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LONG TIME BEHAVIOR IN LOCALLY ACTIVATED RANDOM WALKS*

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Abstract. We consider a 1-dimensional Brownian motion whose diffusion coefficient varies when it crosses the origin. We study the long time behavior and we establish different regimes, depending on the variations of the diffusion coefficient: emergence of a non-Gaussian multipeaked probability distribution and a dynamical transition to an absorbing static state. We compute the generator and we study the partial differential equation which involves its adjoint. We discuss global existence and blow-up of the solution to this latter equation.

Keywords. local time; random walk; dynamical transition; non-Gaussian probability distribution; blow-up.

AMS subject classifications. 35Q92; 60J65.

1. Introduction

In this paper we deal with a new class of one-dimensional linear diffusion problems in which the diffusivity is modified in a prescribed way upon each crossing of the origin. We study both the system of stochastic differential equations satisfied by the position and the diffusion coefficient of a Brownian particle whose diffusion coefficient is modified at each crossing of the origin and the partial differential equation satisfied by the joint distribution of the solution to the stochastic system. In both viewpoints we obtain non-trivial behaviors of the solution: dynamical transition to an absorbing state for the solution to the stochastic system and blow-up of the density of the joint distribution. Global existence versus blow-up has been widely studied for non-linear equations, such as for the Keller-Segel system in two dimensions of space, see e.g. [3]. In our case, the partial differential equation is linear and the instability driving the system towards an inhomogeneous state is the diffusion.

Living systems provide prototypical examples of such a problem: the dynamics of a cell or a bacterium in the presence of a localized patch of nutrients, which enhances its ability to move, as for exemple the dynamics of a macrophage that grows by accumulating smaller and spatially localized particles, such as lipids, Figure 1.1 and [4], or, alternatively, a localized patch of toxins that impairs its mobility. In [1], to describe the movement of such a particle, the following formal system of stochastic differential equations was introduced:

$$\begin{cases} \mathrm{d}X_t &= \sqrt{2A_t}\,\mathrm{d}W_t, \\ \mathrm{d}A_t &= f(A_t)\Delta_{X_t=0}\,\mathrm{d}t, \end{cases} \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a given standard one-dimensional Brownian motion, X_t and A_t respectively denote the position and the diffusion coefficient of the particle at time t, $\Delta_{X_t=0}\,\mathrm{d}t$

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accounts for the local activation, the particle is accelerated or decelerated whenever it crosses the origin, and f is an arbitrary prescribed function. The rigorous definition of the term $\Delta_{X_t=0}\,\mathrm{d}t$ will be given below.

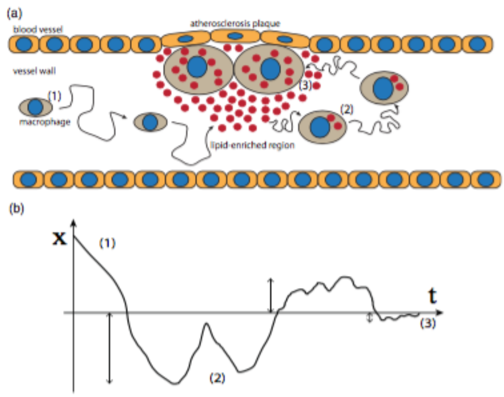

FIG. 1.1. *a) Sketch of the different stages of atherosclerosis plaque formation: (1) rapid diffusion of a free macrophage cell; (2) upon entering a localized lipid-enriched region, the macrophage accumulates lipids, and thereby grows and becomes less mobile; and (3) after many crossings of the lipid-enriched region, the macrophage eventually gets trapped, resulting in the formation of an atherosclerotic plaque. (b) Sketch of a one-dimensional particle trajectory of the model of locally decelerated random walk.*

In [1], the term $\Delta_{X_t=0}$ involved in the previous system was understood in the sense that the dual of the generator of the Markov process $(X_t, A_t)_{t\geq 0}$ was

$$\mathcal{L}^* h(x,a) = a\partial^2_{xx} h(x,a) - \partial_a \left(f(a)h(x,a)\right)\delta_{x=0}, \tag{1.2}$$

for any smooth function h, where $\delta_{x=0}$ is the Dirac delta function. Here, we are first interested to give a correct formulation of the term $\Delta_{X_t=0}$. Intuitively, the term $\Delta_{X_t=0}\,\mathrm{d}t$ should represent a measure on $[0,\infty)$ giving full measure to the set of zeros of the process $(X_t)_{t\geq 0}$. This reminds the notion of local time.

We point out that in the formal system (1.1) at any time, the diffusion coefficient, A_t, depends on the entire history of the trajectory. Thus the evolution of the particle position, X_t, is intrinsically non-Markovian. Despite these considerations, in the particular case where f is a power function, $f(a) = \pm a^\gamma$, $\gamma \geq 0$, we study the long-time behavior of the process $(X_t)_{t\geq 0}$ solution to the system with the correct formulation of the term $\Delta_{X_t=0}$. Our main findings are: (i) The probability distribution of the position has a non-Gaussian tail. (ii) For local acceleration, i.e. f takes nonnegative values, $f(a) = a^\gamma$, a diffusing particle is repelled from the origin, so that the maximum in the probability distribution is at nonzero displacement. (iii) For local deceleration, i.e. f takes negative values, $f(a) = -a^\gamma$, a dynamical transition to an absorbing state occurs: for sufficiently strong deceleration, $\gamma \in (0, 3/2)$, the particle can get trapped at the origin in finite time while if the deceleration process is sufficiently weak, $\gamma \geq 3/2$, the particle never gets trapped.

In a second step, we study the generator of the Markov process $(X_t, A_t)_{t\geq 0}$ solution to the system with the correct formulation of the term $\Delta_{X_t=0}$. In order to do so we first prove that the generator of the Markov process $(W_t, L^W_t)_{t\geq 0}$, in a weak sense, is given by

$$\mathcal{L}_0 h(w,l) = \frac{1}{2}\partial^2_{ww} h(w,l) + \partial_l h(w,l)\delta_{l=0},$$

where L^W_t is the local time at 0 of $(W_t)_{t\geq 0}$. We use this result to prove that the density $u(t,x,a)$ of the joint distribution $\mu_t(x,a)$ of $(X_t, A_t)_{t\geq 0}$, defined on $t \geq 0$, $x \in \mathbb{R}$, $a \geq 0$, satisfies, in a weak sense the parabolic equation

$$\partial_t u(t,x,a) = \mathcal{L}^* u(t,x,a) = a\partial^2_{xx} u(t,x,a) - \partial_a \left(f(a)u(t,x,a)\right)\delta_{x=0}, \tag{1.3}$$

with initial condition $u_0(x,a)$.

Next we study the partial differential Equation (1.3) and the general questions we are concerned with are the following. By studying the regularity of the solution to (1.3), do we recover the results observed during the probabilistic study? In particular, if $f(a) = -a^\gamma$ with $\gamma \geq 3/2$, can we prove global existence? In the case $\gamma \in (0, 3/2)$ can we prove that the solution to (1.3) becomes unbounded in finite time in any L^p space (so-called blow-up)?

We describe the organization of the paper. In Section 2 we build and study the correct equation associated with (1.1). Section 3 is devoted to the computation of the generator of $(X_t, L^W_t)_{t\geq 0}$ from which we deduce the weak formulation satisfied by the joint distribution associated to (X_t, A_t) solution to the correct version of (1.1). In Section 4 we study Equation (1.3).

2. Mathematical study of a correct version of (1.1)

Let us first define the term $\Delta_{X_t=0}\,\mathrm{d}t$. Intuitively, it should represent a time dependent measure of the set of zeros of the process $(X_t)_{t\geq 0}$. This reminds the notion of local time whose definition we recall here for completeness.

For any continuous local martingale $(M_t)_{t\geq 0}$, one can define the *local time at* 0 of $(M_t)_{t\geq 0}$ by:

$$L^M_t := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{|M_s|\leq\varepsilon}\,\mathrm{d}\left\langle M\right\rangle_s,$$

where $\langle M\rangle_s$ is the quadratic variation of the process $(M_t)_{t\geq 0}$ (see [8], Chapter IV). Namely, $\langle M\rangle_t = \lim \sum_{i=1}^n |M_{t_i} - M_{t_{i-1}}|^2$, where the limit (in probability) holds for subdivisions $0 = t_0 < t_1 < ... < t_n = t$ with $\max_{i=1}^n |t_i - t_{i-1}| \to 0$.

The local time satisfies the scaling property

$$L^{\lambda M}_t = \lambda L^M_t \text{ a.s. for any } \lambda > 0.$$

In particular, the process $(\lambda^{-1} L^{\lambda M}_t)_{t\geq 0}$ does not actually depend on λ. Since the process $\left(L^M_t\right)_{t\geq 0}$ is continuous and nondecreasing, we can associate to it a measure $\mathrm{d}L^M_t$ without atoms on $\mathbb{R}_+$. This measure is supported by the set $\{t \geq 0 : M_t = 0\}$. For more details on the theory of local times, we refer to [8], chapter VI.

The formal term $\Delta_{X_t=0}\,\mathrm{d}t$ should satisfy the scaling invariance property

$$\Delta_{\lambda X_t=0}\,\mathrm{d}t = \Delta_{X_t=0}\,\mathrm{d}t.$$

Comparing this formal property to the properties of local time, it seems natural to replace the term $\Delta_{X_t=0}\,\mathrm{d}t$ by the renormalized local time $\mathrm{d}L^X_t/2A_t$. As a consequence, in this work, instead of (1.1), we will study the system

$$\begin{cases} \mathrm{d}X_t &= \sqrt{2A_t}\,\mathrm{d}W_t, \\ \mathrm{d}A_t &= f(A_t)\dfrac{\mathrm{d}L^X_t}{2A_t}, \end{cases} \tag{2.1}$$

with a given initial condition (X_0, A_0). In the sequel, f will be a locally Lipschitz continuous function from $(0, \infty)$ to $\mathbb{R}$, and the initial condition will be assumed to satisfy $A_0 > 0$ almost surely. Note that if $(X_t)_{t\geq 0}$ solves (2.1), then $(X_t)_{t\geq 0}$ is a local martingale as the stochastic integral of a continuous process against Brownian motion (see [8], Chapter IV).

More precisely, we are interested in proving that Equation (2.1) defines a Markov process whose generator is given by (1.2). In order to do so we start by studying a simpler problem

$$\begin{cases} \mathrm{d}X_t &= \sqrt{2A_t}\,\mathrm{d}W_t, \\ \mathrm{d}A_t &= f(A_t)\dfrac{\mathrm{d}L_t^W}{\sqrt{2A_t}}, \end{cases} \tag{2.2}$$

and we prove that the solutions to systems (2.1) and (2.2) coincide when the initial condition satisfies $X_0 = 0$.

REMARK 2.1. In the case where $X_0 \neq 0$ the solution of (2.1) is the solution of (2.2) with W replaced by $\tilde{W}$ where $\tilde{W}_t = W_t + X_0/\sqrt{2A_0}$.

Equation (2.1) may not admit solutions for all positive times, because solutions might blow up in finite time. For example if f is a positive function, the process $(A_t)_{t\geq 0}$ will be nondecreasing, and nothing will a priori prevent it to go to infinity in a finite time τ. In that case, the diffusion coefficient of $(X_t)_{t\geq 0}$ will blow up in finite time, and $(X_t)_{t\geq 0}$ will not admit any extension after time τ.

As a consequence, in the sequel, we will call a *(strong) solution* to Equation (2.1) (resp. (2.2)), a triple $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$, where τ is a stopping time of the Brownian motion $(W_t)_{t\geq 0}$ and $(X_t, A_t)_{0\leq t<\tau}$ is a continuous process adapted to $(W_t)_{t\geq 0}$ satisfying Equation (2.1) (resp. (2.2)) until time τ.

We will say that such a solution is *maximal*, when the process $(A_t)_{0\leq t<\tau}$ converges either to 0 or ∞ as $t \to \tau$ on the event $\{\tau < \infty\}$. Indeed, in those two cases, the term $f(A_t)$ appearing in the equation becomes ill-defined at time τ, since $f(a)$ is only assumed to make sense for $a \in (0, \infty)$.

2.1. Well-posedness of (2.2).

The first equation in (2.2) is explicit in $(W_t, A_t)_{t\geq 0}$: for given $(X_0, (A_t)_{0\leq t<\tau}, (W_t)_{t\geq 0})$, its unique solution is given by

$$\forall t < \tau,\ X_t = X_0 + \int_0^t \sqrt{2A_s}\,\mathrm{d}W_s.$$

Moreover, the second equation in (2.2) is a closed equation on $(A_t)_{t\geq 0}$ and does not depend on $(X_t)_{t\geq 0}$. Thus, studying existence and uniqueness for the equation $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$ is enough to obtain existence and uniqueness for system (2.2).

Recalling that $f : (0, \infty) \to \mathbb{R}$ is assumed to be locally Lipschitz continuous, we obtain the following result.

PROPOSITION 2.1. *Let (X_0, A_0) be a random couple, independent of $(W_t)_{t\geq 0}$ and such that $A_0 > 0$. Then, there exists a unique maximal strong solution $(\tau, (X_t)_{0\leq t\leq\tau}, (A_t)_{0\leq t\leq\tau})$ to Equation (2.2) with initial condition (X_0, A_0).*

Proof. As explained before, we only need to show existence and uniqueness for the equation $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$, which is closely related to the ordinary differential equation $y' = f(y)/\sqrt{2y}$.

First, consider the flow Φ associated with $y' = f(y)/\sqrt{2y}$. Namely,

$$t \mapsto \Phi_x(t) \tag{2.3}$$

is the unique maximal solution to

$$y' = f(y)/\sqrt{2y} \quad \text{satisfying } \Phi_x(0) = x. \tag{2.4}$$

This flow is well defined from the local Lipschitz continuity of $y \mapsto f(y)/\sqrt{2y}$. The flow Φ_x is only defined up to a time $T(x)$. Moreover, in the case $T(x) < \infty$, one necessarily has either $\lim_{t\to T(x)} \Phi_x(t) = \infty$ or $\lim_{t\to T(x)} \Phi_x(t) = 0$.

Then, one can check that $A_t = \Phi_{A_0}(L_t^W)$ is defined up to the time $\tau = \sup\{t \geq 0, L_t^W < T(A_0)\}$ and satisfies the equation $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$. Indeed, since Φ_x is continuously differentiable and $(L_t^W)_{t\geq 0}$ is a continuous nondecreasing process, then the usual chain rule holds, namely one has $\mathrm{d}\Phi_x(L_t^W) = \Phi_x'(L_t^W)\,\mathrm{d}L_t^W$. Moreover, on the event $\{\tau < \infty\}$, $\lim_{t\to\tau} A_t$ exists with value 0 or ∞.

For uniqueness of the solution to $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$, consider two solutions $(\tau, (A_t)_{0\leq t<\tau})$ and $(\tilde\tau, (\tilde A_t)_{0\leq t<\tilde\tau})$, with $A_0 = \tilde A_0$. Then one has the following Grönwall-type inequality: $\forall t < \tau \wedge \tilde\tau$

$$|A_t - \tilde A_t| = \left| \int_0^t \left(f(A_s)/\sqrt{2A_s} - f(\tilde A_s)/\sqrt{2\tilde A_s} \right) \mathrm{d}L_s^W \right| \leq C \int_0^t |A_s - \tilde A_s|\,\mathrm{d}L_s^W.$$

Hence, from the expression

$$\begin{aligned} &e^{-CL_t^W} \int_0^t \left|A_s - \tilde A_s\right| \mathrm{d}L_s^W \\ &= \int_0^t e^{-CL_s^W} \left(|A_s - \tilde A_s| - C \int_0^s \left|A_u - \tilde A_u\right| \mathrm{d}L_u^W \right) \mathrm{d}L_s^W \\ &\leq 0, \end{aligned}$$

it follows that $|A_t - \tilde A_t| = 0$ for $\mathrm{d}L_t^W$-almost all $0 \leq t < \tau \wedge \tilde\tau$. □

Remark 2.2. As it appears in the proof of Proposition 2.1, existence for the stochastic differential Equation (2.2) still holds true provided the ordinary differential equation $y' = f(y)/\sqrt{2y}$ admits a (non necessarily unique) solution.

2.2. Link with system (2.1). In this section, we provide a link between solutions to systems (2.1) and (2.2). Consider a solution $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$ to (2.1) starting from $X_0 = 0$. We prove that the two processes $(X_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ vanish at exactly the same times, until the explosion time τ. Indeed, we show that the measure $\mathrm{d}L_t^X$ has a density with respect to $\mathrm{d}L_t^W$.

Proposition 2.2. *Let (X_0, A_0) be a random variable independent of $(W_t)_{t\geq 0}$ with $A_0 > 0$. Let $(\tau, (A_t)_{0\leq t<\tau}, (X_t)_{0\leq t<\tau})$ be a strong solution to (2.1). Then,*

$$X_t = \sqrt{2A_t}\left(W_t + \frac{X_0}{\sqrt{2A_0}} \right).$$

If in addition we assume that $X_0 = 0$, then

$$\forall t < \tau,\ \mathrm{d}L_t^X = \sqrt{2A_t}\,\mathrm{d}L_t^W \quad \textit{and} \quad X_t = \sqrt{2A_t}W_t. \tag{2.5}$$

Proof. Let the stopping time τ_n be the first positive time for which $(A_t)_{0\le t<\tau}$ reaches $[0, \frac{1}{n}]$. It is defined by

$$\tau_n = \tau \wedge \inf\left\{t \in [0,\tau), A_t \le \frac{1}{n}\right\}.$$

Almost surely, $\tau_n \to \tau$ as n goes to infinity and $A_t > \frac{1}{n}$ on $[0, \tau_n)$. Then, on the time interval $[0, \tau_n)$, one has

$$\begin{aligned} \mathrm{d}W_t - \mathrm{d}\left(\frac{X_t}{\sqrt{2A_t}}\right) &= \left(\mathrm{d}W_t - \frac{\mathrm{d}X_t}{\sqrt{2A_t}}\right) - X_t\,\mathrm{d}\left(\frac{1}{\sqrt{2A_t}}\right) \\ &= 0 + (2A_t)^{-3/2} X_t\,\mathrm{d}A_t \\ &= 0, \end{aligned} \tag{2.6}$$

where the last equality is a consequence of $X_t\,\mathrm{d}L_t^X = 0$. As a consequence, we deduce that $W_t - \frac{X_t}{\sqrt{2A_t}} = W_0 - \frac{X_0}{\sqrt{2A_0}} = -\frac{X_0}{\sqrt{2A_0}}$ up to time τ_n. Letting n go to infinity, we obtain $X_t = \sqrt{2A_t}\left(W_t + \frac{X_0}{\sqrt{2A_0}}\right)$ for all $0 \le t < \tau$.

According to Tanaka's formula (see for example [8], chapter VI), the local time of a local martingale is given by $\mathrm{d}L_t^M = \mathrm{d}|M_t| - \mathrm{sign}(M_t)dM_t$. Hence, it the case $X_0 = 0$, we get

$$\begin{aligned} \mathrm{d}L_t^X &= \mathrm{d}|X_t| - \mathrm{sign}(X_t)\,\mathrm{d}X_t = \mathrm{d}(\sqrt{2A_t}|W_t|) - \mathrm{sign}(W_t)\,\mathrm{d}\left(\sqrt{2A_t}W_t\right) \\ &= \sqrt{2A_t}\Big(\mathrm{d}|W_t| - \mathrm{sign}(W_t)\,\mathrm{d}W_t\Big) + (|W_t| - \mathrm{sign}(W_t)W_t)\frac{\mathrm{d}A_t}{\sqrt{2A_t}} \\ &= \sqrt{2A_t}\,\mathrm{d}L_t^W. \end{aligned}$$

□

Proposition 2.2 is what we needed to establish a link between solutions to (2.1) and (2.2).

COROLLARY 2.1. *A continuous process $(X_t, A_t)_{0\le t<\tau}$ defined up to time τ and satisfying $X_0 = 0$ is a strong solution to Equation (2.2) if and only if it is a strong solution to Equation (2.1).*

Proof. This is a direct consequence of the second equality in (2.5). □

COROLLARY 2.2. *For any initial condition (X_0, A_0) independent of $(W_t)_{t\ge0}$, there exists a unique maximal solution to Equation (2.1).*

Proof. If $X_0 = 0$, there exists a unique maximal solution to (2.2) from Proposition 2.1. From Corollary 2.1, it is also the unique maximal solution to (2.1).

For a general initial condition, up to the time $\zeta = \inf\{t \ge 0, X_t = 0\}$, system (2.1) clearly admits a unique solution $X_t = X_0 + \sqrt{2A_0}W_t$, $A_t = A_0$. After ζ, the Markov property allows to apply existence and uniqueness starting from $X_0 = 0$. □

2.3. A discrete time approximation. In this section, we construct an approximation to the process $(X_t, A_t)_{t\ge0}$. This will give a heuristic justification to equation (2.5).

The Brownian motion will be discretized by a simple random walk

$$Y_n = \sum_{k=1}^{n} U_k, \quad n \in \mathbb{N},$$

where $(U_k)_{k\in\mathbb{N}^*}$ is a sequence of independent random variables, uniformly distributed on $\{1,-1\}$. We will need the *discrete local time* of $(Y_n)_{n\in\mathbb{N}}$, defined by

$$\Lambda_n = \sum_{k=1}^{n} \mathbf{1}_{Y_k=0}.$$

For all $T \geq 0$, one has the trajectorial convergence in distribution

$$\left(\sqrt{\frac{1}{n}}Y_{\lfloor nt\rfloor}, \sqrt{\frac{1}{n}}\Lambda_{\lfloor nt\rfloor}\right)_{0\leq t\leq T} \to (W_t, L_t^W)_{0\leq t\leq T},$$

as $n\to\infty$.

At each crossing of the origin, the approximation has to take into account the modification of the diffusion coefficient. To do so we modify the step size as in [2]. Starting from the initial condition $\hat{X}_0^n = 0$ and $\hat{A}_0^n = A_0$, the sequences $(\hat{X}_k^n)_k$ and $(\hat{A}_k^n)_k$ are defined by induction as follows:

$$\begin{cases} \hat{X}_k^n &= \hat{X}_{k-1}^n + U_k\sqrt{\frac{2}{n}\hat{A}_{k-1}^n}, \\ \hat{A}_k^n &= \hat{A}_{k-1}^n + \frac{f(\hat{A}_{k-1}^n)}{\sqrt{n}}\mathbf{1}_{\hat{X}_k^n=0}. \end{cases}$$

We first state a discrete analog to Equation (2.5).

Lemma 2.1. *Let n be a fixed integer. For all integers k, one has the equalities*

$$\hat{X}_k^n = Y_k\sqrt{\frac{2}{n}\hat{A}_k^n}, \ \textit{and}\ \mathbf{1}_{\hat{X}_k^n=0} = \mathbf{1}_{Y_k=0}.$$

Consider the Euler scheme associated to $y' = f(y)$, namely

$$y_{n+1}^\delta = y_n^\delta + \delta f(y_n^\delta)$$

where δ is some time step. Then, when $n\to\infty$ and $\delta\to 0$ in the regime $n\delta\to t$, one has

$$y_n^\delta \to \Phi_{y_0}(t),$$

with Φ defined by (2.3) and (2.4).

Lemma 2.2. *$\hat{A}_k^n$ is given by $y_{\Lambda_k}^{\sqrt{1/n}}$.*

Theorem 2.3. *For all $T\geq 0$, as n goes to infinity, the trajectory $(\hat{X}_{\lfloor nt\rfloor}^n, \hat{A}_{\lfloor nt\rfloor}^n)_{0\leq t\leq T}$ converges in distribution to $(X_t, A_t)_{0\leq t\leq T}$.*

Proof. For simplicity, we only prove convergence for a fixed time t.

One has $\hat{A}_{\lfloor nt\rfloor}^n = y_{\Lambda_{\lfloor nt\rfloor}}^{\sqrt{1/n}}$. Here $\sqrt{\frac{1}{n}}$ and $\Lambda_{\lfloor nt\rfloor}$ respectively converge to 0 and ∞ in the regime $\sqrt{\frac{1}{n}}\Lambda_{\lfloor nt\rfloor}\to L_t^W$. As a consequence, $\hat{A}_{\lfloor nt\rfloor}^n$ converges to $\Phi_{A_0}(L_t^W) = A_t$.

On the other hand, one has

$$\hat{X}_{\lfloor nt\rfloor}^n = \sqrt{2\hat{A}_{\lfloor nt\rfloor}^n}\times\sqrt{\frac{1}{n}}Y_{\lfloor nt\rfloor}\to\sqrt{2A_t}\times W_t = X_t.$$

∎

2.4. A particular case: f is a power function. From the second equality in (2.5), one can expect at least four different long-time behaviors for the process $(X_t)_{t\geq 0}$. Indeed, the process can stop in finite time, in the case where there exists a finite time t such that $A_t = 0$. On the contrary, the process can perform very large oscillations if $(A_t)_{t\geq 0}$ tends to infinity in finite time. Last, when $(A_t)_{t\geq 0}$ takes its values in $(0,\infty)$, we can expect the process $(X_t)_{t\geq 0}$ either to go asymptotically to 0, if $(A_t)_{t\geq 0}$ decreases fast enough, or to be recurrent in $\mathbb{R}$, in the case where $(A_t)_{t\geq 0}$ remains large enough.

Those four behaviors actually do occur in the case of a power function $f(a) = \pm a^\gamma$. The advantage of such a function is that the expression of A_t can explicitly be computed as a function of L_t^W.

When $(A_t)_{t\geq 0}$ remains in $(0,\infty)$ for all positive t, one can derive a polynomial behavior for $(X_t)_{t\geq 0}$. Indeed, up to renormalisation by a power of t, we prove convergence in law to a non-Gaussian distribution for $(X_t)_{t\geq 0}$.

In a first place, we give an explicit expression of A_t, and then we give the asymptotic behavior of X_t, depending on the sign of f.

2.4.1. Explicit expression of A_t. The following lemma gives the expression of A_t as a function of L_t^W. For simplicity we assume that $X_0 = 1$ and $A_0 = 1$.

LEMMA 2.3. *Let $f(a) = \sigma a^\gamma$, with $\sigma = \pm 1$. The solution $(X_t, A_t)_{0\leq t<\tau}$ to (2.1) exists up to the time τ defined by*

$$\tau := \begin{cases} +\infty & \text{if } \sigma(3/2-\gamma) \geq 0, \\ \inf\left\{t \geq 0,\ L_t^W = \frac{\sqrt{2}}{\sigma(\gamma-3/2)}\right\} & \text{if } \sigma(3/2-\gamma) < 0. \end{cases} \tag{2.7}$$

For all $t \in [0,\tau)$, one has

$$A_t = \begin{cases} e^{\sigma L_t^W/\sqrt{2}} & \text{if } \gamma = 3/2, \\ (1 + \frac{\sigma}{\sqrt{2}(3/2-\gamma)} L_t^W)^{\frac{1}{3/2-\gamma}} & \text{if } \gamma \neq 3/2. \end{cases}$$

Proof. As we have seen in the proof of Proposition 2.1, A_t is given by $A_t = \Phi_{A_0}(L_t^W)$ where Φ_x is the solution to $y'(t) = f(y(t))/\sqrt{2y}$ with initial condition $\Phi_x(0) = x$. Here, Φ_1 is given by

$$\Phi_1(t) = \begin{cases} e^{\sigma t/\sqrt{2}} & \text{if } \gamma = 3/2, \\ (1 + \frac{\sigma}{\sqrt{2}(3/2-\gamma)} t)^{\frac{1}{3/2-\gamma}} & \text{if } \gamma \neq 3/2, \end{cases}$$

which lies in $(0,\infty)$ for $0 \leq t < \sqrt{2}(\gamma - 3/2)/\sigma$ if $\sigma(3/2-\gamma) < 0$, and for all $t \geq 0$ otherwise. The expressions of τ and A_t easily follow. ∎

2.4.2. Local deceleration: $f(a) = -a^\gamma$. In that case the process $(X_t)_{t\geq 0}$ is slowing down and we obtain a dynamical transition to an absorbing state. For sufficiently strong deceleration, the particle might get trapped at the origin in finite time while if the deceleration process is sufficiently weak the particle never gets trapped.

PROPOSITION 2.4. *Assume that $f(a) = -a^\gamma$, then*

- *if $\gamma < 3/2$, the stopping time τ defined in (2.7) is almost surely finite, and one has $\lim_{t\to\tau}(X_t, A_t) = (0,0)$;*
- *if $3/2 \leq \gamma < 2$, $\tau = \infty$ and $X_t \to 0$ as t goes to ∞. However, almost surely, for all $t > 0$, there exists $s > t$ such that $X_s \neq 0$;*

- *if $2 \leq \gamma$, then $\tau = \infty$ and the process $(X_t)_{t\geq 0}$ is recurrent in $\mathbb{R}$.*

Proof. The case $\gamma < 3/2$ is a consequence of the fact that $(A_t)_{t\geq 0}$ is absorbed by 0 in finite time.

The case $\gamma \geq 3/2$, $\gamma \neq 2$ follows from the almost sure asymptotic behavior $t^{1/2-\varepsilon} = o(L_t^W)$ and $L_t^W = o(t^{1/2+\varepsilon})$, for any $\varepsilon > 0$, and from the law of iterated logarithm for Brownian motion.

In the limit case $\gamma = 2$, $(X_t)_{t\geq 0}$ is equivalent in the long time to $(4W_t/L_t^W)_{t\geq 0}$, and it is enough to consider this latter process. From [7], Theorem 4.5, in the case $\beta = 2$, and $f(x) = 1/(x \log x)$, there exist random times $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that

$$\forall n \in \mathbb{N}, \; \frac{\sup_{s\leq t_n} |W_s|}{L_{t_n}^W} \geq \log(L_{t_n}^W).$$

Let $(\tilde{t}_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence satisfying $W_{\tilde{t}_n} = \sup_{s\leq t_n} |W_s|$, in particular one has $|W_{\tilde{t}_n}|/L_{\tilde{t}_n}^W \to \infty$. Since $\limsup_{t\to\infty} |W_t| = \infty$, up to a subsequence, one has $\tilde{t}_n \to \infty$. Moreover, one can find another sequence of random times t'_n going to ∞ and satisfying $W_{t'_n} = 0$. As a consequence, one obtains

$$\liminf_{t\to\infty} \frac{|W_t|}{L_t^W} = 0 \text{ and } \limsup_{t\to\infty} \frac{|W_t|}{L_t^W} = \infty.$$

Hence, $(|X_t|)_{t\geq 0}$ is recurrent in $[0,\infty)$, and by symmetry this implies that $(X_t)_{t\geq 0}$ is recurrent in $\mathbb{R}$. ∎

Using the equality in distribution

$$\begin{aligned} X_t = \sqrt{2A_t} W_t &= \sqrt{2}\left(1 - \frac{3/2-\gamma}{\sqrt{2}} L_t^W\right)^{\frac{1}{3-2\gamma}} W_t \\ &\overset{(d)}{=} \sqrt{2}\left(1 + \frac{\sqrt{t}\,(\gamma - 3/2)}{\sqrt{2}} L_1^W\right)^{\frac{1}{3-2\gamma}} \sqrt{t} W_1, \end{aligned} \tag{2.8}$$

for $\gamma \neq 3/2$, we deduce that, when $\gamma > 1$, the decreasing rate of the process $(X_t)_{t\geq 0}$ is $t^{\frac{2-\gamma}{3-2\gamma}}$. In this expression of the rate, the exponent may be nonpositive or nonnegative. More precisely, one has:

PROPOSITION 2.5. *If $f(a) = -a^\gamma$, with $\gamma > 3/2$, then the convergence in distribution,*

$$t^{\frac{\gamma-2}{3-2\gamma}} X_t \overset{(d)}{\to} C_\gamma (L_1^W)^{\frac{1}{3-2\gamma}} W_1 \quad \text{as } t \to \infty,$$

holds true where $C_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}} (\gamma - 3/2)^{\frac{1}{3-2\gamma}}$.

One can also give the asymptotic behavior as $t \to \tau$ when $\tau < \infty$.

PROPOSITION 2.6. *If $f(a) = -a^\gamma$, with $\gamma < 3/2$, then, as $t \to 0$,*

$$\mathbf{1}_{t<\tau} t^{\frac{\gamma-2}{3-2\gamma}} X_{\tau-t} \overset{(d)}{\to} C'_\gamma (L_1^W)^{\frac{1}{2(3-2\gamma)}} W_1,$$

where $C'_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}} (3/2 - \gamma)^{\frac{1}{2(3-2\gamma)}}$.

Proof. We use the reversibility property of the Brownian motion that we recall, for $T > 0$, setting $\zeta = \inf\{t > 0, L_t^W = T\}$, the equality in distribution

$$(W_t, L_t^W)_{0\leq t\leq\zeta} \overset{(d)}{=} (W_{\zeta-t}, T - L_{\zeta-t}^W)_{0\leq t\leq\zeta},$$

holds true. In our situation, we apply this property to the stopping time $\zeta = \tau$, and we then use (2.8). ∎

REMARK 2.3. The fact that $(X_t)_{t\geq 0}$ can be trapped at 0 for $\gamma < 3/2$ was already noticed in [1]. However, the different behavior for $\gamma \in [3/2, 2)$ was not observed.

REMARK 2.4. As a consequence of Lemma 2.3, the survival probability of $(X_t)_{t\geq 0}$ at time t is given for $\gamma < 3/2$ by

$$S(t) = \mathbb{P}\left(L_t^W \leq \frac{\sqrt{2}}{3/2-\gamma}\right) = \mathbb{P}\left(|W_1| \leq \frac{\sqrt{2}}{\sqrt{t}(3/2-\gamma)}\right) \underset{t\to\infty}{\sim} \frac{2}{(3/2-\gamma)\sqrt{\pi t}},$$

where we used the equalities in distribution $L_t^W = |W_t| = \sqrt{t}|W_1|$. This fact was already observed in [1].

2.4.3. Local acceleration: $f(a) = a^\gamma$. In such a case the diffusion coefficient of $(X_t)_{t\geq 0}$ is nondecreasing. Again, the proof of the following result follows from Lemma 2.3 and the relation $X_t = \sqrt{2A_t}W_t$.

PROPOSITION 2.7. *Assume that $f(a) = a^\gamma$, then*

- *if $\gamma \geq 2$, the stopping time τ defined in (2.7) is almost surely finite and $\lim_{t\to\tau} A_t = \infty$. Moreover, $\lim_{t\to\tau} X_t = 0$;*
- *if $3/2 < \gamma < 2$, τ is almost surely finite and $\lim_{t\to\tau} A_t = \infty$. Moreover, $\liminf_{t\to\tau} X_t = -\infty$ and $\limsup_{t\to\tau} X_t = \infty$;*
- *if $\gamma \leq 3/2$, the time τ satisfies $\tau = \infty$, and $A_t \to \infty$ when $t \to \infty$.*

Furthermore, when $\gamma < 3/2$, from equality in distribution (2.8), one can deduce that the long time behavior of $(X_t)_{t\geq 0}$ is of order $t^{\frac{2-\gamma}{3-2\gamma}}$, where the exponent is positive. More precisely the following result holds true.

PROPOSITION 2.8. *If $f(a) = a^\gamma$, with $\gamma < 1$, then, as $t \to \infty$, one has the convergence in distribution*

$$t^{\frac{\gamma-2}{3-2\gamma}} X_t \overset{(d)}{\to} C'_\gamma (L_1^W)^{\frac{1}{3-2\gamma}} W_1,$$

where $C'_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}}(3/2-\gamma)^{\frac{1}{2(3-2\gamma)}}$.

We can also describe the rate of explosion of $(X_t, A_t)_{t\geq 0}$ as goes to τ, in the case $\tau < \infty$.

PROPOSITION 2.9. *If $f(a) = a^\gamma$, with $\gamma > 1$, then, as $t \to 0$, one has the convergence in distribution*

$$\mathbf{1}_{t<\tau} t^{\frac{\gamma-2}{3-2\gamma}} X_{\tau-t} \overset{(d)}{\to} C_\gamma (L_1^W)^{\frac{1}{3-2\gamma}} W_1,$$

where $C_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}}(\gamma-3/2)^{\frac{1}{2(3-2\gamma)}}$.

REMARK 2.5. The case $\gamma = 0$ was treated by deterministic methods in [1], through an approximation of the Laplace transform of the distribution of X_t. Here, by using the stochastic differential Equation (2.1), we were able to compute the exact asymptotic behavior. We obtain that the growth rate of $(X_t)_{t\geq 0}$ is given by $t^{2/3}$, and that its diffusion coefficient, given by $\left(L_t^W\right)^{3/2}$, behaves as $t^{1/3}$. Those exponents were correctly predicted in [1].

3. Generator and "weak generator" of the process $(X_t, A_t)_{t\geq 0}$

In this section, in order to give a rigorous meaning to the generator given in [1], thereby establishing that the process is really the same as the one considered there, we investigate the weak generator of the Markov process $(X_t, A_t)_{t\geq 0}$ solution to system (2.1). The expression of the generator will follow from the generator of the process $(W_t, L_t^W)_{t\geq 0}$, where W_t is a standard Brownian motion and L_t^W is its local time at 0.

Consider the unique maximal solution $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$ of (2.1), whose existence is ensured by Corollary 2.2. The first step is to extend its state space in order to define a continuous Markov process for all positive times.

3.1. Extended state space.

In the proof of Proposition 2.1, we mentionned that, when τ is finite, A_t necessarily converges as t goes to τ, either toward 0 or ∞. In the case $A_t \to 0$, one can also determine the behavior of X_t, as stated in the following lemma.

LEMMA 3.1. *On the event $\{\tau < \infty, \lim_{t\to\tau} A_t = 0\}$, X_t converges to 0 as t goes to τ.*

Proof. First, one notices that on the set $\{\tau < \infty, \lim_{t\to\tau} A_t = 0\}$ the limit

$$\lim_{t\to\tau} X_t = X_0 + \int_0^\tau \sqrt{2A_s}\, \mathrm{d}W_s$$

exists almost surely. It is thus enough to prove that the event $E = \{\tau < \infty, \lim_{t\to\tau} A_t = 0, \lim_{t\to\tau} X_t \neq 0\}$ is a null set.

On E, there exists a random variable $h > 0$ such that $X_t \neq 0$ for all $t \in (\tau - h, \tau)$. Hence, on E, one has for all $t \in (\tau - h, \tau)$

$$A_\tau - A_t = \int_t^\tau \mathrm{d}A_s = \int_t^\tau \frac{f(A_s)}{2A_s}\, \mathrm{d}L_s^X = 0.$$

However, on E, one also has $\tau = \inf\{t \geq 0, A_t = 0\}$, which contradicts the fact that A_t is constant on $(\tau - h, \tau)$. As a consequence, E has probability 0, which concludes the proof. ∎

From Lemma 3.1, the maximal solution $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$ to Equation (2.1) can be extended to a process defined for all positive times by setting

$$(X_t, A_t) = \begin{cases} (0,0) & \text{on } \{\tau \leq t, \lim_{s\to\tau} A_s = 0\}, \\ (0,\infty) & \text{on } \{\tau \leq t, \lim_{s\to\tau} A_s = \infty\}. \end{cases}$$

For notational simpliciy the extended process will still be denoted by $(X_t, A_t)_{t\geq 0}$. This will define a Markov process with state space $\mathcal{E} = (\mathbb{R} \times [0,\infty)) \cup \{(0,\infty)\}$, which is the half plane $\mathbb{R} \times [0,\infty)$ augmented with an additional point $(0,\infty)$. We define the following topology on $\mathcal{E}$: the subset $\mathbb{R} \times [0,\infty)$ is endowed with its usual topology, and we choose the family $(\mathbb{R} \times [\alpha,\infty))_{\alpha>0}$ as a neighborhood basis of $(0,\infty)$. In other words, any sequence $(x_n, a_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times [0,\infty)$ with $a_n \to \infty$ will satisfy $(x_n, a_n) \to (0,\infty)$ in $\mathcal{E}$.

With these conventions, $(X_t, A_t)_{t\geq 0}$ defines a continuous Markov process with values in $\mathcal{E}$, defined for all positive times. A natural question is then to investigate its generator, and to compute the distribution of (X_t, A_t) for a given $t > 0$. Note that the two points $(0,0)$ and $(0,\infty)$ are absorbing points for the Markov process $(X_t, A_t)_{t\geq 0}$.

3.2. Shape of the distribution of (X_t, A_t)**.** In this part, we give the general form of the distribution μ_t at time t of the solution $(X_t, A_t)_{t\geq 0}$ to (2.1). In particular we prove that its restriction to $\mathbb{R} \times \mathbb{R}_+ \setminus \{(0,0)\}$ has a density, denoted by n_t, with respect to the Lebesgue measure when considering the particular initial condition $\mu_0 = \delta_{(x_0,a_0)}$.

Since the two points $(0,0)$, $(0,\infty)$ are absorbing points for the process $(X_t, A_t)_{t\geq 0}$, the distribution starting from those points will be constant, equal to $\delta_{(0,0)}$ or $\delta_{(0,\infty)}$ respectively.

We first consider the case where the initial condition is $\mu_0 = \delta_{(0,a_0)}$, with $a_0 > 0$.

Lemma 3.2. *Assume that the initial condition (x_0, a_0) satisfies $f(a_0) \neq 0$ and $x_0 = 0$. Then, for all $t > 0$, there exists a measurable function $n_t : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ and two real numbers $p_t \in [0,1]$ and $q_t \in [0,1]$ such that*

$$\mu_t(\,\mathrm{d}x,\,\mathrm{d}a) = n_t(x,a)\,\mathrm{d}x\,\mathrm{d}a + p_t\delta_{(0,0)} + q_t\delta_{(0,\infty)}, \tag{3.1}$$

where $p_t + q_t + \int_{\mathbb{R}\times[0,\infty)} n_t(x,a)\,\mathrm{d}x\,\mathrm{d}a = 1$. Furthermore, if f is nonnegative, one has $p_t = 0$ for all $t > 0$, while if f is nonpositive, one has $q_t = 0$ for all $t > 0$.

Proof. All we need to prove is that the restriction of μ_t to the set $\mathbb{R} \times (0,\infty)$ admits a density with respect to the Lebesgue measure.

First, since $t^{-1/2}(W_t, L_t^W)$ has the same distribution as (W_1, L_1^W), which has a density, see [6], page 45, it follows that (W_t, L_t^W) with $W_0 = 0$ admits a density γ_t on $\mathbb{R}\times(0,\infty)$. Moreover, as $(X_t)_{0\leq t<\tau}$ starts from the initial condition $x_0 = 0$, Proposition 2.2 states that $(X_t, A_t) = \left(\sqrt{2\Phi_{a_0}(L_t^W)}W_t, \Phi_{a_0}(L_t^W)\right)$, where Φ_{a_0} is the flow of the differential equation $y' = f(y)$ starting at a_0. For a given a_0, $\Phi_{a_0}(l)$ is defined for all $l \in [0, T(a_0))$ for some $T(a_0) \in [0,\infty]$.

Let us define the mapping $\Psi(w,l) := \left(\sqrt{2\Phi_{a_0}(l)}w, \Phi_{a_0}(l)\right)$, for (w,l) in $\mathbb{R} \times (0, T(a_0))$. To conclude, it is enough to show that Ψ is a local diffeomorphism from $\mathbb{R}\times(0,T(a_0))$ to $\mathbb{R}\times(0,\infty)$. The Jacobian determinant of the $\mathcal{C}^1$ function Ψ is given by

$$J_\Psi(w,l) = \Phi'_{a_0}(l)\sqrt{2\Phi_{a_0}(l)} = f\left(\Phi_{a_0}(l)\right).$$

From uniqueness in the Cauchy-Lipschitz theorem, if $f(\Phi_{a_0}(l)) = 0$ for some l, then $f(\Phi_{a_0}(l)) = 0$ for all $l \in [0, T(a_0))$, but this contradicts the assumption $f(a_0) \neq 0$. As a consequence, J_Ψ does not vanish on $\mathbb{R}\times(0,T(a_0))$, so that Ψ is a local diffeomorphism, and μ_t has a density on $\mathbb{R}\times(0,\infty)$. ∎

Without the assumption $x_0 = 0$, $(X_t)_{t\geq 0}$ will stay away from 0 for a positive time ζ. In that case $(A_t)_{t\geq 0}$ remains constant on the interval $[0,\zeta]$, and this results in a more complicated expression for μ_t, as stated in the following lemma.

Lemma 3.3. *Assume that $f(a_0) \neq 0$ and $x_0 \neq 0$. Then, μ_t has the form*

$$\mu_t(\,\mathrm{d}x,\,\mathrm{d}a) = m_t(x)\,\mathrm{d}x \otimes \delta_{a_0} + n_t(x,a)\,\mathrm{d}x\,\mathrm{d}a + p_t\delta_{(0,0)} + q_t\delta_{(0,\infty)},$$

where m_t, n_t, are measurable functions respectively defined on $\mathbb{R}$ and $\mathbb{R}\times(0,\infty)$.

Proof. This relies on the strong Markov property used at time $\inf\{t > 0, X_t = 0\}$ together with Lemma 3.2. ∎

The last case to consider is when the process starts from a point where its diffusion coefficient does not change. In that case, $(X_t)_{t\geq 0}$ exists for all positive times, and behaves as a Brownian motion multiplied by some constant.

LEMMA 3.4. *Assume that* $f(a_0) = 0$. *Then,* μ_t *is given by*

$$\mu_t(\,\mathrm{d}x,\,\mathrm{d}a) = \gamma^{x_0}_{2ta_0}(x)\,\mathrm{d}x \otimes \delta_{a_0},$$

where $\gamma^{x_0}_{\sigma^2}$ *denotes the Gaussian distribution with mean* x_0 *and variance* σ^2.

Proof. In that case $(X_t, A_t)_{t\geq 0}$ is given by $(X_t, A_t) = (x_0 + \sqrt{2a_0}W_t, a_0)$. ◻

3.3. Generator and "weak generator" of the process $(W_t, L^W_t)_{t\geq 0}$. We start this section by giving the generator and its domain of the Markov process $(W_t, L^W_t)_{t\geq 0}$. Although it is classical we did not find any references for it, therefore we give the proof here.

PROPOSITION 3.1. *The generator* $\mathcal{L}_W$ *of the Markov process* $(W_t, L^W_t)_{t\geq 0}$ *is given by*

$$\mathcal{L}_W h(w, l) = \frac{1}{2}\partial^2_{ww} h(w, l),$$

and its domain $\mathcal{D}$ *contains the following set* $\mathcal{X}$.

$$\mathcal{X} = \Big\{ h \in \mathcal{C}^0(\mathbb{R}\times[0,\infty)),\ h \in \mathcal{C}^{2,1}([0,+\infty)\times[0,\infty)) \cap \mathcal{C}^{2,1}((-\infty,0]\times[0,\infty)),$$
$$\partial_l h \in \mathcal{C}^0(\mathbb{R}\times[0,\infty)),\ \partial^2_w h \in \mathcal{C}^0(\mathbb{R}\times[0,\infty)),\ \partial_w h(w=0^+, l) = -\partial_l h(w=0^+, l)$$
$$\text{and } \partial_w h(w=0^-, l) = \partial_l h(w=0^-, l) \quad \forall l > 0 \Big\}.$$

Proof. From translation invariance in $(0, l)$, it is enough to perform a Taylor expansion in $(0, 0)$. For all $h \in \mathcal{X}$, one has, with obvious notations,

$$\begin{cases} \forall w > 0 \quad h(w,l) &= h(0,0) + l\partial_l h(0,0) + w\partial^+_w h(0,0) \\ &\quad + \frac{w^2}{2}\partial^2_w h(0,0) + \text{ higher order terms}, \\ \forall w < 0 \quad h(w,l) &= h(0,0) + l\partial_l h(0,0) + w\partial^-_w h(0,0) \\ &\quad + \frac{w^2}{2}\partial^2_w h(0,0) + \text{ higher order terms}, \end{cases}$$

hence using the definition of the set $\mathcal{X}$ we obtain

$$\frac{h(W_t, L^W_t) - h(0,0)}{t} = \frac{L^W_t}{t}\partial_l h(0,0) - \frac{|W_t|}{t}\partial_l h(0,0) + \frac{W^2_t}{2t}\partial^2_w h(0,0) + R(t).$$

Using the properties of W_t and L^W_t, we see that $|\mathbb{E}(R(t))| = O(\sqrt{t})$, hence

$$\mathbb{E}\left[\frac{h(W_t, L^W_t) - h(0,0)}{t}\right] = \partial_l h(0,0)\frac{\mathbb{E}[L^W_t - |W_t|]}{t} + \frac{\mathbb{E}[W^2_t]}{2t}\partial^2_w h(0,0) + O(\sqrt{t}).$$

Then, the conclusion follows by recalling that the law of L^W_t is the same as the law of $|W_t|$ (see [8], chapter VI). ◻

The previous notion of generator (in particular for Feller processes, like here) is an operator from its domain $\mathcal{D} \subset \mathcal{C}^0$ to $\mathcal{C}^0$.

Let us now introduce a "weaker notion of generator". Such a weak generator is weaker than the one introduced in Dynkin [5] and corresponds to an extension of the classical generator to a bigger space by letting $\mathcal{L}h$ be a distribution. This is the object of the following Proposition.

PROPOSITION 3.2. *The weak generator $\mathcal{L}_0$ of the Markov process $(W_t, L_t^W)_{t\geq 0}$ is given by*

$$\mathcal{L}_0 h(w,l) = \frac{1}{2}\partial^2_{ww} h(w,l) + \partial_l h(w,l)\delta_{w=0},$$

for h continuously differentiable in the l-variable and twice continuously differentiable in the w-variable with bounded derivatives.

REMARK 3.1. Since the coefficient $\delta_{w=0}$ is singular, the previous definition of the generator has to be understood in a weak sense. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function with bounded support, then, for h continuously differentiable in the l-variable and twice continuously differentiable in the w-variable with bounded derivatives, for all $l \in [0,\infty)$, one has

$$\lim_{t\to 0}\int_{\mathbb{R}} \varphi(w)\mathbb{E}^{w,l}\left[\frac{h(W_t, L_t^W) - h(w,l)}{t}\right] \mathrm{d}w = \int_{\mathbb{R}} \varphi(w)\frac{1}{2}\partial^2_{ww} h(w,l)\,\mathrm{d}w + \varphi(0)\partial_l h(0,l).$$

Here, $\mathbb{E}^{w,l}$ stands for the expectation conditionally to $\{W_0 = w,\ L_0^W = l\}$. Equivalently, one has, in the distributional sense,

$$\lim_{t\to 0}\mathbb{E}^{w,l}\left[\frac{h(W_t, L_t^W) - h(w,l)}{t}\right] = \frac{1}{2}\partial^2_{ww} h(w,l) + \partial_l h(0,l)\delta_{w=0}.$$

Proof. From time invariance of $(L_t^W)_{t\geq 0}$, one can assume that $l = 0$. From the equality $h(W_t, L_t^W) = h(W_t, 0) + L_t^W \int_0^1 \partial_l h(W_t, sL_t^W)\,\mathrm{d}s$, one obtains

$$\mathbb{E}^{w,0}\left[\frac{h(W_t, L_t^W) - h(w,0)}{t}\right] = \mathbb{E}^{w,0}\left[\frac{h(W_t, 0) - h(w,0)}{t}\right] + \mathbb{E}^{w,0}\left[\frac{L_t^W}{t}\int_0^1 \partial_l h(W_t, sL_t^W)\,\mathrm{d}s\right].$$

The first term in the right-hand side converges to $\frac{1}{2}\partial^2_{ww} h(w,0)$ as $t \to 0$, since $\frac{1}{2}\partial^2_{ww}$ is the generator of the Brownian motion, using the boundedness of $\partial^2_{ww} h$.

Then, using the fact that the law of (W_t, L_t^W) under $\mathbb{E}^{w,0}$ is the same as the law of $(\sqrt{t}W_1, \sqrt{t}L_1^W)$ under $\mathbb{E}^{w/\sqrt{t},0}$, one gets

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(w)\mathbb{E}^{w,0}\left[\frac{L_t^W}{t}\int_0^1 \partial_l h(W_t, sL_t^W)\,\mathrm{d}s\right]\mathrm{d}w \\
&= \int_{\mathbb{R}} \varphi(w)\mathbb{E}^{w/\sqrt{t},0}\left[\frac{\sqrt{t}L_1^W}{t}\int_0^1 \partial_l h(\sqrt{t}W_1, s\sqrt{t}L_1^W)\,\mathrm{d}s\right]\mathrm{d}w \\
&= \int_{\mathbb{R}} \varphi(\sqrt{t}w)\mathbb{E}^{w,0}\left[L_1^W\int_0^1 \partial_l h(\sqrt{t}W_1, s\sqrt{t}L_1^W)\,\mathrm{d}s\right]\mathrm{d}w.
\end{aligned}$$

From the dominated convergence theorem, this converges as t goes to 0 to

$$\varphi(0)\partial_l h(0,0)\int_{\mathbb{R}} \mathbb{E}^{w,0}[L_1^W]\,\mathrm{d}w.$$

Finally, from the occupation time formulation $\int_{\mathbb{R}} L_t^{W+w}\,\mathrm{d}w = t$ (see [8], chapter VI) one obtains

$$\int_{\mathbb{R}} \mathbb{E}^{w,0}[L_1^W]\,\mathrm{d}w = \int_{\mathbb{R}} \mathbb{E}^{0,0}[L_1^{W+w}]\,\mathrm{d}w = 1.$$

□

3.4. Generator and "weak generator" of the process defined by system (2.1). Since $(X_t, A_t)_{t\geq 0}$ can be obtained as a function of $(W_t, L_t^W)_{t\geq 0}$, we can compute the generator of the former from the generator of the latter. We start with the classical generator and its domain and then we describe its "weak generator". Since the proof is similar to the one of Proposition 3.1 we omit it.

PROPOSITION 3.3. *The generator $\mathcal{L}_x$ of the Markov process $(X_t, A_t)_{t\geq 0}$ defined by system (2.1) is given by*

$$\mathcal{L}_x h(x,a) = a\partial^2_{xx} h(x,a),$$

and its domain $\mathcal{D}_x$ contains the following space $\mathcal{X}_x$

$$\mathcal{X}_x = \Big\{ h \in \mathcal{C}^0\left(\mathbb{R}\times[0,\infty)\right), \, h \in \mathcal{C}^{2,1}\left([0,+\infty)\times[0,\infty)\right) \cap \mathcal{C}^{2,1}\left((-\infty,0]\times[0,\infty)\right),$$
$$\partial_a h \in \mathcal{C}^0\left(\mathbb{R}\times[0,\infty)\right), \, \partial^2_{xx} h \in \mathcal{C}^0\left(\mathbb{R}\times[0,\infty)\right),$$
$$\partial_x h(x=0^+,a) = -\frac{f(a)}{2a}\partial_a h(x=0^+,a)$$
$$\text{and } \partial_x h(x=0^-,a) = \frac{f(a)}{2a}\partial_a h(x=0^-,a) \quad \forall a>0 \Big\}.$$

PROPOSITION 3.4. *The "weak generator" $\mathcal{L}$ of the Markov process $(X_t, A_t)_{t\geq 0}$ solution to system (2.1) is given by*

$$\mathcal{L}h(x,a) = a\partial^2_{xx}h(x,a) + f(a)\partial_a h(x,a)\delta_{x=0},$$

for h continuously differentiable in the a-variable and twice continuously differentiable in the x-variable with bounded derivatives.

REMARK 3.2. Again, the previous definition has to be understood in a weak sense. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function with bounded support, then, for h continuously differentiable in the a-variable and twice continuously differentiable in the x-variable with bounded derivatives, for all $a \in [0,\infty)$ one obtains

$$\lim_{t\to 0} \int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}\left[\frac{h(X_t,A_t)-h(x,a)}{t}\right] \mathrm{d}x$$
$$= \int_{\mathbb{R}} \varphi(x) a\partial^2_{xx}h(x,a)\,\mathrm{d}x + \varphi(0)f(a)\partial_a h(0,a).$$

Here, $\tilde{\mathbb{E}}^{x,a}$ stands for the expectation conditionally to $\{X_0 = x,\ A_0 = a\}$.

REMARK 3.3. Since (X_t, A_t) is a Markov process, the following identity holds for any probability density φ

$$\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}[h(X_t,A_t)]\,\mathrm{d}x = \tilde{\mathbb{E}}^{\varphi,a}[h(X_t,A_t)],$$

where $\tilde{E}^{\varphi,a}$ denotes the expectation for an initial condition satisfying $A_0 = a$ and such that X_0 admits φ as density. In other word, when X_0 admits a continuous density v_0, replacing φ by v_t in Remark 3.2 yields the following time-derivative at $t=0$:

$$\partial_t \tilde{\mathbb{E}}^{v_0,a}[h(X_t,A_t)]_{|t=0} = \partial_t \iint_{\mathbb{R}\times\mathbb{R}^+} h(x,a)\,\mathrm{d}\mu_t(x,a)_{|t=0}$$

$$= \int_{\mathbb{R}} a\partial^2_{xx} h(x,a) v_0(x)\,\mathrm{d}x + v_0(0) f(a)\partial_a h(0,a).$$

This is a weak formulation of Equation (4.1) below.

Proof. Let $(W_t)_{t\geq 0}$ be a Brownian motion started at $X_0/\sqrt{2A_0}$. From the proof of Proposition 2.1, we know that the process $(X_t, A_t)_{0\leq t<\tau}$ is given by

$$\forall 0 \leq t < \tau, \quad \begin{cases} X_t &= \sqrt{2\Phi_{A_0}(L^W_t)} W_t, \\ A_t &= \Phi_{A_0}(L^W_t), \end{cases}$$

where $t \to \Phi_x(t)$ is the flow of the differential equation $y' = f(y)/\sqrt{2y}$ with initial condition x. Then, setting $x = \sqrt{2a}w$, one has (if we still denote by $\mathbb{E}^{w,l}$ the expectation conditionally to $\{W_0 = w, L^W_0 = l\}$)

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}\left[\frac{h(X_t,A_t) - h(x,a)}{t}\right]\mathrm{d}x \\
=&\int_{\mathbb{R}} \varphi(x)\mathbb{E}^{\frac{x}{\sqrt{2a}},0}\left[\frac{h(\sqrt{2\Phi_a(L^W_t)}W_t, \Phi_a(L^W_t)) - h(x,a)}{t}\right]\mathrm{d}x \\
=&\sqrt{2a}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)\mathbb{E}^{w,0}\left[\frac{h(\sqrt{2\Phi_a(L^W_t)}W_t, \Phi_a(L^W_t)) - h(\sqrt{2a}w,a)}{t}\right]\mathrm{d}w.
\end{aligned}$$

Applying Proposition 3.2 to the function $F(w,l) = h\left(\sqrt{2\Phi_a(l)}w, \Phi_a(l)\right)$ one obtains

$$\begin{aligned}
&\lim_{t\to 0}\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}\left[\frac{h(X_t,A_t) - h(x,a)}{t}\right]\mathrm{d}x \\
=&\lim_{t\to 0}\sqrt{2a}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)\mathbb{E}^{w,0}\left[\frac{F(W_t,L^W_t) - F(w,0)}{t}\right]\mathrm{d}w \\
=&\sqrt{\frac{a}{2}}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)\partial^2_{ww}F(w,0)\,\mathrm{d}w + \sqrt{2a}\varphi(0)\partial_l F(0,0) \\
=&\sqrt{\frac{a}{2}}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)(2a)\partial^2_{xx}h(\sqrt{2a}w,a)\,\mathrm{d}w + \Phi'_a(0)\sqrt{2a}\varphi(0)\partial_a h(0,a) \\
=&\int_{\mathbb{R}} \varphi(x)a\partial^2_{xx}h(x,a)\,\mathrm{d}x + f(a)\varphi(0)\partial_a h(0,a).
\end{aligned}$$

∎

4. Mathematical study of the PDE (1.3)

In this section we study the PDE (1.3) that we recall now

$$\partial_t u_t(x,a) = a\partial^2_{xx}u_t(x,a) - \partial_a\left(f(a)\,u_t(x,a)\right)\delta_{x=0}, \qquad (x,a) \in \mathbb{R}\times[0,\infty). \tag{4.1}$$

In particular, if $f(a) = -a^\gamma$, by studying the regularity of the solution in a L^p framework, we recover the results observed during the probabilistic study: global existence if $\gamma \geq 3/2$, while, in the case $\gamma < 3/2$, the solution becomes unbounded in finite time (so-called blow-up). Moreover, as in [1], using Laplace and Fourier transforms, for a particular initial condition we explicitly compute the solution to (4.1).

First, in the case where $f(0) = 0$, recalling Proposition 3.4 and the two Remarks 3.2 and 3.3, by performing some integration by parts we can make the link between the absolutely continuous part n_t of the distribution μ_t, if it is regular enough, see Section 3.2, and the PDE (4.1).

4.1. Basic facts about weak solution to (4.1). This is a linear equation on $u = u(t, x, a)$ defined on $t \geq 0$, $x \in \mathbb{R}$, $a \geq 0$. We begin with a definition of weak solutions, adapted to the study of (4.1). We recall that f is assumed to be locally Lipschitz continuous from $(0, \infty)$ to $\mathbb{R}$.

DEFINITION 4.1. *Let $p \in [1, +\infty)$. Let $u_0 \in L^1(\mathbb{R} \times \mathbb{R}_+)$. We say that u is a weak solution to (4.1) on $[0, T)$ if it satisfies:*

$$u \in L^\infty([0,T); L^1(\mathbb{R} \times \mathbb{R}_+) \cap L^p(\mathbb{R} \times \mathbb{R}_+)), \quad \partial_x u \in L^1([0,T) \times \mathbb{R} \times \mathbb{R}_+),$$
$$\text{and } \partial^2_{xx} u \in L^1((0,T) \times \mathbb{R} \times \mathbb{R}_+) \tag{4.2}$$

and for any test function φ in $\mathcal{C}^\infty_c(\mathbb{R} \times \mathbb{R}_+)$ and a.e. $t \in (0, T)$,

$$\iint_{\mathbb{R}\times\mathbb{R}_+} u(t,x,a)\varphi(x,a)\,\mathrm{d}x\,\mathrm{d}a = \iint_{\mathbb{R}\times\mathbb{R}_+} u_0(x,a)\varphi(x,a)\,\mathrm{d}x\,\mathrm{d}a$$
$$- \int_0^t \iint_{\mathbb{R}\times\mathbb{R}_+} a\partial_x u(s,x,a)\partial_x\varphi(x,a)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}_+} f(a)u(s,0,a)\partial_a\varphi(0,a)\,\mathrm{d}a\,\mathrm{d}s.$$

Since $\partial_x u(t, x, a)$ belongs to $L^1((0, T) \times \mathbb{R} \times \mathbb{R}_+)$, the solution is well-defined in the distributional sense under assumption (4.2). In fact we can write $\int_0^T u(t, 0, a)\,\mathrm{d}t = -\int_0^T \int_{x>0} \partial_x u(t, x, a)\,\mathrm{d}x\,\mathrm{d}t$.

Weak solutions in the sense of Definition 4.1 are mass-preserving:

$$M = \iint u_0(x,a)\,\mathrm{d}x\,\mathrm{d}a = \iint u(t,x,a)\,\mathrm{d}x\,\mathrm{d}a.$$

Let us first prove that non-negativity is preserved.

LEMMA 4.1. *Assume that u is a weak solution to (4.1). If $|u_0| = u_0$ almost everywhere (initial data non-negative). Then $|u(t, \cdot)| = u(t, \cdot)$ almost everywhere for later times.*

Proof. Observe that if u is solution in L^1 then $|u|$ is subsolution in L^1 since $\mathrm{sgn}(u)\partial^2_{xx} u \leq \partial^2_{xx}|u|$ and $\mathrm{sgn}(u)\partial_a(f(a)\,u)\delta_{x=0} = \partial_a(f(a)\,|u|)\delta_{x=0}$. Hence $|u| - u$ is a subsolution, and

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint (|u| - u)\,\mathrm{d}x\,\mathrm{d}a \leq 0.$$

□

Let us next prove that in the case $f(a) \leq 0$ (deceleration) and $u_0 \geq 0$, the compact support in a is preserved along time.

LEMMA 4.2. *Assume u is a weak solution to (4.1) with $f(a) \leq 0$. Assume in addition that $supp(u_0) \subset \mathbb{R} \times [0, a_0]$ for some $a_0 > 0$. Then $supp(u) \subset \mathbb{R} \times [0, a_0]$ up to the existence time.*

Proof. Consider any non-negative non-decreasing function $\varphi = \varphi(a)$ smooth on $\mathbb{R}_+$ with support included in $(a_0, +\infty)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint u(t,x,a)\varphi(a)\,\mathrm{d}x\,\mathrm{d}a = \int_a f(a)u(t,0,a)\varphi'(a)\,\mathrm{d}a \leq 0,$$

which proves that $u\varphi = 0$ for later times. Varying the φ as defined above we conclude that $u = 0$ on $\mathbb{R} \times [a_0, +\infty)$ for later times. □

Similarly, in the case $f(a) \geq 0$ (acceleration) and $u_0 \geq 0$, we can prove that if the support of u_0 is included in $\mathbb{R} \times [a_0, +\infty)$ for some $a_0 > 0$, then the same fact is true for all $t > 0$.

4.2. The different cases for the law of change in the particular case where $f(a) = \pm a^\gamma$. Following the probabilistic study performed in Section 2.4 we consider the three following cases:

(a) acceleration at $x = 0$: $f(a) = a^\gamma$ with $\gamma \geq 0$,

(b) subcritical deceleration at $x = 0$: $f(a) = -a^\gamma$ with $\gamma \geq 3/2$,

(c) supercritical deceleration at $x = 0$: $f(a) = -a^\gamma$ with $\gamma \in [0, 3/2)$.

In this part we will prove the following result:

THEOREM 4.2. *Assume that the initial datum u_0 belongs to $L^p \cap L^1$, $p \geq 1$, then*

(a) in the acceleration case, there exists a unique weak solution to (4.1) that satisfies for all $T > 0$, $\sup_{t\in[0,T]} \iint_{\mathbb{R}\times\mathbb{R}_+} |u(t,x,a)|^p \, \mathrm{d}x \, \mathrm{d}a < +\infty$,

(b) in the subcritical deceleration case, for $p = 2$, there exists a unique weak solution to (4.1) that satisfies for all $T > 0$,
$\sup_{t\in[0,T]} \iint_{\mathbb{R}\times\mathbb{R}_+} |u(t,x,a)|^2 \, \mathrm{d}x \, \mathrm{d}a < +\infty$,

(c) in the supercritical deceleration case, any weak solution of (4.1) blows-up in finite time.

Proof. In the first two cases (a) and (b), we prove the propagation of L^p bounds, which is the crucial a priori estimate. To prove that solutions blow-up in finite time in the supercritical case (c), we show that for an appropriate value of M, the momentum $\int a^M u \, \mathrm{d}a$ becomes infinite in finite time.

Case (a): global existence and uniqueness. Let $p \in [1, +\infty)$, we have the following a priori estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}\times\mathbb{R}_+} |u(t,x,a)|^p \, \mathrm{d}x \, \mathrm{d}a \leq -\, p(p-1) \iint_{\mathbb{R}\times\mathbb{R}_+} a \left|\partial_x u(t,x,a)\right|^2 |u(t,x,a)|^{p-2} \, \mathrm{d}x \, \mathrm{d}a$$
$$- \gamma(p-1) \int_{\mathbb{R}_+} a^{\gamma-1} |u(t,0,a)|^p \, \mathrm{d}a \leq 0,$$

which proves that L^p norms remain finite for all times provided they are finite initially (no finite time appearance of a singularity). By applying the same argument to the modulus of the difference of two solutions one proves similarly uniqueness in L^p.

Case (b): global existence and uniqueness. In this case the a priori estimate writes for $p = 2$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}\times\mathbb{R}_+} u^2(t,x,a) \, \mathrm{d}x \, \mathrm{d}a \leq & -2 \iint_{\mathbb{R}\times\mathbb{R}_+} a|\partial_x u(t,x,a)|^2 \, \mathrm{d}x \, \mathrm{d}a \\
& + \gamma \int_{\mathbb{R}_+} a^{\gamma-1} u^2(t,0,a) \, \mathrm{d}a - \limsup_{a\to 0} a^\gamma u^2(t,0,a) \\
\leq & -2 \iint_{\mathbb{R}\times\mathbb{R}_+} a|\partial_x u(t,x,a)|^2 \, \mathrm{d}x \, \mathrm{d}a \\
& + \gamma \int_{\mathbb{R}_+} a^{\gamma-1} u^2(t,0,a) \, \mathrm{d}a
\end{aligned}$$

and we control (with $I_\varepsilon := [-\varepsilon, \varepsilon]$)

$$
\begin{aligned}
|u(t,0,a)| &\le \left| u(t,0,a) - \frac{1}{|I_\varepsilon|}\int_{x\in I_\varepsilon} u(t,x,a)\,\mathrm{d}x \right| + \left| \frac{1}{|I_\varepsilon|}\int_{x\in I_\varepsilon} u(t,x,a)\,\mathrm{d}x \right| \\
&\le \left| \frac{1}{|I_\varepsilon|}\int_{x\in I_\varepsilon} \Big(u(t,0,a) - u(t,x,a) \Big)\,\mathrm{d}x \right| + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} \\
&\le \left| \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_0^x \partial_y u(t,y,a)\,\mathrm{d}y\,\mathrm{d}x \right| + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} \\
&\le \left| \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \partial_y u(t,y,a)(\varepsilon - |y|)\,\mathrm{d}y \right| + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} \\
&\le \frac{\varepsilon^{1/2}}{\sqrt{6}} \|\partial_x u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})}
\end{aligned}
$$

and conclude that

$$
\begin{aligned}
\int_{\mathbb{R}_+} a^{\gamma-1} u^2(t,0,a)\,\mathrm{d}a \le \frac{1}{6} &\left\| \varepsilon^{1/2} a^{(\gamma-1)/2} \partial_x u(t,\cdot,\cdot) \right\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)} \\
&+ \left\| \varepsilon^{-1/2} a^{(\gamma-1)/2} u(t,\cdot,\cdot) \right\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)}.
\end{aligned}
$$

We choose ε depending on a as $\varepsilon = \eta a^{\gamma-1}$ with η small to be fixed, and deduce

$$
\int_{\mathbb{R}_+} a^{\gamma-1} u^2(t,0,a)\,\mathrm{d}a \le \frac{\eta}{6} \left\| a^{(\gamma-1)} \partial_x u(t,\cdot,\cdot) \right\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)} + \frac{1}{\eta} \left\| u(t,\cdot,\cdot) \right\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)}.
$$

Finally we use that for $\gamma \ge 3/2$ we have $2(\gamma-1) \ge 1$ and therefore on $[0, a_0]$ (remember that the support condition on a is propagated according to Lemma 4.2) we have $a^{2(\gamma-1)} \le Ca$. Plugging above we get

$$
\int_{\mathbb{R}_+} a^{\gamma-1} u^2(t,0,a)\,\mathrm{d}a \le \frac{C\eta}{6} \left\| a^{1/2} \partial_x u(t,\cdot,\cdot) \right\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)} + \frac{1}{\eta} \left\| u(t,\cdot,\cdot) \right\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)}.
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}\times\mathbb{R}_+} u^2\,\mathrm{d}x\,\mathrm{d}a \le \left(\frac{C\eta\gamma}{6} - 2 \right) \iint_{\mathbb{R}\times\mathbb{R}_+} a|\partial_x u|^2\,\mathrm{d}x\,\mathrm{d}a + \frac{\gamma}{\eta} \|u\|^2_{L^2_{x,a}(\mathbb{R}\times\mathbb{R}_+)}.
$$

By choosing $\eta < 3/(C\gamma)$, this proves that the L^2 norm exists for all times if it is finite initially.

Case (c): blow-up. First easy step is to compute the evolution for $v(t,a) := u(t,0,a)$. We Fourier transform Equation (4.1) in x:

$$
\begin{aligned}
\partial_t \hat{u}(t,\xi,a) &= -a|\xi|^2 \hat{u}(t,\xi,a) + \partial_a \left(a^\gamma u(t,0,a) \right) \\
&= -a|\xi|^2 \hat{u}(t,\xi,a) + \partial_a \left(\frac{a^\gamma}{2\pi} \int_{\mathbb{R}} \hat{u}(t,\eta,a)\,\mathrm{d}\eta \right),
\end{aligned}
$$

and use Duhamel principle where the last term in the right-hand side is treated as a source term:

$$
\hat{u}(t,\xi,a) = e^{-ta|\xi|^2} \hat{u}(0,\xi,a) + \int_0^t \frac{e^{-(t-s)a|\xi|^2}}{2\pi} \partial_a \left(a^\gamma \int_{\mathbb{R}} \hat{u}(s,\eta,a)\,\mathrm{d}\eta \right) \mathrm{d}s. \tag{4.3}
$$

Let

$$v(t,a) := \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(t,\eta,a)\,\mathrm{d}\eta = u(t,0,a),$$

from (4.3) we deduce the identity:

$$v(t,a) = w(t,a) + C_1\int_0^t \frac{1}{\sqrt{a(t-s)}}\partial_a\left(a^\gamma v(s,a)\right)\,\mathrm{d}s, \tag{4.4}$$

where $C_1 > 0$ is an explicit constant and w is defined by

$$w(t,a) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ta|\xi|^2}\hat{u}(0,\xi,a)\,\mathrm{d}\xi.$$

Then, since $u(s,0,a) \geq 0$, the key remark is that $v(s,a) \geq 0$. Moreover, since the second term in the right-hand side of (4.4) has zero integral against $a^{1/2}$, one has

$$\int_{\mathbb{R}_+} a^{1/2}v(t,a)\,\mathrm{d}a = \int_{\mathbb{R}_+} a^{1/2}w(t,a)\,\mathrm{d}a \leq C_2.$$

We have therefore a moment bound to start with: $\int a^{1/2}v(t,a)\,\mathrm{d}a$ remains bounded for all times. Recalling that for $x \neq 0$ the diffusion in x is at play, we deduce that a singularity, if it exists, can only form at $(0,0)$. Hence, we are here looking only at the value at $x = 0$, since for $x \neq 0$ the evolution is a diffusion in x which does not create singularities. Therefore, choosing $M \in (0,1/2)$ so that $0 < M + \gamma - 1/2$, we have

$$\begin{aligned}\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a &= \int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a + C_1\int_0^t\int_{\mathbb{R}_+}\frac{a^{M-1/2}}{\sqrt{(t-s)}}\partial_a\left(a^\gamma v(s,a)\right)\,\mathrm{d}a\,\mathrm{d}s\\ &= \int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a\\ &\quad - C_1\left(M-\frac{1}{2}\right)\int_0^t\int_{\mathbb{R}_+}\frac{a^{M+\gamma-3/2}}{\sqrt{(t-s)}}v(s,a)\,\mathrm{d}a\,\mathrm{d}s.\end{aligned}$$

Note that since M satisfies $M - 1/2 + \gamma > 0$, in the previous computation the boundary term due to the integration by parts vanishes:

$$\int_0^t \left[\frac{a^{M+\gamma-1/2}}{\sqrt{(t-s)}}v(s,a)\right]_{a=0}^{a=+\infty}\,\mathrm{d}s = 0.$$

Furthermore, there exists some positive constants C_3 and η, with $M - \eta \in (0,1/2)$, such that $-(M-1/2)a^{M+\gamma-3/2} \geq C_3 a^{M-\eta}$ on the compact support $[0,a_0]$. And for all $t \leq T$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a \geq{}& \int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a\\ &+ C_4(1+T)^{-1/2}\int_0^t\int_{\mathbb{R}_+} a^{M-\eta}v(s,a)\,\mathrm{d}a\,\mathrm{d}s\end{aligned}$$

for a constant C_4. We have by interpolation (using $M - \eta < M < 1/2$)

$$\int_{\mathbb{R}_+} a^M v(s,a)\,\mathrm{d}a \leq \left(\int_{\mathbb{R}_+} a^{1/2}v(s,a)\,\mathrm{d}a\right)^\theta\left(\int_{\mathbb{R}_+} a^{M-\eta}v(s,a)\,\mathrm{d}a\right)^{1-\theta}$$

for some $\theta \in (0,1)$. Finally, using the bound on the 1/2-moment, we obtain that

$$\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a \geq \int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a + C_5(1+T)^{-1/2} \int_0^t \left(\int_{\mathbb{R}_+} a^M v(s,a)\,\mathrm{d}a \right)^{1/(1-\theta)} \mathrm{d}s$$

for some constant $C_5 > 0$. This means that $Y(t) := \int a^M v(t,a)\,\mathrm{d}a$ satisfies on $[0,T]$:

$$Y(0) > 0 \quad \text{and} \quad Y'(t) \geq C_5(1+T)^{-1/2} Y(t)^{1+\alpha},$$

where $1+\alpha := 1/(1-\theta) > 1$. This implies on $[0,T]$ that

$$Y(t) \geq \left[\frac{1}{Y(0)^{-\alpha} - \alpha C_5 (1+T)^{-1/2} t} \right]^{1/\alpha}.$$

At $t = T/2$ we have $\alpha C_5(1+T)^{-1/2} t = \alpha C_5 (1+T)^{-1/2} T/2$ goes to infinity as T goes to infinity, hence by taking T large enough, we find that $Y(t)$ becomes infinite in finite time.

Using next that

$$\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a \leq \int_{\mathbb{R}_+} v(t,a)\,\mathrm{d}a + \int_{a\geq 1} a^{1/2} v(t,a)\,\mathrm{d}a,$$

together with $\int_{\mathbb{R}_+} a^{1/2} v(t,a)\,\mathrm{d}a < \infty$, we deduce that the L^1 norm can not stay finite for all time. ∎

4.3. Boundary value problem associated to (4.1). In this part we give some perspectives. In the previous part we studied the solution to (4.1) starting from a smooth initial condition. We proved that in certain cases a singularity appears, namely a Dirac mass. It would be interesting to study measure solutions to (4.1), i.e. after the first singularity occured. We leave this rigorous study for a further work and we only discuss here some points in an informal way.

First, in order to handle measure solutions we heuristically transform Equation (4.1), which includes a singular coefficient, namely $\delta_{x=0}$, into a boundary value problem with regular coefficients. We leave the rigorous justification of this result for a forthcoming work.

In this part, we will use both notations $n(t,x,a), p(t)$ and $n_t(x,a), p_t$. Assume that $f(0) = 0$, $a_0 > 0$ and that $f(a_0) \neq 0$. Assume in addition that $n_0(x,a) = \delta_{(0,a_0)}$, $p_0 = 0$ and $q_0 = 0$, then we believe that (n_t, p_t, q_t) given in Lemma 3.2 satisfies the following problem in the classical sense:

$$\begin{cases} \partial_t n(t,x,a) = a\, \partial_x^2 n(t,x,a) \ \text{ for } (t,x,a) \in \mathbb{R}_+^* \times \mathbb{R}^* \times \mathbb{R}_+^*, \\ a\left(\partial_x n(t,0^+,a) - \partial_x n(t,0^-,a)\right) = \partial_a \left(f(a) n(t,0,a)\right), a \in \mathbb{R}_+^*, \end{cases} \tag{4.5}$$

and that one has

$\lim_{a\to 0} \left(f(a) n(t,0,a)\right) = -p'(t)$, with $p'(t) = 0$ if $f(a) > 0$,
$\lim_{a\to +\infty} \left(f(a) n(t,0,a)\right) = q'(t)$, with $q'(t) = 0$ if $f(a) < 0$.

The heuristics is the following. We use Lemma 3.2 with $n_0(x,a) = \delta_{(0,a_0)}$ and we assume that $f(0) = 0$. From the space symmetry of the process with respect to the origin, we first notice that $n_t(x,a) = n_t(-x,a)$ for all $(t,x,a) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}_+$. Next, as it is classical in such situations, in order to obtain (4.5), we multiply Equation (4.1) by particular test functions φ and we integrate by parts: respectively $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}_\pm^* \times \mathbb{R}_+^*\right)$ and $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R} \times \mathbb{R}_+^*\right)$. Using the smoothing effect of the heat equation, we can see that Equation (4.5) is satisfied in the classical sense.

Let us now introduce some notations that will be useful for the computation of the explicit solution to problem (4.5): H will denote the Heaviside function, $H(a-a_0) = 0$ if $a < a_0$, $H(a-a_0) = 1$ if $a > a_0$ and Z will be the function defined by

$$Z(x,a) = \frac{|x|}{\sqrt{a}} + 2\int_{a_0}^{a} \frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'. \tag{4.6}$$

Proposition 4.3. *Informally, the boundary value problem (4.5) with $n_0(x,a) = \delta_{(0,a_0)}$ and $p_0 = 0$ admits the following explicit solution (n_t, p_t):*

(1) Local deceleration ($f(a) < 0$):

$$n_t(x,a) = H(a_0 - a)\,\frac{Z(x,a)}{|f(a)|\sqrt{4\pi t^3}}\,e^{-\frac{Z(x,a)^2}{4t}} \tag{4.7}$$
$$\text{and}\quad p_t = \operatorname{erfc}\Big(\frac{1}{\sqrt{t}}\int_0^{a_0} \frac{\sqrt{a'}}{|f(a')|}\,\mathrm{d}a'\Big),$$

(2) Local acceleration ($f(a) > 0$):

$$n_t(x,a) = H(a-a_0)\frac{Z(x,a)}{f(a)\sqrt{4\pi t^3}}\,e^{-\frac{Z(x,a)^2}{4t}} \quad \text{and} \quad p_t = 0. \tag{4.8}$$

Proof. Taking the Laplace transform in t, denoted

$$\mathcal{L}^t(n)(x,a,\lambda) = \int_{\mathbb{R}_+} n_t(x,a)e^{-\lambda t}\,\mathrm{d}t,$$

the Fourier transform in x of (4.5) with $n_0(x,a) = \delta_{(0,a_0)}$ and using the equality

$$\mathcal{L}^t(n)(0,a,\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{n}_\lambda(\xi,a)e^{i\xi\times 0}\,\mathrm{d}\xi,$$

we can compute

$$\tilde{n}_\lambda(\xi,a) = \frac{1}{\lambda + a|\xi|^2}\partial_a\left(f(a)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \tilde{n}_\lambda(\xi',a)\,\mathrm{d}\xi'\right)\right) + \frac{\delta_{a=a_0}}{\lambda + a|\xi|^2}, \tag{4.9}$$

where we have denoted by $\tilde{n}_\lambda(\xi,a) = \int_{\mathbb{R}} \mathcal{L}^t(n)(x,a,\lambda)e^{-i\xi x}\,\mathrm{d}x$ the Fourier transform in x of the Laplace transform in t of n_t.

Consequently after integration we obtain

$$\begin{aligned}\int_{\mathbb{R}} \tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi &= \left[\partial_a\left(f(a)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \tilde{n}_\lambda(\xi',a)\,\mathrm{d}\xi'\right)\right) + \delta_{a=a_0}\right]\int_{\mathbb{R}} \frac{1}{\lambda + a|\xi|^2}\,\mathrm{d}\xi \\ &= \frac{\pi}{\sqrt{\lambda a}}\left[\partial_a\left(f(a)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi\right)\right) + \delta_{a=a_0}\right].\end{aligned} \tag{4.10}$$

With the previous expression, $\int_{\mathbb{R}} \tilde{n}_\lambda(\xi, a)\,\mathrm{d}\xi$ can be computed, hence the expression of $\tilde{n}_\lambda$ can be deduced by using (4.9). Finally, by inverting first the Fourier transform and then the Laplace tranform, we can compute n_t. From (4.9) and (4.10) it follows that

$$\begin{aligned}\tilde{n}_\lambda(\xi, a) &= \frac{1}{\pi}\frac{\sqrt{\lambda a}}{\lambda + a\xi^2}\int_{\mathbb{R}} \tilde{n}_\lambda(\xi, a)\,\mathrm{d}\xi \\ &= C(a, a_0, \lambda)\frac{f(a_0)}{2\pi f(a)}\frac{\sqrt{\lambda a}}{\lambda + a\xi^2}e^{2\sqrt{\lambda}\int_{a_0}^{a}\frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'},\end{aligned}$$

where

$$C(a, a_0, \lambda) = \begin{cases} C(\lambda)H(a - a_0) & \text{if } f > 0, \\ C(\lambda)H(a_0 - a) & \text{if } f < 0. \end{cases}$$

$C(\lambda)$ is determined by the jump of $\int_{\mathbb{R}} \tilde{n}_\lambda(\xi, a)\,\mathrm{d}\xi$ at $a = a_0$:

$$\int_{\mathbb{R}} \tilde{n}_\lambda(\xi, a_0^+)\,\mathrm{d}\xi - \int_{\mathbb{R}} \tilde{n}_\lambda(\xi, a_0^-)\,\mathrm{d}\xi = \frac{1}{f(a_0)}, \tag{4.11}$$

from which we deduce that $C(\lambda) = 1/|f(a_0)|$, hence

$$\tilde{n}_\lambda(\xi, a) = C(a, a_0)\frac{1}{2\pi|f(a)|}\frac{\sqrt{\lambda a}}{\lambda + a\xi^2}e^{2\sqrt{\lambda}\int_{a_0}^{a}\frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'}.$$

Next, using the Fourier inverse transform, it yields that

$$\begin{aligned}\mathcal{L}^t(n)(x, a, \lambda) &= C(a, a_0)\frac{1}{|f(a)|}e^{-\left|\sqrt{\frac{\lambda}{a}}x\right| + 2\sqrt{\lambda}\int_{a_0}^{a}\frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'} \\ &= C(a, a_0)\frac{1}{|f(a)|}e^{-\sqrt{\lambda}Z(x,a)},\end{aligned}$$

where Z is given by (4.6).

Laplace inverting this latter expression, we obtain the joint distribution given by (4.7) if $f < 0$ and by (4.8) if $f > 0$. The expression of p_t is straigthforward. ∎

REMARK 4.1. Unsurprisingly the blow-up character of the solution to (4.1) given in Theorem 4.2 can also be observed on the explicit solution given in Proposition 4.3. Indeed the symptom of blow-up in a L^p framework corresponds to $p_t \neq 0$. In the case where $f(a) = -a^\gamma$, we see that $\int_0^a a'^{1/2-\gamma}\,\mathrm{d}a' < \infty$ if $\gamma \in (0, 3/2)$. Since Laplace transform inversion requires a specific initial condition the result given in Theorem 4.2 is more general.

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