}^{n})|v^L|^2\,dv^L=0,$$

and

$$\begin{aligned}
&\int\left[\mathcal{Q}_1^{LH}(f_{L,1}^{*,n},M_{H,0}^{n})+\mathcal{Q}_2^{LH}(M_{L,0}^{n+1},M_{H,0}^{n+1})\right]|v^L|^2\,dv^L \\
&=\int\left[\mathcal{Q}_0^{HL}(M_{H,0}^{n},f_{L,1}^{*,n})+\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1})\right]|v^H|^2\,dv^H \\
&=\int\left[\mathcal{Q}_0^{HL}(M_{H,0}^{n+1},f_{L,1}^{*,n+1})+\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1})\right]|v^H|^2\,dv^H+O(\Delta t) \\
&=3\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}n_{L,0}^{n+1}n_{H,0}^{n+1}(T_{H,0}^{n+1}-T_{L,0}^{n+1})+O(\Delta t),
\end{aligned}$$

where analogous calculation of the integrals for the continuous case is shown in [7]. Denote $\mathcal{D}_t(u^n)$ the discrete time derivative of the numerical quantity of interest u^n:

$$\mathcal{D}_t(u^n):=\frac{u^{n+1}-u^n}{\Delta t}.$$

Integrating both sides of (3.46) on v^L gives

$$\mathcal{D}_t(n_{L,0}^n)=O(\Delta t),$$

by using (2.12) in Theorem 2.1. Integrals of the left-hand side of (3.46) against 1 and $|v^L|^2$ on v^L are

$$\mathcal{D}_t\left(n_{L,0}^n,n_{L,0}^n\big(\frac{1}{2}|u_{L,0}^n|^2+\frac{3}{2}T_{L,0}^n\big)\right)^T.$$

Therefore, the limit of our discretized numerical scheme is given by

$$\mathcal{D}_t(n^n_{L,0}) = O(\Delta t),$$
$$\mathcal{D}_t\left(\frac{3}{2}n^n_{L,0}T^n_{L,0}\right) = 3\,\frac{\lambda(T^{n+1}_{L,0})}{T^{n+1}_{L,0}}\,n^{n+1}_{L,0}\,n^{n+1}_{H,0}\,(T^{n+1}_{H,0} - T^{n+1}_{L,0}) + O(\Delta t),$$

which is consistent with the implicit discretization of the continuous limit (2.28), up to a numerical error of $O(\Delta t)$.

Now we examine the system for the heavy particles $f_{H,0}$, $f_{H,1}$. Multiplying (3.16) by ε, adding it up with (3.15) and using the expansion

$$\mathcal{Q}^{HL}_{\varepsilon} = \mathcal{Q}^{HL}_0 + \varepsilon\mathcal{Q}^{HL}_1 + \varepsilon^2\mathcal{Q}^{HL}_2 + O(\varepsilon^3),$$

one gets

$$\begin{aligned}
&\frac{f^{n+1}_{H,0} - f^n_{H,0}}{\Delta t} + \varepsilon\,\frac{f^{n+1}_{H,1} - f^n_{H,1}}{\Delta t} \\
&= \frac{1}{\varepsilon}\left[\mathcal{Q}^{HH}(f^{n+1}_{H,0}, f^{n+1}_{H,0}) + \mathcal{Q}^{HL}_0(f^n_{H,0}, f^n_{L,0})\right] \\
&\quad + \mathcal{Q}^{HL}_0(f^{n+1}_{H,0}, f^{n+1}_{L,1}) + \mathcal{Q}^{HL}_0(f^n_{H,1}, f^n_{L,0}) + 2\mathcal{Q}^{HH}(f^{n+1}_{H,0}, f^{n+1}_{H,1}) + \mathcal{Q}^{HL}_1(f^{n+1}_{H,0}, f^{n+1}_{L,0}) \\
&\quad + \varepsilon\Big[\mathcal{Q}^{HH}(f^n_{H,1}, f^n_{H,1}) + \mathcal{Q}^{HL}_1(f^{n+1}_{H,0}, f^{n+1}_{L,1}) + \mathcal{Q}^{HL}_1(f^n_{H,1}, f^n_{L,0}) \\
&\qquad + \mathcal{Q}^{HL}_0(f^n_{H,1}, f^n_{L,1}) + \mathcal{Q}^{HL}_2(f^{n+1}_{H,0}, f^{n+1}_{L,0})\Big] \\
&\quad + \varepsilon^2\left[\mathcal{Q}^{HL}_1(f^n_{H,1}, f^n_{L,1}) + \mathcal{Q}^{HL}_2(f^{n+1}_{H,0}, f^{n+1}_{L,1}) + \mathcal{Q}^{HL}_2(f^{n+1}_{H,1}, f^{n+1}_{L,0})\right].
\end{aligned} \tag{3.47}$$

Plug in the leading order term of (3.42) and compare the $O(1)$ terms on both sides, then

$$\begin{aligned}
\frac{M^{n+1}_{H,0} - M^n_{H,0}}{\Delta t} &= 2\mathcal{Q}^{HH}(M^{n+1}_{H,0}, f^{*,n+1}_{H,1}) + \mathcal{Q}^{HL}_0(M^{n+1}_{H,0}, f^{*,n+1}_{L,1}) + \underbrace{\mathcal{Q}^{HL}_0(f^{*,n}_{H,1}, M^n_{L,0})}_{=0} \\
&\quad + \mathcal{Q}^{HL}_1(M^{n+1}_{H,0}, M^{n+1}_{L,0}) + O(\Delta t).
\end{aligned}$$

It is an equation for $f^{*,n+1}_{H,1}$ and can be equivalently written in terms of

$$\phi^{n+1}_H = f^{*,n+1}_{H,1}\,(M^{n+1}_{H,0})^{-1}$$

according to

$$\Gamma_{H,0}\,\phi^{n+1}_H = (M^{n+1}_{H,0})^{-1}\left[\mathcal{D}_t M^n_{H,0} - \mathcal{Q}^{HL}_0(M^{n+1}_{H,0}, f^{*,n+1}_{L,1}) - \mathcal{Q}^{HL}_1(M^{n+1}_{H,0}, M^{n+1}_{L,0})\right] + O(\Delta t), \tag{3.48}$$

$\Gamma_{H,0}$ is a linearization operator given by

$$\Gamma_{H,0}\,\phi^{n+1}_H = 2\,(M^{n+1}_{H,0})^{-1}\,\mathcal{Q}^{HH}(M^{n+1}_{H,0}, M^{n+1}_{H,0}\,\phi^{n+1}_H).$$

The necessary and sufficient condition for the solvability of Equation (3.48) is given by

$$\int_{\mathbb{R}^3}\left[D_t M^n_{H,0} - \mathcal{Q}^{HL}_0(M^{n+1}_{H,0}, f^{*,n+1}_{L,1}) - \mathcal{Q}^{HL}_1(M^{n+1}_{H,0}, M^{n+1}_{L,0})\right]\begin{pmatrix}1\\ v^H\\ |v^H|^2\end{pmatrix}dv^H = O(\Delta t)\,\mathbb{I}_3, \tag{3.49}$$

where $\mathbb{I}_3=(1,1,1)^T$. Analogous to the calculation in [7] for the continuous equations,

$$\int_{\mathbb{R}^3}\left[\mathcal{Q}_0^{HL}(M_{H,0}^{n+1},f_{L,1}^{*,n+1})+\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1})\right](v^H)\begin{pmatrix}1\\ v^H\\ \frac{|v^H|^2}{2}\end{pmatrix}dv^H$$
$$=\begin{pmatrix}0\\0\\-3\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}n_{L,0}^{n+1}n_{H,0}^{n+1}(T_{H,0}^{n+1}-T_{L,0}^{n+1})\end{pmatrix}+O(\Delta t)\mathbb{I}_3. \tag{3.50}$$

Insert (3.50) into (3.49), then

$$\mathcal{D}_t\begin{pmatrix}n_{H,0}^n\\ n_{H,0}^n u_{H,0}^n\\ n_{H,0}^n(\frac{1}{2}|u_{H,0}^n|^2+\frac{3}{2}T_{H,0}^n)\end{pmatrix}=\begin{pmatrix}0\\0\\-3\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}n_{L,0}^{n+1}n_{H,0}^{n+1}(T_{H,0}^{n+1}-T_{L,0}^{n+1})\end{pmatrix}+O(\Delta t)\mathbb{I}_3. \tag{3.51}$$

This shows that $n_{H,0}^n$, $u_{H,0}^n$ are constant in time with a numerical error of $O(\Delta t)$,

$$\mathcal{D}_t(n_{H,0}^n)=O(\Delta t),\qquad \mathcal{D}_t(n_{H,0}^n u_{H,0}^n)=O(\Delta t),$$

while $T_{H,0}^n$ evolves according to

$$\mathcal{D}_t\left(\frac{3}{2}n_{H,0}^n T_{H,0}^n\right)=-3\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}n_{L,0}^{n+1}n_{H,0}^{n+1}(T_{H,0}^{n+1}-T_{L,0}^{n+1})+O(\Delta t),$$

which is consistent with the discretized implicit scheme of the limiting system (2.29), up to a numerical error of $O(\Delta t)$.

We conclude our AP analysis with the following theorem.

THEOREM 3.1. *The time discretized numerical schemes given by (3.13)–(3.14) and (3.15)–(3.16), as $\varepsilon\to 0$, approaches the system*

$$\begin{aligned}
&n_{L,0}^{n+1}=n_{L,0}^n+O(\Delta t),\\
&n_{H,0}^{n+1}=n_{H,0}^n+O(\Delta t),\qquad n_{H,0}^{n+1}u_{H,0}^{n+1}=n_{H,0}^n u_{H,0}^n+O(\Delta t),\\
&\frac{d}{dt}\left(\frac{3}{2}n_{L,0}^n T_{L,0}^n\right)=3\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}n_{L,0}^{n+1}n_{H,0}^{n+1}(T_{H,0}^{n+1}-T_{L,0}^{n+1})+O(\Delta t),\\
&\frac{d}{dt}\left(\frac{3}{2}n_{H,0}^n T_{H,0}^n\right)=-3\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}n_{L,0}^{n+1}n_{H,0}^{n+1}(T_{H,0}^{n+1}-T_{L,0}^{n+1})+O(\Delta t),
\end{aligned}$$

which are consistent with the implicit discretization of the continuous limit shown in (2.28)–(2.29), with a numerical error of $O(\Delta t)$.

REMARK 3.2. We would also like to point out that our AP analysis for the scheme does not include the penalty method, namely the schemes (3.30)–(3.33) that one actually uses in practice, since it is hard to prove a scheme is AP with all the penalty terms included, not done even for the single species Boltzmann (or FPL) equation [10, 21].

4. The space inhomogeneous systems

In the space inhomogeneous case, the evolution equations are given by system (2.5)–(2.6). We first recall the main results in [8] in the following Theorem:

THEOREM 4.1. *As $\varepsilon \to 0$, the limit distributions and limit systems are given by*

$$f_0^L(x,v,t) = n_0^L(x,t)\, M_{0,T_0^L(x,t)}, \qquad f_0^H(x,v,t) = n_0^H(x,t)\, M_{u_0^H(x,t),T_0^H(x,t)},$$

where n_0^L, T_0^L, n_0^H, T_0^H satisfy the coupled system:

$$\frac{\partial n_0^L}{\partial t} + \nabla_x\cdot(n_0^L u_0^H) - \nabla_x\cdot\left[D_{11}\left(\nabla_x n_0^L - \frac{F^L n_0^L}{T_0^L}\right) + D_{12}\left(n_0^L \frac{\nabla_x T_0^L}{T_0^L}\right)\right] = 0, \tag{4.1}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{3}{2} n_0^L T_0^L\right) + \nabla_x\cdot\left(\frac{5}{2} n_0^L T_0^L u_0^H\right) - n_0^L F^L\cdot u_0^H \\
&\qquad - \nabla_x\cdot\left[D_{21}\left(\nabla_x n_0^L - \frac{F^L n_0^L}{T_0^L}\right) + D_{22}\left(n_0^L \frac{\nabla_x T_0^L}{T_0^L}\right)\right] \\
&\qquad + F^L\cdot\left[D_{11}\left(\nabla_x n_0^L - \frac{F^L n_0^L}{T_0^L}\right) + D_{12}\left(n_0^L \frac{\nabla_x T_0^L}{T_0^L}\right)\right] \\
&= u_0^H\cdot\left[\nabla_x(n_0^L T_0^L) - F^L n_0^L\right] + 3\frac{\lambda(T_0^L)}{T_0^L} n_0^L n_0^H (T_0^H - T_0^L),
\end{aligned} \tag{4.2}$$

and

$$\frac{\partial n_0^H}{\partial t} + \nabla_x\cdot(n_0^H u_0^H) = 0, \tag{4.3}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}(n_0^H u_0^H) + \nabla_x\cdot(n_0^H u_0^H \otimes u_0^H) + \nabla_x(n_0^H T_0^H) - n_0^H F^H \\
&= -\left(\nabla_x(n_0^L T_0^L) - F^L n_0^L\right),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{n_0^H |u_0^H|^2}{2} + \frac{3}{2} n_0^H T_0^H\right) + \nabla_x\cdot\left(\left(\frac{n_0^H |u_0^H|^2}{2} + \frac{5}{2} n_0^H T_0^H\right) u_0^H\right) - n_0^H F^H\cdot u_0^H \\
&= -u_0^H\cdot\left[\nabla_x(n_0^L T_0^L) - F^L n_0^L\right] - 3\frac{\lambda(T_0^L)}{T_0^L} n_0^L n_0^H (T_0^H - T_0^L),
\end{aligned} \tag{4.5}$$

where D_{ij} $(i,j=1,2)$ and $\lambda(T)$ are given in the Appendix.

Insert the expansion

$$f^L = f_0^L + \varepsilon f_1^L, \qquad f^H = f_0^H + \varepsilon f_1^H$$

into (2.5) and (2.6), then

$$\begin{aligned}
&\frac{\partial}{\partial t}(f_0^L + \varepsilon f_1^L) + \frac{1}{\varepsilon}\left(v^L\cdot\nabla_x f_0^L + F^L\cdot\nabla_{v^L} f_0^L\right) + \left(v^L\cdot\nabla_x f_1^L + F^L\cdot\nabla_{v^L} f_1^L\right) \\
&= \frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f_0^L + \varepsilon f_1^L, f_0^L + \varepsilon f_1^L) + \mathcal{Q}_\varepsilon^{LH}(f_0^L + \varepsilon f_1^L, f_0^H + \varepsilon f_1^H)\right] \\
&= \frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_0^L, f_0^L) + 2\varepsilon \mathcal{Q}^{LL}(f_0^L, f_1^L) + \varepsilon^2 \mathcal{Q}^{LL}(f_1^L, f_1^L) \\
&\quad + \mathcal{Q}_\varepsilon^{LH}(f_0^L, f_0^H) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_0^L, f_1^H) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_0^H) + \varepsilon^2 \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_1^H)\Big].
\end{aligned}$$

We design the scheme by letting f_0^L, f_1^L satisfy the system

$$\frac{\partial}{\partial t} f_0^L + \left(v^L \cdot \nabla_x f_1^L + F^L \cdot \nabla_{v^L} f_1^L\right) = \frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_0^L, f_0^L) + \mathcal{Q}_0^{LH}(f_0^L, f_0^H)\Big], \tag{4.6}$$

$$\begin{aligned}&\frac{\partial}{\partial t} f_1^L + \frac{1}{\varepsilon^2}\left(v^L \cdot \nabla_x f_0^L + F^L \cdot \nabla_{v^L} f_0^L\right)\\ &= \frac{1}{\varepsilon^2}\Big[\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{LH}(f_0^L, f_0^H) - \mathcal{Q}_0^{LH}(f_0^L, f_0^H)\right) + 2\mathcal{Q}^{LL}(f_0^L, f_1^L) + \varepsilon \mathcal{Q}^{LL}(f_1^L, f_1^L)\\ &\quad + \mathcal{Q}_\varepsilon^{LH}(f_0^L, f_1^H) + \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_0^H) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_1^H)\Big],\end{aligned} \tag{4.7}$$

and letting f_0^H, f_1^H satisfy the following system

$$\frac{\partial}{\partial t} f_0^H + \varepsilon\left(v^H \cdot \nabla_x f_1^H + F^H \cdot \nabla_{v^H} f_1^H\right) = \frac{1}{\varepsilon}\Big[\mathcal{Q}^{H}(f_0^H, f_0^H) + \mathcal{Q}_0^{HL}(f_0^H, f_0^L)\Big], \tag{4.8}$$

$$\begin{aligned}&\frac{\partial}{\partial t} f_1^H + \frac{1}{\varepsilon}\left(v^H \cdot \nabla_x f_0^H + F^H \cdot \nabla_{v^H} f_0^H\right)\\ &= \frac{1}{\varepsilon}\Big[\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{HL}(f_0^H, f_0^L) - \mathcal{Q}_0^{HL}(f_0^H, f_0^L)\right) + 2\mathcal{Q}^{HH}(f_0^H, f_1^H) + \varepsilon \mathcal{Q}^{HH}(f_1^H, f_1^H)\\ &\quad + \mathcal{Q}_\varepsilon^{HL}(f_0^H, f_1^L) + \mathcal{Q}_\varepsilon^{HL}(f_1^H, f_0^L) + \varepsilon \mathcal{Q}_\varepsilon^{HL}(f_1^H, f_1^L)\Big].\end{aligned} \tag{4.9}$$

4.1. Time discretization. Following [19], we rewrite (4.7) into the diffusive relaxation system

$$\begin{aligned}&\frac{\partial}{\partial t} f_1^L + \psi_1\left(v^L \cdot \nabla_x f_0^L + F^L \cdot \nabla_{v^L} f_0^L\right)\\ &= \frac{1}{\varepsilon^2}\Big[\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{LH}(f_0^L, f_0^H) - \mathcal{Q}_0^{LH}(f_0^L, f_0^H)\right) + 2\mathcal{Q}^{LL}(f_0^L, f_1^L) + \varepsilon \mathcal{Q}^{LL}(f_1^L, f_1^L)\\ &\quad + \mathcal{Q}_\varepsilon^{LH}(f_0^L, f_1^H) + \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_0^H) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_1^L, f_1^H)\\ &\quad - (1-\varepsilon^2\psi_1)\left(v^L \cdot \nabla_x f_0^L + F^L \cdot \nabla_{v^L} f_0^L\right)\Big],\end{aligned} \tag{4.10}$$

where a simple choice of ψ_1 is

$$\psi_1 = \min\{1, \frac{1}{\varepsilon^2}\}.$$

Note that when ε is small, $\psi_1 = 1$. The collision operators on the right-hand side are discretized exactly the same as the space homogeneous case. Then the time discretizations for (4.6) and (4.10) are

$$\begin{aligned}&\frac{f_{L,0}^{n+1} - f_{L,0}^n}{\Delta t} + \left(v^L \cdot \nabla_x f_{L,1}^n + F_L^n \cdot \nabla_{v^L} f_{L,1}^n\right)\\ &= \frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f_L^{n+1}, f_{L,0}^{n+1}) + \mathcal{Q}_0^{LH}(f_{L,0}^n, f_{H,0}^n)\right],\end{aligned} \tag{4.11}$$

$$\begin{aligned}&\frac{f_{L,1}^{n+1} - f_{L,1}^n}{\Delta t} + \psi_1\left(v^L \cdot \nabla_x f_{L,0}^n + F_L^n \cdot \nabla_{v^L} f_{L,0}^n\right)\\ &= \frac{1}{\varepsilon^2}\Big[\frac{1}{\varepsilon}\Big(\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1}) - \mathcal{Q}_0^{LH}(f_{L,0}^{n+1}, f_{H,0}^{n+1})\Big) + 2\mathcal{Q}^{LL}(f_{L,0}^{n+1}, f_{L,1}^{n+1}) + \varepsilon \mathcal{Q}^{LL}(f_{L,1}^n, f_{L,1}^n)\\ &\quad + \mathcal{Q}_\varepsilon^{LH}(f_{L,0}^n, f_{H,1}^n) + \mathcal{Q}_\varepsilon^{LH}(f_{L,1}^{n+1}, f_{H,0}^{n+1}) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_{L,1}^n, f_{H,1}^n)\end{aligned}$$

$$-(1-\varepsilon^2\psi_1)(v^L\cdot\nabla_x f_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L} f_{L,0}^{n+1})\Big]. \tag{4.12}$$

Using the same technique, time discretizations for the systems (4.8) and (4.9) are given by

$$\begin{aligned}&\frac{f_{H,0}^{n+1}-f_{H,0}^n}{\Delta t}+\varepsilon\Big(v^H\cdot\nabla_x f_{H,1}^n+F_H^n\cdot\nabla_{v^H} f_{H,1}^n\Big)\\ =&\frac{1}{\varepsilon}\Big[\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_0^{HL}(f_{H,0}^n,f_{L,0}^n)\Big],\end{aligned} \tag{4.13}$$

$$\begin{aligned}&\frac{f_{H,1}^{n+1}-f_{H,1}^n}{\Delta t}+\psi_2\Big(v^H\cdot\nabla_x f_{H,0}^n+F_H^n\cdot\nabla_{v^H} f_{H,0}^n\Big)\\ =&\frac{1}{\varepsilon}\Big[\frac{1}{\varepsilon}\Big(\mathcal{Q}_\varepsilon^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})-\mathcal{Q}_0^{HL}(f_{H,0}^{n+1},f_{L,0}^{n+1})\Big)+2\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,1}^{n+1})+\varepsilon\mathcal{Q}^{HH}(f_{H,1}^n,f_{H,1}^n)\\ &\quad+\mathcal{Q}_\varepsilon^{HL}(f_{H,0}^{n+1},f_{L,1}^{n+1})+\mathcal{Q}_\varepsilon^{HL}(f_{H,1}^n,f_{L,0}^n)+\varepsilon\mathcal{Q}_\varepsilon^{HL}(f_{H,1}^n,f_{L,1}^n)\\ &\quad-(1-\varepsilon\psi_2)\Big(v^H\cdot\nabla_x f_{H,0}^{n+1}+F_H^n\cdot\nabla_{v^H} f_{H,0}^{n+1}\Big)\Big],\end{aligned} \tag{4.14}$$

where

$$\psi_2=\min\{1,\frac{1}{\varepsilon}\}.$$

We will use the penalties exactly the same as discussed in Subsection 3.2, namely the right-hand side of the schemes (3.30)–(3.33). We omit repeating it here.

4.2. The AP property. First, for the light particles, inserting the expansion

$$\mathcal{Q}_\varepsilon^{LH}=\mathcal{Q}_0^{LH}+\varepsilon\mathcal{Q}_1^{LH}+\varepsilon^2\mathcal{Q}_2^{LH}+O(\varepsilon^3)$$

into (4.12), one has

$$\begin{aligned}&\frac{f_{L,1}^{n+1}-f_{L,1}^n}{\Delta t}+\big(v^L\cdot\nabla_x f_{L,0}^n+F_L^n\cdot\nabla_{v^L} f_{L,0}^n\big)\\ =&\frac{1}{\varepsilon^2}\Big[2\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^n,f_{H,1}^n)+\mathcal{Q}_0^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_1^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\\ &\quad-(v^L\cdot\nabla_x f_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L} f_{L,0}^{n+1})\Big]\\ &+\frac{1}{\varepsilon}\Big[\mathcal{Q}^{LL}(f_{L,1}^n,f_{L,1}^n)+\mathcal{Q}_0^{LH}(f_{L,1}^n,f_{H,1}^n)+\mathcal{Q}_1^{LH}(f_{L,0}^n,f_{H,1}^n)\\ &\quad+\mathcal{Q}_1^{LH}(f_{L,1}^n,f_{H,0}^n)+\mathcal{Q}_2^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\Big]\\ &+\mathcal{Q}_1^{LH}(f_{L,1}^n,f_{H,1}^n)+\mathcal{Q}_2^{LH}(f_{L,0}^n,f_{H,1}^n)+\mathcal{Q}_2^{LH}(f_{L,1}^n,f_{H,0}^n)\\ &+\big(v^L\cdot\nabla_x f_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L} f_{L,0}^{n+1}\big).\end{aligned} \tag{4.15}$$

From (4.11), we have

$$\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^n,f_{H,0}^n)=O(\varepsilon^2),$$

which gives

$$f_{L,0}^{n+1}=n_{L,0}^{n+1}M_{0,T_{L,0}^{n+1}}+O(\varepsilon^2+\Delta t):=M_{L,0}^{n+1}+O(\varepsilon^2+\Delta t). \tag{4.16}$$

From (4.13),

$$\mathcal{Q}^{HH}(f_{H,0}^{n+1},f_{H,0}^{n+1})+\underbrace{\mathcal{Q}_0^{HL}(f_{H,0}^{n},f_{L,0}^{n})}_{=O(\varepsilon^2+\Delta t)}=O(\varepsilon),$$

thus

$$f_{H,0}^{n+1}=n_{H,0}^{n+1}M_{u_{H,0}^{n+1},T_{H,0}^{n+1}}+O(\Delta t):=M_{H,0}^{n+1}+O(\varepsilon+\Delta t). \tag{4.17}$$

From (4.15),

$$\begin{aligned}&2\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})+\underbrace{\mathcal{Q}_0^{LH}(f_{L,0}^{n},f_{H,1}^{n})}_{=O(\varepsilon^2+\Delta t)}+\mathcal{Q}_0^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_1^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\\&=v^L\cdot\nabla_x f_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L}f_{L,0}^{n+1}+O(\varepsilon^2).\end{aligned} \tag{4.18}$$

(4.18) is an equation for $f_{L,1}^{n+1}$ and can be equivalently written in terms of

$$\phi_L^{n+1}=f_{L,1}^{n+1}(M_{L,0}^{n+1})^{-1}$$

according to:

$$\Gamma_{L,0}\phi_L^{n+1}=-(M_{L,0}^{n+1})^{-1}\left[v^L\cdot\nabla_x M_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L}M_{L,0}^{n+1}-\mathcal{Q}_1^{LH}(M_{L,0}^{n+1},M_{H,0}^{n+1})\right]+O(\varepsilon+\Delta t),$$

where $\Gamma_{L,0}$ is the linearized operator given by

$$\Gamma_{L,0}\phi_L^{n+1}=(M_{L,0}^{n+1})^{-1}\left[2\mathcal{Q}^{LL}(M_{L,0}^{n+1},M_{L,0}^{n+1}\phi_L^{n+1})+\mathcal{Q}_0^{LH}(M_{L,0}^{n+1}\phi_L^{n+1},M_{H,0}^{n+1})\right].$$

As proved in [8], the unique solution in $(\ker(\Gamma_{L,0}))^{\perp}$ is given by

$$\begin{aligned}&\phi_L^{n+1}\\&=\frac{1}{n_{L,0}^{n+1}}\left(-\left(\nabla_x n_{L,0}^{n+1}-\frac{F_L^{n+1}n_{L,0}^{n+1}}{T_{L,0}^{n+1}}\right)\Psi_1(|v^L|)-n_{L,0}^{n+1}\frac{\nabla_x T_{L,0}^{n+1}}{T_{L,0}^{n+1}}\Psi_2(|v^L|)+\frac{n_{L,0}^{n+1}u_{H,0}^{n+1}}{T_{L,0}^{n+1}}\right)\cdot v^L\\&\qquad+O(\varepsilon+\Delta t),\end{aligned}$$

thus

$$f_{L,1}^{n+1}=\underbrace{M_{L,0}^{n+1}\phi_L^{n+1}}_{:=f_{L,1}^{*,n+1}}+O(\varepsilon+\Delta t). \tag{4.19}$$

We multiply (4.15) by ε and add it to (4.11), then get

$$\begin{aligned}&\frac{f_{L,0}^{n+1}-f_{L,0}^{n}}{\Delta t}+\varepsilon\frac{f_{L,1}^{n+1}-f_{L,1}^{n}}{\Delta t}\\&\quad+\left(v^L\cdot\nabla_x f_{L,1}^{n}+F_L^{n}\cdot\nabla_{v^L}f_{L,1}^{n}\right)+\varepsilon\left(v^L\cdot\nabla_x f_{L,0}^{n}+F_L^{n}\cdot\nabla_{v^L}f_{L,0}^{n}\right)\\&=\frac{1}{\varepsilon^2}\left[\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,0}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^{n},f_{H,0}^{n})\right]\\&\quad+\frac{1}{\varepsilon}\Big[2\mathcal{Q}^{LL}(f_{L,0}^{n+1},f_{L,1}^{n+1})+\mathcal{Q}_0^{LH}(f_{L,0}^{n},f_{H,1}^{n})+\mathcal{Q}_0^{LH}(f_{L,1}^{n+1},f_{H,0}^{n+1})+\mathcal{Q}_1^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})\end{aligned}$$

$$
\begin{aligned}
&-\big(v^L\cdot\nabla_x f_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L} f_{L,0}^{n+1}\big)\Big]\\
&+\mathcal{Q}^{LL}(f_{L,1}^n,f_{L,1}^n)+\mathcal{Q}_0^{LH}(f_{L,1}^n,f_{H,1}^n)+\mathcal{Q}_1^{LH}(f_{L,0}^n,f_{H,1}^n)\\
&+\mathcal{Q}_1^{LH}(f_{L,1}^n,f_{H,0}^n)+\mathcal{Q}_2^{LH}(f_{L,0}^{n+1},f_{H,0}^{n+1})+\varepsilon\Big[\mathcal{Q}_1^{LH}(f_{L,1}^n,f_{H,1}^n)+\mathcal{Q}_2^{LH}(f_{L,0}^n,f_{H,1}^n)\\
&+\mathcal{Q}_2^{LH}(f_{L,1}^n,f_{H,0}^n)+\big(v^L\cdot\nabla_x f_{L,0}^{n+1}+F_L^{n+1}\cdot\nabla_{v^L} f_{L,0}^{n+1}\big)\Big]. \qquad (4.20)
\end{aligned}
$$

Plugging in the leading order term of (4.16), (4.19) and comparing the $O(1)$ terms on both sides gives

$$
\begin{aligned}
&\frac{M_{L,0}^{n+1}-M_{L,0}^n}{\Delta t}+v^L\cdot\nabla_x f_{L,1}^{*,n}+F_L^n\cdot\nabla_{v^L} f_{L,1^{*,n}}\\
&=\mathcal{Q}^{LL}(f_{L,1}^{*,n},f_{L,1}^{*,n})+\mathcal{Q}_0^{LH}(f_{L,1}^{*,n},f_{H,1}^{*,n})+\mathcal{Q}_1^{LH}(f_{L,0}^{*,n},f_{H,1}^{*,n})+\mathcal{Q}_1^{LH}(f_{L,1}^{*,n},M_{H,0}^n)\\
&+\mathcal{Q}_2^{LH}(M_{L,0}^{n+1},M_{H,0}^{n+1})+O(\Delta t). \qquad (4.21)
\end{aligned}
$$

Integrate both sides of (4.20) against 1, v^L, $|v^L|^2$ on v^L, by the statement (2)(i) and (2.13) in Theorem 2.1, thus

$$
\begin{aligned}
&\int \mathcal{Q}^{LL}(f_{L,1}^{*,n},f_{L,1}^{*,n})\,|v^L|^2\,dv^L=\int \mathcal{Q}_0^{LH}(f_{L,1}^{*,n},f_{H,1}^{*,n})\,|v^L|^2\,dv^L=0,\\
&\int \mathcal{Q}_1^{LH}(M_{L,0}^n,f_{H,1}^{*,n})\,|v^L|^2\,dv^L=\int \mathcal{Q}_0^{HL}(f_{H,1}^{*,n},M_{L,0}^n)\,|v^L|^2\,dv^L=0.
\end{aligned}
$$

Integrals of $\frac{M_{L,0}^{n+1}-M_{L,0}^n}{\Delta t}$ are

$$
\frac{d}{dt}\left(n_{L,0}^n,\,n_{L,0}^n u_{L,0}^n,\,n_{L,0}^n(\frac{1}{2}|u_{L,0}^n|^2+\frac{3}{2}T_{L,0}^n)\right)^T.
$$

Analogous to the calculation in [8], then

$$
\begin{aligned}
&\int\Big[\mathcal{Q}_1^{LH}(f_{L,1}^{*,n},M_{H,0}^n)+\mathcal{Q}_2^{LH}(M_{L,0}^{n+1},M_{H,0}^{n+1})\Big]\,|v^L|^2\,dv^L\\
&=\int\Big[\mathcal{Q}_0^{HL}(M_{H,0}^n,f_{L,1}^{*,n})+\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1})\Big]\,|v^H|^2\,dv^H\\
&=\int\Big[\mathcal{Q}_0^{HL}(M_{H,0}^{n+1},f_{L,1}^{*,n+1})+\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1})\Big]\,|v^H|^2\,dv^H+O(\Delta t)\\
&=u_{H,0}^{n+1}\cdot\Big[\nabla_x(n_{L,0}^{n+1}T_{L,0}^{n+1})-F_L^{n+1}n_{L,0}^{n+1}\Big]+3\,\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}\,n_{L,0}^{n+1}\,n_{H,0}^{n+1}\,(T_{H,0}^{n+1}-T_{L,0}^{n+1})+O(\Delta t).
\end{aligned}
$$

Therefore, the limit of our scheme is given by

$$
\begin{aligned}
&\frac{\partial n_{L,0}^n}{\partial t}+\nabla_x\cdot(n_{L,0}^n u_{H,0}^n)\\
&\qquad-\nabla_x\cdot\left[D_{11}\left(\nabla_x n_{L,0}^n-\frac{F_L^n n_{L,0}^n}{T_{L,0}^n}\right)+D_{12}\left(n_{L,0}^n\,\frac{\nabla_x T_{L,0}^n}{T_{L,0}^n}\right)\right]=O(\Delta t), \qquad (4.22)\\
&\frac{\partial}{\partial t}\left(\frac{3}{2}n_{L,0}^n T_{L,0}^n\right)+\nabla_x\cdot\left(\frac{5}{2}n_{L,0}^n T_{L,0}^n u_{H,0}^n\right)-n_{L,0}^n F_L^n u_{H,0}^n
\end{aligned}
$$

$$
\begin{aligned}
&-\nabla_x\cdot\left[D_{21}\left(\nabla_x n^n_{L,0}-\frac{F^n_L n^n_{L,0}}{T^n_{L,0}}\right)+D_{22}\left(n^n_{L,0}\frac{\nabla_x T^n_{L,0}}{T^n_{L,0}}\right)\right]\\
&+F^n_L\cdot\left[D_{11}\left(\nabla_x n^n_{L,0}-\frac{F^n_L n^n_{L,0}}{T^n_{L,0}}\right)+D_{12}\left(n^n_{L,0}\frac{\nabla_x T^n_{L,0}}{T^n_{L,0}}\right)\right]\\
=&u^{n+1}_{H,0}\cdot\left[\nabla_x(n^{n+1}_{L,0}T^{n+1}_{L,0})-F^{n+1}_L n^{n+1}_{L,0}\right]+3\frac{\lambda(T^{n+1}_{L,0})}{T^{n+1}_{L,0}}n^{n+1}_{L,0}n^{n+1}_{H,0}(T^{n+1}_{H,0}-T^{n+1}_{L,0})+O(\Delta t).
\end{aligned}
\tag{4.23}
$$

This is first order (in Δt) consistent to the implicit numerical discretization of the limit Equation (4.1)–(4.2).

Next we look at the system for the heavy particles. Inserting the expansion

$$
\mathcal{Q}^{HL}_0+\varepsilon\mathcal{Q}^{HL}_1+\varepsilon^2\mathcal{Q}^{HL}_2+O(\varepsilon^3)
$$

into (4.14), one has

$$
\begin{aligned}
&\frac{f^{n+1}_{H,1}-f^n_{H,1}}{\Delta t}+\left(v^H\cdot\nabla_x f^n_{H,0}+F^n_H\cdot\nabla_{v^H}f^n_{H,0}\right)\\
=&\frac{1}{\varepsilon}\Big[\mathcal{Q}^{HL}_0(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_0(f^n_{H,1},f^n_{L,0})+2\mathcal{Q}^{HH}(f^{n+1}_{H,0},f^{n+1}_{H,1})+\mathcal{Q}^{HL}_1(f^{n+1}_{H,0},f^{n+1}_{L,0})\\
&-\left(v^H\cdot\nabla_x f^{n+1}_{H,0}+F^n_H\cdot\nabla_{v^H}f^{n+1}_{H,0}\right)\Big]\\
&+\mathcal{Q}^{HH}(f^n_{H,1},f^n_{H,1})+\mathcal{Q}^{HL}_1(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_1(f^n_{H,1},f^n_{L,0})+\mathcal{Q}^{HL}_0(f^n_{H,1},f^n_{L,1})\\
&+\mathcal{Q}^{HL}_2(f^{n+1}_{H,0},f^{n+1}_{L,0})+\left(v^H\cdot\nabla_x f^{n+1}_{H,0}+F^n_H\cdot\nabla_{v^H}f^{n+1}_{H,0}\right)+\varepsilon\Big[\mathcal{Q}^{HL}_1(f^n_{H,1},f^n_{L,1})\\
&+\mathcal{Q}^{HL}_2(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_2(f^{n+1}_{H,1},f^{n+1}_{L,0})\Big].
\end{aligned}
\tag{4.24}
$$

We multiply (4.24) by ε and add it up with (4.13), then get

$$
\begin{aligned}
&\frac{f^{n+1}_{H,0}-f^n_{H,0}}{\Delta t}+\varepsilon\frac{f^{n+1}_{H,1}-f^n_{H,1}}{\Delta t}+\varepsilon\big(v^H\cdot\nabla_x f^n_{H,1}\\
&\quad+F^n_H\cdot\nabla_{v^H}f^n_{H,1}\big)+\left(v^H\cdot\nabla_x f^n_{H,0}+F^n_H\cdot\nabla_{v^H}f^n_{H,0}\right)\\
=&\frac{1}{\varepsilon}\left[\mathcal{Q}^{HH}(f^{n+1}_{H,0},f^{n+1}_{H,0})+\mathcal{Q}^{HL}_0(f^n_{H,0},f^n_{L,0})\right]+\mathcal{Q}^{HL}_0(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_0(f^n_{H,1},f^n_{L,0})\\
&+2\mathcal{Q}^{HH}(f^{n+1}_{H,0},f^{n+1}_{H,1})+\mathcal{Q}^{HL}_1(f^{n+1}_{H,0},f^{n+1}_{L,0})-\left(v^H\cdot\nabla_x f^{n+1}_{H,0}+F^n_H\cdot\nabla_{v^H}f^{n+1}_{H,0}\right)\\
&+\varepsilon\Big[\mathcal{Q}^{HH}(f^n_{H,1},f^n_{H,1})+\mathcal{Q}^{HL}_1(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_1(f^n_{H,1},f^n_{L,0})\\
&+\mathcal{Q}^{HL}_0(f^n_{H,1},f^n_{L,1})+\mathcal{Q}^{HL}_2(f^{n+1}_{H,0},f^{n+1}_{L,0})+\left(v^H\cdot\nabla_x f^{n+1}_{H,0}+F^n_H\cdot\nabla_{v^H}f^{n+1}_{H,0}\right)\Big]\\
&+\varepsilon^2\left[\mathcal{Q}^{HL}_1(f^n_{H,1},f^n_{L,1})+\mathcal{Q}^{HL}_2(f^{n+1}_{H,0},f^{n+1}_{L,1})+\mathcal{Q}^{HL}_2(f^{n+1}_{H,1},f^{n+1}_{L,0})\right].
\end{aligned}
\tag{4.25}
$$

Plugging in the leading order term of (4.17) and comparing the $O(1)$ terms on both sides, one gets

$$
\frac{M^{n+1}_{H,0}-M^n_{H,0}}{\Delta t}=2\mathcal{Q}^{HH}(M^{n+1}_{H,0},f^{*,n+1}_{H,1})+\mathcal{Q}^{HL}_0(M^{n+1}_{H,0},f^{*,n+1}_{L,1})+\underbrace{\mathcal{Q}^{HL}_0(f^{*,n}_{H,1},M^n_{L,0})}_{=0}
$$

$$+\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1})-\left(v^H\cdot\nabla_x M_{H,0}^n+F_H^n\cdot\nabla_{v^H}M_{H,0}^n\right)+O(\Delta t). \tag{4.26}$$

(4.26) can be equivalently written for

$$\phi_{H,1}^{n+1}=(M_{H,0}^{n+1})\,f_{H,1}^{*,n+1}$$

with

$$\Gamma_{H,0}\,\phi_{H,1}^{n+1}=(M_{H,0}^{n+1})^{-1}\,S_{H,1}^{n+1}+O(\Delta t). \tag{4.27}$$

$\Gamma_{H,0}$ is a linearization operator given by

$$\Gamma_{H,0}\,\phi_{H,1}^{n+1}=2\,(M_{H,0}^{n+1})^{-1}\,\mathcal{Q}^{HH}(M_{H,0}^{n+1},\,M_{H,0}^{n+1}\,\phi_{H,1}^{n+1}),$$

and $S_{H,1}^{n+1}$ is

$$\begin{aligned}S_{H,1}^{n+1}=&\left(D_t+v^H\cdot\nabla_x+F_H^n\cdot\nabla_{v^H}\right)M_{H,0}^n\\&-\mathcal{Q}_0^{HL}(M_{H,0}^{n+1},f_{L,1}^{n+1})-\mathcal{Q}_1^{HL}(M_{H,0}^{n+1},M_{L,0}^{n+1}).\end{aligned} \tag{4.28}$$

The necessary and sufficient condition of solvability of Equation (4.27) is

$$\int_{\mathbb{R}^3}S_{H,1}^{n+1}\begin{pmatrix}1\\ v^H\\ |v^H|^2\end{pmatrix}dv^H=O(\Delta t)\,\mathbb{I}_3. \tag{4.29}$$

The following is analogous to the proof shown in [8], except that we have a discrete counterpart here. With details omitted, (4.29) thus gives

$$\frac{\partial n_{H,0}^n}{\partial t}+\nabla_x\cdot(n_{H,0}^n\,u_{H,0}^n)=O(\Delta t), \tag{4.30}$$

$$\begin{aligned}&\frac{\partial}{\partial t}(n_{H,0}^n\,u_{H,0}^n)+\nabla_x\cdot(n_{H,0}^n\,u_{H,0}^n\otimes u_{H,0}^n)+\nabla_x(n_{H,0}^n\,T_{H,0}^n)-n_{H,0}^n\,F_H^n\\&=-\left(\nabla_x(n_{L,0}^{n+1}\,T_{L,0}^{n+1})-F_L^{n+1}\,n_{L,0}^{n+1}\right)+O(\Delta t),\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{n_{H,0}^n\,|u_{H,0}^n|^2}{2}+\frac{3}{2}n_{H,0}^n\,T_{H,0}^n\right)+\nabla_x\cdot\left(\left(\frac{n_{H,0}^n\,|u_{H,0}^n|^2}{2}+\frac{5}{2}n_{H,0}^n\,T_{H,0}^n\right)u_{H,0}^n\right)\\&\qquad-n_{H,0}^n\,F_H^n\cdot u_{H,0}^n\\&=-u_{H,0}^{n+1}\cdot\left[\nabla_x(n_{L,0}^{n+1}\,T_{L,0}^{n+1})-F_L^{n+1}\,n_{L,0}^{n+1}\right]-3\,\frac{\lambda(T_{L,0}^{n+1})}{T_{L,0}^{n+1}}\,n_{L,0}^{n+1}\,n_{H,0}^{n+1}\,(T_{H,0}^{n+1}-T_{L,0}^{n+1})\\&\qquad+O(\Delta t).\end{aligned} \tag{4.32}$$

Therefore, (4.30), (4.31) and (4.32) are consistent with the discrete scheme of the hydrodynamic limit system (4.3)–(4.5), up to a numerical error of $O(\Delta t)$.

4.3. Splitting of convection from the collision. As in [18, 19], we adopt a first-order time splitting approach to separate the convection from the collision operators. To summarize, our scheme is given by the following equations:

Moment equations for $f_{L,0}$ and $f_{H,0}$:

$$(P_0)_{L,0}^{n+1}=(P_0)_{L,0}^n+\Delta t\int_{\mathbb{R}^3}v^L\cdot\nabla_x f_{L,1}^n\,dv^L, \tag{4.33}$$

$$(P_1)_{L,0}^{n+1} = (P_1)_{L,0}^n + \frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3} \phi_1^L\, \mathcal{Q}_0^{LH}(f_{L,0}^n, f_{H,0}^n)(v^L)\, dv^L + \Delta t \int_{\mathbb{R}^3} \phi_1^L\, v^L \cdot \nabla_x f_{L,1}^n\, dv^L, \tag{4.34}$$

$$(P_2)_{L,0}^{n+1} = (P_2)_{L,0}^n + \Delta t \int_{\mathbb{R}^3} \phi_2^L\, v^L \cdot \nabla_x f_{L,1}^n\, dv^L, \tag{4.35}$$

$$(P_0)_{H,0}^{n+1} = (P_0)_{H,0}^n + \varepsilon \Delta t \int_{\mathbb{R}^3} v^H \cdot \nabla_x f_{H,1}^n\, dv^H, \tag{4.36}$$

$$(P_i)_{H,0}^{n+1} = (P_i)_{H,0}^n + \frac{\Delta t}{\varepsilon^2}\int_{\mathbb{R}^3} \phi_i^H\, \mathcal{Q}_0^{HL}(f_{H,0}^n, f_{L,0}^n)(v^H)\, dv^H + \varepsilon \Delta t \int_{\mathbb{R}^3} \phi_i^H\, v^H \cdot \nabla_x f_{H,1}^n\, dv^H, \tag{4.37}$$

where ϕ_i^L, ϕ_i^H are defined in (3.25) and $i = 1, 2$.

The scheme for $f_{L,0}$, $f_{L,1}$, $f_{H,0}$, $f_{H,1}$ are given by:

Step 1: The implicit collision step

$$\frac{f_{L,0}^* - f_{L,0}^n}{\Delta t} = \frac{1}{\varepsilon^2}\Big[\mathcal{Q}^{LL}(f_{L,0}^n, f_{L,0}^n) - \mathcal{P}(f_{L,0}^n) + \mathcal{P}(f_{L,0}^*) + \mathcal{Q}_0^{LH}(f_{L,0}^n, f_{H,0}^n)\Big], \tag{4.38}$$

$$\begin{aligned}
\frac{f_{L,1}^* - f_{L,1}^n}{\Delta t} = \frac{1}{\varepsilon^2}\Big[&\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{LH}(f_{L,0}^*, f_{H,0}^*) - \mathcal{Q}_0^{LH}(f_{L,0}^*, f_{H,0}^*)\right)\\
&+ \frac{1}{2}\Big[\mathcal{Q}^{LL}(f_{L,0}^n + f_{L,1}^n, f_{L,0}^n + f_{L,1}^n) + \mu(f_{L,0}^n + f_{L,1}^n) - \mu(f_{L,0}^* + f_{L,1}^*)\\
&- \left(\mathcal{Q}^{LL}(f_{L,0}^n - f_{L,1}^n, f_{L,0}^n - f_{L,1}^n) + \mu(f_{L,0}^n - f_{L,1}^n) - \mu(f_{L,0}^* - f_{L,1}^*)\right)\Big]\\
&+ \varepsilon \mathcal{Q}^{LL}(f_{L,1}^n, f_{L,1}^n) + \mathcal{Q}_\varepsilon^{LH}(f_{L,0}^n, f_{H,1}^n) + \left(\mathcal{Q}_\varepsilon^{LH}(f_{L,1}^n, f_{H,0}^n) - \mathcal{Q}_0^{LH}(f_{L,1}^n, f_{H,0}^n)\right)\\
&+ \mathcal{Q}_0^{LH}(f_{L,1}^*, f_{H,0}^*) + \varepsilon \mathcal{Q}_\varepsilon^{LH}(f_{L,1}^n, f_{H,1}^n) - (1 - \varepsilon^2 \psi_1)\left(v^L \cdot \nabla_x f_{L,0}^* + F^L \cdot \nabla_{v^L} f_{L,0}^*\right)\Big],
\end{aligned} \tag{4.39}$$

$$\frac{f_{H,0}^* - f_{H,0}^n}{\Delta t} = \frac{1}{\varepsilon}\Big[\mathcal{Q}^{HH}(f_{H,0}^*, f_{H,0}^*) - \mathcal{P}(f_{H,0}^n) + \mathcal{P}(f_{H,0}^*) + \mathcal{Q}_0^{HL}(f_{H,0}^n, f_{L,0}^n)\Big], \tag{4.40}$$

$$\begin{aligned}
\frac{f_{H,1}^* - f_{H,1}^n}{\Delta t} = \frac{1}{\varepsilon}\Big[&\frac{1}{\varepsilon}\left(\mathcal{Q}_\varepsilon^{HL}(f_{H,0}^*, f_{L,0}^*) - \mathcal{Q}_0^{HL}(f_{H,0}^*, f_{L,0}^*)\right)\\
&+ \frac{1}{2}\Big[\mathcal{Q}^{HH}(f_{H,0}^n + f_{H,1}^n, f_{H,0}^n + f_{H,1}^n) + \mu(f_{H,0}^n + f_{H,1}^n) - \mu(f_{H,0}^* + f_{H,1}^*)\\
&- \left(\mathcal{Q}^{HH}(f_{H,0}^n - f_{H,1}^n, f_{H,0}^n - f_{H,1}^n) + \mu(f_{H,0}^n - f_{H,1}^n) - \mu(f_{H,0}^* - f_{H,1}^*)\right)\Big]\\
&+ \varepsilon \mathcal{Q}^{HH}(f_{H,1}^n, f_{H,1}^n) + \mathcal{Q}_\varepsilon^{HL}(f_{H,0}^*, f_{L,1}^*) + \mathcal{Q}_\varepsilon^{HL}(f_{H,1}^n, f_{L,0}^n) + \varepsilon \mathcal{Q}_\varepsilon^{HL}(f_{H,1}^n, f_{L,1}^n)\\
&- (1 - \varepsilon \psi_2)\left(v^H \cdot \nabla_x f_{H,0}^* + F_H^n \cdot \nabla_{v^H} f_{H,0}^*\right)\Big].
\end{aligned} \tag{4.41}$$

The order is to first solve (4.38), (4.40), then solve (4.39) and (4.41).

Step 2: The explicit transport step

$$\frac{f_{L,0}^{n+1} - f_{L,0}^*}{\Delta t} + \left(v^L \cdot \nabla_x f_{L,1}^* + F_L^* \cdot \nabla_{v^L} f_{L,1}^*\right) = 0, \tag{4.42}$$

$$\frac{f_{L,1}^{n+1}-f_{L,1}^{*}}{\Delta t}+\psi_1\left(v^L\cdot\nabla_x f_{L,0}^{*}+F_L^{*}\cdot\nabla_{v^L} f_{L,0}^{*}\right)=0. \tag{4.43}$$

and

$$\frac{f_{H,0}^{n+1}-f_{H,0}^{*}}{\Delta t}+\varepsilon\left(v^H\cdot\nabla_x f_{H,1}^{*}+F_H^{*}\cdot\nabla_{v^H} f_{H,1}^{*}\right)=0, \tag{4.44}$$

$$\frac{f_{H,1}^{n+1}-f_{H,1}^{*}}{\Delta t}+\psi_2\left(v^H\cdot\nabla_x f_{H,0}^{*}+F_H^{*}\cdot\nabla_{v^H} f_{H,0}^{*}\right)=0, \tag{4.45}$$

where

$$\psi_1=\min\{1,\frac{1}{\varepsilon^2}\},\qquad \psi_2=\min\{1,\frac{1}{\varepsilon}\}.$$

5. Conclusion and future work

In this paper, we develop asymptotic-preserving time discretizations for disparate mass binary gas or plasma for both the homogeneous and inhomogeneous cases, at the relaxation time scale, for both the Boltzmann and Fokker-Planck-Landau collision operators. We introduce a novel splitting of the system and a carefully designed explicit-implicit time discretization so as to first guarantee the correct asymptotic behavior at the relaxation time limit and also significantly reduce the algebraic complexity which will be comparable to their single species counterparts. The design of the AP schemes are strongly guided by the asymptotic behavior of the system studied in [7, 8]. We also prove that a simplied version of the time discretization is asymptotic-preserving.

In the follow-up work, spatial and velocity discretizations will be discussed, along with extensive numerical simulations and experiments. Moreover, we plan to address the issue of uncertainty quantification (UQ), by adding random inputs into the system, and develop efficient numerical methods for such uncertain kinetic system.

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Appendix A. Definitions of $\mathcal{Q}$. In the Fokker-Planck-Landau case, the collision operators are given by

$$\mathcal{Q}_{\mathcal{L}}^{LL}:=\mathcal{Q}_{\mathcal{L}}^{LL}(f^L,f^L)(v^L)=\nabla_{v^L}\cdot\int_{\mathbb{R}^3}B(v^L-v_*^L)S(v^L-v_*^L)(\nabla_{v^L}f^Lf_*^L-\nabla_{v_*^L}f^Lf^L)\,dv_*^L,$$

$$\mathcal{Q}_{\mathcal{L}}^{HH}:=\mathcal{Q}_{\mathcal{L}}^{HH}(f^H,f^H)(v^H)=\nabla_{v^H}\cdot\int_{\mathbb{R}^3}B(v^H-v_*^H)S(v^H-v_*^H)(\nabla_{v^H}f^Hf_*^H-\nabla_{v_*^H}f^Hf^H)\,dv_*^H,$$

$$\begin{aligned}\mathcal{Q}_{\mathcal{L},\varepsilon}^{LH}&:=\mathcal{Q}_{\mathcal{L},\varepsilon}^{LH}(f^L,f^H(\varepsilon v^H))(v^L)\\&=(1+\varepsilon^2)^{\frac{\gamma+2}{2}}\nabla_{v^L}\cdot\int_{\mathbb{R}^3}B(\frac{v^L-\varepsilon v^H}{\sqrt{1+\varepsilon^2}})S(v^L-\varepsilon v^H)(\nabla_{v^L}f^Lf^H-\varepsilon\nabla_{v^H}f^Hf^L)\,dv^H,\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_{\mathcal{L},\varepsilon}^{HL}&:=\mathcal{Q}_{\mathcal{L},\varepsilon}^{HL}(f^H(\varepsilon v^H),f^L)(v^H)\\&=(1+\varepsilon^2)^{\frac{\gamma+2}{2}}\nabla_{v^H}\cdot\int_{\mathbb{R}^3}B(\frac{v^L-\varepsilon v^H}{\sqrt{1+\varepsilon^2}})S(v^L-\varepsilon v^H)(\nabla_{v^L}f^Lf^H-\varepsilon\nabla_{v^H}f^Hf^L)\,dv^L,\end{aligned}$$

where the matrix $S(w)$ and the intra-molecular potential $B(w)$, respectively, are given by

$$S(w)=\mathrm{Id}-\frac{w\otimes w}{|w|^2},\qquad B(w)=\frac{1}{2}|w|^{\gamma+2}.$$

In particular, $B\left(\frac{1}{\sqrt{1+\varepsilon^2}}w\right)=\frac{1}{2}(1+\varepsilon^2)^{-\frac{\gamma+2}{2}}|w|^{\gamma+2}$, and the value $\gamma=-3$ corresponds to Coulomb interactions.

In the Boltzmann case, the collision operators in center of mass – relative velocity coordinates expressed in the angular scattering direction σ, are given by

$$\mathcal{Q}_{\mathcal{B}}^{LL}:=\mathcal{Q}^{LL}(f^L,f^L)(v^L)=\int_{\mathbb{R}^3}\int_{S^2}B^L(v^L-v_*^L,\sigma)(f'^{,L}f_*'^{,L}-f^Lf_*^L)\,d\sigma dv_*^L,$$

$$\mathcal{Q}_{\mathcal{B}}^{HH}:=\mathcal{Q}^{HH}(f^H,f^H)(v^H)=\int_{\mathbb{R}^3}\int_{S^2}B^H(v^H-v_*^H,\sigma)(f'^{,H}f_*'^{,H}-f^Hf_*^H)\,d\sigma dv_*^H,$$

$$\mathcal{Q}_{\mathcal{B},\varepsilon}^{LH}:=\mathcal{Q}_{\varepsilon}^{LH}(f^L,f^H(\varepsilon v^H))(v^L)=(1+\varepsilon^2)^{\frac{\gamma}{2}}\int_{\mathbb{R}^3}\int_{S^2}B(\frac{v^L-\varepsilon v^H}{\sqrt{1+\varepsilon^2}},\sigma)(f'^{L,\varepsilon}f'^{H,\varepsilon}-f^Lf^H)\,d\sigma dv^H,$$

$$\mathcal{Q}_{\mathcal{B},\varepsilon}^{HL}:=\mathcal{Q}_{\varepsilon}^{HL}(f^H(\varepsilon v^H),f^L)(v^H)$$
$$=\left(\frac{1+\varepsilon^2}{\varepsilon^2}\right)^{\frac{\gamma}{2}}\int_{\mathbb{R}^3}\int_{S^2}B(\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}(v^L-\varepsilon v^H),\sigma)(f'^{L,\varepsilon}f'^{H,\varepsilon}-f^Lf^H)\,d\sigma dv^L,$$

with

$$v'^{L,\varepsilon}=v^L+\frac{1}{1+\varepsilon^2}\left(|v^L-\varepsilon v^H|\sigma-(v^L-\varepsilon v^H)\right)=\frac{\varepsilon^2v^L+\varepsilon v^H+|v^L-\varepsilon v^H|\sigma}{1+\varepsilon^2},$$

and

$$v'^{H,\varepsilon}=\varepsilon v^H-\frac{\varepsilon^2}{1+\varepsilon^2}\left(|v^L-\varepsilon v^H|\sigma-(v^L-\varepsilon v^H)\right)=\frac{\varepsilon^2v^L+\varepsilon v^H-\varepsilon^2|v^L-\varepsilon v^H|\sigma}{1+\varepsilon^2}.$$

Here the collision kernel B is assumed to be in the form

$$B(w,\sigma)=\frac{1}{2}|w|^\gamma b(\frac{w}{|w|}\cdot\sigma).$$

Appendix B. The penalty methods. For the Boltzmann equation, the best choice of this relaxation operator shown in [10] is

$$P(f)=\beta(\mathcal{M}_{\rho,u,T}-f),$$

where $\beta>0$ is an upper bound of $||\nabla\mathcal{Q}(\mathcal{M}_{\rho,u,T})||$. Another simple example of β at time t^n is

$$\beta^n=\sup\left|\frac{\mathcal{Q}(f^n,f^n)-\mathcal{Q}(f^{n-1},f^{n-1})}{f^n-f^{n-1}}\right|.$$

We briefly review the penalty method introduced in [10] for the Boltzmann equation in the form:

$$\partial_t f+v\cdot\nabla_x f=\frac{1}{\varepsilon}\mathcal{Q}_B(f,f),$$

the discretized scheme is given by

$$\frac{f^{n+1}-f^n}{\Delta t}+v\cdot\nabla_x f^n=\frac{\mathcal{Q}_B(f^n,f^n)-P(f^n)}{\varepsilon}+\frac{P(f^{n+1})}{\varepsilon},\tag{B.1}$$

where $P(f)=\beta[\mathcal{M}_{\rho,u,T}(v)-f(v)]$. Multiplying (B.1) by $\phi(v)=(1,v,|v|^2)^T$, one gets the macroscopic quantities $U:=(\rho,\rho u,T)$:

$$U^{n+1}=\int\phi(v)(f^n-\Delta t v\cdot\nabla_x f^n)\,dv.$$

U^{n+1} is obtained explicitly, which defines $\mathcal{M}^{n+1}$, thus f^{n+1} can be computed explicitly.

On the other hand, [21] discusses the penalty method for solving the multiscale Fokker-Planck-Landau equation:

$$\partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}\mathcal{Q}_L(f,f), \tag{B.2}$$

The authors in [21] demonstrate analytically and numerically that the best choice of the penalization operator is the linear Fokker-Planck (FP) operator,

$$P_{FP}(f) = \nabla_v\cdot\left(M\nabla_v\left(\frac{f}{M}\right)\right), \tag{B.3}$$

where

$$M(x,v) = \frac{\rho(x)}{(2\pi T(x))^{N_v/2}}\exp\left(-\frac{(v-u(x))^2}{2T(x)}\right).$$

The first order AP scheme for (B.2) is given by

$$\frac{f^{n+1}-f^n}{\Delta t} = \frac{1}{\varepsilon}\left(\mathcal{Q}(f^n,f^n) - \beta P^n f^n + \beta P^{n+1}f^{n+1}\right),$$

where β is chosen large enough to ensure stability. For example, let $\beta = \beta_0 \max_v \lambda(D_A(f))$, with $\beta_0 > \frac{1}{2}$ and $\lambda(D_A)$ is the spectral radius of the positive symmetric matrix D_A, defined by

$$D_A(f) = \int_{\mathbb{R}^3} A(v-v_*)f_*\,dv_*,$$

with

$$A(z) = |z|^{\gamma+2}\left(\mathbb{I} - \frac{z\otimes z}{|z|^2}\right).$$

Compute the moments of f^n by

$$(\rho,\rho u,\rho T)^{n+1} = \int_{\mathbb{R}^3}\left(1, v, \frac{(v-u)^2}{2}\right)f^n\,dv,$$

and update M^{n+1}. One can then solve f^{n+1} by

$$f^{n+1} = \left(1 - \frac{\beta\Delta t}{\varepsilon}P^{n+1}\right)^{-1}\left(f^n + \frac{\Delta t}{\varepsilon}(\mathcal{Q}(f^n) - \beta P^n f^n)\right). \tag{B.4}$$

Introduce the symmetrized operator [21]

$$\tilde{P}h = \frac{1}{\sqrt{M}}\nabla_v\cdot\left(M\nabla_v\left(\frac{h}{\sqrt{M}}\right)\right),$$

then the penalty operator is $P_{FP}f = \sqrt{M}\tilde{P}\left(\frac{f}{\sqrt{M}}\right)$. Rewrite (B.4) as

$$\left(\frac{f}{\sqrt{M}}\right)^{n+1} = \left(1 - \frac{\beta\Delta t}{\varepsilon}P^{n+1}\right)^{-1}\left\{\frac{1}{\sqrt{M^{n+1}}}\left[f^n + \frac{\Delta t}{\varepsilon}\left(\mathcal{Q}(f^n) - \beta\sqrt{M^n}\tilde{P}^n\left(\frac{f^n}{\sqrt{M^n}}\right)\right)\right]\right\}.$$

The discretization of $\tilde{P}$ in one dimension is given by

$$(\tilde{P}h)_j = \frac{1}{(\Delta v)^2}\frac{1}{\sqrt{M_j}}\left\{\sqrt{M_j M_{j+1}}\left(\left(\frac{h}{\sqrt{M}}\right)_{j+1} - \left(\frac{h}{\sqrt{M}}\right)_j\right)\right.$$
$$\left. - \sqrt{M_j M_{j-1}}\left(\left(\frac{h}{\sqrt{M}}\right)_j - \left(\frac{h}{\sqrt{M}}\right)_{j-1}\right)\right\}$$
$$= \frac{1}{(\Delta v)^2}\left(h_{j+1} - \frac{\sqrt{M_{j+1}}+\sqrt{M_{j-1}}}{\sqrt{M_j}}h_j + h_{j-1}\right).$$

Since the new operator $\tilde{P}$ is symmetric, the Conjugate Gradient (CG) method can be used to get $\left(\frac{f}{\sqrt{M}}\right)^{n+1}$. See section 3 in [21] on details for the full discretization.

Definitions of $\lambda(T)$ and coefficients D^ε_{ij} $(i,j=1,2)$.

We recall some definitions given in [8]. In the Boltzmann case, $\lambda(T)$ is given by

$$\lambda(T) = \frac{2}{3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v,\Omega)(v,\Omega)^2 M_{0,T}(v)\, d\Omega dv,$$

and in the FPL case,

$$\lambda(T) = \frac{2}{3}\int_{\mathbb{R}^3} B(v)\, M_{0,T}(v)\, dv.$$

The coefficients D^ε_{ij} are given by

$$D^\varepsilon_{1j} = \frac{1}{3}\int_{\mathbb{R}^3} M_{0,T^L_\varepsilon}(v)\,\Psi_{j\varepsilon}(|v|)\,|v|^2\, dv,$$
$$D^\varepsilon_{2j} = \frac{1}{6}\int_{\mathbb{R}^3} M_{0,T^L_\varepsilon}(v)\,\Psi_{j\varepsilon}(|v|)\,|v|^4\, dv.$$

Ψ_i is given by the following: The unique solutions ψ^L_i, $i=1,2$, in $\left(\ker \Gamma^L_0\right)^\perp$, of the equations

$$\Gamma^L_0\,\psi^L_1 = v^L, \qquad \Gamma^L_0\,\psi^L_2 = \left(\frac{1}{2}\frac{|v^L|^2}{T^L_0} - \frac{3}{2}\right)v^L$$

are of the form:

$$\psi^L_i = -\Psi_i(|v^L|)\,v^L.$$

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