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HYPERBOLIC CONSENSUS GAMES*

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Abstract. We introduce the use of conservation laws in multi-player consensus games. Indeed, a general non-anticipative strategy is proposed within a rigorous analytic framework. By means of numerical integrations, we describe peculiar features of this strategy, such as its effectiveness, the automatic formation of coalitions and the effects of competitions.

Keywords. Hyperbolic Consensus Game; Multi-agent Consensus Strategies; Differential Games; Instantaneous Control.

AMS subject classifications. 35L65; 91A23; 93B52.

1. Introduction

A group of *"leaders"*, or broadcasting agents, aims at getting the consensus of a variety of individuals. We identify each individual's opinion with a *"position"* p moving in $\mathbb{R}^N$. It is then natural to describe the leaders through their *"positions"* $P_1, P_2, \ldots, P_k$, also in $\mathbb{R}^N$. We are thus lead to the general system of ordinary differential equation

$$\dot{p} = v(t, p, P_1(t), \ldots, P_k(t))$$

t being time. The vector field v describes the interaction among individuals and agents, which can be attractive, repulsive, or a mixture of the two. Clearly, no linearity assumption can be reasonably required on v, otherwise the agent – individuals interaction increases as their distance increases.

The task of the agent P_i, be it attractive or repulsive, is to maximize its own consensus, i.e., to drive the maximal amount of individuals (or their opinions) as near as possible to its own target region $\mathcal{T}_i$ at time T, for a suitable non-empty $\mathcal{T}_i \subset \mathbb{R}^N$. The time horizon T is finite and the same for all agents.

The presence of a high number of individuals, as well as of uncertainties in their initial positions or specific movements, suggests to describe the dynamics underneath the present problem through the continuity equation

$$\partial_t \rho + \operatorname{div}_x (\rho\, v(t, x, P_1(t), \ldots, P_k(t))) = 0, \tag{1.1}$$

where the description of each individual is substituted by that of the individuals' density distribution $\rho = \rho(t,x)$, while the goal of the i–th leader is formalized through the minimization of the quantity

$$\mathcal{J}_i = \int_{\mathbb{R}^N} \rho(T,x)\, d(x, \mathcal{T}_i)\, \mathrm{d}x \tag{1.2}$$

where $d(x, \mathcal{T}_i) = \inf_{y \in \mathcal{T}_i} \|x - y\|$ is the distance between the position x and the target $\mathcal{T}_i$.

Aim of this paper is to formalize the above setting, to provide basic well posedness theorems and to initiate the search for controls/strategies to tackle the above problem.

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Indeed, the case $k=1$ of a single broadcasting agent leads to a control problem, while the case $k>1$ of k possibly competing agents fits into game theory.

As it is usual in control theory, rather than the agents' positions P_i, it is preferable to use as controls/strategies the agents' speeds u_i, with $u_i \in \mathbb{R}^N$ subject to a boundedness constraint of the type $\|u_i\| \leq U$, for a positive U. Introducing the initial individuals' distribution $\bar{\rho}$ and agents' positions $\bar{P}_1, \ldots, \bar{P}_k$, the dynamics is then described by the Cauchy Problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\rho\, v(t,x,P_1(t),\ldots,P_k(t))) = 0 \\ \rho(0,x) = \bar{\rho}(x) \end{cases} \quad \text{where} \quad \begin{cases} \dot{P}_i = u_i(t) \\ P_i(0) = \bar{P}_i \end{cases} \quad i = 1,\ldots,k \tag{1.3}$$

where the cost functionals $\mathcal{J}_i$ are as in (1.2). This structure is amenable to the introduction of several control/game theoretic concepts, from optimal controls to Nash equilibria, and to the search for their existence. Below we initiate this study providing the basic analytic framework and tackling the problem of control/strategies that minimize costs of the type (1.2). In particular, we introduce a control/strategy with the following properties:

(1) it is *non-anticipative*, i.e. its value at time t requires knowledge neither of the state of the systems nor of any others' strategy at times greater than t;

(2) it is *explicit*, i.e. we provide it through a fully computable closed-form integral formula, see Section 2;

(3) it is *effective*, i.e. not only is it competitive with other simple strategies, but it also leads to the automatic cooperation of agents having the same goal, see Section 3;

(4) it is *versatile*, i.e. it applies to any sort of agent–individuals interaction: long/short range, attractive/repulsive, … as well as to both competitive or cooperative agents.

A posteriori, the strategy defined below can be seen as a version of the *instantaneous control*, see [1, 13, 16, 17, 19], specialized to the present setting (1.3).

Note that this framework, restricted to the case $N=2$, allows also to describe the individual–continuum interactions considered, for instance, in [10], see also [8, 9], and [11], where an entirely different analytic structure is exploited. From this point of view, the present results are related to the vast literature on crowd and swarm dynamics, see the recent works [1, 2, 6, 7, 13, 15, 23–25, 28] or the review [3] and the references therein.

Concerning our choice of the conservation law (1.1), we stress that typical of equations of this kind is the finite speed both of propagation of information and of the support of the density. This is in contrast with the typical situation in standard differential games ruled by parabolic equations. The present setting is completely independent of any kinetic approach such as that pursued in [1] and does not require any moment closure procedure. Indeed, similarly to the well known situation in fluid dynamics, it can be of interest to address a further justification of (1.3) from a kinetic point of view.

In the next section we first provide the basic notation and definitions, then we provide basic well posedness results and introduce a reasonable non-anticipative strategy. Section 3 is devoted to sample applications, while all analytic proofs are deferred to Section 4.

2. A non-anticipative strategy

Throughout, the positive time T and the maximal speed U are fixed. The closed ball in $\mathbb{R}^m$ centered at u with radius U is $\overline{B_{\mathbb{R}^m}(u,U)}$ and, when the space is clear, we shorten it to $\overline{B(u,U)}$.

Introduce $P \equiv (P_1, \ldots, P_k)$, so that $P \in \mathbb{R}^m$ with $m = kN$. Below, a recurrent assumption on the function v in (1.3) is the following:

***(v)*:** The vector field $v \in \mathbf{C^0}([0,T]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^N)$ is such that

- for all $t\in[0,T]$ and $P\in\mathbb{R}^m$, the map $x\to v(t,x,P)$ is in $\mathbf{C^{1,1}}(\mathbb{R}^N;\mathbb{R}^N)$;
- for all $t\in[0,T]$ and $x\in\mathbb{R}^N$, the map $P\to v(t,x,P)$ is in $\mathbf{C^{0,1}}(\mathbb{R}^m;\mathbb{R}^N)$.

With reference to control or game theoretic problem presented in the Introduction, below we consider the slightly more general expression

$$\mathcal{J}_i = \int_{\mathbb{R}^N} \rho(T,x)\,\psi_i(x)\,\mathrm{d}x \tag{2.1}$$

which reduces to (1.2) in the case $\psi_i(x) = d(x,\mathcal{T}_i)$.

The i-th leader P_i seeks a control $u_i\in\mathbf{L}^\infty\left([0,T];\overline{B(0,U)}\right)$ that minimizes the cost (2.1). Assume first that P_i knows in advance the strategies u_j, for $j\neq i$, of the other controllers P_j. Then, the existence of an optimal control follows thanks to the well posedness of (1.3).

PROPOSITION 2.1. *Fix positive T and U. Let v be bounded and satisfy **(v)**, $\bar P_j\in\mathbb{R}^N$ for $j=1,\ldots,k$ and $u_j\in\mathbf{L}^\infty([0,T];\overline{B(0,U)})$ for $j=1,\ldots,k$ with $j\neq i$. In each of the two cases*

$$\begin{array}{lll} \psi_i\in\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}) & \textit{and} \quad \bar\rho\in\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) & \textit{or}\\ \psi_i\in\mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R}) & \textit{and} \quad \bar\rho\in\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) & \textit{compactly supported,}\end{array}$$

call ρ the solution to (1.3). Then, the map

$$\begin{array}{rcl} \mathcal{J}_i \;:\; \mathbf{L}^\infty\left([0,T];\overline{B(0,U)}\right) & \to & \mathbb{R}\\ u_i & \to & \displaystyle\int_{\mathbb{R}^N}\rho(T,x)\,\psi_i(x)\,\mathrm{d}x \end{array} \tag{2.2}$$

is weak$\star$ sequentially continuous in $\mathbf{L}^\infty$ and there exists an optimal control u_i^* minimizing $\mathcal{J}_i$.

We refer to Corollary 4.1 for the analytic details.

Note however that the approach on which Proposition 2.1 is based can hardly be used in the present game theoretic setting. Indeed, the possible necessary conditions for a control u^* to minimize $\mathcal{J}$ in (2.2) could not be used in a game theoretic setting. Such conditions would require that P_i is aware of all other strategies u_j, $j\neq i$, on the whole time interval $[0,T]$. This ability to foresee the future choices of the competitors is unreasonable whenever different agents are confronting with each other.

We now proceed towards the definition of a non-anticipative strategy. To this aim, always considering an arbitrary number of controllers, we fix our attention on P_i and we simplify the notation setting $P=P_i$, $u=u_i$, $\mathcal{J}=\mathcal{J}_i$ and comprising within the time dependence of the function v all the other strategies u_j, for $j\neq i$, obtaining the problem

$$\left\{\begin{array}{l}\partial_t\rho+\mathrm{div}_x\left(\rho\, v(t,x,P(t))\right)=0\\ \rho(0,x)=\bar\rho(x)\end{array}\right. \quad\text{where}\quad \left\{\begin{array}{l}\dot P=u(t)\\ P(0)=\bar P.\end{array}\right. \tag{2.3}$$

In this setting, we construct below a *non-anticipative* strategy u for the controller P, i.e., a strategy $u=u(t)$ that depends only on ρ at times $s\in[0,t[$. For a wider discussion about non-anticipative strategies in the framework of sub–optimal techniques, we refer to [13].

For a positive (suitably small) Δt, we seek the best choice of a speed $w \in \overline{B(0,U)}$ on the interval $[t, t+\Delta t]$ such that the solution $\rho_w = \rho_w(\tau, x)$ to

$$\begin{cases} \partial_\tau \rho_w + \operatorname{div}_x \left(\rho_w\, v(t,x,P(t)+(\tau - t)w)\right) = 0 \\ \rho_w(t,x) = \rho(t,x) \end{cases} \qquad \tau \in [t, t+\Delta t] \tag{2.4}$$

is likely to best contribute to decreasing the value of $\mathcal{J}$. Remark that the dependence of v on t in (2.4) is *frozen* at time t. It is this choice that will later lead to a non-anticipative strategy. The proof that (2.4) is well posed is deferred to Lemma 4.4.

In the case of the functional (1.2), a natural choice for the agent P at time t is then to choose a speed w on the time interval $[t, t+\Delta t]$ to *myopically* minimize the *myopic* functional

$$\begin{array}{rccc} \mathcal{J}_{t,\Delta t} : & \mathbb{R}^N & \to & \mathbb{R} \\ & w & \to & \displaystyle\int_{\mathbb{R}^N} \rho_w(t+\Delta t, x)\, \psi(x)\, \mathrm{d}x \end{array} \tag{2.5}$$

see also [17, 22]. The main theorem now follows, providing an effective hint on a non-anticipative optimal choice of w.

THEOREM 2.1. *Fix $T>0$ and $U>0$. Let $v \in \mathbf{C^{0,1}}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)$ and $\psi \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$. As initial data in (2.3), choose a boundedly supported $\bar\rho \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$ and a $\bar P \in \mathbb{R}^N$. Define ρ as the solution to (2.3) and ρ_w as the solution to (2.4), for a $w \in \overline{B(0,U)}$.*

Then, for any $t \in [0,T[$ and $\Delta t \in]0, T-t]$ the map (2.5) is well defined and Lipschitz continuous.

Moreover, if $v \in \mathbf{C^2}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)$, the map (2.5) admits the expansion

$$\mathcal{J}_{t,\Delta t}(w+\delta_w) = \mathcal{J}_{t,\Delta t}(w) + \operatorname{grad}_w \mathcal{J}_{t,\Delta t}(w)\cdot \delta_w + o(\delta_w) \qquad \text{as } \delta_w \to 0 \tag{2.6}$$

where, as $\Delta t \to 0$,

$$\begin{aligned} \operatorname{grad}_w \mathcal{J}_{t,\Delta t}(w) = {} & \frac{(\Delta t)^2}{2} \int_{\mathbb{R}^N} \operatorname{grad}_x \rho(t,x) D_P v(t,x,P(t))\, \psi(x)\, \mathrm{d}x \\ & - \frac{(\Delta t)^2}{2} \int_{\mathbb{R}^N} \rho(t,x) \operatorname{grad}_P \operatorname{div}_x v(t,x,P(t))\, \psi(x)\, \mathrm{d}x + o(\Delta t)^2. \end{aligned} \tag{2.7}$$

The proof is deferred to Section 4. On the basis of Theorem 2.1, the definition of an effective non-anticipative strategy for P_i can be easily achieved as follows. Split the interval $[0,T]$ in smaller portions $[t_\ell, t_{\ell+1}[$, where $t_\ell = \ell \Delta t$. On each of them, define $u_i(t) = w_\ell$, where w_ℓ minimizes on $\overline{B(0,U)}$ the cost $\mathcal{J}_{t_\ell,\Delta t}$ defined in (2.5). The leading term in the right-hand side of (2.7) is independent of w, so that for Δt small it is reasonable to choose

$$w_\ell = -\frac{U \displaystyle\int_{\mathbb{R}^N} \left[\operatorname{grad}_x \rho(t_\ell,x)\, D_P v(t_\ell,x,P_i(t_\ell)) - \rho(t_\ell,x)\, \operatorname{grad}_P \operatorname{div}_x v(t_\ell,x,P_i(t_\ell))\right] \psi(x)\, \mathrm{d}x}{\left\| \displaystyle\int_{\mathbb{R}^N} \left[\operatorname{grad}_x \rho(t_\ell,x)\, D_P v(t_\ell,x,P_i(t_\ell)) - \rho(t_\ell,x)\, \operatorname{grad}_P \operatorname{div}_x v(t_\ell,x,P_i(t_\ell))\right] \psi(x)\, \mathrm{d}x \right\|}$$

as long as the denominator above does not vanish, in which case we set $w_\ell = 0$. Remark that, through the term ρ_ℓ, the right-hand side above depends on all the past values $w_0, \ldots, w_{\ell-1}$ attained by u_i. Formally, in the limit $\Delta t \to 0$, the above relation thus leads to an integro-differential equation.

3. Qualitative features of the non-anticipative strategy

This section shows that the strategy (2.7) can be effectively used in various situations. To this aim, we present below some numerical integrations of the game (1.3)–(1.2).

For the function v in (1.3), we typically choose an expression of the form

$$v(t,x,P) = \sum_{i=1}^{k} a_i\left(\|x-P_i\|\right)(P_i - x), \tag{3.1}$$

where $P \equiv (P_1,\dots,P_k)$ and $a_i : \mathbb{R}^+ \to \mathbb{R}$, $i \in \{1,\cdots,k\}$, is chosen so that ***(v)*** holds. In other words, at time t, the velocity $v(t,x,P)$ of the individual at x is the sum of k vectors, each of them parallel to the straight line through x and the agent's position P_i and its strength depends on the distance between x and P_i. Typically, the functions a_i are chosen so that for all t and P, the map $x \to v(t,x,P)$ is either compactly supported, or vanishes as $\|x\| \to +\infty$. Note that $a_i > 0$ whenever P_i is attractive, while $a_i < 0$ in the repulsive case. In the examples below, each target $\mathcal{T}_i$ is a single point and, correspondingly, the cost ψ_i is the distance from that point.

With reference to (2.3), in each of the integrations below we use the Lax–Friedrichs algorithm [18, Example 3.2] or [21, Section 4.6] with dimensional splitting [18, Paragraph 4.1] or [21, Section 19.5] to integrate the conservation law, while the usual explicit forward Euler method provides the *"exact"* (up to rounding errors) solutions to the ordinary differential equation. To ease the presentations of the results, we fix the space dimension $N = 2$. Correspondingly, in each of the rectangular domains Ω considered below, we fix a uniform regular rectangular grid consisting of $n_x \times n_y$ points.

3.1. Consensus pursued by a single attractive agent. Consider (2.3) in the numerical domain $\Omega = [0,10] \times [0,10]$, with

$$\begin{array}{llll} N = 2, & a_1(\xi) = \frac{1}{0.1+\xi} e^{-0.1\xi^2}, & \bar\rho = \chi_{[6,8]\times[2,8]}, & \mathcal{T}_1 = \{(1,8)\}, \\ k = 1, & v(t,x,P) = (3.1), & \bar P_1 \equiv (3,2), & T = 10. \\ m = 2, & U = 1.5, & & \end{array} \tag{3.2}$$

We now compute the solution to (1.3) with u piecewise constant given by the strategy (2.7), constant on intervals $[j\Delta t, (j+1)\Delta t]$, where $\Delta t = 0.01$. The resulting solution, obtained on a grid of $n_x \times n_y = 6000 \times 6000$ cells, is displayed in Figure 3.1.

The strategy relying on Theorem 2.1 can be seen as *myopic*, in the sense that it is based on an optimization over a short time interval, namely from t to $t + \Delta t$. However, remarkably, in the present case the leader P_1 does not move directly towards the target $\mathcal{T}_1$. On the contrary, it first moves to the right to *collect* a higher quantity of individuals and then moves back to the left; see Figure 3.1.

The resulting cost (1.2) is $\mathcal{J}_1 = 25.4$. Choosing a constant u, so that P_1 moves along a straight line, leaves $\bar P_1$ at $t = 0$ and reaches $\mathcal{T}$ at time $t = T$, leads to the higher cost 69.6.

3.2. Two competing attractive agents. We now test how effective the strategy (2.7) can be when competing against an agent driven by an *a priori* assigned strategy. More precisely, we let $\Omega = [0,10] \times [0,10]$, with

$$\begin{array}{llll} N = 2, & a_1(\xi) = \frac{1}{0.1+\xi} e^{-0.2\xi^2}, & \bar\rho = \chi_{[7,9]\times[3,7]}, & \mathcal{T}_1 = \{(1,9)\}, \\ k = 2, & a_2(\xi) = \frac{1}{0.1+\xi} e^{-0.2\xi^2}, & \bar P_1 = (8,5), & \mathcal{T}_2 = \{(1,1)\}, \\ m = 4, & v(t,x,P) = (3.1), & \bar P_2 = (8,5), & T = 10. \\ & U = 1.5, & & \end{array} \tag{3.3}$$

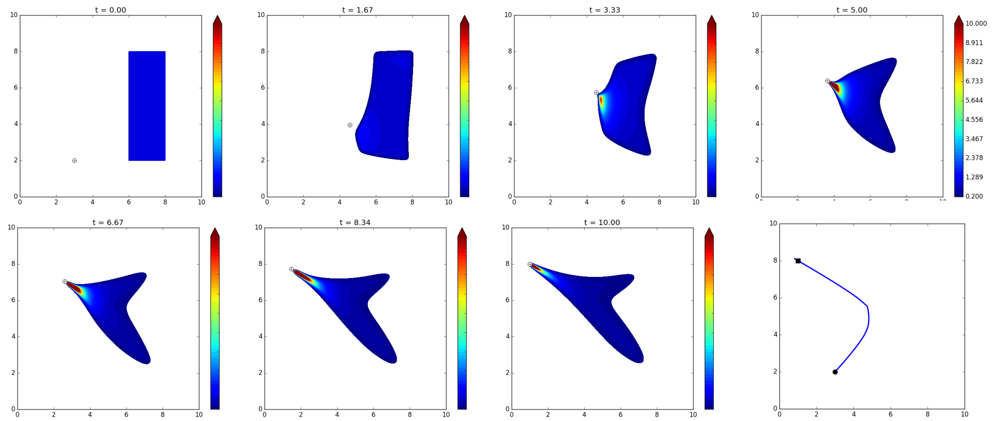

FIG. 3.1. *Numerical integration of (1.3) with the strategy (2.7) and the parameters (3.2). The first 7 figures depict the contour plots of the solution ρ and the position of P_1, the bottom right diagram displays the trajectory of P_1, whose initial position is $(3,2)$, drawn as a black circle. Note that the leader first moves to the right and then turns to the left.*

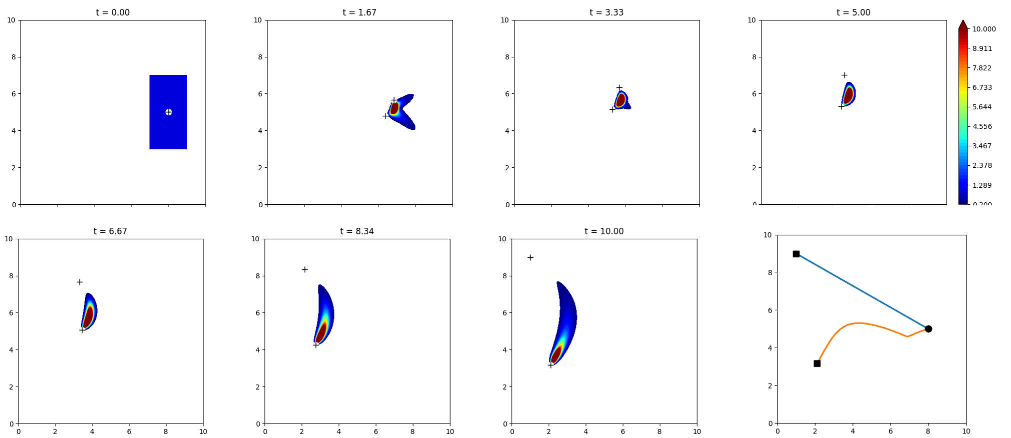

FIG. 3.2. *Numerical integration of (1.3)–(3.3) with two players. P_1 is assigned strategy (3.4), while P_2 uses (2.7) with $\Delta t = 0{,}01$. The first 7 figures depict the contour plots of the solution ρ, the bottom right diagram displays the trajectories of P_1 and P_2, whose initial positions are as in (3.3). The resulting cost of P_2 is lower than that of P_1, see Table 3.1.*

The agent P_1 is assigned the rectilinear trajectory with constant velocity

$$P_1(t) = \begin{bmatrix} 8 \\ 5 \end{bmatrix} + \begin{bmatrix} -0.7 \\ 0.4 \end{bmatrix} t, \qquad \text{corresponding to} \qquad u_1(t) = \begin{bmatrix} -0.7 \\ 0.4 \end{bmatrix}, \tag{3.4}$$

towards its target located at the point $(1,9)$.

The player P_2 is assigned the strategy u_2 by means of (2.7), with $\Delta t = 0.01$. The result, obtained on a grid of $n_x \times n_y = 3000 \times 3000$ cells, is shown in Figure 3.2: strategy (2.7) leads to the victory of P_2. Here, P_2 first moves slightly up, superimposing its attraction to that of P_1. Then, it bends downwards towards $\mathcal{T}_2$, see the last picture in Figure 3.2. As a result, P_2 attracts more individuals than P_1 and wins the game, see the costs in the first line of Table 3.1.

For completeness, we compare the situation described above with that of P_1 playing alone, P_2 being absent. Note the significant variation in the cost $\mathcal{J}_1$ of P_1 due to P_2 entering the game, see the costs in the second line of Table 3.1.

As a further test, we consider the case of both players using strategy (2.7). The corresponding result, see the last line in Table 3.1, confirms that if the two players have the same effect on the individuals, if the initial configuration is symmetric and if both players use strategy (2.7), then the players break even.

Strategy of P_1	Strategy of P_2	Cost $\mathcal{J}_1$ of P_1	Cost $\mathcal{J}_2$ of P_2
(3.4)	(2.7)	40.2	29.2
(3.4)	(absent)	19.9	//
(2.7)	(2.7)	33.2	33.2

TABLE 3.1. *Values of the costs $\mathcal{J}_1$ and $\mathcal{J}_2$ resulting from (1.3)–(3.3) with different strategies. The first line shows that strategy (2.7) wins against (3.4). On the second line, P_1 plays alone. The third line correctly shows that, in a symmetric situation, if both players use strategy (2.7), then the result is even.*

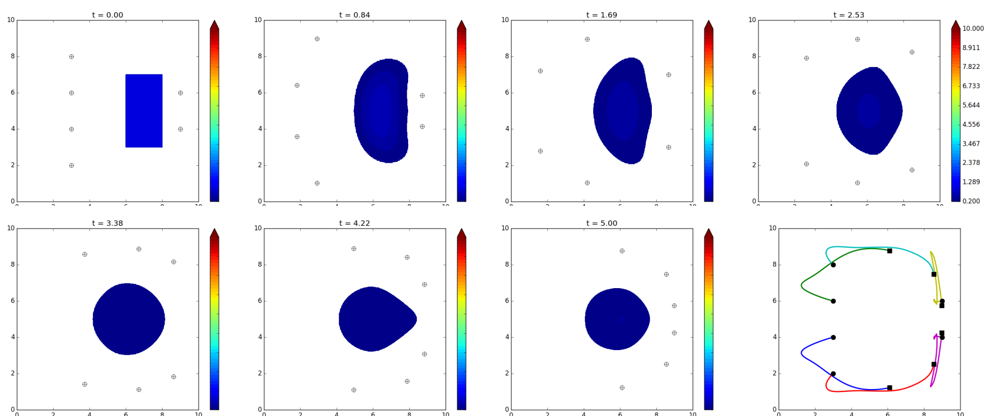

FIG. 3.3. *Integration of (2.3) with parameters (3.5). The 6 players are assigned the same target and automatically cooperate. After time $T = 5$, a portion of the individuals escapes the numerical domain, distorting the computation of the cost.*

3.3. Automatic cooperation among repulsive agents. The strategy introduced in Section 2 fosters a sort of automatic *cooperation* among agents having the same goal. Consider (2.3) with cost (1.2) and parameters, where $i = 1, \ldots, 6$,

$$\begin{array}{cccc} N = 2, & a_i(\xi) = -\frac{1}{0.1+\xi} e^{-0.2\xi^2}, & \bar\rho = \chi_{[6,8]\times[3,7]}, & \bar P_4 = (3,8), \\ k = 6, & v(t,x,P) = (3.1), & \bar P_1 = (3,2), & \bar P_5 = (9,4), \\ m = 12, & U = 1.5, & \bar P_2 = (3,4), & \bar P_6 = (9,6), \\ T = 5, & & \bar P_3 = (3,6), & \mathcal{T}_i = \{(5,5)\}. \end{array} \tag{3.5}$$

Then, the application of the strategy defined in Section 2, with $\Delta t = 0.01$, automatically results in an apparently effective team play, see Figure 3.3.

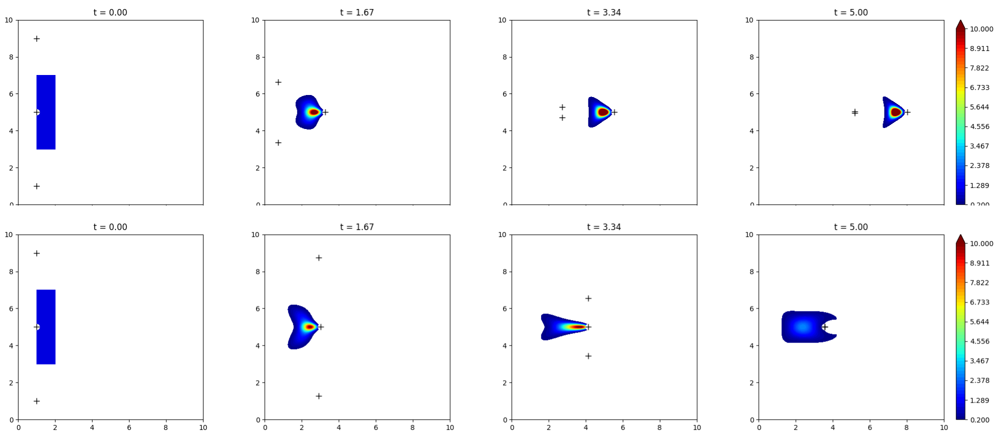

FIG. 3.4. *Upper line, integration of* (2.3) *with parameters* (3.6) *and with the same cost for all players* $\psi_1(x)=\psi_2(x)=\psi_3(x)=d(x,\mathcal{T}_1)$. *In the lower line, we set* $\psi_1=\psi_3=-\psi_2$ *as in* (3.7). *As a result,* P_1 *and* P_3 steal *most of the followers to* P_2. *In both cases,* P_1 *and* P_3 *are repulsive, while* P_2 *is attracting. On the first line,* P_2 *and* P_3 *end up superimposing to each other while, on the second line, all* 3 *controllers are superimposed at the final time.*

This integration is computed on the numerical domain $\Omega=[0,10]\times[0,10]$ through a grid of $n_x\times n_y=3000\times 3000$ cells. The resulting final cost, common to all players, is 15.5.

We stress that here we use the terms *"automatic cooperation"* with a naive, though meaningful, content. Indeed, the different players do not superimpose and their positions evolve with an apparently combined motion resulting in a effective confinement of the individuals. Remark that each controller moves according to the strategy outlined in Section 2, without any knowledge of each other's future trajectory.

3.4. Competition/cooperation among attractive/repulsive agents. Finally, the following integrations of (2.3) show first that cooperation arises also between attractive and repulsive agents. Then, it emphasizes the clear difference between cooperation and competition. Consider first the case

$$\begin{array}{rrll} N=2, & a_1(\xi)=a_3(\xi)=-\frac{1}{0.1+\xi}\,e^{-0.2\xi^2}, & \bar\rho=\chi_{[1,2]\times[3,7]}, & \\ k=3, & a_2(\xi)=\frac{1}{0.1+\xi}\,e^{-0.2\xi^2}, & \bar P_1=(1,1), & \mathcal{T}_1=\{(9,5)\}, \\ m=6, & v(t,x,P)=(3.1), & \bar P_2=(1,5), & \mathcal{T}_2=\{(9,5)\}, \\ T=5, & U=1.5, & \bar P_3=(1,9), & \mathcal{T}_3=\{(9,5)\}. \end{array} \tag{3.6}$$

whose solution is depicted in Figure 3.4, first line.

The final cost is 7.0, the density ρ being highly concentrated near to the target $\mathcal{T}_1$. Then, we keep the same parameters, but modify the costs of P_1 and P_3 setting

$$\psi_1(x)=\psi_3(x)=-d(x,\mathcal{T}_1) \quad \text{and} \quad \psi_2(x)=d(x,\mathcal{T}_1). \tag{3.7}$$

The resulting evolution is in Figure 3.4, second line. Note that P_1 and P_3 now follow a quite different trajectory, *"cutting"* the density ρ so that the final cost of P_2 raises to 26.0. In both integrations, the mesh consists of $n_x\times n_y=3000\times 3000$ points and $\Delta t=0.01$.

4. Analytic proofs

Throughout, for $a,b\in\mathbb{R}$, denote $\langle a,b\rangle=[\min\{a,b\},\max\{a,b\}]$. By $\mathcal{L}^N$ we mean the Lebesgue measure in $\mathbb{R}^N$. In $\mathbb{R}$, $|\cdot|$ is the absolute value, while $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$. The norm in the functional space $\mathcal{F}$ is denoted $\|\cdot\|_{\mathcal{F}}$. The space $\mathbf{C^0}(A;\mathbb{R}^n)$ of the $\mathbb{R}^n$-valued functions defined on the subset A of $\mathbb{R}^m$ is equipped with $\|f\|_{\mathbf{C^0}(A;\mathbb{R}^n)}=\sup_{x\in A}\|f(x)\|$. Throughout, $\mathrm{TV}(\cdot)$ stands for the total variation, see [14, Chapter 5]. For a measurable function ρ defined on $\mathbb{R}^N$, $\operatorname{spt}\rho$ is its support, see [5, Proposition 4.17]. Throughout, the continuous dependence of v on t, as required in ***(v)***, can be easily relaxed to mere measurability.

The following result on ordinary differential equations deserves being recalled.

LEMMA 4.1 ([4, Chapter 3]). *Let $V_1,V_2\in\mathbf{C^0}([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$ be such that the maps $x\to V_i(t,x)$ are in $\mathbf{C^{0,1}}(\mathbb{R}^N;\mathbb{R}^N)$ for $i=1,2$ and for all $t\in[0,T]$. Then, for all $(\bar t,\bar x)\in[0,T]\times\mathbb{R}^N$ and $i\in\{1,2\}$, the Cauchy Problem*

$$\begin{cases}\dot x=V_i(t,x)\\ x(\bar t)=\bar x\end{cases}\tag{4.1}$$

admits, on the interval $[0,T]$, the unique solution $t\to X_i(t;\bar t,\bar x)$ and the following estimate holds, for all $t\in[0,T]$:

$$\begin{aligned}\|X_1(t;\bar t,\bar x)-X_2(t;\bar t,\bar x)\|\le{}&\|V_1-V_2\|_{\mathbf{L^1}(\langle\bar t,t\rangle;\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))}\\ &\times\exp\left(\|D_xV_2\|_{\mathbf{L}^\infty(\langle\bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^{N\times N})}|t-\bar t|\right).\end{aligned}\tag{4.2}$$

If moreover $x\to V_i(t,x)\in\mathbf{C^1}(\mathbb{R}^N;\mathbb{R}^N)$ for all $t\in[0,T]$, the map $x\to X_i(t;\bar t,\bar x)$ is differentiable and its derivative $t\to D_xX_i(t;\bar t,\bar x)$ solves the linear matrix ordinary differential equation

$$\begin{cases}\dot Y=D_xV_i(t,X_i(t;\bar t,\bar x))\,Y\\ Y(\bar t)=\mathbf{Id}.\end{cases}\tag{4.3}$$

LEMMA 4.2. *Let $V\in\mathbf{C^0}([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$ be such that the map $x\to V(t,x)$ is in $\mathbf{C^1}(\mathbb{R}^N;\mathbb{R}^N)$ for $i=1,2$ and for all $t\in[0,T]$. Then, for all $\bar t\in[0,T[$, $i\in\{1,2\}$, and $\bar\rho\in\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$, the Cauchy Problem*

$$\begin{cases}\partial_t\rho+\operatorname{div}_x(\rho V(t,x))=0\\ \rho(\bar t,x)=\bar\rho(x)\end{cases}\tag{4.4}$$

admits, on the interval $[\bar t,T]$, the unique Kružkov solution

$$\rho(t,x)=\bar\rho(X(\bar t;t,x))\exp\left(-\int_{\bar t}^t\operatorname{div}_xV(\tau,X(\tau;t,x))\,\mathrm{d}\tau\right)\tag{4.5}$$

and if $\operatorname{spt}\bar\rho$ is bounded, then

$$\operatorname{spt}\rho(t)\subseteq B\left(\operatorname{spt}\rho(\bar t),\|V\|_{\mathbf{L}^\infty(\langle\bar t,t\rangle\times\operatorname{spt}\bar\rho;\mathbb{R}^N)}|t-\bar t|\,e^{\|D_xV\|_{\mathbf{L}^\infty(\langle\bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^{N\times N})}|t-\bar t|}\right).\tag{4.6}$$

The fact that (4.5) solves (4.4) in Kružkov sense follows from [9, Lemma 5.1], see also [12, Lemma 2.7]. The bound (4.6) is a direct consequence of [12, Proposition 2.8].

LEMMA 4.3. *Let $V_1,V_2\in\mathbf{C^0}([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$ be such that both maps $x\to V_i(t,x)$, $i=1,2$, are in $\mathbf{C^{1,1}}([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$. If $\bar\rho\in\mathbf{C^{0,1}}(\mathbb{R}^N;\mathbb{R})$, then*

$$\|\rho_1(t)-\rho_2(t)\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}$$

$$
\begin{aligned}
&\leq \|\mathrm{grad}_x \bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \mathcal{L}^N\left(\mathrm{spt}\,\bar\rho, Ce^{C|t-\bar t|}|t-\bar t|\right) e^{2C|t-\bar t|} \|V_1 - V_2\|_{\mathbf{L}^1(\langle \bar t, t\rangle;\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))}\\
&\quad + \left(\|\mathrm{div}_x (V_1 - V_2)\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R})} + C\|V_1-V_2\|_{\mathbf{L}^1(\langle \bar t, t\rangle;\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))}\right)\\
&\qquad \times \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} e^{2C|t-\bar t|}|t-\bar t|,
\end{aligned}
$$

where

$$
C = \max_{i=1,2} \left\{ \begin{array}{l} \|V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^N)} \\ \|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^{N\times N})} \\ \|\mathrm{grad}_x \,\mathrm{div}_x V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^N)} \end{array} \right\}. \tag{4.7}
$$

Proof. Using (4.5) and the triangle inequality, we have

$$
\|\rho_1(t) - \rho_2(t)\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \leq (I) + (II) + (III)
$$

where

$$
\begin{aligned}
(I) &= \int_{\mathbb{R}^N} |\bar\rho(X_1(\bar t;t,x)) - \bar\rho(X_2(\bar t;t,x))| \exp\left|\int_{\bar t}^t \mathrm{div}_x V_1(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right| \mathrm{d}x\\
(II) &= \int_{\mathbb{R}^N} \bar\rho(X_2(\bar t;t,x))\\
&\quad \times \left|\exp\left[-\int_{\bar t}^t \mathrm{div}_x V_1(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right] - \exp\left[-\int_{\bar t}^t \mathrm{div}_x V_2(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right]\right| \mathrm{d}x\\
(III) &= \int_{\mathbb{R}^N} \bar\rho(X_2(\bar t;t,x))\\
&\quad \times \left|\exp\left[-\int_{\bar t}^t \mathrm{div}_x V_2(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right] - \exp\left[-\int_{\bar t}^t \mathrm{div}_x V_2(\tau, X_2(\tau;t,x))\,\mathrm{d}\tau\right]\right| \mathrm{d}x
\end{aligned}
$$

and we now bound the three terms separately. To estimate (I), observe that by (4.6)

$$
\bigcup_{i=1}^2 \mathrm{spt}\,\rho_i(t) \subseteq B\left(\mathrm{spt}\,\bar\rho, \max_{i=1,2}\|V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times \mathrm{spt}\,\bar\rho;\mathbb{R}^N)} \exp\left(\|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^N)}|t-\bar t|\right)|t-\bar t|\right)
$$

and, using (4.2),

$$
\begin{aligned}
(I) &= \int_{\bigcup_{i=1}^2 \mathrm{spt}\,\rho_i(t)} |\bar\rho(X_1(\bar t;t,x)) - \bar\rho(X_2(\bar t;t,x))| \exp\left|\int_{\bar t}^t \mathrm{div}_x V_1(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right| \mathrm{d}x\\
&\leq \int_{\bigcup_{i=1}^2 \mathrm{spt}\,\rho_i(t)} \|\mathrm{grad}_x\bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \|X_1(\bar t;t,x) - X_2(\bar t;t,x)\|\\
&\qquad\qquad \times \exp\left(\|D_x V_1\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^N)}|t-\bar t|\right)\mathrm{d}x\\
&\leq \|\mathrm{grad}_x\bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)}\\
&\quad \times \mathcal{L}^N\left(\mathrm{spt}\,\bar\rho, \max_{i=1,2}\|V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathrm{spt}\,\bar\rho;\mathbb{R}^N)} \exp\left(\|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^{N\times N})}|t-\bar t|\right)|t-\bar t|\right)\\
&\quad \times \|V_1 - V_2\|_{\mathbf{L}^1(\langle \bar t,t\rangle;\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))}\\
&\quad \times \exp\left(\left(\|D_x V_1\|_{\mathbf{L}^\infty((\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^{N\times N})} + \|D_x V_2\|_{\mathbf{L}^\infty((\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R}^{N\times N})}\right)|t-\bar t|\right).
\end{aligned}
$$

Passing to the estimate of (II), using the inequality $\left|e^a - e^b\right| \leq e^{\max\{a,b\}} |a-b|$,

$$\begin{aligned}
&\left| \exp\left(-\int_{\bar t}^t \operatorname{div}_x V_1(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right) - \exp\left(-\int_{\bar t}^t \operatorname{div}_x V_2(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right)\right| \\
&\leq \exp\left(\max_{i=1,2} \|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \\
&\qquad \times \left| \int_{\bar t}^t \left|\operatorname{div}_x V_2(\tau, X_1(\tau;t,x)) - \operatorname{div}_x V_1(\tau, X_1(\tau;t,x))\right| \mathrm{d}\tau \right| \\
&\leq \exp\left(\max_{i=1,2} \|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \|\operatorname{div}_x V_1 - \operatorname{div}_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R})} |t-\bar t|
\end{aligned}$$

so that

$$\begin{aligned}
(II) \leq{}& \exp\left(\max_{i=1,2} \|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \|\operatorname{div}_x (V_1 - V_2)\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R})} \\
&\times \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} |t-\bar t|.
\end{aligned}$$

To bound (III), use (4.2) and proceed similarly:

$$\begin{aligned}
&\left| \exp\left(-\int_{\bar t}^t \operatorname{div}_x V_2(\tau, X_1(\tau;t,x))\,\mathrm{d}\tau\right) - \exp\left(-\int_{\bar t}^t \operatorname{div}_x V_2(\tau, X_2(\tau;t,x))\,\mathrm{d}\tau\right)\right| \\
&\leq \exp\left(\|D_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \\
&\qquad \times \left| \int_{\bar t}^t \left|\operatorname{div}_x V_2(\tau, X_1(\tau;t,x)) - \operatorname{div}_x V_2(\tau, X_2(\tau;t,x))\right| \mathrm{d}\tau \right| \\
&\leq \exp\left(2\|D_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \|\operatorname{grad}_x \operatorname{div}_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^N)} \\
&\qquad \times \|V_1 - V_2\|_{\mathbf{L}^1(\langle \bar t, t\rangle; \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))} |t-\bar t|
\end{aligned}$$

so that

$$\begin{aligned}
(III) \leq{}& \exp\left(2\|D_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \\
&\times \|\operatorname{grad}_x \operatorname{div}_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^N)} \|V_1 - V_2\|_{\mathbf{L}^1(\langle \bar t, t\rangle; \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))} |t-\bar t|.
\end{aligned}$$

Summing up the expressions obtained:

$$\begin{aligned}
&\|\rho_1(t) - \rho_2(t)\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \\
&\leq \|\operatorname{grad}_x \bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \|V_1 - V_2\|_{\mathbf{L}^1(\langle \bar t, t\rangle; \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))} \\
&\quad \times \mathcal{L}^N\left(\operatorname{spt}\bar\rho, \max_{i=1,2} \|V_i\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \operatorname{spt}\bar\rho)} \exp\left(\|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N)} |t-\bar t|\right) |t-\bar t|\right) \\
&\quad \times \exp\left(\left(\|D_x V_1\|_{\mathbf{L}^\infty((\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} + \|D_x V_2\|_{\mathbf{L}^\infty((\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})}\right) |t-\bar t|\right) \\
&\quad + \exp\left[\max_{i=1,2} \|D_x V_i\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N)} |t-\bar t|\right] \|\operatorname{div}_x (V_1 - V_2)\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N;\mathbb{R})} \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} |t-\bar t| \\
&\quad + \exp\left(2\|D_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^{N\times N})} |t-\bar t|\right) \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \|\operatorname{grad}_x \operatorname{div}_x V_2\|_{\mathbf{L}^\infty(\langle \bar t, t\rangle \times \mathbb{R}^N; \mathbb{R}^N)} \\
&\qquad \times \|V_1 - V_2\|_{\mathbf{L}^1(\langle \bar t, t\rangle; \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}))} |t-\bar t|.
\end{aligned}$$

Introduce C as in (4.7). Then,

$$
\begin{aligned}
& \|\rho_1(t)-\rho_2(t)\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \\
\le & \|\mathrm{grad}_x \bar\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^N)} \mathcal{L}^N\left(\mathrm{spt}\,\bar\rho, Ce^{C|t-\bar t|}|t-\bar t|\right) e^{2C|t-\bar t|}\|V_1-V_2\|_{\mathbf{L^1}(\langle \bar t,t\rangle;\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}))} \\
& +\|\bar\rho\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} e^{2C|t-\bar t|} \|\mathrm{div}_x(V_1-V_2)\|_{\mathbf{L^\infty}(\langle \bar t,t\rangle\times\mathbb{R}^N;\mathbb{R})} |t-\bar t| \\
& +\|\bar\rho\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} C\, e^{2C|t-\bar t|} \|V_1-V_2\|_{\mathbf{L^1}(\langle \bar t,t\rangle;\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}))} |t-\bar t|
\end{aligned}
$$

completing the proof. □

We now prove well posedness and basic estimates for (1.3) or, equivalently, (2.3).

Proposition 4.1. *Fix positive T and U. Let $v\in\mathbf{C^0}([0,T]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^N)$ be such that for all $t\in[0,T]$ and $P\in\mathbb{R}^m$, the map $x\to v(t,x,P)$ is in $\mathbf{C^{0,1}}(\mathbb{R}^N;\mathbb{R}^N)$. For any $\bar\rho\in\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$, $\bar P\in\mathbb{R}^m$ and $u\in\mathbf{L^\infty}([0,T];\overline{B_{\mathbb{R}^m}(0,U)})$, problem (2.3) admits the unique solution*

$$
\rho(t,x)=\bar\rho(X(0;t,x))\exp\left(-\int_0^t \mathrm{div}_x v(\tau,X(\tau;t,x),P(\tau))\,\mathrm{d}\tau\right)
$$

*where $t\to X(t;\bar t,\bar x)$ solves $\begin{cases}\dot x = v(t,x,P(t))\\ x(\bar t)=\bar x\end{cases}$ and $P(t)=\bar P+\int_0^t u(\tau)\,\mathrm{d}\tau$ for $t\in[0,T]$. Moreover, if v satisfies **(v)** and $u_1,u_2\in\mathbf{L^\infty}([0,T];\overline{B_{\mathbb{R}^m}(0,U)})$, then (with obvious notation) for all $t\in[0,T]$,*

$$
\|X_1(t;0,\bar x)-X_2(t;0,\bar x)\| \le Cte^{Ct}\|P_1-P_2\|_{\mathbf{C^0}([0,t];\mathbb{R}^m)} \tag{4.8}
$$

$$
\begin{aligned}
\|\rho_1(t)-\rho_2(t)\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \le C\Big(&\|\mathrm{grad}_x\bar\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^N)}\mathcal{L}^N\left(B(\mathrm{spt}\,\bar\rho, Cte^{Ct})\right) \\
&+\|\bar\rho\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}(1+Ct)\Big)te^{2Ct}\|P_1-P_2\|_{\mathbf{C^0}([0,t];\mathbb{R}^m)}
\end{aligned} \tag{4.9}
$$

where C is independent of the initial datum, more precisely:

$$
C=\max\left\{\begin{array}{ll}\|v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^N)}, & \|D_x v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^{N\times N})}, \\ \|D_P v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^{N\times m})}, & \|\mathrm{grad}_x\,\mathrm{div}_x v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^N)}\end{array}\right\}. \tag{4.10}
$$

Here, the term *"solution"* means Kružkov solution [20, Definition 1], which is also a strong solution as soon as $\bar\rho$ is smooth, see [26].

Proof. The first statement follows from Lemma 4.2. Define $V_i(t,x)=v(t,x,P_i(t))$, with $P_i(t)=\bar P+\int_0^t u_i(\tau)\,\mathrm{d}\tau$, for $i=1,2$. Then, direct computations yield:

$$
\begin{aligned}
\|V_i\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N;\mathbb{R}^N)} &\le \|v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^N)}. \\
\|D_xV_i\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N;\mathbb{R}^{N\times N})} &= \|D_x v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^{N\times N})}. \\
\|\mathrm{grad}_x\,\mathrm{div}_x V_i\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N;\mathbb{R}^N)} &\le \|\mathrm{grad}_x\,\mathrm{div}_x v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^N)}. \\
\|V_1-V_2\|_{\mathbf{L^1}([0,t];\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}))} &= \int_0^t \sup_{x\in\mathbb{R}^N}\|v(\tau,x,P_1(\tau))-v(\tau,x,P_2(\tau))\|\,\mathrm{d}\tau \\
&\le \|D_P v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R}^{N\times m})}\, t\,\|P_1-P_2\|_{\mathbf{L^\infty}([0,t];\mathbb{R}^m)}. \\
\|\mathrm{div}_x(V_1-V_2)\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N;\mathbb{R})} &\le \|\mathrm{div}_x v\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R}^N\times\mathbb{R}^m;\mathbb{R})}\|P_1-P_2\|_{\mathbf{L^\infty}([0,t];\mathbb{R}^N)}.
\end{aligned}
$$

Now, (4.8) directly follows from (4.2) in Lemma 4.1. To prove (4.9) use Lemma 4.3. □

COROLLARY 4.1. *Fix positive T and U. Let v be bounded and satisfy* **(v)**, $\bar\rho \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$ *and* $\bar P \in \mathbb{R}^m$. *If* $u_n, u_* \in \mathbf{L}^\infty([0,T];\overline{B_{\mathbb{R}^m}(0,U)})$ *are such that* $u_n \overset{\star}{\rightharpoonup} u_*$ *in* $\mathbf{L}^\infty([0,T];\mathbb{R}^m)$ *as* $n \to +\infty$, *then, up to subsequences,*

$$P_n \to P_* \text{ in } \mathbf{C^0}([0,T];\mathbb{R}^N) \quad \text{and} \quad \begin{array}{ll} \rho_n(t) \to \rho_*(t) & \text{in } \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \text{ for all } t \in [0,T], \\ \rho_n \to \rho_* & \text{in } \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\right). \end{array}$$

If $\bar\rho \in \mathbf{C^1}(\mathbb{R}^N;\mathbb{R})$, *then*

$$\begin{array}{ll} \operatorname{grad}_x \rho_n(t) \to \operatorname{grad}_x \rho_*(t) & \text{in } \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \text{ for all } t \in [0,T], \\ \operatorname{grad}_x \rho_n \to \operatorname{grad}_x \rho_* & \text{in } \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\right). \end{array}$$

The poof is a straightforward consequence of (4.8) and (4.9) in Proposition 4.1. We now verify that (2.4) is well posed.

LEMMA 4.4. *Fix positive T, U, and* $\Delta t \in]0,T[$. *Let* $v \in \mathbf{C^{0,1}}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)$. *For any* $\bar\rho \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$, $\bar P \in \mathbb{R}^N$, $u \in \mathbf{L}^\infty([0,T];\overline{B(0,U)})$, $t \in [0, T-\Delta t[$ *and* $w \in \mathbb{R}^N$, *problem* (2.4) *admits a unique solution given by*

$$\rho_w(\tau,x) = \rho(t, X_{t,w}(t;\tau,x)) \exp\left(-\int_t^\tau \operatorname{div}_x v(t, X_{t,w}(s;\tau,x), P(t) + (s-t)w)\,\mathrm{d}s\right) \tag{4.11}$$

where

$$\tau \to X_{t,w}(\tau;\bar t,x) \quad \text{solves} \quad \begin{cases} \xi' = v(t,\xi,P(t)+(\tau-t)w) \\ \xi(\bar t) = x \end{cases} \text{for } \bar t,\tau \in [t,t+\Delta t]. \tag{4.12}$$

Moreover, if $\bar\rho \in \mathbf{C^{1,1}}(\mathbb{R}^N;\mathbb{R})$ *and* $\mathcal{L}^N(\operatorname{spt}\bar\rho) < +\infty$, *for all* $w_1, w_2 \in \mathbb{R}^N$

$$\begin{aligned} &\|\rho_{w_1}(t+\Delta t) - \rho_{w_2}(t+\Delta t)\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \\ &\le \left(\|\operatorname{grad}_x \bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \mathcal{L}^N\left(\operatorname{spt}\bar\rho, Ce^{C\Delta t}\Delta t\right) + (1+C\Delta t)\|\bar\rho\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}\right) \\ &\quad \times Ce^{2C\Delta t}(\Delta t)^2\|w_1 - w_2\| \end{aligned} \tag{4.13}$$

where

$$C = \max\left\{ \begin{array}{l} \|v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times B(P(t),U\Delta t);\mathbb{R}^N)}, \|D_x v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times B(P(t),U\Delta t);\mathbb{R}^{N\times N})}, \\ \frac{1}{2}\|D_P v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^{N\times N})}, \|D_P \operatorname{div}_x v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)} \end{array} \right\}. \tag{4.14}$$

(Above and in the sequel, $\xi' = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}$).

Proof. The first statement is a direct consequence of Lemma 4.2. To prove (4.13), we apply Lemma 4.3 with $V(\tau,x) = v(t,x,P(t)+(\tau-t)w)$ for $\tau \in [t,t+\Delta t]$:

$$\begin{aligned} \|V_i\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N;\mathbb{R}^N)} &\le \|v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times B(P(t),\Delta t\|w_i\|);\mathbb{R}^N)}, \\ \|D_x V_i\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N;\mathbb{R}^{N\times N})} &\le \|D_x v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times B(P(t),\Delta t\|w_i\|),\mathbb{R}^{N\times N})}, \\ \|V_1 - V_2\|_{\mathbf{L^1}([t.t+\Delta t],\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N))} &\le \frac{1}{2}\|D_P v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^{N\times N})}(\Delta t)^2\|w_1 - w_2\|, \\ \|\operatorname{div}_x(V_1 - V_2)\|_{\mathbf{L}^\infty([t.t+\Delta t]\times\mathbb{R}^N;\mathbb{R}^N)} &\le \|D_P \operatorname{div}_x v\|_{\mathbf{L}^\infty([t,t+\Delta t]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)}(\Delta t)\|w_1 - w_2\|. \end{aligned}$$

With the notation (4.14), and assuming that $C \ge 1$,

$$\|\rho_1(t+\Delta t) - \rho_2(t+\Delta t)\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}$$

$$\begin{aligned}&\le \|\mathrm{grad}_x \bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \mathcal{L}^N\left(\mathrm{spt}\,\bar\rho, Ce^{C\Delta t}\Delta t\right) e^{2C\Delta t} C(\Delta t)^2 \|w_1-w_2\|\\ &\quad + \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} e^{2C\Delta t} C(\Delta t)^2 \|w_1-w_2\| + \|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} C^2 e^{2C\Delta t} (\Delta t)^3 \|w_1-w_2\|\\ &\le \left(\|\mathrm{grad}_x \bar\rho\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \mathcal{L}^N\left(\mathrm{spt}\,\bar\rho, Ce^{C\Delta t}\Delta t\right) + (1+C\Delta t)\|\bar\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}\right)\\ &\qquad \times Ce^{2C\Delta t}(\Delta t)^2 \|w_1-w_2\|\end{aligned}$$

completing the proof. ∎

For the reader's convenience, we recall here, without proof, the following result.

LEMMA 4.5 ([27, Lemma 1.15]). *Let X,Y be Banach spaces, $A \subseteq X$ be open, $x_o \in A$ and $J\colon A \to Y$ be Gâteaux differentiable at all $x \in A$ in all directions $v \in X$. Assume that*

(1) the map $v \to D_v J(x)$ is linear and continuous for all $x \in A$;

(2) $\lim_{x\to x_o} \sup_{\|v\|_X=1} \|D_v J(x) - D_v J(x_o)\|_Y = 0$.

Then, J is Fréchet differentiable at x_o.

The next result describes the Fréchet differentiability of the characteristic curves.

LEMMA 4.6. *Fix $t \in [0,T[$, $\Delta t \in]0, T-t]$ and $x \in \mathbb{R}^N$. If $v \in \mathbf{C^2}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)$, then the map*

$$\begin{array}{rccc}\mathcal{X}_{t,x} : & \mathbb{R}^N & \to & \mathbf{C^0}([t,t+\Delta t];\mathbb{R}^N)\\ & w & \to & \mathcal{X}_{t,x}(w),\end{array}$$

defined so that $\tau \to (\mathcal{X}_{t,x}(w))(\tau)$ solves the Cauchy problem

$$\begin{cases}\xi' = v(t,\xi,P(t)+(\tau-t)w)\\ \xi(t) = x,\end{cases} \tag{4.15}$$

is Fréchet differentiable in $\mathbb{R}^N$. Moreover $\mathcal{X}_{t,x}$ has the Taylor expansion

$$\mathcal{X}_{t,x}(w+\delta_w) = \mathcal{X}_{t,x}(w) + D\mathcal{X}_{t,x}(w)\,\delta_w + o(\delta_w) \quad \text{in } \mathbf{C^0} \text{ as } \delta_w \to 0$$

where $\tau \to (D\mathcal{X}_{t,x}(w))(\tau)$ solves the linear first order $N\times N$ matrix differential equation

$$\begin{cases}Y' = D_x v(t, \mathcal{X}_{t,x}(w)(\tau), P(t)+(\tau-t)w)\,Y\\ \qquad\qquad + (\tau-t)\,D_P v(t, \mathcal{X}_{t,x}(w)(\tau), P(t)+(\tau-t)w)\\ Y(t) = 0\end{cases} \tag{4.16}$$

and the term $D\mathcal{X}_{t,x}(w)$ satisfies the expansion, as $\tau \to t$,

$$(D\mathcal{X}_{t,x}(w))(\tau) = \frac{(\tau-t)^2}{2} D_P v(t,x,P(t)) + o(\tau-t)^2. \tag{4.17}$$

Proof. Since t and x are kept fixed throughout this proof, we write $\mathcal{X}(w)$ for $\mathcal{X}_{t,x}(w)$. Recall that, for $\tau \in [t, t+\Delta t]$,

$$\mathcal{X}(w)(\tau) = x + \int_t^\tau v(t, \mathcal{X}(w)(s), P(t)+(s-t)w)\,\mathrm{d}s.$$

Fix a direction $\delta_w \in \mathbb{R}^N \setminus \{0\}$. First we show the boundedness of the difference quotient

$$\frac{\|\mathcal{X}(w+\varepsilon\delta_w)(\tau) - \mathcal{X}(w)(\tau)\|}{\varepsilon}.$$

For $\tau \in [t, t+\Delta t]$, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\|\mathcal{X}(w+\varepsilon\delta_w)(\tau)-\mathcal{X}(w)(\tau)\| \\
\leq &\frac{1}{\varepsilon}\int_t^\tau \|v(t,\mathcal{X}(w+\varepsilon\delta_w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))-v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\|\mathrm{d}s \\
\leq &\frac{1}{\varepsilon}\int_t^\tau \Big\|v(t,\mathcal{X}(w+\varepsilon\delta_w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))-v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))\Big\|\mathrm{d}s \\
&+\frac{1}{\varepsilon}\int_t^\tau \|v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))-v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\|\mathrm{d}s \\
\leq &\|v\|_{\mathbf{C^1}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)}\left[\int_t^\tau \frac{\|\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)\|}{\varepsilon}\mathrm{d}s+\int_t^\tau (s-t)\|\delta_w\|\mathrm{d}s\right] \\
\leq &\|v\|_{\mathbf{C^1}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)}\left[\int_t^\tau \frac{\|\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)\|}{\varepsilon}\mathrm{d}s+(\Delta t)^2\|\delta_w\|\right].
\end{aligned}
$$

Hence an application of Grönwall Lemma, see, e.g., [4, Chapter 3, Lemma 3.1] ensures that

$$
\frac{\|\mathcal{X}(w+\varepsilon\delta_w)(\tau)-\mathcal{X}(w)(\tau)\|}{\varepsilon} \leq K_1(\Delta t)^3\|\delta_w\|\exp(K_1\Delta t), \tag{4.18}
$$

where $K_1 = \|v\|_{\mathbf{C^1}([0,T]\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^N)}$. Consequently

$$
\lim_{\varepsilon\to 0}\sup_{\tau\in[t,t+\Delta t]}\|\mathcal{X}(w+\varepsilon\delta_w)(\tau)-\mathcal{X}(w)(\tau)\| = 0. \tag{4.19}
$$

We now prove the existence of directional derivatives of $\mathcal{X}$ along the direction $\delta_w \in \mathbb{R}^N\setminus\{0\}$. Calling $\tau \to Y(\tau)$ the solution to the Cauchy problem (4.16), we have

$$
\begin{aligned}
&\frac{\mathcal{X}(w+\varepsilon\delta_w)(\tau)-\mathcal{X}(w)(\tau)}{\varepsilon}-Y(\tau)\,\delta_w \\
=&\frac{1}{\varepsilon}\int_t^\tau [v(t,\mathcal{X}(w+\varepsilon\delta_w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))-v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)]\mathrm{d}s \\
&-\int_t^\tau D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,Y(s)\,\mathrm{d}s\,\delta_w \\
&-\int_t^\tau (s-t)D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,\mathrm{d}s\,\delta_w \\
=&\frac{1}{\varepsilon}\int_t^\tau \big[v(t,\mathcal{X}(w+\varepsilon\delta_w)(s),P(t)+(s-t)(w+\varepsilon\delta_w)) \\
&\qquad\qquad -v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+\delta_w))\big]\,\mathrm{d}s \\
&-\int_t^\tau D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,Y(s)\,\mathrm{d}s\,\delta_w \\
&+\frac{1}{\varepsilon}\int_t^\tau [v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))-v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)]\mathrm{d}s \\
&-\int_t^\tau (s-t)D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,\mathrm{d}s\,\delta_w \\
=&\int_t^\tau\int_0^1\Big(D_x v(t,\vartheta\mathcal{X}(w+\varepsilon\delta_w)(s)+(1-\vartheta)\mathcal{X}(w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))\,\mathrm{d}\vartheta
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times \frac{\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)}{\varepsilon}\Big)\,\mathrm{d}s \\
&\quad -\int_t^\tau D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,Y(s)\,\mathrm{d}s\,\delta_w \\
&\quad +\int_t^\tau (s-t)\Big(\int_0^1 D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+(1-\vartheta)\varepsilon\delta_w))\,\mathrm{d}\vartheta \\
&\qquad\qquad -D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\Big)\,\mathrm{d}s\,\delta_w \\
&=\int_t^\tau \int_0^1 \Big(D_x v(t,\vartheta\mathcal{X}(w+\varepsilon\delta_w)(s)+(1-\vartheta)\mathcal{X}(w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))\,\mathrm{d}\vartheta \\
&\qquad\qquad \times \frac{\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)}{\varepsilon}\Big)\,\mathrm{d}s \\
&\quad \mp\int_t^\tau D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,\frac{\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)}{\varepsilon}\,\mathrm{d}s \\
&\quad -\int_t^\tau D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\,Y(s)\,\mathrm{d}s\,\delta_w \\
&\quad +\int_t^\tau (s-t)\Big(\int_0^1 D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+(1-\vartheta)\varepsilon\delta_w))\,\mathrm{d}\vartheta \\
&\qquad\qquad -D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\Big)\,\mathrm{d}s\,\delta_w \\
&=\int_t^\tau \int_0^1 \Big(D_x v(t,\vartheta\mathcal{X}(w+\varepsilon\delta_w)(s)+(1-\vartheta)\mathcal{X}(w)(s),P(t)+(s-t)(w+\varepsilon\delta_w))\,\mathrm{d}\vartheta \\
&\qquad\qquad -D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\Big)\,\frac{\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)}{\varepsilon}\,\mathrm{d}s \\
&\quad +\int_t^\tau D_x v(t,\mathcal{X}(w)(s),P(t)+(s-t)w) \\
&\qquad\qquad \times\left(\frac{\mathcal{X}(w+\varepsilon\delta_w)(s)-\mathcal{X}(w)(s)}{\varepsilon}-Y(s)\delta_w\right)\mathrm{d}s \\
&\quad +\int_t^\tau (s-t)\Big(\int_0^1 D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)(w+(1-\vartheta)\varepsilon\delta_w))\,\mathrm{d}\vartheta \\
&\qquad\qquad -D_P v(t,\mathcal{X}(w)(s),P(t)+(s-t)w)\Big)\,\mathrm{d}s\,\delta_w .
\end{aligned}
$$

Calling $\mathcal{O}(1)$ a constant dependent on the $\mathbf{C^2}$ norm of v and on the right-hand side of (4.18), the above equality leads to

$$
\begin{aligned}
&\left\|\frac{\mathcal{X}(w+\varepsilon\delta_w)(\tau)-\mathcal{X}(w)(\tau)}{\varepsilon}-Y(\tau)\,\delta_w\right\| \\
&\le \mathcal{O}(1)+\int_t^\tau \mathcal{O}(1)\left\|\frac{\mathcal{X}(w+\varepsilon\delta_w)(\tau)-\mathcal{X}(w)(\tau)}{\varepsilon}-Y(\tau)\,\delta_w\right\|\mathrm{d}s \\
&\quad +\int_t^\tau \mathcal{O}(1)\,(s-t)\varepsilon\,\mathrm{d}s\,\delta_w .
\end{aligned}
$$

Thanks to (4.19), an application of Grönwall Lemma proves the directional differentiability of $w\to\mathcal{X}(w)$ in the direction δ_w.

To prove the differentiability of $\mathcal{X}$, we are left to verify that (1) and (2) in Lemma 4.5 hold. The linearity of $\delta_w\to D\mathcal{X}(w)(\delta_w)$ is immediate, thanks to the homogeneous initial datum in (4.16). The assumed $\mathbf{C^2}$ regularity of v ensures the $\mathbf{C^1}$ regularity of the right-hand side in (4.16) and, hence, the boundedness of $\delta_w \to D\mathcal{X}(w)(\delta_w)$ (in the sense of linear operators), completing the proof of 1. Standard theorems on the continuous dependence of solutions to ordinary differential equations on parameters, see [4, Theorem 4.2], ensure that also 2. in Lemma 4.5 holds, completing the proof of the differentiability of $\mathcal{X}$.

The proof of the Taylor expansion (4.17) follows easily using (4.16). Indeed, by (4.16), we deduce that $Y(t) = Y'(t) = 0$, while $Y''(t) = D_P v(t,x,P(t))$, so that, if $\tau \in [t, t+\Delta t]$, then

$$Y(\tau) = \frac{(\tau - t)^2}{2} D_P v(t,x,P(t)) + o\left((\tau - t)^2\right).$$

This completes the proof of (4.17) and of the lemma. ∎

Proof. **(Proof of Theorem 2.1.)** The map $\mathcal{J}_{t,\Delta t}$ is well defined by Lemma 4.4. To prove its Lipschitz continuity, let $w_1, w_2 \in \mathbb{R}^N$. Denote $V_i(\tau,x) = v(t,x,P(t) + (\tau - t)w_i)$; $X_i = X_{t,w_i}$ the solution to (4.1) and $\rho_i = \rho_{w_i}$ the corresponding solution to (4.4). Straightforward computations yield

$$\begin{aligned} |\mathcal{J}_{t,\Delta t}(w_1) - \mathcal{J}_{t,\Delta t}(w_2)| &\le \int_{\mathbb{R}^n} |\rho_1(t+\Delta t,x) - \rho_2(t+\Delta t,x)| |\psi(x)| \,\mathrm{d}x \\ &\le \|\rho_1(t+\Delta t) - \rho_2(t+\Delta t)\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})}. \end{aligned}$$

and the proof of Lipschitzeanity is completed thanks to (4.13).

To prove (2.7), recall (2.4)–(4.11). Fix $t, t+\Delta t$ in $[0,T]$. The solution $\tau \to \mathcal{X}_w(\tau; t+\Delta t, x)$ to

$$\begin{cases} \xi' = v(t,\xi,P(t) + (\tau - t)w) \\ \xi(t+\Delta t) = x \end{cases} \qquad \tau \in [t, t+\Delta t] \tag{4.20}$$

will be shortened to $\tau \to \mathcal{X}_w(\tau;x)$. By Lemma 4.6, we have the expansion

$$\mathcal{X}_{w+\varepsilon\delta_w}(\tau;x) = \mathcal{X}_w(\tau;x) + \varepsilon D_w \mathcal{X}_w(\tau;x)\delta_w + o(\varepsilon) \quad \text{in } \mathbf{C^0} \text{ as } \varepsilon \to 0, \tag{4.21}$$

where $\tau \to D_w\mathcal{X}_w(\tau; t+\Delta t, x)$, or $\tau \to D_w\mathcal{X}_w(\tau;x)$ for short, solves the Cauchy Problem

$$\begin{cases} Y' = D_x v(t, \mathcal{X}_w(\tau;x), P(t) + (\tau - t)w)Y + (\tau - t) D_P v(t, \mathcal{X}_w(\tau;x), P(t) + (\tau - t)w) \\ Y(t+\Delta t) = 0 \end{cases} \tag{4.22}$$

for $\tau \in [t, t+\Delta t]$. With reference to (2.5), denote for simplicity $\mathcal{J} = \mathcal{J}_{t,\Delta t}$, $\psi = \psi$ and compute:

$$\begin{aligned} &\frac{1}{\varepsilon}(\mathcal{J}(w + \varepsilon\delta_w) - \mathcal{J}(w)) \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N} (\rho_{w+\varepsilon\delta_w}(t+\Delta t,x) - \rho_w(t+\Delta t,x))\psi(x)\,\mathrm{d}x \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N} \Bigg[\rho(t, \mathcal{X}_{w+\varepsilon\delta_w}(t;x)) \\ &\qquad\qquad \times \exp\left(-\int_t^{t+\Delta t} \operatorname{div}_x v(s, \mathcal{X}_{w+\varepsilon\delta_w}(s;x), P(t) + (s-t)(w+\varepsilon\delta_w))\,\mathrm{d}s\right) \end{aligned}$$

$$
\begin{aligned}
&\quad -\rho(t,\mathcal{X}_w(t;x)) \\
&\qquad \times \exp\left(-\int_t^{t+\Delta t} \operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right)\Big]\psi(x)\,\mathrm{d}x \\
&=(I)+(II)+(III)
\end{aligned}
$$

where

$$
\begin{aligned}
(I) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}[\rho(t,\mathcal{X}_{w+\varepsilon\delta_w}(t;x))-\rho(t,\mathcal{X}_w(t;x))] \\
&\quad \times \exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_{w+\varepsilon\delta_w}(s;x),P(t)+(s-t)(w+\varepsilon\delta_w))\,\mathrm{d}s\right)\psi(x)\,\mathrm{d}x \\
(II) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\rho(t,\mathcal{X}_w(t;x)) \\
&\quad \times \Big[\exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_{w+\varepsilon\delta_w}(s;x),P(t)+(s-t)(w+\varepsilon\delta_w))\,\mathrm{d}s\right) \\
&\qquad -\exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_{w+\varepsilon\delta_w}(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right)\Big]\psi(x)\,\mathrm{d}x \\
(III) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\rho(t,\mathcal{X}_w(t;x)) \\
&\quad \times \Big[\exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_{w+\varepsilon\delta_w}(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right) \\
&\qquad -\exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right)\Big]\psi(x)\,\mathrm{d}x.
\end{aligned}
$$

The following estimate uses $D_w\mathcal{X}_w$ as defined in (4.22) and is of use to compute (I):

$$
\begin{aligned}
&\frac{1}{\varepsilon}(\rho(t,\mathcal{X}_{w+\varepsilon\delta_w}(t;x))-\rho(t,\mathcal{X}_w(t;x))) \\
&= \int_0^1 \operatorname{grad}_x\rho(t,\vartheta\mathcal{X}_{w+\varepsilon\delta_w}(t;x)+(1-\vartheta)\mathcal{X}_w(t;x))\,\mathrm{d}\vartheta\;\frac{\mathcal{X}_{w+\varepsilon\delta_w}(t;x)-\mathcal{X}_w(t;x)}{\varepsilon} \\
&\overset{\varepsilon\to 0}{\to} \operatorname{grad}_x\rho(t,\mathcal{X}_w(t;x))\,D_w\mathcal{X}_w(t;x)\,\delta_w,
\end{aligned}
$$

so that,

$$
\begin{aligned}
(I) &\overset{\varepsilon\to 0}{\to} \int_{\mathbb{R}^N}\operatorname{grad}_x\rho(t,\mathcal{X}_w(t;x))\,D_w\mathcal{X}_w(t;x)\,\delta_w \\
&\qquad\qquad \times\exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right)\psi(x)\,\mathrm{d}x
\end{aligned}
$$

while

$$
\begin{aligned}
(II) &\overset{\varepsilon\to 0}{\to} -\int_{\mathbb{R}^N}\rho(t,\mathcal{X}_w(t;x))\exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right) \\
&\quad \times\int_t^{t+\Delta t}\operatorname{grad}_P\operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)(s-t)\,\mathrm{d}s\,\delta_w\psi(x)\,\mathrm{d}x
\end{aligned}
$$

and similarly, using $D_w\mathcal{X}_w$ defined as solution to (4.22),

$$(III) \overset{\varepsilon\to 0}{\to} -\int_{\mathbb{R}^n} \rho(t,\mathcal{X}_w(s;x)) \exp\left(-\int_t^{t+\Delta t} \operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right) \times \int_t^{t+\Delta t} \operatorname{grad}_x \operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\, D_w\mathcal{X}_w(s;x)\,\mathrm{d}s\,\psi(x)\,\mathrm{d}x\,\delta_w.$$

Adding the three terms we get:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathcal{J}(w+\varepsilon\delta_w)-\mathcal{J}(w)\right)\\
&=\int_{\mathbb{R}^N}\Big[\operatorname{grad}_x \rho(t,\mathcal{X}_w(t;x))\, D_w\mathcal{X}_w(t;x)\\
&\qquad -\rho(t,\mathcal{X}_w(t;x))\int_t^{t+\Delta t}\Big(\operatorname{grad}_P \operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,(s-t)\\
&\qquad\qquad +\operatorname{grad}_x \operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\, D_w\mathcal{X}_w(s;x)\Big)\,\mathrm{d}s\Big]\\
&\quad\times \exp\left(-\int_t^{t+\Delta t}\operatorname{div}_x v(s,\mathcal{X}_w(s;x),P(t)+(s-t)w)\,\mathrm{d}s\right)\psi(x)\,\mathrm{d}x\,\delta_w.
\end{aligned}$$

To compute the limit as $\Delta t\to 0$ of the expression above, recall that as $\Delta t\to 0$,

$$\begin{aligned}
\mathcal{X}_w(t;t+\Delta t,x) &= x - v(t,x,P(t))\,\Delta t + o(\Delta t) && \text{[by (4.20)]}\\
D_w\mathcal{X}_w(t;t+\Delta t,x) &= \tfrac{1}{2}\, D_P v(t,x,P(t))\,(\Delta t)^2 + o(\Delta t)^2 && \text{[by (4.17)]}
\end{aligned}$$

so that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathcal{J}(w+\varepsilon\delta_w)-\mathcal{J}(w)\right)\\
&=\frac{(\Delta t)^2}{2}\int_{\mathbb{R}^N}\left(\operatorname{grad}_x\rho(t,x)\, D_P v(t,x,P(t)) - \rho(t,x)\operatorname{grad}_P \operatorname{div}_x v(s,x,P(t))\right)\psi(x)\,\mathrm{d}x\,\delta_w\\
&\quad + o(\Delta t)^2
\end{aligned}$$

completing the proof. ∎

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