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A VARIATIONAL STRUCTURE FOR INTERACTING PARTICLE SYSTEMS AND THEIR HYDRODYNAMIC SCALING LIMITS*

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Abstract. We consider hydrodynamic scaling limits for a class of reversible interacting particle systems, which includes the symmetric simple exclusion process and certain zero-range processes. We study a (non-quadratic) microscopic action functional for these systems. We analyse the behaviour of this functional in the hydrodynamic limit and we establish conditions under which it converges to the (quadratic) action functional of Macroscopic Fluctuation Theory. We discuss the implications of these results for rigorous analysis of hydrodynamic limits.

Keywords. Interacting Particle Systems; Macroscopic Fluctuation Theory; Large Deviations; action functionals; Γ-convergence.

AMS subject classifications. 82C22; 35Q35; 76M28.

1. Introduction

Recently, a *canonical structure* has been introduced [29, 30] to describe dynamical fluctuations in stochastic systems. The resulting theory has several attractive features: Firstly, it applies to a wide range of systems, including finite-state Markov chains and Macroscopic Fluctuation Theory (MFT) [5], see [21]. Secondly, it is based on an *action functional* which is a relative entropy between probability measures on path spaces — this means that it provides a variational description of the systems under consideration, and the action can be related to large deviation rate functionals. Thirdly, it extends the classical Onsager-Machlup theory [34] in a natural way, by replacing the quadratic functionals that appear in that theory with a pair of convex but *non-quadratic* Legendre duals Ψ and $\Psi^\star$. (This is sometimes called a Ψ-$\Psi^\star$ representation [31].) In Onsager-Machlup theory and in MFT, the minimiser of the action describes the most probable evolution of a macroscopic system, either in terms of thermodynamic forces and fluxes (in Onsager-Machlup theory) or densities and fluxes (in MFT): this feature is maintained in the canonical structure.

This structure can be applied to any finite-state Markov chain and provides a unifying formulation of a wide range of systems [21]. In particular, lattice systems of interacting particles can be described by canonical structures in two ways: either on the microscopic (Markov chain) level via non-quadratic Legendre duals, or as a coarse-grained version through the hydrodynamic limit, where the action reduces to a quadratic MFT functional. One therefore expects that in the hydrodynamic scaling limit, the microscopic (non-quadratic) structure should converge (in some suitable sense) to the macroscopic one. Such a convergence would offer a new way to understand and derive hydrodynamic limits. The main question of this article is whether this natural conjecture holds.

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We give a partial (positive) answer, by proving several theorems that relate the microscopic and macroscopic action functionals for interacting particle systems. Specifically, we consider a class of systems on periodic lattices with *gradient dynamics* and a conserved number of particles, which includes as special cases the symmetric simple exclusion process and a large class of reversible zero-range processes. In the hydrodynamic limit, the number of lattice sites and the number of particles go to infinity together, at fixed density, and the microscopic transition rates have a parabolic scaling. (These are among the simplest models for which one can rigorously establish a hydrodynamic limit [22].)

Our analysis is based on the microscopic action, which is a relative entropy between two probability measures: one measure encodes the dynamics of the particle system itself (the *reference process*) and the other represents some other *observed process*, which is to be compared with the reference process. We consider observed processes that concentrate (in the hydrodynamic limit) on deterministic paths. By comparing different processes, we can extract information about the hydrodynamic limit of the reference process (if this limit exists). That is, the reference process and the observed process have different hydrodynamic limits in general, and the macroscopic action functional measures the difference between them. It is minimised in the case where the observed process and the reference process coincide, in which case the action is zero — under suitable assumptions, this means that the hydrodynamic limit of the reference process can be characterised as the minimiser of the macroscopic action. Moreover, the macroscopic action can be represented as a sum of three terms — we show that these individual contributions are asymptotically dominated by corresponding contributions to the microscopic action, see Theorem 3.3. Then, for a specific choice of the observed process (which is related to the hydrodynamic limit of the reference process), we show that the microscopic action converges to the macroscopic one, see Theorems 3.4 and 3.5.

The inspiration for this study comes from [18] and [16], which derive hydrodynamic (or mean-field) limits as minimisers of macroscopic action functionals, for the simple exclusion process [18] and for a McKean-Vlasov equation on a finite graph [16]. In common with these works, our approach is (loosely) based on the Sandier-Serfaty approach [37] to study sequences of gradient flows via Γ-convergence. However, our approach is different from [16, 18] because it starts from the (non-quadratic) canonical structure, instead of the *quadratic* structure for time-reversal symmetric Markov chains, that was independently derived by Maas [28] and Mielke [32]. A similar structure to the canonical one exploited here was recently used in [2] to derive a diffusive limit for the linear Boltzmann equation. All of these approaches have in common that they consider time-reversal symmetric systems for which the dynamics can be identified with gradient flows of a free energy functional, so that the limiting probability measure concentrates on curves of maximal slope, which can be identified as minimisers of the macroscopic action. Further, our approach is also closely related to EDP-convergence, where EDP stands for Energy-Dissipation-Principle, see e.g. [8, 14, 24, 33].

Compared with previous studies, our work has two novel features. First, we do not restrict to curves of maximal slope (which follow the gradient of the free energy): instead we consider a class of paths for which the microscopic action functional stays controlled, in the hydrodynamic limit. In principle, this means that our methods are not limited to time-reversal symmetric systems: the corresponding action functional can be defined for a large class of Markov chains in a meaningful way. However, in order to reduce the number of technical issues we have to deal with, we limit ourselves to reversible systems in this work. (More precisely, we consider Markov chains with (in general) time-dependent rates, where the rates at every time obey detailed balance with respect to an invariant measure that also (in general) depends on time. This means that we can exploit readily-available tools from the theory of hydrodynamic limits for these processes, notably the replacement lemma.) An extension to systems without detailed balance is left for future work.

The second novel aspect is that we consider particle systems for which the hydrodynamic limit is a *non-linear* diffusion equation, in contrast (for example) to the symmetric exclusion process studied in [18], whose hydrodynamic limit is linear diffusion. This is a significant difference for rigorous results: within the canonical structure one sees naturally that the hydrodynamic limit is a (generalised) gradient flow, as expected on physical grounds. However, in contrast to (linear) diffusion with a linear mobility, where the (now-)classic Wasserstein evolution provides the natural geometrical setting for the gradient flow, the analogous setting for diffusions with non-linear mobility is not so well-developed. In particular, a key challenge is to establish the validity of a chain rule for the macroscopic entropy functional, which is known for linear diffusion [1], but whose extension to the non-linear setting is not at all straightforward. We show here that (with some technical effort) the required results for non-linear diffusion can be obtained by casting the evolution into the classic Wasserstein setting (Theorem 4.2): this is not the most natural (physical) setting for the process of interest, but it is sufficient to establish the required results.

This line of research — linking Markov chains and partial differential equations via canonical structures — is quite recent. Consequently, a number of problems remain open. In particular, our approach is not yet a hydrodynamic limit passage: for this, the macroscopic concentration of the limiting path measure would have to be proved. Also, the microscopic action converges in the hydrodynamic limit to a macroscopic action functional that turns out to coincide with a large deviation rate functional [5]. However, in this work we do not establish any links to large deviation theory; this could be a natural future line of research (e.g. one could consider similar calculations to the ones in [15] for independent particles with Langevin dynamics). Another question is whether (and how) the method presented here can provide guidance for limit passages for non-reversible systems.

Our study combines techniques from a number of different fields: we have attempted to make it self-contained (and hence accessible to a general reader), at the expense of including some classical material (which expert readers may prefer to skip). This is indicated in the beginning of the relevant sections. In Section 2, we describe the particle systems and their canonical structure. Section 3 states the main results. Section 4 is entirely devoted to technical questions of regularity and a proof of the chain rule, while Section 5 contains the proofs of the main theorems.

2. Interacting particle systems

2.1. Particle systems on the discrete torus. The setting we analyse covers a broad class of particle models, as we now describe. This section also collects some classic facts on particle models. We consider systems with a fixed number of indistinguishable particles, distributed over the L^d sites of the flat torus $\mathbb{T}_L^d := \mathbb{Z}^d/(L\mathbb{Z}^d)$. Let $\eta(i)$ be the number of particles on site $i \in \mathbb{T}_L^d$, so the configuration space of the system is $\Omega_L \subseteq \mathbb{N}_0^{\mathbb{T}_L^d}$. Configurations are denoted with $\eta = (\eta(i))_{i\in\mathbb{T}_L^d}$. Let $\eta^{i,i'}$ be the configuration obtained from η by moving a particle from site i to site i'. The total number of particles on each site may be bounded by $N_{\max} \in \mathbb{N}_0$, that is, $\Omega_L = \{0,\dots,N_{\max}\}^{\mathbb{T}_L^d}$, or unbounded. We fix $T>0$ and consider the time interval $[0,T]$. The (random) state of the system at time

$t\in[0,T]$ is denoted by η_t.

The particles hop between sites of the lattice with some rate $\hat{r}_{\eta,\eta^{i,i'}}$, which is assumed to be non-zero only if i and i' are neighbours, $|i-i'|=1$. We consider a parabolic scaling, so the hydrodynamic limit is obtained by rescaling time by a factor L^2, such that the transition rates for the Markov chain are $r_{\eta,\eta^{i,i'}}=L^2\hat{r}_{\eta,\eta^{i,i'}}$. Let Λ be the flat torus $\mathbb{T}^d=[0,1)^d$. The jump rates for the particle models considered in this article depend on an external potential $V\in C^2(\Lambda;\mathbb{R})$, and two functions $g_1,g_2\colon \mathbb{N}_0\to[0,\infty)$, such that

$$\hat{r}^V_{\eta,\eta^{i,i'}}=g_1(\eta(i))g_2(\eta(i'))\mathrm{e}^{-\frac{1}{2}(V(i'/L)-V(i/L))}. \tag{2.1}$$

We also consider time-dependent potentials $\tilde{V}\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ which lead to a time-heterogeneous Markov chain with transition rates $r^{\tilde{V}_t}$ at time $t\in[0,T]$. We write $\tilde{V}$ for a time-dependent potential and V for a time-independent potential. The choice in (2.1) includes many particle processes, such as the zero-range process and the simple exclusion process. This specific form was chosen to enable the use of existing results from the theory of hydrodynamic limits, notably the replacement lemma employed below.

An interacting particle system has *gradient dynamics* (or is of *gradient type*) if there exists a function $\mathrm{d}\colon \mathbb{N}_0\to[0,\infty)$ such that (for $V=0$) $r^0_{\eta,\eta^{i,i'}}-r^0_{\eta,\eta^{i',i}}=\mathrm{d}(\eta(i))-\mathrm{d}(\eta(i'))$. In this case we define $\hat{\phi}_i(\mu):=\sum_{\eta\in\Omega_L}\mu(\eta)\mathrm{d}(\eta(i))$. (Note that this is the simplest form of a gradient system, which in more generality can consist of differences of finite cylinder functions, cf. [22]).

2.1.1. Invariant measures, initial conditions, and microscopic free energy. The number of particles is conserved by the dynamics, so these systems have many possible invariant measures. The hydrodynamic limit relies on a particular structure for these measures, as follows. Let ν_* be a (not necessarily normalised) reference measure on Ω_L, with $\nu_*(\eta)>0$ for all $\eta\in\Omega_L$, which is assumed to have a product structure in the sense that $\nu_*(\eta)=\prod_{i\in\mathbb{T}^d_L}\nu_{*,1}(\eta(i))$ for some probability measure $\nu_{*,1}$ on $\mathbb{N}_0$. We assume that the process with rates $\hat{r}^0$ satisfies the *detailed balance condition*

$$\nu_*(\eta)\hat{r}^0_{\eta,\eta^{i,i+e_k}}=\nu_*(\eta^{i,i+e_k})\hat{r}^0_{\eta^{i,i+e_k},\eta} \tag{2.2}$$

for all $\eta\in\Omega_L$, $i\in\mathbb{T}^d_L$ and $k=1,\dots,d$. This implies that ν_* is invariant for the dynamics $\hat{r}^0$ and that these dynamics are time reversal-symmetric with respect to ν_*. To avoid technical difficulties, we further assume that the one site *partition function* is finite, i.e. for all $\theta\in\mathbb{R}$

$$Z_1(\theta):=\sum_{n\in\mathbb{N}_0}\mathrm{e}^{\theta n}\nu_{*,1}(n)<\infty. \tag{2.3}$$

In classical statistical mechanics (see for example [4, Section 3] or [9]), the *local free energy density* is given by the Legendre dual of the *cumulant generating function (or pressure)* of $\nu_{*,1}$, i.e.

$$f(a)=\sup_{\theta\in\mathbb{R}}\big(a\theta-\log Z_1(\theta)\big)=af'(a)-\log Z_1(f'(a)), \tag{2.4}$$

which implies that f is convex. In the following, we will assume that $f\in C^2([0,N_{\max}];\mathbb{R})$ and that a.e. $f''>0$, see Section 2.4.2). Now, for $\alpha\in(0,N_{\max})$, we define the probability measures

$$\nu_{\alpha,1}(n):=\frac{\mathrm{e}^{f'(\alpha)n}}{Z_1(f'(\alpha))}\nu_{*,1}(n) \tag{2.5}$$

and $\nu_\alpha := \prod_{i\in\mathbb{T}_L^d} \nu_{\alpha,1}$. For each $\alpha \in (0, N_{\max})$ this choice implies that $E_{\nu_\alpha}\left[\sum_{i\in\mathbb{T}_L^d} \eta(i)/L^d\right] = \alpha$ (where E_{ν_α} denotes the expectation with respect to ν_α) and that ν_α is stationary and satisfies (2.2) for the process with rates $\hat{r}^0$. For an external potential $V \in C^2(\Lambda;\mathbb{R})$ the process with rates $\hat{r}^V$ satisfies detailed balance with respect to the probability measures $\nu_\alpha^V(\eta) \propto \nu_\alpha(\eta)\mathrm{e}^{-\sum_{i\in\mathbb{T}_L^d} V(i/L)\eta(i)}$. For the measure ν_α^V, the expected number of particles at $u \in \Lambda$ is defined as

$$\bar{\rho}_{\alpha,V}(u) := \frac{E_{\nu_{\alpha,1}}\left[\eta(0)\mathrm{e}^{-V(u)\eta(0)}\right]}{E_{\nu_{\alpha,1}}\left[\mathrm{e}^{-V(u)\eta(0)}\right]} < \infty. \tag{2.6}$$

Combining (2.6) with (2.5) allows to show that $\bar{\rho}_{\alpha,V}(u) = (f')^{-1}(-V(u) + f'(\alpha))$, or equivalently $f'(\bar{\rho}_{\alpha,V}(u)) = -V(u) + f'(\alpha)$. Consequently (2.6) is strictly monotonically increasing in α. Since the number of particles is conserved, its distribution is fully determined by the initial condition for the model. In everything that follows, we restrict to initial distributions $(\mu_0^L)_{L\in\mathbb{N}}$ for which the total density of particles is bounded uniformly: there exists $C_{\mathrm{tot}} \in (0, N_{\max}]$ such that for all $L \in \mathbb{N}$

$$\mu_0^L\left(\eta \in \Omega_L \,\middle|\, \frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d} \eta(i) \le C_{\mathrm{tot}}\right) = 1. \tag{2.7}$$

This means that the Markov chain is supported on finitely many configurations, allowing us to treat each particle system as a finite state Markov chain. Finally, for any $V \in C^2(\Lambda;\mathbb{R})$ and any α, define the relative entropy (or *microscopic free energy*) as

$$\mathcal{F}_{L,\alpha}^V(\mu) := \mathcal{H}(\mu|\nu_\alpha^V) = \sum_{\eta\in\Omega_L} \mu(\eta)\log\left(\frac{\mu(\eta)}{\nu_\alpha^V(\eta)}\right), \tag{2.8}$$

where μ is a probability measure (on Ω_L). If μ is the probability measure for our interacting particle system at some time t then $\mathcal{F}_{L,\alpha}^V(\mu) < \infty$, by (2.7), since $\nu_*(\eta) > 0$ for all $\eta \in \Omega_L$.

2.1.2. Canonical structure for Markov chains. We now describe a Ψ-$\Psi^\star$ structure for finite state Markov chains which is related to a relative entropy between path measures [21]. This structure is central to this article (see also [29, 30]). Let μ be a probability measure on Ω_L supported on finitely many configurations. We think of this measure as a (generic) distribution of the particle system. For $\eta, \eta' \in \Omega_L$ we define the *probability current* from η to η' as

$$J_{\eta,\eta'}(\mu) := \mu(\eta) r_{\eta,\eta'}^V - \mu(\eta') r_{\eta',\eta}^V. \tag{2.9}$$

The divergence at η is $\operatorname{div} J(\mu)(\eta) := \sum_{\eta'\in\Omega_L} J_{\eta,\eta'}(\mu)$. Following [21], define a *mobility*

$$a_{\eta,\eta'}(\mu) := 2\left[\mu(\eta) r_{\eta,\eta'}^V \mu(\eta') r_{\eta',\eta}^V\right]^{1/2} \tag{2.10}$$

which is independent of V since $\hat{r}_{\eta,\eta'}^V \hat{r}_{\eta',\eta}^V = \hat{r}_{\eta,\eta'}^0 \hat{r}_{\eta',\eta}^0$. Let the discrete gradient of a function h on Ω_L be $\nabla^{\eta,\eta'} h := h(\eta') - h(\eta)$ and define a *thermodynamic force* (cf. [21, 29, 30]) as

$$F_{\eta,\eta'}^V(\mu) := -\nabla^{\eta,\eta'} \log\left(\frac{\mu}{\nu_\alpha^V}\right), \tag{2.11}$$

which is in fact independent of α, as $\nu_\alpha(\eta)/\nu_\alpha(\eta^{i,i'}) = \nu_*(\eta)/\nu_*(\eta^{i,i'})$. For a general interpretation of the mobility and the force and their physical relation to thermodynamic quantities, such as entropy production and housekeeping heat, we refer the reader to [21].

The canonical structure is based on a dual paring between currents and thermodynamic forces. We consider generic currents j and forces F, which are arbitrary anti-symmetric functions on $\Omega_L \times \Omega_L$ with $j_{\eta,\eta'} = -j_{\eta',\eta}$ and $F_{\eta,\eta'} = -F_{\eta',\eta}$. The dual pairing is $\langle j, F\rangle_L := \frac{1}{2}\sum_{\eta,\eta'\in\Omega_L} j_{\eta,\eta'} F_{\eta,\eta'} \mathbf{1}_{\{a_{\eta,\eta'}(\mu)>0\}}$ (which implicitly depends on μ). Here $\mathbf{1}_A$ is the indicator function of the event A, which is given by $\mathbf{1}_A = 1$ if the statement A is satisfied and $\mathbf{1}_A = 0$ otherwise. Now define

$$\Psi_L^\star(\mu, F) := \sum_{\eta,\eta'\in\Omega_L} a_{\eta,\eta'}(\mu)\Big[\cosh\big(\tfrac{1}{2}F_{\eta,\eta'}\big) - 1\Big] \tag{2.12}$$

and

$$\begin{aligned}\Psi_L(\mu, j) := \sum_{\eta,\eta'\in\Omega_L} a_{\eta,\eta'}(\mu)\Bigg[\frac{j_{\eta,\eta'}}{a_{\eta,\eta'}(\mu)} \operatorname{arcsinh}\Big(\frac{j_{\eta,\eta'}}{a_{\eta,\eta'}(\mu)}\Big) \\ - \cosh\Bigg(\operatorname{arcsinh}\Big(\frac{j_{\eta,\eta'}}{a_{\eta,\eta'}(\mu)}\Big)\Bigg) + 1\Bigg],\end{aligned} \tag{2.13}$$

where the summands in (2.13) have to be interpreted as being equal to zero whenever $a_{\eta,\eta'}(\mu) = 0$. The two functions (2.12) and (2.13) are both symmetric and strictly convex in their second argument. Moreover, they are Legendre dual with respect to the dual pairing $\langle j, F\rangle_L$ and give rise to the *Onsager-Machlup functional*,

$$\Phi_L(\mu, j, F) := \Psi_L(\mu, j) - \langle j, F\rangle_L + \Psi_L^\star(\mu, F) \geq 0, \tag{2.14}$$

where the inequality follows from the Fenchel-Young inequality (which directly follows from the Legendre duality of Ψ and $\Psi^\star$). This functional will be used in the following to characterise the relative entropy between path measures. In particular, we will study the convergence of the *non-quadratic* functionals Ψ and $\Psi^\star$ to their quadratic counterparts to a macroscopic quadratic functional, which has the form of the macroscopic Onsager-Machlup functional.

2.1.3. Projection onto the physical domain. So far we considered currents and densities on the full configuration space Ω_L. To obtain hydrodynamic behaviour, we 'project' the system onto the physical domain $\mathbb{T}_L^d$ and also embed the sequence of these domains (indexed by L) into the flat torus Λ. This section introduces the associated notation.

For a (generic) probability measure μ on Ω_L (which we again think of as the current distribution of the particle system), we can define the averaged number of particles $\hat\rho_i(\mu)$ at site $i \in \mathbb{T}_L^d$ and an averaged particle current $\hat\jmath^V_{i,i'}(\mu)$, as

$$\hat\rho_i(\mu) := \sum_{\eta\in\Omega_L} \mu(\eta)\eta(i) \qquad \text{and} \qquad \hat\jmath^V_{i,i'}(\mu) := \sum_{\eta\in\Omega_L} \mu(\eta)\big(\hat r^V_{\eta,\eta^{i,i'}} - \hat r^V_{\eta,\eta^{i',i}}\big). \tag{2.15}$$

The current $\hat\jmath^V_{i,i'}(\mu)$ describes the expected net flow of particles from site i to site i' if the distribution of the particle system is given by μ. For gradient dynamics and $V = 0$ the current (2.15) is

$$\hat\jmath^0_{i,i'}(\mu) = \hat\phi_i(\mu) - \hat\phi_{i'}(\mu) = -\nabla^{i,i'}\hat\phi(\mu), \tag{2.16}$$

where the discrete gradient on $\mathbb{T}_L^d$ is (for $h\colon \mathbb{T}_L^d \to \mathbb{R}$) defined as $\nabla^{i,i'} h = h(i') - h(i)$. Similar to (2.15), define also two (averaged) mobilities for the edge connecting i and i' as

$$\hat{a}_{i,i'}(\mu) := \sum_{\eta\in\Omega_L} 2\left[\mu(\eta)\hat{r}^V_{\eta,\eta^{i,i'}}\mu(\eta^{i,i'})\hat{r}^V_{\eta^{i,i'},\eta}\right]^{1/2}, \quad \hat{\chi}^V_{i,i'}(\mu) := \frac{1}{2}\sum_{\eta\in\Omega_L}\mu(\eta)\left(\hat{r}^V_{\eta,\eta^{i,i'}} + \hat{r}^V_{\eta,\eta^{i',i}}\right), \tag{2.17}$$

which are related by $\hat{a}_{i,i'}(\mu) \le 2\hat{\chi}^V_{i,i'}(\mu)$ (with equality for $\mu = \nu^V_\alpha$). Note that the two mobilities characterise the average particle jumps between i and i' and are therefore symmetric in i and i'.

For the embedding on the flat torus, let $\mathcal{M}_+(\Lambda)$ be the set of finite and non-negative Radon measures on Λ, endowed with the weak topology. Define the empirical measure $\Theta_L\colon \Omega_L \to \mathcal{M}_+(\Lambda)$ as

$$\Theta_L(\eta) := \frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\eta(i)\delta_{i/L}. \tag{2.18}$$

Thus, each configuration η of an interacting particle system of size L corresponds to a measure $\Theta_L(\eta) \in \mathcal{M}_+(\Lambda)$.

2.1.4. Reference process and observed process. We analyse hydrodynamic limits by comparing different (microscopic) processes. For any given L, the *reference process* is an interacting particle system on the discrete torus, as defined in Section 2.1. The *observed process* is another interacting particle system on the same space, whose path measure (see below) is absolutely continuous with respect to the reference process. Hydrodynamic limits are analysed by considering sequences of observed and reference processes, indexed by L. With slight abuse of terminology, we sometimes refer to the sequence of observed processes as simply "the observed process", and similarly for the reference process.

We consider observed processes with unique hydrodynamic limits. This leads to a variational characterisation of the hydrodynamic limit of the reference process, by minimising the relative entropy between the reference process and the observed process. This follows the usual approach in the calculus of variations: one considers observed processes with (known) hydrodynamic limits, which are candidates for the hydrodynamic limit of the reference process. The optimal candidate is the one that minimises the relative entropy, and the hydrodynamic limit of this optimal candidate matches the hydrodynamic limit of the reference process (assuming that it exists).

2.2. Path measures on the microscopic scale.

2.2.1. Path measures for the reference and observed processes. Our analysis of the hydrodynamic limit is based on the convergence of path measures. In this section, we introduce the notation that allows us to define the path measures Q_L and limit measures Q^* studied in the remainder of the article.

For any topological space $\mathcal{S}$ we denote with $\mathcal{D}([0,T];\mathcal{S})$ the set of $\mathcal{S}$ valued càdlàg paths (right-continuous paths with left limits) on $[0,T]$. For details, see [7, Chapter 3], as well as [22, Chapter 4.1] and [6]. For $t\in[0,T]$ let $X_t\colon \mathcal{D}([0,T];\mathcal{S}) \to \mathcal{S}$ be the marginal at time t, which evaluates a path $\gamma = (\gamma_t)_{t\in[0,T]} \in \mathcal{D}([0,T];\mathcal{S})$ at time t: $X_t(\gamma) = \gamma_t$. We recall that whilst X_t is measurable for all $t\in[0,T]$, it is continuous only for almost all $t\in(0,T)$, as well as $t=0$ and $t=T$. In the following, the expression *path measure* will refer to a probability distribution on $\mathcal{D}([0,T];\mathcal{S})$ for some $\mathcal{S}$.

Given some L, the *reference process* is a particle system with a time-dependent potential $\tilde{V}\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$, whose path measure [on $\mathcal{D}([0,T];\Omega_L)$] is denoted by $P_L^{\tilde{V}}$. We can recover the distribution of this Markov chain at time t from $P_L^{\tilde{V}}$ via the push-forward measure $(X_t)_\# P_L^{\tilde{V}}$.

The *observed process* can be any (possibly time-heterogeneous) Markov chain on Ω_L, whose path measure [on $\mathcal{D}([0,T];\Omega_L)$] is denoted by P_L. This process is assumed to have the following properties: the path measure P_L is absolutely continuous with respect to $P_L^{\tilde{V}}$, the initial condition of P_L coincides with the one of $P_L^{\tilde{V}}$, that is, $(X_0)_\# P_L = (X_0)_\# P_L^{\tilde{V}} = \mu_0^L$, and the transition rates r_t^L are bounded in time, i.e. for each $L\in\mathbb{N}$, we assume that $\sup_{t\in[0,T]}(r_t^L)_{\eta,\eta'}<\infty$ for all $\eta,\eta'\in\Omega_L$.

We can assign to P_L a unique path $(\mu_t^L, \jmath_t^L)_{t\in[0,T]}$ consisting of the density $\mu_t^L := (X_t)_\# P_L$ and the current $(\jmath_t^L)_{\eta,\eta'} := \mu_t^L(\eta)(r_t^L)_{\eta,\eta'} - \mu_t^L(\eta')(r_t^L)_{\eta',\eta}$, which are again linked by a continuity equation $\partial_t \mu_t^L = -\operatorname{div} \jmath_t^L$.

We remark that for the choice $P_L = P_L^{\tilde{V}}$ the current $\jmath_t^L$ simply coincides with the probability current (2.9) for the time-dependent rate $r^{\tilde{V}_t}$. In this case, one can further show that the associated density and current (2.15) satisfy the continuity equation $\partial_t \hat{\rho}_i(\mu_t^L) = -\operatorname{div} \hat{\jmath}^V(\mu_t^L)(i)$, where the divergence on the physical domain $\mathbb{T}_L^d$ is defined as $\operatorname{div}\hat{\jmath}^V(\mu)(i) := \sum_{i'\in\mathbb{T}_L^d} \hat{\jmath}^V_{i,i'}(\mu)$.

Since every Ω_L can be embedded into the flat torus Λ (as a map from Ω_L to $\mathcal{M}_L(\Lambda)$), there is a corresponding embedding of the path space $\mathcal{D}([0,T];\Omega_L)$ into $\mathcal{D}([0,T];\mathcal{M}_L(\Lambda))$. In particular, each path measure Q_L on $\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$ that is supported on $\mathcal{M}_L(\Lambda) := \{L^{-d}\sum_{i\in\mathbb{T}_L^d} k_i\delta_{i/L} \,|\, k_i\in\mathbb{N}_0, k_i\le N_{\max}\}$ can be identified with a unique measure P_L on $\mathcal{D}([0,T];\Omega_L)$. The measure on $\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$ that corresponds to the reference process $P_L^{\tilde{V}}$ is denoted with $Q_L^{\tilde{V}}$. Similarly, for the observed process, there is a Q_L corresponding to P_L. No information is lost on embedding the processes into Λ, so $\mathcal{H}(Q_L|Q_L^{\tilde{V}}) = \mathcal{H}(P_L|P_L^{\tilde{V}})$, which can be proved by two applications of Lemma 9.4.5 in [1] with the bijection from $\mathcal{M}_L(\Lambda)$ to Ω_L.

We summarise this notation, which will be used extensively below: the reference process and the observed processes can be fully characterised by their path measures [both on $\mathcal{D}([0,T];\mathcal{M}_L(\Lambda))$], which are denoted by $Q_L^{\tilde{V}}$ and Q_L respectively. There are corresponding path measures on $\mathcal{D}([0,T];\Omega_L)$ which are denoted by $P_L^{\tilde{V}}$ and P_L.

2.2.2. Microscopic action functional. To compare the reference and the observed process, consider the thermodynamic force for the reference process at time t, which is $F^{\tilde{V}_t}(\mu_t^L)$, evaluated from (2.11) with $\mu_t^L = (X_t)_\# P_L$. Since P_L is absolutely continuous with respect to $P_L^{\tilde{V}}$, the relative entropy $\mathcal{H}(P_L|P_L^{\tilde{V}})$ is, under the assumptions in Section 2.2.1, finite and (cf. [21, Appendix]) coincides with

$$\mathcal{H}\big(P_L|P_L^{\tilde{V}}\big) = \mathcal{H}\big(\mu_0^L|(X_0)_\# P_L^{\tilde{V}}\big) + \frac{1}{2}\int_0^T \Phi_L\big(\mu_t^L, \jmath_t^L, F^{\tilde{V}_t}(\mu_t^L)\big)\,\mathrm{d}t. \tag{2.19}$$

Moreover, $\mathcal{H}(\mu_0^L|(X_0)_\# P_L^{\tilde{V}}) = 0$, since P_L and $P_L^{\tilde{V}}$ share the same initial condition. We interpret $\frac{1}{2}\Phi_L(\mu_t^L, \jmath_t^L, F_\alpha^{\tilde{V}_t}(\mu_t^L))$ as an extended Lagrangian [21] and define the *microscopic action* of the path measure Q_L as the relative entropy

$$\mathbb{A}_L^{\tilde{V}}(Q_L) := \mathcal{H}\big(Q_L|Q_L^{\tilde{V}}\big) = \mathcal{H}\big(P_L|P_L^{\tilde{V}}\big) = \frac{1}{2}\int_0^T \Phi_L\big(\mu_t^L, \jmath_t^L, F^{\tilde{V}_t}(\mu_t^L)\big)\,\mathrm{d}t. \tag{2.20}$$

This is the central functional defined on the discrete (lattice) level studied in this article.

2.3. Macroscopic quantities. In the hydrodynamic scaling limit, the microscopic action (2.20) will converge to a macroscopic action, which is (2.30). (For the macroscopic setting, we restrict our considerations to potentials V that are constant in time.) We now show how the macroscopic action functional is constructed.

2.3.1. The macroscopic free energy. For $\alpha \in (0, N_{\max}]$ and $V \in C^2(\Lambda;\mathbb{R})$, we define the *macroscopic free energy* $\mathcal{F}_\alpha^V : \mathcal{M}_+(\Lambda) \to [0,\infty]$ as

$$\mathcal{F}_\alpha^V(\pi) := \sup_{h \in C(\Lambda;\mathbb{R})} \left[\langle \pi, h \rangle - \int_\Lambda \log\left(\frac{Z_1(f'(a)+h(u)-V(u))}{Z_1(f'(a)-V(u))} \right) \mathrm{d}u \right]. \tag{2.21}$$

This free energy coincides with a rate function: there is a large-deviation principle for the particle configuration Θ_L sampled from the steady state ν_α^V; the speed of this LDP is L^d and its rate function is $\mathcal{F}_\alpha^V(\pi)$, (see e.g. Section 5.1, page 75 in [22] for the special case of a zero-range process). From (2.3), $\mathcal{F}_\alpha^V(\pi)$ is finite only if $\pi(\mathrm{d}u) = \rho(u)\mathrm{d}u$ for some density $\rho \in \mathcal{L}^1(\Lambda;[0,\infty))$. In the following we thus write $\mathcal{F}_\alpha^V(\rho)$ for $\mathcal{F}_\alpha^V(\pi)$. As in Macroscopic Fluctuation Theory [5, Section 5.A], we can represent $\mathcal{F}_\alpha^V$ for reversible systems as

$$\mathcal{F}_\alpha^V(\rho) = \int_\Lambda \Big[f(\rho(u)) - f(\bar\rho_{\alpha,V}(u)) - f'(\bar\rho_{\alpha,V}(u))\big(\rho(u) - \bar\rho_{\alpha,V}(u)\big) \Big] \mathrm{d}u, \tag{2.22}$$

where $\bar\rho_{\alpha,V} \in \mathcal{L}^1(\Lambda;[0,\infty))$, introduced in (2.6), is the steady state density for the dynamics of the macroscopic system. Note that (2.22) inherits the convexity of f.

2.3.2. The hydrodynamic current and the hydrodynamic equation. In the hydrodynamic limit, the particle density at time t is given by some $\rho_t \in \mathcal{L}^1(\Lambda;[0,\infty))$. The hydrodynamic current describes the resulting particle flow:

$$J(\rho) := -\nabla\phi(\rho) - \chi(\rho)\nabla V, \tag{2.23}$$

where ϕ and χ are functions that depend on the system of interest and are discussed later in this section. The hydrodynamic equation is then

$$\dot\rho_t = -\nabla\cdot J(\rho_t) = \Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V). \tag{2.24}$$

In this article, we consider weak solutions to (2.24), in the sense that for all $G \in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$

$$\begin{aligned} &\int_\Lambda \rho_T G_T \,\mathrm{d}u - \int_\Lambda \rho_0 G_0 \,\mathrm{d}u - \int_0^T \int_\Lambda \rho_t \partial_t G_t \,\mathrm{d}u\mathrm{d}t \\ &= \int_0^T \int_\Lambda \phi(\rho_t)\Delta G_t \,\mathrm{d}u\mathrm{d}t - \int_0^T \int_\Lambda \chi(\rho_t)\nabla V\cdot\nabla G_t \,\mathrm{d}u\mathrm{d}t. \end{aligned} \tag{2.25}$$

The dynamics on the macroscopic scale are characterised by the functions ϕ, χ in (2.24). To relate these quantities to the microscopic dynamics, we consider the case $V=0$, so that $E_{\nu_{\alpha,1}}[\eta(0)] = \alpha$. Define the macroscopic mobility $\chi\colon [0,N_{\max}] \to [0,\infty)$ as

$$\chi(\alpha) := \hat\chi^0_{i,i+e_k}(\nu_\alpha) = \frac{1}{2}\hat a_{i,i+e_k}(\nu_\alpha), \tag{2.26}$$

which is independent of i and e_k (and thus well-defined). To see this, note from (2.2) and (2.17) that $\hat\chi^0_{i,i+e_k}(\nu_\alpha) = \sum_{\eta\in\Omega_L} \nu_\alpha(\eta)\hat r^0_{\eta,\eta^{i,i+e_k}} = E_{\nu_{\alpha,1}}[g_1(\eta(0))]E_{\nu_{\alpha,1}}[g_2(\eta(0))]$, where we used (2.1) and the product structure of ν_α. Similarly, define $\phi\colon [0,N_{\max}]\to[0,\infty)$ by $\phi(\alpha):=\hat\phi_i(\nu_\alpha)=E_{\nu_{\alpha,1}}[\mathrm{d}(\eta(0))]$, which is by construction independent of i. One then can prove the *local Einstein relation*

$$\phi'(\alpha)=f''(\alpha)\chi(\alpha), \tag{2.27}$$

which relates ϕ and χ to the free energy f from Section 2.3.1. Equation (2.27) can be obtained by differentiating $\phi(\alpha)=E_{\nu_{*,1}}[\mathrm{d}(\eta(0))\mathrm{e}^{f'(\alpha)\eta(0)}]/E_{\nu_{*,1}}[\mathrm{e}^{f'(\alpha)\eta(0)}]$. Note that $\phi'(\alpha)=\frac{1}{2}f''(\alpha)\sum_\eta\nu_\alpha(\eta)\big[\mathrm{d}(\eta(i))-\mathrm{d}(\eta(i'))\big](\eta(i)-\eta(i'))$ (for $i,i'\in\mathbb{T}_L^d$ arbitrary with $i\neq i'$). Further, the gradient structure and detailed balance yield $\frac{1}{2}\sum_\eta\nu_\alpha(\eta)\big[\hat r^0_{\eta,\eta^{i,i'}}-\hat r^0_{\eta,\eta^{i',i}}\big](\eta(i)-\eta(i'))=\frac{1}{2}\sum_\eta\nu_\alpha(\eta)\big[\hat r^0_{\eta,\eta^{i,i'}}+\hat r^0_{\eta,\eta^{i',i}}\big]=\chi(\alpha)$.

2.3.3. The macroscopic action functional and the chain rule. For $\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ and $h\colon\Lambda\to\mathbb{R}^d$, we introduce the norm $\|h\|^2_{\chi(\rho)}:=\int_\Lambda\chi(\rho(u))|h(u)|^2\mathrm{d}u$ (for full details and associated spaces, see Section 4 below). The macroscopic analogues of the (time integrals of the) microscopic functions Ψ_L and $\Psi_L^\star$ from (2.12), (2.13) are

$$\mathcal{E}\big((\rho_t)_{t\in[0,T]}\big):=\sup_G\bigg[\bigg(\int_\Lambda\rho_T G_T\,\mathrm{d}u-\int_\Lambda\rho_0G_0\,\mathrm{d}u-\int_0^T\int_\Lambda\rho_t\partial_tG_t\,\mathrm{d}u\mathrm{d}t\bigg)-\frac{1}{2}\int_0^T\|\nabla G_t\|^2_{\chi(\rho_t)}\mathrm{d}t\bigg] \tag{2.28}$$

and

$$\mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big):=\sup_G\bigg[\bigg(\int_0^T\int_\Lambda\phi(\rho_t)\Delta G_t\,\mathrm{d}u\mathrm{d}t-\int_0^T\int_\Lambda\chi(\rho_t)\nabla V\cdot\nabla G_t\,\mathrm{d}u\mathrm{d}t\bigg)-\frac{1}{2}\int_0^T\|\nabla G_t\|^2_{\chi(\rho_t)}\mathrm{d}t\bigg], \tag{2.29}$$

where the supremum is in both cases over $C^{1,2}([0,T]\times\Lambda;\mathbb{R})$. We will show in Propositions 4.1 and 4.3 that, under certain assumptions, these functionals can be expressed as time integrals of suitably defined norms

$$\mathcal{E}\big((\rho_t)_{t\in[0,T]}\big)=\frac{1}{2}\int_0^T\|\dot\rho_t\|^2_{-1,\chi(\rho_t)}\mathrm{d}t$$

and

$$\begin{aligned}\mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big)&=\frac{1}{2}\int_0^T\|\Delta\phi(\rho_t)+\nabla\cdot(\chi(\rho_t)\nabla V)\|^2_{-1,\chi(\rho_t)}\mathrm{d}t\\&=\frac{1}{2}\int_0^T\|f''(\rho_t)\nabla\rho_t+\nabla V\|^2_{\chi(\rho_t)}\mathrm{d}t.\end{aligned}$$

In particular, we will show that non-quadratic Ψ and $\Psi^\star$ of (2.13) and (2.12) can be bounded by the quadratic expressions $\mathcal{E}$ and $\mathcal{E}^\star$, respectively.

Finally, for $(\pi_t)_{t\in[0,T]}$ absolutely continuous with respect to the Lebesgue measure, we define the *macroscopic action* as

$$\mathbb{A}\big((\pi_t)_{t\in[0,T]}\big):=\frac{1}{2}\big[\mathcal{F}^V_\alpha(\rho_T)-\mathcal{F}^V_\alpha(\rho_0)+\mathcal{E}\big((\rho_t)_{t\in[0,T]}\big)+\mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big)\big]. \tag{2.30}$$

If $(\pi_t)_{t\in[0,T]}$ is not absolutely continuous with respect to the Lebesgue measure, we set $\mathbb{A}\big((\pi_t)_{t\in[0,T]}\big)=+\infty$.

In a nutshell, the main results of this article are twofold: Firstly, we establish relations between suitably scaled $\mathbb{A}_L^{\tilde{V}}$ of (2.20) and the continuum limit (2.30): see Theorems 3.3 to 3.5. Secondly, we show that under suitable regularity assumptions, in particular if the free energy $\mathcal{F}_\alpha^V$ satisfies a chain rule (see Equation (3.4)), the macroscopic action can be re-written in a way which reveals the hydrodynamic limit as minimiser of this functional, see (3.5) below.

2.4. Assumptions on the particle systems studied.

2.4.1. Local equilibrium assumption and the replacement lemma. When taking the hydrodynamic limit, one must prove a local equilibration condition, which means that the system resembles — in a small neighbourhood around any point — an equilibrium system. To make this precise, take $\ell\in\mathbb{N}$ and define the average number of particles in a box with diameter $2\ell+1$ as

$$\eta^\ell(i):=\frac{1}{(2\ell+1)^d}\sum_{|m|\le\ell}\eta(i+m).$$

Similarly, we also define the averages $\hat{\chi}^\ell_{i,i+e_k}(\mu):=(2\ell+1)^{-d}\sum_{|m|\le\ell}\hat{\chi}_{i+m,i+m+e_k}(\mu)$ and $\hat{\phi}^\ell_i(\mu):=(2\ell+1)^{-d}\sum_{|m|\le\ell}\hat{\phi}_{i+m}(\mu)$.

Now assume that $L\gg1$ and $\epsilon\ll1$ and that the state of the system is given by $\eta\in\Omega_L$. Define $\ell=\lfloor\epsilon L\rfloor$, which is the size of a macroscopic box with diameter $\approx2\epsilon$ (measured on the macroscopic scale). Hence $\hat{\chi}^{\lfloor\epsilon L\rfloor}_{i,i+e_k}(\delta_\eta)$ is a locally averaged mobility. *Local equilibration* means that $\hat{\chi}_{i,i+e_k}(\nu_{\eta^{\lfloor\epsilon L\rfloor}(i)})$ is close to the expected mobility for an equilibrium distribution ν_α with the same (locally-averaged) particle density. That is, the time averaged distributions $\mu^L_{[0,T]}:=\frac{1}{T}\int_0^T\mu^L_t\,dt$ satisfy in local equilibrium

$$\limsup_{\epsilon\to0}\limsup_{L\to\infty}\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\sum_{\eta\in\Omega_L}\mu^L_{[0,T]}(\eta)\left|\hat{\chi}^{\lfloor\epsilon L\rfloor}_{i,i+e_k}(\delta_\eta)-\hat{\chi}_{i,i+e_k}(\nu_{\eta^{\lfloor\epsilon L\rfloor}(i)})\right|=0,\tag{2.31}$$

as well as

$$\limsup_{\epsilon\to0}\limsup_{L\to\infty}\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{\eta\in\Omega_L}\mu^L_{[0,T]}(\eta)\left|\hat{\phi}^{\lfloor\epsilon L\rfloor}_i(\delta_\eta)-\hat{\phi}_i(\nu_{\eta^{\lfloor\epsilon L\rfloor}(i)})\right|=0.\tag{2.32}$$

Remark 2.1 (Replacement Lemma). Note that results like (2.31) and (2.32) are classically obtained by proving the stronger replacement lemma, which in our notation amounts to proving for $\hat{\chi}$ (and analogously for $\hat{\phi}$)

$$\limsup_{\epsilon\to0}\limsup_{L\to\infty}\sup_{\mu}\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\sum_{\eta\in\Omega_L}\mu(\eta)\left|\hat{\chi}^{\lfloor\epsilon L\rfloor}_{i,i+e_k}(\delta_\eta)-\hat{\chi}_{i,i+e_k}(\nu_{\eta^{\lfloor\epsilon L\rfloor}(i)})\right|=0,\tag{2.33}$$

where the supremum is taken over a class of measures μ satisfying certain bounds on the relative entropy (i.e. the free energy) and the Dirichlet form, which can be identified with $\frac{1}{2}\Psi^\star(\mu,F^V(\mu))$ (see e.g. the remark in the proof of Proposition 5.3 below). In the following, we will follow the classical approach and work with (2.33). We state sufficient conditions for the replacement lemma in Section 3.2 below and establish in this way the validity of (2.31) and (2.32).

2.4.2. Assumptions on the path measures $P_L^{\tilde V}$. We have presented a general framework for interacting particles on lattices and their hydrodynamic scaling limits. The results of the next section are similarly general and can be applied to a range of systems, including the symmetric simple exclusion process and certain zero-range processes, as discussed in Section 3.4 below. However, our results for hydrodynamic limits clearly do not apply to all interacting-particle systems. We summarise here the main assumptions on the reference process $P_L^{\tilde V}$ required in the following analysis: these need to be verified in order to apply our results to a particular system.

On the microscopic scale, we assume that the transition rates are given by (2.1) and are of gradient type. The initial conditions and invariant measures are as described in Section 2.1.1. We note that many of the proofs given below make use of assumption (2.7). Despite the fact that it is a non-standard assumption for hydrodynamic limits (unless $N_{\max}<\infty$, in which case (2.7) holds trivially), it is not too restrictive, in the sense that the typical initial conditions $(\mu_0^L)_{L\in\mathbb{N}}$ can be shown to satisfy (cf. equation (1.4) in Section 5.1 on page 71 in [22]) $\lim_{A\to\infty}\limsup_{L\to\infty}\mu_0^L(\eta\in\Omega_L\,|\,L^{-d}\sum_{i\in\mathbb{T}_L^d}\eta(i)\geq A)=0$.

When taking the hydrodynamic limit, we assume that for any sequence of measures $(\mu^L)_{L\in\mathbb{N}}$ satisfying (2.7), it holds that

$$C_{\hat\chi}:=\limsup_{L\to\infty}\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^{d}\hat\chi_{i,i+e_k}(\mu^L)<\infty, \tag{2.34}$$

which ensures that the total rate of particle jumps for the reference process stays controlled as $L\to\infty$. Similarly we suppose that any sequence of measures $(\mu^L)_{L\in\mathbb{N}}$ obeying (2.7) also satisfies

$$C_{\hat\phi}:=\limsup_{L\to\infty}\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\hat\phi_i(\mu^L)<\infty. \tag{2.35}$$

In addition, our proofs require the following technical assumptions on the functions f, ϕ and χ that characterise the hydrodynamic limit itself: We assume that $f\in C^2([0,N_{\max}];\mathbb{R})$ with $f(0)=0$, $f''>0$ a.e. and that $\lim_{r\to0}f'(r)=-\infty$ and $\lim_{r\to N_{\max}}f'(r)=\infty$. Note that this implies by (2.5) that $\phi(0)=0=\chi(0)$. Further, we assume that $\phi,\chi>0$ on $(0,N_{\max})$ and that both ϕ and χ are Lipschitz continuous on $[0,N_{\max}]$, without loss of generality with common Lipschitz constant $C_{\mathrm{Lip}}>0$. Since $\phi(0)=\chi(0)=0$, we have in particular $0<\phi(a),\chi(a)\leq C_{\mathrm{Lip}}a$ for $a\in(0,N_{\max}]$. We further assume that ϕ is continuously differentiable on $(0,N_{\max})$ (by the above Lipschitz condition with bounded derivative) and also strictly monotonically increasing. This implies the existence of a continuous inverse $\phi^{-1}:\phi([0,N_{\max}])\to[0,N_{\max}]$, where $\phi([0,N_{\max}])=\{\phi(a):a\in[0,N_{\max}]\}$. We also suppose that ϕ^{-1} has a bounded derivative (which is by the inverse function theorem equivalent to saying that there exists $C_*>0$ such that $\phi'(a)\geq C_*$ for all $a\in(0,N_{\max}]$).

3. Statement of the results

In this section, we discuss the behaviour of the microscopic action in the limit $L\to\infty$, and the implications of this behaviour for hydrodynamic limits. Sections 3.1 and 3.2 derive preliminary results, which establish properties of the action functionals and sufficient conditions for local equilibration. Section 3.3 states the main results, consisting of three theorems (Theorems 3.3–3.5). Finally Section 3.4 discusses the applications of these theorems in two specific particle systems, and their implications for hydrodynamic limits.

3.1. Properties of the microscopic and macroscopic action functions.

3.1.1. Chain rule on microscopic scale. Consider $(\mu_t^L, \jmath_t^L)_{t\in[0,T]}$ as in Section 2.2.1. The force $F^V(\mu_t^L)$ can be linked to the free energy (2.8) via the classical chain rule formula (cf. Theorem 9.2 of Appendix 1 in [22], Proposition 2.2 in [18] and also [21]) $\mathcal{F}_{L,\alpha}^V(\mu_{t_2}^L) - \mathcal{F}_{L,\alpha}^V(\mu_{t_1}^L) = -\int_{t_1}^{t_2} \langle \jmath_t^L, F^V(\mu_t^L)\rangle_L \mathrm{d}t$, which is a special case of the following result (proved in Section 5.1 below).

Proposition 3.1 (Chain rule for the microscopic free energy). *Let $\tilde{V} \in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ and consider a path measure P_L on Ω_L, as described in Section 2.2.1, with associated density and current $(\mu_t^L, \jmath_t^L)_{t\in[0,T]}$. Then the map $t\mapsto \mathcal{F}_{L,\alpha}^{\tilde{V}_t}(\mu_t^L)$ is absolutely continuous for $t\in[0,T]$ and satisfies the following chain rule. For all $0\le t_1 < t_2 \le T$*

$$\mathcal{F}_{L,\alpha}^{\tilde{V}_{t_2}}(\mu_{t_2}^L) - \mathcal{F}_{L,\alpha}^{\tilde{V}_{t_1}}(\mu_{t_1}^L) = -\int_{t_1}^{t_2} \langle \jmath_t^L, F^{\tilde{V}_t}(\mu_t^L)\rangle_L \mathrm{d}t + \int_{t_1}^{t_2} \sum_{i\in\mathbb{T}_L^d} \big(\hat{\rho}_i(\mu_t^L) - \bar{\rho}_{\alpha,\tilde{V}_t}(i)\big)\partial_t \tilde{V}_t(\tfrac{i}{L})\mathrm{d}t. \tag{3.1}$$

Now fix some $\alpha\in(0,N_{\max})$ and combine Proposition 3.1 with (2.14) and (2.19), which yields

$$\begin{aligned}\mathbb{A}_L^{\tilde{V}}(Q_L) = \frac{1}{2}\big[\mathcal{F}_{L,\alpha}^{\tilde{V}_T}(\mu_T^L) - \mathcal{F}_{L,\alpha}^{\tilde{V}_0}(\mu_0^L)\big] + \frac{1}{2}\int_0^T \Psi_L(\mu_t^L, \jmath_t^L)\mathrm{d}t + \frac{1}{2}\int_0^T \Psi_L^\star\big(\mu_t^L, F^{\tilde{V}_t}(\mu_t^L)\big)\mathrm{d}t \\ - \frac{1}{2}\int_0^T \sum_{i\in\mathbb{T}_L^d} \big(\hat{\rho}_i(\mu_t^L) - \bar{\rho}_{\alpha,\tilde{V}_t}(i)\big)\partial_t\tilde{V}_t(\tfrac{i}{L})\mathrm{d}t \ge 0.\end{aligned} \tag{3.2}$$

3.1.2. Macroscopic action. We now establish some properties of $\mathbb{A}$, as defined in (2.30). If $\mathbb{A}\big((\pi_t)_{t\in[0,T]}\big) < \infty$ one can show that

$$\begin{aligned}\mathbb{A}\big((\pi_t)_{t\in[0,T]}\big) = \frac{1}{2}\big[\mathcal{F}_\alpha^V(\rho_T) - \mathcal{F}_\alpha^V(\rho_0)\big] \\ + \frac{1}{4}\int_0^T \big(\|\dot{\rho}_t\|_{-1,\chi(\rho_t)}^2 + \|\Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\|_{-1,\chi(\rho_t)}^2\big)\mathrm{d}t,\end{aligned} \tag{3.3}$$

see Proposition 4.1 and Proposition 4.3. For a definition of the norm $\|\cdot\|_{-1,\chi(\rho_t)}$ (and the associated inner product $\langle\cdot,\cdot\rangle_{-1,\chi(\rho_t)}$) we also refer to Section 4 below.

Note that $\mathbb{A}((\pi_t)_{t\in[0,T]})$ as defined here might in general be negative. A sufficient condition for non-negativity of $\mathbb{A}((\pi_t)_{t\in[0,T]})$ is ensured by the validity of the following chain rule, which can be seen as a macroscopic counterpart to (3.1) for potentials constant in time. A *formal* calculation yields for $0\le t_1 < t_2 \le T$ the chain rule

$$\mathcal{F}_\alpha^V(\rho_{t_2}) - \mathcal{F}_\alpha^V(\rho_{t_1}) = \int_{t_1}^{t_2} \langle \dot{\rho}_t, \frac{\delta\mathcal{F}_\alpha^V}{\delta\rho_t}\rangle \mathrm{d}t = -\int_{t_1}^{t_2} \langle\dot{\rho}_t, \Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\rangle_{-1,\chi(\rho_t)}\mathrm{d}t. \tag{3.4}$$

Combined with (3.3) this allows us to (formally!) rewrite the macroscopic action functional (3.3) as

$$\mathbb{A}\big((\pi_t)_{t\in[0,T]}\big) = \frac{1}{4}\int_0^T \big\|\dot{\rho}_t - \Delta\phi(\rho_t) - \nabla\cdot(\chi(\rho_t)\nabla V)\big\|_{-1,\chi(\rho_t)}^2 \mathrm{d}t. \tag{3.5}$$

In Section 4.2 we summarise some geometrical properties of the relevant function spaces and we establish sufficient conditions for the chain rule:

THEOREM 3.1. *Let the assumptions from Section 2.4.2 hold and additionally assume that* $\chi'(a) \geq C_*$ *for all* $a \in (0, N_{\max}]$ *(for some* $C_* > 0$*). If* $d > 1$*, then further assume that the free energy density* f *satisfies the McCann condition for geodesic convexity (stated in Equation (4.13) below). Then any path* $(\pi_t)_{t\in[0,T]}$ *with* $\mathbb{A}((\pi_t)_{t\in[0,T]}) < \infty$ *and* $\mathcal{F}_\alpha^V(\rho_0) < \infty$ *satisfies the identities in Equation (3.4).*

Note that the McCann condition is always satisfied in one spatial dimension (where it reduces to convexity of f). We further stress that in Macroscopic Fluctuation Theory the validity of the chain rule is implicitly assumed by Equation (2.15) in [5], which relates the large deviation rate for a forward path to its time-reversed counterpart.

3.2. Sufficient Conditions for local equilibration. The following theorem, proved in Section 5.1 below, yields a sufficient condition for the local equilibration discussed in Section 2.4.1 in terms of the free energy (2.8) of the initial condition and the action functional (2.20).

THEOREM 3.2. *Let* $(P_L)_{L\in\mathbb{N}}$ *be as in Section 2.2.1 with densities* $(\mu_t^L)_{t\in[0,T]}$*, for* $L \in \mathbb{N}$*, and associated path measures* $(Q_L)_{L\in\mathbb{N}}$ *on* $\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$*. Assume there exist* $V \in C^2(\Lambda;\mathbb{R})$ *and* $\alpha \in [0, N_{\max})$ *such that*

$$\limsup_{L\to\infty} \frac{1}{L^d} \mathcal{F}_{L,\alpha}^V(\mu_0^L) < \infty \tag{3.6}$$

and $\tilde{V} \in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ *such that*

$$\limsup_{L\to\infty} \frac{1}{L^d} \mathbb{A}_L^{\tilde{V}}(Q_L) < \infty. \tag{3.7}$$

Then $(\mu_{[0,T]}^L)_{L\in\mathbb{N}}$ *satisfies the local equilibrium assumption, (2.31) and (2.32). Moreover, Equations (3.6) and (3.7) are independent of* V*,* $\tilde{V}$ *and* α*, such that these conditions can equivalently be stated as* $\limsup_{L\to\infty} L^{-d}\mathcal{H}(Q_L|Q_{\nu_\alpha}) < \infty$*, where* Q_{ν_α} *denotes the measure on* $\mathcal{D}([0,T];\Omega_L)$ *with marginals equal to* ν_α*, in the sense that* $(X_t)_\# Q_{\nu_\alpha} = (\Theta_L)_\# \nu_\alpha$ *for all* $t \in [0,T]$*.*

3.3. Particle systems on hydrodynamic scale. We now present our main results. We consider sequences of path measures $(Q_L^V)_{L\in\mathbb{N}}$ and $(Q_L)_{L\in\mathbb{N}}$ on $\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$, as defined in Section 2.2.1, as well as the corresponding sequences $(P_L^V)_{L\in\mathbb{N}}$ and $(P_L)_{L\in\mathbb{N}}$. We define Q^* as a (possibly non-unique) limit point of the sequence of observed processes $(Q_L)_{L\in\mathbb{N}}$ and we establish various properties of this limit. The physical idea is that the path on which Q^* is supported is a *candidate* for the hydrodynamic limit for the reference process $(Q_L^V)_{L\in\mathbb{N}}$. By analysing the large-L behaviour of the microscopic action $\mathbb{A}_L^V(Q_L)$, the aim is to show that the only admissible candidate path is the true hydrodynamic limit. For specific examples, see Section 3.4, below.

3.3.1. Assumptions for scaling limits. To apply the results of this section to a specific interacting particle system (reference process), several assumptions have to be satisfied. We assume that the conditions given in Section 2.4.2 have been verified. We assume also that the initial distributions $(\mu_0^L)_{L\in\mathbb{N}}$ of $(P_L^V)_{L\in\mathbb{N}}$ converge to a fixed density $\rho_0 \in \mathcal{L}^1(\Lambda;[0,\infty))$ in the sense that $(\Theta_L)_\#\mu_0^L \to \delta_{\pi_0}$ with $\pi_0(\mathrm{d}u) = \rho_0(u)\mathrm{d}u$. For the rest of this Section 3.3, we fix α uniquely by requiring that $\int_\Lambda \rho_0(u)\mathrm{d}u = \int_\Lambda \bar{\rho}_{\alpha,V}(u)\mathrm{d}u$.

Further, we assume that the observed processes $(Q_L)_{L\in\mathbb{N}}$ are relatively compact [7, 22]. Then there is a measure Q^* on $\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$ and a subsequence of $(Q_L)_{L\in\mathbb{N}}$ converging to Q^* (such that the marginal at time $t=0$ satisfies $(X_0)_\# Q^* = \delta_{\pi_0}$). Finally, we assume that the measure Q^* is concentrated on paths that are absolutely continuous with respect to the Lebesgue measure,

$$Q^*\Big((\pi_t)_{t\in[0,T]}\in\mathcal{D}([0,T];\mathcal{M}_+(\Lambda)):\pi_t(\mathrm{d}u)=\rho_t(u)\mathrm{d}u \text{ for a.a. } t\in[0,T]\Big)=1. \tag{3.8}$$

We note that the paths in (3.8) satisfy $\rho_t\in\mathcal{L}^1(\Lambda;[0,\infty))$. Moreover, if $N_{\max}<\infty$, then clearly also $\rho_t\leq N_{\max}$ a.e. on Λ for almost all $t\in[0,T]$. However, the limit Q^* is not assumed to be unique: there could exist other subsequences of $(Q_L)_{L\in\mathbb{N}}$ with different limits.

Given a specific model, the compactness of the sequence $(Q_L)_{L\in\mathbb{N}}$ and the support on absolutely continuous paths (3.8) often follow from (3.6) in combination with an assumption on the transition rates of the particle system. This is the case for the examples considered in Section 3.4 below.

3.3.2. Comparison with classical proofs of the hydrodynamic limit. To provide context for our analysis, we briefly summarise the classical approach to hydrodynamic limits. Here, we consider separately the observed process and the reference process, but the classical approach takes $(P_L)_{L\in\mathbb{N}}=(P_L^V)_{L\in\mathbb{N}}$. The task of proving a hydrodynamic limit for $(Q_L)_{L\in\mathbb{N}}$ then consists of characterising all limiting distributions. The first step is to establish relative compactness [7, 22], which ensures the existence of a (possibly non-unique) limit Q^*. One then shows that Q^* is unique and that it is concentrated on a single path $(\rho_t)_{t\in[0,T]}$ (i.e. $Q^*=\delta_{(\pi_t)_{t\in[0,T]}}$ and $\pi_t(\mathrm{d}u)=\rho_t(u)\mathrm{d}u$ for almost all $t\in[0,T]$). This general approach includes both the entropy method and the relative entropy method [22]: note that it *first* establishes that Q^* is supported on weak solutions to (2.25) and *then* uses a uniqueness result for this solution to infer that Q^* is supported on this unique solution, see e.g. [22, Chapter 4].

Our approach here differs in two main points: We consider an observed process that is different from the reference process ($P_L\neq P_L^V$ in general) and we assume that the sequence $(Q_L)_L$ has a unique limiting distribution Q^* that is concentrated on a single path, as in (3.8). (As a special case, one may take $P_L=P_L^V$, under the assumption that the hydrodynamic limit exists, but the following results are not restricted to this case.) These assumptions mean that the results in this work do not prove the existence of a hydrodynamic limit, neither for the observed process nor the reference process. Rather, they assume the existence of such a limit, and they establish properties of the associated path $(\pi_t)_{t\in[0,T]}$ and its macroscopic action $\mathbb{A}\big((\pi_t)_{t\in[0,T]}\big)$.

3.3.3. Convergence of free energy and action for deterministic limits. The following first main theorem yields regularity results for $(P_L)_{L\in\mathbb{N}}$ under the assumptions of Section 3.3.1 and those of Theorem 3.2. In particular, it shows that the macroscopic action (and its individual contributions) are asymptotically dominated by their (more detailed) microscopic counterparts.

THEOREM 3.3 (Regularity of the limit and asymptotic lower bounds). *Let $(P_L)_{L\in\mathbb{N}}$ be a sequence as in Section 3.3.1, with density and current $(\mu_t^L,\jmath_t^L)_{t\in[0,T]}$, for $L\in\mathbb{N}$. We suppose that the associated sequence $(Q_L)_{L\in\mathbb{N}}$ has a unique limit point $Q^*=\delta_{(\pi_t)_{t\in[0,T]}}$ for some $(\pi_t)_{t\in[0,T]}\in\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$ and that the initial condition is well prepared in the sense that the free energies converge (cf. [18, 33, 37])*

$$\lim_{L\to\infty}\frac{1}{L^d}\mathcal{F}_{L,\alpha}^V(\mu_0^L)=\mathcal{F}_\alpha^V(\rho_0). \tag{3.9}$$

Further assume that $(Q_L)_{L\in\mathbb{N}}$ satisfies (3.7) for $\tilde V_t = V$, such that

$$\limsup_{L\to\infty} \frac{1}{L^d} \mathbb{A}_L^V(Q_L) < \infty. \tag{3.10}$$

Then $(\pi_t)_{t\in[0,T]}$ is narrowly continuous, i.e. $(\pi_t)_{t\in[0,T]} \in C([0,T];\mathcal{M}_+(\Lambda))$ and the action satisfies the lower bound

$$\liminf_{L\to\infty} \frac{1}{L^d} \mathbb{A}_L^V(Q_L) \geq \mathbb{A}((\pi_t)_{t\in[0,T]}). \tag{3.11}$$

Further, the free energy satisfies for all $t\in[0,T]$

$$\liminf_{L\to\infty} \frac{1}{L^d} \mathcal{F}_{L,\alpha}^V(\mu_t^L) \geq \mathcal{F}_\alpha^V(\rho_t), \tag{3.12}$$

as well as

$$\liminf_{L\to\infty} \frac{1}{L^d} \int_0^T \Psi_L\big(\mu_t^L, \jmath_t^L\big)\,\mathrm{d}t \geq \frac{1}{2}\int_0^T \|\dot\rho_t\|_{-1,\chi(\rho_t)}^2\,\mathrm{d}t \tag{3.13}$$

and

$$\liminf_{L\to\infty} \frac{1}{L^d} \int_0^T \Psi_L^\star\big(\mu_t^L, F^V(\mu_t^L)\big)\,\mathrm{d}t \geq \frac{1}{2}\int_0^T \|\Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\|_{-1,\chi(\rho_t)}^2\,\mathrm{d}t. \tag{3.14}$$

In this theorem, we see for the first time a connection between the non-quadratic microscopic functionals Ψ and $\Psi^\star$ and their macroscopic quadratic counterparts, see (3.13) and (3.14).

Proof. Note that the assumptions of Theorem 3.2 are satisfied, so that the local equilibration assumptions (2.31) and (2.32) hold. The result (3.11) follows from the representation of $\mathbb{A}_L^V$ in (3.2), the definition of $\mathbb{A}$ in (2.30) combined with (3.9) and the following three inequalities (for which the proofs will be given in Section 5.2). Firstly, for the free energy at the final time T, we obtain from Proposition 5.4 and the continuity of X_T (the evaluation of the path at the final time $t=T$) that

$$\liminf_{L\to\infty} \frac{1}{L^d} \mathcal{F}_{L,\alpha}^V(\mu_T^L) \geq \mathcal{F}_\alpha^V(\rho_T). \tag{3.15}$$

Secondly,

$$\liminf_{L\to\infty} \frac{1}{L^d} \int_0^T \Psi_L\big(\mu_t^L, \jmath_t^L\big)\,\mathrm{d}t \geq \mathcal{E}\big((\rho_t)_{t\in[0,T]}\big), \tag{3.16}$$

which follows from Proposition 5.5, and thirdly

$$\liminf_{L\to\infty} \frac{1}{L^d} \int_0^T \Psi_L^\star\big(\mu_t^L, F^V(\mu_t^L)\big)\,\mathrm{d}t \geq \mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big), \tag{3.17}$$

which is proved in Proposition 5.6. Proposition 4.1 and Proposition 4.3 then yield (3.13) and (3.14), respectively. Proposition 4.2 further shows that the path is 2-absolutely continuous in the Wasserstein sense (see (4.6) in Section 4), from which we can deduce the narrow continuity using Lemma 4.1. The inequality (3.12) for the free energy at any time $t\in[0,T]$ then follows from another application of Proposition 5.4. ∎

It is instructive to consider Theorem 3.3 in the case where the observed process is equal to the reference process $P_L = P_L^V$. In this case the microscopic action $\mathbb{A}_L^V(Q_L) = 0$ and the theorem has implications for the hydrodynamic limit of the reference process, as follows. Either Q^* does not concentrate on a single path, in which case the theorem is inapplicable; or Q^* does concentrate on a single path, and the theorem shows that the macroscopic action of that path satisfies $\mathbb{A}((\pi_t)_{t\in[0,T]}) \le 0$, by (3.11). In the examples that we consider below, this macroscopic action is zero, see below.

We consider a special case for the observed process P_L. We keep the reference process P_L^V as outlined in Section 3.3.1 and consider for some (possibly time-dependent) potential $\tilde{H} \in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ the process $P_L = P_L^{\tilde{V}}$ for the potential $\tilde{V}_t = V + \tilde{H}_t$ as defined in Section 2.2.1. Note that both processes have the same initial condition μ_0^L and their transition rates $r^{V+\tilde{H}_t}$ and r^V coincide up to a change of the external potential (i.e. the functions g_1 and g_2 in (2.1) coincide for both processes). We assume that the corresponding path measures $(Q_L^{V+\tilde{H}})_{L\in\mathbb{N}}$ satisfy, as in Section 2.3.2 above, a hydrodynamic limit with hydrodynamic equation

$$\dot{\rho}_t = \Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla(V+\tilde{H}_t)). \tag{3.18}$$

In this case one can improve the result (3.11) from Theorem 3.3 by showing that the action functionals $\mathbb{A}_L^V(Q_L^{V+\tilde{H}})$ converge, as described by the following second main theorem.

THEOREM 3.4. *Assume that $P_L = P_L^{V+\tilde{H}}$ for some $\tilde{H} \in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ and that $(P_L)_{L\in\mathbb{N}}$ satisfies the assumptions in Theorem 3.3. Moreover, assume that the density of the path $(\pi_t)_{t\in[0,T]}$ is a weak solution to (3.18), in the sense of (2.25). Then*

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{L^d}\mathbb{A}_L^V(Q_L^{V+\tilde{H}}) &= \frac{1}{4}\int_0^T \left\|\nabla\tilde{H}_t\right\|^2_{\chi(\rho_t)}\mathrm{d}t \\ &= \frac{1}{4}\int_0^T \left\|\dot{\rho}_t - \Delta\phi(\rho_t) - \nabla\cdot(\chi(\rho_t)\nabla V)\right\|^2_{-1,\chi(\rho_t)}\mathrm{d}t.\end{aligned} \tag{3.19}$$

We postpone the proof of Theorem 3.4 to Section 5.3 below. See also Section 10 in [22] for the specific calculations for the simple exclusion process, which can be seen as a special case of our computations. We further stress that for measures of the form $(P_L^{\tilde{V}})_{L\in\mathbb{N}}$ the assumption on (3.10) in Theorem 3.3 is satisfied trivially, since $\mathbb{A}_L^{\tilde{V}}(Q_L^{\tilde{V}}) = 0$.

Theorem 3.4 clarifies the relationship between the microscopic and macroscopic action functionals. It shows how the non-quadratic (Ψ-$\Psi^\star$) form of the microscopic action $\mathbb{A}_L^V$ converges to a (simpler) quadratic form, when viewed on the macroscopic scale. Of course, this convergence requires some information about the regularity of the path that dominates Q^*: this comes from the assumption (3.18).

Recall that the lower bound (3.11) in Theorem 3.3 and the limit (3.19) in Theorem 3.4 coincide (by (3.5)) if and only if the chain rule (3.4) holds. The validity of the chain rule (3.4) for the path $(\pi_t)_{t\in[0,T]}$ in Theorem 3.4 can be shown to be equivalent to the case where the limits in (3.12), (3.13) and (3.14) exist and all three inequalities are equalities.

THEOREM 3.5. *Let the assumptions in Theorem 3.4 hold. Further assume that $\mathcal{F}_\alpha^V$ satisfies the chain rule (3.4) for the path $(\rho_t)_{t\in[0,T]}$. Then the free energy converges for* *all* $t \in [0,T]$,

$$\lim_{L\to\infty} \frac{1}{L^d} \mathcal{F}^V_{L,\alpha}(\mu^L_t) = \mathcal{F}^V_\alpha(\rho_t). \tag{3.20}$$

Moreover,

$$\lim_{L\to\infty} \frac{1}{L^d} \int_0^T \Psi_L(\mu^L_t, \jmath^L_t)\,\mathrm{d}t = \frac{1}{2}\int_0^T \|\dot\rho_t\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t \tag{3.21}$$

and

$$\lim_{L\to\infty} \frac{1}{L^d} \int_0^T \Psi^\star_L(\mu^L_t, F^V(\mu^L_t))\,\mathrm{d}t = \frac{1}{2}\int_0^T \|\Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t. \tag{3.22}$$

Also the opposite implication holds: If (3.20), (3.21) *and* (3.22) *are satisfied, then* $\mathcal{F}^V_\alpha$ *satisfies the chain rule* (3.4) *for* $(\rho_t)_{t\in[0,T]}$.

Proof. This proof is similar to calculations performed in [23] and [18], where the authors establish (3.20) for the hydrodynamic limit of the simple exclusion process. Note that (3.19), (3.3), (3.5) and the chain rule (3.4) imply

$$\begin{aligned}&\lim_{L\to\infty} \frac{1}{L^d}\left(\mathcal{F}^V_{L,\alpha}(\mu^L_T) + \int_0^T \Psi_L(\mu^L_t, \jmath^L_t)\,\mathrm{d}t + \int_0^T \Psi^\star_L(\mu^L_t, F^V(\mu^L_t))\,\mathrm{d}t\right)\\ &= \mathcal{F}^V_\alpha(\rho_T) + \frac{1}{2}\int_0^T \|\dot\rho_t\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t.\end{aligned}$$

We apply the inequality $\limsup_{n\to\infty}(a_n + b_n + c_n) \ge \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n + \liminf_{n\to\infty} c_n$ to the expression on the left-hand side to obtain the inequality

$$\limsup_{L\to\infty} \frac{1}{L^d}\mathcal{F}^V_{L,\alpha}(\mu^L_T) \le \mathcal{F}^V_\alpha(\rho_T).$$

The result for an arbitrary time $t \in [0,T]$ then follows by repeating the above proof for the time interval $[0,t]$. The remaining two limits (3.21) and (3.22) follow in a similar way by a slight modification of the above steps.

For the opposite implication, we assume that (3.20), (3.21) and (3.22) hold. In this case we have

$$\begin{aligned}&\frac{1}{2}\int_0^T \left\|\dot\rho_t - \Delta\phi(\rho_t) - \nabla\cdot(\chi(\rho_t)\nabla V)\right\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t\\ &= \mathcal{F}^V_\alpha(\rho_T) - \mathcal{F}^V_\alpha(\rho_0) + \frac{1}{2}\int_0^T \|\dot\rho_t\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t\\ &\quad + \frac{1}{2}\int_0^T \|\Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t,\end{aligned}$$

which is equivalent to (3.4) for $t_1 = 0$ and $t_2 = T$. Repeating the above steps for $[0,t]$ (for any $t \in [0,T]$) then finishes the proof. ∎

REMARK 3.1 (Remark on Chain Rule). In summary, we have seen that there are at least three ways to verify the chain rule (3.4). One way is to prove the assumptions of Theorem 3.1. Alternatively, one can derive a Large Deviation Principle, as in Macroscopic Fluctuation Theory (cf. the discussion below Theorem 3.1); or one can directly calculate the limits in Theorem 3.5.

Now recall the case where the observed process and the reference process coincide, $P_L = P_L^V$. One sees that (3.20)–(3.22) in Theorem 3.5 are similar to (3.12)–(3.14) in Theorem 3.3, but Theorem 3.5 is stronger, in that the limits have been shown to exist. To prove this, the additional assumption (3.18) was required, as well as (3.4). For the example systems considered below, these assumptions can be proven by other means. This establishes that the macroscopic action $\mathbb{A}((\pi_t)_{t\in[0,T]})$ is non-negative, as long as the density ρ associated with π is a solution of (3.18), for some $\tilde{H}$. In this case one sees that the hydrodynamic limit of the reference system can be characterised as the unique zero of $\mathbb{A}$, within this class of paths.

Moreover, the quadratic structure of $\mathbb{A}$ together with the macroscopic chain rule means that the minimiser of $\mathbb{A}$ can be identified as a gradient flow for the free energy. Such gradient flows are widespread in macroscopic descriptions of physical systems: we speculate that the structure presented here is similarly general. That is, it is natural to expect gradient flows as macroscopic descriptions of physical systems whose microscopic descriptions are reversible Markov chains, because the non-quadratic Ψ-$\Psi^\star$ form of the microscopic action often converges to a quadratic functional on the macroscopic scale.

3.4. Examples

Standard examples of particle models described by the class of models in Section 2.1 are (i) the zero-range process (ZRP) for which $\Omega_L = \mathbb{N}_0^{\mathbb{T}_L^d}$, and g_1 is a function that satisfies $g_1(0)=0$ and $g_2=1$; and (ii) the (symmetric) simple exclusion process (SEP), where $\Omega_L = \{0,1\}^{\mathbb{T}_L^d}$, $g_1(n) = \mathbf{1}_{\{n=1\}}$ and $g_2(n) = \mathbf{1}_{\{n=0\}}$; and (iii) the generalised exclusion processes, where $\Omega_L = \{0,\cdots,m\}^{\mathbb{T}_L^d}$, $g_1(n) = \mathbf{1}_{\{n\geq 1\}}$ and $g_2(n) = \mathbf{1}_{\{n\leq m\}}$ for some fixed $m\in\mathbb{N}$ [22]. The latter is an example of a non-gradient system. We focus on the two gradient models ZRP and SEP, which have $\mathrm{d}(k) = g_1(k)$ and $\mathrm{d}(k) = k$, respectively.

3.4.1. Zero-Range Process

The ZRP satisfies the assumptions of Section 2.4.2 if we assume that the rates are strictly monotonically increasing and sub-linear. That is, we assume that there exists $g^* > 0$ such that $0 < g_1(k+1) - g_1(k) \leq g^*$. Since $g_1(0) = 0$ we have $g_1(k) \leq g^* k$. The mobility for the ZRP is given by $\chi(a) = \phi(a)$, where $E_{\nu_\alpha}[g_1(\eta(0))] = \phi(\alpha)$. The reference measure is $\nu_{*,1}(n) = 1/(\prod_{k=1}^n g(k))$ and the α-dependent invariant distribution is for $z(\phi(\alpha)) := \sum_{n=0}^\infty \phi(\alpha)^n \nu_{*,1}(n)$ given by

$$\nu_{\alpha,1}(\eta(0)) = \frac{\phi(\alpha)^{\eta(0)}}{z(\phi(\alpha))}\nu_{*,1}(\eta(0)).$$

Finally, the free energy is

$$\mathcal{F}_\alpha^V(\rho) = \int_\Lambda \left[\rho(u)\log\left(\frac{\phi(\rho(u))}{\mathrm{e}^{-V(u)}\phi(\alpha)}\right) - \log\left(\frac{z(\phi(\rho(u)))}{z(\mathrm{e}^{-V(u)}\phi(\alpha))}\right)\right]\mathrm{d}u$$

for $f(a) = \rho\log\phi(a) - \log z(\phi(a))$ and $\bar{\rho}_{\alpha,V}(u) = \phi^{-1}(\mathrm{e}^{-V(u)}\phi(\alpha))$.

These considerations establish that Theorems 3.3 to 3.5 can be applied to the ZRP. We now consider the implications of these theorems for hydrodynamic limits. We first compare the path measures for the ZRP (that is, the sequence of P_L^V indexed by L) with some sequence of path measures P_L which concentrate on an absolutely continuous path $(\pi_t)_{t\in[0,T]}$ and satisfies the assumptions of Theorem 3.2. In this case one may apply Theorem 3.3, which establishes an asymptotic lower bound on the rescaled microscopic action $L^{-d}\mathbb{A}_L^V(Q_L)$. If $(\pi_t)_{t\in[0,T]}$ is the hydrodynamic limit of the ZRP then P_L^V has to concentrate on $(\pi_t)_{t\in[0,T]}$, but one also has (in general) that $L^{-d}\mathbb{A}_L^V(Q_L^V) = 0$. Hence, if $L^{-d}\mathbb{A}_L^V(Q_L)$ is bounded away from zero then the path $(\pi_t)_{t\in[0,T]}$ associated with P_L can be ruled out as a possible hydrodynamic limit.

In fact the hydrodynamic limit of the ZRP is known to be given by (3.18) with $\tilde{H}=0$ (see Section 5 in [22]), in which case Theorem 3.3 bounds the macroscopic action by zero: $\mathbb{A}((\pi_t)_{t\in[0,T]})\leq 0$. However this bound is not yet sufficient to show that P_L^V concentrates on $(\pi_t)_{t\in[0,T]}$, so it does not prove the hydrodynamic limit.

We now restrict our consideration to measures of the form $P_L=P_L^{V+\tilde{H}}$ that concentrate on paths which satisfy (3.18), for some $\tilde{H}$. In this case, Theorem 3.4 may be applied. This establishes that the limit of $L^{-d}\mathbb{A}_L^V(Q_L^{V+\tilde{H}})$ exists. We moreover can verify the assumptions of Theorem 3.1 (at least for $d=1$) or alternatively rely on the existence of the pathwise LDP (see [3]), which shows that also Theorem 3.5 holds – this establishes a lower bound $\mathbb{A}((\pi_t)_{t\in[0,T]})\geq 0$ for any path $(\pi_t)_{t\in[0,T]}$ that solves (3.18), with some $\tilde{H}$. This means that $(\pi_t)_{t\in[0,T]}$ is only admissible as a candidate for the hydrodynamic limit of the ZRP, if it is a (weak) solution to (3.18) with $\tilde{H}=0$ (otherwise one has the contradiction $0=\lim_{L\to\infty}L^{-d}\mathbb{A}_L^V(Q_L^V)=\mathbb{A}((\pi_t)_{t\in[0,T]})>0$).

3.4.2. Simple Exclusion Process

For the SEP the invariant reference measure is $\nu_{*,1}(0)=\nu_{*,1}(1)=1$ and the α-dependent invariant product measure are Bernoulli distributed $\nu_{\alpha,1}(\eta(0))=\alpha^{\eta(0)}(1-\alpha)^{1-\eta(0)}$. The functions ϕ and χ are given by $\phi(\alpha)=\alpha$ and $\chi(\alpha)=\alpha(1-\alpha)$. The free energy is given by

$$\mathcal{F}_\alpha^V(\rho)=\int_\Lambda\Bigg[\rho(u)\log\left(\frac{\rho(u)}{\alpha\mathrm{e}^{-V(u)}}\right)+(1-\rho(u))\log\left(\frac{1-\rho(u)}{1-\alpha}\right)\\ +\log\Big(\alpha\mathrm{e}^{-V(u)}+(1-\alpha)\Big)\Bigg]\mathrm{d}u,$$

which is of the form (2.22) for the free energy density $f(a)=a\log a+(1-a)\log(1-a)$ and the stationary density is $\bar{\rho}_{\alpha,V}(u)=\alpha\mathrm{e}^{-V(u)}/(\alpha\mathrm{e}^{-V(u)}+(1-\alpha))$.

For the sequence $P_L^{V+\tilde{H}}$ the hydrodynamic limit is again given in (3.18), which has, for suitable initial condition, a unique weak solution (see Proposition 5.1 on page 273 in [22]). We can proceed as for the ZRP and can establish (under suitable assumptions) that the results of Theorem 3.3 and Theorem 3.4 hold.

Note that this process does not satisfy the assumptions of Theorem 3.1 (as the assumption $\chi'(a)\geq C_*$ is not satisfied). Nonetheless, we can establish the chain rule (3.4) if the pathwise LDP holds (cf. the discussion at the end of Section 3.1). This was e.g. proved in [22, Chapter 10] (see also [6]), such that also in this case the results of Theorem 3.5 hold.

4. Regularity of paths and the chain rule

The main aim of this section is to prove Theorem 3.1. The central difficulty is that classical approaches to establish chain rules in metric spaces rely on λ-convexity of the functional under consideration; this property is delicate and apparently not sufficiently well understood in a context other than the classic (unweighted) Wasserstein setting. The processes considered here are, however, naturally linked to weighted Wasserstein spaces, where important elements of the classic Wasserstein theory are still missing. We circumvent this problem by showing that while the classic Wasserstein space is not the natural space for the processes we study, they can be cast in this setting. The analysis is then somewhat technical, but largely follows arguments in [1]. The novel Ψ-$\Psi^\star$-structure is thus less relevant in this section than for the proofs in Section 5.

In the following, we consider paths with conserved volume, for which also the action is finite: $\mathbb{A}((\rho_t)_{t\in[0,T]})<\infty$. Combined with $\mathcal{F}_\alpha^V(\rho_0)<\infty$ and (2.30), this implies that $\mathcal{E}((\rho_t)_{t\in[0,T]})<\infty$ and $\mathcal{E}^\star((\rho_t)_{t\in[0,T]})<\infty$. We will see that the former of the two implies regularity in time (that $(\rho_t)_{t\in[0,T]}$ is absolutely-continuous in the Wasserstein sense) and the latter yields certain regularity in space (such that e.g. the weak gradient $\nabla\phi(\rho)$ exists a.e. in Λ).

The following steps are based on ideas from Section 4 in [10]. For a more recent and concise representation of the following material, we refer to Appendices D.5 and D.6) in [19]. A discussion of similar content in terms of interacting particle systems can e.g. be found in [6].

For any topological space $\mathcal{S}$, we denote with $\mathscr{D}(\mathcal{S};\mathbb{R})=C_c^\infty(\mathcal{S};\mathbb{R})$ the vector space of real-valued infinitely often differentiable and compactly supported functions on $\mathcal{S}$ and equip $\mathscr{D}(\mathcal{S};\mathbb{R})$ with the usual topology for test functions, see e.g. [19, Appendix D.1]. Its topological dual, the space of (Schwartz) distributions, will be denoted with $\mathscr{D}'(\mathcal{S};\mathbb{R})$. The application of $g\in\mathscr{D}(\mathcal{S};\mathbb{R})$ to a distribution $\vartheta\in\mathscr{D}'(\mathcal{S};\mathbb{R})$ is denoted by $\langle\vartheta,g\rangle$.

The Otto calculus yields a formal interpretation of $\mathcal{M}_+(\Lambda)$ as an infinite dimensional Riemannian manifold (see for example Chapter 15 in [39] or Section 8.1.2 in [38]). For a measure $\pi\in\mathcal{M}_+(\Lambda)$, one can define three isometric spaces $H^1_\pi(\Lambda;\mathbb{R})$, $H^{-1}_\pi(\Lambda;\mathbb{R})$ and $\mathcal{L}^2_{\nabla,\pi}(\Lambda;\mathbb{R}^d)$, which all can play the role of the 'tangent space' at π. We next give precise definitions of all three spaces. For $h\colon\Lambda\to\mathbb{R}^d$, we define the norm $\|h\|_\pi^2:=\int_\Lambda|h(u)|^2\pi(\mathrm{d}u)$. For $g\in W^1_{\mathrm{loc}}(\Lambda;\mathbb{R})$ this norm gives rise to the semi-norm $\|g\|_{1,\pi}:=\|\nabla g\|_\pi$, where ∇g denotes the weak derivative of g. Since $\{g\in\mathscr{D}(\Lambda;\mathbb{R})\colon\int_\Lambda g\,\mathrm{d}u=0\}$ equipped with $\|\cdot\|_{1,\pi}$ is a normed space, we can define its completion to be $H^1_\pi(\Lambda;\mathbb{R})$. For $\vartheta\in\mathscr{D}'(\Lambda;\mathbb{R})$ the dual norm, which is defined as

$$\|\vartheta\|^2_{-1,\pi}:=\sup_{g\in H^1_\pi(\Lambda;\mathbb{R})}\big(2\langle\vartheta,g\rangle-\|g\|^2_{1,\pi}\big), \tag{4.1}$$

gives rise to $H^{-1}_\pi(\Lambda;\mathbb{R}):=\{\vartheta\in\mathscr{D}'(\Lambda;\mathbb{R})\colon\|\vartheta\|_{-1,\pi}<\infty\}$, the dual of $H^1_\pi(\Lambda;\mathbb{R})$. Note that $H^1_\pi(\Lambda;\mathbb{R})$ is a Hilbert space (with inner product $\langle\cdot,\cdot\rangle_{1,\pi}$ defined in the obvious way using the polarisation identity for inner products); it therefore is reflexive, which implies the existence of a linear and isometric map from $H^1_\pi(\Lambda;\mathbb{R})$ to $H^{-1}_\pi(\Lambda;\mathbb{R})$, formally given by $g\mapsto-\nabla\cdot(\pi\nabla g)$. The inner product on $H^{-1}_\pi(\Lambda;\mathbb{R})$ will be denoted with $\langle\cdot,\cdot\rangle_{-1,\pi}$. Finally, let $\mathcal{L}^2_{\nabla,\pi}(\Lambda;\mathbb{R}^d)$ be the completion of $\{\nabla\zeta:\zeta\in\mathscr{D}(\Lambda;\mathbb{R})\}$ with respect to $\|\cdot\|_\pi$. It is then easy to see that $H^1_\pi(\Lambda;\mathbb{R})$ is also isometric to $\mathcal{L}^2_{\nabla,\pi}(\Lambda;\mathbb{R}^d)$ (cf. page 379 in [19]). We will denote the map from $H^1_\pi(\Lambda;\mathbb{R})$ to $\mathcal{L}^2_{\nabla,\pi}(\Lambda;\mathbb{R}^d)$ with ∇.

For our purposes, the spaces $H^{-1}_\pi(\Lambda;\mathbb{R})$ and $\mathcal{L}^2_{\nabla,\pi}(\Lambda;\mathbb{R}^d)$ yield the more relevant representations. The two prominent cases that will appear in the following are $\pi(\mathrm{d}u)=\rho(u)\mathrm{d}u$ and $\pi(\mathrm{d}u)=\chi(\rho(u))\mathrm{d}u$. In these cases we will identify the densities ρ and $\chi(\rho)$ as measures and write $H^1_\rho(\Lambda;\mathbb{R})$ and $H^1_{\chi(\rho)}(\Lambda;\mathbb{R})$ instead of $H^1_\pi(\Lambda;\mathbb{R})$ (and similar for the other spaces we just introduced).

4.1. Regularity of paths on the hydrodynamic scale. Now, fix a path $(\pi_t)_{t\in[0,T]}\in\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$ that is absolutely continuous with respect to the Lebesgue measure with density $(\rho_t)_{t\in[0,T]}$. We equip $C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ with the $(\rho_t)_{t\in[0,T]}$ dependent semi-norm $G\mapsto(\int_0^T\|\nabla G_t\|^2_{\chi(\rho_t)}\mathrm{d}t)^{1/2}$, on which we define the two real valued linear operators

$$L_{\mathcal{E}}(G):=\int_\Lambda\rho_TG_T\,\mathrm{d}u-\int_\Lambda\rho_0G_0\,\mathrm{d}u-\int_0^T\int_\Lambda\rho_t\partial_tG_t\,\mathrm{d}u\mathrm{d}t$$

and

$$L_{\mathcal{E}^\star}(G) := \int_0^T \int_\Lambda \phi(\rho_t)\nabla\cdot\nabla G_t \,\mathrm{d}u\mathrm{d}t - \int_0^T \int_\Lambda \chi(\rho_t)\nabla V\cdot\nabla G_t \,\mathrm{d}u\mathrm{d}t.$$

Note that these two operators coincide with the left and right-hand side of (2.25), respectively. Moreover, the corresponding operator norms are given by $\mathcal{E}((\rho_t)_{t\in[0,T]})$ in (2.28) and $\mathcal{E}^\star((\rho_t)_{t\in[0,T]})$ in (2.29), respectively (cf. e.g. [10, 19]).

Under the assumptions of Theorem 3.3, we have prior information on the regularity of the path $(\rho_t)_{t\in[0,T]}$, i.e. we can assume that $\mathcal{E}((\rho_t)_{t\in[0,T]}), \mathcal{E}^\star((\rho_t)_{t\in[0,T]}) < \infty$ (such that $L_{\mathcal{E}}$ and $L_{\mathcal{E}^\star}$ are bounded linear operators).

Note that $L_{\mathcal{E}}$ and $L_{\mathcal{E}^\star}$ are both invariant under addition of a constant in the sense that $L_{\mathcal{E}^\star}(G) = L_{\mathcal{E}^\star}(G+c)$ for any $c\in\mathbb{R}$. We thus can (with slight abuse of notation) redefine $L_{\mathcal{E}}$ and $L_{\mathcal{E}^\star}$ as operators on $\{\nabla G : G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})\}$, equipped with $\nabla G \mapsto (\int_0^T \|\nabla G_t\|^2_{\chi(\rho_t)}\mathrm{d}t)^{1/2}$, as

$$L_{\mathcal{E}}(\nabla G) := L_{\mathcal{E}}(G) \quad \text{and} \quad L_{\mathcal{E}^\star}(\nabla G) := L_{\mathcal{E}^\star}(G).$$

Let $\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$ be the $(\rho_t)_{t\in[0,T]}$ dependent completion of $\{\nabla G : G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})\}$ with respect to $\nabla G\mapsto (\int_0^T \|\nabla G_t\|^2_{\chi(\rho_t)}\mathrm{d}t)^{1/2}$. Note that if $h = (h_t)_{t\in[0,T]} \in \mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$, then $h_t \in \mathcal{L}^2_{\nabla,\chi(\rho_t)}(\Lambda;\mathbb{R}^d)$ for a.a. $t\in[0,T]$. In Section 4.2 we will also consider $\mathcal{L}^2_{\nabla,\mathrm{id}}([0,T]\times\Lambda;\mathbb{R}^d)$, where the norm is replaced with $\nabla G\mapsto (\int_0^T \|\nabla G_t\|^2_{\rho_t}\mathrm{d}t)^{1/2}$.

Since $\mathcal{E}((\rho_t)_{t\in[0,T]}), \mathcal{E}^\star((\rho_t)_{t\in[0,T]}) < \infty$ the Bounded Linear Transformation Theorem (see e.g. Theorem I.6 in [35]), allows us to extend $L_{\mathcal{E}}(\nabla G)$ and $L_{\mathcal{E}^\star}(\nabla G)$ to bounded linear operators on $\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$ with the same operator norms as above. For $h\in\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$ we have

$$L_{\mathcal{E}}(h) = \int_\Lambda \rho_T \nabla^{-1}h_T\,\mathrm{d}u - \int_\Lambda \rho_0\nabla^{-1}h_0\,\mathrm{d}u - \int_0^T\int_\Lambda \rho_t\partial_t(\nabla^{-1}h_t)\,\mathrm{d}u\mathrm{d}t,$$

where ∇^{-1} denotes (for each $t\in[0,T]$) the isometric map from $\mathcal{L}^2_{\nabla,\chi(\rho_t)}(\Lambda;\mathbb{R}^d)$ to $H^1_{\chi(\rho_t)}(\Lambda;\mathbb{R})$. Further

$$L_{\mathcal{E}^\star}(h) = \int_0^T\int_\Lambda \phi(\rho_t)\nabla\cdot h_t\,\mathrm{d}u\mathrm{d}t - \int_0^T\int_\Lambda \chi(\rho_t)\nabla V\cdot h_t\,\mathrm{d}u\mathrm{d}t.$$

By Riesz' representation theorem (e.g. Theorem II.4 in [35]), there exist unique elements $v, w\in\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$, with $v = (v_t)_{t\in[0,T]}$ and $w = (w_t)_{t\in[0,T]}$, for which these two bounded operators can be represented by

$$L_{\mathcal{E}}(h) = \int_0^T\int_\Lambda \chi(\rho_t)v_t\cdot h_t\,\mathrm{d}u\mathrm{d}t, \qquad L_{\mathcal{E}^\star}(h) = \int_0^T\int_\Lambda \chi(\rho_t)w_t\cdot h_t\,\mathrm{d}u\mathrm{d}t. \tag{4.2}$$

Substituting (4.2) in (2.28) and (2.29) yields (c.f. Lemma 4.8 in [10])

$$\mathcal{E}\big((\rho_t)_{t\in[0,T]}\big) = \frac{1}{2}\int_0^T \|v_t\|^2_{\chi(\rho_t)}\mathrm{d}t, \qquad \mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big) = \frac{1}{2}\int_0^T \|w_t\|^2_{\chi(\rho_t)}\mathrm{d}t. \tag{4.3}$$

PROPOSITION 4.1. *Assume that $\mathcal{E}((\rho_t)_{t\in[0,T]})<\infty$ and that χ satisfies the assumptions of Section 2.4.2. Then the weak time derivative of ρ_t, denoted $\dot\rho_t$, exists in $H^{-1}_{\chi(\rho_t)}(\Lambda;\mathbb{R})$ for a.a. $t\in[0,T]$. Moreover,*

$$\mathcal{E}\big((\rho_t)_{t\in[0,T]}\big)=\frac{1}{2}\int_0^T\|\dot\rho_t\|^2_{-1,\chi(\rho_t)}\,\mathrm{d}t. \tag{4.4}$$

Proof. Results of this kind are standard and we hence only sketch the proof. Consider the unique $v\in\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$ from (4.2) and recall that $v_t\in\mathcal{L}^2_{\nabla,\chi(\rho_t)}(\Lambda;\mathbb{R}^d)$ for a.a. $t\in[0,T]$.

Following e.g. Lemma 4.8 in [10] (see also [13]), one shows that $\mathcal{E}((\rho_t)_{t\in[0,T]})<\infty$ implies that $t\mapsto\langle\rho_t,\cdot\rangle$ is absolutely continuous in the sense of distributions, such that the distributional derivative $\dot\rho_t\in\mathscr{D}'(\Lambda;\mathbb{R})$ exists for a.a. $t\in(0,T)$. In our case, the latter satisfies for $G\in\mathscr{D}(\Lambda;\mathbb{R})$ and a.a. $t\in(0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Lambda\rho_t G\,\mathrm{d}u=\langle\dot\rho_t,G\rangle=\int_\Lambda\chi(\rho_t)v_t\cdot\nabla G\,\mathrm{d}u. \tag{4.5}$$

Thus $\dot\rho_t=-\nabla\cdot(\chi(\rho_t)v_t)$ in the distributional sense for a.a. $t\in(0,T)$, such that $v_t\in\mathcal{L}^2_{\nabla,\chi(\rho_t)}(\Lambda;\mathbb{R}^d)$ can uniquely be identified with $\dot\rho_t$. Further the isometry from $\mathcal{L}^2_{\nabla,\chi(\rho_t)}(\Lambda;\mathbb{R}^d)$ to $H^{-1}_{\chi(\rho_t)}(\Lambda;\mathbb{R})$ (for a.a. $t\in[0,T]$) implies that $\dot\rho_t\in H^{-1}_{\chi(\rho_t)}(\Lambda;\mathbb{R})$ and (4.4) also follows. ∎

Let $p\in[1,\infty]$. We say a path $(\pi_t)_{t\in[0,T]}$ is *p-absolutely continuous (in the Wasserstein sense)*, if there exists a function $m\in\mathcal{L}^p([0,T];\mathbb{R})$, such that for any $0\le t_1<t_2\le T$

$$W_2(\pi_{t_1},\pi_{t_2})\le\int_{t_1}^{t_2}m(s)\,\mathrm{d}s, \tag{4.6}$$

where W_2 denotes the 2-Wasserstein distance [1,38]. In this case, the metric derivative (cf. equation (1.1.3) in [1]) exists for a.a. $t\in(0,T)$,

$$|\pi_t'|:=\limsup_{h\to0}\left(\frac{W_2(\pi_t,\pi_{t+h})}{h}\right)<\infty$$

and $t\mapsto|\pi_t'|$ is the minimal function that satisfies (4.6), see Theorem 1.1.2 in [1]. In other words, $(\pi_t)_{t\in[0,T]}$ is p-absolutely continuous if and only if the map $t\mapsto|\pi_t'|$ is an element of $\mathcal{L}^p([0,T];\mathbb{R})$. From now on we consider the case $p=2$.

LEMMA 4.1. *A path $(\pi_t)_{t\in[0,T]}\in\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))$ is 2-absolutely continuous if and only if there exists a vector field $\tilde v=(\tilde v_t)_{t\in[0,T]}$ with $\tilde v_t\in\mathcal{L}^2_{\nabla,\pi_t}(\Lambda;\mathbb{R}^d)$ and $\int_0^T\|\tilde v_t\|_{\pi_t}\mathrm{d}t<\infty$ that satisfies $\dot\pi_t+\nabla\cdot(\pi_t\tilde v_t)=0$ in the distributional sense for almost all $t\in[0,T]$. In this case we have in particular $(\pi_t)_{t\in[0,T]}\in C([0,T];\mathcal{M}_+(\Lambda))$.*

Proof. The result follows from a modification of Lemma 8.1.2 and Theorem 8.3.1 in [1] to the domain Λ. Assume first that $(\pi_t)_{t\in[0,T]}$ is 2-absolutely continuous. Then Theorem 8.3.1 implies that the continuity equation $\dot\pi_t+\nabla\cdot(\pi_t\tilde v_t)=0$ holds for some $\tilde v_t$, which can, by Lemma 8.4.2 in [1], without loss of generality be chosen to satisfy $\tilde v_t\in\mathcal{L}^2_{\nabla,\pi_t}(\Lambda;\mathbb{R}^d)$.

For the opposite implication we assume that the continuity equation holds and that moreover $\int_0^T\|\tilde v_t\|_{\pi_t}\mathrm{d}t<\infty$. An application of the Hölder inequality combined with

$\sup_{t\in[0,T]}\pi_t(\Lambda)<\infty$ ensures that $\int_0^T\int_\Lambda|\tilde{v}_t(u)|\pi_t(\mathrm{d}u)\mathrm{d}t<\infty$. Lemma 8.1.2 thus implies that the curve has a weakly continuous modification $(\tilde{\pi}_t)_{t\in[0,T]}\in C([0,T];\mathcal{M}_+(\Lambda))$. Now, since every right-continuous path that admits a continuous modification already has to be continuous, we have $(\pi_t)_{t\in[0,T]}=(\tilde{\pi}_t)_{t\in[0,T]}$. This allows us to apply the reverse implication of Theorem 8.3.1 to $(\pi_t)_{t\in[0,T]}$, which yields that $(\pi_t)_{t\in[0,T]}$ is 2-absolutely continuous. ∎

The Wasserstein distance W_2 has a fluid dynamical representation in terms of the Brenier-Benamou formula (compare Equation (8.0.3) in [1] and Section 8.1 in [38]). The distance of two measures $\pi,\hat{\pi}\in\mathcal{M}_+(\Lambda)$ with $\pi(\Lambda)=\hat{\pi}(\Lambda)>0$ is given by

$$W_2^2(\pi,\hat{\pi})=\inf\left\{\int_0^1\|\tilde{v}_t\|_{\mu_t}^2\,\mathrm{d}t\;\middle|\;\mu_0=\pi,\ \mu_1=\hat{\pi},\ \dot{\mu}_t+\nabla\cdot(\mu_t\tilde{v}_t)=0\right\},$$

where the infimum is taken over all 2-absolutely continuous paths of measures $(\mu_t)_{t\in[0,T]}$ and velocities $\tilde{v}_t\in\mathcal{L}^2_{\nabla,\mu_t}(\Lambda;\mathbb{R}^d)$ satisfying the continuity equation above.

Let $(\pi_t)_{t\in[0,T]}$ be absolutely continuous with respect to the Lebesgue measure with density $(\rho_t)_{t\in[0,T]}$. We say that $(\rho_t)_{t\in[0,T]}$ is 2-absolutely continuous if $(\pi_t)_{t\in[0,T]}$ is 2-absolutely continuous. Moreover, we will identify densities with their associated measures. In particular, we write $W_2^2(\rho,\hat{\rho})=W_2^2(\pi,\hat{\pi})$ for $\pi(\mathrm{d}u)=\rho(u)\mathrm{d}u$ and $\pi(\mathrm{d}u)=\rho(u)\mathrm{d}u$.

Proposition 4.2. *Assume that $\mathcal{E}((\rho_t)_{t\in[0,T]})<\infty$ and that χ satisfies the assumptions of Section 2.4.2. Then $(\rho_t)_{t\in[0,T]}$ is 2-absolutely continuous in the Wasserstein sense.*

Proof. We choose the time rescaling $\bar{t}=t(t_2-t_1)+t_1$ and set $\mu_t=\rho_{\bar{t}}$ and $\tilde{v}_t=(t_2-t_1)(\chi(\rho_{\bar{t}})v_{\bar{t}})/\rho_{\bar{t}}$, such that $\dot{\mu}_t+\nabla\cdot(\mu_t\tilde{v}_t)=0$ by construction. We obtain for all $0\le t_1<t_2\le T$

$$W_2^2(\rho_{t_1},\rho_{t_2})\le(t_2-t_1)\int_{t_1}^{t_2}\|(\chi(\rho_t)v_t)/\rho_t\|_{\rho_t}^2\,\mathrm{d}t\le(t_2-t_1)\int_{t_1}^{t_2}C_{\mathrm{Lip}}\|v_t\|_{\chi(\rho_t)}^2\,\mathrm{d}t<\infty,$$

such that the metric derivative satisfies for almost all $t\in[0,T)$

$$|\rho_t'|=\limsup_{h\to0}\left(\frac{W_2(\rho_t,\rho_{t+h})}{h}\right)\le\sqrt{C_{\mathrm{Lip}}}\|v_t\|_{\chi(\rho_t)}. \tag{4.7}$$

The square integrability of the right-hand side now implies that $(\rho_t)_{t\in[0,T]}$ is 2-absolutely continuous. ∎

Proposition 4.3. *Assume that $\mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big)<\infty$ and that f,ϕ and χ satisfy the assumptions of Section 2.4.2. Then*

$$\begin{aligned}\mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big)&=\frac{1}{2}\int_0^T\|\Delta\phi(\rho_t)+\nabla\cdot(\chi(\rho_t)\nabla V)\|_{-1,\chi(\rho_t)}^2\,\mathrm{d}t\\&=\frac{1}{2}\int_0^T\|f''(\rho_t)\nabla\rho_t+\nabla V\|_{\chi(\rho_t)}^2\,\mathrm{d}t.\end{aligned}\tag{4.8}$$

Proof. $\mathcal{E}^\star\big((\rho_t)_{t\in[0,T]}\big)<\infty$ implies that the distributional derivative of $\phi(\rho_t)\in\mathcal{L}^1_{\mathrm{loc}}(\Lambda;\mathbb{R})$ satisfies $\nabla\phi(\rho_t)\in\mathcal{L}^1_{\mathrm{loc}}(\Lambda;\mathbb{R}^d)$ for a.a. $t\in[0,T]$ (cf. Appendix D.6 in [19]). Equivalently, $\phi(\rho_t)\in W^{1,1}_{\mathrm{loc}}(\Lambda;\mathbb{R})$ for a.a. $t\in[0,T]$. The first identity in (4.8) can be established as in Appendix D.6 in [19] (for the choice $\mu(\mathrm{d}u)=\chi(\rho_t(u))\mathrm{d}u$). We turn to the second identity. Since ϕ^{-1} is continuously differentiable with bounded derivative, we obtain by the chain rule for functions in $W^{1,1}_{\mathrm{loc}}(\Lambda;\mathbb{R})$ with bounded derivative (see e.g. Theorem 4 (ii) in [17]) that also $\nabla\rho_t\in\mathcal{L}^1_{\mathrm{loc}}(\Lambda;\mathbb{R})$, and thus $\rho_t\in W^{1,1}_{\mathrm{loc}}(\Lambda;\mathbb{R})$, for almost all $t\in[0,T]$. The derivative is for almost all $u\in\Lambda$ given by

$$\nabla\rho_t(u)=(\phi^{-1})'(\phi(\rho_t(u)))\nabla\phi(\rho_t(u))=\frac{\nabla\phi(\rho_t(u))}{\phi'(\rho_t(u))}, \tag{4.9}$$

where the last identity follows from the Implicit Function Theorem. Multiplying with $\phi'(\rho_t)$ and using the local Einstein relation (2.27) we obtain that almost everywhere

$$\nabla\phi(\rho_t)=\phi'(\rho_t)\nabla\rho_t=\chi(\rho_t)f''(\rho_t)\nabla\rho_t. \tag{4.10}$$

Combined with w in (4.3), we have for any $G\in\mathscr{D}(\Lambda;\mathbb{R})$ and almost all $t\in[0,T]$ that

$$\int_\Lambda\chi(\rho_t)w_t\cdot\nabla G\mathrm{d}u=\int_\Lambda\big(\nabla\phi(\rho_t)+\chi(\rho_t)\nabla V\big)\cdot\nabla G\mathrm{d}u=\int_\Lambda\chi(\rho_t)[f''(\rho_t)\nabla\rho_t+\nabla V]\cdot\nabla G\mathrm{d}u$$

such that we can identify $w_t=f''(\rho_t)\nabla\rho_t+\nabla V$. Substituting this identity in (4.3) yields the final result. □

4.2. Chain rule for the free energy. In this section, we prove Theorem 3.1, which establishes rigorously the validity of the macroscopic chain rule (3.4), for which we so far gave only a formal derivation. Consider a given path $(\rho_t)_{t\in[0,T]}$ that satisfies $\mathbb{A}((\rho_t)_{t\in[0,T]})<\infty$. We restrict ourselves to densities $\rho,\hat\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ s.t. $\int_\Lambda\rho\mathrm{d}u=\int_\Lambda\hat\rho\mathrm{d}u>0$ and continue to identify densities with measures. The constant volume implies that free energy differences do not depend on α. Indeed, defining $\mathcal{F}(\rho):=\int_\Lambda f(\rho(u))\mathrm{d}u$ and $\mathcal{V}(\rho):=\int_\Lambda V(u)\rho(u)\mathrm{d}u$ (for $V\in C^2(\Lambda;\mathbb{R})$), we can define an α-independent modification of the free energy

$$\mathcal{F}^V(\rho):=\mathcal{F}(\rho)+\mathcal{V}(\rho), \tag{4.11}$$

which is (with (2.22)) easily seen to satisfy $\mathcal{F}^V_\alpha(\hat\rho)-\mathcal{F}^V_\alpha(\rho)=\mathcal{F}^V(\hat\rho)-\mathcal{F}^V(\rho)$.

We assume that $f\in C^2([0,\infty);\mathbb{R})$ satisfies the assumptions in Section 2.4.2, such that the functional $\mathcal{F}\colon\mathcal{L}^1(\Lambda;[0,\infty))\to(-\infty,\infty]$ is proper and lower-semicontinuous (see Remark 9.3.8 in [1]). Note that for $N_{\max}=\infty$ the assumption $\lim_{r\to N_{\max}}f'(r)=\infty$ implies super linearity of f.

We set

$$L_f(a):=af'(a)-f(a)=\int_0^a rf''(r)\mathrm{d}r$$

and note the similarity to $\phi(a)=\int_0^a\phi'(r)\mathrm{d}r=\int_0^a\chi(r)f''(r)\mathrm{d}r$ (where we again used the local Einstein relation (2.27)); in particular $L_f'(a)/a=f''(a)=\phi'(a)/\chi(a)$. The quantity L_f is sometimes referred to as a 'pressure' function due to its relation to the thermodynamic pressure in classical thermodynamics, see e.g. Remark 5.18 (ii) in [38].

We denote the (2-)Wasserstein distance between ρ and $\hat\rho$ with $W_2(\rho,\hat\rho)$. A constant speed geodesic (connecting ρ to $\hat\rho$) is a curve $(\rho_t)_{t\in[0,1]}$ such that ($\rho_0=\rho$, $\rho_1=\hat\rho$ and) $W_2(\rho_s,\rho_t)=|t-s|W_2(\rho,\hat\rho)$ for all $s,t\in[0,T]$. With this, a functional $\mathcal{G}$ is called λ-convex (also called semi-convex) for $\lambda\in\mathbb{R}$ if the inequality

$$\mathcal{G}(\rho_t)\le(1-t)\mathcal{G}(\rho_0)+t\mathcal{G}(\rho_1)-\frac{\lambda}{2}t(1-t)W_2^2(\rho_0,\rho_1) \tag{4.12}$$

holds for each constant speed geodesic $(\rho_t)_{t\in[0,1]}$. Note that if two functionals $\mathcal{G}_i$ are λ_i-convex for $i=1,2$, then clearly $\mathcal{G}_1+\mathcal{G}_2$ is λ-convex with $\lambda=\min(\lambda_1,\lambda_2)$.

We call $\mathcal{G}$ geodesically convex if the map $t\mapsto\mathcal{G}(\rho_t)$ is convex for any geodesic $(\rho_t)_{t\in[0,1]}$ (which is equivalent to λ-convexity for $\lambda=0$). A useful criterion for geodesic convexity of the free energy $\mathcal{F}$ is the McCann condition (see Proposition 9.3.9 and equation (9.3.11) in [1]): A convex function $f\in C^2([0,\infty);\mathbb{R})$ with $f(0)=0$ satisfies the McCann condition (in d dimensions) if the map

$$s\mapsto s^d f(s^{-d}) \tag{4.13}$$

is convex on $(0,\infty)$ (cf. the discussion in Section 9.3 in [1]). In the case $d=1$, convexity of f is sufficient to establish geodesic convexity. For a potential energy of the form $\mathcal{V}(\rho)=\int_\Lambda V(u)\rho(u)\mathrm{d}u$ λ-convexity is equivalent to λ-convexity (also called strong convexity) of V on Λ (see equation (9.3.3) and Proposition 9.3.2 in [1]), which is $V((1-t)x+ty)\le(1-t)V(x)+tV(y)-(\lambda/2)t(1-t)\|x-y\|^2$. For $V\in C^2(\Lambda;\mathbb{R})$ the Hessian matrix is bounded and this assumption is trivially satisfied. Note that under the assumption that $\mathcal{F}$ is geodesically-convex and $\mathcal{V}$ is λ-convex for some $\lambda\le 0$, also $\mathcal{F}^V$ is λ-convex.

4.2.1. Assumptions for chain rule. To our knowledge, minimal sufficient conditions for the validity of a chain rule of the form (3.4) are still an open question. One difficulty is that the existing theory requires λ-convexity of the functional in question. In the case of independent particles (with $\chi(a)=\phi(a)=a$) sufficient conditions for λ-convex functionals can be obtained from the general theory for gradient flows in Wasserstein spaces, which was established in [1] (see also [36,38]). We note that generalisations of the gradient flow theory in Wasserstein spaces with non-linear (usually concave) mobilities have been considered in the literature, see e.g. [11,12,25–27]. Yet, establishing the chain rule in a weighted Wasserstein metric is fraught with technical difficulties, in particular λ-convexity of the functional. We overcome this difficulty here by showing that in the setting studied here, where a weighted Wasserstein metric is the natural space, the chain rule can be established in an unweighted (classical) Wasserstein setting, where strong tools are available.

In this section, we establish the chain rule (3.4) in the special case that the density f of the free energy $\mathcal{F}^V$ satisfies the McCann condition for geodesic convexity (4.13) and the particle process is 'not too far away' from the process with independent particles (where $\chi(a)=\phi(a)=a$): We consider the case $N_{\max}=\infty$ and assume there exists $C_*>0$ (without loss of generality the same constant which bounds $\phi'(a)$ from below) such that

$$C_*\le\chi'(a) \tag{4.14}$$

for almost all $a\in(0,\infty)$. This implies that $C_*\le\chi'(a),\phi'(a)\le C_{\mathrm{Lip}}$, such that also $C_*a\le\chi(a),\phi(a)\le C_{\mathrm{Lip}}a$. We obtain for any $\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ that the norms $\|\cdot\|_\rho$ and $\|\cdot\|_{\chi(\rho)}$ are equivalent,

$$C_*\|\cdot\|_\rho\le\|\cdot\|_{\chi(\rho)}\le C_{\mathrm{Lip}}\|\cdot\|_\rho. \tag{4.15}$$

In this case also the limit points coincide such that $\mathcal{L}^2_{\nabla,\chi(\rho)}(\Lambda;\mathbb{R}^d)=\mathcal{L}^2_{\nabla,\rho}(\Lambda;\mathbb{R}^d)$. This will allow us to leverage results from the classical Wasserstein framework in [1].

Remark 4.1. The Lipschitz continuity of $\chi(a)$ implies that $\mathcal{L}^2_{\nabla,\rho}(\Lambda;\mathbb{R}^d)\subseteq\mathcal{L}^2_{\nabla,\chi(\rho)}(\Lambda;\mathbb{R}^d)$. In general, this is a strict inclusion (consider e.g. the case of the SEP with $\chi(a)=a(1-a)$ and $\rho=1$ on a subset $O\subseteq\Lambda$ with positive Lebesgue measure). A (weaker, density ρ dependent) condition for the opposite inclusion to hold is

$$\inf_{u\in\Lambda}\frac{\chi(\rho(u))}{\rho(u)}>0,$$

which can in this case replace the constant in the lower bound of (4.15). Note that this is a density specific condition, whereas the above condition (4.14) is a model specific condition (which is independent of ρ). For the SEP, this condition is satisfied precisely in the case when ρ is bounded away from the maximal possible local particle density, i.e. $\rho\le N_{\max}-\epsilon$ (for some $\epsilon>0$). The same considerations show that in general $\mathcal{L}^2_{\nabla,\mathrm{id}}([0,T]\times\Lambda;\mathbb{R}^d)\subseteq\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$ and that (4.14), or alternatively

$$\inf_{(t,u)\in[0,T]\times\Lambda}\frac{\chi(\rho_t(u))}{\rho_t(u)}>0,$$

ensures that $\mathcal{L}^2_{\nabla,\mathrm{id}}([0,T]\times\Lambda;\mathbb{R}^d)=\mathcal{L}^2_{\nabla,\chi}([0,T]\times\Lambda;\mathbb{R}^d)$.

4.2.2. Validity of the chain rule. The following results, which are mainly based on Chapter 9 and 10 in [1], relate $L_f(\rho)$ to the directional derivative, the Fréchet-subdifferential, and the metric slope of $\mathcal{F}(\rho)$. Below we sketch results which can be obtained by a suitable modification of the results in [1]. More precisely, we are interested in the case where the domain is $\Lambda=\mathbb{T}^d$ and the measures of interest are absolutely continuous with respect to the Lebesgue measure.

As shown in Theorem 1.25 in [36] there exists for any $\rho,\hat\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ with $\int_\Lambda\rho\mathrm{d}u=\int_\Lambda\hat\rho\mathrm{d}u>0$ a unique optimal transport map from ρ to $\hat\rho$ of the form $r=i-\nabla\varphi$, where φ is semi-concave (i.e. there exists a constant $C>0$ such that $\varphi(u)-C|u|^2$ is concave). Moreover, the interpolation $r_t:=(1-t)i+tr$ between r and the identity i on Λ is such that $(r_t)_\#\rho$ has a Lebesgue density for all $t\in[0,1]$ (which can e.g. be shown by a modification of the proof of Proposition 9.3.9. in [1]).

Now, assume that f satisfies the McCann condition for geodesic convexity (4.13), that $\mathcal{F}(\rho),\mathcal{F}(\hat\rho)<\infty$, and that $L_f(\rho)\in W^{1,1}(\Lambda;\mathbb{R})$. Then

$$\int_\Lambda\nabla[L_f(\rho)]\cdot(r-i)\mathrm{d}u\le-\int_\Lambda L_f(\rho)\,\mathrm{tr}\,\tilde\nabla(r-i)\mathrm{d}u=\lim_{t\searrow0}\frac{\mathcal{F}((r_t)_\#\rho)-\mathcal{F}(\rho)}{t}<\infty,$$

where $\tilde\nabla r$ denotes the approximate derivative (see Definition 5.5.1 in [1]) and i is the identity on Λ. This result can be obtained from a modification of the proofs of Lemma 10.4.4 and Lemma 10.4.5 in [1].

For a λ-convex functional $\mathcal{G}$, the Fréchet-subdifferential $\partial\mathcal{G}(\rho)$ at $\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ with $\int_\Lambda\rho\mathrm{d}u>0$ consists of all vectors $\zeta\in\mathcal{L}^2_\rho(\Lambda;\mathbb{R}^d):=\{\zeta\colon\Lambda\to\mathbb{R}^d:\|\zeta\|_\rho<\infty\}$ such that for all $\hat\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ with $\int_\Lambda\rho\mathrm{d}u=\int_\Lambda\hat\rho\mathrm{d}u$

$$\mathcal{G}(\hat\rho)-\mathcal{G}(\rho)\ge\int_\Lambda\zeta\cdot(r-i)\rho\mathrm{d}u+\frac{\lambda}{2}W_2^2(\rho,\hat\rho),\tag{4.16}$$

where r is the optimal transport map from ρ to $\hat\rho$ (see Equation (10.1.7) in [1]).

LEMMA 4.2 (Slope and subdifferential, cf. Theorem 10.4.6 in [1]). *Assume that f satisfies the McCann condition for geodesic convexity (4.13). For $\rho\in\mathcal{L}^1(\Lambda;[0,\infty))$ with $\int_\Lambda\rho\mathrm{d}u>0$ and $\mathcal{F}(\rho)<\infty$ the following statements are equivalent.*

(1) The Fréchet-subdifferential (4.16) is non-empty, $\partial\mathcal{F}^V(\rho)\neq\emptyset$.

(2) The metric derivative at ρ is finite,

$$|\partial \mathcal{F}^V|(\rho) := \limsup_{W_2(\rho,\hat{\rho})\to 0} \frac{(\mathcal{F}^V(\rho) - \mathcal{F}^V(\hat{\rho}))^+}{W_2(\rho,\hat{\rho})} < \infty.$$

(3) $L_f(\rho) \in W^{1,1}_{\mathrm{loc}}(\Lambda;\mathbb{R})$ with $\nabla[L_f(\rho)] + \rho\nabla V = \rho w$ for some $w \in \mathcal{L}^2_{\nabla,\rho}(\Lambda;\mathbb{R}^d)$.

If either of the above holds we have $w \in \partial\mathcal{F}(\rho)$ and $\|w\|_\rho = |\partial\mathcal{F}|(\rho)$. Moreover, if the additional assumption (4.14) holds, then the above conditions are also equivalent to

(4) $\phi(\rho) \in W^{1,1}_{\mathrm{loc}}(\Lambda;\mathbb{R})$ with $\nabla[\phi(\rho)] + \chi(\rho)\nabla V = \chi(\rho) w$ for some $w \in \mathcal{L}^2_{\nabla,\chi(\rho)}(\Lambda;\mathbb{R}^d)$.

Proof. The equivalence between (1) and (2) holds since (by Lemma 10.1.5 in [1]) the metric slope for (regular and thus in particular) λ-convex functionals is given by

$$|\partial\mathcal{F}|(\rho) = \min\{\|\zeta\|_\rho : \zeta \in \partial\mathcal{F}(\rho)\}. \tag{4.17}$$

We next show that (2) implies (3). The result follows from a standard calculation, cf. e.g. the proof of Lemma 3.5 in [26]. Consider a smooth function $\xi \in C^\infty_c(\Lambda;\mathbb{R})$. We define the flow associated to $\nabla\xi$ as the unique solution $X(t,u)$ to $\dot{X}(t,u) = \nabla\xi(X(t,u))$, $X(0,u) = u$ for $u \in \Lambda$ and $t \in (0,1)$. For $\rho^\xi_t := X(t,\cdot)_\#\rho$ we have (cf. (3.32) in [26])

$$W_2^2(\rho,\rho^\xi_t) \le t\int_0^t \|\nabla\xi\|^2_{\rho^\xi_s}\,\mathrm{d}s = t^2(\|\nabla\xi\|^2_\rho + o(1)). \tag{4.18}$$

Similar to (3.35) and (3.36) in [26] one finds

$$\lim_{t\to 0}\frac{\mathcal{F}(\rho^\xi_t) - \mathcal{F}(\rho)}{t} = \int_\Lambda \nabla[L_f(\rho)]\cdot\nabla\xi\,\mathrm{d}u \quad \text{and} \quad \lim_{t\to 0}\frac{\mathcal{V}(\rho^\xi_t) - \mathcal{V}(\rho)}{t} = \int_\Lambda \rho\nabla V\cdot\nabla\xi\,\mathrm{d}u. \tag{4.19}$$

Using (4.18) and $\mathcal{F}^V = \mathcal{F} + \mathcal{V}$ we obtain (cf. (3.33) in [26])

$$|\partial\mathcal{F}^V|(\rho) \ge \frac{1}{\|\nabla\xi\|_\rho}\lim_{t\to 0}\frac{\mathcal{F}^V(\rho^\xi_t) - \mathcal{F}^V(\rho)}{t} = \frac{1}{\|\nabla\xi\|_\rho}\int_\Lambda (\nabla[L_f(\rho)] + \rho\nabla V)\cdot\nabla\xi\,\mathrm{d}u.$$

Similar to the discussion at the beginning of Section 4.1, $|\partial\mathcal{F}^V|(\rho) < \infty$ implies that the linear operator $v \mapsto \int_\Lambda (\nabla[L_f(\rho)] + \rho\nabla V)\cdot v\,\mathrm{d}u$ from $\mathcal{L}^2_{\nabla,\rho}(\Lambda;\mathbb{R}^d)$ to $\mathbb{R}$ is bounded, such that Riesz' representation theorem implies the existence of $w \in \mathcal{L}^2_{\nabla,\rho}(\Lambda;\mathbb{R}^d)$ for which $\nabla[L_f(\rho)] + \rho\nabla V = \rho w$, such that $L_f(\rho) \in W^{1,1}_{\mathrm{loc}}(\Lambda;\mathbb{R})$. In particular $|\partial\mathcal{F}^V|(\rho) \ge \|w\|_\rho$.

For the implication (3) to (2) consider any $\hat{\rho} \in \mathcal{L}^1(\Lambda;[0,\infty))$ with $\int_\Lambda \rho\,\mathrm{d}u = \int_\Lambda \hat{\rho}\,\mathrm{d}u$ and $\mathcal{F}(\hat{\rho}) < \infty$. Then

$$\mathcal{F}(\hat{\rho}) - \mathcal{F}(\rho) \ge \lim_{t\to 0}\frac{\mathcal{F}((r_t)_\#\rho) - \mathcal{F}(\rho)}{t} \ge \int_\Lambda \nabla[L_F(\rho)]\cdot(r - i)\,\mathrm{d}u,$$

where the first inequality follows from the monotonicity of the difference quotient (see Equation (10.4.24) in [1]). The λ-convexity of $\mathcal{V}$ yields (cf. (4.12))

$$\mathcal{V}(\hat{\rho}) - \mathcal{V}(\rho) \ge \lim_{t\to 0}\frac{\mathcal{V}((r_t)_\#\rho) - \mathcal{V}(\rho)}{t} + \frac{\lambda}{2}W_2^2(\rho,\hat{\rho}) = \int_\Lambda \rho\nabla V\cdot(r - i)\,\mathrm{d}u + \frac{\lambda}{2}W_2^2(\rho,\hat{\rho}).$$

This implies that $w = (\nabla[L_F(\rho)]/\rho + \nabla V) \in \partial\mathcal{F}^V(\rho)$ and thus $|\partial\mathcal{F}^V|(\rho) \le \|w\|_\rho < \infty$ by Equation (4.17).

The equivalence between (3) and (4) can be seen as follows: Recall that $C_* L_f'(a) \leq \phi'(a) \leq C_{\mathrm{Lip}} L_f'(a)$ and also $C_* L_f(a) \leq \phi(a) \leq C_{\mathrm{Lip}} L_f(a)$. With the same argument as in the proof of Proposition 4.3 we obtain that the chain rule holds as in (4.9), i.e. $L_f'(\rho)\nabla\rho = \nabla[L_f(\rho)]$ and $\phi'(\rho)\nabla\rho = \nabla[\phi(\rho)]$, such that $C_*\|\nabla[L_f(\rho)]\| \leq \|\nabla[\phi(\rho)]\| \leq C_{\mathrm{Lip}}\|\nabla[L_f(\rho)]\|$. This proves that $\phi(\rho) \in W^{1,1}(\Lambda;\mathbb{R})$ if and only if $L_f(\rho) \in W^{1,1}(\Lambda;\mathbb{R})$. Moreover $w = \nabla[L_f(\rho)]/\rho = \nabla[\phi(\rho)]/\chi(\rho)$. ∎

Finally, we can outline a proof for Theorem 3.1, which follows ideas from [1, 26]. Since we work on the torus $\Lambda = \mathbb{T}^d$ (rather than $\mathbb{R}^d$), we sketch the argument.

Proof. (**Proof of Theorem 3.1.**) Since $\mathbb{A}$ is finite and the assumptions of Section 2.4.2 are valid, Propositions 4.1 and 4.2 and 4.3 hold. Moreover, since f satisfies the McCann condition (4.13) and also the assumption (4.14) on χ' holds, we can apply Lemma 4.2. Combining all these results we have that the map $t \mapsto |\rho_t'||\partial\mathcal{F}^V|(\rho_t)$ is in $\mathcal{L}^1_{\mathrm{loc}}([0,T];\mathbb{R})$. This then implies that $t \mapsto \mathcal{F}^V(\rho_t)$ is locally absolutely continuous (see e.g. Lemma 3.4 in [26]), with a.e. derivative

$$\frac{d}{dt}\mathcal{F}^V(\rho_t) = -\langle v_t, w_t\rangle_{\chi(\rho_t)} = -\langle \dot\rho_t, \Delta(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla V)\rangle_{-1,\chi(\rho_t)},$$

which implies the chain rule (3.4). ∎

5. Proofs and supplementary content

For nearest neighbour transitions, the following proposition yields a special representation for symmetric summands.

PROPOSITION 5.1. *Let $A_{\eta,\eta'}$ be a symmetric function (such that $A_{\eta,\eta'} = A_{\eta',\eta}$) with $A_{\eta,\eta} = 0$ and $A_{\eta,\eta^{i,j}} = 0$ whenever $|i-j| \neq 1$. If either $\sum_{\eta,\eta'\in\Omega_L}|A_{\eta,\eta'}| < \infty$ or $A_{\eta,\eta'} \geq 0$ for all $\eta,\eta' \in \Omega_L$, then*

$$\sum_{\eta,\eta'\in\Omega_L} A_{\eta,\eta'} = 2\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\sum_{\eta\in\Omega_L} A_{\eta,\eta^{i,i+e_k}}\mathbf{1}_{\{\eta(i)>0\}}. \tag{5.1}$$

Proof. Note that by definition $\sum_{\eta,\eta'\in\Omega_L} A_{\eta,\eta'} = \sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\sum_{\eta\in\Omega_L}\big(A_{\eta,\eta^{i,i+e_k}} + A_{\eta,\eta^{i,i-e_k}}\big)\mathbf{1}_{\{\eta(i)>0\}}$. Using symmetry, the second summand is equal to $A_{\eta^{i,i-e_k},\eta}$, such that first replacing the configuration η with $\eta^{i-e_k,i}$ before replacing the index i with $i+e_k$ yields (5.1). ∎

Following [22] Chapter 5, we define for $\epsilon > 0$ the approximation of the identity $\iota_\epsilon := (2\epsilon)^{-d}\mathbf{1}_{[-\epsilon,\epsilon)^d}(\cdot)$. Recall that the convolution of a measure $\pi \in \mathcal{M}_+(\Lambda)$ with a function $f \in \mathcal{L}^1(\Lambda;\mathbb{R})$ is defined as $[\pi * f](u) := \int_\Lambda f(u'-u)\pi(\mathrm{d}u')$. The convolution of ι_ϵ with the empirical measure (2.18) is the function

$$[\Theta_L(\eta) * \iota_\epsilon](u) = (2\epsilon L)^{-d}\sum_{i\in\mathbb{T}_L^d}\mathbf{1}_{[\frac{2i-1}{2L},\frac{2i+1}{2L})^d}(u)\sum_{j:|i-j|\leq\lfloor\epsilon L\rfloor}\eta(j), \tag{5.2}$$

which is piecewise constant on $\{[\frac{2i-1}{2L},\frac{2i+1}{2L})^d\}_{i\in\mathbb{T}_L^d}$. This allows us to represent the averaged particle density as a function of the empirical distribution, i.e.

$$[\Theta_L(\eta) * \iota_\epsilon](i/L) = \Big(\frac{2\lfloor\epsilon L\rfloor + 1}{2\epsilon L}\Big)^d \eta^{\lfloor\epsilon L\rfloor}(i).$$

For $\pi(\mathrm{d}u) = \rho(u)\mathrm{d}u$ the convolution yields $[\pi * \iota_\epsilon](u) = (2\epsilon)^{-d}\int_{[u-\epsilon,u+\epsilon)^d}\rho(u')\mathrm{d}u'$. Since $\lim_{\epsilon\to 0}[\pi * \iota_\epsilon](u) = \rho(u)$ for almost all $u \in \Lambda$, we define $[\pi * \iota_0](u) := \rho(u)$.

5.1. Proofs of the statements in Section 3.1.

Proof. (**Proof of Proposition 3.1.**) Recall that $(\mu_t^L)_{t\in[0,T]}$ is finitely supported in the sense that the set $\mathcal{N}_0 := \{\eta\in\Omega_L | \mu_t^L(\eta)>0 \text{ for some } t\in[0,T]\}$ is finite. Since r_t^L consists of nearest neighbour transitions, also the set $\mathcal{N}_1 := \{(\eta,\eta')\in\Omega_L\times\Omega_L | \mu_t^L(\eta)(r_t^L)_{\eta,\eta'}>0 \text{ or } \mu_t^L(\eta')(r_t^L)_{\eta',\eta}>0 \text{ for some } t\in[0,T]\}$ is finite. Thus the left-hand side of (3.1) is equal to

$$\sum_{\eta\in\mathcal{N}_0}\left[\mu_{t_2}^L(\eta)\log\left(\frac{\mu_{t_2}^L(\eta)}{\nu_\alpha(\eta)}\right)-\mu_{t_1}^L(\eta)\log\left(\frac{\mu_{t_1}^L(\eta)}{\nu_\alpha(\eta)}\right)\right]$$
$$+\sum_{\eta\in\mathcal{N}_0}\sum_{i\in\mathbb{T}_L^d}\left(\mu_{t_2}^L(\eta)\eta(i)\tilde V_{t_2}(\tfrac{i}{L})-\mu_{t_1}^L(\eta)\eta(i)\tilde V_{t_1}(\tfrac{i}{L})\right)$$
$$+\log\left(\sum_{\eta\in\Omega_L}\nu_\alpha(\eta)\mathrm{e}^{-\sum_{i\in\mathbb{T}_L^d}\tilde V_{t_2}(i/L)\eta(i)}\right)-\log\left(\sum_{\eta\in\Omega_L}\nu_\alpha(\eta)\mathrm{e}^{-\sum_{i\in\mathbb{T}_L^d}\tilde V_{t_1}(i/L)\eta(i)}\right).$$

Similar to Theorem 9.2 of Appendix 1 in [22], one then shows using (2.3) that the latter is equal to

$$\sum_{\eta\in\mathcal{N}_0}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_t^L(\eta)\log\left(\frac{\mu_t^L(\eta)}{\nu_\alpha(\eta)}\right)\right]\mathrm{d}t+\sum_{\eta\in\mathcal{N}_0}\sum_{i\in\mathbb{T}_L^d}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_t^L(\eta)\eta(i)\tilde V_t(\tfrac{i}{L})\right]\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\sum_{\eta\in\Omega_L}\nu_\alpha^{\tilde V_t}(\eta)\sum_{i\in\mathbb{T}_L^d}\eta(i)\partial_t\tilde V_t(\tfrac{i}{L})\mathrm{d}t.$$

A straightforward calculation (using $\partial_t\mu_t^L(\eta)=-\operatorname{div}\jmath_t^L(\eta)$, the fact that the transition rates r_t^L are bounded, and the fact that μ_t^L is supported on a finite number of configurations) allows to show that

$$\mathcal{F}_{L,\alpha}^{\tilde V_{t_2}}(\mu_{t_2}^L)-\mathcal{F}_{L,\alpha}^{\tilde V_{t_1}}(\mu_{t_1}^L)=-\sum_{\eta\in\mathcal{N}_0}\int_{t_1}^{t_2}\operatorname{div}\jmath_t^L(\eta)\left(\log\left(\frac{\mu_t^L(\eta)}{\nu_\alpha(\eta)}\right)+1\right)\mathrm{d}t$$
$$-\sum_{\eta\in\mathcal{N}_0}\int_{t_1}^{t_2}\operatorname{div}\jmath_t^L(\eta)\sum_{i\in\mathbb{T}_L^d}\eta(i)\tilde V_t(\tfrac{i}{L})\mathrm{d}t+\int_{t_1}^{t_2}\sum_{\eta\in\Omega_L}\left(\mu_t^L(\eta)-\nu_\alpha^{\tilde V_t}(\eta)\right)\sum_{i\in\mathbb{T}_L^d}\eta(i)\partial_t\tilde V_t(\tfrac{i}{L})\mathrm{d}t. \tag{5.3}$$

Using once more the boundedness of the nearest neighbour transition rates and that μ_0 is supported on finitely many configurations, we can show, employing the bound $\log(\mu_t^L(\eta)/\nu_\alpha(\eta))\le|\log(\nu_\alpha(\eta))|$, that

$$\int_0^T\sum_{\eta,\eta'\in\Omega_L}\left|(\jmath_t^L)_{\eta,\eta'}\log\left(\frac{\mu_t^L(\eta)}{\nu_\alpha(\eta)}\right)\right|\mathrm{d}t$$
$$\le\int_0^T\sum_{(\eta,\eta')\in\mathcal{N}_1}\left(\mu_t^L(\eta)(r_t^L)_{\eta,\eta'}+\mu_t^L(\eta')(r_t^L)_{\eta',\eta}\right)|\log(\nu_\alpha(\eta))|\mathrm{d}t<\infty.$$

The latter allows us to combine the first two summands on the right-hand side of (5.3), which are equal to $-\sum_{\eta\in\Omega_L}\operatorname{div}\jmath_t^L(\eta)\log(\mu_t^L(\eta)/\nu_\alpha^{\tilde V_t}(\eta))=-\langle\jmath_t^L,F^{\tilde V_t}(\mu_t^L)\rangle_L$, where the last identity follows by a summation by parts (cf. Equation (15) in [21]). This finishes the proof. ∎

The proof of Theorem 3.2 relies on an auxiliary statement of independent interest, which we prove first. The result gives sufficient conditions for local equilibration.

LEMMA 5.1. *Consider $(P_L)_{L\in\mathbb{N}}$ from Section 2.2.1 with associated density $(\mu_t^L)_{t\in[0,T]}$. Assume there exists $\tilde V\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ such that the inequalities*

$$\limsup_{L\to\infty}\frac{1}{L^d}\int_0^T \mathcal{F}_{L,\alpha}^{\tilde V_t}(\mu_t^L)\,\mathrm{d}t<\infty \tag{5.4}$$

and

$$\limsup_{L\to\infty}\frac{1}{L^d}\int_0^T \Psi_L^\star\big(\mu_t^L,F^{\tilde V_t}(\mu_t^L)\big)\,\mathrm{d}t<\infty \tag{5.5}$$

are satisfied. Then $(\mu_{[0,T]}^L)_{L\in\mathbb{N}}$ (where again $\mu_{[0,T]}^L:=\frac{1}{T}\int_0^T\mu_t^L\,\mathrm{d}t$) is in the class considered by the replacement lemma (2.33). In particular (2.31) and (2.32) are satisfied for $(\mu_{[0,T]}^L)_{L\in\mathbb{N}}$. Moreover, these assumptions are independent of the choices of $\tilde V$ and α: We can replace $\tilde V$ with $\tilde V+\tilde H$ for some $\tilde H\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ and also replace α with $\alpha'\in(0,N_{\max})$ in (5.4) arbitrary. Then (5.4) and (5.5) are satisfied for $\tilde V$ and α if and only if they are satisfied for $\tilde V+\tilde H$ and α'.

Proof. The bound (5.4) for $\tilde V+\tilde H$ and α' follows similar to Remark 1.2 on page 70 of [22]. For (5.5) note that the basic estimate $\cosh(x+y)\le\cosh(x)\mathrm{e}^{|y|}$ combined with (2.34) yields

$$\frac{1}{L^d}\int_0^T \Psi_L^\star\big(\mu_t^L,F_\alpha^{\tilde V_t+\tilde H_t}(\mu_t^L)\big)\,\mathrm{d}t\le\frac{C_{\tilde H}}{L^d}\int_0^T \Psi_L^\star\big(\mu_t^L,F^{\tilde V_t}(\mu_t^L)\big)\,\mathrm{d}t+2(C_{\tilde H}-1)TC_{\hat\chi} \tag{5.6}$$

for some $C_{\tilde H}>0$ that only depends on H. We thus can restrict to the special case $\tilde V_t=0$. The two bounds needed for the replacement lemma (2.33) then follow from convexity, i.e. $\mathcal{F}_{L,\alpha}^0(\mu_{[0,T]}^L)\le\frac{1}{T}\int_0^T\mathcal{F}_{L,\alpha}^0(\mu_t^L)\mathrm{d}t$ and $\Psi_L^\star\big(\mu_{[0,T]}^L,F^0(\mu_{[0,T]}^L)\big)\le\frac{1}{T}\int_0^T\Psi_L^\star\big(\mu_t^L,F^0(\mu_t^L)\big)\,\mathrm{d}t$ (cf. the discussion in Chapter 5.3 near equation (3.1) on page 81 in [22]). ∎

With this result at hand, we can turn to the proof of Theorem 3.2.

Proof. (**Proof of Theorem 3.2.**) Since the relative entropy is non-negative, we obtain with a modification of (3.2) to the time interval $[t,T]$ (for each $t\in[0,T]$) that

$$\begin{aligned}\mathcal{F}_{L,\alpha}^{\tilde V_t}(\mu_t^L)&\le\mathcal{F}_{L,\alpha}^{\tilde V_T}(\mu_T^L)+\int_t^T\Psi_L(\mu_s^L,\jmath_s^L)\,\mathrm{d}s+\int_t^T\Psi_L^\star\big(\mu_s^L,F_\alpha^{\tilde V_s}(\mu_s^L)\big)\,\mathrm{d}s\\&\qquad-\int_t^T\sum_{i\in\mathbb{T}_L^d}\big(\hat\rho_i(\mu_s^L)-\bar\rho_{\alpha,\tilde V_s}(i)\big)\,\partial_s\tilde V_s(\tfrac{i}{L})\,\mathrm{d}s\\&\le\mathbb{A}_L^{\tilde V}(Q_L)+\mathcal{F}_{L,\alpha}^{\tilde V_0}(\mu_0^L)+C_{\tilde V}\bigg(TL^dC_{\mathrm{tot}}+\int_0^T\sum_{i\in\mathbb{T}_L^d}\bar\rho_{\alpha,\tilde V_t}(i/L)\,\mathrm{d}t\bigg),\end{aligned} \tag{5.7}$$

where $C_{\tilde V}$ is a constant that only depends on $\tilde V$. Thus

$$\limsup_{L\to\infty}\frac{1}{L^d}\int_0^T\mathcal{F}_{L,\alpha}^{\tilde V_t}(\mu_t^L)\,\mathrm{d}t\le\limsup_{L\to\infty}\frac{T}{L^d}\mathbb{A}_L^{\tilde V}(Q_L)+\limsup_{L\to\infty}\frac{T}{L^d}\mathcal{F}_{L,\alpha}^{\tilde V_0}(\mu_0^L)$$

$$+T^2 C_{\tilde V} C_{\mathrm{tot}} + T C_{\tilde V} \int_0^T \int_\Lambda \bar\rho_{\alpha,\tilde V_t}(u)\,\mathrm{d}u\mathrm{d}t < \infty. \tag{5.8}$$

The second inequality follows from a similar estimate to (5.7): Consider the second inequality in (5.7) for $t=0$ and drop the term $\mathcal{F}^V_{L,\alpha}(\mu^L_T) + \int_0^T \Psi_L(\mu^L_t, \jmath^L_t)\mathrm{d}t \geq 0$. Then

$$\int_0^T \Psi^\star_L\big(\mu^L_t, F^{\tilde V_t}(\mu^L_t)\big)\,\mathrm{d}t$$
$$\leq \mathbb{A}^{\tilde V}_L(Q_L) + \mathcal{F}^{\tilde V_0}_{L,\alpha}(\mu^L_0) + 2C_{\tilde V}\left(TL^d C_{\mathrm{tot}} + \int_0^T \sum_{i\in\mathbb{T}^d_L} \bar\rho_{\alpha,\tilde V_t}(i/L)\,\mathrm{d}t\right)$$

and we can conclude as in (5.8). We then apply Lemma 5.1 to obtain that the equations (2.31) and (2.32) are satisfied for $(\mu^L_{[0,T]})_{L\in\mathbb{N}}$. The independence of V, $\tilde V$ and α follows from the considerations in Lemma 5.1. ∎

5.2. Proofs of liminf inequalities. This section is devoted to the proof of the liminf inequalities in the proof of Theorem 3.3. Many of the ideas of the following proofs are borrowed from the entropy method developed in [20]. We here follow the presentation of this method in Chapter 5 of the book by Kipnis and Landim [22]. The results we want to prove are of the form $\liminf_{L\to\infty} B_L \geq B_*$. The general strategy involves replacing B_L by some (possibly ϵ dependent) C^ϵ_L and to show that

$$\liminf_{\epsilon\to 0}\liminf_{L\to\infty} C^\epsilon_L \geq B_* \qquad \text{and} \qquad \limsup_{\epsilon\to 0}\limsup_{L\to\infty} |B_L - C^\epsilon_L| = 0.$$

5.2.1. Bounds for Ψ_L and $\Psi^\star_L$. In order to achieve the projection to the physical domain anticipated in Section 2.1 we consider functions which are linear in η. For this we fix a function $G\in C^1(\Lambda;\mathbb{R})$ and define $\tilde G_L\colon \Omega_L\to\mathbb{R}$ by $\tilde G_L(\eta) := L^d\langle\Theta_L(\eta), G\rangle = \sum_{i\in\mathbb{T}^d_L} G(i/L)\eta(i)$, for which the discrete derivative satisfies the identity $\nabla^{\eta,\eta^{i,i+e_k}}\tilde G_L = \nabla^{i,i+e_k} G(\cdot/L)$. Note that this last identity allows us to reduce the dependence on the configuration space to a dependence on the physical domain. Choosing the 'force' $F = \nabla\tilde G_L$, we obtain with Proposition 5.1 (since all summands are non-negative) that

$$\Psi^\star_L(\mu, \nabla\tilde G_L) = 2\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d \hat a_{i,i+e_k}(\mu) L^2 \Big[\cosh\big(\tfrac12 \nabla^{i,i+e_k} G(\cdot/L)\big) - 1\Big] \tag{5.9}$$

and similarly, for the current $j^G_{\eta,\eta'} = a_{\eta,\eta'}(\mu)\sinh\big(\tfrac12\nabla^{\eta,\eta'}\tilde G_L\big)$ associated with the above force (cf. [21])

$$\Psi_L(\mu, j^G) = 2\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d \hat a_{i,i+e_k}(\mu) L^2 \Big[\sinh\big(\tfrac12\nabla^{i,i+e_k}G(\cdot/L)\big)\tfrac12\nabla^{i,i+e_k}G(\cdot/L)$$
$$- \Big(\cosh\big(\tfrac12\nabla^{i,i+e_k}G(\cdot/L)\big) - 1\Big)\Big]. \tag{5.10}$$

We next derive upper bounds for (5.9) and (5.10) and a lower bound for $\Psi^\star_L(\mu, F^V(\mu))$.

PROPOSITION 5.2 (Upper bounds for Ψ_L and $\Psi_L^\star$). *Let μ be a measure on Ω_L. Further let $f_{\eta,\eta'} := \nabla^{\eta,\eta'}\tilde{G}_L$ for some $G\colon \Lambda\to\mathbb{R}$ and $j^G_{\eta,\eta'} := a_{\eta,\eta'}(\mu)\sinh\left(\frac{1}{2}\nabla^{\eta,\eta'}\tilde{G}_L\right)$. Then*

$$\Psi_L^\star(\mu,\nabla\tilde{G}_L) \le \Psi_L(\mu,j^G) \le \frac{1}{2}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d \hat{\chi}^0_{i,i+e_k}(\mu)\left[2L\sinh\left(\tfrac{1}{2}\nabla^{i,i+e_k}G(\cdot/L)\right)\right]^2. \tag{5.11}$$

Proof. The proof follows from the basic inequalities $\cosh(x)-1 \le x\sinh(x) - (\cosh(x)-1) \le \frac{1}{2}\sinh(x)^2$ applied to (5.9) and (5.10), together with the inequality $\hat{a}_{i,i+e_k}(\mu) \le 2\hat{\chi}^0_{i,i+e_k}(\mu)$ stated below (2.17). ∎

PROPOSITION 5.3 (Lower bound for $\Psi_L^\star$). *Let μ be a measure on Ω_L, $\alpha\in(0,N_{\max})$ and $V\in C^2(\Lambda;\mathbb{R})$. Then, for any $G\colon\Lambda\to\mathbb{R}$ we have the following lower bound on $\Psi_L^\star\big(\mu,F^V(\mu)\big)$ uniform in α*

$$\begin{aligned}&\Psi_L^\star\big(\mu,F^V(\mu)\big)\\ &\ge \sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\Big[\big(L\hat{j}^V_{i,i+e_k}(\mu)\big)\big(L\nabla^{i,i+e_k}G(\cdot/L)\big) - \frac{1}{2}\hat{\chi}^V_{i,i+e_k}(\mu)\big[L\nabla^{i,i+e_k}G(\cdot/L)\big]^2\Big].\end{aligned} \tag{5.12}$$

Proof. We use the notation $\rho := \mu/\nu^V_\alpha$ (s.t. ρ is the density of μ with respect to ν^V_α) and $q_{\eta,\eta'} := \nu^V_\alpha(\eta) r^V_{\eta,\eta'}$, such that the relation $q_{\eta,\eta'} = q_{\eta',\eta}$ (detailed balance) holds. Then $F^V_{\eta,\eta'}(\mu) = -\nabla^{\eta,\eta'}\log\rho$ and $a_{\eta,\eta'}(\mu) = 2\sqrt{\rho(\eta)q_{\eta,\eta'}\rho(\eta')q_{\eta',\eta}}$. Further, $a_{\eta,\eta'}(\mu)[\cosh(\frac{1}{2}F^V_{\eta,\eta'}(\mu))-1] = \sqrt{q_{\eta,\eta'}q_{\eta',\eta}}(\sqrt{\rho(\eta)}-\sqrt{\rho(\eta')})^2$. Using the representation in Proposition 5.1 and $q_{\eta,\eta'} = \sqrt{q_{\eta,\eta'}q_{\eta',\eta}} = q_{\eta',\eta}$, we obtain

$$\Psi_L^\star\big(\mu,F^V(\mu)\big) = \sum_{\eta\in\Omega_L}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d 2q_{\eta,\eta^{i,i+e_k}}\left(\sqrt{\rho(\eta)}-\sqrt{\rho(\eta^{i,i+e_k})}\right)^2.$$

Define $H_{\eta,\eta'} = \frac{1}{4}\big(\sqrt{\rho(\eta)}+\sqrt{\rho(\eta')}\big)\nabla^{\eta,\eta'}\tilde{G}_L$. Using $\nabla^{\eta,\eta^{i,i+e_k}}\tilde{G}_L = \nabla^{i,i+e_k}G(\cdot/L)$ one easily establishes

$$\begin{aligned}2\Big(\sqrt{\rho(\eta)}-\sqrt{\rho(\eta^{i,i+e_k})}\Big)^2 &\ge 4\Big(\sqrt{\rho(\eta)}-\sqrt{\rho(\eta^{i,i+e_k})}\Big)H_{\eta,\eta^{i,i+e_k}} - 2H^2_{\eta,\eta^{i,i+e_k}}\\ &= \big(\rho(\eta)-\rho(\eta^{i,i+e_k})\big)\nabla^{i,i+e_k}G(\cdot/L) - \frac{1}{8}\big(\sqrt{\rho(\eta)}+\sqrt{\rho(\eta^{i,i+e_k})}\big)^2(\nabla^{i,i+e_k}G(\cdot/L))^2.\end{aligned}$$

Using $q_{\eta,\eta'} = q_{\eta',\eta}$, the inequality $\frac{1}{2}(x+y)^2 \le x^2+y^2$, and $\mu(\eta)r^V_{\eta,\eta'} = \rho(\eta)q_{\eta,\eta'}$ thus allows to bound $2q_{\eta,\eta^{i,i+e_k}}(\sqrt{\rho(\eta)}-\sqrt{\rho(\eta^{i,i+e_k})})^2$ from below by

$$\begin{aligned}&\big(\mu(\eta)r^V_{\eta,\eta^{i,i+e_k}} - \mu(\eta^{i,i+e_k})r^V_{\eta^{i,i+e_k},\eta}\big)\nabla^{i,i+e_k}G(\cdot/L)\\ &\quad - \frac{1}{4}\big(\mu(\eta)r^V_{\eta,\eta^{i,i+e_k}} + \mu(\eta^{i,i+e_k})r^V_{\eta^{i,i+e_k},\eta}\big)(\nabla^{i,i+e_k}G(\cdot/L))^2.\end{aligned}$$

Note that $\sum_{\eta\in\Omega_L}\mu(\eta)r^V_{\eta,\eta^{i+e_k,i}} = \sum_{\eta\in\Omega_L}\mu(\eta^{i,i+e_k})r^V_{\eta^{i,i+e_k},\eta}$ implies that

$$\Psi_L^\star\big(\mu,F^V(\mu)\big) \ge \sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\Bigg[\Big(\sum_{\eta\in\Omega_L}\mu(\eta)\big(r^V_{\eta,\eta^{i,i+e_k}} - r^V_{\eta,\eta^{i+e_k,i}}\big)\Big)\nabla^{i,i+e_k}G(\cdot/L)$$

$$-\frac{1}{4}\Big(\sum_{\eta\in\Omega_L}\mu(\eta)\big(r^V_{\eta,\eta^{i,i+e_k}}+r^V_{\eta,\eta^{i+e_k,i}}\big)\Big)\big(\nabla^{i,i+e_k}G(\cdot/L)\big)^2\Big], \qquad (5.13)$$

which coincides by (2.15) and (2.17) with the right-hand side of (5.12). ∎

5.2.2. Asymptotic lower bound for the free energy.

PROPOSITION 5.4. *Let the assumptions of Theorem 3.3 hold and let $t\in[0,T]$ be such that the path $(\pi_t)_{t\in[0,T]}$ is continuous at t. Then*

$$\liminf_{L\to\infty}\frac{1}{L^d}\mathcal{F}^V_{L,\alpha}(\mu^L_t)\geq\mathcal{F}^V_\alpha(\rho_t). \qquad (5.14)$$

Proof. For each $h\in C(\Lambda;\mathbb{R})$ the entropy inequality (a special case of the Fenchel inequality, see Proposition 8.1 and page 340 in Appendix 1 in [22]) implies

$$\frac{1}{L^d}\mathcal{F}^V_{L,\alpha}(\mu^L_t)\geq\frac{1}{L^d}\Big[\sum_{\eta\in\Omega_L}\mu^L_t(\eta)\sum_{i\in\mathbb{T}^d_L}h(i/L)\eta(i)-\log\Big(\sum_{\eta\in\Omega_L}\nu^V_\alpha(\eta)\mathrm{e}^{\sum_{i\in\mathbb{T}^d_L}h(i/L)\eta(i)}\Big)\Big]$$
$$=\sum_{\eta\in\Omega_L}\mu^L_t(\eta)\langle\Theta_L(\eta),h\rangle-\frac{1}{L^d}\sum_{i\in\mathbb{T}^d_L}\log\Big(\frac{E_{\nu_{\alpha,1}}[\mathrm{e}^{(h(i/L)-V(i/L))\eta(0)}]}{E_{\nu_{\alpha,1}}[\mathrm{e}^{-V(i/L)\eta(0)}]}\Big).$$

By the assumption of finite moments in (2.3) the dominated convergence theorem yields that $u\mapsto E_{\nu_{\alpha,1}}[\mathrm{e}^{(h(u)-V(u))\eta(0)}]$ is continuous.

By (2.7), we can restrict to measures with bounded volume, such that a truncation argument, combined with the weak convergence $Q^L\to Q^*=\delta_{(\pi_t)_{t\in[0,T]}}$ and the continuity of the projection/evaluation at time t implies $\sum_{\eta\in\Omega_L}\mu^L_t(\eta)\langle\Theta_L(\eta),h\rangle=\mathbb{E}_{Q_L}[\langle\pi_t,h\rangle]\to\mathbb{E}_{Q_L}[\langle\pi_t,h\rangle]=\langle\pi_t,h\rangle$. Thus

$$\liminf_{L\to\infty}\frac{1}{L^d}\mathcal{F}^V_{L,\alpha}(\mu^L_t)\geq\langle\pi_t,h\rangle-\int_\Lambda\log\Big(\frac{E_{\nu_{\alpha,1}}[\mathrm{e}^{(h(u)-V(u))\eta(0)}]}{E_{\nu_{\alpha,1}}[\mathrm{e}^{-V(u)\eta(0)}]}\Big)\mathrm{d}u. \qquad (5.15)$$

Taking the supremum with respect to $h\in C(\Lambda;\mathbb{R})$ combined with (2.21) then finishes the proof. ∎

5.2.3. Asymptotic lower bound for Ψ. The following proofs will depend on uniform continuity of functions (which follows here from continuity and the compactness of the domain Λ (or $[0,T]\times\Lambda$)).

LEMMA 5.2. *Under the assumptions of Theorem 3.3, we have for any $G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$*

$$\limsup_{\epsilon\to0}\limsup_{L\to\infty}\Big|\int_0^T\frac{1}{L^d}\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d\hat{\chi}_{i,i+e_k}(\mu^L_t)\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2\mathrm{d}t$$
$$-\int_0^T\int_\Lambda\sum_{\eta\in\Omega_L}\mu^L_t(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](u)\big)|\nabla G_t(u)|^2\mathrm{d}u\mathrm{d}t\Big|=0. \qquad (5.16)$$

Proof. We first show that without loss of generality we can set $V=0$ for the rates (2.1). We denote with $\hat{\chi}^V$ the mobility for a smooth potential V and with $\hat{\chi}^0$ the mobility for $V=0$. Note that

$$\begin{aligned}&\left|\int_0^T \frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\Big(\hat\chi^V_{i,i+e_k}(\mu_t^L)-\hat\chi^0_{i,i+e_k}(\mu_t^L)\Big)\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2\mathrm{d}t\right|\\&\le\int_0^T\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\hat\chi^0_{i,i+e_k}(\mu_t^L)2\big(\cosh\big(\tfrac12\nabla^{i,i+e_k}V(\cdot/L)\big)-1\big)\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2\mathrm{d}t.\end{aligned} \tag{5.17}$$

Taylor's theorem enables us to find for each $t\in[0,T]$ a number $\xi\in(i/L,(i+e_k)/L)$ for which $L\nabla^{i,i+e_k}G_t(\cdot/L)=\partial_kG_t(\xi)$. Defining $C_G:=\sum_{k=1}^d\sup_{t\in[0,T]}\|\partial_kG_t\|_\infty^2<\infty$ allows us to bound the right-hand side of (5.17) from above by

$$\frac{2C_GT}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\big(\cosh\big(\tfrac12\nabla^{i,i+e_k}V(\cdot/L)\big)-1\big)\hat\chi^0_{i,i+e_k}\left(\frac1T\int_0^T\mu_t^L\,\mathrm{d}t\right). \tag{5.18}$$

Using the uniform continuity of V (on the compact set Λ), we obtain for each $\epsilon>0$ that $|\nabla^{i,i+e_k}V(\cdot/L)|<\epsilon$ as $L\to\infty$ independent of i and e_k, such that (5.18) is (for L large enough) with (2.34) bounded by $2C_GC_{\hat\chi}T(\cosh(\epsilon/2)-1)$. Thus, taking the limit superior $\epsilon\to0$ after taking $L\to\infty$ in (5.18) shows that the left-hand side of (5.17) vanishes. This justifies the replacement of V with $V=0$ in the mobility. We thus drop the indices V and 0 and simply write $\hat\chi$ for the mobility with $V=0$.

To prove (5.16) it is sufficient to show that

$$\begin{aligned}&\limsup_{\epsilon\to0}\limsup_{L\to\infty}\\&\left|\int_0^T\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\frac{\hat\chi_{i,i+e_k}(\mu_t^L)}{(2\lfloor\epsilon L\rfloor+1)^d}\sum_{|m|\le\lfloor\epsilon L\rfloor}\Big(\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2-\big[\partial_kG_t((i+m)/L)\big]^2\Big)\mathrm{d}t\right|\\&+\frac{C_GT}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\sum_{\eta\in\Omega_L}\left(\frac1T\int_0^T\mu_t^L(\eta)\mathrm{d}t\right)\left|\hat\chi^{\lfloor\epsilon L\rfloor}_{i,i+e_k}(\delta_\eta)-\hat\chi_{i,i+e_k}(\nu_{\eta^{\lfloor\epsilon L\rfloor}(i)})\right|\\&+\frac{C_GT}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{\eta\in\Omega_L}\left(\frac1T\int_0^T\mu_t^L(\eta)\mathrm{d}t\right)\left|\chi(\eta^{\lfloor\epsilon L\rfloor}(i))-\chi\Big(\Big(\frac{2\epsilon L}{2\lfloor\epsilon L\rfloor+1}\Big)^d\eta^{\lfloor\epsilon L\rfloor}(i)\Big)\right|\\&+\left|\int_0^T\frac1{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](i/L)\big)|\nabla G_t(i/L)|^2\mathrm{d}t\right.\\&\qquad\left.-\int_0^T\int_\Lambda\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](u)\big)|\nabla G_t(u)|^2\mathrm{d}u\mathrm{d}t\right|=0.\end{aligned} \tag{5.19}$$

By uniform continuity of $(\partial_kG_t)^2$ for each $\delta>0$ there exists an $\epsilon>0$ such that $|u-u'|<\epsilon$ implies that $|(\partial_kG_t(u))^2-(\partial_kG_t(u'))^2|<\delta$ uniformly in $t\in[0,T]$. Thus, by (2.34), the first term in (5.19) is, for ϵ small enough, bounded by

$$\begin{aligned}&\int_0^T\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\frac{\hat\chi_{i,i+e_k}(\mu_t^L)}{(2\lfloor\epsilon L\rfloor+1)^d}\sum_{|m|\le\lfloor\epsilon L\rfloor}\Big|\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2-\big[\partial_kG_t((i+m)/L)\big]^2\Big|\mathrm{d}t\\&\le T\delta C_{\hat\chi}.\end{aligned}$$

Letting $\delta\to 0$ shows that the first term in (5.19) vanishes.

The second term is controlled by the local equilibrium assumption (2.31); the third term vanishes using the Lipschitz continuity of χ and the bound on the expected number of particles: The Lipschitz continuity yields that the third summand in (5.19) is bounded by

$$C_G C_{\mathrm{Lip}} T\left|1-\left(\frac{2\epsilon L}{2\lfloor\epsilon L\rfloor+1}\right)^d\right|\sum_{\eta\in\Omega_L}\left(\frac{1}{T}\int_0^T\mu_t^L(\eta)\,\mathrm{d}t\right)\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\eta^{\lfloor\epsilon L\rfloor}(i).$$

By the conservation of particles, the last expression can be bounded by $C_G C_{\mathrm{Lip}} C_{\mathrm{tot}} T\,\big|1-\big(\frac{2\epsilon L}{2\lfloor\epsilon L\rfloor+1}\big)^d\big|$, which vanishes as $L\to\infty$.

For the last term in (5.19) recall that $[\Theta_L(\eta)*\iota_\epsilon](u)$ is piecewise constant on $\{[\frac{2i-1}{2L},\frac{2i+1}{2L})^d\}_{i\in\mathbb{T}_L^d}$ (cf. (5.2)). The proof thus reduces to establishing a bound for

$$\int_0^T\sum_{i\in\mathbb{T}_L^d}\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](i/L)\big)\left|\int_{[\frac{2i-1}{2L},\frac{2i+1}{2L})^d}\big(|\nabla G_t(i/L)|^2-|\nabla G_t(u)|^2\big)\,\mathrm{d}u\right|\mathrm{d}t,$$

which is easily obtained, as the last expression is, by the Lipschitz continuity, (2.7), and (5.2) bounded above by

$$C_{\mathrm{Lip}} C_{\mathrm{tot}}(2\epsilon)^{-d}\int_0^T\sum_{i\in\mathbb{T}_L^d}\int_{[\frac{2i-1}{2L},\frac{2i+1}{2L})^d}\big||\nabla G_t(i/L)|^2-|\nabla G_t(u)|^2\big|\,\mathrm{d}u\mathrm{d}t,$$

which converges by the uniform continuity of ∇G to zero for $L\to\infty$. ∎

Note that the above proof does not depend on the fact that we consider the square gradient of a function G. We can replace the square by the product of two different gradients and immediately obtain the following results.

LEMMA 5.3. *Under the assumptions of Theorem 3.3 we have for any $G,H\in C^1([0,T]\times\Lambda;\mathbb{R})$ that*

$$\begin{aligned}\limsup_{\epsilon\to0}\limsup_{L\to\infty}\Bigg|\int_0^T\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\hat{\chi}_{i,i+e_k}(\mu_t^L)\big[L\nabla^{i,i+e_k}H_t(\cdot/L)\big]\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]\,\mathrm{d}t\\ -\int_0^T\int_\Lambda\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](u)\big)\nabla H_t(u)\cdot\nabla G_t(u)\,\mathrm{d}u\mathrm{d}t\Bigg|=0.\end{aligned}\tag{5.20}$$

COROLLARY 5.1. *Under the assumptions of Theorem 3.3 we have for any $G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ that*

$$\begin{aligned}\limsup_{\epsilon\to0}\limsup_{L\to\infty}\Bigg|\int_0^T\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\hat{\chi}_{i,i+e_k}(\mu_t^L)\big[2L\sinh\big(\tfrac{1}{2}\nabla^{i,i+e_k}G_t(\cdot/L)\big)\big]^2\,\mathrm{d}t\\ -\int_0^T\int_\Lambda\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](u)\big)|\nabla G_t(u)|^2\,\mathrm{d}u\mathrm{d}t\Bigg|=0\end{aligned}\tag{5.21}$$

and for any $G,H\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$

$$\limsup_{\epsilon\to 0}\limsup_{L\to\infty}$$
$$\Bigg|\int_0^T \frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d \hat{\chi}_{i,i+e_k}(\mu_t^L)\big[2L\sinh\big(\tfrac{1}{2}\nabla^{i,i+e_k}G_t(\cdot/L)\big)\big]\big[L\nabla^{i,i+e_k}H_t(\cdot/L)\big]\mathrm{d}t$$
$$-\int_0^T\int_\Lambda\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](u)\big)\nabla G_t(u)\cdot\nabla H_t(u)\mathrm{d}u\mathrm{d}t\Bigg| = 0. \tag{5.22}$$

We now turn to the proof of the lower bound in (3.16).

PROPOSITION 5.5. *Let the assumptions of Theorem 3.3 hold. Then (3.16) is satisfied.*

Proof. For any $G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ we have

$$\sum_{\eta\in\Omega_L}\tilde{G}_L(T,\eta)\mu_T^L(\eta)-\sum_{\eta\in\Omega_L}\tilde{G}_L(0,\eta)\mu_0^L(\eta)-\int_0^T\sum_{\eta\in\Omega_L}\partial_t\tilde{G}_L(t,\eta)\mu_t^L(\eta)\mathrm{d}t$$
$$=\int_0^T\langle \jmath_t^L,\nabla\tilde{G}_L(t,\cdot)\rangle_L\mathrm{d}t\le\int_0^T\Psi_L(\mu_t^L,\jmath_t^L)\mathrm{d}t+\int_0^T\Psi_L^\star(\mu_t^L,\nabla\tilde{G}_L(t,\cdot))\mathrm{d}t. \tag{5.23}$$

Combined with Proposition 5.2 we obtain that $\frac{1}{L^d}\int_0^T\Psi_L(\mu_t^L,\jmath_t^L)\mathrm{d}t$ is bounded below by

$$\sum_{\eta\in\Omega_L}\mu_T^L(\eta)\langle\Theta_L(\eta),G_T\rangle-\sum_{\eta\in\Omega_L}\mu_0^L(\eta)\langle\Theta_L(\eta),G_0\rangle-\int_0^T\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\langle\Theta_L(\eta),\partial_t G_t\rangle\mathrm{d}t$$
$$-\frac{1}{2L^d}\int_0^T\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\hat{\chi}^0_{i,i+e_k}(\mu_t^L)\big[2L\sinh\big(\tfrac{1}{2}\nabla^{i,i+e_k}G_t(\cdot/L)\big)\big]^2\mathrm{d}t. \tag{5.24}$$

For $\epsilon>0$ and G fixed we define the function $f^{\epsilon,G}\colon\mathcal{D}([0,T];\mathcal{M}_+(\Lambda))\to\mathbb{R}$ which assigns to a path $(\tilde{\pi}_t)_{t\in[0,T]}$ the value

$$f^{\epsilon,G}((\tilde{\pi}_t)_{t\in[0,T]}):=\langle\tilde{\pi}_T,G_T\rangle-\langle\tilde{\pi}_0,G_0\rangle-\int_0^T\langle\tilde{\pi}_t,\partial_t G_t\rangle\mathrm{d}t$$
$$-\frac{1}{2}\int_0^T\int_\Lambda\chi\big([\tilde{\pi}_t*\iota_\epsilon](u)\big)|\nabla G_t(u)|^2\mathrm{d}u\mathrm{d}t.$$

By (2.7), we can restrict $f^{\epsilon,G}$ to measures with bounded volume. In this case $f^{\epsilon,G}$ is continuous and bounded, which follows from dominated convergence using the estimate $\chi\big([\pi_t*\iota_\epsilon](u)\big)|\nabla G_t(u)|^2\le C_G C_{\mathrm{Lip}}C_{\mathrm{tot}}/(2\epsilon)^d<\infty$. We can rewrite (5.24) as

$$\mathbb{E}_{Q_L}\big[f^{\epsilon,G}\big]+\frac{1}{2}\int_0^T\int_\Lambda\sum_{\eta\in\Omega_L}\mu_t^L(\eta)\chi\big([\Theta_L(\eta)*\iota_\epsilon](u)\big)|\nabla G_t|^2\mathrm{d}u\mathrm{d}t$$
$$-\frac{1}{2L^d}\int_0^T\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\hat{\chi}_{i,i+e_k}(\mu_t^L)\big[2L\sinh\big(\tfrac{1}{2}\nabla^{i,i+e_k}G_t(\cdot/L)\big)\big]^2\mathrm{d}t$$

and define the remainder

$$R_L^\epsilon := \frac{1}{2}\left|\int_0^T \frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d \hat\chi_{i,i+e_k}(\mu_t)\left[2L\sinh\left(\tfrac{1}{2}\nabla^{i,i+e_k}G_t(\cdot/L)\right)\right]^2\mathrm{d}t\right.$$
$$\left. -\int_0^T\int_\Lambda \sum_{\eta\in\Omega_L}\mu_t(\eta)\chi\left([\Theta_L(\eta)*\iota_\epsilon](u)\right)|\nabla G_t|^2\mathrm{d}u\mathrm{d}t\right|$$

to obtain $L^{-d}\int_0^T\Psi_L(\mu_t,j_t)\mathrm{d}t \ge \mathbb{E}_{Q_L}\left[f^{\epsilon,G}\right]-R_L^\epsilon$.

Since $f^{\epsilon,G}$ is continuous and bounded, the weak convergence $Q_L\to Q^*=\delta_{(\pi_t)_{t\in[0,T]}}$ implies that $\lim_{L\to\infty}\mathbb{E}_{Q_L}\left[f^{\epsilon,G}\right]=\mathbb{E}_{Q^*}\left[f^{\epsilon,G}\right]=f^{\epsilon,G}((\pi_t)_{t\in[0,T]})$. Furthermore $\limsup_{\epsilon\to0}\limsup_{L\to\infty}R_L^\epsilon=0$ by Corollary 5.1. Thus $\liminf_{L\to\infty}L^{-d}\int_0^T\Psi_L(\mu_t,j_t)\mathrm{d}t\ge \liminf_{\epsilon\to0}f^{\epsilon,G}((\pi_t)_{t\in[0,T]})$.

For $\pi_t(\mathrm{d}u)=\rho_t(u)\mathrm{d}u$ the distance $|f^{\epsilon,G}((\pi_t)_{t\in[0,T]})-f^{0,G}((\pi_t)_{t\in[0,T]})|$ is bounded from above by

$$\frac{C_G}{2}\int_0^T\int_\Lambda\left|\chi([\rho_t*\iota_\epsilon](u))-\chi(\rho_t(u))\right|\mathrm{d}u\mathrm{d}t \le \frac{C_GC_{\mathrm{Lip}}}{2}\int_0^T\int_\Lambda\left|[\rho_t*\iota_\epsilon](u)-\rho_t(u)\right|\mathrm{d}u\mathrm{d}t, \tag{5.25}$$

which is integrable. The dominated convergence theorem then implies that $f^{\epsilon,G}((\pi_t)_{t\in[0,T]})\to f^{0,G}((\pi_t)_{t\in[0,T]})$, which proves $\liminf_{L\to\infty}L^{-d}\int_0^T\Psi_L(\mu_t,j_t)\mathrm{d}t\ge f^{0,G}((\pi_t)_{t\in[0,T]})$. Taking the supremum over all $G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ finally yields (3.16). ∎

5.2.4. Asymptotic lower bound for $\Psi^\star$. The proofs in this section are very similar to the proofs in Section 5.2.3. We will therefore be brief.

LEMMA 5.4. *Suppose the assumptions of Theorem 3.3 hold. Then*

$$\limsup_{\epsilon\to0}\limsup_{L\to\infty}$$
$$\left|\int_0^T\left(\frac{1}{L^d}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\left[\left(L\hat j^V_{i,i+e_k}(\mu_t^L)\right)\left(L\nabla^{i,i+e_k}G_t(\cdot/L)\right)-\frac{1}{2}\hat\chi^V_{i,i+e_k}(\mu_t^L)\left[L\nabla^{i,i+e_k}G_t(\cdot/L)\right]^2\right]\right.\right.$$
$$-\mathbb{E}_{Q_L}\left[\int_\Lambda\phi\left([\pi_t*\iota_\epsilon](u)\right)\Delta G_t\mathrm{d}u-\int_\Lambda\chi\left([\pi_t*\iota_\epsilon](u)\right)\nabla V\cdot\nabla G_t\mathrm{d}u\right.$$
$$\left.\left.\left.-\frac{1}{2}\int_\Lambda\chi\left([\pi_t*\iota_\epsilon](u)\right)|\nabla G_t|^2\mathrm{d}u\right]\right)\mathrm{d}t\right|=0. \tag{5.26}$$

Proof. Note that

$$\hat j^V_{i,i+e_k}(\mu)=\hat j^0_{i,i+e_k}(\mu)\cosh\left(\tfrac{1}{2}\nabla^{i,i+e_k}V(\cdot/L)\right)+\hat\chi^0_{i,i+e_k}(\mu)2\sinh\left(-\tfrac{1}{2}\nabla^{i,i+e_k}V(\cdot/L)\right). \tag{5.27}$$

Using (2.16) and (5.27), a discrete integration by parts (i.e. a shift of the index) yields

$$\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\left(L\hat j^V_{i,i+e_k}(\mu)\right)\left(L\nabla^{i,i+e_k}G_t(\cdot/L)\right)-\frac{1}{2}\hat\chi_{i,i+e_k}(\mu)\left[L\nabla^{i,i+e_k}G_t(\cdot/L)\right]^2$$
$$=\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\hat\phi_i(\mu)L^2\left[\cosh\left(\tfrac{1}{2}\nabla^{i,i+e_k}V(\cdot/L)\right)\nabla^{i,i+e_k}G_t(\cdot/L)\right.$$
$$\left.-\cosh\left(\tfrac{1}{2}\nabla^{i-e_k,i}V(\cdot/L)\right)\nabla^{i-e_k,i}G_t(\cdot/L)\right]$$

$$+\hat{\chi}^0_{i,i+e_k}(\mu)2L\sinh\big(-\tfrac{1}{2}\nabla^{i,i+e_k}V(\cdot/L)\big)\big(L\nabla^{i,i+e_k}G_t(\cdot/L)\big) \\ -\frac{1}{2}\hat{\chi}_{i,i+e_k}(\mu)\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2.$$

Combining this with the expression in (5.26), it is sufficient to show that

$$\limsup_{\epsilon\to 0}\limsup_{L\to\infty}\bigg|\int_0^T \frac{1}{L^d}\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d \hat{\phi}_i(\mu^L_t)L^2\Big[\cosh\big(\tfrac{1}{2}\nabla^{i,i+e_k}V(\cdot/L)\big)\nabla^{i,i+e_k}G_t(\cdot/L) \\ -\cosh\big(\tfrac{1}{2}\nabla^{i-e_k,i}V(\cdot/L)\big)\nabla^{i-e_k,i}G_t(\cdot/L)\Big]-\mathbb{E}_{Q_L}\bigg[\int_\Lambda \phi\big([\pi_t*\iota_\epsilon](u)\big)\Delta G_t(u)\mathrm{d}u\bigg]\mathrm{d}t\bigg|=0, \tag{5.28}$$

as well as

$$\limsup_{\epsilon\to 0}\limsup_{L\to\infty}\bigg|\int_0^T \frac{1}{L^d}\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d \hat{\chi}^0_{i,i+e_k}(\mu^L_t)2L\sinh\big(\tfrac{1}{2}\nabla^{i,i+e_k}V(\cdot/L)\big)\big(L\nabla^{i,i+e_k}G_t(\cdot/L)\big) \\ -\mathbb{E}_{Q_L}\bigg[\int_\Lambda \chi\big([\pi_t*\iota_\epsilon](u)\big)\nabla V(u)\cdot\nabla G_t(u)\mathrm{d}u\bigg]\mathrm{d}t\bigg| \\ +\frac{1}{2}\bigg|\int_0^T \frac{1}{L^d}\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d \hat{\chi}_{i,i+e_k}(\mu^L_t)\big[L\nabla^{i,i+e_k}G_t(\cdot/L)\big]^2 \\ -\mathbb{E}_{Q_L}\bigg[\int_\Lambda \chi\big([\pi_t*\iota_\epsilon](u)\big)|\nabla G_t(u)|^2\mathrm{d}u\bigg]\mathrm{d}t\bigg|=0. \tag{5.29}$$

Note that (5.29) follows from the above considerations (Lemma 5.3 and Corollary 5.1), such that we are only left to prove (5.28), which can be proven with the same calculations as above (with $\hat{\chi}$ replaced by $\hat{\phi}$ combined with (2.34) and using (2.32) instead of (2.31)). ∎

PROPOSITION 5.6. *Under the assumptions of Theorem 3.3 the inequality (3.17) holds.*

Proof. We only sketch the proof, which is very similar to the one of Proposition 5.5. For

$$f^{\epsilon,G}((\tilde{\pi}_t)_{t\in[0,T]}):=\int_0^T\int_\Lambda \phi\big([\tilde{\pi}_t*\iota_\epsilon](u)\big)\Delta G_t\mathrm{d}u\mathrm{d}t \\ -\int_0^T\int_\Lambda \chi\big([\tilde{\pi}_t*\iota_\epsilon](u)\big)\nabla V\cdot\nabla G_t\mathrm{d}u\mathrm{d}t-\frac{1}{2}\int_0^T\int_\Lambda \chi\big([\tilde{\pi}_t*\iota_\epsilon](u)\big)|\nabla G_t|^2\mathrm{d}u\mathrm{d}t.$$

Proposition 5.3 implies that

$$\Psi^\star_L\big(\mu,F^V(\mu)\big) \\ \ge\sum_{i\in\mathbb{T}^d_L}\sum_{k=1}^d\Big[\big(L\hat{\jmath}^V_{i,i+e_k}(\mu)\big)\big(L\nabla^{i,i+e_k}G(\cdot/L)\big)-\frac{1}{2}\hat{\chi}^V_{i,i+e_k}(\mu)\big[L\nabla^{i,i+e_k}G(\cdot/L)\big]^2\Big].$$

As in the proof of Proposition 5.5, one obtains $\frac{1}{L^d}\int_0^T\Psi^\star_L(\mu^L_t,F^S(\mu^L_t))\mathrm{d}t\ge\mathbb{E}_{Q_L}\big[f^{\epsilon,G}\big]-R^\epsilon_L$, where R^ϵ_L coincides with (5.26) in Lemma 5.4.

The latter implies that $\limsup_{\epsilon\to 0}\limsup_{L\to\infty} R_L^{\epsilon}=0$, such that again by weak convergence with $\epsilon\to 0$

$$\liminf_{L\to\infty}\frac{1}{L^d}\int_0^T \Psi_L^{\star}(\mu_t^L, F^S(\mu_t^L))\,\mathrm{d}t \geq f^{0,G}((\pi_t)_{t\in[0,T]}).$$

Taking the supremum with respect to $G\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$ yields (3.17). ∎

5.3. Proof of Theorem 3.4.

Proof. We extend the proof in [3]. We will skip some details, as they are similar to the above calculations. Let $\tilde{H}\in C^{1,2}([0,T]\times\Lambda;\mathbb{R})$. The log density of $P_L^{V+\tilde{H}}$ with respect to P_L^V (where both measures have the same initial condition μ_0^L) has the explicit representation (cf. [3] and the Appendix in [21])

$$\begin{aligned}\log\frac{dP_L^{V+\tilde{H}}}{dP_L^V}((\eta_t)_{t\in[0,T]}) = \frac{L^d}{2}\Big[\langle\Theta_L(\eta_T),\tilde{H}_T\rangle - \langle\Theta_L(\eta_0),\tilde{H}_0\rangle - \int_0^T \langle\Theta_L(\eta_t),\partial_t\tilde{H}_t\rangle\,\mathrm{d}t\Big]\\ -\int_0^T\sum_{i\in\mathbb{T}_L^d}\sum_{i':|i-i'|=1}\hat{r}^V_{\eta_t,\eta_t^{i,i'}}L^2\big(\mathrm{e}^{-\frac{1}{2}(\tilde{H}_t(i'/L)-\tilde{H}_t(i/L))}-1\big)\mathrm{d}t.\end{aligned}$$

Using $2(ac+bd)=(a-b)(c-d)+(a+b)(c+d)$ we can represent the expression in the last line as

$$\begin{aligned}\int_0^T\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\Big[L\big(\hat{r}^V_{\eta_t,\eta_t^{i,i+e_k}}-\hat{r}^V_{\eta_t,\eta_t^{i+e_k,i}}\big)\big(L\sinh\big(-\tfrac{1}{2}\nabla^{i,i+e_k}\tilde{H}_t(\tfrac{\cdot}{L})\big)\big)\\ +\big(\hat{r}^V_{\eta_t,\eta_t^{i,i+e_k}}+\hat{r}^V_{\eta_t,\eta_t^{i+e_k,i}}\big)L^2\big(\cosh\big(-\tfrac{1}{2}\nabla^{i,i+e_k}\tilde{H}_t(\tfrac{\cdot}{L})\big)-1\big)\Big]\mathrm{d}t.\end{aligned}$$

Taking the expected value of this expression with respect to P_L^V, in combined with (2.15) and (2.17), yields

$$\begin{aligned}\int_0^T\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\Big[\big(L\hat{j}^V_{i,i+e_k}(\mu_t^L)\big)\big(L\sinh\big(-\tfrac{1}{2}\nabla^{i,i+e_k}\tilde{H}_t(\tfrac{\cdot}{L})\big)\big)\\ +2\hat{\chi}^V_{i,i+e_k}(\mu_t^L)L^2\big(\cosh\big(\tfrac{1}{2}\nabla^{i,i+e_k}\tilde{H}_t(\tfrac{\cdot}{L})\big)-1\big)\Big]\mathrm{d}t,\end{aligned}\tag{5.30}$$

which is asymptotically equivalent to

$$\int_0^T\frac{1}{2}\sum_{i\in\mathbb{T}_L^d}\sum_{k=1}^d\Big[-\big(L\hat{j}^V_{i,i+e_k}(\mu_t^L)\big)\big(L\nabla^{i,i+e_k}\tilde{H}_t(\tfrac{\cdot}{L})\big)+\frac{1}{2}\hat{\chi}^V_{i,i+e_k}(\mu_t^L)L^2\big|\nabla^{i,i+e_k}\tilde{H}_t(\tfrac{\cdot}{L})\big|^2\Big]\mathrm{d}t.$$

A result similar to Lemma 5.4 yields

$$\lim_{L\to\infty}\frac{1}{L^d}\mathbb{A}_L^V\big(Q_L^{V+\tilde{H}}\big)=\lim_{\epsilon\to 0}\lim_{L\to\infty}\frac{1}{2}\mathbb{E}_{Q_L}\big[f^{\epsilon,\tilde{H}}\big]=\frac{1}{2}f^{0,\tilde{H}}((\pi_t)_{t\in[0,T]}),$$

where the functional $f^{\epsilon,\tilde{H}}$ is given by

$$f^{\epsilon,\tilde{H}}((\pi_t)_{t\in[0,T]}):=\langle\pi_T,\tilde{H}_T\rangle-\langle\pi_0,\tilde{H}_0\rangle-\int_0^T\langle\pi_t,\partial_t\tilde{H}_t\rangle\mathrm{d}t-\int_0^T\int_\Lambda\phi([\pi_t*\iota_\epsilon](u))\Delta\tilde{H}_t\,\mathrm{d}u\,\mathrm{d}t$$

$$+\int_0^T\int_\Lambda \chi([\pi_t * \iota_\epsilon](u))\nabla V\cdot\nabla\tilde{H}_t\mathrm{d}u\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Lambda \chi([\pi_t*\iota_\epsilon](u))|\nabla\tilde{H}_t|^2\mathrm{d}u\mathrm{d}t.$$

Finally, since the hydrodynamic path $(\pi_t)_{t\in[0,T]}$ solves $\dot{\rho}_t = \Delta\phi(\rho_t) + \nabla\cdot(\chi(\rho_t)\nabla(V + \tilde{H}_t))$, we obtain

$$f^{0,\tilde{H}}((\pi_t)_{t\in[0,T]}) = \frac{1}{2}\int_0^T \|\tilde{H}_t\|^2_{1,\chi(\rho_t)} = \frac{1}{2}\int_0^T \|\dot{\rho}_t - \Delta\phi(\rho_t) - \nabla\cdot(\chi(\rho_t)\nabla V)\|^2_{-1,\chi(\rho_t)}.$$

□

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