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CONDITIONAL REGULARITY FOR THE 3D INCOMPRESSIBLE MHD EQUATIONS VIA PARTIAL COMPONENTS*

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Abstract. In this paper we establish some new regularity criteria for the three dimensional incompressible magnetohydrodynamic (MHD) equations. Particularly, we prove that if ∇u_3 and the horizontal magnetic field $b_h=(b_1,b_2)$ satisfy certain integrable conditions with respect to space and time variables in Lebesgue spaces, then a weak solution (u,b) is actually regular. Moreover, we obtain a regularity criterion in the framework of scaling invariance.

Keywords. MHD equations; Regularity criteria; Partial components.

AMS subject classifications. 35Q35; 35B65; 76D05.

1. Introduction

We are concerned with the following 3D viscous incompressible magnetohydrodynamic (MHD) equations:

$$\begin{cases} \partial_t u+u\cdot\nabla u=\Delta u-\nabla p+b\cdot\nabla b, \\ \partial_t b+u\cdot\nabla b=\Delta b+b\cdot\nabla u, \\ \operatorname{div}u=\operatorname{div}b=0, \\ u(x,0)=u_0(x),\ b(x,0)=b_0(x), \end{cases} \tag{1.1}$$

where $u=(u_1(x,t),u_2(x,t),u_3(x,t))$, $b=(b_1(x,t),b_2(x,t),b_3(x,t))$ and $p=p(x,t)$, $x\in\mathbb{R}^3$, $t>0$ denote the unknown velocity field, magnetic field and pressure, respectively. $u_0(x)$ and $b_0(x)$ are the given initial data. In the early 1970s, Duvaut and Lions [7] constructed a class of global weak solutions, similarly to the Leray-Hopf weak solutions to the three dimensional Navier-Stokes equations. But the strong solution is generally local. It is not known whether the smooth solution of Cauchy problem in three dimensions exists for all time for given sufficiently smooth and divergence-free initial data. For the two-dimensional case, the smoothness of solutions has been shown. The same results hold in the case of three dimensions under the assumption that (u,b) belongs to $L^\infty(0,T;H^1(\mathbb{R}^3))$, see Sermange and Temam [20] for the details. The main difference between the two-dimensional and three-dimensional cases can be well understood by considering the dynamics of the fluid vorticity as pointed out by Constantin and Fefferman in [6] for the Navier-Stokes equations. Therefore, the global regularity of weak solutions in three dimensions is still an outstanding challenging open problem.

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It seems to be beyond the scope of the present techniques. Nevertheless, there exist many criteria to guarantee the global regularity of weak solutions of (1.1), see for example [5,9,18,19,26,27] and references therein. We mention here the following classical results from He et al. [9] and Zhou [26]; of course, some other results which may also be very interesting cannot be mentioned due to the length of this article. Motivated by the regularity criteria of the Navier-Stokes equations, He et al. and Zhou established the fundamental Serrin-type regularity criteria only in terms of the velocity field, independently. Precisely, they showed if the velocity field satisfies

$$u\in L^{\alpha,\gamma} \text{ with } \frac{2}{\alpha}+\frac{3}{\gamma}\leq 1, \quad 3<\gamma\leq\infty, \tag{1.2}$$

or

$$\nabla u\in L^{\alpha,\gamma} \text{ with } \frac{2}{\alpha}+\frac{3}{\gamma}\leq 2, \quad \frac{3}{2}<\gamma\leq\infty, \tag{1.3}$$

then the weak solution of (1.1) is smooth on $\mathbb{R}^3\times(0,T]$. The main interest of this Serrin-type regularity criteria in [9,26] is that there are no assumptions on the magnetic field after the initial time. As stated in [9], this indicates that the fluid velocity may play a dominant role in the evolution of MHD flows. Note that the condition (1.3) requires that ∇u should satisfy some integrability with suitable indices α and γ in Lebesgue spaces, and the number on the right-hand side of the first inequality is 2. This result is optimal from the scaling-invariant point of view. However, it seems more restrictive since ∇u is actually a 3×3 matrix and 9 elements of it are supposed to satisfy (1.3). Since u is divergence-free, we may expect regularity conditions imposed on the gradient of only one velocity component, say ∇u_3, instead of the gradient of all u, moreover, these conditions are expected to be scaling invariant. Although it is a straightforward idea, this can not be reached very easily without the help of magnetic field for the time being. It is obvious that one cannot see any contribution of the magnetic field or the interplay between the fluid velocity and the magnetic field in the criteria of [9,26]. In order to capture the nature of coupling effects between the fluid velocity and the magnetic fields in the magneto-fluid motion, the role of magnetic field should not be neglected. Then the problem of so-called "regularity criteria for the MHD equations via partial components" has drawn many researchers' interest in the last ten years, see for example [1,10–13,16,17,23,32,33] and references therein. Particularly, Yamazaki in [25] provided a regularity criterion in terms of u_3 and b both in scaling-invariant norms using anisotropic Littlewood-Paley theory. Recently, Zhang in [31] proved the global regularity based on ∇u_3 and some current density. It seems that this kind of investigation on only one velocity component is more difficult than the one of akin criteria based on one partial derivative of the velocity field; in this regard one can see for example the discussions [1,4,12,24,31] on the MHD equations, and also [2,3,15,21,22,28–30] for the Navier-Stokes equations ($b\equiv 0$ in (1.1)). In this paper, we are interested in the conditions which can guarantee the regularity of weak solutions in view of the gradient of only one velocity component and the magnetic field. Namely, we will first prove the following result:

THEOREM 1.1. *Assume that (u,b) is a global weak solution to the MHD Equations (1.1) corresponding to the initial data $(u_0,b_0)\in H^s(\mathbb{R}^3)$, $s\geq 3$ with $\nabla\cdot u_0=\nabla\cdot b_0=0$ and satisfies the energy inequality. Suppose $\nabla u_3\in L^{\alpha_1}(0,T;L^{\beta_1})$ and $b\in L^{\alpha_2}(0,T;L^{\beta_2})$ with*

$$\frac{2}{\alpha_1}+\frac{3}{\beta_1}\leq\frac{7}{4}+\frac{1}{2\beta_1}, \quad 2<\beta_1<\infty, \tag{1.4}$$

and

$$\frac{2}{\alpha_2}+\frac{3}{\beta_2}\le\frac{3}{4}+\frac{1}{2\beta_2},\quad \frac{10}{3}\le\beta_2\le\infty, \tag{1.5}$$

Then (u,b) *remains smooth on* $(0,T]$.

REMARK 1.1. We note that the authors in [17] have proven if

$$\nabla_h u\in L^{s_1,t_1} \text{ with } \frac{2}{s_1}+\frac{3}{t_1}\le 2,\quad \frac{3}{2}<t_1\le\infty, \tag{1.6}$$

$$\nabla_h b\in L^{s_2,t_2} \text{ with } \frac{2}{s_2}+\frac{3}{t_2}\le 2,\quad \frac{3}{2}<t_2\le\infty, \tag{1.7}$$

then the corresponding solution is actually regular, where $\nabla_h=(\partial_1,\partial_2)$ is the horizontal gradient. It was improved recently by Jia [14] where the assumptions on $\nabla_h u_3$ and $\nabla_h b_3$ of (1.6) and (1.7) were removed. However, we find that more information on the unknowns is still required such as $u_h=(u_1,u_2)$ and $b_h=(b_1,b_2)$, although the spaces from (1.6) and (1.7) are scaling invariant. If one imposes some conditions only on the partial derivative of the velocity field, say $\partial_3 u$, then regularity can also be guaranteed. Jia et al. in [12] proved the following criterion

$$\partial_3 u\in L^{\beta,\alpha} \text{ with } \frac{2}{\beta}+\frac{3}{\alpha}\le\frac{3(\alpha+2)}{4\alpha} \text{ and } \alpha>2. \tag{1.8}$$

It is obvious that they removed restraints on the magnetic field b, and the number of the first inequality on the right-hand side of (1.8) is not scaling invariant $\left(\lim_{\alpha\to 2}\frac{3(\alpha+2)}{4\alpha}=\frac{3}{2}\right)$. Moreover, all the components of u are involved, i.e., on one hand, they gain less restrictions on the unknowns (no assumptions on b), but, on the other hand, lose the nice scaling-invariant property. Based on the above comparison, there might be a kind of balance between the scaling-invariant norms and the unknown variables in regularity theory, and the loss of regularity in velocity turns out to be balanced by some additional regularity in magnetic fields. This can be exactly used to demonstrate the coupling effect of velocity and magnetic fields. Therefore, in this paper, we make full consideration of it and try to make balance between them to guarantee the regularity of weak solutions, this is shown in Theorem 1.1.

REMARK 1.2. Conditions in Theorem 1.1 are not optimal since the spaces from (1.4) and (1.5) are not scaling invariant in Serrin's framework. We find that in (1.4) $\frac{7}{4}+\frac{1}{2\beta_1}$ goes to 2 as β_1 tends to 2, i.e., it is almost optimal for ∇u_3 as β_1 changes around 2. So the possible improvement of conditions in Theorem 1.1 is to achieve a balance by using some more information on velocity field in the framework of scaling invariance, see conditions in Theorem 1.2 for the details, additional conditions on partial derivative of the horizontal velocity are imposed.

THEOREM 1.2. *Assume that* (u,b) *is a global weak solution to the MHD Equations (1.1) corresponding to the initial data* $(u_0,b_0)\in H^s(\mathbb{R}^3)$, $s\ge 3$ *with* $\nabla\cdot u_0=\nabla\cdot b_0=0$ *and satisfies the energy inequality. Suppose that* $\partial_3 u_h\in L^{\alpha_3}(0,T;L^{\beta_3})$ *with*

$$\frac{2}{\alpha_3}+\frac{3}{\beta_3}\le 2,\quad \frac{3}{2}<\beta_3\le\infty.$$

Furthermore, assume that $\nabla u_3 \in L^{\alpha_4}(0,T;L^{\beta_4})$ *and* $b\in L^{\alpha_5}(0,T;L^{\beta_5})$ *with*

$$\frac{2}{\alpha_4}+\frac{3}{\beta_4}\le 2,\quad \frac{3}{2}<\beta_4\le\infty,$$

and

$$\frac{2}{\alpha_5}+\frac{3}{\beta_5}\le 1,\quad 3<\beta_5\le\infty.$$

Moreover, for the case β_i $(i=3,4,5)$ *being the left endpoint of the above intervals, let the norms* $||\partial_3 u_h||_{L^{\infty,\frac{3}{2}}}$, $||\nabla u_3||_{L^{\infty,\frac{3}{2}}}$ *and* $||b||_{L^{\infty,3}}$ *be sufficiently small, respectively. Then* (u,b) *remains smooth on* $(0,T]$.

Considering the balance between the scaling-invariance norms and the unknown variables which are involved in the conditions, if we focus on the partial components of the unknowns u and b, it is possible to establish the following regularity criterion.

Theorem 1.3. *Assume that* (u,b) *is a global weak solution to the MHD Equations (1.1) corresponding to the initial data* $(u_0,b_0)\in H^s(\mathbb{R}^3)$, $s\ge 3$ *with* $\nabla\cdot u_0=\nabla\cdot b_0=0$ *and satisfies the energy inequality. Suppose* $\nabla u_3\in L^{\alpha_6}(0,T;L^{\beta_6})$ *and* $b_h\in L^{\alpha_7}(0,T;L^{\beta_7})$ *either with*

$$\frac{2}{\alpha_6}+\frac{3}{\beta_6}\le\frac{7}{4}+\frac{1}{2\beta_6},\quad \beta_6>2, \tag{1.9}$$

and

$$\frac{2}{\alpha_7}+\frac{3}{\beta_7}\le\frac{2}{5},\quad \beta_7>\frac{15}{2}, \tag{1.10}$$

or

$$\frac{2}{\alpha_6}+\frac{3}{\beta_6}\le\frac{7}{5},\quad \beta_6>\frac{15}{7}, \tag{1.11}$$

and

$$\frac{2}{\alpha_7}+\frac{3}{\beta_7}\le 1,\quad \beta_7>3, \tag{1.12}$$

Then (u,b) *remains smooth on* $(0,T]$.

Remark 1.3. Theorem 1.3 implies that what we earn is less restriction on the unknowns, and what we lose is the right-hand side numbers of (1.10) and (1.11); they are smaller than what we expected. Note that, as we know, the ultimate goal is to establish criteria with conditions only on $\partial_i u_j$ and b_k for $i,j,k\in\{1,2,3\}$ in Serrin's regularity class. To our knowledge, it is still very challenging.

2. Preliminaries

We can take a forward step based on the following anisotropic Lebesgue spaces and some technical inequalities.

Definition 2.1. *Let* $\bar{p}=(p_1,p_2,p_3)$, $p_i\in[1,\infty]$, $i=1,2,3$. *We say a function* f *belongs to* $L^{\bar{p}}$ *if* f *is measurable on* $\mathbb{R}^3$ *and the following norm is finite:*

$$\|f\|_{\bar{p}}=\left\|\left\|\|f\|_{L_1^{p_1}}\right\|_{L_2^{p_2}}\right\|_{L_3^{p_3}}:=\left(\int\left(\int\left(\int|f(x_1,x_2,x_3)|^{p_1}dx_1\right)^{\frac{p_2}{p_1}}dx_2\right)^{\frac{p_3}{p_2}}dx_3\right)^{\frac{1}{p_3}}.$$

These anisotropic Lebesgue spaces seem to be convenient for our purposes, since they differentiate between different directions and better estimates can be expected.

LEMMA 2.1. *Let p, q, $r\in[2,\infty)$ and $1/p+1/q+1/r-1/2\geq 0$. Then there exists a constant c such that for every $f\in L^2\cap C^\infty$*

$$\left\|\left\|\|f\|_{L_1^p}\right\|_{L_2^q}\right\|_{L_3^r}\leq c\|\partial_3 f\|_2^{\frac{r-2}{2r}}\|\partial_2 f\|_2^{\frac{q-2}{2q}}\|\partial_1 f\|_2^{\frac{p-2}{2p}}\|f\|_2^{\frac{1}{r}+\frac{1}{q}+\frac{1}{p}-\frac{1}{2}}.$$

The proof of Lemma 2.1 was given in [8].

In the following, we denote $\int f(x)dx$ the integral over the whole three dimensional space, use the standard notation for Lebesgue spaces $L^p(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_p:=\|\cdot\|_{L^p(\mathbb{R}^3)}$ and for Sobolev spaces $W^{k,p}(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_{k,p}$. C denotes a generic constant which might change from line to line even within the same line. Moreover, denote

$$J^2(T_2)=\sup_{T_1<\tau<T_2}\left(\|\nabla_h u(\tau)\|_2^2+\|\nabla_h b(\tau)\|_2^2\right)+\int_{T_1}^{T_2}\left(\|\nabla\nabla_h u(\tau)\|_2^2+\|\nabla\nabla_h b(\tau)\|_2^2\right)d\tau,$$

$$L^2(T_2)=\sup_{T_1<\tau<T_2}\left(\|\partial_3 u(\tau)\|_2^2+\|\partial_3 b(\tau)\|_2^2\right)+\int_{T_1}^{T_2}\left(\|\nabla\partial_3 u(\tau)\|_2^2+\|\nabla\partial_3 b(\tau)\|_2^2\right)d\tau.$$

and

$$K^2(T_2)=\sup_{T_1<\tau<T_2}\left(\|\nabla u(\tau)\|_2^2+\|\nabla b(\tau)\|_2^2\right)+\int_{T_1}^{T_2}\left(\|\Delta u(\tau)\|_2^2+\|\Delta b(\tau)\|_2^2\right)d\tau.$$

We also work with

$$E^2(T_2)\overset{\text{def}}{=}\sup_{0<\tau<T_2}\left(\|u(\tau)\|_2^2+\|b(\tau)\|_2^2\right)+\int_0^{T_2}\left(\|\nabla u(\tau)\|_2^2+\|\nabla b(\tau)\|_2^2\right)d\tau.$$

Note that $E(T_2)\leq E(0)$ due to the energy inequality.

3. Proof of Theorem 1.1

Let $T^*=\sup\{\tau>0;\ (u,b)$ is regular on $(0,\tau)\}$. Since $(u_0,b_0)\in W^{1,2}(\mathbb{R}^3)$, (u,b) is regular on some positive time interval and T^* is either equal to infinity (in which case the proof is finished) or it is a positive number and (u,b) is regular on $(0,T^*)$. It is sufficient to prove that $T^*>T$. We proceed by contradiction and suppose that $T^*\leq T$. We take $\epsilon>0$ sufficiently small (it will be specified later) and fix $T_1\in(0,T^*)$ such that

$$\|\nabla u_3\|_{L^{\alpha_1}(T_1,T^*;L^{\beta_1})}<\epsilon \ \text{ and } \ \|b\|_{L^{\alpha_2}(T_1,T^*;L^{\beta_2})}<\epsilon. \tag{3.1}$$

Taking arbitrarily $T_2\in(T_1,T^*)$ the proof will be finished if we show that

$$\|\nabla u(T_2)\|_2+\|\nabla b(T_2)\|_2\leq C<\infty, \tag{3.2}$$

where C is independent of T_2. Actually, the standard extension argument then shows that the regularity of (u,b) can be extended beyond T^* and it contradicts the definition of T^*. As a matter of fact, it now suffices to show that

$$J^2(T_2)+L^2(T_2)\leq Const.<\infty, \tag{3.3}$$

uniformly in T_2. Firstly, we prove the following estimate.

LEMMA 3.1. *There holds*

$$L(T_2) \le C + CJ(T_2)^{\frac{4}{3}}, \tag{3.4}$$

where C is a constant independent of T_2.

Proof. The inequality (3.4) is crucial for (3.3). Multiplying the first equation of (1.1) by $-\partial_{33}u$ and integrating over $\mathbb{R}^3$ from (T_1,T_2), we obtain

$$\begin{aligned}&\frac{1}{2}||\partial_3 u(T_2)||_2^2 + \int_{T_1}^{T_2} ||\nabla\partial_3 u(\tau)||_2^2 d\tau \\ =&\frac{1}{2}||\partial_3 u(T_1)||_2^2 + \int_{T_1}^{T_2}\int (u\cdot\nabla)u\cdot\partial_{33}u\ dxd\tau - \int_{T_1}^{T_2}\int (b\cdot\nabla)b\cdot\partial_{33}u\ dxd\tau.\end{aligned} \tag{3.5}$$

Likewise, for the second equation of (1.1), we multiply it by $-\partial_{33}b$ and integrate over $\mathbb{R}^3$ and (T_1,T_2), it follows that

$$\begin{aligned}&\frac{1}{2}||\partial_3 b(T_2)||_2^2 + \int_{T_1}^{T_2} ||\nabla\partial_3 b(\tau)||_2^2 d\tau \\ =&\frac{1}{2}||\partial_3 b(T_1)||_2^2 + \int_{T_1}^{T_2}\int (u\cdot\nabla)b\cdot\partial_{33}b\ dxd\tau - \int_{T_1}^{T_2}\int (b\cdot\nabla)u\cdot\partial_{33}b\ dxd\tau.\end{aligned} \tag{3.6}$$

Now we denote the four terms on the right-hand side of (3.5) and (3.6) by RHS, i.e.,

$$\begin{aligned}RHS :=& \int_{T_1}^{T_2}\int (u\cdot\nabla)u\cdot\partial_{33}u\ dxd\tau - \int_{T_1}^{T_2}\int (b\cdot\nabla)b\cdot\partial_{33}u\ dxd\tau \\ &+ \int_{T_1}^{T_2}\int (u\cdot\nabla)b\cdot\partial_{33}b\ dxd\tau - \int_{T_1}^{T_2}\int (b\cdot\nabla)u\cdot\partial_{33}b\ dxd\tau \\ :=& RHS_1 + RHS_2 + RHS_3 + RHS_4.\end{aligned}$$

Firstly, we estimate RHS_1:

$$\begin{aligned}&\int (u\cdot\nabla)u\cdot\partial_{33}u dx \\ =& -\sum_{j,k=1}^{3}\int \partial_3 u_j\partial_j u_k\partial_3 u_k\ dx - \sum_{j,k=1}^{3}\int u_j\partial_{j3}^2 u_k\partial_3 u_k\ dx \\ =& -\sum_{j,k=1}^{3}\int \partial_3 u_j\partial_j u_k\partial_3 u_k\ dx \\ =& \sum_{j=1}^{2}\sum_{k=1}^{3}\int u_k(\partial_{3j}^2 u_j\partial_3 u_k + \partial_{3j}^2 u_k\partial_3 u_j)\ dx + \sum_{k=1}^{3}\int (\partial_1 u_1 + \partial_2 u_2)\partial_3 u_k\partial_3 u_k\ dx \\ =& \sum_{j=1}^{2}\sum_{k=1}^{3}\int u_k(\partial_{3j}^2 u_j\partial_3 u_k + \partial_{3j}^2 u_k\partial_3 u_j)\ dx - \sum_{k=1}^{3} 2\int (u_1\partial_3 u_k\partial_{31}^2 u_k + u_2\partial_3 u_k\partial_{32}^2 u_k)\ dx \\ \le& C\int |u||\partial_3 u||\nabla\nabla_h u| dx\end{aligned}$$

$$\leq C\|\partial_1 u\|_2^{\frac{1}{3}}\|\partial_2 u\|_2^{\frac{1}{3}}\|\partial_3 u\|_2^{\frac{1}{3}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\partial_{13}^2 u\|_2^{\frac{1}{6}}\|\partial_{23}^2 u\|_2^{\frac{1}{6}}\|\partial_{33}^2 u\|_2^{\frac{1}{6}}\|\nabla\nabla_h u\|_2$$
$$\leq C\|\nabla_h u\|_2^{\frac{2}{3}}\|\partial_3 u\|_2^{\frac{1}{3}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\nabla\nabla_h u\|_2^{\frac{4}{3}}\|\nabla\partial_3 u\|_2^{\frac{1}{6}}.$$

Then we have

$$\begin{aligned}
&\left|\int_{T_1}^{T_2}\int (u\cdot\nabla)u\cdot\partial_{33}u\ dxd\tau\right| \\
&\leq C\int_{T_1}^{T_2}\|\nabla_h u\|_2^{\frac{2}{3}}\|\partial_3 u\|_2^{\frac{1}{3}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\nabla\nabla_h u\|_2^{\frac{4}{3}}\|\nabla\partial_3 u\|_2^{\frac{1}{6}}d\tau \\
&\leq C\|\nabla_h u\|_{\infty,2}^{\frac{2}{3}}\|\partial_3 u\|_{\infty,2}^{\frac{1}{3}}\|\partial_3 u\|_{2,2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{2,2}^{\frac{4}{3}}\|\nabla\partial_3 u\|_{2,2}^{\frac{1}{6}} \\
&\leq CJ(T_2)^2 L(T_2)^{\frac{1}{2}}.
\end{aligned} \tag{3.7}$$

Similarly, we deal with RHS_3 as follows.

$$\begin{aligned}
&\int (u\cdot\nabla)b\cdot\partial_{33}b dx \\
=&-\sum_{j,k=1}^{3}\int \partial_3 u_j\partial_j b_k\partial_3 b_k dx-\sum_{j,k=1}^{3}\int u_j\partial_{j3}^2 b_k\partial_3 b_k dx \\
=&-\sum_{j,k=1}^{3}\int \partial_3 u_j\partial_j b_k\partial_3 b_k dx \\
=&\sum_{j=1}^{2}\sum_{k=1}^{3}\int b_k(\partial_{3j}^2 u_j\partial_3 b_k+\partial_{3j}^2 b_k\partial_3 u_j)dx+\sum_{k=1}^{3}\int(\partial_1 u_1+\partial_2 u_2)\partial_3 b_k\partial_3 b_k dx \\
=&\sum_{j=1}^{2}\sum_{k=1}^{3}\int b_k(\partial_{3j}^2 u_j\partial_3 b_k+\partial_{3j}^2 b_k\partial_3 u_j)dx-2\sum_{k=1}^{3}\int(u_1\partial_3 b_k\partial_{31}^2 b_k+u_2\partial_3 b_k\partial_{32}^2 b_k)dx \\
\leq& C\int(|u||\partial_3 b||\nabla\nabla_h b|+|b||\partial_3 b||\nabla\nabla_h u|+|b||\partial_3 u||\nabla\nabla_h b|)dx.
\end{aligned}$$

Then we obtain by a similar way to RHS_1 that

$$\left|\int_{T_1}^{T_2}\int (u\cdot\nabla)b\cdot\partial_{33}b\ dxd\tau\right|\leq CJ(T_2)^2 L(T_2)^{\frac{1}{2}}. \tag{3.8}$$

Finally, we estimate RHS_2+RHS_4:

$$\begin{aligned}
&\int [(b\cdot\nabla)b\cdot\partial_{33}u+(b\cdot\nabla)u\cdot\partial_{33}b]\ dx \\
=&\int (b\cdot\nabla)(b+u)\partial_{33}(u+b)\ dx-\int [(b\cdot\nabla)b\cdot\partial_{33}b+(b\cdot\nabla)u\cdot\partial_{33}u]\ dx \\
=&-\sum_{i,j=1}^{3}\int (\partial_3 b_i\cdot\partial_i b_j\cdot\partial_3 u_j+\partial_3 b_i\cdot\partial_i u_j\cdot\partial_3 b_j)\ dx,
\end{aligned}$$

it follows by similar techniques on RHS_1 and RHS_3 that

$$RHS_2+RHS_4\leq CJ(T_2)^2 L(T_2)^{\frac{1}{2}}. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain (3.4). ∎

3.1. Estimate of J^2. Multiplying the first two equations of (1.1) by $\Delta_h u = \sum_{j=1}^2 \partial_{jj}^2 u$ and $\Delta_h b = \sum_{j=1}^2 \partial_{jj}^2 b$, respectively, integrating by parts and taking the divergence-free conditions into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right) + \left(\|\nabla_h \nabla u\|_2^2 + \|\nabla_h \nabla b\|_2^2\right) \\
&= \int [(u\cdot\nabla u)\Delta_h u - (b\cdot\nabla b)\Delta_h u + (u\cdot\nabla b)\Delta_h b - (b\cdot\nabla u)\Delta_h b]\ dx \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.10}$$

Next, we estimate $J_i\,(i=1,2,3,4)$ one by one. By regrouping the terms of J_1, we have

$$\begin{aligned}
J_1 &= \int (u\cdot\nabla u)\cdot\Delta_h u\ dx \\
&= \sum_{i,j=1}^2 \int u_i\cdot\partial_i u_j\cdot\Delta_h u_j\ dx + \sum_{i=1}^3 \int u_i\cdot\partial_i u_3\cdot\Delta_h u_3\ dx + \sum_{j=1}^2 \int u_3\cdot\partial_3 u_j\cdot\Delta_h u_j\ dx \\
&:= J_1^{(1)} + J_1^{(2)} + J_1^{(3)}.
\end{aligned} \tag{3.11}$$

Using the result given by Kukavica et al. in [15] for $J_1^{(1)}$, one has

$$\begin{aligned}
J_1^{(1)} &= \frac{1}{2}\sum_{i,j=1}^2 \int \partial_3 u_3\cdot\partial_j u_i\cdot\partial_j u_i\ dx - \int \partial_3 u_3\cdot\partial_1 u_1\cdot\partial_2 u_2\ dx + \int \partial_3 u_3\cdot\partial_2 u_1\cdot\partial_1 u_2\ dx \\
&\le C\int |\nabla u_3||\nabla_h u|^2\ dx.
\end{aligned} \tag{3.12}$$

Using the divergence-free condition $\mathrm{div} u = 0$ and integrating by parts, we have

$$J_1^{(2)} = -\sum_{i=1}^3\sum_{j=1}^2 \int \partial_j u_i\cdot\partial_i u_3\cdot\partial_j u_3\ dx \le C\int |\nabla u_3||\nabla_h u|^2\ dx, \tag{3.13}$$

and

$$\begin{aligned}
J_1^{(3)} &= \sum_{j=1}^2\sum_{k=1}^2 \int \left(-\partial_k u_3\cdot\partial_3 u_j\cdot\partial_k u_j + \frac{1}{2}\partial_3 u_3\cdot\partial_k u_j\cdot\partial_k u_j\right)\ dx \\
&\le C\int |\nabla u_3||\partial_3 u||\nabla_h u|\ dx + C\int |\nabla u_3||\nabla_h u|^2\ dx.
\end{aligned} \tag{3.14}$$

Integrating by parts and taking the divergence-free condition for b into account, it follows that

$$\begin{aligned}
J_2 &= -\int (b\cdot\nabla b)\cdot\Delta_h u\ dx = \sum_{i,j=1}^3\sum_{k=1}^2 \int b_i\cdot b_j\cdot\partial_{kk}^2\partial_i u_j\ dx \\
&= -\sum_{i,j=1}^3\sum_{k=1}^2 \int \partial_k(b_i\cdot b_j)\cdot\partial_{ki}^2 u_j\ dx \\
&\le C\int |b||\nabla_h b||\nabla\nabla_h u|\ dx.
\end{aligned} \tag{3.15}$$

Similarly

$$
\begin{aligned}
J_3 &= \int (u\cdot\nabla b)\cdot\Delta_h b\ dx = -\sum_{i,j=1}^{3}\sum_{k=1}^{2}\int \partial_k u_i\cdot\partial_i b_j\cdot\partial_k b_j\ dx \\
&= \sum_{i,j=1}^{3}\sum_{k=1}^{2}\int b_j\cdot\partial_i(\partial_k u_i\cdot\partial_k b_j)\ dx \\
&\le C\int |b||\nabla_h u||\nabla\nabla_h b|\ dx, \qquad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= -\int (b\cdot\nabla u)\cdot\Delta_h b\ dx \\
&\le C\int |b||\nabla_h u||\nabla\nabla_h b|\ dx + C\int |b||\partial_3 u||\nabla\nabla_h b|\ dx. \qquad (3.17)
\end{aligned}
$$

Thus, inserting (3.11)-(3.17) into (3.10), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)+\left(\|\nabla_h\nabla u\|_2^2+\|\nabla_h\nabla b\|_2^2\right) \\
&\le C\int |\nabla u_3||\nabla_h u|^2\ dx + C\int |\nabla u_3||\partial_3 u||\nabla_h u|\ dx \\
&\quad + C\int |b||\nabla_h b||\nabla\nabla_h u|\ dx + C\int |b||\nabla_h u||\nabla\nabla_h b|\ dx + C\int |b||\partial_3 u||\nabla\nabla_h b|\ dx. \qquad (3.18)
\end{aligned}
$$

There are five terms in the above inequality (3.18) to be estimated. We separate the b terms from the u terms in order to obtain better estimates. For the b terms, we estimate as following:

$$
\begin{aligned}
C\int |b||\nabla_h b||\nabla\nabla_h u|\ dx &\le C||b||_{\beta_2}^2||\nabla_h b||_{\frac{2\beta_2}{\beta_2-2}}^2 + \frac{1}{6}\int |\nabla\nabla_h u|^2\ dx \\
&\le C||b||_{\beta_2}^2||\nabla_h b||_2^{2(1-\frac{3}{\beta_2})}||\nabla_h b||_6^{\frac{6}{\beta_2}} + \frac{1}{6}\int |\nabla\nabla_h u|^2\ dx \\
&\le C\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-3}}\|\nabla_h b\|_2^2 + \frac{1}{6}\left(\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h b\|_2^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
C\int |b||\nabla_h u||\nabla\nabla_h b|\ dx &\le C||b||_{\beta_2}^2||\nabla_h u||_{\frac{2\beta_2}{\beta_2-2}}^2 + \frac{1}{6}\int |\nabla\nabla_h b|^2\ dx \\
&\le C||b||_{\beta_2}^2||\nabla_h u||_2^{2(1-\frac{3}{\beta_2})}||\nabla_h u||_6^{\frac{6}{\beta_2}} + \frac{1}{6}\int |\nabla\nabla_h b|^2\ dx \\
&\le C\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-3}}\|\nabla_h u\|_2^2 + \frac{1}{6}\left(\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h b\|_2^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&C\int |b||\partial_3 u||\nabla\nabla_h b|\ dx \\
&\le C\int |b|^2|\partial_3 u|^2\ dx + \frac{1}{6}\int |\nabla\nabla_h b|^2\ dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq C||b||_{\beta_2}^2||\partial_3 u||^2_{\frac{2\beta_2}{\beta_2-2}}+\frac{1}{6}\int|\nabla\nabla_h b|^2\ dx\\
&\leq C||b||_{\beta_2}^2||\partial_3 u||_2^{2(1-\frac{3}{\beta_2})}||\partial_3 u||_6^{\frac{6}{\beta_2}}+\frac{1}{6}\int|\nabla\nabla_h b|^2\ dx\\
&\leq C||b||_{\beta_2}^2||\partial_3 u||_2^{2(1-\frac{3}{\beta_2})}||\nabla\nabla_h u||_2^{\frac{4}{\beta_2}}||\nabla\partial_3 u||_2^{\frac{2}{\beta_2}}+\frac{1}{6}\int|\nabla\nabla_h b|^2\ dx\\
&\leq C\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-2}}\|\partial_3 u\|_2^{\frac{3\beta_2-10}{2\beta_2-4}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\nabla\partial_3 u\|_2^{\frac{2}{\beta_2-2}}+\frac{1}{6}\left(\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h b\|_2^2\right),
\end{aligned}
$$

where the interpolation inequality and Young's inequality are again used. Integrating inequality (3.10) over time interval (T_1,T_2), it follows

$$
\begin{aligned}
J^2(T_2)\leq C(T_1)+C&\int_{T_1}^{T_2}\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-3}}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)\ d\tau\\
&+C\int_{T_1}^{T_2}\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-2}}\|\partial_3 u\|_2^{\frac{3\beta_2-10}{2\beta_2-4}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\nabla\partial_3 u\|_2^{\frac{2}{\beta_2-2}}\ d\tau\\
&+C\int_{T_1}^{T_2}\int|\nabla u_3||\nabla_h u|^2dxd\tau+C\int_{T_1}^{T_2}\int|\nabla u_3||\partial_3 u||\nabla_h u|dxd\tau. \quad (3.19)
\end{aligned}
$$

The first term of (3.19) can be estimated by

$$
\begin{aligned}
&C\int_{T_1}^{T_2}\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-3}}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)\ d\tau\\
&\leq C\sup_{T_1<\tau<T_2}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)\left(\int_{T_1}^{T_2}\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-3}}\ d\tau\right)\\
&\leq C\sup_{T_1<\tau<T_2}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)\left(\int_{T_1}^{T_2}\|b\|_{\beta_2}^{\frac{8\beta_2}{3\beta_2-10}}\ d\tau\right)^{\frac{3\beta_2-10}{4\beta_2-12}}\left(\int_{T_1}^{T_2}1d\tau\right)^{\frac{\beta_2-2}{4\beta_2-12}}\\
&\leq C(T_2-T_1)\epsilon J^2(T_2). \quad (3.20)
\end{aligned}
$$

The second term of (3.19) can be estimated by

$$
\begin{aligned}
&C\int_{T_1}^{T_2}\|b\|_{\beta_2}^{\frac{2\beta_2}{\beta_2-2}}\|\partial_3 u\|_2^{\frac{3\beta_2-10}{2\beta_2-4}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\nabla\partial_3 u\|_2^{\frac{2}{\beta_2-2}}\ d\tau\\
&\leq C\|b\|_{L^{\alpha_2}(T_1,T_2;L^{\beta_2})}^{\frac{2\beta_2}{\beta_2-2}}\|\partial_3 u\|_{L^\infty(T_1,T_2;L^2)}^{\frac{3\beta_2-10}{2\beta_2-4}}\|\nabla u\|_{L^2(T_1,T_2;L^2)}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2(T_1,T_2;L^2)}^{\frac{2}{\beta_2-2}}\\
&\leq C\epsilon J^2(T_2). \quad (3.21)
\end{aligned}
$$

For the endpoint case of b, due to (3.1) and (3.4) we also have

$$
\begin{aligned}
&C\int_{T_1}^{T_2}\|b\|_{\frac{10}{3}}^5\|\partial_3 u\|_2^{\frac{1}{2}}\|\nabla\partial_3 u\|_2^{\frac{3}{2}}\ d\tau\\
&\leq C\|b\|_{L^\infty(T_1,T_2;L^{\frac{10}{3}})}^5\|\nabla u\|_{L^2(T_1,T_2;L^2)}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2(T_1,T_2;L^2)}^{\frac{3}{2}}\\
&\leq C\epsilon(C+CJ(T_2)^{\frac{4}{3}})^{\frac{3}{2}}\leq C\epsilon(1+J^2(T_2)).
\end{aligned}
$$

For the u terms, we have

$$
C\int_{T_1}^{T_2}\int|\nabla u_3||\nabla_h u|^2\ dxd\tau
$$

$$
\begin{aligned}
&\leq C\|\nabla u_3\|_{L^{\alpha_1,\beta_1}}\|\nabla_h u\|^2_{L^{\frac{4\beta_1}{3},\frac{2\beta_1}{\beta_1-1}}}\\
&\leq C\|\nabla u_3\|_{L^{\alpha_1}(T_1,T_2;L^{\beta_1})}\|\nabla_h u\|^{\frac{4\beta_1-6}{2\beta_1}}_{L^\infty(T_1,T_2;L^2)}\|\nabla\nabla_h u\|^{\frac{3}{\beta_1}}_{L^2(T_1,T_2;L^2)}\\
&\leq C\epsilon J^2(T_2).
\end{aligned}
\tag{3.22}
$$

It remains to estimate the following term:

$$
\int_{T_1}^{T_2}\int |\nabla u_3||\partial_3 u||\nabla_h u|\ dxd\tau.
$$

Using Hölder's inequality and Lemma 2.1, we have the following

$$
\begin{aligned}
&\int |\nabla u_3||\partial_3 u||\nabla_h u|\ dx\\
&\leq ||\nabla u_3||_{\beta_1}\left\|\left\|\|\partial_3 u\|_{L_1^{q_1}}\right\|_{L_2^{q_2}}\right\|_{L_3^{q_3}}\left\|\left\|\|\nabla_h u\|_{L_1^{r_1}}\right\|_{L_2^{r_2}}\right\|_{L_3^{r_3}}\\
&\leq ||\nabla u_3||_{\beta_1}||\partial_{13}u||_2^{\frac{q_1-2}{2q_1}}||\partial_{23}u||_2^{\frac{q_2-2}{2q_2}}||\partial_{33}u||_2^{\frac{q_3-2}{2q_3}}||\partial_3 u||_2^{\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}-\frac{1}{2}}\\
&\quad ||\partial_1\nabla_h u||_2^{\frac{r_1-2}{2r_1}}||\partial_2\nabla_h u||_2^{\frac{r_2-2}{2r_2}}||\partial_3\nabla_h u||_2^{\frac{r_3-2}{2r_3}}||\nabla_h u||_2^{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{2}},
\end{aligned}
$$

where β_1, q_i and r_i satisfy that

$$
\frac{1}{\beta_1}+\frac{1}{q_i}+\frac{1}{r_i}=1,\quad i=1,2,3.
$$

Therefore, if we take $q_1=q_2=\dfrac{2\beta_1}{\beta_1-2}$, $q_3=2$, $r_1=r_2=2$, $r_3=\dfrac{2\beta_1}{\beta_1-2}$, then it follows that

$$
\begin{aligned}
&\int_{T_1}^{T_2}\int |\nabla u_3||\partial_3 u||\nabla_h u|\ dxd\tau\\
&\leq \int_{T_1}^{T_2}||\nabla u_3||_{\beta_1}|||\partial_3 u||_2^{\frac{1}{q_1}+\frac{1}{q_2}}|\partial_{13}u||_2^{\frac{q_1-2}{2q_1}}||\partial_{23}u||_2^{\frac{q_2-2}{2q_2}}\\
&\quad ||\partial_3\nabla_h u||_2^{\frac{r_3-2}{2r_3}}||\nabla_h u||_2^{\frac{1}{r_3}+\frac{1}{2}}\ d\tau\\
&\leq \int_{T_1}^{T_2}||\nabla u_3||_{\beta_1}||\partial_3 u||_2^{\frac{3}{4q_1}+\frac{3}{4q_2}}||\nabla u||_2^{\frac{1}{4q_1}+\frac{1}{4q_2}}\\
&\quad ||\nabla_h u||_2^{\frac{1}{r_3}+\frac{1}{2}}||\nabla\nabla_h u||_2^{\frac{3}{2}-\frac{1}{q_1}-\frac{1}{q_2}-\frac{1}{r_3}}\ d\tau\\
&\leq ||\nabla u_3||_{L^{\alpha_1}(T_1,T_2;L^{\beta_1})}||\nabla_h u||^{\frac{1}{r_3}+\frac{1}{2}}_{L^\infty(T_1,T_2;L^2)}||\partial_3 u||^{\frac{3}{4q_1}+\frac{3}{4q_2}}_{L^\infty(T_1,T_2;L^2)}\\
&\quad ||\nabla u||^{\frac{1}{4q_1}+\frac{1}{4q_2}}_{L^2(T_1,T_2;L^2)}||\nabla\nabla_h u||^{\frac{3}{2}-\frac{1}{q_1}-\frac{1}{q_2}-\frac{1}{r_3}}_{L^2(T_1,T_2;L^2)}\\
&\leq CE^{\frac{1}{4q_1}+\frac{1}{4q_2}}\epsilon J^2(T_2),
\end{aligned}
\tag{3.23}
$$

where (3.4) was used. Now if we combine (3.19)-(3.23) together, it is easy to obtain (3.3) with the aid of (3.4). This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Based on the proof of Theorem 1.1, we only need to do the following estimates in a different way. Some notations used here are the same with those in Theorem 1.1. Firstly, we redo estimate for $J_1^{(3)}$ as following

$$\begin{aligned} J_1^{(3)} &= \sum_{j=1}^{2} \int u_3 \cdot \partial_3 u_j \cdot \Delta_h u_j \, dx \\ &= -\sum_{j=1}^{2}\sum_{k=1}^{2} \int \partial_k u_3 \partial_3 u_j \partial_k u_j \, dx - \sum_{j=1}^{2}\sum_{k=1}^{2} \int u_3 \partial_{3k} u_j \partial_k u_j \, dx \\ &= -\sum_{j=1}^{2}\sum_{k=1}^{2} \int \partial_k u_3 \partial_3 u_j \partial_k u_j \, dx + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \int \partial_3 u_3 (\partial_k u_j)^2 \, dx \\ &\le C \int |\partial_3 u_3| |\nabla_h u|^2 \, dx + C \int |\partial_3 u_h| |\nabla_h u|^2 \, dx. \end{aligned}$$

Then

$$\begin{aligned} C \int_{T_1}^{T_2} \int |\partial_3 u_h| |\nabla_h u|^2 \, dx d\tau &\le C \|\partial_3 u_h\|_{L^{\alpha_3}(T_1,T_2;L^{\beta_3})} \|\nabla_h u\|^2_{L^{\frac{4\beta_3}{3}}(T_1,T_2;L^{\frac{2\beta_3}{\beta_3-1}})} \\ \le C \|\partial_3 u_h\|_{L^{\alpha_3}(T_1,T_2;L^{\beta_3})} \|\nabla_h u\|^{\frac{4\beta_3-6}{2\beta_3}}_{L^\infty(T_1,T_2;L^2)} &\|\nabla\nabla_h u\|^{\frac{3}{\beta_3}}_{L^2(T_1,T_2;L^2)} \le C\epsilon J^2(T_2). \end{aligned}$$

Secondly, for the estimate of J_4, we have the following

$$\begin{aligned} J_4 &= -\int (b\cdot\nabla u)\Delta_h b \, dx \\ &= -\sum_{i,j,k=1}^{2} \int b_i \cdot \partial_i u_j \cdot \partial_{kk}^2 b_j \, dx - \sum_{i,k=1}^{2} \int b_i \cdot \partial_i u_3 \cdot \partial_{kk}^2 b_3 \, dx \\ &\quad -\sum_{j=1}^{3}\sum_{k=1}^{2} \int b_3 \cdot \partial_3 u_j \cdot \partial_{kk}^2 b_j \, dx \\ &\le C \int |\partial_3 u_h| |\nabla_h b|^2 \, dx + C \int |b| |\nabla_h b| |\nabla\nabla_h u| \, dx + C \int |b| |\nabla_h u| |\nabla\nabla_h b| \, dx. \end{aligned}$$

Similar techniques on $\int_{T_1}^{T_2} \int |\partial_3 u_h| |\nabla_h b|^2 \, dx d\tau$ yield the following

$$C \int_{T_1}^{T_2} \int |\partial_3 u_h| |\nabla_h b|^2 \, dx d\tau \le C\epsilon J^2(T_2).$$

The remaining parts are very similar to ones in the proof of Theorem 1.1; we skip it for concision.

5. Proof of Theorem 1.3

Based on (3.2) in the proof of Theorem 1.1, this proof will be divided into two steps.

Step 1. The estimation of $\|\nabla_h u\|_2 + \|\nabla_h b\|_2$.

Due to (3.12)-(3.14), we have

$$J_1 = \int (u\cdot\nabla u)\cdot\Delta_h u \, dx \le \int |\nabla u_3| |\partial_3 u| |\nabla_h u| + |\nabla u_3| |\nabla_h u|^2 \, dx, \tag{5.1}$$

and J_2 can be estimated as following:

$$
\begin{aligned}
J_2 =& -\int (b\cdot\nabla b)\cdot\Delta_h u\,dx\\
=& -\sum_{i,j,k=1}^{2}\int b_i\cdot\partial_i b_j\cdot\partial_{kk}^2 u_j\,dx - \sum_{i,k=1}^{2}\int b_i\cdot\partial_i b_3\cdot\partial_{kk}^2 u_3\,dx - \sum_{j=1}^{3}\sum_{k=1}^{2}\int b_3\cdot\partial_3 b_j\cdot\partial_{kk}^2 u_j\,dx\\
=& -\sum_{i,j,k=1}^{2}\int b_i\cdot\partial_i b_j\cdot\partial_{kk}^2 u_j\,dx - \sum_{i,k=1}^{2}\int b_i\cdot\partial_i b_3\cdot\partial_{kk}^2 u_3\,dx +\\
& \sum_{j=1}^{3}\sum_{k=1}^{2}\int \left(b_3\cdot\partial_{k3}^2 b_j\cdot\partial_k u_j + \partial_k b_3\cdot\partial_3 b_j\cdot\partial_k u_j\right)dx\\
\le& C\int_{\mathbb{R}^3}|b_h||\nabla_h b||\nabla\nabla_h u|\ dx + C\int_{\mathbb{R}^3}|b_h||\nabla_h u||\nabla\nabla_h b|\ dx\\
& + C\int_{\mathbb{R}^3}|b_3||\nabla_h b||\nabla\nabla_h u|\ dx + C\int_{\mathbb{R}^3}|b_3||\nabla_h u||\nabla\nabla_h b|\ dx.
\end{aligned}
\tag{5.2}
$$

Similarly

$$
\begin{aligned}
J_3 =& \int_{\mathbb{R}^3}(u\cdot\nabla b)\cdot\Delta_h b\,dx\\
=& \sum_{i,j,k=1}^{2}\int_{\mathbb{R}^3} u_i\cdot\partial_i b_j\cdot\partial_{kk}^2 b_j\,dx + \sum_{i,k=1}^{2}\int_{\mathbb{R}^3} u_i\cdot\partial_i b_3\cdot\partial_{kk}^2 b_3\,dx\\
& + \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3\cdot\partial_3 b_j\cdot\partial_{kk}^2 b_j\,dx\\
:=& J_3^{(1)} + J_3^{(2)} + J_3^{(3)}.
\end{aligned}
$$

By integration by parts and taking the divergence-free condition into account, we obtain

$$
\begin{aligned}
J_3^{(1)} =& \sum_{i,j,k=1}^{2}\int u_i\cdot\partial_i b_j\cdot\partial_{kk}^2 b_j\ dx\\
=& -\sum_{i,j,k=1}^{2}\int \partial_k u_i\cdot\partial_i b_j\cdot\partial_k b_j\ dx - \sum_{i,j,k=1}^{2}\int u_i\cdot\partial_{ki}^2 b_j\cdot\partial_k b_j\ dx\\
=& \sum_{i,j,k=1}^{2}\int \left(\partial_{kk}^2 u_i\cdot\partial_i b_j\cdot b_j + \partial_k u_i\cdot\partial_{ki}^2 b_j\cdot b_j\right)\ dx + \frac{1}{2}\sum_{i,j,k=1}^{2}\int \partial_i u_i\cdot\partial_k b_j\cdot\partial_k b_j\ dx\\
\le& C\int |b_h||\nabla_h u||\nabla\nabla_h b|\ dx + C\int |b_h||\nabla_h b||\nabla\nabla_h u|\ dx.
\end{aligned}
$$

In the same way, we get the estimates for $J_3^{(2)}$ and $J_3^{(3)}$ as follows:

$$
\begin{aligned}
J_3^{(2)} =& \sum_{i,k=1}^{2}\int u_i\cdot\partial_i b_3\cdot\partial_{kk}^2 b_3\ dx\\
=& -\sum_{i,k=1}^{2}\int \partial_k u_i\cdot\partial_i b_3\cdot\partial_k b_3\ dx - \sum_{i,k=1}^{2}\int u_i\cdot\partial_{ki}^2 b_3\cdot\partial_k b_3\ dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i,k=1}^{2} \int \left(\partial_k u_i \cdot \partial_{ki}^2 b_3 \cdot b_3 + \partial_{kk}^2 u_i \cdot \partial_i b_3 \cdot b_3\right)\, dx + \frac{1}{2} \sum_{i,k=1}^{2} \int \partial_i u_i \cdot \partial_k b_3 \cdot \partial_k b_3 \, dx \\
&= \sum_{i,k=1}^{2} \int \left(\partial_k u_i \cdot \partial_{ki}^2 b_3 \cdot b_3 + \partial_{kk}^2 u_i \cdot \partial_i b_3 \cdot b_3\right)\, dx \\
&\quad - \frac{1}{2} \sum_{i,k=1}^{2} \int \left(\partial_{ki}^2 u_i \cdot \partial_k b_3 \cdot b_3 + \partial_i u_i \cdot \partial_{kk}^2 b_3 \cdot b_3\right)\, dx \\
&\le C \int |b_3||\nabla_h b||\nabla\nabla_h u| \, dx + C \int |b_3||\nabla_h u||\nabla\nabla_h b| \, dx,
\end{aligned}
$$

and

$$
\begin{aligned}
J_3^{(3)} &= \sum_{j=1}^{3} \sum_{k=1}^{2} \int u_3 \cdot \partial_3 b_j \cdot \partial_{kk}^2 b_j \, dx \\
&= - \sum_{j=1}^{3} \sum_{k=1}^{2} \int \left(\partial_k u_3 \cdot \partial_3 b_j \cdot \partial_k b_j + u_3 \cdot \partial_{k3}^2 b_j \cdot \partial_k b_j\right)\, dx \\
&= \sum_{j=1}^{3} \sum_{k=1}^{2} \int \left(\partial_{3k}^2 u_3 \cdot b_j \cdot \partial_k b_j + \partial_k u_3 \cdot b_j \cdot \partial_{3k}^2 b_j\right)\, dx \\
&\quad + \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{2} \int \partial_3 u_3 \cdot \partial_k b_j \cdot \partial_k b_j \, dx \\
&= \sum_{j=1}^{3} \sum_{k=1}^{2} \int \left(\partial_{3k}^2 u_3 \cdot b_j \cdot \partial_k b_j + \partial_k u_3 \cdot b_j \cdot \partial_{3k}^2 b_j\right)\, dx \\
&\quad + \frac{1}{2} \sum_{j=1}^{3} \sum_{i,k=1}^{2} \int \left(\partial_{ki}^2 u_i \cdot \partial_k b_j \cdot b_j + \partial_i u_i \cdot \partial_{kk}^2 b_j \cdot b_j\right)\, dx \\
&\le C \int |b_3||\nabla_h b||\nabla\nabla_h u| \, dx + C \int |b_3||\nabla_h u||\nabla\nabla_h b| \, dx \\
&\quad + \int |b_h||\nabla_h b||\nabla\nabla_h u| \, dx + C \int |b_h||\nabla_h u||\nabla\nabla_h b| \, dx.
\end{aligned}
$$

Owing to the estimates for $J_3^{(1)}$, $J_3^{(2)}$, $J_3^{(3)}$, we obtain

$$
\begin{aligned}
J_3 &\le C \int |b_3||\nabla_h b||\nabla\nabla_h u| \, dx + C \int |b_3||\nabla_h u||\nabla\nabla_h b| \, dx \\
&\quad + \int |b_h||\nabla_h b||\nabla\nabla_h u| \, dx + C \int |b_h||\nabla_h u||\nabla\nabla_h b| \, dx. \qquad (5.3)
\end{aligned}
$$

Concerning the estimate for J_4, we have

$$
\begin{aligned}
J_4 &= - \int (b \cdot \nabla u) \Delta_h b \, dx \\
&= - \sum_{i,j,k=1}^{2} \int b_i \cdot \partial_i u_j \cdot \partial_{kk}^2 b_j \, dx - \sum_{i,k=1}^{2} \int b_i \cdot \partial_i u_3 \cdot \partial_{kk}^2 b_3 \, dx
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j=1}^{3}\sum_{k=1}^{2}\int b_3\cdot\partial_3 u_j\cdot\partial_{kk}^2 b_j\ dx\\
&\le C\int |b_3||\partial_3 u||\nabla\nabla_h b|\ dx+C\int |b_h||\nabla_h b||\nabla\nabla_h u|\ dx. \qquad (5.4)
\end{aligned}
$$

Thus, inserting (5.1)-(5.4) into (3.10), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)+\left(\|\nabla_h\nabla u\|_2^2+\|\nabla_h\nabla b\|_2^2\right)\\
&\le C\int |\nabla u_3||\partial_3 u||\nabla_h u|+|\nabla u_3||\nabla_h u|^2\ dx+C\int |b_3||\nabla b||\nabla\nabla_h u|\ dx\\
&+C\int |b_3||\nabla u||\nabla\nabla_h b|dx\\
&+\int |b_h||\nabla_h b||\nabla\nabla_h u|dx+\ C\int |b_h||\nabla_h u||\nabla\nabla_h b|dx\\
&:=K_1+K_2+K_3+K_4+K_5.
\end{aligned}
$$

Firstly, suppose that (1.9) and (1.10) are satisfied, we show (u,b) remains smooth on $(0,T]$. Since K_2, K_4 and K_3, K_5 are similar, so we only estimate K_2, K_4.

$$
\begin{aligned}
K_2&=\int |b_3||\nabla b||\nabla\nabla_h u|\ dx\\
&\le C\|b_3\|_{\frac{10}{3}}\|\nabla b\|_5\|\nabla\nabla_h u\|_2\\
&\le C\|b_3\|_{\frac{10}{3}}\|\nabla b\|_2^{\frac{1}{10}}\|\nabla b\|_6^{\frac{9}{10}}\|\nabla\nabla_h u\|_2\\
&\le C\|b_3\|_{\frac{10}{3}}\|\nabla b\|_2^{\frac{1}{10}}\|\nabla\nabla_h b\|_2^{\frac{3}{5}}\|\Delta b\|_2^{\frac{3}{10}}\|\nabla\nabla_h u\|_2\\
&\le C\|b_3\|_{\frac{10}{3}}^5\|\nabla b\|_2^{\frac{1}{2}}\|\Delta b\|_2^{\frac{3}{2}}+\frac{1}{8}\left(\|\nabla\nabla_h b\|_2^2+\|\nabla\nabla_h u\|_2^2\right), \qquad (5.5)
\end{aligned}
$$

and

$$
\begin{aligned}
K_4&=\int |b_h||\nabla_h b||\nabla\nabla_h u|\ dx\\
&\le c\|b_h\|_2^{\frac{6}{11}}\|b_h\|_{\beta_7}^{\frac{5}{11}}\|\nabla_h b\|_q\|\nabla\nabla_h u\|_2\\
&\le c\|b\|_2^{\frac{24q}{11(6-q)}}\|b_h\|_{\beta_7}^{\frac{20q}{11(6-q)}}\|\nabla_h b\|_2^2+\frac{1}{8}\left(\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h b\|_2^2\right), \qquad (5.6)
\end{aligned}
$$

where $\frac{3}{11}+\frac{5}{11\beta_7}+\frac{1}{q}+\frac{1}{2}=1$, $q\in\left(\frac{22}{5},6\right)$, then $\beta_7\in\left(\frac{15}{2},\infty\right)$. It follows from (5.5) and (5.6) that

$$
\begin{aligned}
J^2(T_2)\le& C+\int_{T_1}^{T_2}K_1 d\tau+C\int_{T_1}^{T_2}\|b_3\|_{\frac{10}{3}}^5\|\nabla b\|_2^{\frac{1}{2}}\|\nabla\partial_3 b\|_2^{\frac{3}{2}}d\tau\\
&+C\int_{T_1}^{T_2}\|b_3\|_{\frac{10}{3}}^5\|\nabla u\|_2^{\frac{1}{2}}\|\nabla\partial_3 u\|_2^{\frac{3}{2}}d\tau+C\int_{T_1}^{T_2}\|b\|_2^{\frac{24q}{11(6-q)}}\|b_h\|_{\beta_7}^{\frac{20q}{11(6-q)}}\|\nabla_h b\|_2^2 d\tau\\
&+C\int_{T_1}^{T_2}\|b\|_2^{\frac{24q}{11(6-q)}}\|b_h\|_{\beta_7}^{\frac{20q}{11(6-q)}}\|\nabla_h u\|_2^2 d\tau. \qquad (5.7)
\end{aligned}
$$

We deal with the right-hand side of (5.7) in the following. It follows from (3.22) and (3.23), we have

$$\int_{T_1}^{T_2} K_1 d\tau \leq C\epsilon J^2(T_2).$$

and

$$\int \|b\|_2^{\frac{24q}{11(6-q)}} \|b_h\|_{\beta_7}^{\frac{20q}{11(6-q)}} \|\nabla_h b\|_2^2 d\tau \leq C\|b_h\|_{L^{\alpha_7}(T_1,T_2;L^{\beta_7})}^{\frac{20q}{11(6-q)}} \|\nabla_h b\|_{L^\infty(T_1,T_2;L^2)}^2 \leq C\epsilon J^2(T_2).$$

Similarly,

$$\int \|b\|_2^{\frac{24q}{11(6-q)}} \|b_h\|_{\beta_7}^{\frac{20q}{11(6-q)}} \|\nabla_h u\|_2^2 d\tau \leq C\epsilon J^2(T_2).$$

Next, we estimate

$$\int \|b_3\|_{\frac{10}{3}}^5 \|\nabla b\|_2^{\frac{1}{2}} \|\nabla \partial_3 b\|_2^{\frac{3}{2}} d\tau \leq C \sup_{T_1<\tau<T_2} \|b_3\|_{\frac{10}{3}}^5 K^{\frac{3}{2}}. \tag{5.8}$$

Thus, if we can show that

$$\sup_{T_1<\tau<T_2} \|b_3\|_{\frac{10}{3}} \leq C. \tag{5.9}$$

then we obtain

$$J^2 \leq C + CK^{\frac{3}{2}}. \tag{5.10}$$

Now our main concern is to prove (5.9). We multiply the equation of b_3 in (1.1)

$$\partial_t b_3 + (u\cdot\nabla)b_3 - (b\cdot\nabla)u_3 - \Delta b_3 = 0$$

by $|b_3|^{\frac{4}{3}} b_3$ and integrate over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{3}{10}\frac{d}{dt}\||b_3|^{\frac{5}{3}}\|_2^2 + \frac{9}{25}\|\nabla |b_3|^{\frac{5}{3}}\|_2^2 \\
&\leq \int |b||\nabla u_3|(|b_3|^{\frac{5}{3}})^{\frac{7}{5}}\ dx \leq \int |b_3||\nabla u_3|(|b_3|^{\frac{5}{3}})^{\frac{7}{5}}\ dx + \int |b_h||\nabla u_3|(|b_3|^{\frac{5}{3}})^{\frac{7}{5}}\ dx \\
&\leq \int |\nabla u_3|(|b_3|^{\frac{5}{3}})^2\ dx + \int |b_h||\nabla u_3|(|b_3|^{\frac{5}{3}})^{\frac{7}{5}}\ dx \\
&\leq \|\nabla u_3\|_{\beta_6}\||b_3|^{\frac{5}{3}}\|_{\frac{2\beta_6}{\beta_1-2}}^2 + \|b_h\|_{\beta_7}\|\nabla u\|_2\||b_3|^{\frac{5}{3}}\|_{\frac{14\beta_7}{5\beta_2-10}}^{\frac{7}{5}} \\
&\leq C\|\nabla u_3\|_{\beta_6}^{\frac{2\beta_6}{2\beta_6-3}}\||b_3|^{\frac{5}{3}}\|_2^2 + C\|b_h\|_{\beta_7}^{\frac{10\beta_7}{7\beta_7-15}}\|\nabla u\|_2^{\frac{10\beta_7}{7\beta_7-15}}\||b_3|^{\frac{5}{3}}\|_2^{\frac{8\beta_7-30}{7\beta_7-15}} + \frac{9}{50}\|\nabla |b_3|^{\frac{5}{3}}\|_2^2 \\
&\leq C\|\nabla u_3\|_{\beta_6}^{\frac{2\beta_6}{2\beta_6-3}}\||b_3|^{\frac{5}{3}}\|_2^2 + C\|b_h\|_{\beta_7}^{\frac{10\beta_7}{7\beta_7-15}}\|\nabla u\|_2^{\frac{10\beta_7}{7\beta_7-15}}\left(\||b_3|^{\frac{5}{3}}\|_2^2+1\right) + \frac{9}{50}\|\nabla |b_3|^{\frac{5}{3}}\|_2^2.
\end{aligned} \tag{5.11}$$

Then (5.9) holds true by applying Grönwall's inequality to (5.11).

Step 2. The estimation of $\|\nabla u\|_2 + \|\nabla b\|_2$.

Multiplying the first two equations of (1.1) by Δu and Δb, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right)+\left(\|\Delta u\|_2^2+\|\Delta b\|_2^2\right)\\&=\int[(u\cdot\nabla u)\Delta u-(b\cdot\nabla b)\Delta u+(u\cdot\nabla b)\Delta b-(b\cdot\nabla u)\Delta b]\ dx\\&\le C\int|\nabla_h(u,b)|\cdot|\nabla(u,b)|^2dx\\&\le C\|\nabla_h(u,b)\|_2\|\nabla(u,b)\|_4^2\\&\le C\|\nabla_h(u,b)\|_2\|\nabla(u,b)\|_2^{\frac{1}{2}}\|\nabla\nabla_h(u,b)\|_2\|\Delta(u,b)\|_2^{\frac{1}{2}}.\end{aligned}\tag{5.12}$$

Then integrating in time yields

$$\begin{aligned}K^2(T_2)\le C+C\sup_{T_1<\tau<T_2}\|\nabla_h(u,b)\|_2\left(\int_{T_1}^{T_2}\|\nabla u\|_2^2+\|\nabla b\|^2d\tau\right)^{\frac{1}{4}}\\\cdot\left(\int_{T_1}^{T_2}\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h b\|^2d\tau\right)^{\frac{1}{2}}\left(\int_{T_1}^{T_2}\|\Delta u\|_2^2+\|\Delta b\|^2d\tau\right)^{\frac{1}{4}}\\\le C+CJ\cdot\eta\cdot J\cdot K^{\frac{1}{2}}\le C+CJ^2\cdot\eta K^{\frac{1}{2}}.\end{aligned}\tag{5.13}$$

Gathering (5.10) into (5.13), and taking η sufficiently small, we obtain that $K^2(T_2)\le C$, it implies that (3.2) holds true.

Secondly, we prove regularity in conditions (1.11) and (1.12). As in the first case, it sufficies to estimate $\int_{T_1}^{T_2}K_1d\tau$, $\int_{T_1}^{T_2}K_2d\tau$, $\int_{T_1}^{T_2}K_4d\tau$. In view of (1.10), for any $\beta_6\in\left(\frac{15}{7},\infty\right)$, there exists an α_6 which satisfies the assumption of (1.11). By using (3.22) and (3.23), we obtain

$$\begin{aligned}&\int_{T_1}^{T_2}K_1d\tau\le C\left(\int\|\nabla u_3\|_{\beta_6}^{\alpha}d\tau\right)^{\frac{1}{\alpha}}J^2(T_2)\\&\le C(T_2-T_1)\left(\int\|\nabla u_3\|_{\beta_6}^{\alpha\cdot\frac{\alpha_6}{\alpha}}d\tau\right)^{\frac{1}{\alpha_6}}J^2(T_2)\le C(T_2-T_1)\epsilon J^2(T_2).\end{aligned}\tag{5.14}$$

It is easy to find that (5.8) is also right, we just have some changes in (5.11)

$$\begin{aligned}&\frac{3}{10}\frac{d}{dt}\||b_3|^{\frac{5}{3}}\|_2^2+\frac{9}{25}\|\nabla|b_3|^{\frac{5}{3}}\|_2^2\le\int|b||\nabla u_3|(|b_3|^{\frac{5}{3}})^{\frac{7}{5}}\ dx\\&\le C\|b\|_{\frac{12\beta_6}{5(\beta_6-1)}}\|\nabla u_3\|_{\beta_6}\||b_3|^{\frac{5}{3}}\|_{\frac{12\beta_6}{5(\beta_6-1)}}^{\frac{7}{5}}\\&\le C\|b\|_2^{\frac{3}{4}-\frac{5}{4\beta_6}}\|\nabla b\|_2^{\frac{1}{4}+\frac{5}{4\beta_6}}\|\nabla u_3\|_{\beta_6}\||b_3|^{\frac{5}{3}}\|_2^{\frac{7}{5}\left(\frac{3}{4}-\frac{5}{4\beta_6}\right)}\|\nabla|b_3|^{\frac{5}{3}}\|_2^{\frac{7}{5}\left(\frac{1}{4}+\frac{5}{4\beta_6}\right)}\\&\le C\|\nabla b\|_2^{\frac{10(\beta_6+5)}{33\beta_6-35}}\|\nabla u_3\|_{\beta_6}^{\frac{40\beta_6}{33\beta_6-35}}\||b_3|^{\frac{5}{3}}\|_2^{\frac{14(3\beta_6-5)}{33\beta_1-35}}+\frac{9}{50}\|\nabla|b_3|^{\frac{5}{3}}\|_2^2\\&\le C\|\nabla b\|_2^{\frac{10(\beta_6+5)}{33\beta_6-35}}\|\nabla u_3\|_{\beta_6}^{\frac{40\beta_6}{33\beta_6-35}}\left(\||b_3|^{\frac{5}{3}}\|_2^2+1\right)+\frac{9}{50}\|\nabla|b_3|^{\frac{5}{3}}\|_2^2,\end{aligned}\tag{5.15}$$

then we apply Grönwall's inequality in (5.15), it follows (5.9). For the term $\int_{T_1}^{T_2}K_4d\tau$, we obtain that

$$\int_{T_1}^{T_2}K_4d\tau\le\int\|b_h\|_{\beta_7}\|\nabla_h b\|_{\frac{2\beta_7}{\beta_7-2}}\|\nabla\nabla_h u\|_2\ d\tau$$

$$\leq C\int \|b_h\|_{\beta_7}\|\nabla_h b\|_2^{\frac{3\beta_7-6}{2\beta_7}}\|\nabla\nabla_h u\|_2^{\frac{\beta_7+6}{2\beta_7}}\,d\tau$$
$$\leq C\|b_h\|_{L^{\alpha_7}(T_1,T_2;L^{\beta_7})}\|\nabla_h b\|_{L^\infty(T_1,T_2;L^2)}^{\frac{3\beta_7-6}{2\beta_7}}\|\nabla\nabla_h u\|_{L^2(T_1,T_2;L^2)}^{\frac{\beta_7+6}{2\beta_7}}$$
$$\leq C\epsilon J^2(T_2).$$

Then we can repeat Step 2 in the first case to obtain $K^2(T_2)\leq C$, it implies that (3.2) holds true. This completes the proof of Theorem 1.3.

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1