Commun. Math. Sci.
Vol. 17, No. 4, pp. 989–1004

BOUNDARY BLOW-UP SOLUTIONS OF ELLIPTIC EQUATIONS INVOLVING REGIONAL FRACTIONAL LAPLACIAN*

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Abstract. In this paper, we study existence of boundary blow-up solutions for elliptic equations involving regional fractional Laplacian:

$$\begin{aligned} (-\Delta)^\alpha_\Omega u+f(u)=0 \qquad &\text{in} \quad \Omega,\\ u=+\infty \qquad &\text{on} \quad \partial\Omega, \end{aligned} \tag{0.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N\geq 2)$ with C^2 boundary $\partial\Omega$, $\alpha\in(0,1)$ and the operator $(-\Delta)^\alpha_\Omega$ is the regional fractional Laplacian. When f is a nondecreasing continuous function satisfying $f(0)\geq 0$ and some additional conditions, we address the existence and nonexistence of solutions for problem (0.1). Moreover, we further analyze the asymptotic behavior of solutions to problem (0.1).

Keywords. Regional Fractional Laplacian; Boundary blow-up solution; Asymptotic behavior.

AMS subject classifications. 35J61; 35B44; 35B40.

1. Introduction

The usual Laplacian operator may be thought as a macroscopic manifestation of the Brownian motion, as known from the Fokker-Planck equation for a stochastic differential equation with a Brownian motion (a Gaussian process), whereas the fractional Laplacian operator $(-\Delta)^\alpha$ is associated with a 2α-stable Lévy motion (a non-Gaussian process) $L_t^{2\alpha}$, $\alpha\in(0,1)$, (see [10] for a discussion about this microscopic-macroscopic relation). Given a bounded open domain Ω in $\mathbb{R}^N$, the regional fractional Laplacians defined in Ω are generators of the reflected symmetric 2α-stable processes, see [8, 9, 12]. Motivated by numerous applications related to (0.1) and by the great mathematical interest in solving (0.1), we tackle this rich PDE problem in this paper.

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N\geq 2)$ with C^2 boundary $\partial\Omega$, $\rho(x)=\mathrm{dist}(x,\mathbb{R}^N\setminus\Omega)$ and $f:\mathbb{R}\to\mathbb{R}$ be a nondecreasing, locally Lipschitz continuous function satisfying $f(0)\geq 0$. We are concerned with the existence of boundary blow-up solutions for elliptic equations involving regional fractional Laplacian

$$\begin{aligned} (-\Delta)^\alpha_\Omega u+f(u)=0 \qquad &\text{in} \quad \Omega,\\ u=+\infty \qquad &\text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\alpha\in(0,1)$ and $(-\Delta)^\alpha_\Omega$ is the regional fractional Laplacian defined by

$$(-\Delta)^\alpha_\Omega u(x)=P.V.\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}dy, \qquad x\in\Omega.$$

Here $P.V.$ denotes the principal value of the integral, that for notational simplicity we omit in what follows.

*Received: June 2, 2018; Accepted (in revised form): February 24, 2019. Communicated by Weizhu Bao.

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When $\alpha=1$, in the seminal works by Keller [13] and Osserman [16], the authors studied the boundary blow-up solutions for the nonlinear reaction diffusion equation

$$\begin{aligned} -\Delta u+f(u)=0 \qquad &\text{in} \quad \Omega, \\ u=+\infty \qquad &\text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

They independently proved that this equation admits a solution if and only if f is a nondecreasing positive function satisfying the Keller-Osserman criterion, that is,

$$\int_1^{+\infty} \frac{ds}{\sqrt{\int_0^s f(t)dt}} < +\infty. \tag{1.3}$$

From then on, boundary blow-up problem (1.2) has been extended by numerous mathematicians in various ways: weakening the assumptions on the domain, generalizing the differential operator and the nonlinear term for equations and systems. Moreover, the qualitative properties of boundary blow-up solutions, such as asymptotic behavior, uniqueness and symmetry results, attract a great attention, see the references [1–3, 11, 14, 15].

In a recent work, Chen-Felmer-Quaas [5] considered an analog of (1.2) where the Laplacian is replaced by the fractional Laplacian

$$\begin{aligned} (-\Delta)^\alpha u+f(u)=0 \qquad &\text{in} \quad \Omega, \\ u=0 \qquad &\text{in} \quad \mathbb{R}^N\setminus\Omega, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x)=+\infty, & \end{aligned} \tag{1.4}$$

where the fractional Laplacian operator $(-\Delta)^\alpha$ is defined as

$$(-\Delta)^\alpha u(x)=P.V.\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}dy.$$

They studied the existence, uniqueness and non-existence of boundary blow-up solutions by Perron's method when $f(s)=s^p$ with $p>1$. Later on, the authors and Wang in [7] studied the boundary blow-up solutions of (1.4) which is derived by measure-type data when f is a continuous and increasing function satisfying

$$\int_1^\infty f(s)s^{-1-\frac{1+\alpha}{1-\alpha}}ds<+\infty. \tag{1.5}$$

We obtained a sequence of boundary blow-up solutions of (1.4), which have the asymptotic behavior $\operatorname{dist}(x,\partial\Omega)^{\alpha-1}$ as $x\to\partial\Omega$. In particular, when $f(s)\le c_1 s^q$ for $s\ge 0$, where $q\le 2\alpha+1$ and $c_1>0$, this sequence of solutions blow up every where in Ω.

For a regular function u such that $u=0$ in $\mathbb{R}^N\setminus\bar{\Omega}$, we remark that

$$(-\Delta)^\alpha_\Omega u(x)=(-\Delta)^\alpha u(x)-u(x)\phi(x), \quad \forall x\in\Omega,$$

where

$$\phi(x)=\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-y|^{N+2\alpha}}dy.$$

From the connections between the fractional Laplacian and the regional fractional Laplacian, we observe that the boundary blow-up solution of (1.4) provides a sub-solution for (1.1), then we have following proposition.

Proposition 1.1. *Assume that $\alpha\in(0,1)$ and f is a nondecreasing function satisfying $f(0)\ge 0$ and locally Lipschitz continuous in $\mathbb{R}$.*

(i) If $f(s)\le c_1 s^q$ for $s\ge 0$, where $q\le 2\alpha+1$ and $c_1>0$, then problem (1.1) has no solution u satisfying

$$\lim_{\rho(x)\to 0^+} u(x)\rho(x)^{1-\alpha}=+\infty. \tag{1.6}$$

(ii) If

$$c_2 s^p\le f(s)\le c_3 s^q \quad \text{for} \quad s\ge 1, \tag{1.7}$$

where $2\alpha+1<p\le q\le \frac{1+\alpha}{1-\alpha}$ and $c_2,c_3>0$, then problem (1.1) has a solution u satisfying

$$c_4\rho(x)^{-\frac{2\alpha}{q-1}}\le u(x)\le c_5\rho(x)^{-\frac{2\alpha}{p-1}},\quad \forall x\in\Omega, \tag{1.8}$$

where $c_5\ge c_4>0$.

We notice that Proposition 1.1 can not cover the case where $f(s)\ge s^p$ with $p\ge\frac{1+\alpha}{1-\alpha}$. Our purpose in this note is to solve more general cases. To this end, we first introduce an important proposition on the regional fractional elliptic problem with finite boundary data.

Proposition 1.2. *Let $\alpha\in(\frac12,1)$, $n\in\mathbb{N}$, $g\in C^1(\bar\Omega)$ and f be a locally Lipschitz continuous and nondecreasing function.*

Then problem

$$\begin{aligned} (-\Delta)^\alpha_\Omega u+f(u)&=g \qquad \text{in} \quad \Omega,\\ u&=n \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{1.9}$$

admits a unique solution u_n such that

$$-c_6\left(\|g_-\|_{L^\infty(\Omega)}+f(n)\right)\rho^{2\alpha-1}\le u_n-n\le c_6\|g_+\|_{L^\infty(\Omega)}\rho^{2\alpha-1}\quad \text{in} \quad \Omega, \tag{1.10}$$

where $g_\pm=\max\{\pm g,0\}$ and $c_6>0$ is independent of n, f and g.

Moreover, if $g\ge 0$ and $f(0)\ge 0$, then u_n is positive.

The derivation of the solution of (1.9) makes use of the Green's function of the regional fractional Laplacian and Perron's method. The authors in [8] showed that for $\alpha\in(\frac12,1)$, the Green's function of the regional fractional provides boundary decay estimate, while for $\alpha\in(0,\frac12]$, the Green's function of the regional fractional behaves very different, without any boundary decaying, thus it is even hard to obtain a solution for (1.9).

We call a solution u_m of (1.1) as the minimal solution if for any solution v of (1.1), we have that

$$v\ge u_m \quad \text{in} \quad \Omega.$$

As normal, the minimal boundary blow-up solution with $\alpha\in(\frac12,1)$ is approached by the solutions of (1.9) by taking $n\to+\infty$.

Theorem 1.1. *Assume that $\alpha\in(\frac12,1)$ and f is a nondecreasing continuous function satisfying $f(0)\ge 0$. Furthermore,* *(i) If $f(s)\ge c_7s^p$ for $s\ge 0$, where $p>1+2\alpha$ and $c_7>0$, then problem (1.1) possesses the minimal boundary blow-up solution u_m.*

Assume more that $f(s)\le c_8s^q$ for $s\ge 1$, where $q\ge p$ and $c_8>0$, then u_m has asymptotic behavior near the boundary as

$$c_9\rho(x)^{-\frac{2\alpha-1}{q-1}}\le u_m(x)\le c_{10}\rho(x)^{-\frac{2\alpha}{p-1}}, \tag{1.11}$$

where $c_{10}\ge c_9>0$.

(ii) If $f(s)\le c_{11}s^q$ for $s\ge 0$, where $c_{11}>0$ and

$$q\le 1+2\alpha \quad \text{and} \quad q<\frac{\alpha}{1-\alpha}, \tag{1.12}$$

then problem (1.1) has no solution.

Compared to Proposition 1.1, we notice that (i) when $\alpha\in(\frac{1}{2},1)$, we improve the existence for the case that $f(s)\ge c_7s^p$ for $s\ge 0$ and $p>1+2\alpha$ in Theorem 1.1; (ii) if $\alpha>\frac{\sqrt{2}}{2}$ for $f(s)=s^p$ with $p\le 1+2\alpha$, problem (1.1) has any solution.

The lower bound in (1.11) is derived by the inequality (1.10) and the upper bound in (1.11) is obtained by constructing a suitable super-solution for problem (1.1).

This paper is organized as follows. Section §2 is devoted to present some preliminaries on the definition of viscosity solution, comparison principle, stability theorem, regularity results and to make use of solutions of corresponding problem with the fractional Laplacian to prove Proposition 1.1. In Section §3, we first prove the existence of solutions to problem (1.9), asymptotic behavior and then prove Theorem 1.1.

2. Preliminary

The purpose of this section is to introduce some preliminaries. We start it by defining the notion of viscosity solution, inspired by the definition of viscosity sense for nonlocal problem in [4].

2.1. Viscosity solution.

DEFINITION 2.1. *We say that a continuous function $u\in L^1(\Omega)$ is a viscosity super-solution (sub-solution) of*

$$\begin{aligned} (-\Delta)^\alpha_\Omega u+f(u)&=g \qquad \text{in} \quad \Omega,\\ u&=h \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

if $u\ge h$ (resp. $u\le h$) on $\partial\Omega$ and for every point $x_0\in\Omega$ and some neighborhood V of x_0 with $\bar V\subset\Omega$ and for any $\varphi\in C^2(\bar V)$ such that $u(x_0)=\varphi(x_0)$ and x_0 is the minimum (resp. maximum) point of $u-\varphi$ in V, let

$$\tilde u=\begin{cases}\varphi & \text{in} \quad V,\\ u & \text{in} \quad \Omega\setminus V,\end{cases}$$

we have

$$(-\Delta)^\alpha_\Omega\tilde u(x_0)+f(u(x_0))\ge g(x_0) \quad (\textit{resp. } (-\Delta)^\alpha_\Omega\tilde u(x_0)+f(u(x_0))\le g(x_0)).$$

We say that u is a viscosity solution of (2.1) if it is a viscosity super-solution and also a viscosity sub-solution of (2.1).

Now we introduce the comparison principle.

THEOREM 2.1. *Assume that the functions $g:\Omega\to\mathbb{R}$, $h:\partial\Omega\to\mathbb{R}$ are continuous and $f:\mathbb{R}\to\mathbb{R}$ is nondecreasing. Let u and v be a viscosity super-solution and sub-solution of (2.1), respectively. If*

$$v\le u \quad \text{on} \quad \partial\Omega,$$

then

$$v\le u \quad \text{in} \quad \Omega. \tag{2.2}$$

Proof. Let us define $w=u-v$, then

$$\begin{aligned} (-\Delta)^\alpha_\Omega w\ge f(v)-f(u) \qquad & \text{in} \quad \Omega,\\ w\ge 0 \qquad\qquad & \text{on} \quad \partial\Omega. \end{aligned} \tag{2.3}$$

If (2.2) fails, then there exists $x_0\in\Omega$ such that

$$w(x_0)=u(x_0)-v(x_0)=\min_{x\in\Omega} w(x)<0,$$

by the fact that f is nondecreasing, we have that $f(v(x_0))-f(u(x_0))\ge 0$ and then in the viscosity sense,

$$(-\Delta)^\alpha_\Omega w(x_0)\ge 0. \tag{2.4}$$

Since w is a viscosity super-solution x_0 is the minimum point in Ω and $w\ge 0$ on $\partial\Omega$, then we can take a small neighborhood V_0 of x_0 such that $\tilde{w}=w(x_0)$ in V_0, From (2.4), we have that

$$(-\Delta)^\alpha_\Omega \tilde{w}(x_0)\ge 0.$$

But

$$(-\Delta)^\alpha_\Omega \tilde{w}(x_0)=\int_{\Omega\setminus V_0}\frac{w(x_0)-w(y)}{|x_0-y|^{N+2\alpha}}dy<0,$$

which is impossible. □

For a regular function w such that $w=0$ in $\mathbb{R}^N\setminus\bar{\Omega}$, we observe that

$$(-\Delta)^\alpha_\Omega w(x)=(-\Delta)^\alpha w(x)-w(x)\phi(x), \quad \forall x\in\Omega, \tag{2.5}$$

where

$$\phi(x)=\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-y|^{N+2\alpha}}dy. \tag{2.6}$$

LEMMA 2.1. *Let ϕ be defined in (2.6) and $\rho(x)=dist(x,\partial\Omega)$, then $\phi\in C^{0,1}_{\rm loc}(\Omega)$ and*

$$\frac{1}{c_{12}}\rho(x)^{-2\alpha}\le\phi(x)\le c_{12}\rho(x)^{-2\alpha}, \quad x\in\Omega, \tag{2.7}$$

for some $c_{12}>0$.

Proof. For $x_1,x_2\in\Omega$ and any $z\in\mathbb{R}^N\setminus\Omega$, we have that

$$|z-x_1|\ge\rho(x_1)+\rho(z),\qquad |z-x_2|\ge\rho(x_2)+\rho(z)$$

and

$$||z-x_1|^{N+2\alpha}-|z-x_2|^{N+2\alpha}|\le c_{13}|x_1-x_2|(|z-x_1|^{N+2\alpha-1}+|z-x_2|^{N+2\alpha-1}),$$

for some $c_9>0$ independent of x_1 and x_2. Then

$$|\phi(x_1)-\phi(x_2)|\le\int_{\mathbb{R}^N\setminus\Omega}\frac{||z-x_2|^{N+2\alpha}-|z-x_1|^{N+2\alpha}|}{|z-x_1|^{N+2\alpha}|z-x_2|^{N+2\alpha}}dz$$
$$\le c_{13}|x_1-x_2|\left[\int_{\mathbb{R}^N\setminus\Omega}\frac{dz}{|z-x_1||z-x_2|^{N+2\alpha}}+\int_{\mathbb{R}^N\setminus\Omega}\frac{dz}{|z-x_1|^{N+2\alpha}|z-x_2|}\right].$$

By direct computation, we have that

$$\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|z-x_1||z-x_2|^{N+2\alpha}}dz$$
$$\le\int_{\mathbb{R}^N\setminus B_{\rho(x_1)}(x_1)}\frac{1}{|z-x_1|^{N+2\alpha+1}}dz+\int_{\mathbb{R}^N\setminus B_{\rho(x_2)}(x_2)}\frac{1}{|z-x_2|^{N+2\alpha+1}}dz$$
$$\le c_{14}[\rho(x_1)^{-1-2\alpha}+\rho(x_2)^{-1-2\alpha}]$$

and similarly to obtain that

$$\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|z-x_1|^{N+2\alpha}|z-x_2|}dz\le c_{14}[\rho(x_1)^{-1-2\alpha}+\rho(x_2)^{-1-2\alpha}],$$

where $c_{14}>0$ is independent of x_1,x_2. Then

$$|\phi(x_1)-\phi(x_2)|\le c_{13}c_{14}[\rho(x_1)^{-1-2\alpha}+\rho(x_2)^{-1-2\alpha}]|x_1-x_2|,$$

that is, ϕ is $C^{0,1}$ locally in Ω.

Now we prove (2.7). Without loss of generality, we may assume that $0\in\partial\Omega$, the inside pointing normal vector at 0 is $e_N=(0,\cdots,0,1)\in\mathbb{R}^N$ and let $s\in(0,\frac14)$ such that $\mathbb{R}^N\setminus\Omega\subset\mathbb{R}^N\setminus B_s(se_N)$ and for $c>0$, we denote the cone

$$A_c=\{y=(y',y_N)\in\mathbb{R}^N:y_N\le s-c|y'|\}.$$

We observe that there is $c_{15}>0$ such that

$$[A_{c_{15}}\cap(B_1(se_N)\setminus B_{2s}(se_N))]\subset\mathbb{R}^N\setminus\Omega.$$

By the definition of ϕ, we have that

$$\phi(se_N)=\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|se_N-y|^{N+2\alpha}}dy\le\int_{\mathbb{R}^N\setminus B_s(se_N)}\frac{1}{|se_N-y|^{N+2\alpha}}dy\le c_{16}s^{-2\alpha}$$

for some $c_{16}>0$. On the other hand, we have that

$$\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|se_N-y|^{N+2\alpha}}dy\ge\int_{A_{c_{15}}\cap(B_1(se_N)\setminus B_{2s}(se_N))}\frac{1}{|se_N-y|^{N+2\alpha}}dy\ge c_{17}s^{-2\alpha},$$

for some $c_{17}\in(0,1)$. The proof ends. ∎

The next theorem gives the stability property for viscosity solutions in our setting.

THEOREM 2.2. *Assume that the function $g:\Omega\to\mathbb{R}$ is continuous, $f:\mathbb{R}\to\mathbb{R}$ is nondecreasing and $f(0)\ge 0$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C^1(\Omega)$, uniformly bounded in $L^1(\Omega)$, g_n and g be continuous in Ω such that*

$(-\Delta)^\alpha_\Omega u_n+f(u_n)\ge g_n$ (resp. $(-\Delta)^\alpha_\Omega u_n+f(u_n)\le g_n$) *in Ω in the viscosity sense,*

$u_n\to u$ locally uniformly in Ω,

$u_n\to u$ in $L^1(\Omega)$,

$h_n\to h$ locally uniformly in Ω.

Then $(-\Delta)^\alpha_\Omega u+f(u)\ge g$ (resp. $(-\Delta)^\alpha_\Omega u+f(u)\le g$) in Ω in the viscosity sense.

Proof. We define $\tilde u_n=u_n$ in Ω, $\tilde u_n=0$ in $\mathbb{R}^N\setminus\bar\Omega$ and $\tilde u=u$ in Ω, $\tilde u=0$ in $\mathbb{R}^N\setminus\bar\Omega$, then

$$(-\Delta)^\alpha_\Omega u_n(x)=(-\Delta)^\alpha\tilde u_n(x)-u_n(x)\phi(x),\qquad x\in\Omega.$$

where ϕ is defined as (2.6). By Lemma 2.1, $\phi\in C^{0,1}_{\rm loc}(\Omega)$ and $\phi(x)\le c_8\rho(x)^{-2\alpha}$, $x\in\Omega$. Then we apply [5, Theorem 2.4] to obtain that $(-\Delta)^\alpha\tilde u+f(\tilde u)\ge g+\phi\tilde u$ (resp. $(-\Delta)^\alpha\tilde u+f(\tilde u)\le g+\phi\tilde u$) in Ω in the viscosity sense, which implies$(-\Delta)^\alpha_\Omega u+f(u)\ge g$ (resp. $(-\Delta)^\alpha_\Omega u+f(u)\le g$) in Ω in the viscosity sense. ∎

Next we have an interior regularity result. For simplicity, we denote by C^t the space $C^{t_0,t-t_0}$ for $t\in(t_0,t_0+1)$, t_0 is a positive integer.

PROPOSITION 2.1. *Assume that $\alpha\in(\frac12,1)$, $g\in C^\theta_{\rm loc}(\Omega)$ with $\theta>0$, $w\in C^{2\alpha+\epsilon}_{\rm loc}(\Omega)\cap L^1(\Omega)$, with $\epsilon>0$ and $2\alpha+\epsilon$ not being an integer, is a solution of*

$$(-\Delta)^\alpha_\Omega w=g\quad\text{in}\quad\Omega.\tag{2.8}$$

Let $\mathcal{O}_1,\mathcal{O}_2$ be open C^2 sets such that

$$\bar{\mathcal{O}}_1\subset\mathcal{O}_2\subset\bar{\mathcal{O}}_2\subset\Omega.$$

Then

(i) for any $\gamma\in(0,2\alpha)$ not an integer, there exists $c_{18}>0$ such that

$$\|w\|_{C^\gamma(\mathcal{O}_1)}\le c_{18}\left[\|w\|_{L^\infty(\mathcal{O}_2)}+\|w\|_{L^1(\Omega)}+\|g\|_{L^\infty(\mathcal{O}_2)}\right];\tag{2.9}$$

(ii) for any $\epsilon'\in(0,\min\{\theta,\epsilon\})$, $2\alpha+\epsilon'$ not an integer, there exists $c_{19}>0$ such that

$$\|w\|_{C^{2\alpha+\epsilon'}(\mathcal{O}_1)}\le c_{19}\left[\|w\|_{L^\infty(\mathcal{O}_2)}+\|w\|_{L^1(\Omega)}+\|g\|_{c^\theta(\mathcal{O}_2)}\right].\tag{2.10}$$

Proof. Let $\tilde w=w$ in Ω, $\tilde w=0$ in $\mathbb{R}^N\setminus\bar\Omega$, we have that

$$(-\Delta)^\alpha\tilde w(x)=(-\Delta)^\alpha_\Omega w(x)+w(x)\phi(x),\quad\forall x\in\Omega,$$

where ϕ is defined as (2.6). It follows by Lemma 2.1, $\phi\in C^{0,1}_{\rm loc}(\Omega)$. Combining with (2.8), we have that

$$(-\Delta)^\alpha\tilde w(x)=g(x)+w(x)\phi(x),\quad\forall x\in\Omega.$$

By [6, Lemma 3.1], for any $\gamma\in(0,2\alpha)$, we have that

$$\|w\|_{C^\gamma(\mathcal{O}_1)}\le c_{20}\left[\|w\|_{L^\infty(\mathcal{O}_2)}+\|w\|_{L^1(\Omega)}+\|g+w\phi\|_{L^\infty(\mathcal{O}_2)}\right]$$

$$\le c_{21}\left[\|w\|_{L^\infty(\mathcal{O}_2)}+\|w\|_{L^1(\Omega)}+\|g\|_{L^\infty(\mathcal{O}_2)}\right]$$

and by [17, Lemma 2.10], for any $\epsilon'\in(0,\min\{\theta,\epsilon\})$, we have that

$$\begin{aligned}\|w\|_{C^{2\alpha+\epsilon'}(\mathcal{O}_1)}&\le c_{22}\left[\|w\|_{C^{\epsilon'}(\mathcal{O}_2)}+\|g+w\phi\|_{C^{\epsilon'}(\mathcal{O}_2)}\right]\\&\le c_{23}\left[\|w\|_{L^\infty(\mathcal{O}_2)}+\|w\|_{L^1(\Omega)}+\|g\|_{C^{\epsilon'}(\mathcal{O}_2)}\right],\end{aligned}$$

where $c_{22},c_{23}>0$. This ends the proof. □

2.2. Proof of Proposition 1.1. Basically, the existence for boundary blow-up problem is usually resorted to the Perron's method. In this subsection, we extend the Perron's method to the problem involving regional fractional Laplacian.

To this end, we first introduce the existence of boundary blow-up solution of fractional elliptic problem with locally Lipschitz continuous nonlinearity f, precisely,

$$\begin{cases}(-\Delta)^\alpha u(x)+f(u)=g, & x\in\Omega,\\ u(x)=0, & x\in\bar\Omega^c,\\ \lim_{x\in\Omega,\ x\to\partial\Omega}u(x)=+\infty.\end{cases}\tag{2.11}$$

THEOREM 2.3. *Assume that $f:\mathbb{R}\to\mathbb{R}$ is nondecreasing, C^γ_{loc} and $f(0)=0$, the function $g:\Omega\to\mathbb{R}$ is a C^γ_{loc} in Ω. Suppose that there are super-solution $\bar U$ and sub-solution $\underline{U}$ of (2.11) such that $\bar U$ and $\underline{U}$ are C^2 locally in Ω, bounded in $L^1(\mathbb{R}^N,\frac{dy}{1+|y|^{N+2\alpha}})$ and*

$$\bar U\ge\underline{U}\ \text{ in }\Omega,\qquad \liminf_{x\in\Omega,x\to\partial\Omega}\underline{U}(x)=+\infty,\qquad \bar U=\underline{U}=0\ \text{ in }\bar\Omega^c.$$

Then there exists at least one solution u of (2.11) in the viscosity sense and

$$\underline{U}\le u\le\bar U\ \text{ in }\ \Omega.$$

Additionally, suppose that $g\ge0$ in Ω, then $u>0$ in Ω.

Proof. We follow the proof of [5, Theorem 2.6] replacing $|u|^{p-1}u$ by $f(u)$.

THEOREM 2.4. *Let Ω be an open bounded C^2 domain and $p>0$. Suppose that there are super-solution $\bar U$ and sub-solution $\underline{U}$ of (1.1) such that $\bar U$ and $\underline{U}$ are C^2 locally in Ω,*

$$\bar U\ge\underline{U}\ \text{ in }\Omega,\qquad \liminf_{x\in\Omega,x\to\partial\Omega}\underline{U}(x)=+\infty.$$

Then there exists at least one solution u of (1.1) in the viscosity sense and

$$\underline{U}\le u\le\bar U\ \text{ in }\ \Omega.\tag{2.12}$$

Proof. From (2.5), to search the solution of (1.1) is equivalent to finding out the solution of the fractional problem

$$\begin{aligned}(-\Delta)^\alpha u+f(u)=\phi u &\quad\text{in}\quad \Omega,\\ u=0 &\quad\text{in}\quad \mathbb{R}^N\setminus\Omega,\\ \lim_{x\in\Omega,x\to\partial\Omega}u(x)=+\infty,&\end{aligned}\tag{2.13}$$

where ϕ is given by (2.6). Make zero extensions of $\bar{U}$ and $\underline{U}$ in $\mathbb{R}^N\setminus\Omega$ and still denote them by $\bar{U}$ and $\underline{U}$ respectively, then $\bar{U}$ and $\underline{U}$ are the super and sub-solutions of (2.13). Now we apply Theorem 2.3 to obtain the existence of solution to (2.13).

From Lemma 2.1, ϕ is $C^{0,1}$ locally in Ω, so is $\phi\underline{U}$, then by Theorem 2.3, we obtain that problem (2.13) replaced ϕu by $\phi\underline{U}$ admits a solution u_1 satisfying (2.12). By regularity results in [17], we have that

$$\|u_1\|_{C^{2\alpha+\gamma}(\Omega)}\le c_{24}\|\bar{U}\|_{L^\infty(\Omega)}$$

for some $\gamma\in(0,1)$.

Inductively, by Theorem 2.3, we obtain that problem (2.13) replaced ϕu by ϕu_{n-1} has a solution u_n such that

$$u_{n-1}\le u_n\le \bar{U}\quad \text{in}\quad \Omega. \tag{2.14}$$

Applying stability theorem [5, Theorem 2.4] and regularity result in [17], we obtain that the limit of $\{u_n\}_n$ is a solution of (2.13). □

For $t_0>0$ small, $A_{t_0}=\{x\in\Omega:\ \rho(x)<t_0\}$ is C^2 and let us define

$$V_\tau(x)=\begin{cases}\rho(x)^\tau, & x\in A_{t_0},\\ l(x), & x\in\Omega\setminus A_{t_0},\\ 0, & x\in\Omega^c,\end{cases} \tag{2.15}$$

where $\tau\in(-1,0)$ and the function l is positive such that V_τ is C^2 in Ω.

Proof. **(Proof of Proposition 1.1.)** (i) Now we prove the nonexistence when $q\le 1+2\alpha$. From Theorem 1.1 and Theorem 1.2 in [7], the semilinear fractional problem

$$\begin{aligned}(-\Delta)^\alpha u+c_1u^q&=0 \quad &&\text{in}\quad \Omega,\\ u&=0 \quad &&\text{in}\quad \mathbb{R}^N\setminus\Omega,\\ \lim_{x\in\Omega,x\to\partial\Omega}u(x)&=+\infty\end{aligned} \tag{2.16}$$

admits a sequence of solutions $\{v_k\}_k$ satisfying that the mapping $k\mapsto v_k$ is increasing,

$$v_k(x)\le c_{25}k\rho(x)^{\alpha-1},\quad \forall x\in\Omega$$

and

$$\lim_{k\to\infty}v_k(x)=\infty,\quad \forall x\in\Omega, \tag{2.17}$$

where $c_{25}>0$.

We observe that v_k is a sub-solution of (1.1) for any k.

If (1.1) has a solution u satisfying (1.6), then by the comparison principle, for any k, there holds that

$$v_k(x)\le u(x),\quad \forall x\in\Omega.$$

Then it is impossible that u is a solution of (1.1) by (2.17).

(ii) When $q\in(1+2\alpha,\frac{1+\alpha}{1-\alpha})$, it infers from [5] that there exists a solution v_q of (2.16) replacing c_1 by c_3 from the assumption (1.7) such that

$$\frac{1}{c_{26}}\rho(x)^{-\frac{2\alpha}{q-1}}\le v_q(x)\le c_{26}\rho(x)^{-\frac{2\alpha}{q-1}},\qquad \forall x\in\Omega. \tag{2.18}$$

where $c_{26}>0$. By (1.7), v_p is a sub-solution of

$$\begin{aligned} (-\Delta)^\alpha u+f(u)&=u\phi && \text{in} \quad \Omega,\\ u&=0 && \text{in} \quad \mathbb{R}^N\setminus\Omega,\\ \lim_{x\in\Omega,x\to\partial\Omega} u(x)&=+\infty. \end{aligned} \tag{2.19}$$

So v_p is a sub-solution of (1.1).

We next construct a suitable super-solution of (1.1). From [5, Proposition 3.1], we know that the function V_τ with $\tau=-\frac{2\alpha}{p-1}\in(-1,0)$ satisfies

$$(-\Delta)^\alpha V_\tau(x)\ge c_\tau\rho(x)^{\tau-2\alpha}, \qquad \forall x\in\Omega,$$

where V_τ is given by (2.15).

We consider λV_τ with $\lambda>0$. We observe that

$$\begin{aligned} (-\Delta)^\alpha_\Omega(\lambda V_\tau)+f(\lambda V_\tau)&=(-\Delta)^\alpha(\lambda V_\tau)+f(\lambda V_\tau)-\lambda\phi V_\tau\\ &\ge c_\tau\lambda\rho(x)^{\tau-2\alpha}+c_2c_{26}^{-p}\lambda^p\rho(x)^{-\frac{2\alpha p}{p-1}}-c_{27}\lambda\rho(x)^{\tau-2\alpha}\\ &\ge \left[c_2c_{26}^{-p}\lambda^{p-1}-|c_\tau|-c_{27}\right]\lambda\rho(x)^{\tau-2\alpha}\\ &\ge 0 \end{aligned}$$

if $\lambda>0$ sufficiently big. By Theorem 2.4, it deduces that (1.1) has a solution u such that

$$v_q\le u\le\lambda V_\tau \quad \text{in} \quad \Omega,$$

which implies (1.8). $\square$

3. Boundary blow-up solutions

In this section, we introduce the boundary blow-up solutions for $\alpha\in(\frac12,1)$. We first show the results for the existence. To this end, let us denote $G_{\Omega,\alpha}$ by the Green kernel of $(-\Delta)^\alpha_\Omega$ in $\Omega\times\Omega$ and $\mathbb{G}_{\Omega,\alpha}[\cdot]$ by the Green operator defined as

$$\mathbb{G}_{\Omega,\alpha}[g](x)=\int_\Omega G_{\Omega,\alpha}(x,y)g(y)dy.$$

Proposition 3.1. *Assume that $\alpha\in(\frac12,1)$, $n\in\mathbb{N}$ and $g\in C^\theta(\bar\Omega)$ with $\theta>0$, then*

$$\begin{aligned} (-\Delta)^\alpha_\Omega w&=g && \text{in} \quad \Omega,\\ w&=n && \text{on} \quad \partial\Omega \end{aligned} \tag{3.1}$$

admits a unique solution w_n such that

$$-\mathbb{G}_{\Omega,\alpha}[g_-]\le w_n-n\le\mathbb{G}_{\Omega,\alpha}[g_+] \quad \text{in} \quad \Omega, \tag{3.2}$$

where $g_\pm=\max\{\pm g,0\}$.

Proof. *(Existence.)* Since $\mathbb{G}_{\Omega,\alpha}[g]$ is a solution of

$$(-\Delta)^\alpha_\Omega w=g \quad \text{in} \quad \Omega,$$

From [8], there exists $c_{28}>0$ such that for any $(x,y)\in\Omega\times\Omega$ with $x\ne y$,

$$G_{\Omega,\alpha}(x,y)\le c_{28}\min\left\{\frac{1}{|x-y|^{N-2\alpha}},\frac{\rho(x)^{2\alpha-1}\rho(y)^{2\alpha-1}}{|x-y|^{N-2+2\alpha}}\right\}. \tag{3.3}$$

For $x\in\Omega$, we have that

$$
\begin{aligned}
|\mathbb{G}_{\Omega,\alpha}[g](x)| &\le c_{28}\int_{\Omega}\frac{\rho(x)^{2\alpha-1}\rho(y)^{2\alpha-1}}{|x-y|^{N-2+2\alpha}}|g(y)|dy\\
&\le c_{28}\rho(x)^{2\alpha-1}\|g\|_{L^\infty(\Omega)}\int_{\Omega}\frac{\rho(y)^{2\alpha-1}}{|x-y|^{N-2+2\alpha}}dy\\
&\le c_{29}\|g\|_{L^\infty(\Omega)}\rho(x)^{2\alpha-1},
\end{aligned}
$$

where $c_{29}>0$. Therefore, $\mathbb{G}_{\Omega,\alpha}[g]$ is a solution of

$$
\begin{aligned}
(-\Delta)^\alpha_\Omega w &= g \qquad \text{in} \quad \Omega,\\
w &= 0 \qquad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{3.4}
$$

and $n+\mathbb{G}_{\Omega,\alpha}[g]$ is obviously a solution of (3.1).

Uniqueness. Let v be another solution of (3.1), we observe that $w-v$ is a solution of

$$
\begin{aligned}
(-\Delta)^\alpha_\Omega u &= 0 \qquad \text{in} \quad \Omega,\\
u &= 0 \qquad \text{on} \quad \partial\Omega.
\end{aligned}
$$

Then it follows by maximum principle that $w-v\equiv 0$ in Ω.

Finally, since $\mathbb{G}_{\Omega,\alpha}[g_+]$ is a super-solution of (3.4) and $-\mathbb{G}_{\Omega,\alpha}[g_-]$ is a sub-solution of (3.4), then (3.2) follows. □

We remark that the existence of solution to (3.1) could be extended to general boundary data. Precisely, let $\xi:\partial\Omega\to\mathbb{R}$ be a boundary trace of a $C^2(\bar{\Omega})$ function $\tilde{\xi}$, i.e.

$$\xi=\tilde{\xi} \quad \text{on} \quad \partial\Omega.$$

For $\alpha\in(\frac{1}{2},1)$, problem

$$
\begin{aligned}
(-\Delta)^\alpha_\Omega w &= 0 \qquad \text{in} \quad \Omega,\\
w &= \xi \qquad \text{on} \quad \partial\Omega
\end{aligned}
\tag{3.5}
$$

admits a unique solution

$$w_\xi=\tilde{\xi}-\mathbb{G}_{\Omega,\alpha}[(-\Delta)^\alpha_\Omega\tilde{\xi}] \quad \text{in} \quad \Omega. \tag{3.6}$$

We observe that $\mathbb{G}_{\Omega,\alpha}[(-\Delta)^\alpha_\Omega\tilde{\xi}]$ decays at the rate $\rho^{2\alpha-1}$ and w_ξ is independent of the choice of $\tilde{\xi}$. In fact, let $\tilde{\xi}_1\in C^2(\bar{\Omega})$ have the trace ξ and the corresponding solution v_ξ, then $w:=w_\xi-v_\xi$ is a solution of

$$
\begin{aligned}
(-\Delta)^\alpha_\Omega w &= 0 \qquad \text{in} \quad \Omega,\\
w &= 0 \qquad \text{on} \quad \partial\Omega,
\end{aligned}
$$

which implies by strong maximum principle that

$$w\equiv 0.$$

In the particular case that $\xi=n$, we have that $\tilde{\xi}=n$ in Ω and

$$\mathbb{G}_{\Omega,\alpha}[(-\Delta)^\alpha_\Omega\tilde{\xi}]=0 \quad \text{in} \quad \Omega.$$

This part is devoted to study the existence of solution of (1.9). To this end, we first introduce the following lemma.

LEMMA 3.1. *Let $n\in\mathbb{N}$, $b\geq 0$ and $g\in C^1(\bar{\Omega})$, then*

$$\begin{aligned}(-\Delta)^{\alpha}_{\Omega}u+bu&=g \quad \text{in} \quad \Omega,\\ u&=n \quad \text{on} \quad \partial\Omega\end{aligned} \tag{3.7}$$

admits a unique solution.

Proof. We observe that $n+\mathbb{G}_{\Omega,\alpha}[g_+]$ and $n-\mathbb{G}_{\Omega,\alpha}[g_-]$ are super and sub-solutions of (3.7) respectively. We make an extension of $n+\mathbb{G}_{\Omega,\alpha}[g_+]$ and $n-\mathbb{G}_{\Omega,\alpha}[g_-]$ by n in $\mathbb{R}^N\setminus\Omega$ and still denote $n+\mathbb{G}_{\Omega,\alpha}[g_+]$ and $n-\mathbb{G}_{\Omega,\alpha}[g_-]$. Let $\Omega_t:=\{x\in\Omega:\rho(x)>t\}$ for $t\geq 0$ and then there exists $t_0>0$ such that Ω_t is C^2 for any $t\in[0,t_0]$, since Ω is C^2.

By Perron's method, there exists a unique solution w_t of

$$\begin{aligned}(-\Delta)^{\alpha}u+(b+\phi)u&=g-bn \qquad \text{in} \quad \Omega_t,\\ u&=n-\mathbb{G}_{\Omega,\alpha}[g_-] \quad \text{in} \quad \mathbb{R}^N\setminus\Omega_t,\end{aligned}$$

where ϕ is defined as (2.6). Since $t\in(0,t_0)$, ϕ is positive and $\phi\in C^{0,1}_{\text{loc}}(\Omega_t)$, then w_t is a solution of

$$\begin{aligned}(-\Delta)^{\alpha}_{\Omega}u+bu&=g+bn \qquad \text{in} \quad \Omega_t,\\ u&=n-\mathbb{G}_{\Omega,\alpha}[g_-] \quad \text{in} \quad \Omega\setminus\Omega_t\end{aligned}$$

and by Theorem 2.1, we derive that

$$n-\mathbb{G}_{\Omega,\alpha}[g_-]\leq w_t\leq w_{t'}\leq n+\mathbb{G}_{\Omega,\alpha}[g_+] \quad \text{for } 0<t'<t<t_0.$$

By Proposition 2.1 and Theorem 2.2, the limit of w_t as $t\to 0$ is a classical solution of (3.7). □

Proof. **(Proof of Proposition 1.2.)** *Existence.* Let us define

$$w_+(x)=\int_\Omega G_{\Omega,\alpha}(x,y)g_+(y)dy \quad \text{and} \quad w_-(x)=\int_\Omega G_{\Omega,\alpha}(x,y)g_-(y)dy.$$

By (3.3), there exists $c_{30}>0$ such that

$$0\leq w_+(x)\leq c_{30}\|g\|_{L^\infty(\Omega)}\rho(x)^{2\alpha-1}, \quad x\in\Omega$$

and for $x\in\Omega$,

$$0\leq w_-(x)+f(n)\int_\Omega G_{\Omega,\alpha}(x,y)ndy\leq c_{30}(\|g_-\|_{L^\infty(\Omega)}+f(n))\rho(x)^{2\alpha-1}.$$

Let

$$\bar{w}(x)=n-w_-(x)-f(n)\int_\Omega G_{\Omega,\alpha}(x,y)ndy$$

and

$$b_1=\max\{n+\|w_+\|_{L^\infty(\Omega)},\|\bar{w}\|_{L^\infty(\Omega)}\},$$

then $\varphi(s):=(\|f'\|_{L^\infty([-b_1,b_1])}+b_1)s-f(s)$ is increasing in $[-b_1,b_1]$. It follows by Lemma 3.1 that there exists a unique solution v_m of

$$\begin{aligned} (-\Delta)^\alpha_\Omega v_m+b_2v_m=b_2v_{m-1}-f(v_{m-1})+g \quad &\text{in} \quad \Omega,\\ v_m=n \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{3.8}$$

where $b_2=\|f'\|_{L^\infty([-b_1,b_1])}+b_1$, $m\in\mathbb{N}$ and $v_0=-b_1$. We observe that $\{v_m\}$ is a increasing sequence and uniformly bounded in Ω. Therefore, the limit of $\{v_m\}$ as $m\to\infty$ satisfies (1.9).

To prove (1.10). By direct computation, we have that

$$(-\Delta)^\alpha_\Omega(n+w_+(x))+f(n+w_+(x))\ge g_+(x)+f(n)\ge g(x),\quad x\in\Omega$$

and

$$(-\Delta)^\alpha_\Omega\bar w(x)+f(\bar w(x))\le -g_-(x)-f(n)+f(n)\le g(x),\quad x\in\Omega$$

thus $n+w_+$ and $n-w_--n\int_\Omega G_{\Omega,\alpha}(x,y)ndy$ are the super-solution and sub-solution of (1.9), respectively. It infers (1.10) by Theorem 2.1. □

LEMMA 3.2. *Let $\tau\in(-1,0)$ and V_τ be defined in (2.15), then*

$$|(-\Delta)^\alpha_\Omega V_\tau(x)|\le c_{31}\rho(x)^{\tau-2\alpha},\quad \forall x\in\Omega, \tag{3.9}$$

where $c_{31}>0$.

Proof. We denote $\tilde V_\tau=V_\tau$ in Ω and $\tilde V_\tau=0$ in $\mathbb{R}^N\setminus\Omega$, from [5, Proposition 3.2], there exists $c_{32}>1$ such that

$$|(-\Delta)^\alpha\tilde V_\tau(x)|\le c_{32}\rho(x)^{\tau-2\alpha},\quad \forall x\in\Omega. \tag{3.10}$$

We observe that

$$(-\Delta)^\alpha_\Omega V_\tau(x)=(-\Delta)^\alpha\tilde V_\tau(x)-V_\tau(x)\phi(x),$$

where ϕ is defined as (2.6) and by Lemma 2.1, we have that

$$\phi(x)\le c_{12}\rho(x)^{-2\alpha},\quad \forall x\in\Omega.$$

Together with (3.10), we have that

$$\begin{aligned} |(-\Delta)^\alpha_\Omega V_\tau(x)|&\le|(-\Delta)^\alpha\tilde V_\tau(x)|+c_{12}V_\tau(x)\rho(x)^{-2\alpha}\\ &\le c_{33}\rho(x)^{\tau-2\alpha},\quad \forall x\in\Omega. \end{aligned}$$

The proof ends. □

Proof. **(Proof of Theorem 1.1(i).)** From Proposition 1.2 with $g\equiv 0$, there exists a unique positive solution u_n of

$$\begin{aligned} (-\Delta)^\alpha_\Omega u+h(u)=0 \quad &\text{in} \quad \Omega,\\ u=n \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{3.11}$$

and

$$n-n^p\rho(x)^{\alpha-1}\le u_n(x)\le n,\quad \forall x\in\Omega.$$

By Theorem 2.1, for any $n\in\mathbb{N}$,

$$u_n\le u_{n+1}\quad \text{in}\quad \Omega.$$

From lemma 3.2, there exists $\lambda>0$ such that $\lambda V_{-\frac{2\alpha}{p-1}}$ is a super-solution of (3.11), where $-\frac{2\alpha}{p-1}\in(-1,0)$ for $p>1+2\alpha$. It follows by Theorem 2.1, that for all $n\in\mathbb{N}$,

$$u_n\le \lambda V_{-\frac{2\alpha}{p-1}}\quad \text{in}\quad \Omega.$$

Then the limit of $\{u_n\}$ exists in Ω, denoted by u_∞. Moreover, we have that u_n has uniform bound in L^∞ locally in Ω, and then by regular result, we infer that u_n has uniform bound in $C^{2\alpha+\theta}$ locally in Ω. By Theorem 2.2, u_∞ is a viscosity solution of (1.1).

Lower bound. From Proposition 1.2, we have that

$$u_n\ge n-c_{34}n^q\rho^{2\alpha-1}\quad \text{in}\quad \Omega,$$

then for n big, let $r=(\lambda n)^{-\frac{q-1}{2\alpha-1}}$, where $\lambda=(2^{2\alpha}c_{34})^{\frac{1}{q-1}}$ chosen later, then for $x\in\Omega_r\setminus\Omega_{2r}$, we have that

$$\begin{aligned}
u_n(x)&\ge \frac{1}{\lambda}r^{-\frac{2\alpha-1}{q-1}}-c_{34}\frac{1}{\lambda^p}r^{-\frac{2\alpha-1}{q-1}p}(2r)^{2\alpha-1}\\
&\ge \frac{1}{\lambda}(1-\frac{2^{2\alpha-1}c_{34}}{\lambda^{q-1}})r^{-\frac{2\alpha-1}{q-1}}\\
&\ge \frac{1}{2\lambda}\rho(x)^{-\frac{2\alpha-1}{q-1}}.
\end{aligned}$$

where λ is independent of n. For any $x\in\Omega\setminus\Omega_{r_0}$, there exists n such that

$$u_\infty(x)\ge u_n(x)\ge \frac{1}{2\lambda}\rho(x)^{-\frac{2\alpha-1}{q-1}}.$$

We notice that the solution u_∞ is the minimal solution of (1.1), since for any boundary blow-up solution u, we may imply by comparison principle that $u\ge u_n$ in Ω, which infers that $u_\infty\le u$ in Ω. The proof ends. $\square$

Now we give the proof for the nonexistence part of Theorem 1.1.

Proof. **(Proof of Theorem 1.1** *(ii)***.)** If $q\le 1$, we observe that for $n>1$,

$$u_n\ge nu_1\quad \text{in}\quad \Omega,$$

which implies that (1.1) has no solution.

In what follows, we assume that $q>1$. By contradiction, we may assume that there exists a solution u of (1.1) when $f(s)\le c_{11}s^q$ for $s\ge 0$ and q satisfying (1.12). By Theorem 2.1, we have that

$$u_n\le u\quad \text{in}\quad \Omega.$$

From Proposition 1.2, we have that

$$u_n\ge n-c_{34}n^q\rho^{2\alpha-1}\quad \text{in}\quad \Omega.$$

Then for n big, let $r_n=(\lambda n)^{-\frac{q-1}{2\alpha-1}}$, where $\lambda=(2^{2\alpha}c_{34})^{\frac{1}{q-1}}$ chosen later, then for $x\in\Omega_{r_n}\setminus\Omega_{2r_n}$, we have that

$$u_n(x)\geq\frac{1}{\lambda}r_n^{-\frac{2\alpha-1}{q-1}}-\frac{c_{34}}{\lambda^p}r_n^{-\frac{2\alpha-1}{q-1}p}(2r_n)^{2\alpha-1}\geq\frac{1}{2\lambda}\rho(x)^{-\frac{2\alpha-1}{q-1}}.$$

For any $x\in\Omega\setminus\Omega_{r_0}$, there exists n such that

$$u(x)\geq u_n(x)\geq\frac{1}{2\lambda}\rho(x)^{-\frac{2\alpha-1}{q-1}}. \tag{3.12}$$

When $1<q\leq 2\alpha$, we have that $\rho^{-\frac{2\alpha-1}{q-1}}$ is not in $L^1(\Omega)$, then it follows from (3.12) for any $x\in\Omega$ and any $\epsilon>0$

$$\begin{aligned}(-\Delta)^\alpha_{\Omega,\epsilon}u(x)&\leq-\int_{\Omega\setminus B_\epsilon(0)}\frac{u_n(y)-u(x)}{|x-y|^{N+2\alpha}}dy\\&\leq-\epsilon^{-N-2\alpha}\left[\int_\Omega u_n(y)dy-u(x)|\Omega|\right]\\&\to-\infty\quad\text{as}\quad n\to\infty,\end{aligned}$$

which is impossible.

From (1.12), we have that $-\frac{2\alpha-1}{q-1}<\alpha-1$, then if follows from (3.12) that

$$\lim_{\rho(x)\to 0^+}u(x)\rho^{1-\alpha}(x)=+\infty, \tag{3.13}$$

which contradicts Proposition 1.1 (i). □

Acknowledgements. H. Chen is supported by NSFC (No:11726614, 11661045), by the Jiangxi Provincial Natural Science Foundation, No: 20161ACB20007, and by Doctoral Research Foundation of Jiangxi Normal University.

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