COMMUN. MATH. SCI.
Vol. 17, No. 3, pp. 587–608


# GLOBAL WELL-POSEDNESS OF THE FREE-SURFACE DAMPED INCOMPRESSIBLE EULER EQUATIONS WITH SURFACE TENSION*

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**Abstract.** We consider a layer of an incompressible inviscid fluid, bounded below by a fixed general bottom and above by a free moving boundary, in a horizontally periodic setting. The fluid dynamics is governed by the gravity-driven incompressible Euler equations with damping, and the effect of surface tension is included on the free surface. We prove that the problem is globally well-posed for the small initial data; moreover, the solution decays exponentially to the equilibrium.

**Keywords.** Euler; Free boundary problems; Damping; Surface tension; Global well-posedness.

**AMS subject classifications.** 35L60; 35Q35; 76B15.

## 1. Introduction

**1.1. Eulerian formulation.** We consider an incompressible inviscid fluid, subject to the influence of gravity and surface tension forces, evolving in a moving domain

$$\Omega(t)=\left\{y\in\mathbb{T}^2\times\mathbb{R}\,|\,-b(y_1,y_2)<y_3<h(t,y_1,y_2)\right\}. \tag{1.1}$$

We assume that the domain is horizontally periodic for $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ the usual 1-torus. The lower boundary of $\Omega(t)$ is assumed to be rigid and given by the smooth function $b>0$, but the upper boundary is a free surface that is the graph of the unknown function $h$: $\mathbb{R}_+\times\mathbb{T}^2\to\mathbb{R}$. For each $t>0$, the fluid is described by its velocity and pressure functions, which are given by $u(t,\cdot):\Omega(t)\to\mathbb{R}^3$ and $p(t,\cdot):\Omega(t)\to\mathbb{R}$, respectively. We require that $(u,p,h)$ satisfy the following free boundary problem:

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla p+au=-ge_3 & \text{in } \Omega(t)\\ \operatorname{div} u=0 & \text{in } \Omega(t)\\ \partial_t h=u_3-u_1\partial_1 h-u_2\partial_2 h & \text{on } \{y_3=h(t,y_1,y_2)\}\\ p=p_{atm}-\sigma H & \text{on } \{y_3=h(t,y_1,y_2)\}\\ u\cdot\nu=0 & \text{on } \{y_3=-b\}.\end{cases} \tag{1.2}$$

The first two equations in (1.2) are the incompressible Euler equations with damping (or dissipation) and gravity, where $a>0$ is the damping coefficient and $g>0$ is the strength of gravity; the equations are used in geophysical models for large-scale processes in atmosphere and ocean [11, 26], where $au$ models the friction due to the bottom of the ocean or the Rayleigh friction (or the Ekman pumping/dissipation) in the planetary boundary layer in the presence of rough boundaries [6, 12, 31]. We may refer to, for instance, [4, 7, 9, 19, 27] for some mathematical results of the incompressible damped Euler equations. The third equation in (1.2) states that the free surface moves with the velocity of the fluid. $p_{atm}$ is the constant pressure of the atmosphere, $\sigma>0$ is the

*Received: August 7, 2018; Accepted (in revised form): December 27, 2018. Communicated by Pierre-Emmanuel Jabin.

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surface tension coefficient, $H$ is twice the mean curvature of the free surface given by the formula

$$H = \nabla \cdot \left( \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \tag{1.3}$$

and $\nu$ is the outward normal vector to the lower boundary.

To complete the statement of the problem, we must specify the initial conditions. We suppose that the initial upper boundary is given by the graph of the function $h(0) = h_0 : \mathbb{T}^2 \to \mathbb{R}$, which yields the initial domain $\Omega(0)$ on which the initial velocity $u(0) = u_0 : \Omega(0) \to \mathbb{R}^3$ is specified. We will assume that $h_0 > -b$ on $\mathbb{T}^2$.

**1.2. Background.** The early works for the free-surface incompressible Euler equations (without damping, *i.e.*, $a = 0$) were focused on the irrotational fluids (*i.e.*, $\operatorname{curl} u = 0$), which began with Nalimov [24] of the local well-posedness for the small initial data and was generalized to the general initial data by Wu [34, 35], see also Lannes [22] and Ambrose and Masmoudi [2, 3]. For the full free-surface incompressible Euler equations, the first local well-posedness was obtained by Lindblad [23] for the case without surface tension and by Coutand and Shkoller [10] for the case with (and without) surface tension, see also Christodoulou and Lindblad [8], Schweizer [28], Shatah and Zeng [29] and Zhang and Zhang [38].

For the free-surface incompressible Euler equations with the irrotational assumption, certain dispersive effects can be used to establish the global well-posedness for the small initial data; we refer to Wu [36, 37], Germain, Masmoudi and Shatah [13, 14], Ionescu and Pusateri [20, 21] and Alazard and Delort [1]. For the free-surface incompressible Navier-Stokes equations, due to the dissipation and regularizing effects of the viscosity, the global well-posedness has been established for the small initial data; we refer to Beale [5], Hataya [18], Guo and Tice [15–17], Tan and Wang [32] and Wang, Tice and Kim [33] for instance.

We may refer to the references cited in these works mentioned above for more proper survey of the literature of the local or global well-posedness of free boundary problems. In this paper, we will prove the global well-posedness for the free-surface incompressible Euler equations with damping, gravity and surface tension for the small initial data. The damping effect leads to the global well-posedness for the small initial data, which may be expected; we may refer to, for instance, [25, 30] for the global well-posedness of the compressible damped Euler equations in fixed domains. Nevertheless, as we will see later, the analysis for the free boundary problems is much more involved.

**1.3. Reformulation in flattening coordinates.** In order to transform the free boundary problem (1.2) to be one in the fixed domain, we will use a flattening transformation as [5] rather than the Lagrangian coordinate transformation. To this end, we define the fixed domain

$$\Omega := \mathbb{T}^2 \times (-b, 0), \tag{1.4}$$

for which we will write the coordinates as $x \in \Omega$. We shall write $\Sigma := \{x_3 = 0\}$ for the upper boundary and $\Sigma_b := \{x_3 = -b(x_1, x_2)\}$ for the lower boundary. We will think of $h$ as a function on $\mathbb{R}_+ \times \Sigma$, and flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto (x_1, x_2, \varphi(t,x) := x_3 + \eta(t,x)) =: \Phi(t,x) = y \in \Omega(t), \tag{1.5}$$

where $\eta = (1 + \frac{x_3}{b})\mathcal{P}h$, and $\mathcal{P}h$ is the harmonic extension of $h$ onto $\{x_3 \le 0\}$ with $\mathcal{P}$ defined by (7.1).

Note that if $h$ is sufficiently small in an appropriate Sobolev space, then the Jacobian $\partial_3\varphi = 1 + \partial_3\eta > 0$ and hence the mapping $\Phi$ is a diffeomorphism. This allows us to transform the free boundary problem to one on the fixed domain $\Omega$. We define

$$v(t,x) = u(t,\Phi(t,x)),\ q(t,x) = (p + gy_3 - p_{atm})(t,\Phi(t,x)) \text{ in } \Omega. \tag{1.6}$$

Set

$$\partial_i^\varphi = \partial_i - \frac{\partial_i\varphi}{\partial_3\varphi}\partial_3, \quad i = t,1,2, \quad \partial_3^\varphi = \frac{1}{\partial_3\varphi}\partial_3 \tag{1.7}$$

such that

$$\partial_i u \circ \Phi(t,\cdot) = \partial_i^\varphi v, \quad i = t,1,2,3.$$

Then in the new coordinates, the problem (1.2) becomes

$$\begin{cases} \partial_t^\varphi v + v\cdot\nabla^\varphi v + \nabla^\varphi q + av = 0 & \text{in } \Omega \\ \nabla^\varphi \cdot v = 0 & \text{in } \Omega \\ \partial_t h = v\cdot \mathbf{N} & \text{on } \Sigma \\ q = gh - \sigma H & \text{on } \Sigma \\ v\cdot\nu = 0 & \text{on } \Sigma_b \\ (v,h)\,|_{t=0} = (v_0,h_0). \end{cases} \tag{1.8}$$

Here we have written $(\nabla^\varphi)_i = \partial_i^\varphi,\ i=1,2,3,\ \nabla^\varphi\cdot v = \partial_i^\varphi v_i$ and $\mathbf{N} = (-\partial_1\eta, -\partial_2\eta, 1)$.

We assume that the initial surface function satisfies the zero-average condition

$$\int_\Sigma h_0 = 0. \tag{1.9}$$

For sufficiently regular solutions, the condition (1.9) persists in time, that is,

$$\int_\Sigma h(t) = 0. \tag{1.10}$$

Indeed, by the third, fifth and second equations in (1.8), we have

$$\frac{d}{dt}\int_\Sigma h = \int_\Sigma \partial_t h = \int_\Sigma v\cdot\mathbf{N} = \int_\Omega \nabla^\varphi \cdot v d\mathcal{V}_t = 0, \tag{1.11}$$

where $d\mathcal{V}_t$ stands for the volume element induced by the change of variable (1.5):

$$d\mathcal{V}_t = \partial_3\varphi dx. \tag{1.12}$$

The condition (1.10) will allow us to apply Poincaré's inequality for $h$ (and hence $q$) on $\Sigma$ for all $t \geq 0$. If it happens that the initial surface does not satisfy the zero average condition (1.9), then we can shift the data and the coordinate system so that (1.9) is satisfied, provided that the extra condition $\inf b + \int_\Sigma h_0 > 0$ is satisfied; see [16] for instance.

The problem (1.8) possesses a natural physical energy. For sufficiently regular solutions, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation induced by the damping:

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |v|^2 d\mathcal{V}_t + \int_\Sigma \left(g|h|^2 + \sigma\left(\sqrt{1+|\nabla h|^2} - 1\right)\right)\right) + a\int_\Omega |v|^2 d\mathcal{V}_t = 0. \tag{1.13}$$

**1.4. Main results.** We denote $H^k(\Omega)$ with $k\ge 0$ and $H^s(\Sigma)$ with $s\in\mathbb{R}$ for the usual Sobolev spaces on $\Omega$ and $\Sigma$, whose norms denoted by $\|\cdot\|_k$ and $|\cdot|_s$, respectively. For a generic integer $n\ge 3$, we define the energy as

$$\mathcal{E}_n := \sum_{j=0}^{n} \left\|\partial_t^j v\right\|_{\frac{3}{2}(n-j)}^2 + \sum_{j=0}^{n-1} \left\|\partial_t^j q\right\|_{\frac{3}{2}(n-j)-\frac{1}{2}}^2 + \sum_{j=0}^{n+1} \left|\partial_t^j h\right|_{\frac{3}{2}(n-j)+1}^2 . \tag{1.14}$$

Our main results of this paper are stated as follows.

THEOREM 1.1. *Let $n\ge 3$ be an integer. Assume that $v_0\in H^{\frac{3}{2}n}(\Omega)$, $h_0\in H^{\frac{3}{2}n+1}(\Sigma)$ satisfy $\nabla^{\varphi_0}\cdot v_0=0$ in $\Omega$, $v_0\cdot\nu=0$ on $\Sigma_b$ and the zero-average condition (1.9). There exists a universal constant $\varepsilon_0>0$ such that if $\mathcal{E}_n(0)\le\varepsilon_0$, then there exists a global unique solution $(v,q,h)$ solving (1.8) on $[0,\infty)$. Moreover, there exists universal constants $C,\gamma>0$ such that for all $t\ge 0$,*

$$\mathcal{E}_n(t)+\int_0^t \mathcal{E}_n(\tau)\,d\tau \le C\mathcal{E}_n(0) \tag{1.15}$$

*and*

$$\mathcal{E}_n(t)\le C\mathcal{E}_n(0)e^{-\gamma t}. \tag{1.16}$$

REMARK 1.1. Since $h$ is such that the mapping $\Phi(t,\cdot)$, defined in (1.5), is a diffeomorphism for each $t\ge 0$. As such, one may change coordinates to $y\in\Omega(t)$ to produce a global-in-time, exponentially decaying solution to (1.2). Also, our theorem holds for any $n\ge 3$, which implies that the sufficiently small smooth initial data leads to the global smooth solutions to (1.2).

We now present a sketch of the proof of Theorem 1.1. The local well-posedness of the problem (1.8) in our energy functional $\mathcal{E}_n$ $(n\ge 3)$ can follow exactly in the same way as that of the problem without damping and gravity; indeed, the local well-posedness of the incompressible Euler equations in the energy functional $\mathcal{E}_3$ (*i.e.*, $n=3$) was proved by Coutand and Shkoller [10], and, if we could derive the a priori estimates in the energy functional $\mathcal{E}_n$ $(n\ge 3)$, there would not be any essential difficulties to generalize the local well-posedness result in [10] to the case $n\ge 3$ or with damping and gravity. Therefore, to prove Theorem 1.1, it suffices to derive the a priori estimates as recorded in Theorem 6.1.

The first step is to utilize the geometric structure of (1.8) and the energy-dissipation structure (1.13) to derive the following temporal energy evolution estimate:

$$\bar{\mathcal{E}}_n(t)+\int_0^t \bar{\mathcal{D}}_n(\tau)\,d\tau \lesssim \mathcal{E}_n(0)+(\mathcal{E}_n(t))^{3/2}+\int_0^t (\mathcal{E}_n(\tau))^{3/2}\,d\tau, \tag{1.17}$$

where

$$\bar{\mathcal{E}}_n := \sum_{j=0}^{n}\left(\left\|\partial_t^j v\right\|_0^2+\left|\partial_t^j h\right|_1^2\right) \text{ and } \bar{\mathcal{D}}_n := \sum_{j=0}^{n}\left\|\partial_t^j v\right\|_0^2. \tag{1.18}$$

Note that $\partial_t^{n-1}v\in H^{3/2}(\Omega)$ in $\mathcal{E}_n$ (cf. (1.14)) rather than $H^1(\Omega)$ is important for closing the highest order temporal derivative estimate (see [10]). Such a regularity gains from the regularizing effect of surface tension and the vorticity estimates. We remark here that, required not only by even the local well-posedness with surface tension (see [10]) but also by the derivation of the dissipation estimates of $q$ and $h$ (required also for the global well-posedness without surface tension), we need to study the time-differential problems.

The next goal is to replace $\bar{\mathcal{E}}_n$ and $\bar{\mathcal{D}}_n$ in the left-hand side of (1.17) by the full one $\mathcal{E}_n$. To derive the estimates for the normal derivatives of $v$, the natural way is to estimate instead the vorticity $\operatorname{curl}^\varphi v = \nabla^\varphi \times v$ to get rid of the pressure term $\nabla^\varphi q$, and the conclusion is that

$$\begin{aligned}&\sup_{[0,T]}\sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1}+a\int_0^T\sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1}\\&\lesssim \mathcal{E}_n(0)+\sup_{[0,T]}(\mathcal{E}_n)^2+\int_0^T(\mathcal{E}_n)^2.\end{aligned}\tag{1.19}$$

Note that to utilize (1.19) so as to employ the Hodge-type elliptic estimates of $\partial_t^j v$ for $j=0,\dots,n-1$, we need to show that $\partial_t^j v_3 \in H^{\frac{3}{2}(n-j)-\frac{1}{2}}(\Sigma)$. This requires us to chain the Hodge-type elliptic estimates of $v$, the regularizing elliptic estimates of $h$ by the presence of surface tension and the derivation of the estimates of $q$. Indeed, first we have

$$\sum_{j=1}^{n+1}\left|\partial_t^j h\right|^2_{-\frac{1}{2}}\lesssim \bar{\mathcal{D}}_n+(\mathcal{E}_n)^2 \text{ and hence } \sum_{j=1}^{n-1}\left|\partial_t^j q\right|^2_{-\frac{5}{2}}\lesssim \bar{\mathcal{D}}_n+(\mathcal{E}_n)^2.\tag{1.20}$$

We then use the following diagram to provide a sketch of the chaining procedure:

$$\begin{aligned}&\partial_t^n v\Longrightarrow \partial_t^{n-1}q\Longrightarrow \left.\begin{matrix}\partial_t^n h\\ \partial_t^{n-1}h\end{matrix}\right\}\cdots\cdots\partial_t^j h\xRightarrow{(1.8)_3+Hodge}\partial_t^{j-1}v\xRightarrow{(1.8)_1}\partial_t^{j-2}q\\&\qquad\qquad\xRightarrow{(1.8)_4}\partial_t^{j-2}h\cdots\cdots\begin{cases}\partial_t^2 h\Longrightarrow \partial_t v\Longrightarrow \nabla q\Longrightarrow \nabla h\\ \partial_t h\Longrightarrow v.\end{cases}\end{aligned}\tag{1.21}$$

The diagram (1.21) means that if at the beginning $\partial_t^n h\in H^1$ and $\partial_t^n v\in L^2$, then we can derive the desired regularity of the following terms as described in $\mathcal{E}_n$.

Since $|\partial_t^n h|_1^2\le \bar{\mathcal{E}}_n$, the diagram (1.21) implies that for the estimates of $\mathcal{E}_n$ in the energy, we have (since $|h|_1\le \bar{\mathcal{E}}_n$, $\nabla q$ and $\nabla h$ in (1.21) can be then replaced by $q$ and $h$)

$$\mathcal{E}_n\lesssim \bar{\mathcal{E}}_n+\sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1}+(\mathcal{E}_n)^2.\tag{1.22}$$

However, for the estimates of $\mathcal{E}_n$ in the dissipation, the difficulty is that $|\partial_t^n h|_1^2$ is not controlled by $\bar{\mathcal{D}}_n$. The most crucial point of our global analysis is to discover the following:

$$\int_0^T|\partial_t^n h|_1^2\lesssim \sup_{[0,T]}\left(\left|\partial_t^{n-1}h\right|_1^2+|\partial_t^n h|_1^2\right)+\int_0^T\left(\left|\partial_t^{n-1}h\right|^2_{\frac{5}{2}}+\left|\partial_t^{n+1}h\right|^2_{-\frac{1}{2}}\right).\tag{1.23}$$

Hence, we have the following time-integrated dissipation estimate (the $L^2$ norms of $q$ and $h$ are controlled by using Poincaré's inequality since $\int_\Sigma h=0$)

$$\int_0^T\mathcal{E}_n\lesssim \sup_{[0,T]}\bar{\mathcal{E}}_n+\int_0^T\left(\bar{\mathcal{D}}_n+\sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1}+(\mathcal{E}_n)^2\right).\tag{1.24}$$

Consequently, combining (1.17), (1.19), (1.22) and (1.24), we can close the a priori estimates, as recorded in Theorem 6.1. The exponential decay estimate can also be derived. Hence, the proof of Theorem 1.1 is completed.

**1.5. Notation.** We use the Einstein convention of summing over repeated indices. Throughout the paper $C>0$ denotes a generic constant that does not depend on the data and time, but can depend on the the parameters of the problem, e.g., $g,a$, $\sigma$, $b$, $n$. We refer to such constants as "universal". Such constants are allowed to change from line to line. We employ $A_1 \lesssim A_2$ to mean that $A_1 \le C A_2$ for a universal constant $C>0$.

We write $\mathbb{N}=\{0,1,2,\dots\}$ for the collection of non-negative integers. When using space-time differential multi-indices, we write $\mathbb{N}^{1+d}=\{\alpha=(\alpha_0,\alpha_1,\dots,\alpha_d)\}$ to emphasize that the $0-$index term is related to temporal derivatives. For $\alpha\in\mathbb{N}^{1+d}$, $\partial^\alpha=\partial_t^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$. For any differential operator $\partial^\alpha$, we use the commutators

$$[\partial^\alpha,f]g=\partial^\alpha(fg)-f\partial^\alpha g \text{ and } [\partial^\alpha,f,g]=\partial^\alpha(fg)-f\partial^\alpha g-\partial^\alpha fg. \tag{1.25}$$

We denote $x_*=(x_1,x_2)$ for the horizontal coordinates and $v_*=(v_1,v_2)$ for the horizontal components of $v$. We denote $\nabla_*$ for the horizontal gradient, $\mathrm{div}_*$ for the horizontal divergence and $\Delta_*$ for the horizontal Laplacian. To simplify the notations, we still use $\nabla f$ to mean $\nabla_* f$, etc., when without confusion, for function $f$ defined on $\Sigma$. We omit the differential elements of the integrals over $\Omega$, $\Sigma$ and $\Sigma_b$, and also sometimes the time differential elements.

## 2. Preliminaries

In this section we record some preliminary results that will be used in the derivation of the a priori estimates for the solutions to (1.8). We will assume throughout the rest of the paper that the solutions are given on the interval $[0,T]$ and obey the a priori assumption

$$\mathcal{E}_n(t)\le\delta,\quad \forall t\in[0,T] \tag{2.1}$$

for an integer $n\ge 3$ and a sufficiently small constant $\delta>0$. This implies in particular that

$$\frac{1}{2}\le\partial_3\varphi(t,x)\le\frac{3}{2},\quad \forall(t,x)\in[0,T]\times\bar{\Omega}. \tag{2.2}$$

(2.1) and (2.2) will be used frequently, without mentioning explicitly.

In order to derive the estimates for the time derivatives of the solutions to (1.8) (essentially for the highest order), even for the local well-posedness of (1.8), it is natural to utilize the geometric structure of the equations given in (1.8); this is also motivated by the works [15–17] of viscous surface waves in which such geometric structure is crucial for handling the pressure term when estimating the highest order time derivatives of the solutions. We apply $\partial_t^j$ for $j=0,\dots,n$ to (1.8) to find that

$$\begin{cases}\partial_t^\varphi\partial_t^j v+v\cdot\nabla^\varphi\partial_t^j v+\nabla^\varphi\partial_t^j q+a\partial_t^j v=F^{1,j} & \text{in } \Omega\\ \nabla^\varphi\cdot\partial_t^j v=F^{2,j} & \text{in } \Omega\\ \partial_t\partial_t^j h=\partial_t^j v\cdot\mathbf{N}+F^{3,j} & \text{on } \Sigma\\ \partial_t^j q=g\partial_t^j h-\sigma\partial_t^j H & \text{on } \Sigma\\ \partial_t^j v\cdot\nu=0 & \text{on } \Sigma_b,\end{cases} \tag{2.3}$$

where

$$F^{1,j} = -\left[\partial_t^j, \partial_t^\varphi + v\cdot\nabla^\varphi\right] v - \left[\partial_t^j, \nabla^\varphi\right] q, \tag{2.4}$$

$$F^{2,j} = -\left[\partial_t^j, \nabla^\varphi\right]\cdot v, \tag{2.5}$$

$$F^{3,j} = \left[\partial_t^j, \mathbf{N}\right]\cdot v. \tag{2.6}$$

Furthermore, we may write that for $j \geq 1$,

$$\begin{aligned}\partial_t^j H &\equiv \nabla\cdot\left(\partial_t^j\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)\right)\\ &= \nabla\cdot\left(\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}} + \nabla h\partial_t^j\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\right) + \left[\partial_t^j, \nabla h, \frac{1}{\sqrt{1+|\nabla h|^2}}\right]\right)\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\partial_t^j\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\right) &= \partial_t^{j-1}\partial_t\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\right) = -\partial_t^{j-1}\left(\frac{\nabla h\cdot\nabla\partial_t h}{\sqrt{1+|\nabla h|^2}^3}\right)\\ &= -\frac{\nabla h\cdot\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}^3} - \left[\partial_t^{j-1}, \frac{\nabla h}{\sqrt{1+|\nabla h|^2}^3}\right]\cdot\nabla\partial_t h.\end{aligned} \tag{2.8}$$

It thus follows that

$$\partial_t^j H = \nabla\cdot\left(\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\cdot\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}^3}\nabla h + F^{4,j}\right), \tag{2.9}$$

where

$$F^{4,j} = -\left[\partial_t^{j-1}, \frac{\nabla h}{\sqrt{1+|\nabla h|^2}^3}\right]\cdot\nabla\partial_t h\nabla h + \left[\partial_t^j, \frac{1}{\sqrt{1+|\nabla h|^2}}, \nabla h\right]. \tag{2.10}$$

We present the estimates of these nonlinear terms $F^{i,j}$.

LEMMA 2.1. *It holds that*

$$\left\|F^{1,j}\right\|_0^2 + \left\|F^{2,j}\right\|_{1/2}^2 + \left|F^{3,j}\right|_0^2 + \left|F^{4,j}\right|_{3/2}^2 \lesssim (\mathcal{E}_n)^2. \tag{2.11}$$

*Proof.* We expand these commutators in $F^{i,j}$ into a sum of products and then control each product with the highest order derivative term in $H^r$ ($r = 0, 1/2, 3/2$, accordingly) and the lower order derivative terms in $H^m$ for $m$ depending on $r$, using Lemmas 7.2 and 7.1, the trace theory along with the definition (1.14) of $\mathcal{E}_n$; all of them are bounded by $\mathcal{E}_n$. We remark that it is needed to have included the term $\left|\partial_t^{n+1}h\right|_{-1/2}^2$ in the definition of $\mathcal{E}_n$ so that when estimating $F^{1,n}$, by Lemma 7.1,

$$\left\|\partial_t^{n+1}\varphi\right\|_0^2 = \left\|\partial_t^{n+1}\eta\right\|_0^2 \lesssim \left\|\partial_t^{n+1}\mathcal{P}h\right\|_0^2 \lesssim \left|\partial_t^{n+1}h\right|_{-1/2}^2 \leq \mathcal{E}_n. \tag{2.12}$$

The estimate (2.11) follows by summing. ∎

In order to utilize the linear structure of (1.8), it is convenient to write the equations in the linear perturbed form

$$\begin{cases} \partial_t v + \nabla q + av = G^1 & \text{in } \Omega \\ \operatorname{div} v = G^2 & \text{in } \Omega \\ \partial_t h = v_3 + G^3 & \text{on } \Sigma \\ q = gh - \sigma\Delta h + G^4 & \text{on } \Sigma \\ v\cdot\nu = 0 & \text{on } \Sigma_b, \end{cases} \tag{2.13}$$

where

$$G^1 = \partial_t\eta\partial_3^\varphi v + \nabla\eta\partial_3^\varphi q - v\cdot\nabla^\varphi v, \tag{2.14}$$

$$G^2 = \nabla\eta\cdot\partial_3^\varphi v, \tag{2.15}$$

$$G^3 = -v\cdot\nabla h, \tag{2.16}$$

$$G^4 = -\sigma\nabla\cdot(((1+|\nabla h|^2)^{-1/2}-1)\nabla h). \tag{2.17}$$

Here we have used the fact that, by (1.7), $\partial_i^\varphi - \partial_i = -\partial_i\eta\partial_3^\varphi$ for $i = t, 1, 2, 3$.

We present the estimates of these nonlinear terms $G^i$.

LEMMA 2.2. *It holds that*

$$\sum_{j=0}^{n-1}\left(\left\|\partial_t^j G^1\right\|^2_{\frac{3}{2}(n-j)-\frac{3}{2}} + \left\|\partial_t^j G^2\right\|^2_{\frac{3}{2}(n-j)-1} + \left|\partial_t^j G^3\right|^2_{\frac{3}{2}(n-j)-\frac{1}{2}}\right) + \sum_{j=0}^{n}\left|\partial_t^j G^4\right|^2_{\frac{3}{2}(n-j)-1}$$
$$\lesssim (\mathcal{E}_n)^2. \tag{2.18}$$

*Proof.* The proof follows in the same way as Lemma 2.1, expect when estimating $\partial_t^n G^4$ we need to use first the structure of $G^4$:

$$\left|\partial_t^n G^4\right|^2_{-1} \lesssim \left|\partial_t^n\left(((1+|\nabla h|^2)^{-1/2}-1)\nabla h\right)\right|^2_0 \tag{2.19}$$

and then estimate in the same way as Lemma 2.1. □

Since the Jacobian of the change of variable (1.5) is $\partial_3\varphi$, when performing energy estimates it is natural to use on $\Omega$ the weighted $L^2$ product $\int_\Omega fg\,d\mathcal{V}_t$, where $d\mathcal{V}_t$ was defined by (1.12). We have the following integration by parts identities for the operators $\partial_i^\varphi$ and the above weighted product:

LEMMA 2.3. *It holds that*

$$\int_\Omega \partial_i^\varphi f g\,d\mathcal{V}_t = -\int_\Omega f\partial_i^\varphi g\,d\mathcal{V}_t + \int_\Sigma fg\mathbf{N}_i + \int_{\Sigma_b} fg\nu_i, \quad i=1,2,3, \tag{2.20}$$

$$\int_\Omega \partial_t^\varphi f g\,d\mathcal{V}_t = \frac{d}{dt}\int_\Omega fg\,d\mathcal{V}_t - \int_\Omega f\partial_t^\varphi g\,d\mathcal{V}_t - \int_\Sigma fg\partial_t h \tag{2.21}$$

*and*

$$\int_\Omega (\partial_t^\varphi f + v\cdot\nabla^\varphi f)f\,d\mathcal{V}_t = \frac{1}{2}\frac{d}{dt}\int_\Omega |f|^2\,d\mathcal{V}_t. \tag{2.22}$$

*Proof.* The formulas (2.20) and (2.21) are straightforward consequences of the standard integration by parts formulas. Now, by (2.20) and (2.21) and using the second, third and fifth equations in (1.8), we obtain

$$\begin{aligned}\int_\Omega (\partial_t^\varphi f + v\cdot\nabla^\varphi f) f \, d\mathcal{V}_t &= \frac{1}{2}\frac{d}{dt}\int_\Omega |f|^2 \, d\mathcal{V}_t - \frac{1}{2}\int_\Omega \nabla^\varphi\cdot v |f|^2 \, d\mathcal{V}_t \\ &\quad - \frac{1}{2}\int_\Sigma |f|^2 (\partial_t h - v\cdot\mathbf{N}) + \frac{1}{2}\int_{\Sigma_b} |f|^2 v\cdot\nu \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega |f|^2 \, d\mathcal{V}_t. \end{aligned} \tag{2.23}$$

This proves (2.22). □

## 3. Temporal estimates

In this section we will derive the energy evolution estimates for the temporal derivatives of the solutions to (1.8). For a generic integer $n \geq 3$, we define the temporal energy by

$$\bar{\mathcal{E}}_n := \sum_{j=0}^n \left( \left\| \partial_t^j v \right\|_0^2 + \left| \partial_t^j h \right|_1^2 \right) \tag{3.1}$$

and the corresponding dissipation by

$$\bar{\mathcal{D}}_n := \sum_{j=0}^n \left\| \partial_t^j v \right\|_0^2. \tag{3.2}$$

We derive the following time-integrated temporal energy evolution estimate.

PROPOSITION 3.1. *It holds that*

$$\bar{\mathcal{E}}_n(t) + \int_0^t \bar{\mathcal{D}}_n(\tau)\, d\tau \lesssim \mathcal{E}_n(0) + (\mathcal{E}_n(t))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2}\, d\tau. \tag{3.3}$$

*Proof.* Let $j = 1, \ldots, n$. Taking the inner product of the first equation in (2.3) with $\partial_t^j v$ and then integrating by parts over $\Omega$ by using the second, third and fifth equations in (1.8), we obtain , by Lemma 2.3,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\partial_t^j v\right|^2 d\mathcal{V}_t + \int_\Omega \nabla^\varphi \partial_t^j q \cdot \partial_t^j v \, d\mathcal{V}_t + a\int_\Omega \left|\partial_t^j v\right|^2 d\mathcal{V}_t = \int_\Omega F^{1,j}\cdot \partial_t^j v \, d\mathcal{V}_t. \tag{3.4}$$

By (2.11), we have

$$\int_\Omega F^{1,j}\cdot\partial_t^j v \, d\mathcal{V}_t \lesssim \|F^{1,j}\|_0 \left\|\partial_t^j v\right\|_0 \lesssim \mathcal{E}_n^{3/2}. \tag{3.5}$$

Using the fifth, fourth, third and second equations in (2.3), we obtain

$$\begin{aligned} -\int_\Omega \nabla^\varphi \partial_t^j q \cdot \partial_t^j v \, d\mathcal{V}_t &= -\int_\Sigma \partial_t^j q \partial_t^j v \cdot \mathbf{N} + \int_\Omega \partial_t^j q \nabla^\varphi \cdot \partial_t^j v \, d\mathcal{V}_t \\ &= -\int_\Sigma (g\partial_t^j h - \sigma \partial_t^j H)(\partial_t^{j+1} h - F^{3,j}) + \int_\Omega \partial_t^j q F^{2,j} \, d\mathcal{V}_t. \end{aligned} \tag{3.6}$$

Integrating by parts in $t$ leads to

$$-\int_\Sigma g\partial_t^j h\partial_t^{j+1}h=-\frac{1}{2}\frac{d}{dt}\int_\Sigma g\left|\partial_t^j h\right|^2. \tag{3.7}$$

By (2.11), we have

$$\int_\Sigma g\partial_t^j hF^{3,j}\lesssim\left|\partial_t^j h\right|_0\left|F^{3,j}\right|_0\lesssim\mathcal{E}_n^{3/2}. \tag{3.8}$$

By (2.9), we may write

$$\int_\Sigma \sigma\partial_t^j H\partial_t^{j+1}h=\int_\Sigma\sigma\nabla\cdot\left(\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}-\frac{\nabla h\cdot\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}^3}\nabla h+F^{4,j}\right)\partial_t^{j+1}h. \tag{3.9}$$

Integrating by parts in both $x_*$ and $t$, we find that

$$\begin{aligned}\int_\Sigma\sigma\nabla\cdot\left(\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}\right)\partial_t^{j+1}h&=-\int_\Sigma\sigma\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}\cdot\nabla\partial_t^{j+1}h\\
=-\frac{1}{2}\frac{d}{dt}\int_\Sigma\sigma\frac{|\nabla\partial_t^j h|^2}{\sqrt{1+|\nabla h|^2}}&+\frac{1}{2}\int_\Sigma\sigma\partial_t\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\right)\left|\nabla\partial_t^j h\right|^2\\
\le-\frac{1}{2}\frac{d}{dt}\int_\Sigma\sigma\frac{|\nabla\partial_t^j h|^2}{\sqrt{1+|\nabla h|^2}}&+C\mathcal{E}_n^{3/2}.\end{aligned} \tag{3.10}$$

Similarly, we have

$$-\int_\Sigma\sigma\nabla\cdot\left(\frac{\nabla h\cdot\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}^3}\nabla h\right)\partial_t^{j+1}h\le-\frac{1}{2}\frac{d}{dt}\int_\Sigma\sigma\frac{|\nabla h\cdot\nabla\partial_t^j h|^2}{\sqrt{1+|\nabla h|^2}}+C\mathcal{E}_n^{3/2}. \tag{3.11}$$

By (2.11), we obtain

$$\int_\Sigma\sigma\nabla\cdot F^{4,j}\partial_t^{j+1}h\le\left|F^{4,j}\right|_{3/2}\left|\partial_t^{j+1}h\right|_{-1/2}\lesssim\mathcal{E}_n^{3/2}. \tag{3.12}$$

Now we write

$$-\int_\Sigma\sigma\partial_t^j HF^{3,j}=-\int_\Sigma\sigma\partial_t^j H\left(\partial_t^j\mathbf{N}\cdot v+\left[\partial_t^j,\mathbf{N},\cdot v\right]\right). \tag{3.13}$$

The integration by parts in $x_*$ yields, by estimating $\left|\left[\partial_t^j,\mathbf{N},\cdot v\right]\right|_1\lesssim\mathcal{E}_n$ as Lemma 2.1,

$$\begin{aligned}-\int_\Sigma\sigma\partial_t^j H\left[\partial_t^j,\mathbf{N},\cdot v\right]&=\int_\Sigma\sigma\partial_t^j\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)\cdot\nabla\left[\partial_t^j,\mathbf{N},\cdot v\right]\\
\lesssim\left|\partial_t^j\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)\right|_0&\left|\left[\partial_t^j,\mathbf{N},\cdot v\right]\right|_1\lesssim\mathcal{E}_n^{3/2}.\end{aligned} \tag{3.14}$$

It follows from (2.9) that

$$-\int_\Sigma\sigma\partial_t^j H\partial_t^j\mathbf{N}\cdot v=\int_\Sigma\sigma\nabla\cdot\left(\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}-\frac{\nabla h\cdot\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}^3}\nabla h+F^{4,j}\right)v\cdot\nabla\partial_t^j h. \tag{3.15}$$

By (2.11), we directly have

$$\int_\Sigma \sigma\nabla\cdot F^{4,j} v\cdot\nabla\partial_t^j h \lesssim \left|F^{4,j}\right|_1 \left|v\cdot\nabla\partial_t^j h\right|_0 \lesssim \mathcal{E}_n^{3/2}. \tag{3.16}$$

Upon an integration by parts in $x_*$, we find that

$$\begin{aligned}&\int_\Sigma \sigma\nabla\cdot\left(\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}\right) v\cdot\nabla\partial_t^j h = -\int_\Sigma \sigma\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}\cdot\nabla(v\cdot\nabla\partial_t^j h)\\ &= -\int_\Sigma \sigma\frac{\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}}\cdot\nabla v\cdot\nabla\partial_t^j h + \frac{1}{2}\int_\Sigma \sigma\,\mathrm{div}_*\left(\frac{v_*}{\sqrt{1+|\nabla h|^2}}\right)\left|\nabla\partial_t^j h\right|^2 \lesssim \mathcal{E}_n^{3/2}.\end{aligned} \tag{3.17}$$

Similarly, we have

$$-\int_\Sigma \sigma\nabla\cdot\left(\frac{\nabla h\cdot\nabla\partial_t^j h}{\sqrt{1+|\nabla h|^2}^3}\nabla h\right) v\cdot\nabla\partial_t^j h \lesssim \mathcal{E}_n^{3/2}. \tag{3.18}$$

Finally, we turn to the last $F^{2,j}$ term in (3.6). If $j\le n-1$, then we have

$$\int_\Omega \partial_t^j q F^{2,j}\, d\mathcal{V}_t \le \left\|\partial_t^j q\right\|_0 \left\|F^{2,j}\right\|_0 \lesssim \mathcal{E}_n^{3/2}. \tag{3.19}$$

For $j=n$, since $\partial_t^n q$ is out of control, we need to integrate by parts in $t$. We may use the following decomposition

$$F^{2,n} = -\partial_3\varphi\tilde{F}^{2,n} \text{ with } \tilde{F}^{2,n} = \sum_{i=1}^5 \tilde{F}_i^{2,n}, \tag{3.20}$$

where

$$\tilde{F}_1^{2,n} = n\partial_t\mathbf{N}\cdot\partial_t^{n-1}\partial_3 v,\quad \tilde{F}_2^{2,n} = n\partial_t\partial_3\eta\partial_t^{n-1}\mathrm{div}_* v_*,\quad \tilde{F}_3^{2,n} = \partial_t^n\mathbf{N}\cdot\partial_3 v, \tag{3.21}$$

$$\tilde{F}_4^{2,n} = \partial_t^n\partial_3\eta\,\mathrm{div}_* v_*,\quad \tilde{F}_5^{2,n} = \sum_{\ell=2}^{n-1} C_n^\ell\left(\partial_t^\ell\mathbf{N}\cdot\partial_t^{n-\ell}\partial_3 v + \partial_t^\ell\partial_3\eta\cdot\partial_t^{n-\ell}\mathrm{div}_* v_*\right). \tag{3.22}$$

Accordingly,

$$\int_\Omega \partial_t^n q F^{2,n}\, d\mathcal{V}_t = -\sum_{i=1}^5\int_\Omega \partial_t^n q\tilde{F}_i^{2,n}. \tag{3.23}$$

By the integration by parts in $t$, we have

$$-\int_\Omega \tilde{F}_5^{2,n}\partial_t^n q = -\frac{d}{dt}\int_\Omega \tilde{F}_5^{2,n}\partial_t^{n-1} q + \mathcal{R}_5^n, \tag{3.24}$$

where, estimating as Lemma 2.1,

$$\mathcal{R}_5^n = \int_\Omega \partial_t\tilde{F}_5^{2,n}\partial_t^{n-1} q \lesssim \left\|\partial_t\tilde{F}_5^{2,n}\right\|_0 \left\|\partial_t^{n-1} q\right\|_0 \lesssim \mathcal{E}_n^{3/2}. \tag{3.25}$$

Integrate by parts in $t$ to write

$$-\int_\Omega \tilde{F}_4^{2,n}\partial_t^n q = -\frac{d}{dt}\int_\Omega \tilde{F}_4^{2,n}\partial_t^{n-1} q + \mathcal{R}_4^n, \tag{3.26}$$

where, by further integrating by parts in $x_3$ and the trace theory,

$$\begin{aligned}\mathcal{R}_4^n &= \int_\Omega \left(\partial_t^{n+1}\partial_3\eta \operatorname{div}_* v_* + \partial_t^n \partial_3\eta\partial_t \operatorname{div}_* v_*\right)\partial_t^{n-1}q \\ &= \int_\Sigma \partial_t^{n+1} h \operatorname{div}_* v_* \partial_t^{n-1} q - \int_\Omega \partial_t^{n+1}\eta\partial_3\left(\operatorname{div}_* v_* \partial_t^{n-1} q\right) + \int_\Omega \partial_t^n\partial_3\eta\partial_t \operatorname{div}_* v_* \partial_t^{n-1} q \\ &\lesssim \mathcal{E}_n^{3/2}. \end{aligned} \tag{3.27}$$

Similarly, integrate by parts in both $t$ and $x_*$ to write

$$-\int_\Omega \left(\tilde{F}_2^{2,n} + \tilde{F}_3^{2,n}\right)\partial_t^n q = -\frac{d}{dt}\int_\Omega \left(\tilde{F}_2^{2,n} + \tilde{F}_3^{2,n}\right)\partial_t^{n-1} q + \mathcal{R}_{2,3}^n, \tag{3.28}$$

where

$$\mathcal{R}_{2,3}^n = \int_\Omega \partial_t \left(\tilde{F}_2^{2,n} + \tilde{F}_3^{2,n}\right)\partial_t^{n-1} q \lesssim \mathcal{E}_n^{3/2}. \tag{3.29}$$

Now we turn to the most delicate term that involves $\tilde{F}_1^{2,n}$. Integrate by parts in $x_3$ first to get

$$-\int_\Omega \tilde{F}_1^{2,n}\partial_t^n q = -\int_\Sigma n\partial_t \mathbf{N}\cdot\partial_t^{n-1} v \partial_t^n q + \int_\Omega n\partial_3\left(\partial_t^n q\partial_t \mathbf{N}\right)\cdot\partial_t^{n-1} v. \tag{3.30}$$

Integrating by parts in $t$, we obtain

$$\int_\Omega n\partial_3\left(\partial_t^n q\partial_t\mathbf{N}\right)\cdot\partial_t^{n-1} v = \frac{d}{dt}\int_\Omega n\partial_3\left(\partial_t^{n-1} q\partial_t \mathbf{N}\right)\cdot\partial_t^{n-1} v + \mathcal{R}_1^n, \tag{3.31}$$

with

$$\mathcal{R}_1^n = -\int_\Omega n\partial_3\left(\partial_t^{n-1} q\partial_t\mathbf{N}\right)\cdot\partial_t^n v + n\partial_3\left(\partial_t^{n-1} q\partial_t^2 \mathbf{N}\right)\cdot\partial_t^{n-1} v \lesssim \mathcal{E}_n^{3/2}. \tag{3.32}$$

By $q = gh - \sigma H$, we have

$$-\int_\Sigma n\partial_t\mathbf{N}\cdot\partial_t^{n-1} v\partial_t^n q = -\int_\Sigma n\partial_t \mathbf{N}\cdot\partial_t^{n-1} v(\partial_t^n h - \sigma\partial_t^n H) \lesssim \mathcal{E}_n^{3/2}. \tag{3.33}$$

Hence, (3.24)–(3.33) implies that

$$\int_\Omega \partial_t^n q F^{2,n}\, d\mathcal{V}_t \le -\frac{d}{dt}\mathcal{B}_n + C\mathcal{E}_n^{3/2}, \tag{3.34}$$

where

$$\mathcal{B}_n = \sum_{j=2}^5 \int_\Omega \tilde{F}_j^{2,n}\partial_t^{n-1} q + \int_\Omega n\partial_3\left(\partial_t^{n-1} q\partial_t\mathbf{N}\right)\cdot\partial_t^{n-1} v \lesssim \mathcal{E}_n^{3/2}. \tag{3.35}$$

Consequently, in light of the estimates (3.5)–(3.19) and (3.34), we deduce from (3.4) by summing over $j = 1,\dots,n$ and (1.13), by integrating in time from 0 to $t$, that, by the definitions (3.1) of $\bar{\mathcal{E}}_n$ and (3.2) of $\bar{\mathcal{D}}_n$,

$$\bar{\mathcal{E}}_n(t) + \int_0^t \bar{\mathcal{D}}_n(\tau)\, d\tau \lesssim \bar{\mathcal{E}}_n(0) + (\mathcal{E}_n(0))^{3/2} + (\mathcal{E}_n(t))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2}\, d\tau. \tag{3.36}$$

The estimate (3.3) thus follows. □

## 4. Vorticity estimates

In this section we will derive the estimates for the normal derivatives of $v$. The natural way is to estimate instead the vorticity $\mathrm{curl}^\varphi v = \nabla^\varphi \times v$ to get rid of the pressure term $\nabla^\varphi q$; applying $\mathrm{curl}^\varphi$ to the first equation in (1.8), we have

$$\partial_t^\varphi \mathrm{curl}^\varphi v + v\cdot\nabla^\varphi \mathrm{curl}^\varphi v + a\mathrm{curl}^\varphi v = -[\mathrm{curl}^\varphi, v]\cdot\nabla^\varphi v. \tag{4.1}$$

Note that our energy functional $\mathcal{E}_n$ involves the fractional Sobolev regularity of the solutions, and we can not apply the fractional spatial derivatives to the equations in a domain with boundary. Traditionally, the fractional Sobolev regularity estimates follow by the interpolation from the ones of integers. For this, we let $j=0,\dots,n-1$ and then apply $\mathfrak{D}_j$, where $\mathfrak{D}_j$ denotes for any $\partial^\alpha, \alpha\in\mathbb{N}^{1+3}$ with $\alpha_0=j$ and $\alpha_1+\alpha_2+\alpha_3\le \left\lfloor \frac{3}{2}(n-j)\right\rfloor -1$, to (4.1) to find that

$$\partial_t^\varphi \mathfrak{D}_j \mathrm{curl}^\varphi v + v\cdot\nabla^\varphi \mathfrak{D}_j \mathrm{curl}^\varphi v + a\mathfrak{D}_j \mathrm{curl}^\varphi v = \mathfrak{F}^j, \tag{4.2}$$

where

$$\mathfrak{F}^j = -[\mathfrak{D}_j, \partial_t^\varphi + v\cdot\nabla^\varphi]\mathrm{curl}^\varphi v - \mathfrak{D}_j[\mathrm{curl}^\varphi, v]\cdot\nabla^\varphi v. \tag{4.3}$$

The estimate of $\mathfrak{F}^j$ is recorded as follows.

Lemma 4.1. *Denote $r_j = \frac{3}{2}(n-j) - \left\lfloor \frac{3}{2}(n-j)\right\rfloor \in [0,1)$. It holds that*

$$\left\|\mathfrak{F}^j\right\|_{r_j}^2 \lesssim (\mathcal{E}_n)^2. \tag{4.4}$$

*Proof.* The proof follows in the same way as Lemma 2.1. □

We first prove a general estimate for the following damping-transport equation

$$\partial_t^\varphi f + v\cdot\nabla^\varphi f + af = \mathfrak{F}. \tag{4.5}$$

Lemma 4.2. *For any $r\in[0,1]$, it holds that*

$$\sup_{[0,T]}\|f\|_r^2 + a\int_0^T \|f\|_r^2 \lesssim \|f(0)\|_r^2 + \int_0^T \|\mathfrak{F}\|_r^2. \tag{4.6}$$

*Proof.* The standard energy estimate of (4.5) implies , by Lemma 2.3,

$$\frac{d}{dt}\int_\Omega |f|^2 d\mathcal{V}_t + a\int_\Omega |f|^2 d\mathcal{V}_t \lesssim \|\mathfrak{F}\|_0^2. \tag{4.7}$$

Similarly, applying $\nabla$ to (4.5), we find that

$$\frac{d}{dt}\int_\Omega |\nabla f|^2 d\mathcal{V}_t + a\int_\Omega |\nabla f|^2 d\mathcal{V}_t \lesssim \|\nabla\mathfrak{F}\|_0^2 + \|[\nabla, \partial_t^\varphi + v\cdot\nabla^\varphi]f\|_0^2. \tag{4.8}$$

Note that

$$\partial_t^\varphi + v\cdot\nabla^\varphi = \partial_t + v_*\cdot\nabla_* + \frac{\partial_t\varphi - v\cdot\mathbf{N}}{\partial_3\varphi}\partial_3, \tag{4.9}$$

and it is straightforward to estimate as in Lemma 2.1 that

$$\|[\nabla, \partial_t^\varphi + v\cdot\nabla^\varphi]f\|_0^2 \lesssim \mathcal{E}_n\|\nabla f\|_0^2. \tag{4.10}$$

Since $\mathcal{E}_n \le \delta$ is small, we then have from (4.8) that

$$\frac{d}{dt}\int_\Omega |\nabla f|^2\, d\mathcal{V}_t + a\int_\Omega |\nabla f|^2\, d\mathcal{V}_t \lesssim \|\mathfrak{F}\|_1^2. \tag{4.11}$$

By integrating directly in time (4.7) and (4.11), we may conclude that the estimate (4.6) holds for $r=0,1$. The estimate (4.6) for $r\in(0,1)$ then follows by the interpolation. ☐

We now derive the following energy estimates of the vorticity.

PROPOSITION 4.1. *It holds that*

$$\sup_{[0,T]}\sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac32(n-j)-1} + a\int_0^T \sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac32(n-j)-1}$$
$$\lesssim \mathcal{E}_n(0) + \sup_{[0,T]}(\mathcal{E}_n)^2 + \int_0^T (\mathcal{E}_n)^2. \tag{4.12}$$

*Proof.* Let $j=0,\dots,n-1$. Applying Lemma 4.1 to (4.2) with $f=\mathfrak{D}^j\operatorname{curl} v$, $\mathfrak{F}=\mathfrak{F}^j$ and $r=r_j$, by (4.4), we have

$$\sup_{[0,T]}\|\mathfrak{D}_j \operatorname{curl}^\varphi v\|^2_{r_j} + a\int_0^T \|\mathfrak{D}_j \operatorname{curl}^\varphi v\|^2_{r_j} \lesssim \|\mathfrak{D}_j \operatorname{curl}^\varphi v(0)\|^2_{r_j} + \int_0^T \|\mathfrak{F}^j\|^2_{r_j}$$
$$\lesssim \|\mathfrak{D}_j \operatorname{curl}^\varphi v(0)\|^2_{r_j} + \int_0^T (\mathcal{E}_n)^2. \tag{4.13}$$

This implies that, by summing over such $\alpha$ in $\mathfrak{D}_j$,

$$\sup_{[0,T]}\left\|\partial_t^j \operatorname{curl}^\varphi v\right\|^2_{\frac32(n-j)-1} + a\int_0^T \left\|\partial_t^j \operatorname{curl}^\varphi v\right\|^2_{\frac32(n-j)-1}$$
$$\lesssim \left\|\partial_t^j \operatorname{curl}^\varphi v(0)\right\|^2_{\frac32(n-j)-1} + \int_0^T (\mathcal{E}_n)^2. \tag{4.14}$$

Since $\left\|\partial_t^j(\operatorname{curl}^\varphi - \operatorname{curl})v\right\|^2_{\frac32(n-j)-1} \lesssim (\mathcal{E}_n)^2$ as Lemma 2.1, we deduce from (4.14) that

$$\sup_{[0,T]}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac32(n-j)-1} + \int_0^T \left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac32(n-j)-1} \lesssim \mathcal{E}_n(0) + (\mathcal{E}_n)^2 + \int_0^T (\mathcal{E}_n)^2. \tag{4.15}$$

This yields the estimate (4.12). ☐

## 5. Elliptic regularity

In this section we will chain the Hodge-type elliptic estimates of $v$, the regularizing elliptic estimates of $h$ by the presence of surface tension and the derivation of the estimates of $q$ by using directly the fourth and first equations in (2.13), to get the full energy estimates $\mathcal{E}_n$ in terms of $\bar{\mathcal{E}}_n$ and $\bar{\mathcal{D}}_n$ (recalling the definitions (1.14), (3.1) and (3.2) of them).

We first present the result in the "energy".

PROPOSITION 5.1. *It holds that*

$$\mathcal{E}_n \lesssim \bar{\mathcal{E}}_n + \sum_{j=0}^{n-1}\left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac32(n-j)-1} + (\mathcal{E}_n)^2. \tag{5.1}$$

*Proof.* We first derive some preliminary estimates resulting from the control of $\bar{\mathcal{D}}_n$ (weaker than $\bar{\mathcal{E}}_n$ , cf. (3.1) and (3.2)). By the normal trace estimate (7.6), the second equation in (2.3) and (2.11), we obtain

$$\begin{aligned}\sum_{j=0}^{n}\left|\partial_t^j v\cdot\mathbf{N}\right|_{-1/2}^2 &\lesssim \sum_{j=0}^{n}\left(\left\|\partial_t^j v\right\|_0^2+\left\|\nabla^{\varphi}\cdot\partial_t^j v\right\|_0^2\right)\\ &=\sum_{j=0}^{n}\left(\left\|\partial_t^j v\right\|_0^2+\left\|F^{2,j}\right\|_0^2\right)\lesssim \bar{\mathcal{D}}_n+(\mathcal{E}_n)^2. \end{aligned}\tag{5.2}$$

By the third equation in (2.3), (2.11) and (5.2), we have

$$\sum_{j=1}^{n+1}\left|\partial_t^j h\right|_{-1/2}^2\le\sum_{j=1}^{n+1}\left(\left|\partial_t^{j-1}v\cdot\mathbf{N}\right|_{-1/2}^2+\left|F^{3,j-1}\right|_{-1/2}^2\right)\lesssim\bar{\mathcal{D}}_n+(\mathcal{E}_n)^2.\tag{5.3}$$

By the fourth equation in (2.13), (5.3) and (2.18), we obtain

$$\sum_{j=1}^{n}\left|\partial_t^j q\right|_{-\frac{5}{2}}^2\lesssim\sum_{j=1}^{n}\left(\left|\partial_t^j h\right|_{-\frac{1}{2}}^2+\left|\partial_t^j G^4\right|_{-\frac{5}{2}}^2\right)\lesssim\bar{\mathcal{D}}_n+(\mathcal{E}_n)^2.\tag{5.4}$$

On the other hand, by the first equation in (2.13) and (2.18), we have

$$\left\|\partial_t^{n-1}\nabla q\right\|_0^2\lesssim\|\partial_t^n v\|_0^2+\left\|\partial_t^{n-1}v\right\|_0^2+\left\|\partial_t^{n-1}G^1\right\|_0^2\le\bar{\mathcal{D}}_n+(\mathcal{E}_n)^2.\tag{5.5}$$

Then by the Poincaré-type inequality, (5.5) and (5.4), we obtain

$$\left\|\partial_t^{n-1}q\right\|_1^2\lesssim\left\|\partial_t^{n-1}\nabla q\right\|_0^2+\left|\partial_t^{n-1}q\right|_{-\frac{5}{2}}^2\le\bar{\mathcal{D}}_n+(\mathcal{E}_n)^2.\tag{5.6}$$

By the standard elliptic regularity of $h$ of the fourth equation in (2.13), the trace theory, (2.18) and (5.6), we have

$$\left|\partial_t^{n-1}h\right|_{\frac{5}{2}}^2\lesssim\left|\partial_t^{n-1}q\right|_{\frac{1}{2}}^2+\left|\partial_t^{n-1}G^4\right|_{\frac{1}{2}}^2\lesssim\left\|\partial_t^{n-1}q\right\|_1^2+(\mathcal{E}_n)^2\le\bar{\mathcal{D}}_n+(\mathcal{E}_n)^2.\tag{5.7}$$

Next, we will prove an iteration-type argument: we assume the control of $\left|\partial_t^j h\right|_{\frac{3}{2}(n-j)+1}^2$ for $j=1,\dots,n$, then we derive the estimates of $\left|\partial_t^{j-2}h\right|_{\frac{3}{2}(n-(j-2))+1}^2$ (for $j-2\ge0$). We write compactly

$$\mathcal{X}_n:=\bar{\mathcal{D}}_n+\sum_{j=0}^{n-1}\left\|\partial_t^j\operatorname{curl}v\right\|_{\frac{3}{2}(n-j)-1}^2+\sum_{j=0}^{n-2}\left\|\partial_t^j v\right\|_{\frac{3}{2}(n-j)-\frac{3}{2}}^2.\tag{5.8}$$

By the third equation in (2.13) and (2.18), we have

$$\begin{aligned}\left|\partial_t^{j-1}v_3\right|_{\frac{3}{2}(n-(j-1))-\frac{1}{2}}^2&\le\left|\partial_t^j h\right|_{\frac{3}{2}(n-(j-1))-\frac{1}{2}}^2+\left|\partial_t^{j-1}G^3\right|_{\frac{3}{2}(n-(j-1))-\frac{1}{2}}^2\\ &\lesssim\left|\partial_t^j h\right|_{\frac{3}{2}(n-j)+1}^2+(\mathcal{E}_n)^2.\end{aligned}\tag{5.9}$$

Then employing the Hodge-type elliptic estimates (7.7), by (5.8), (5.9) and (2.18), we obtain

$$\begin{aligned}\left\|\partial_t^{j-1}v\right\|^2_{\frac32(n-(j-1))}&\lesssim\left\|\partial_t^{j-1}v\right\|^2_0+\left\|\partial_t^{j-1}\operatorname{curl}v\right\|^2_{\frac32(n-(j-1))-1}\\&\quad+\left\|\partial_t^{j-1}\operatorname{div}v\right\|^2_{\frac32(n-(j-1))-1}+\left|\partial_t^{j-1}v_3\right|^2_{\frac32(n-(j-1))-\frac12}\\&\lesssim\mathcal{X}_n+\left\|\partial_t^{j-1}G^2\right\|^2_{\frac32(n-(j-1))-1}+\left|\partial_t^{j}h\right|^2_{\frac32(n-j)+1}+(\mathcal{E}_n)^2\\&\lesssim\left|\partial_t^{j}h\right|^2_{\frac32(n-j)+1}+\mathcal{X}_n+(\mathcal{E}_n)^2.\end{aligned}\tag{5.10}$$

By the first equation in (2.13), (5.10), (5.8) and (2.18), we have (for $j-2\geq 0$)

$$\begin{aligned}\left\|\partial_t^{j-2}\nabla q\right\|^2_{\frac32(n-(j-1))}&\lesssim\left\|\partial_t^{j-1}v\right\|^2_{\frac32(n-(j-1))}+\left\|\partial_t^{j-2}v\right\|^2_{\frac32(n-(j-1))}+\left\|\partial_t^{j-2}G^1\right\|^2_{\frac32(n-(j-1))}\\&\lesssim\left|\partial_t^{j}h\right|^2_{\frac32(n-j)+1}+\mathcal{X}_n+(\mathcal{E}_n)^2.\end{aligned}\tag{5.11}$$

We then use Poincaré's inequality to estimate $\left\|\partial_t^{j-2}q\right\|^2_0$. If $j-2\geq 1$, then we use the boundary estimate (5.4); if $j-2=0$, then we need to estimate for $|q|^2_{-\frac52}$ in a different way. For this, we recall here that $|h|_1^2\leq\bar{\mathcal{E}}_n$. By the fourth equation in (2.13) and (2.18), we obtain

$$|q|^2_{-\frac52}\leq|h|^2_{-\frac12}+\left|G^4\right|^2_{-\frac52}\lesssim|h|^2_{-\frac12}+(\mathcal{E}_n)^2.\tag{5.12}$$

Hence, by the Poincaré-type inequality, (5.11), (5.4) and (5.12), we have that for $j-2\geq 1$,

$$\begin{aligned}\left\|\partial_t^{j-2}q\right\|^2_{\frac32(n-(j-2))-\frac12}&\lesssim\left\|\partial_t^{j-2}\nabla q\right\|^2_{\frac32(n-(j-1))}+\left|\partial_t^{j-2}q\right|^2_{-\frac52}\\&\lesssim\left|\partial_t^{j}h\right|^2_{\frac32(n-j)+1}+\mathcal{X}_n+(\mathcal{E}_n)^2\end{aligned}\tag{5.13}$$

and that

$$\|q\|^2_{\frac32n-\frac12}\lesssim\|\nabla q\|^2_{\frac32(n-1)}+|q|^2_{-\frac52}\lesssim|h|^2_{-\frac12}+\left|\partial_t^2h\right|^2_{\frac32(n-2)+1}+\mathcal{X}_n+(\mathcal{E}_n)^2.\tag{5.14}$$

By the elliptic regularity of the fourth equation in (2.13), the trace theory, (2.18), (5.13) and (5.14), we have that for $j-2\geq 1$,

$$\begin{aligned}\left|\partial_t^{j-2}h\right|^2_{\frac32(n-(j-2))+1}&\lesssim\left|\partial_t^{j-2}q\right|^2_{\frac32(n-(j-2))-1}+\left|\partial_t^{j-2}G^4\right|^2_{\frac32(n-(j-2))-1}\\&\lesssim\left\|\partial_t^{j-2}q\right\|^2_{\frac32(n-(j-2))-\frac12}+(\mathcal{E}_n)^2\\&\lesssim\left|\partial_t^{j}h\right|^2_{\frac32(n-j)+1}+\mathcal{X}_n+(\mathcal{E}_n)^2\end{aligned}\tag{5.15}$$

and that

$$|h|^2_{\frac32n+1}\lesssim|q|^2_{\frac32n-1}+\left|G^4\right|^2_{\frac32n-1}\lesssim\|q\|^2_{\frac32n-\frac12}+(\mathcal{E}_n)^2$$

$$\leq |h|^2_{-\frac{1}{2}} + \left|\partial_t^2 h\right|^2_{\frac{3}{2}(n-2)+1} + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.16}$$

We have thus arrived at the iterated estimates.

By a simple inductive argument on (5.15), combining with (5.16), we have

$$\sum_{j=0}^{n-2} \left|\partial_t^j h\right|^2_{\frac{3}{2}(n-j)+1} \lesssim |\partial_t^n h|_1^2 + \left|\partial_t^{n-1} h\right|^2_{\frac{5}{2}} + |h|^2_{-\frac{1}{2}} + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.17}$$

This together with (5.7) and (5.3) implies

$$\sum_{j=0}^{n+1} \left|\partial_t^j h\right|^2_{\frac{3}{2}(n-j)+1} \lesssim |\partial_t^n h|_1^2 + |h|^2_{-\frac{1}{2}} + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.18}$$

We then obtain that by (5.10) and (5.18),

$$\sum_{j=0}^{n} \left|\partial_t^j v\right|^2_{\frac{3}{2}(n-j)} \lesssim |\partial_t^n h|_1^2 + |h|^2_{-\frac{1}{2}} + \mathcal{X}_n + (\mathcal{E}_n)^2 \tag{5.19}$$

and that by (5.6), (5.13), (5.14) and (5.18),

$$\sum_{j=0}^{n-1} \left|\partial_t^j q\right|^2_{\frac{3}{2}(n-j)-1} \lesssim |\partial_t^n h|_1^2 + |h|^2_{-\frac{1}{2}} + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.20}$$

Consequently, combining (5.18)–(5.20) and recalling (5.8), by the definitions (1.14) of $\mathcal{E}_n$ and (3.1) of $\bar{\mathcal{E}}_n$, we have

$$\begin{aligned} \mathcal{E}_n &\lesssim |\partial_t^n h|_1^2 + |h|^2_{-\frac{1}{2}} + \mathcal{X}_n + (\mathcal{E}_n)^2 \\ &\lesssim \bar{\mathcal{E}}_n + \sum_{j=0}^{n-1} \left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1} + \sum_{j=0}^{n-2} \left\|\partial_t^j v\right\|^2_{\frac{3}{2}(n-j)-\frac{3}{2}} + (\mathcal{E}_n)^2. \end{aligned} \tag{5.21}$$

We may use the Sobolev interpolation to improve the above to be

$$\begin{aligned} \mathcal{E}_n &\lesssim \bar{\mathcal{E}}_n + \sum_{j=0}^{n-1} \left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1} + \sum_{j=0}^{n-2} \left\|\partial_t^j v\right\|^2_0 + (\mathcal{E}_n)^2 \\ &\lesssim \bar{\mathcal{E}}_n + \sum_{j=0}^{n-1} \left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1} + (\mathcal{E}_n)^2. \end{aligned} \tag{5.22}$$

This is the estimate (5.1). □

Now we present the time-integrated result in the "dissipation".

PROPOSITION 5.2. *It holds that*

$$\int_0^T \mathcal{E}_n \lesssim \sup_{[0,T]} \bar{\mathcal{E}}_n + \int_0^T \left( \bar{\mathcal{D}}_n + \sum_{j=0}^{n-1} \left\|\partial_t^j \operatorname{curl} v\right\|^2_{\frac{3}{2}(n-j)-1} + (\mathcal{E}_n)^2 \right). \tag{5.23}$$

*Proof.* We remark that we may not be able to show a pointwise-in-time estimate of $\mathcal{E}_n$ in the dissipation as that in the energy of Proposition 5.1 since we can not control

$|\partial_t^n h|_1^2$ and $|h|_{-\frac{1}{2}}^2$ by $\bar{\mathcal{D}}_n$ (cf. the first inequality in the estimate (5.21)). To get around that $|\partial_t^n h|_1^2$ is not controlled by $\bar{\mathcal{D}}_n$, our crucial observation is that we can show an $L^1$-in-time estimate of $|\partial_t^n h|_1^2$ by the control of the $L^1$-in-time bound of $\left|\partial_t^{n+1} h\right|_{-1/2}^2 + \left|\partial_t^{n-1} h\right|_{5/2}^2$ and $L^\infty$-in-time bound of $|\partial_t^n h|_1^2 + \left|\partial_t^{n-1} h\right|_1^2$; while to get around that $|h|_{-\frac{1}{2}}^2$ is not controlled by $\bar{\mathcal{D}}_n$, we will modify those estimates in Proposition 5.1 that involve $|h|_{-1/2}^2$ so as to remove $|h|_{-1/2}^2$ from the right-hand sides of those estimates to have the improvements.

We first derive the $L^1$-in-time estimate of $|\partial_t^n h|_1^2$. We can still use the estimates (5.2)–(5.7) in Proposition 5.1. By Parseval's theorem, we have that, by (5.3) with $j=n+1$ and (5.7), using Cauchy's inequality, by the definition (3.1) of $\bar{\mathcal{E}}_n$,

$$
\begin{aligned}
\int_0^T |\partial_t^n h|_1^2 &= \int_0^T \sum_{\xi\in\mathbb{Z}^2} (1+|\xi|^2)\left|\partial_t^n \hat{h}\right|^2 \\
&= \int_0^T \sum_{\xi\in\mathbb{Z}^2} \partial_t \left((1+|\xi|^2)\partial_t^{n-1}\hat{h}\partial_t^n\hat{h}\right) - \sum_{\xi\in\mathbb{Z}^2}(1+|\xi|^2)\partial_t^{n-1}\hat{h}\partial_t^{n+1}\hat{h} \\
&= \sum_{\xi\in\mathbb{Z}^2}\left((1+|\xi|^2)\partial_t^{n-1}\hat{h}\partial_t^n\hat{h}\right)(T) - \sum_{\xi\in\mathbb{Z}^2}\left((1+|\xi|^2)\partial_t^{n-1}\hat{h}\partial_t^n\hat{h}\right)(0) \\
&\quad - \int_0^T \sum_{\xi\in\mathbb{Z}^2}(1+|\xi|^2)^{5/2}\partial_t^{n-1}\hat{h}(1+|\xi|^2)^{-1/2}\partial_t^{n+1}\hat{h} \\
&\lesssim \sup_{[0,T]}\left(\left|\partial_t^{n-1}h\right|_1^2 + |\partial_t^n h|_1^2\right) + \int_0^T \left(\left|\partial_t^{n-1}h\right|_{\frac{5}{2}}^2 + \left|\partial_t^{n+1}h\right|_{-\frac{1}{2}}^2\right) \\
&\lesssim \sup_{[0,T]}\bar{\mathcal{E}}_n + \int_0^T \left(\bar{\mathcal{D}}_n + (\mathcal{E}_n)^2\right).
\end{aligned}
\tag{5.24}
$$

Now we continue to proceed with the estimates in Proposition 5.1. We can use the estimates (5.8)–(5.11). But we can not use the estimate (5.12) since $|h|_{-\frac{1}{2}}^2$ is not controlled by $\bar{\mathcal{D}}_n$. We can use the estimates (5.13) and (5.15), and we need to remove $|h|_{-\frac{1}{2}}^2$ from the right-hand sides of the estimates (5.14) and (5.16). Indeed, by the trace theory and (5.11) with $j=2$, we obtain

$$
|\nabla_* q|_{\frac{3}{2}n-2}^2 \lesssim \|\nabla_* q\|_{\frac{3}{2}(n-1)}^2 \lesssim \left|\partial_t^2 h\right|_{\frac{3}{2}(n-2)+1}^2 + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.25}
$$

But since $\int_\Sigma h=0$ and hence $\int_\Sigma q=0$ by the fourth equation of (1.2), by Poincaré's inequality and (5.25), we have

$$
|q|_{\frac{3}{2}n-1}^2 \lesssim |\nabla_* q|_{\frac{3}{2}n-2}^2 \lesssim \left|\partial_t^2 h\right|_{\frac{3}{2}(n-2)+1}^2 + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.26}
$$

Hence, we have improved (5.14) to be (5.26), and thus we can improve (5.16) to be

$$
|h|_{\frac{3}{2}n+1}^2 \lesssim \left|\partial_t^2 h\right|_{\frac{3}{2}(n-2)+1}^2 + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.27}
$$

Therefore, comparing the estimates (5.26) and (5.27) with the estimates (5.14) and (5.16) implies that we can remove $|h|_{-1/2}^2$ from the right-hand sides of the estimates

(5.17)–(5.20); summing the resultings in the estimates (5.18)–(5.20) then yield, by the definition (1.14) of $\mathcal{E}_n$,

$$\mathcal{E}_n \lesssim |\partial_t^n h|_1^2 + \mathcal{X}_n + (\mathcal{E}_n)^2. \tag{5.28}$$

Consequently, combining (5.28) and (5.24), we deduce that

$$\int_0^T \mathcal{E}_n \lesssim \int_0^T \left(|\partial_t^n h|_1^2 + \mathcal{X}_n + (\mathcal{E}_n)^2\right) \lesssim \sup_{[0,T]} \bar{\mathcal{E}}_n + \int_0^T \left(\mathcal{X}_n + (\mathcal{E}_n)^2\right). \tag{5.29}$$

This thus implies (5.23) in the same way as in Proposition 5.1 that (5.21) implies (5.1). □

## 6. Global energy estimates

Now the proof of Theorem 1.1 follows, in a standard way, by the local well-posedness theory, a continuity argument and the following a priori estimates.

THEOREM 6.1. *Let $n \geq 3$ be an integer. There exists a universal constant $\delta > 0$ such that if*

$$\mathcal{E}_n(t) \leq \delta, \quad \forall t \in [0,T], \tag{6.1}$$

*then*

$$\mathcal{E}_n(t) + \int_0^t \mathcal{E}_n(\tau)\,d\tau \lesssim \mathcal{E}_n(0), \quad \forall t \in [0,T]. \tag{6.2}$$

*Moreover, there exists a universal constant $\gamma > 0$ such that*

$$\mathcal{E}_n(t) \lesssim \mathcal{E}_n(0)e^{-\gamma t}, \quad \forall t \in [0,T]. \tag{6.3}$$

*Proof.* By (6.1), combining Propositions 3.1, 4.1, 5.1 and 5.2, we deduce that

$$\begin{aligned}\sup_{0\leq\tau\leq T} \mathcal{E}_n(\tau) + \int_0^T \mathcal{E}_n(\tau)\,d\tau &\lesssim \mathcal{E}_n(0) + \sup_{0\leq s\leq T} (\mathcal{E}_n(\tau))^{3/2} + \int_0^t (\mathcal{E}_n(\tau))^{3/2}\,d\tau \\ &\lesssim \mathcal{E}_n(0) + \sqrt{\delta}\left(\sup_{0\leq\tau\leq T} \mathcal{E}_n(\tau) + \int_0^t \mathcal{E}_n(\tau)\,d\tau\right). \end{aligned}\tag{6.4}$$

This implies (6.2) since $\delta$ is small.

Now we prove the decay estimate (6.3). Following the proof of (6.2), we may deduce that

$$\mathcal{E}_n(t) + \int_s^t \mathcal{E}_n(\tau)\,d\tau \lesssim \mathcal{E}_n(s), \quad \forall t \geq s \geq 0. \tag{6.5}$$

If we define

$$V(s) = \int_s^t \mathcal{E}_n(\tau)\,d\tau, \tag{6.6}$$

then by (6.5), we have that there exists a universal constant $\gamma > 0$ such that

$$2\gamma V(s) \leq \mathcal{E}_n(s) = -\frac{d}{ds}V(s). \tag{6.7}$$

Applying the Grönwall lemma to (6.7), we obtain

$$V(s) \le V(0)e^{-2\gamma s} \le \frac{1}{2\gamma}\mathcal{E}_n(0)e^{-2\gamma s}. \tag{6.8}$$

Now integrating (6.5) in $s$ from $t/2$ to $t$ implies that, by (6.8),

$$\frac{t}{2}\mathcal{E}_n(t) \lesssim \int_{\frac{t}{2}}^{t} \mathcal{E}_n(s)\,ds = V\left(\frac{t}{2}\right) \le \frac{1}{2\gamma}\mathcal{E}_n(0)e^{-\gamma t}. \tag{6.9}$$

This together with (6.2) yields (6.3). □

**Appendix. Analytic tools.**

**7.1. Harmonic extension.** We define the Poisson integral in $\mathbb{T}^2 \times (-\infty, 0)$ by

$$\mathcal{P}f(x) = \sum_{\xi \in \mathbb{Z}^2} e^{2\pi i \xi \cdot x_*} e^{2\pi |\xi| x_3} \hat{f}(\xi), \tag{7.1}$$

where

$$\hat{f}(\xi) = \int_{\mathbb{T}^2} f(x_*) e^{-2\pi i \xi \cdot x_*}. \tag{7.2}$$

We have the following boundedness of $\mathcal{P}$.

Lemma 7.1. *It holds that*

$$\|\mathcal{P}f\|_s \lesssim |f|_{s-1/2}, \quad s \in \mathbb{R}. \tag{7.3}$$

*Proof.* We may refer to Lemma A.9 of [15] for instance. □

**7.2. Estimates in Sobolev spaces.** We will need some estimates of the product of functions in Sobolev spaces.

Lemma 7.2. *Let the domain be either $\Omega$ or $\Sigma$, and $d$ be the dimension.*

1. *Let $0 \le r \le s_1 \le s_2$ be such that $s_1 > d/2$. Then*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{7.4}$$

2. *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + d/2$. Then*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{7.5}$$

*Proof.* These results are standard and may be derived, for example, by use of the Fourier characterization of the $H^s$ spaces and the extension if the domain is $\Omega$. □

**7.3. Normal trace estimates.** We need the following $H^{-1/2}(\Sigma)$ boundary estimates for functions satisfying $v \in L^2(\Omega)$ and $\nabla^\varphi \cdot v \in L^2(\Omega)$.

Lemma 7.3. *Assume that $\|\nabla\varphi\|_{L^\infty} \le C$, then*

$$|v \cdot \mathbf{N}|_{-1/2} \lesssim \|v\|_0 + \|\nabla^\varphi \cdot v\|_0. \tag{7.6}$$

*Proof.* We may refer to Lemma 3.3 of [15]. □

**7.4. Elliptic estimates.** Our derivation of high order energy estimates for the velocity $v$ is based on the following Hodge-type elliptic estimates.

LEMMA 7.4. *Let $r \geq 1$, then it holds that for $v \cdot \nu = 0$ on $\Sigma_b$,*

$$\|v\|_r \lesssim \|v\|_0 + \|\mathrm{curl}\, v\|_{r-1} + \|\mathrm{div}\, v\|_{r-1} + |v_3|_{r-1/2}. \tag{7.7}$$

*Proof.* The estimate is well-known and follows from the identity $-\Delta v = \mathrm{curl}\,\mathrm{curl}\, v - \nabla \mathrm{div}\, v$. $\square$

**Acknowledgements.** The author is deeply grateful to the referee for the valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (No. 11771358).

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