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SPLITTING UP METHOD FOR 2D STOCHASTIC PRIMITIVE EQUATIONS WITH MULTIPLICATIVE NOISE*

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Abstract. This paper concerns the convergence of an iterative scheme for 2D stochastic primitive equations on a bounded domain. The stochastic system is split into two equations: a deterministic 2D primitive equations with random initial value and a linear stochastic parabolic equation, which are both simpler for numerical computations. An estimate of approximation error is given, which implies that the strong speed rate of the convergence in probability is almost $\frac{1}{2}$.

Keywords. splitting up method; primitive equations; approximation error; speeding of convergence in probability; stopping time.

AMS subject classifications. 60H15; 60H30; 76D06; 76M35.

1. Introduction

In this paper, we focus on the convergence of some iterative schemes for 2D stochastic primitive equations, which is helpful for numerical approximation. As a fundamental model in meteorology, the primitive equations were derived from the Navier-Stokes equations, with rotation, coupled with thermodynamics and salinity diffusion-transport equations (see [14, 15, 18]). This model in the deterministic case has been intensively investigated because of the interests stemmed from physics and mathematics. For example, the mathematical study of the primitive equations originated in a series of articles by Lions, Temam, and Wang in the early 1990s (see [14–17] and the references therein), where they set up the mathematical framework and showed the global existence of weak solutions. Cao and Titi [3] developed an approach to dealing with the L^6-norm of the fluctuation $\tilde{v}$ of horizontal velocity and obtained the global well-posedness for the 3D viscous primitive equations.

Along with the great successful developments of deterministic primitive equations, the random situation has also been developed rapidly. For 3D stochastic primitive equations, Guo and Huang [11] obtained the existence of universal random attractor of strong solution under the assumptions that the momentum equation is driven by an additive stochastic forcing and the thermodynamical equation is under a fixed heat source. Debussche, Glatt-Holtz, Temam and Ziane [4] established the global well-posedness of the strong solution when this model is driven by multiplicative random noises. Dong et al. [5] studied its ergodic theory and proved that all weak solutions which are limits of spectral Galerkin approximations share the same invariant measure. Moreover, they established a large deviation principle for this model in [6]. For 2D stochastic primitive equations, Gao and Sun [9] obtained its global well-posedness and Freidlin-Wentzell's large deviations.

The aim of this paper is to study numerical approximations to 2D stochastic primitive equations. There are many literature on this topic for stochastic parabolic differential equations. For example, using the semigroup and the cubature techniques,

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Dörsek [7] studied the weak speed of convergence of a certain time-splitting scheme combining with a Galerkin approximation in the space variable for the stochastic Navier-Stokes equations with an additive noise. The strong convergence of the splitting up method has already been studied in a series of papers by Gyöngy and Krylov (see [12,13] etc.), where the rate of convergence is obtained based on stochastic calculus. However, the linear setting used in their papers does not cover some hydrodynamical models, such as stochastic Navier-Stokes equations, stochastic primitive equations and so on. Recently, Bessaih, Brzeźniak and Millet [2] studied the splitting up method for the strong solution of 2D stochastic Navier-Stokes equations on a torus in the space $L^2([0,T];V)$ and proved that the strong speed of convergence in probability is almost $\frac{1}{2}$.

In this paper, we devote to obtaining the strong speed of the convergence in probability for 2D stochastic primitive equations using the splitting up method from [2]. The splitting up method is implemented by using two consecutive steps on each time interval. The first step is to solve the deterministic 2D primitive equations with random initial value. The second step is to solve a stochastic parabolic equation. The corresponding solutions are denoted by v^n and η^n (see (4.1) and (4.2)), respectively. Our aim is to establish the approximation error of $v^n - v$ and $\eta^n - v$ in the space $L^\infty([0,T];H)\cap L^2([0,T];V)$. During the proof process, the uniform $V-$norm estimates $\mathbb{E}\sup_{t\in[0,T]}\|v(t)\|^2$ of strong solution play a key role (see Proposition 5.1). In [2], the authors obtained such estimates of 2D stochastic Navier-Stokes equations by transforming this model into a curvature equation and utilizing its cancellation property in $H\subset L^2$. However, for 2D stochastic primitive equations, we have no uniform $V-$norm estimates, only $\mathbb{E}\int_0^T\|v(t)\|^2dt\le C$ is available, which leads to some difficulties. For example, during the proof process of Proposition 5.1, the index of $\|v(t)\|$ in $I(t)$ has to be strictly less than 2. Otherwise, we will encounter $\mathbb{E}\int_0^T\|v(t)\|^\alpha dt$ for $\alpha>2$ after using Hölder's inequality. To overcome this difficulty, we divide $\|v(t)\|$ into several parts with small index and make use of uniform $V-$estimates of v^n and uniform $H-$estimates of $v,\partial_z v$ (for details, see Proposition 5.1). Moreover, in order to obtain the uniform $V-$estimates of v^n, an appropriate stopping time is introduced (see Lemma 4.5). Besides, compared with 2D stochastic Navier-Stokes equations, we need to make additional $H-$estimates of $\partial_z v^n$ and $\partial_z\eta^n$ appeared in the estimations of nonlinear terms (see Lemma 2.1).

Specifically, for any $n\ge 1$, set the error term

$$e_n(T)\stackrel{\text{def}}{=}\sup_{k=0,\cdots,n}\Big(|v^n(t_k^+)-v(t_k)|+|\eta^n(t_k^-)-v(t_k)|\Big)$$
$$+\left(\int_0^T\|v^n(s)-v(s)\|^2ds\right)^{\frac{1}{2}}+\left(\int_0^T\|\eta^n(s)-v(s)\|^2ds\right)^{\frac{1}{2}}.$$

Under some conditions, the main result we obtain is

THEOREM 1.1. *Let $\varepsilon\in[0,1)$. Under Hypotheses A-C, the error term $e_n(T)$ converges to 0 in probability with the speed almost $\frac{1}{2}$. Precisely, for any sequence $l(n)_{n\ge1}$ converging to ∞, we have*

$$\lim_{n\to\infty}\mathbb{P}\Big(e_n(T)\ge\frac{l(n)}{\sqrt{n}}\Big)=0.$$

Here, ε is a parameter appeared in Hypotheses A–C, which will be described in Section 4 and 5.

This paper is organized as follows: In Section 2, the mathematical framework is introduced. We obtain the global well-posedness of strong solution in Section 3. In

Section 4, the splitting up method is presented, where two approximation equations to the primitive equations are constructed. Further, the H and $V-$norm estimates of the difference between the two approximation equations are established, respectively. Finally, an auxiliary process is introduced for technical reasons. In Section 5, the speed rate of the convergence in probability is obtained.

2. The mathematical framework

The two dimensional primitive equations can be formally derived from the full three dimensional system under the assumption of invariance with respect to the second horizontal variable y as in [10]. The 2D primitive equations driven by a stochastic forcing in a Cartesian system can be written as

$$\frac{\partial v}{\partial t}-\mu\Delta v+v\partial_x v+\theta\partial_z v+\partial_x p=\psi(t,v)\frac{dW}{dt}, \tag{2.1}$$

$$\partial_x v+\partial_z\theta=0, \tag{2.2}$$

where the velocity $v=v(t,x,z)\in\mathbb{R}$, the vertical velocity θ and the pressure p are all unknown functionals. $(x,z)\in\mathcal{M}=[0,L]\times[-h,0]$. W is a cylindrical Winner process, which will be given in Section 2.2. $\Delta=\partial_x^2+\partial_z^2$ is the Laplacian operator. Note that p is independent of the vertical variable z.

We impose the following boundary conditions:

$$\partial_z v=0,\ \theta=0 \text{ on } \Gamma_u=(0,L)\times\{0\}, \tag{2.3}$$

$$\partial_z v=0,\ \theta=0 \text{ on } \Gamma_b=(0,L)\times\{-h\}, \tag{2.4}$$

$$v=0 \text{ on } \Gamma_l=\{0,L\}\times(-h,0). \tag{2.5}$$

Without loss of generality, we assume that

$$\mu=1,\quad \int_{-h}^0 v dz=0.$$

Integrating (2.2) from $-h$ to z and using (2.3), (2.4), we have

$$\theta(t,x,z)\overset{\text{def}}{=}\Phi(v)(t,x,z)=-\int_{-h}^z\partial_x v(t,x,z')dz'.$$

Then, (2.1)-(2.5) can be rewritten as

$$\frac{\partial v}{\partial t}-\Delta v+v\partial_x v+\Phi(v)\partial_z v+\partial_x p=\psi(t,v)\frac{dW}{dt}, \tag{2.6}$$

$$\partial_z v|_{\Gamma_u}=0,\quad \partial_z v|_{\Gamma_b}=0,\quad v|_{\Gamma_l}=0. \tag{2.7}$$

The initial condition is given by

$$v(0)=v_0. \tag{2.8}$$

2.1. Some functional spaces. Let $\mathcal{L}(K_1;K_2)$ (resp. $\mathcal{L}_2(K_1;K_2)$) be the space of bounded (resp. Hilbert-Schmidt) linear operators from the Hilbert space K_1 to K_2, whose norm is denoted by $\|\cdot\|_{\mathcal{L}(K_1;K_2)}(\|\cdot\|_{\mathcal{L}_2(K_1;K_2)})$. For $p\in\mathbb{Z}^+$, set

$$|\phi|_p=\begin{cases}\left(\int_{\mathcal{M}}|\phi(x,z)|^p dxdz\right)^{\frac{1}{p}}, \forall\phi\in L^p(\mathcal{M}),\\ \left(\int_0^l|\phi(x)|^p dx\right)^{\frac{1}{p}}, \qquad \forall\phi\in L^p((0,L)).\end{cases}$$

In particular, $|\cdot|$ and $(\cdot,\cdot)$ represent norm and inner product of $L^2(\mathcal{M})$ (or $L^2((0,L))$), respectively. For $m\in\mathbb{N}_+$, $(W^{m,p}(\mathcal{M}),\|\cdot\|_{m,p})$ stands for the classical Sobolev space, see [1]. When $p=2$, we denote by $H^m(\mathcal{M})=W^{m,2}(\mathcal{M})$,

$$\begin{cases} H^m(\mathcal{M})=\left\{v\Big|\partial_\alpha v\in(L^2(\mathcal{M}))^2 \text{ for } |\alpha|\le m\right\}, \\ |v|^2_{H^m(\mathcal{M})}=\sum_{0\le|\alpha|\le m}|\partial_\alpha v|^2. \end{cases}$$

It's known that $(H^m(\mathcal{M}),|\cdot|_{H^m(\mathcal{M})})$ is a Hilbert space. $|\cdot|_{H^p((0,L))}$ stands for the norm of $H^p((0,L))$ for $p\in\mathbb{Z}^+$.

Defining our working spaces for (2.6)-(2.8)

$$H\overset{\text{def}}{=}\left\{v\in L^2(\mathcal{M})^2:\ \int_{-h}^0 vdz=0\right\},$$
$$V\overset{\text{def}}{=}\left\{v\in(H^1(\mathcal{M}))^2:\ \int_{-h}^0 vdz=0,\ v=0 \text{ on } \Gamma_l\right\},$$

The space H is endowed with the L^2 inner product

$$(v,\tilde{v})=\int_{\mathcal{M}} v\tilde{v}dxdz.$$

The norm of H is denoted by $|v|=(v,v)^{\frac{1}{2}}$. The inner product and norm in the space V are given by

$$((v,\tilde{v}))=\int_{\mathcal{M}}(\partial_x v\partial_x\tilde{v}+\partial_z v\partial_z\tilde{v})dxdz,$$

and taking $\|\cdot\|=\sqrt{((\cdot,\cdot))}$. Note that under the above definition, a Poincaré inequality $|v|\le C\|v\|$ holds for all $v\in V$.

Define the intermediate space

$$\mathcal{H}\overset{\text{def}}{=}\{v\in H,\partial_z v\in H\}.$$

Let V' be the dual space of V. We have the dense and continuous embeddings

$$V\hookrightarrow H=H'\hookrightarrow V',$$

and denote by $\langle x,y\rangle$ the duality between $x\in V$ and $y\in V'$.

2.2. Some functionals. The Leray operator P_H is the orthogonal projection of $L^2(\mathcal{M})$ onto H. Define a Stokes-type operator A as a bounded map from V to V' as $\langle v,Au\rangle=((v,u))$. A can be extended to an unbounded operator from H to H according to $Av=-P_H\Delta v$ with the domain:

$$D(A)=\left\{v\in H^2(\mathcal{M}):\int_{-h}^0 vdz=0,\ v=0 \text{ on } \Gamma_l,\ \partial_z v=0 \text{ on } \Gamma_u\cup\Gamma_b\right\}.$$

It's well-known that A is a self-adjoint and positive definite operator. Due to the regularity results of the Stokes problem of geophysical fluid dynamics, we have $|Av|\cong|v|_{H^2(\mathcal{O})}$, see [19].

For the nonlinear terms, let

$$B(v,\tilde{v})\overset{\text{def}}{=}P_H(v\partial_x\tilde{v}+\Phi(v)\partial_z\tilde{v}).$$

We establish that B is a well-defined and continuous mapping from $V\times V\to V'$ according to

$$\langle B(u,v),\phi\rangle = b(u,v,\phi),$$

where the associated trilinear form is given by

$$b(u,v,\phi) \overset{\text{def}}{=} \int_{\mathcal{M}} (u\partial_x v\phi + \Phi(u)\partial_z v\phi)d\mathcal{M}.$$

This is contained in the following lemma, which is established in [10].

LEMMA 2.1. *b is a continuous linear form on $V\times V\times V$ and satisfies*

$$|b(v,\tilde{v},\hat{v})| = |\langle B(v,\tilde{v}),\hat{v}\rangle| \le C\|\tilde{v}\||v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}|\hat{v}|^{\frac{1}{2}}\|\hat{v}\|^{\frac{1}{2}} + C|\partial_z\tilde{v}|\|v\||\hat{v}|^{\frac{1}{2}}\|\hat{v}\|^{\frac{1}{2}} \tag{2.9}$$

$$|\langle B(\tilde{v},\tilde{v}) - B(\hat{v},\hat{v}),\tilde{v}-\hat{v}\rangle| \le C\|\tilde{v}\||\tilde{v}-\hat{v}|\|\tilde{v}-\hat{v}\| + C|\partial_z\tilde{v}|\|\tilde{v}-\hat{v}\|^{\frac{3}{2}}|\tilde{v}-\hat{v}|^{\frac{1}{2}}, \tag{2.10}$$

for any $v,\tilde{v},\hat{v}\in V$. Moreover, b satisfies the cancellation property $b(u,v,v)=0$ and

$$b(v,\tilde{v},\hat{v}) = -b(v,\hat{v},\tilde{v}). \tag{2.11}$$

REMARK 2.1. It is obvious that the above estimates of nonlinear terms of primitive equations are of higher order than 2D Navier-Stokes equations, which results in difficulties stated in the introduction.

For the stochastic forcing, we fix a single stochastic basis $\mathcal{T} \overset{\text{def}}{=} (\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P},W)$ with the expectation $\mathbb{E}$. Here, W is a cylindrical Wiener process with the form $W(t,\omega) = \sum_{i\ge1} r_i w_i(t,\omega)$, where $\{r_i\}_{i\ge1}$ is a complete orthonormal basis of a Hilbert space U, $\{w_i\}_{i\ge1}$ is a sequence of independent one-dimensional standard Brownian motions on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$.

Set

$$F(t,v(t)) = Av(t) + B(v(t),v(t)),$$

using the above functionals, it yields

$$\begin{cases} dv(t) + F(t,v(t))dt = \psi(t,v(t))dW(t), \\ v(0) = v_0. \end{cases} \tag{2.12}$$

3. Global well-posedness. In this part, we aim to obtain a priori estimates of the strong solution of (2.12). Firstly, we state the following definition introduced by [10].

DEFINITION 3.1. *Let $\mathcal{T} = (\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P},W)$ be a fixed stochastic basis, $T>0$ and $p\ge2$. Assume the initial data $v_0\in L^p(\Omega;H)$ and is $\mathcal{F}_0-$measurable. An $\mathcal{F}_t-$predictable stochastic process $v(t,\omega)$ is called a strong solution of (2.12) on $[0,T]$ with the initial value v_0 if*

$$v\in C([0,T];H) \quad \mathbb{P}-a.s., \quad v\in L^p(\Omega;C([0,T];H))\bigcap L^p(\Omega;L^2([0,T];V)),$$

and satisfies

$$(v(t),\phi) - (v_0,\phi) + \int_0^t \Big[\langle v(s),A\phi\rangle + \langle B(v,v),\phi\rangle\Big]ds = \int_0^t (\psi(s,v(s))dW(s),\phi), \quad \mathbb{P}-a.s.$$

for all $\phi \in D(A)$.

In order to obtain the global well-posedness of (2.12), we need the following Hypotheses:

Hypothesis A: ψ is a continuous mapping, $\psi:[0,T]\times V \to \mathcal{L}_2(U;H)$ (resp. $\psi:[0,T]\times H \to \mathcal{L}_2(U;H)$ for $\varepsilon=0$) satisfies that there exist positive constants $K_i, i=0,\cdots,4$, such that for $t\in[0,T]$, $0\le\varepsilon<1$,

(A.1) $\|\psi(t,\phi)\|^2_{\mathcal{L}_2(U;H)} \le K_0+K_1|\phi|^2+\varepsilon K_2\|\phi\|^2, \quad \phi\in V,$

(A.2) For $\phi_1,\phi_2\in V$,

$$\|\psi(t,\phi_1)-\psi(t,\phi_2)\|^2_{\mathcal{L}_2(U;H)} \le K_3|\phi_1-\phi_2|^2+\varepsilon K_4\|\phi_1-\phi_2\|^2.$$

Hypothesis B: There exist constants $L_i, i=0,\cdots,2$, such that for $t\in[0,T]$, $0\le\varepsilon<1$,

$$\|\partial_z\psi(t,\phi)\|^2_{\mathcal{L}_2(U;H)} \le L_0+L_1|\partial_z\phi|^2+\varepsilon L_2\|\partial_z\phi\|^2, \quad \partial_z\phi\in V.$$

THEOREM 3.1. *Assume* $v_0\in\mathcal{H}$, *Hypotheses A–B hold with* $K_2\le K_4<2$ *and* $L_2<2$, *there exists a unique global solution* v *of (2.12) in the sense of Definition 3.1 with* $v(0)=v_0$. *Furthermore, if* $q\in[2,\frac{1}{37}+\frac{2}{37K_2})$, *there exists a constant* $C=C(\varepsilon,q,K_0,K_1,K_2,T)$ *such that*

$$\mathbb{E}\Big(\sup_{0\le s\le T}|v(s)|^q+\int_0^T\|v(s)\|^2|v(s)|^{q-2}ds\Big)\le C(1+\mathbb{E}|v_0|^q). \tag{3.1}$$

If $K_2<\frac{2}{147}$, *then*

$$\mathbb{E}\int_0^T|v(s)|^2\|v(s)\|^2ds\le C(1+\mathbb{E}|v_0|^4). \tag{3.2}$$

Similarly, if $q\in[2,\frac{1}{37}+\frac{2}{37L_2})$, *there exists a constant* $C=C(\varepsilon,q,R_0,R_1,R_2,T)$ *such that*

$$\mathbb{E}\Big(\sup_{0\le s\le T}|\partial_z v(s)|^q+\int_0^T\|\partial_z v(s)\|^2|\partial_z v(s)|^{q-2}ds\Big)\le C(1+\mathbb{E}|\partial_z v_0|^q).$$

In particular, if $L_2<\frac{2}{147}$, *we have*

$$\mathbb{E}\Big(\sup_{0\le s\le T}|\partial_z v(s)|^4+\int_0^T|\partial_z v(s)|^2\|\partial_z v(s)\|^2ds\Big)\le C(1+\mathbb{E}|\partial_z v_0|^4).$$

Proof. When $K_2\le K_4<2$ and $L_2<2$, the global well-posedness of strong solution to (2.12) in the sense of Definition 3.1 has been proved by [10], we omit it. Let v be the strong solution of (2.12). For any $q\ge2$, applying Itô's formula to $|v(t)|^q$, we have

$$\begin{aligned}
&d|v(t)|^q+q|v(t)|^{q-2}\|v(t)\|^2dt\\
&=-q|v(t)|^{q-2}\langle v(t),B(v(t),v(t))\rangle dt\\
&\quad+q|v(t)|^{q-2}\langle v(t),\psi(t,v(t))dW(t)\rangle+\frac{q(q-1)}{2}|v(t)|^{q-2}\|\psi(t,v(t))\|^2_{\mathcal{L}_2(U;H)}dt.
\end{aligned}$$

Using (2.11), we obtain

$$d|v(t)|^q+q|v(t)|^{q-2}\|v(t)\|^2dt$$

$$\leq q|v(t)|^{q-2}\langle v(t),\psi(t,v(t))dW(t)\rangle+\frac{q(q-1)}{2}|v(t)|^{q-2}\|\psi(t,v(t))\|^2_{\mathcal{L}_2(U;H)}dt. \tag{3.3}$$

Then, it follows that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|v(t)|^q+q\mathbb{E}\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt\\
\leq&\mathbb{E}\sup_{t\in[0,T]}\int_0^t|v(s)|^{q-2}\langle v(s),\psi(s,v(s))dW(s)\rangle\\
&+\frac{q(q-1)}{2}\mathbb{E}\int_0^T|v(t)|^{q-2}\|\psi(t,v(t))\|^2_{\mathcal{L}_2(U;H)}dt\\
\overset{\text{def}}{=}&I_1+I_2.
\end{aligned}$$

With the help of Hypothesis A, we get

$$\begin{aligned}
I_2\leq&\frac{q(q-1)}{2}K_0\mathbb{E}\int_0^T|v(t)|^{q-2}dt+\frac{q(q-1)}{2}K_1\mathbb{E}\int_0^T\sup_{s\in[0,t]}|v(s)|^qdt\\
&+\varepsilon\frac{q(q-1)}{2}K_2\mathbb{E}\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt.
\end{aligned}$$

Utilizing the Burkholder-Davies-Gundy inequality and Hypothesis A, we have

$$\begin{aligned}
I_1\leq&6q\mathbb{E}\left(\int_0^T|v(t)|^{2(q-2)}|v(t)|^2\|\psi(t,v(t))\|^2_{\mathcal{L}_2(U;H)}dt\right)^{\frac{1}{2}}\\
\leq&6q\mathbb{E}\left(\int_0^T|v(t)|^{2q-2}(K_0+K_1|v(t)|^2+\varepsilon K_2\|v(t)\|^2)dt\right)^{\frac{1}{2}}\\
\leq&6q\mathbb{E}\left(K_0\int_0^T|v(t)|^{2q-2}dt\right)^{\frac{1}{2}}+6q\mathbb{E}\left(K_1\int_0^T|v(t)|^{2q}dt\right)^{\frac{1}{2}}\\
&+6q\mathbb{E}\left(\varepsilon K_2\int_0^T|v(t)|^{2q-2}\|v(t)\|^2dt\right)^{\frac{1}{2}}\\
\overset{\text{def}}{=}&I_1^1+I_1^2+I_1^3.
\end{aligned}$$

Using the Young's inequality, it gives

$$\begin{aligned}
I_1^1\leq&6qK_0^{\frac{1}{2}}\mathbb{E}\left(\sup_{t\in[0,T]}|v(t)|^q\int_0^T|v(t)|^{q-2}dt\right)^{\frac{1}{2}}\\
\leq&6qK_0^{\frac{1}{2}}\mathbb{E}\left[\sup_{t\in[0,T]}|v(t)|^{\frac{q}{2}}\left(\int_0^T|v(t)|^{q-2}dt\right)^{\frac{1}{2}}\right]\\
\leq&\frac{1}{6}\mathbb{E}(\sup_{t\in[0,T]}|v(t)|^q)+18q^2K_0\mathbb{E}\int_0^T\sup_{s\in[0,t]}|v(s)|^qdt+18q^2K_0T.
\end{aligned}$$

I_1^2 can be bounded as

$$I_1^2\leq6qK_1^{\frac{1}{2}}\mathbb{E}\left(\int_0^T|v(t)|^{2q}dt\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{6}\mathbb{E}(\sup_{t\in[0,T]}|v(t)|^q)+18q^2K_1\mathbb{E}\int_0^T\sup_{s\in[0,t]}|v(s)|^q dt.$$

By the Young's inequality, we have

$$\begin{aligned}I_1^3 &\leq 6q\varepsilon^{\frac{1}{2}}K_2^{\frac{1}{2}}\mathbb{E}\left(\sup_{t\in[0,T]}|v(t)|^q\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt\right)^{\frac{1}{2}}\\ &\leq \frac{1}{6}\mathbb{E}(\sup_{t\in[0,T]}|v(t)|^q)+18q^2\varepsilon K_2\mathbb{E}\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt.\end{aligned}$$

Based on the above inequalities, we have

$$\begin{aligned}I_1 \leq &\frac{1}{2}\mathbb{E}(\sup_{t\in[0,T]}|v(t)|^q)+18q^2(K_0+K_1)\mathbb{E}\int_0^T\sup_{s\in[0,t]}|v(s)|^q dt\\ &+18q^2\varepsilon K_2\mathbb{E}\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt+18q^2K_0T.\end{aligned}$$

Collecting the above estimates, we conclude that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}|v(t)|^q+2\left(q-\varepsilon\frac{q(q-1)}{2}K_2-18q^2\varepsilon K_2\right)\mathbb{E}\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt\\ \leq &2\left(\frac{q(q-1)}{2}K_0+\frac{q(q-1)}{2}K_1+18q^2(K_0+K_1)\right)\mathbb{E}\int_0^T\sup_{s\in[0,t]}|v(s)|^q dt\\ &+q(q-1)K_0T+36q^2K_0T.\end{aligned}\tag{3.4}$$

When $q\in[2,\frac{1}{37}+\frac{2}{37K_2})$, we have

$$q-\varepsilon\frac{q(q-1)}{2}K_2-18q^2\varepsilon K_2>0.$$

Applying Grönwall's inequality to (3.4), we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|v(t)|^q\leq C(q,K_0,K_1,K_2,T)(1+\mathbb{E}|v_0|^q).\tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\mathbb{E}\sup_{t\in[0,T]}|v(t)|^q+\mathbb{E}\int_0^T|v(t)|^{q-2}\|v(t)\|^2dt\leq C(q,K_0,K_1,K_2,T)(1+\mathbb{E}|v_0|^q).\tag{3.6}$$

Let $r=\partial_z v$. From (2.12), we have

$$dr+Ardt+(v\partial_x r+\Phi(v)\partial_z r)dt=\partial_z\psi(t,v(t))dW(t).\tag{3.7}$$

Applying Itô's formula to (3.7), we obtain

$$\begin{aligned}&d|r(t)|^q+q|r(t)|^{q-2}\|r(t)\|^2dt\\ =&-q|r(t)|^{q-2}\langle r(t),(v\partial_x r+\Phi(v)\partial_z r)\rangle dt\\ &+q|r(t)|^{q-2}\langle r(t),\partial_z\psi(t,v(t))dW(t)\rangle+\frac{q(q-1)}{2}|r(t)|^{q-2}\|\partial_z\psi(t,v(t))\|^2_{\mathcal{L}_2(U;H)}dt.\end{aligned}$$

We deduce from (2.11) that

$$\begin{aligned}&d|r(t)|^q+q|r(t)|^{q-2}\|r(t)\|^2dt\\ =&q|r(t)|^{q-2}\langle r(t),\partial_z\psi(t,v(t))dW(t)\rangle+\frac{q(q-1)}{2}|r(t)|^{q-2}\|\partial_z\psi(t,v(t))\|^2_{\mathcal{L}_2(U;H)}dt.\end{aligned}\tag{3.8}$$

Note that (3.8) is similar to (3.3). Following the same process exactly as above, we conclude the result. □

REMARK 3.1. For (2.12) with $v_0\in L^p(\Omega;V)$, we have no uniform $V-$norm estimates of v. That is, we can not find a positive constant C such that $\mathbb{E}\sup_{t\in[0,T]}\|v(t)\|^2\leq C$.

4. Splitting up method

Let $\prod^n=\{0=t_0<t_1<\cdots<t_n=T\}$ be a finite partition of a given interval $[0,T]$ with a constant mesh $h=\frac{T}{n}$. Let $\varepsilon\in[0,1)$ and let $F_\varepsilon:[0,T]\times V\to V'$ be defined by

$$F_\varepsilon(t,v)=(1-\varepsilon)Av+B(v,v).$$

It's easy to know $F_0(t,v)=F(t,v)$.

Set $t_{-1}=-\frac{T}{n}$. For $t\in[t_{-1},0)$, define

$$v^n(t)=\eta^n(t)=v_0,\quad \mathcal{F}_t=\mathcal{F}_0.$$

The scheme is defined by induction as follows. Suppose we have defined processes $v^n(t)$ and $\eta^n(t)$ for $t\in[t_{i-1},t_i)$, $i=0,\cdots,n-1$, such that $\eta^n(t_i^-)$ is an $H-$valued $\mathcal{F}_{t_i}-$measurable function. This clearly holds for $i=0$. Then we define $v^n(t),t\in[t_i,t_{i+1})$ as the unique solution of the (deterministic) problem with positive viscosity $1-\varepsilon$ and with initial condition $\eta^n(t_i^-)$ at time t_i, that is,

$$\begin{cases}\frac{dv^n(t)}{dt}+F_\varepsilon(t,v^n(t))=0,\quad t\in[t_i,t_{i+1}),\\ v^n(t_i)=v^n(t_i^+)=\eta^n(t_i^-),\end{cases}\tag{4.1}$$

Note that $v^n(t_{i+1}^-)$ is a well-defined $H-$valued $\mathcal{F}_{t_i}-$measurable random variable. Then we can define $\eta^n(t),t\in[t_i,t_{i+1})$ as the unique solution of the random problem with initial condition $v^n(t_{i+1}^-)$ at time t_i:

$$\begin{cases}d\eta^n(t)+\varepsilon A\eta^n(t)dt=\psi(t,\eta^n(t))dW(t),\ t\in[t_i,t_{i+1}),\\ \eta^n(t_i)=\eta^n(t_i^+)=v^n(t_{i+1}^-),\end{cases}\tag{4.2}$$

We claim that $\eta^n(t_{i+1}^-)$ defined above is a well-defined $H-$valued $\mathcal{F}_{t_{i+1}}-$measurable random variable. In fact, when $\varepsilon>0$, it's classical that (4.2) has a unique weak solution provided the stochastic parabolic condition holds (K_2, K_4, L_2 are small enough). When $\varepsilon=0$, the smoothing effect of A does not act anymore, but ψ satisfies the usual growth and Lipschitz conditions for the $H-$norm. Finally, let $v^n(T^+)=\eta^n(T^-)$.

REMARK 4.1. As stated in [2], v^n and η^n constructed above are not continuous, only right continuous.

In order to prove the convergence of the above scheme, we will need to establish a priori estimates on v^n and η^n. Firstly, we introduce some notations. Recall that $\prod^n=\{0=t_0<t_1<\cdots<t_n=T\}$. Set

$$\begin{cases}d_n(t)\overset{\text{def}}{=}t_i,\ d_n^*(t)\overset{\text{def}}{=}t_{i+1},\quad \text{for } t\in[t_i,t_{i+1}),\quad i=0,1,\cdots,n-2,\\ d_n(t)\overset{\text{def}}{=}t_{n-1},\ d_n^*(t)\overset{\text{def}}{=}t_n,\ \text{for } t\in[t_{n-1},t_n].\end{cases}\tag{4.3}$$

Then, the processes $v^n(t),\eta^n(t)$ can be rewritten in a way close to the continuous equation:

$$v^n(t)=v_0-\int_0^t F_\varepsilon(s,v^n(s))ds+\int_0^{d_n(t)}[-\varepsilon A\eta^n(s)ds+\psi(s,\eta^n(s))dW(s)], \tag{4.4}$$

$$\eta^n(t)=v_0-\int_0^{d_n^*(t)} F_\varepsilon(s,v^n(s))ds+\int_0^t[-\varepsilon A\eta^n(s)ds+\psi(s,\eta^n(s))dW(s)]. \tag{4.5}$$

In the following, we aim to establish both $H-$norm and $V-$norm estimates of the difference between v^n and η^n.

4.1. $H-$norm of $v^n-\eta^n$. To achieve it, we firstly need to make a priori estimates on v^n and η^n.

LEMMA 4.1. *Let $v_0\in\mathcal{H}$. Fix $\varepsilon\in[0,1)$. Let Hypotheses A–B hold with $K_2\le K_4<2$ and $L_2<2$. Then there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}|v_0|^2,K_i,L_i)$ such that for every integer $n\ge 1$,*

$$\sup_{t\in[0,T]}\mathbb{E}\Big(|\eta^n(t)|^2+\sup_{s\in[d_n(t),d_n^*(t))}|v^n(s)|^2\Big)+\mathbb{E}\int_0^T\|v^n(s)\|^2ds\le C. \tag{4.6}$$

Moreover, if $\varepsilon\in(0,1)$, there exists a constant C such that

$$\sup_n\mathbb{E}\int_0^T\|\eta^n(s)\|^2ds\le C. \tag{4.7}$$

Proof. Taking the scalar product of (4.1) by v^n and integrating over $(t_i,t]$ for $t\in[t_i,t_{i+1})$, we have

$$|v^n(t)|^2+2(1-\varepsilon)\int_{t_i}^t\|v^n(s)\|^2ds=|\eta^n(t_i^-)|^2-2\int_{t_i}^t\langle B(v^n(s),v^n(s)),v^n(s)\rangle ds.$$

By (2.11), we obtain

$$|v^n(t)|^2+2(1-\varepsilon)\int_{t_i}^t\|v^n(s)\|^2ds\le|\eta^n(t_i^-)|^2. \tag{4.8}$$

Taking expectation of (4.8), we get

$$\mathbb{E}(\sup_{t_i\le t<t_{i+1}}|v^n(t)|^2)\le\mathbb{E}|\eta^n(t_i^-)|^2. \tag{4.9}$$

Applying Itô's formula to (4.2) and by Hypothesis A, it yields that for $t\in[t_i,t_{i+1})$,

$$\begin{aligned}&\mathbb{E}|\eta^n(t)|^2+2\varepsilon\mathbb{E}\int_{t_i}^t\|\eta^n(s)\|^2ds\\&=\mathbb{E}|v^n(t_{i+1}^-)|^2+\mathbb{E}\int_{t_i}^t\|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;H)}ds\\&\le\mathbb{E}|v^n(t_{i+1}^-)|^2+\mathbb{E}\int_{t_i}^t(K_0+K_1|\eta^n(s)|^2+\varepsilon K_2\|\eta^n(s)\|^2)ds.\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}|\eta^n(t)|^2+\varepsilon(2-K_2)\mathbb{E}\int_{t_i}^t\|\eta^n(s)\|^2ds\\ \leq&\mathbb{E}|v^n(t_{i+1}^-)|^2+\frac{K_0T}{n}+K_1\int_{t_i}^t\mathbb{E}|\eta^n(s)|^2ds.\end{aligned}\tag{4.10}$$

Since $K_2<2$, we can neglect the integral of $V-$norm in (4.10) to obtain

$$\sup_{t_i\leq t<t_{i+1}}\mathbb{E}|\eta^n(t)|^2\leq(\mathbb{E}|v^n(t_{i+1}^-)|^2+\frac{K_0T}{n})e^{\frac{K_1T}{n}}.\tag{4.11}$$

Putting (4.9) to (4.11), it gives

$$\sup_{t_i\leq t<t_{i+1}}\mathbb{E}|\eta^n(t)|^2\leq(\mathbb{E}|\eta^n(t_i^-)|^2+\frac{K_0T}{n})e^{\frac{K_1T}{n}}.\tag{4.12}$$

Set

$$\tilde{r}_1\overset{\text{def}}{=}K_1,\quad \tilde{r}_2\overset{\text{def}}{=}K_0,$$

then, by a mathematical induction argument, we infer that for $i=0,\cdots,n-1$,

$$\mathbb{E}(\sup_{t_i\leq t<t_{i+1}}|v^n(t)|^2)\vee(\sup_{t_i\leq t<t_{i+1}}\mathbb{E}|\eta^n(t)|^2)\leq\mathbb{E}|v_0|^2e^{(i+1)\frac{\tilde{r}_1T}{n}}+\frac{\tilde{r}_2T}{n}\sum_{j=1}^{i+1}e^{j\frac{\tilde{r}_1T}{n}}.$$

Hence, we deduce that

$$\begin{aligned}&\Big[\sup_{t\in[0,T]}\mathbb{E}(\sup_{d_n(t)\leq s<d_n^*(t)}|v^n(s)|^2)\Big]\vee\Big[\sup_{t\in[0,T]}\mathbb{E}|\eta^n(t)|^2\Big]\\ \leq&\mathbb{E}|v_0|^2e^{\tilde{r}_1T}+\frac{\tilde{r}_2T}{n}\sum_{j=1}^{n}e^{j\frac{\tilde{r}_1T}{n}}\\ \leq&\mathbb{E}|v_0|^2e^{\tilde{r}_1T}+\frac{\tilde{r}_2}{\tilde{r}_1}e^{2\tilde{r}_1T},\end{aligned}\tag{4.13}$$

which proves part of (4.6). Moreover, from (4.8), (4.10), and using (4.13), we obtain that for every $i=0,\cdots,n-1$,

$$\begin{aligned}\mathbb{E}|v^n(t_{i+1}^-)|^2+(1-\varepsilon)\mathbb{E}\int_{t_i}^{t_{i+1}}\|v^n(s)\|^2ds\leq\mathbb{E}|\eta^n(t_i^-)|^2+\frac{CT}{n},\\ \mathbb{E}|\eta^n(t_{i+1}^-)|^2+\varepsilon(2-K_2)\int_{t_i}^{t_{i+1}}\|\eta^n(s)\|^2ds\leq\mathbb{E}|v^n(t_{i+1}^-)|^2+\frac{CT}{n}.\end{aligned}$$

Adding all these inequalities from $i=0$ to $n-1$, we conclude the proof of (4.6). At the same time, when $\varepsilon>0$, it gives (4.7). □

Referring to [2] and similar to Lemma 4.1, we have the following higher moments of $H-$norm.

LEMMA 4.2. *Let $v_0\in\mathcal{H}$ be $\mathcal{F}_0-$measurable. Fix $\varepsilon\in[0,1)$. Let Hypotheses A–B hold with $K_2<\frac{2}{2p-1}$, for some $p\geq2$ and $K_4\leq L_2<2$. Then there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}|v_0|^{2p},K_i,L_i)$ such that for every integer $n\geq1$,*

$$\sup_{t\in[0,T]}\mathbb{E}\Big(|\eta^n(t)|^{2p}+\sup_{s\in[d_n(t),d_n^*(t))}|v^n(s)|^{2p}\Big)+\mathbb{E}\int_0^T\|v^n(s)\|^2|v^n(s)|^{2(p-1)}ds\leq C.\tag{4.14}$$

In particular, when $p=2$, it gives

$$\mathbb{E}\int_0^T |v^n(s)|^4 ds \le \mathbb{E}\int_0^T \|v^n(s)\|^2 |v^n(s)|^2 ds \le C. \tag{4.15}$$

Moreover, if $\varepsilon\in(0,1)$, there exists a constant C such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \|\eta^n(s)\|^2 |\eta^n(s)|^{2(p-1)} ds \le C. \tag{4.16}$$

Compared with the 2D stochastic Navier-Stokes equations studied by [2], we need the following additional estimates of $\partial_z v^n$ and $\partial_z \eta^n$. Define

$$r^n = \partial_z v^n, \quad q^n = \partial_z \eta^n.$$

From (4.1), we have for $t\in[t_i,t_{i+1})$,

$$dr^n + (1-\varepsilon)Ar^n dt + \Big(v^n \partial_x r^n + \Phi(v^n)\partial_z r^n\Big) dt = 0. \tag{4.17}$$

Moreover, we deduce from (4.2) that for $t\in[t_i,t_{i+1})$,

$$dq^n + \varepsilon A q^n dt = \partial_z \psi(t,\eta^n(t)) dW(t). \tag{4.18}$$

The initial conditions for (4.17) and (4.18) are $r^n(t_i)=q^n(t_i^-)$, $q^n(t_i)=r^n(t_{i+1}^-)$, respectively.

LEMMA 4.3. *Let $v_0\in\mathcal{H}$ be $\mathcal{F}_0-$measurable random variable. Fix $\varepsilon\in[0,1)$. Let Hypotheses A–B hold with $K_2\le K_4\le L_2<2$. Then there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}|\partial_z v_0|^2,K_i,L_i)$ such that for every integer $n\ge 1$,*

$$\sup_{t\in[0,T]} \mathbb{E}\Big(|q^n(t)|^2 + \sup_{s\in[d_n(t),d_n^*(t))} |r^n(s)|^2\Big) + \mathbb{E}\int_0^T \|r^n(s)\|^2 ds \le C. \tag{4.19}$$

Moreover, if $\varepsilon\in(0,1)$, there exists a constant C such that

$$\sup_n \mathbb{E}\int_0^T \|q^n(s)\|^2 ds \le C. \tag{4.20}$$

Proof. Taking the scalar product of (4.17) with r^n in H and integrating over $(t_i,t]$ for $t\in[t_i,t_{i+1})$. With the help of the cancellation property, we have

$$\frac{d|r^n|^2}{dt} + 2(1-\varepsilon)\|r^n\|^2 \le 0, \tag{4.21}$$

that is,

$$|r^n(t)|^2 + 2(1-\varepsilon)\int_{t_i}^t \|r^n(s)\|^2 ds \le |q^n(t_i^-)|^2. \tag{4.22}$$

Taking the expectation of (4.22), we deduce that

$$\mathbb{E}\left(\sup_{t\in[t_i,t_{i+1})} |r^n(t)|^2\right) \le \mathbb{E}|q^n(t_i^-)|^2. \tag{4.23}$$

Using Itô's formula to (4.18) and by Hypothesis B, we have for $t\in[t_i,t_{i+1})$,

$$\mathbb{E}|q^n(t)|^2+\varepsilon(2-L_2)\mathbb{E}\int_{t_i}^t \|q^n(s)\|^2ds\le \mathbb{E}|r^n(t_{i+1}^-)|^2+\frac{L_0T}{n}+L_1\int_{t_i}^t\mathbb{E}|q^n(s)|^2ds.$$

When $L_2<2$, ignoring the $V-$norm and by Grönwall's inequality, we get

$$\sup_{t\in[t_i,t_{i+1})}\mathbb{E}|q^n(t)|^2\le\Big(\mathbb{E}|r^n(t_{i+1}^-)|^2+\frac{L_0T}{n}\Big)\mathrm{e}^{\frac{L_1T}{n}}. \tag{4.24}$$

Putting (4.23) into (4.24), we obtain

$$\sup_{t\in[t_i,t_{i+1})}\mathbb{E}|q^n(t)|^2\le\Big(\mathbb{E}|q^n(t_i^-)|^2+\frac{L_0T}{n}\Big)\mathrm{e}^{\frac{L_1T}{n}}.$$

Set $\tilde r_3=L_1,\tilde r_4=L_0$, by the induction argument, we have for $i=0,\cdots,n-1$,

$$\mathbb{E}(\sup_{t_i\le t<t_{i+1}}|r^n(t)|^2)\vee(\sup_{t_i\le t<t_{i+1}}\mathbb{E}|q^n(t)|^2)\le\mathbb{E}|\partial_zv_0|^2\mathrm{e}^{(i+1)\frac{\tilde r_3T}{n}}+\frac{\tilde r_4T}{n}\sum_{j=1}^{i+1}\mathrm{e}^{j\frac{\tilde r_3T}{n}}.$$

Hence, we deduce that

$$\begin{aligned}&\Big[\sup_{t\in[0,T]}\mathbb{E}(\sup_{d_n(t)\le s<d_n^*(t)}|r^n(s)|^2)\Big]\vee\Big[\sup_{t\in[0,T]}\mathbb{E}|q^n(t)|^2\Big]\\ \le&\mathbb{E}|\partial_zv_0|^2\mathrm{e}^{\tilde r_3T}+\frac{\tilde r_4}{\tilde r_3}\mathrm{e}^{2\tilde r_3T}.\end{aligned}\tag{4.25}$$

Using the same argument as Lemma 4.1, we conclude the rest of the result. ∎

Similarly to Lemma 4.1, we can obtain the following higher order norm estimates.

LEMMA 4.4. *Let $v_0\in\mathcal{H}$ be $\mathcal{F}_0-$measurable random variable. Fix $\varepsilon\in[0,1)$. Let Hypotheses A–B hold with $K_2\le K_4<2$ and $L_2<\frac{2}{2p-1}$ for some $p\ge2$. Then there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}|\partial_zv_0|^2,K_i,L_i)$ such that for every integer $n\ge1$,*

$$\sup_{t\in[0,T]}\mathbb{E}\Big(|q^n(t)|^{2p}+\sup_{s\in[d_n(t),d_n^*(t))}|r^n(s)|^{2p}\Big)+\mathbb{E}\int_0^T\|r^n(s)\|^2|r^n(s)|^{2(p-1)}ds\le C. \tag{4.26}$$

Moreover, if $\varepsilon\in(0,1)$, there exists a constant C such that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^T\|q^n(s)\|^2|q^n(s)|^{2(p-1)}ds\le C. \tag{4.27}$$

Based on the above, we are ready to prove an upper bound of the $H-$norm of the difference between v^n and η^n.

PROPOSITION 4.1. *Let $v_0\in\mathcal{H}$ be $\mathcal{F}_0-$measurable random variable. For any $\varepsilon\in[0,1)$. Assume Hypotheses A–B hold with $K_2<\frac{2}{3}$, $K_4<2$ and $L_2<\frac{2}{3}$, there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}|\partial_zv_0|^4,K_i,L_i)$ such that for any $n\in\mathbb{N}$,*

$$\mathbb{E}\int_0^T|v^n(t)-\eta^n(t)|^2dt\le\frac{CT}{n}. \tag{4.28}$$

Proof. **Case 1:** $\varepsilon=0$. For any $t\in[0,T)$, by (4.2) and Hypothesis A, we have

$$\mathbb{E}|\eta^n(t)-v^n(d_n^*(t))|^2=\mathbb{E}\int_{d_n(t)}^t \|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;H)}ds\le\mathbb{E}\int_{d_n(t)}^t (K_0+K_1|\eta^n(s)|^2)ds.$$

Then, by Fubini's theorem and Lemma 4.1, we obtain

$$\mathbb{E}\int_0^T |\eta^n(t)-v^n(d_n^*(t))|^2dt\le C\mathbb{E}\int_0^T (1+|\eta^n(s)|^2)\left(\int_s^{d_n^*(s)} dt\right)ds\le C\frac{T}{n}. \tag{4.29}$$

From (4.1), we have

$$|v^n(d_n^*(t)^-)-v^n(t)|^2=2\int_t^{d_n^*(t)}\langle v^n(s)-v^n(t),dv^n(s)\rangle=\sum_{i=1}^2 I_i(t),$$

where

$$I_1(t)\stackrel{\text{def}}{=}-2(1-\varepsilon)\int_t^{d_n^*(t)}\langle v^n(s)-v^n(t),Av^n(s)\rangle ds,$$
$$I_2(t)\stackrel{\text{def}}{=}-2\int_t^{d_n^*(t)}\langle v^n(s)-v^n(t),B(v^n(s),v^n(s))\rangle ds.$$

Using Lemma 4.1 and the Young's inequality, we have

$$\begin{aligned}\left|\mathbb{E}\int_0^T I_1(t)dt\right|&=\left|(1-\varepsilon)\mathbb{E}\int_0^T\int_t^{d_n^*(t)}(-2\|v^n(s)\|^2+2\|v^n(s)\|\|v^n(t)\|)dsdt\right|\\&\le\left|(1-\varepsilon)\mathbb{E}\int_0^T\int_t^{d_n^*(t)}(-2\|v^n(s)\|^2+2\|v^n(s)\|^2+\frac{1}{2}\|v^n(t)\|^2)dsdt\right|\\&\le\frac{1-\varepsilon}{2}\mathbb{E}\int_0^T\|v^n(t)\|^2\left(\int_t^{d_n^*(t)}ds\right)dt\le\frac{CT}{n}.\end{aligned}$$

By (2.9), we have

$$\begin{aligned}\left|\mathbb{E}\int_0^T I_2(t)dt\right|\le&2\mathbb{E}\int_0^T\int_t^{d_n^*(t)}\|v^n(t)\||v^n(s)|_4^2dsdt\\&+2\mathbb{E}\int_0^T\int_t^{d_n^*(t)}|r^n(t)|\|v^n(s)\||v^n(s)|^{\frac{1}{2}}\|v^n(s)\|^{\frac{1}{2}}dsdt\\&\stackrel{\text{def}}{=}K_1+K_2.\end{aligned}$$

By Lemma 4.2, we deduce that

$$\begin{aligned}K_1&=2\mathbb{E}\int_0^T\|v^n(t)\|\left(\int_t^{d_n^*(t)}|v^n(s)|_4^2ds\right)dt\\&\le2\left(\mathbb{E}\int_0^T\|v^n(t)\|^2dt\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T\left(\int_t^{d_n^*(t)}|v^n(s)|_4^2ds\right)^2dt\right)^{\frac{1}{2}}\\&\le C\left(\mathbb{E}\int_0^T\frac{T}{n}\int_t^{d_n^*(t)}|v^n(s)|_4^4dsdt\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\frac{T}{n}\mathbb{E}\int_0^T |v^n(s)|_4^4\Big(\int_{d_n(s)}^s dt\Big)ds\right)^{\frac{1}{2}}\\
&\le \frac{CT}{n}.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, Fubini's theorem and Lemmas 4.1, 4.2, 4.4, we get

$$
\begin{aligned}
K_2 &\le \mathbb{E}\int_0^T\int_t^{d_n^*(t)} |r^n(t)|\|v^n(s)\||v^n(s)|^{\frac{1}{2}}\|v^n(s)\|^{\frac{1}{2}}dsdt\\
&\le \mathbb{E}\int_0^T\int_t^{d_n^*(t)} |r^n(t)|^2|v^n(s)|\|v^n(s)\|dsdt+\mathbb{E}\int_0^T\int_t^{d_n^*(t)}\|v^n(s)\|^2dsdt\\
&\le \mathbb{E}\int_0^T |r^n(t)|^4\int_t^{d_n^*(t)} dsdt+\mathbb{E}\int_0^T\int_t^{d_n^*(t)}(1+|v^n(s)|^2)\|v^n(s)\|^2dsdt\\
&\le \mathbb{E}\int_0^T |r^n(t)|^4\int_t^{d_n^*(t)} dsdt+\mathbb{E}\int_0^T(1+|v^n(s)|^2)\|v^n(s)\|^2(\int_{d_n(s)}^s dt)ds\\
&\le \frac{CT}{n}.
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\int_0^T |v^n(d_n^*(t)^-)-v^n(t)|^2dt\le \frac{CT}{n}. \tag{4.30}
$$

Combining (4.29) and (4.30), we conclude the result when $\varepsilon=0$.

Case 2: $\varepsilon\in(0,1)$. For any $t\in[0,T]$, from (4.4) and (4.5), we have

$$
\eta^n(t)-v^n(t)=-\int_t^{d_n^*(t)} F_\varepsilon(s,v^n(s))ds-\varepsilon\int_{d_n(t)}^t A\eta^n(s)ds+\int_{d_n(t)}^t \psi(s,\eta^n(s))dW(s).
$$

Applying Itô's formula to $|\eta^n(t)-v^n(t)|^2$, we obtain

$$
\mathbb{E}\int_0^T |\eta^n(t)-v^n(t)|^2dt=\sum_{i=1}^4 J_i,
$$

where

$$
\begin{aligned}
J_1(t)&\overset{\text{def}}{=}-2(1-\varepsilon)\mathbb{E}\int_0^T\int_t^{d_n^*(t)}\langle Av^n(s),\eta^n(s)-v^n(s)\rangle dsdt,\\
J_2(t)&\overset{\text{def}}{=}-2\mathbb{E}\int_0^T\int_t^{d_n^*(t)}\langle B(v^n(s),v^n(s)),\eta^n(s)-v^n(s)\rangle dsdt,\\
J_3(t)&\overset{\text{def}}{=}-2\varepsilon\mathbb{E}\int_0^T\int_{d_n(t)}^t\langle A\eta^n(s),\eta^n(s)-v^n(s)\rangle dsdt,\\
J_4(t)&\overset{\text{def}}{=}\mathbb{E}\int_0^T\int_{d_n(t)}^t\|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;H)}dsdt.
\end{aligned}
$$

Exactly as pages 12-13 in [2], we have

$$
J_1(t)\le\frac{C(1-\varepsilon)T}{n},\quad J_3(t)\le\frac{C\varepsilon T}{n},\quad J_4(t)\le\frac{CT}{n}.
$$

By (2.9), the Cauchy-Schwarz inequality, Fubini's theorem and Lemmas 4.1, 4.2 and 4.4, we have

$$\begin{aligned}J_2(t) &\le 2\mathbb{E}\int_0^T\int_t^{d_n^*(t)}(\|\eta^n(s)\||v^n(s)|\|v^n(s)\|+|\partial_z\eta^n(s)|\|v^n(s)\|^{\frac{3}{2}}|v^n(s)|^{\frac{1}{2}})ds\\ &\le C\mathbb{E}\int_0^T(\|\eta^n(s)\|^2+|v^n(s)|^2\|v^n(s)\|^2+|q^n(s)|^4+\|v^n(s)\|^2\\ &\quad +|v^n(s)|^2\|v^n(s)\|^2)\Big(\int_{d_n(s)}^s dt\Big)ds\\ &\le \frac{CT}{n}.\end{aligned}$$

The above estimates imply that (4.28) holds when $\varepsilon\in(0,1)$. □

4.2. $V-$norm of $v^n-\eta^n$. In order to obtain $V-$norm of $v^n-\eta^n$, we need an additional hypothesis.

Hypothesis C: There exist constants $R_i, i=0,1,2$, such that for $t\in[0,T]$, $0\le\varepsilon<1$,

$$\|\psi(t,\phi)\|^2_{\mathcal{L}_2(U;V)}\le R_0+R_1\|\phi\|^2+\varepsilon R_2|A\phi|^2,\quad \phi\in D(A).$$

Fix n, for some $N>0$, define the stopping time

$$\tau_n^N\overset{\text{def}}{=}\inf\Big\{t:\sup_{i=0,\cdots,n-1}\int_{t_i\wedge t}^{t_{i+1}\wedge t}(|v^n(s)|^2\|v^n(s)\|^2+|r^n(s)|\|r^n(s)\|)ds>\frac{N}{n}\Big\}. \tag{4.31}$$

Then, we obtain

Lemma 4.5. *Let $v_0\in V$. Fix $\varepsilon\in[0,1)$. Let Hypotheses A–C hold with $K_2<\frac{2}{3}$, $K_4<2$ and $L_2<\frac{2}{3}$, $R_2<2$, then there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}\|v_0\|^2,K_i,L_i,R_i)$ such that for any integer $n\ge1$,*

$$\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}\Big(\|\eta^n(t)\|^2+\sup_{s\in[d_n(t)\wedge\tau_n^N,d_n^*(t)\wedge\tau_n^N)}\|v^n(s)\|^2\Big)+\mathbb{E}\int_0^{T\wedge\tau_n^N}\|v^n(s)\|_2^2ds\le C\tilde{K}(N), \tag{4.32}$$

where $\tilde{K}(N)=\frac{1}{N}e^{C(T)N}$.

Moreover, if $\varepsilon\in(0,1)$, we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^{T\wedge\tau_n^N}\|\eta^n(s)\|_2^2ds\le C\tilde{K}(N). \tag{4.33}$$

Proof. Taking the scalar product of (4.1) by Av^n in H and integrating over $(t_i,t]$ for $t\in[t_i,t_{i+1})$, we have

$$\|v^n(t)\|^2+2(1-\varepsilon)\int_{t_i}^t\|v^n(s)\|_2^2ds=\|\eta^n(t_i^-)\|^2-2\int_{t_i}^t\langle B(v^n(s),v^n(s)),Av^n(s)\rangle ds.$$

Applying the chain rule to $e^{\phi(t)}\|v^n(t)\|^2$, we reach

$$e^{\phi(t)}\|v^n(t)\|^2+2(1-\varepsilon)\int_{t_i}^t e^{\phi(s)}\|v^n(s)\|_2^2ds$$

$$=e^{\phi(t_i^-)}\|\eta^n(t_i^-)\|^2-2\int_{t_i}^t e^{\phi(s)}\langle B(v^n(s),v^n(s)),Av^n(s)\rangle ds+\int_{t_i}^t \phi'(s)\|v^n(s)\|^2e^{\phi}ds.$$

Using Hölder's inequality and interpolation inequality, we deduce that

$$\begin{aligned}&|\langle B(v^n(s),v^n(s)),Av^n(s)\rangle|\\ \leq& C|Av^n||v^n|^{\frac{1}{2}}\|v^n\|\|v^n\|_2^{\frac{1}{2}}+C|Av^n|\|v^n\||r^n|^{\frac{1}{2}}\|r^n\|^{\frac{1}{2}}\\ \leq&\frac{(1-\varepsilon)}{2}\|v^n\|_2^2+C_1(|v^n|^2\|v^n\|^2+|r^n|\|r^n\|)\|v^n\|^2.\end{aligned}$$

Then, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N)}e^{\phi(t)}\|v^n(t)\|^2+(1-\varepsilon)\int_{t_i\wedge\tau_n^N}^{t_{i+1}\wedge\tau_n^N}e^{\phi(s)}\|v^n(s)\|_2^2ds\right)\\ \leq&\mathbb{E}(e^{\phi(t_i^-\wedge\tau_n^N)}\|\eta^n(t_i^-\wedge\tau_n^N)\|^2)+2C_1\mathbb{E}\int_{t_i\wedge\tau_n^N}^{t_{i+1}\wedge\tau_n^N}e^{\phi(s)}(|v^n|^2\|v^n\|^2+|r^n|\|r^n\|)\|v^n\|^2ds\\ &+\int_{t_i\wedge\tau_n^N}^{t_{i+1}\wedge\tau_n^N}\phi'(s)\|v^n(s)\|^2e^{\phi}ds.\end{aligned}\tag{4.34}$$

For $t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N]$, set

$$\phi(t)\overset{\text{def}}{=}-C_1\int_{t_i^-\wedge\tau_n^N}^t(|v^n|^2\|v^n\|^2+|r^n|\|r^n\|)ds,$$

where C_1 is the constant appeared in (4.34).

Based on the previous estimates and by Grönwall's inequality, we deduce that

$$\mathbb{E}\left(\sup_{t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N)}e^{\phi(t)}\|v^n(t)\|^2\right)\leq\mathbb{E}(\|\eta^n(t_i^-\wedge\tau_n^N)\|^2).\tag{4.35}$$

Since $e^{\phi(t_{i+1}\wedge\tau_n^N)}\geq e^{-C_1\frac{N}{n}}$ $\mathbb{P}-a.s.$, we deduce from (4.35) that

$$\mathbb{E}\left(\sup_{t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N)}\|v^n(t)\|^2\right)\leq\mathbb{E}(\|\eta^n(t_i^-\wedge\tau_n^N)\|^2)e^{C_1\frac{N}{n}}.\tag{4.36}$$

Applying Itô's formula to (4.2), by Hypothesis B, we have for $t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N)$,

$$\begin{aligned}&\mathbb{E}\|\eta^n(t)\|^2+\varepsilon(2-R_2)\mathbb{E}\int_{t_i\wedge\tau_n^N}^t\|\eta^n(s)\|_2^2ds\\ \leq&\mathbb{E}\|v^n(t_{i+1}^-\wedge\tau_n^N)\|^2+\frac{R_0T}{n}+R_1\int_{t_i\wedge\tau_n^N}^t\mathbb{E}|\eta^n(s)|^2ds.\end{aligned}$$

When $R_2<2$, we can ignore the $V-$norm. Then, by (4.1) and Grönwall's inequality, we get

$$\sup_{t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N)}\mathbb{E}\|\eta^n(t)\|^2\leq\left(\mathbb{E}\|v^n(t_{i+1}^-\wedge\tau_n^N)\|^2+\frac{R_0T}{n}\right)\mathrm{e}^{\frac{R_1T}{n}}.\tag{4.37}$$

Putting (4.36) into (4.37), we deduce that

$$\sup_{t\in[t_i\wedge\tau_n^N,t_{i+1}\wedge\tau_n^N)}\mathbb{E}\|\eta^n(t)\|^2\le\Big(\mathbb{E}\|\eta^n(t_i^-\wedge\tau_n^N)\|^2+\frac{R_0T}{n}\Big)e^{\frac{(C_1N+R_1T)}{n}}.$$

Set $\tilde{r}_5=C_1N+R_1T,\tilde{r}_6=R_0T$, by the induction argument, we have for $i=0,\cdots,n-1$,

$$\begin{aligned}&\mathbb{E}(\sup_{t_i\wedge\tau_n^N\le t<t_{i+1}\wedge\tau_n^N}\|v^n(t)\|^2)\vee(\sup_{t_i\wedge\tau_n^N\le t<t_{i+1}\wedge\tau_n^N}\mathbb{E}\|\eta^n(t)\|^2)\\&\le\mathbb{E}\|v_0\|^2e^{(i+1)\frac{\tilde{r}_5}{n}}+\frac{\tilde{r}_6}{n}\sum_{j=1}^{i+1}e^{j\frac{\tilde{r}_5}{n}}.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}&\Big[\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}(\sup_{d_n(t)\wedge\tau_n^N\le s<d_n^*(t)\wedge\tau_n^N}\|v^n(s)\|^2)\Big]\vee\Big[\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}\|\eta^n(t)\|^2\Big]\\&\le\mathbb{E}\|v_0\|^2e^{\tilde{r}_5}+\frac{\tilde{r}_6}{\tilde{r}_5}e^{2\tilde{r}_5}.\end{aligned}\tag{4.38}$$

Exactly following the same procedure as Lemma 4.1, we can obtain the result. ∎

Similarly to Lemma 4.1, it gives that

LEMMA 4.6. *Let $v_0\in V$ be $\mathcal{F}_0-$measurable random variable. Fix $\varepsilon\in[0,1)$. Let Hypotheses A–C hold with $K_2<\frac{2}{3}$, $K_4<2$ and $L_2<\frac{2}{2p-1}$, $R_2<\frac{2}{2p-1}$ for some $p\ge 2$. Then there exists a positive constant $C=C(\varepsilon,T,\mathbb{E}\|v_0\|^2,K_2,K_4,L_2,R_2)$ such that for every integer $n\ge 1$,*

$$\begin{aligned}&\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}\Big(\|\eta^n(t)\|^{2p}+\sup_{s\in[d_n(t)\wedge\tau_n^N,d_n^*(t)\wedge\tau_n^N)}\|v^n(s)\|^{2p}\Big)\\&+\mathbb{E}\int_0^{T\wedge\tau_n^N}\|v^n(s)\|_2^2\|v^n(s)\|^{2(p-1)}ds\le C\tilde{K}(N),\end{aligned}\tag{4.39}$$

where $\tilde{K}(N)$ is the same as Lemma 4.5.

Moreover, if $\varepsilon\in(0,1)$, we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^{T\wedge\tau_n^N}\|\eta^n(s)\|_2^2\|\eta^n(s)\|^{2(p-1)}ds\le C\tilde{K}(N).\tag{4.40}$$

Up to now, we are ready to obtain an upper bound of the $V-$norm of the difference between v^n and η^n.

PROPOSITION 4.2. *Let $v_0\in V$ be $\mathcal{F}_0-$measurable random variable. Fix $\varepsilon\in[0,1)$. Assume Hypotheses A–C hold with $K_2<\frac{2}{3}$, $K_4<2$ and $L_2<\frac{2}{3}$, $R_2<\frac{2}{3}$, there exists a positive constant C such that for any $n\in\mathbb{N}$,*

$$\mathbb{E}\int_0^{T\wedge\tau_n^N}\|v^n(t)-\eta^n(t)\|^2dt\le\frac{C(T)\tilde{K}(N)}{n}.\tag{4.41}$$

Proof. **Case 1:** $\varepsilon=0$. For any $t\in[0,T\wedge\tau_n^N)$, by (4.2) and Hypothesis C, we have

$$\mathbb{E}\|\eta^n(t)-v^n(d_n^*(t))\|^2=\mathbb{E}\int_{d_n(t)}^t\|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;V)}ds\le\mathbb{E}\int_{d_n(t)}^t(R_0+R_1\|\eta^n(s)\|^2)ds.$$

Then, by Fubini's theorem and Lemma 4.1, we obtain

$$\begin{aligned}&\mathbb{E}\int_0^{T\wedge\tau_n^N}\|\eta^n(t)-v^n(d_n^*(t))\|^2dt\\ \leq& C\mathbb{E}\int_0^{T\wedge\tau_n^N}(1+\|\eta^n(s)\|^2)\left(\int_s^{d_n^*(s)}dt\right)ds\\ \leq& C\frac{T}{n}.\end{aligned}\tag{4.42}$$

From (4.1), we have

$$\|v^n(d_n^*(t)^-)-v^n(t)\|^2=2\int_t^{d_n^*(t)}\langle A(v^n(s)-v^n(t)),dv^n(s)\rangle=\sum_{i=1}^2 I_i(t),$$

where

$$\begin{aligned}I_1(t)&=-2(1-\varepsilon)\int_t^{d_n^*(t)}\langle A(v^n(s)-v^n(t)),Av^n(s)\rangle ds,\\ I_2(t)&=-2\int_t^{d_n^*(t)}\langle A(v^n(s)-v^n(t)),B(v^n(s),v^n(s))\rangle ds,\end{aligned}$$

Using Lemma 4.5 and the Young's inequality, we have

$$\begin{aligned}&\left|\mathbb{E}\int_0^{T\wedge\tau_n^N}I_1(t)dt\right|\\ =&\left|(1-\varepsilon)\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_t^{d_n^*(t)}(-2|Av^n(s)|^2+2|Av^n(s)||Av^n(t)|)dsdt\right|\\ \leq&\left|(1-\varepsilon)\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_t^{d_n^*(t)}(-2|Av^n(s)|^2+2|Av^n(s)|^2+\frac{1}{2}|Av^n(t)|^2)dsdt\right|\\ \leq&\frac{1-\varepsilon}{2}\mathbb{E}\int_0^{T\wedge\tau_n^N}|Av^n(t)|^2\left(\int_t^{d_n^*(t)}ds\right)dt\leq\frac{C\tilde{K}(N)T}{n}.\end{aligned}$$

Using Hölder's inequality, interpolation inequality and Young's inequality, we obtain

$$\begin{aligned}&|\langle Av^n(s),B(v^n(s),v^n(s))\rangle|\leq\frac{1}{4}|Av^n(s)|^2+C(|v^n|^2\|v^n\|^2+|r^n|\|r^n\|)\|v^n\|^2,\\ &|\langle Av^n(t),B(v^n(s),v^n(s))\rangle|\leq\frac{1}{4}|Av^n(t)|^2+\frac{1}{4}|Av^n(s)|^2+C(|v^n|^2\|v^n\|^2+|r^n|\|r^n\|)\|v^n\|^2,\end{aligned}$$

Hence, by Lemmas 4.1, 4.4, 4.5 and 4.6, we deduce that

$$\begin{aligned}&\left|\mathbb{E}\int_0^{T\wedge\tau_n^N}I_2(t)dt\right|\\ \leq&\frac{1}{2}\mathbb{E}\int_0^{T\wedge\tau_n^N}|Av^n(s)|^2\left(\int_{d_n(s)}^s dt\right)ds+\frac{1}{4}\left|\mathbb{E}\int_0^{T\wedge\tau_n^N}|Av^n(t)|^2\left(\int_t^{d_n^*(t)}ds\right)dt\right|\\ &+C\left|\mathbb{E}\int_0^{T\wedge\tau_n^N}(|v^n|^2\|v^n\|^4+|r^n|^2\|r^n\|^2+\|v^n\|^4)\left(\int_{d_n(s)}^s dt\right)ds\right|\end{aligned}$$

$$\leq \frac{C\tilde{K}(N)T}{n}.$$

Therefore, based on the above, we conclude that (4.41) holds when $\varepsilon=0$.

Case 2: $\varepsilon\in(0,1)$. We have

$$\begin{aligned}
&\mathbb{E}\int_0^{T\wedge\tau_n^N}\|\eta^n(t)-v^n(t)\|^2dt\\
=&-2(1-\varepsilon)\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_t^{d_n^*(t)}\langle Av^n(s),A(\eta^n(s)-v^n(s))\rangle dsdt\\
&-2\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_t^{d_n^*(t)}\langle B(v^n(s),v^n(s)),A(\eta^n(s)-v^n(s))\rangle dsdt\\
&-2\varepsilon\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_{d_n(t)}^{t}\langle A(\eta^n(s)),A(\eta^n(s)-v^n(s))\rangle dsdt\\
&+\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_{d_n(t)}^{t}\|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;V)}dsdt\\
&\overset{\text{def}}{=}J_1+J_2+J_3+J_4.
\end{aligned}$$

Note that

$$\begin{aligned}
\langle Au,A(y-u)\rangle&=\langle A(u-y),A(y-u)\rangle+\langle Ay,A(y-u)\rangle\\
&=-|A(y-u)|^2+\langle Ay,A(y-u)\rangle\leq\langle Ay,A(y-u)\rangle, &(4.43)\\
2\langle Au,A(y-u)\rangle&\leq\langle Au,A(y-u)\rangle+\langle Ay,A(y-u)\rangle\\
&=\langle A(y+u),A(y-u)\rangle\leq\langle Ay,Ay\rangle. &(4.44)
\end{aligned}$$

By (4.44) and Fubini's theorem, we have

$$\begin{aligned}
J_1&\leq(1-\varepsilon)\mathbb{E}\int_0^{T\wedge\tau_n^N}|A\eta^n(s)|^2\left(\int_{d_n(s)}^{s}dt\right)ds\\
&\leq\frac{(1-\varepsilon)T}{n}\mathbb{E}\int_0^{T\wedge\tau_n^N}|A\eta^n(s)|^2ds\leq\frac{C\tilde{K}(N)(1-\varepsilon)T}{n}. \qquad (4.45)
\end{aligned}$$

Similar to the above, we have

$$J_2\leq C\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_t^{d_n^*(t)}(|Av^n(s)|^2+|A\eta^n(s)|^2+|v^n|^2\|v^n\|^4+|r^n|^2\|r^n\|^2+\|v^n\|^4)dsdt.$$

With the help of Fubini's theorem, Lemma 4.2, we get

$$\begin{aligned}
J_2&\leq C\mathbb{E}\int_0^{T\wedge\tau_n^N}(|Av^n(s)|^2+|A\eta^n(s)|^2+|v^n|^2\|v^n\|^4+|r^n|^2\|r^n\|^2+\|v^n\|^4)\left(\int_{d_n(s)}^{s}dt\right)ds\\
&\leq\frac{C\tilde{K}(N)T}{n}. \qquad (4.46)
\end{aligned}$$

From Lemma 4.5, it's easy to obtain

$$J_3\leq C\varepsilon\mathbb{E}\int_0^{T\wedge\tau_n^N}\int_{d_n(t)}^{t}(|Av^n(s)|^2+|A\eta^n(s)|^2)dsdt$$

$$\leq C\mathbb{E}\int_0^{T\wedge\tau_n^N}(|Av^n(s)|^2+|A\eta^n(s)|^2)\left(\int_{d_n(s)}^s dt\right)ds$$
$$\leq \frac{C\tilde{K}(N)T}{n}. \tag{4.47}$$

We deduce from Hypothesis C, Lemma 4.1 and Lemma 4.5 that

$$J_4\leq \mathbb{E}\int_0^{T\wedge\tau_n^N}\int_{d_n(t)}^t(R_0+R_1\|\eta^n(s)\|^2+\varepsilon R_2|A\eta^n(s)|^2)dsdt$$
$$\leq C\frac{R_0T}{n}+C\mathbb{E}\int_0^{T\wedge\tau_n^N}(R_1\|\eta^n(s)\|^2+\varepsilon R_2|A\eta^n(s)|^2)\left(\int_{d_n(s)}^s dt\right)ds$$
$$\leq \frac{C\tilde{K}(N)T}{n}. \tag{4.48}$$

Combining (4.45)-(4.48), we complete the proof of (4.41) when $\varepsilon\in(0,1)$. □

4.3. Auxiliary process. For technical reasons, consider an auxiliary process $Z^n(t),t\in[0,T]$ defined by

$$Z^n(t)=v_0-\int_0^t F_\varepsilon(s,v^n(s))ds-\varepsilon\int_0^{d_n(t)}A\eta^n(s)ds+\int_0^t\psi(s,\eta^n(s))dW(s).$$

When $\varepsilon=0$, we have

$$Z^n(t_k)=\eta^n(t_k^-)=v^n(t_k^+)\quad\text{for}\quad k=0,1,\cdots,n.$$

The following lemma gives an estimate of the difference between Z^n and v^n in different topologies.

LEMMA 4.7. *Let $v_0\in V$ be $\mathcal{F}_0-$measurable random variable. Fix $\varepsilon\in[0,1)$.*

(i) *Suppose that Hypotheses A–B hold with $K_2<\frac{2}{2p-1}$, $K_4<2$ and $L_2<\frac{2}{3}$, $R_2<\frac{2}{2p-1}$. Then there exists a positive constant $C=C(T,\varepsilon,\mathbb{E}|v_0|^{2p})$ such that for every integer $n\geq 1$,*

$$\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}|Z^n(t)-v^n(t)|^{2p}\leq\frac{C\tilde{K}(N)}{n^p}. \tag{4.49}$$

Moreover, if $L_2=0$, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}|\partial_z(Z^n(t)-v^n(t))|^{2p}\leq\frac{C}{n^p}. \tag{4.50}$$

(ii) *Assume that Hypothesis A–C hold with $K_2<\frac{2}{3}$, $K_4<2$ and $L_2<\frac{2}{3}$, $R_2<2$. Then there exists a positive constant $C=C(T,\varepsilon,\mathbb{E}\|v_0\|^{2p})$ such that for every integer $n\geq 1$,*

$$\mathbb{E}\int_0^{T\wedge\tau_n^N}\|Z^n(t)-v^n(t)\|^2dt\leq\frac{C\tilde{K}(N)}{n}.$$

Moreover, if $L_2<\frac{2}{2p-1}$ and $R_2=0$, we have

$$\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}\|Z^n(t)-v^n(t)\|^{2p}\leq\frac{C\tilde{K}(N)}{n^p}.$$

Proof. For $t\in[0,T\wedge\tau_n^N]$, we have

$$Z^n(t)-v^n(t)=\int_{d_n(t)}^t \psi(s,\eta^n(s))dW(s).$$

(i) Applying the Burkholder-Davies-Gundy inequality, Hypothesis A, Lemma 4.2 and Lemma 4.6, we obtain

$$\begin{aligned}
&\mathbb{E}|Z^n(t)-v^n(t)|^{2p}\\
&\le C_p\mathbb{E}|\int_{d_n(t)}^t \|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;H)}ds|^p\\
&\le C_p(\frac{T}{n})^{p-1}\mathbb{E}|\int_{d_n(t)}^t |K_0+K_1|\eta^n(s)|^2+\varepsilon K_2\|\eta^n(s)\|^2|^p ds\\
&\le \frac{C_p(T)}{n^p}\left(K_0^p+K_1^p\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}|\eta^n(t)|^{2p}+\varepsilon^p K_2^p\sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}\|\eta^n(t)\|^{2p}\right)\\
&\le \frac{C_p(T)\tilde{K}(N)}{n^p}.
\end{aligned}$$

Note that

$$\partial_z(Z^n(t)-v^n(t))=\int_{d_n(t)}^t \partial_z\psi(s,\eta^n(s))dW(s).$$

When $L_2=0$, using Hypothesis B and Lemma 4.4, we deduce that

$$\begin{aligned}
\mathbb{E}|\partial_z(Z^n(t)-v^n(t))|^{2p}&\le C_p\mathbb{E}|\int_{d_n(t)}^t \|\partial_z\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;H)}ds|^p\\
&\le C_p(\frac{T}{n})^{p-1}\mathbb{E}|\int_{d_n(t)}^t |L_0+L_1|\partial_z\eta^n(s)|^2|^p ds\\
&\le \frac{C_p(T)}{n^p}(L_0^p+L_1^p\sup_{t\in[0,T]}\mathbb{E}|\partial_z\eta^n(t)|^{2p})\le \frac{C_p(T)}{n^p}.
\end{aligned}$$

(ii) With the aid of Hypothesis C, the Burkholder-Davies-Gundy inequality, the Fubini's theorem and Lemmas 4.1, 4.5, we get

$$\begin{aligned}
&\mathbb{E}\int_0^{T\wedge\tau_n^N}\|Z^n(t)-v^n(t)\|^2dt\\
&\le \int_0^{T\wedge\tau_n^N}\mathbb{E}\int_{d_n(t)}^t \|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;V)}dsdt\\
&\le \mathbb{E}\int_0^T (R_0+R_1\|\eta^n(s)\|^2+\varepsilon R_2|A\eta^n(s)|^2)\Big(\int_s^{d_n^*(s)}dt\Big)ds\\
&\le \frac{T}{n}[R_0T+R_1\mathbb{E}\int_0^{T\wedge\tau_n^N}\|\eta^n(s)\|^2ds+\varepsilon R_2\mathbb{E}\int_0^{T\wedge\tau_n^N}|A\eta^n(s)|^2ds]\\
&\le \frac{C(T)\tilde{K}(N)}{n}.
\end{aligned}$$

If $R_2=0$, by Hypothesis C and Lemma 4.6, it gives

$$\begin{aligned}\mathbb{E}\|Z^n(t)-v^n(t)\|^{2p} &\le C_p\mathbb{E}|\int_{d_n(t)}^t \|\psi(s,\eta^n(s))\|^2_{\mathcal{L}_2(U;V)}ds|^p \\ &\le C_p(\frac{T}{n})^{p-1}\mathbb{E}|\int_{d_n(t)}^t |R_0+R_1\|\eta^n(s)\|^2|^p ds \\ &\le \frac{C_p(T)}{n^p}(R_0^p+R_1^p \sup_{t\in[0,T\wedge\tau_n^N]}\mathbb{E}\|\eta^n(t)\|^{2p}) \\ &\le \frac{C_p(T)\tilde{K}(N)}{n^p}.\end{aligned}$$

□

From Propositions 4.1, 4.2 and Lemma 4.7, we deduce that

Corollary 4.1. *There exists a positive constant $C=C(T,\varepsilon)$ such that for every integer $n\ge 1$,*

$$\mathbb{E}\int_0^{T\wedge\tau_n^N} |Z^n(t)-\eta^n(t)|^2dt \le \frac{C\tilde{K}(N)}{n},$$
$$\mathbb{E}\int_0^{T\wedge\tau_n^N} \|Z^n(t)-\eta^n(t)\|^2dt \le \frac{C\tilde{K}(N)}{n}.$$

5. Speed of convergence

In this section, we devote to prove Theorem 1.1.

For the strong solution v of (2.12), v^n of (4.1), r^n of (4.17) and some $M>0$, define the stopping time

$$\varsigma_n^M \overset{\text{def}}{=} \inf\Big\{t\in[0,T]: \int_0^t (\|v(s)\|+\|v^n(s)\|^2+|r^n|^4)ds>M\Big\}.$$

Let $\tau \overset{\text{def}}{=} \varsigma_n^M\wedge\tau_n^N$, where τ_n^N is defined by (4.31).

The following proposition states that the strong speed of convergence of Z^n to v (resp. v^n and η^n to v) in $L^\infty([0,T\wedge\tau];H)$ (resp. $L^\infty([0,T\wedge\tau];V)$) is $\frac{1}{2}$.

Proposition 5.1. *Let $v_0\in V$ be $\mathcal{F}_0-$measurable random variable. For any $\varepsilon\in[0,1)$, assume Hypotheses A–C hold with $K_2<\frac{2}{147}$, $L_2=R_2=0$ and εK_4 strictly smaller than $2(1-\varepsilon)$, then there exists positive constant $C(T)$ such that for every $M>0$ and $n\in\mathbb{N}$, we have*

$$\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau]}|Z^n(t)-v(t)|^2+\int_0^{T\wedge\tau}\Big(\|v^n(t)-v(t)\|^2+\|\eta^n(t)-v(t)\|^2\Big)dt\right) \le\frac{K(T,M,N)}{n}, \tag{5.1}$$

where

$$K(T,M,N)=C(T)\tilde{K}(N)\exp\{C(T)\mathrm{e}^{C(b_0)M}\},\quad C(b_0) \text{ is a positive constant,}$$
and $\tilde{K}(N)=\frac{1}{N}e^{C(T)N}$.

Proof. Fix $M>0$ and $n\geq 1$. Then for any $t\in[0,T]$, we have

$$Z^n(t\wedge\tau)-v(t\wedge\tau)=-\int_0^{t\wedge\tau}[F_\varepsilon(s,v^n(s))-F(s,v(s))]ds-\varepsilon\int_0^{d_n(t\wedge\tau)}A\eta^n(s)ds$$
$$+\int_0^{t\wedge\tau}[\psi(s,\eta^n(s))-\psi(s,v(s))]dW(s).$$

Applying Itô's formula to $|Z^n(t\wedge\tau)-v(t\wedge\tau)|^2$, we get

$$|Z^n(t\wedge\tau)-v(t\wedge\tau)|^2=\sum_{i=1}^5 J_i(t),$$

where

$$J_1(t)\overset{\text{def}}{=}-2\int_0^{t\wedge\tau}\langle F_\varepsilon(s,v^n(s))-F_\varepsilon(s,v(s)),Z^n(s)-v(s)\rangle ds,$$
$$J_2(t)\overset{\text{def}}{=}-2\varepsilon\int_0^{d_n(t\wedge\tau)}\langle A\eta^n(s)-Av(s),Z^n(s)-v(s)\rangle ds,$$
$$J_3(t)\overset{\text{def}}{=}-2\varepsilon\int_{d_n(t\wedge\tau)}^{t\wedge\tau}\langle Av(s),Z^n(s)-v(s)\rangle ds,$$
$$J_4(t)\overset{\text{def}}{=}\int_0^{t\wedge\tau}\|\psi(s,\eta^n(s))-\psi(s,v(s))\|^2_{\mathcal{L}_2(U;H)}ds,$$
$$J_5(t)\overset{\text{def}}{=}2\int_0^{t\wedge\tau}\langle[\psi(s,\eta^n(s))-\psi(s,v(s))]dW(s),Z^n(s)-v(s)\rangle.$$

Using (2.11), $J_1(t)$ can be rewritten as

$$\begin{aligned}J_1(t)=&-2(1-\varepsilon)\int_0^{t\wedge\tau}\langle Av^n(s)-Av(s),v^n(s)-v(s)\rangle ds\\&-2(1-\varepsilon)\int_0^{t\wedge\tau}\langle Av^n(s)-Av(s),Z^n(s)-v^n(s)\rangle ds\\&-2\int_0^{t\wedge\tau}\langle B(v^n(s)-v(s),v^n(s)),v^n(s)-v(s)\rangle ds\\&-2\int_0^{t\wedge\tau}\langle[B(v^n(s)-v(s),v^n(s))+B(v(s),v^n(s)-v(s))],Z^n(s)-v^n(s)\rangle ds\\&\overset{\text{def}}{=}J_{1,1}(t)+J_{1,2}(t)+J_{1,3}(t)+J_{1,4}(t).\end{aligned}$$

Referring to pages 21-23 in [2], the following estimates hold:

$$J_{1,1}(t)=-2(1-\varepsilon)\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds,$$
$$J_{1,2}(t)\leq b_0(1-\varepsilon)\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+\frac{1-\varepsilon}{b_0}\int_0^{t\wedge\tau}\|Z^n(s)-v^n(s)\|^2ds.$$

Using (2.11) and Young's inequality, we obtain

$$J_{1,3}(t)\leq 2\int_0^{t\wedge\tau}|\langle B(v^n(s)-v(s),v^n(s)),v^n-v(s)\rangle|ds$$

$$
\begin{aligned}
&\leq 2C\int_0^{t\wedge\tau}(\|v^n(s)\||v^n(s)-v(s)|\|v^n(s)-v(s)\|\\
&\quad+|\partial_z v^n|\|v^n(s)-v(s)\|^{\frac{3}{2}}|v^n(s)-v(s)|^{\frac{1}{2}})ds\\
&\leq b_0\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(b_0)\int_0^{t\wedge\tau}(\|v^n(s)\|^2+|\partial_z v^n|^4)|Z^n(s)-v(s)|^2ds\\
&\quad+C(b_0)\int_0^{t\wedge\tau}(\|v^n(s)\|^2+|\partial_z v^n|^4)|Z^n(s)-v^n(s)|^2ds.
\end{aligned}
$$

$J_{1,4}(t)$ can be rewritten as

$$
\begin{aligned}
J_{1,4}(t)=&-2\int_0^{t\wedge\tau}\langle B(v^n(s)-v(s),v^n(s)),Z^n(s)-v^n(s)\rangle ds\\
&-2\int_0^{t\wedge\tau}\langle B(v(s),v^n(s)-v(s)),Z^n(s)-v^n(s)\rangle ds\\
&\overset{\text{def}}{=}\tilde{J}_1(t)+\tilde{J}_2(t).
\end{aligned}
$$

Using (2.9) and Young's inequality, we get

$$
\begin{aligned}
&\tilde{J}_1(t)\\
&\leq 2\int_0^{t\wedge\tau}(\|v^n(s)\|^{\frac{3}{4}}|Z^n(s)-v^n(s)|^{\frac{1}{2}}\|Z^n(s)-v^n(s)\|^{\frac{1}{2}}\|v^n(s)\|^{\frac{1}{4}}|v^n(s)-v(s)|^{\frac{1}{2}}\\
&\quad\times\|v^n(s)-v(s)\|^{\frac{1}{2}}+|\partial_z v^n|\|v^n(s)-v(s)\||Z^n(s)-v^n(s)|^{\frac{1}{2}}\|Z^n(s)-v^n(s)\|^{\frac{1}{2}})ds\\
&\leq C\int_0^{t\wedge\tau}(\|v^n(s)\|^{\frac{3}{2}}|Z^n(s)-v^n(s)|\|Z^n(s)-v^n(s)\|\\
&\quad+\|v^n(s)\|^{\frac{1}{2}}|v^n(s)-v(s)|\|v^n(s)-v(s)\|+\frac{b_0}{2}\|v^n(s)-v(s)\|^2\\
&\quad+C(b_0)|\partial_z v^n|^4|Z^n(s)-v^n(s)|^2+C(b_0)\|Z^n(s)-v^n(s)\|)ds\\
&\leq b_0\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(b_0)\int_0^{t\wedge\tau}\|v^n(s)\||Z^n(s)-v(s)|^2ds\\
&\quad+C(b_0)\int_0^{t\wedge\tau}(1+\|v^n(s)\|^3+|\partial_z v^n|^4)|Z^n(s)-v^n(s)|^2ds\\
&\quad+C(b_0)\int_0^{t\wedge\tau}\|Z^n(s)-v^n(s)\|^2ds.
\end{aligned}
$$

We deduce from (2.9) and (2.11) that

$$
\begin{aligned}
\tilde{J}_2(t)\leq&2\int_0^{t\wedge\tau}|\langle B(v(s),Z^n(s)-v^n(s)),v^n(s)-v(s)\rangle|ds\\
\leq&2\int_0^{t\wedge\tau}\|Z^n(s)-v^n(s)\||v(s)|^{\frac{1}{2}}\|v(s)\|^{\frac{1}{2}}|v^n(s)-v(s)|^{\frac{1}{2}}\|v^n(s)-v(s)\|^{\frac{1}{2}}ds\\
&+2\int_0^{t\wedge\tau}|\partial_z(Z^n(s)-v^n(s))|\|v(s)\||v^n(s)-v(s)|^{\frac{1}{2}}\|v^n(s)-v(s)\|^{\frac{1}{2}}ds\\
\overset{\text{def}}{=}&\tilde{J}_{2,1}+\tilde{J}_{2,2}.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$
\tilde{J}_{2,1}(t)\leq\int_0^{t\wedge\tau}(b_0\|v^n(s)-v(s)\|^2+C|v(s)|\|v(s)\|^{\frac{1}{2}}\|Z^n(s)-v^n(s)\|^2
$$

$$
\begin{aligned}
&+C(b_0)\|v(s)\||v^n(s)-v(s)|^2)ds\\
\le & b_0\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(b_0)\int_0^{t\wedge\tau}\|v(s)\||Z^n(s)-v(s)|^2ds\\
&+C(b_0)\int_0^{t\wedge\tau}\|v(s)\||Z^n(s)-v^n(s)|^2ds\\
&+C\int_0^{t\wedge\tau}(|v(s)|^2+\|v(s)\|)\|Z^n(s)-v^n(s)\|^2ds.
\end{aligned}
$$

By the Hölder inequality and Young's inequality, we deduce that

$$
\begin{aligned}
\tilde{J}_{2,2}(t)\le & \int_0^{t\wedge\tau}(|\partial_z(Z^n-v^n)|^2\|v(s)\|^{\frac{3}{2}}+\|v(s)\|^{\frac{1}{2}}|v^n(s)-v(s)|\|v^n(s)-v(s)\|)ds\\
\le & \int_0^{t\wedge\tau}\Big(b_0\|v^n(s)-v(s)\|^2+|\partial_z(Z^n-v^n)|^2\|v(s)\|^{\frac{3}{2}}\\
&+C(b_0)\|v(s)\||v^n(s)-v(s)|^2\Big)ds\\
\le & b_0\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(b_0)\int_0^{t\wedge\tau}(1+\|v(s)\|)|Z^n(s)-v(s)|^2ds\\
&+C(b_0)\int_0^{t\wedge\tau}(1+\|v(s)\|)|Z^n(s)-v^n(s)|^2ds+C\int_0^{t\wedge\tau}|\partial_z(Z^n-v^n)|^2\|v(s)\|^{\frac{3}{2}}ds.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
J_{1,4}(t)\le & 3b_0\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(b_0)\int_0^{t\wedge\tau}(1+\|v(s)\|+\|v^n(s)\|)|Z^n(s)-v(s)|^2ds\\
&+C(b_0)\int_0^{t\wedge\tau}(1+\|v^n(s)\|^3+|\partial_z v^n|^4+\|v(s)\|)|Z^n(s)-v^n(s)|^2ds\\
&+C(b_0)\int_0^{t\wedge\tau}(1+|v(s)|^2+\|v(s)\|)\|Z^n(s)-v^n(s)\|^2ds\\
&+C\int_0^{t\wedge\tau}\|v(s)\|^{\frac{3}{2}}|\partial_z(Z^n-v^n)|^2ds.
\end{aligned}
$$

Replacing v by v^n, and using the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$
\begin{aligned}
J_2(t)\le & -2\varepsilon\int_0^{d_n(t\wedge\tau)}\|v^n(s)-v(s)\|^2ds+2\varepsilon\int_0^{d_n(t\wedge\tau)}\|\eta^n(s)-v^n(s)\|\|Z^n(s)-v^n(s)\|ds\\
&+2\varepsilon\int_0^{d_n(t\wedge\tau)}\|v^n(s)-v(s)\|(\|\eta^n(s)-v^n(s)\|+\|Z^n(s)-v^n(s)\|)ds\\
\le & C(\varepsilon)\int_0^{d_n(t\wedge\tau)}\|\eta^n(s)-v^n(s)\|^2ds+C(\varepsilon)\int_0^{d_n(t\wedge\tau)}\|Z^n(s)-v^n(s)\|^2ds.
\end{aligned}
$$

We deduce from the Cauchy-Schwarz inequality and Young's inequality that

$$
\begin{aligned}
J_3(t)\le & 2\varepsilon\int_{d_n(t\wedge\tau)}^{t\wedge\tau}\|v(s)\|(\|Z^n(s)-v^n(s)\|+\|v^n(s)-v(s)\|)ds\\
\le & b_0\varepsilon\int_{d_n(t\wedge\tau)}^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(\varepsilon)\int_{d_n(t\wedge\tau)}^{t\wedge\tau}\|v^n(s)-v(s)\|\|Z^n(s)-v^n(s)\|ds
\end{aligned}
$$

$$
\begin{aligned}
&+C(\varepsilon)\int_{d_n(t\wedge\tau)}^{t\wedge\tau}(\|v^n(s)\|\|Z^n(s)-v^n(s)\|+\|v^n(s)\|\|v^n(s)-v(s)\|)ds\\
\le&\, 2b_0\varepsilon\int_{d_n(t\wedge\tau)}^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds+C(\varepsilon)\int_{d_n(t\wedge\tau)}^{t\wedge\tau}\|Z^n(s)-v^n(s)\|^2ds\\
&+C(\varepsilon)\frac{T}{n}\sup_{d_n(t\wedge\tau)\le s\le t\wedge\tau}\|v^n(s)\|^2.
\end{aligned}
$$

Using Hypothesis A, we obtain

$$
\begin{aligned}
J_4(t)\le&\int_0^{t\wedge\tau}(K_3|\eta^n(s)-v(s)|^2+\varepsilon K_4\|\eta^n(s)-v(s)\|^2)ds\\
\le&\, 2K_3\int_0^{t\wedge\tau}|Z^n(s)-v(s)|^2ds+\varepsilon K_4 b_0\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds\\
&+2K_3\int_0^{t\wedge\tau}|\eta^n(s)-Z^n(s)|^2ds+\varepsilon C\int_0^{t\wedge\tau}\|\eta^n(s)-v^n(s)\|^2ds.
\end{aligned}
$$

Choosing $b_0>0$ satisfies

$$2(1-\varepsilon)-b_0(1-\varepsilon)-3b_0-2b_0\varepsilon-\varepsilon K_4 b_0>\alpha>0, \text{ for some } \alpha>0.$$

For $t\in[0,T]$, let

$$X(t)\overset{\text{def}}{=}\sup_{s\in[0,t\wedge\tau]}|Z^n(s)-v(s)|^2,\quad Y(t)\overset{\text{def}}{=}\int_0^{t\wedge\tau}\|v^n(s)-v(s)\|^2ds.$$

Then,

$$X(t)+\alpha Y(t)\le\int_0^{t\wedge\tau}\Theta_1(s)X(s)ds+\Theta_2(t),$$

where the processes are defined as follows:

$$
\begin{aligned}
\Theta_1(s)&\overset{\text{def}}{=}C(b_0)(1+\|v(s)\|+\|v^n(s)\|^2+|r^n|^4),\\
\Theta_2(t)&\overset{\text{def}}{=}\sup_{s\in[0,t\wedge\tau]}|J_5(s)|+I(t),\\
I(t)&\overset{\text{def}}{=}C(b_0)\int_0^{t\wedge\tau}(1+\|v^n(s)\|^3+|\partial_z v^n|^4+\|v(s)\|)|Z^n(s)-v^n(s)|^2ds\\
&+C(b_0)\int_0^{t\wedge\tau}(1+|v|^2+\|v\|)\|Z^n(s)-v^n(s)\|^2ds\\
&+C\int_0^{t\wedge\tau}|\partial_z(Z^n-v^n)|^2\|v(s)\|^{\frac{3}{2}}ds\\
&+C(\varepsilon)\frac{T}{n}\sup_{t\wedge\tau\le s\le d_n(t\wedge\tau)}\|v^n(s)\|^2+2K_3\int_0^{t\wedge\tau}|\eta^n(s)-Z^n(s)|^2ds\\
&+\varepsilon C\int_0^{t\wedge\tau}\|\eta^n(s)-v^n(s)\|^2ds.
\end{aligned}
$$

The definition of τ implies that

$$\int_0^{\tau}\Theta_1(s)ds\le C(b_0)(T+M)\overset{\text{def}}{=}C_0,\ \mathbb{P}-a.s..$$

By the Burkholder-Davies-Gundy inequality, Hypothesis A, Proposition 4.2 and Corollary 4.1, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau}|J_5(s)|\Big)\\
=&C\mathbb{E}\Big(\int_0^{t\wedge\tau}\|\psi(s,\eta^n(s))-\psi(s,v(s))\|^2_{\mathcal{L}_2(U;H)}|Z^n(s)-v(s)|^2ds\Big)^{\frac{1}{2}}\\
\le&\beta\mathbb{E}\big(\sup_{0\le s\le t\wedge\tau}|Z^n(s)-v(s)|^2\big)+C(\beta)\mathbb{E}\int_0^{t\wedge\tau}(K_3|\eta^n(s)-v(s)|^2+\varepsilon K_4\|\eta^n(s)-v(s)\|^2)ds\\
\le&\beta\mathbb{E}\big(\sup_{0\le s\le t\wedge\tau}|Z^n(s)-v(s)|^2\big)+C(\beta)K_3\int_0^{t\wedge\tau}\mathbb{E}|Z^n(s)-v(s)|^2ds\\
&+C(\beta)K_3\int_0^{t\wedge\tau}\mathbb{E}|\eta^n(s)-Z^n(s)|^2ds+C(\beta)\varepsilon K_4\int_0^{t\wedge\tau}\mathbb{E}\|\eta^n(s)-v^n(s)\|^2ds\\
&+C(\beta)\varepsilon K_4\int_0^{t\wedge\tau}\mathbb{E}\|v^n(s)-v(s)\|^2ds\\
\le&\beta\mathbb{E}X(t)+C(\beta)K_3\int_0^t\mathbb{E}X(s)ds+C(\beta)\varepsilon K_4\mathbb{E}Y(t)+\frac{C(T)\tilde{K}(N)}{n},
\end{aligned}$$

where $\beta>0$ will be chosen later. Using Theorem 3.1 and Lemmas 4.1, 4.4, 4.5, 4.6, we have

$$\begin{aligned}
\mathbb{E}I(t)\le&C(b_0)T^{\frac{1}{2}}\Big(\mathbb{E}\sup_{s\in[0,t\wedge\tau]}|Z^n(s)-v^n(s)|^4\Big)^{\frac{1}{2}}\\
&\times\Big[\mathbb{E}\int_0^{t\wedge\tau}(1+\|v^n(s)\|^6+|r^n(s)|^8+\|v(s)\|^2)ds\Big]^{\frac{1}{2}}\\
&+C(b_0)T^{\frac{1}{2}}\Big(\mathbb{E}\sup_{s\in[0,t\wedge\tau]}\|Z^n(s)-v^n(s)\|^4\Big)^{\frac{1}{2}}\Big[\mathbb{E}\int_0^{t\wedge\tau}(1+|v(s)|^4+\|v(s)\|^2)ds\Big]^{\frac{1}{2}}\\
&+C(T)\Big(\mathbb{E}\sup_{s\in[0,t\wedge\tau]}|\partial_z(Z^n(s)-v^n(s))|^8\Big)^{\frac{1}{4}}\Big(\mathbb{E}\int_0^{t\wedge\tau}\|v(s)\|^2ds\Big)^{\frac{3}{4}}\\
&+C(\varepsilon)\frac{T}{n}\mathbb{E}\sup_{d_n(t\wedge\tau)\le s\le t\wedge\tau}\|v^n(s)\|^2+2K_3\mathbb{E}\int_0^{t\wedge\tau}|\eta^n(s)-Z^n(s)|^2ds\\
&+\varepsilon C\mathbb{E}\int_0^{t\wedge\tau}\|\eta^n(s)-v^n(s)\|^2ds\le\frac{C(T)\tilde{K}(N)}{n}.
\end{aligned}$$

Choosing $\beta>0$ such that

$$2\beta(1+C_0\mathrm{e}^{C(b_0)M})\le1,$$

then suppose K_4 is small enough to ensure that

$$C(\beta)\varepsilon K_4(1+C_0\mathrm{e}^{C(b_0)M})\le\frac{\alpha}{4}.$$

Then, using similar argument as Lemma 3.9 in [8], we deduce that

$$X(t)+\frac{\alpha}{2}Y(t)\le[I(t)+\sup_{0\le s\le t\wedge\tau}|M(s)|](1+C_0\mathrm{e}^{C(b_0)M}).$$

Taking expectation and by estimates of $\mathbb{E}I(t)$, we obtain

$$\begin{aligned}&\mathbb{E}X(T)+\frac{\alpha}{4}\mathbb{E}Y(T)\\ \leq& 2\frac{C(T)\tilde{K}(N)}{n}(1+C_0\mathrm{e}^{C(b_0)M})+C(\beta)K_3(1+C_0\mathrm{e}^{C(b_0)M})\int_0^t \mathbb{E}X(s)ds.\end{aligned}$$

Applying the Grönwall inequality, we have

$$\mathbb{E}X(T)+\frac{\alpha}{4}\mathbb{E}Y(T)\leq 2\frac{C(T)\tilde{K}(N)}{n}(1+C_0\mathrm{e}^{C(b_0)M})\cdot\exp\Big\{C(\beta)K_3T(1+C_0\mathrm{e}^{C(b_0)M})\Big\}.$$

where $C(T),C_0,C(b_0),C(\beta)$ is independent of n.

Finally, with the aid of Proposition 4.2, we have

$$\begin{aligned}\mathbb{E}\int_0^{T\wedge\tau}\|\eta^n(t)-v(t)\|^2dt&\leq\mathbb{E}\int_0^{T\wedge\tau}\|\eta^n(t)-v^n(t)\|^2dt+\mathbb{E}Y(T)\\ &\leq\frac{C(T)\tilde{K}(N)}{n}\exp\Big\{C(T)\mathrm{e}^{C(b_0)M}\Big\}.\end{aligned}$$

We complete the proof. □

REMARK 5.1. As explained in the introduction, the index of $\|v\|$ appeared in $I(t)$ has to be strictly less than 2. Otherwise, $\mathbb{E}I(t)$ can not be controlled because of the lack of uniform $V-$norm estimates of v.

For every $M=M(n)>0, N=N(n)>0$, $t\in[0,T]$ and any integer $n\geq 1$, let

$$\begin{aligned}\Omega_n^{M,N}(t)\overset{\text{def}}{=}\Big\{\omega\in\Omega:&\sup_{i=0,\cdots,n-1}\int_{t_i\wedge t}^{t_{i+1}\wedge t}(|v^n(s)|^2\|v^n(s)\|^2+|r^n(s)|\|r^n(s)\|)ds\leq\frac{N}{n}\\ &\textit{and}\int_0^t(\|v(s)\|+\|v^n(s)\|^2+|r^n|^4)ds\leq M\Big\}.\end{aligned}$$

THEOREM 5.1. *Under the same conditions as Proposition 5.1, we have*

$$\mathbb{E}\left[I_{\Omega_n^{M,N}(t)}\sup_{k=0,\cdots,n}\Big(|v^n(t_k^+)-v(t_k)|+|\eta^n(t_k^+)-v(t_k)|\Big)\right]\leq\frac{K(M,N,T)}{n},\tag{5.2}$$

$$\mathbb{E}\left[I_{\Omega_n^{M,N}(t)}\int_0^t(\|v^n(s)-v(s)\|^2+\|\eta^n(s)-v(s)\|^2)ds\right]\leq\frac{K(M,N,T)}{n}.\tag{5.3}$$

where $K(M,N,T)=C(T)\tilde{K}(N)\exp\Big\{C(T)\mathrm{e}^{C(b_0)M}\Big\}$, $\tilde{K}(N)=\frac{1}{N}e^{C(T)N}$.

Proof. On $\Omega_n^{M,N}(t)$, we have $\tau\geq t$. With the aid of Proposition 5.1, we deduce that (5.3) holds. For (5.2), by the Hölder inequality and Lemma 4.6, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{k=0,\cdots,n}|Z^n(t_k\wedge\tau)-\eta^n(t_k^-\wedge\tau)|^2\right)&=\mathbb{E}\left(\sup_{k=0,\cdots,n}\varepsilon^2|\int_{t_k\wedge\tau}^{t_{k+1}\wedge\tau}A\eta^n(s)ds|^2\right)\\ &\leq\varepsilon^2\frac{T}{n}\mathbb{E}\left(\sup_{k=0,\cdots,n}\int_{t_k\wedge\tau}^{t_{k+1}\wedge\tau}|A\eta^n(s)|^2ds\right)\end{aligned}$$

$$\leq \frac{C(T)\tilde{K}(N)\varepsilon^2}{n}.$$

In view of $Z^n(t_k)=v^n(t_k^+)=\eta^n(t_k^-)$, we deduce from Proposition 5.1 that

$$\mathbb{E}\Big[I_{\Omega_n^{M,N}(T)}\sup_{k=0,\cdots,n}(|v^n(t_k^+)-v(t_k)|^2+|\eta^n(t_k^-)-v(t_k)|^2)\Big]\leq \frac{K(M,N,T)}{n}. \tag{5.4}$$

Using Hypothesis A and Lemma 4.5, for $k=0,\cdots,n-1$, we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|\eta^n(t)-\eta^n(t_k^+)|^2\right)\\
&\leq \mathbb{E}\left[\int_{t_k\wedge\tau}^{t_{k+1}\wedge\tau}(\frac{\varepsilon}{2}\|\eta^n(t_k^+)\|^2+K_0+K_1|\eta^n(s)|^2+\varepsilon K_2\|\eta^n(s)\|^2)ds\right]\\
&\quad+\mathbb{E}\left[\int_{t_k\wedge\tau}^{t_{k+1}\wedge\tau}|\eta^n(s)-\eta^n(t_k^+)|^2(K_0+K_1|\eta^n(s)|^2+\varepsilon K_2\|\eta^n(s)\|^2)ds\right]^{\frac{1}{2}}\\
&\leq \frac{1}{2}\mathbb{E}\left(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|\eta^n(t)-\eta^n(t_k^+)|^2\right)+C\frac{T}{n}\sup_{s\in[0,T\wedge\tau]}\mathbb{E}\|\eta^n(s)\|^2.
\end{aligned}$$

It follows that

$$\mathbb{E}\left(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|\eta^n(t)-\eta^n(t_k^+)|^2\right)\leq \frac{C(T)\tilde{K}(N)}{n}.$$

Using Hypothesis A and Hypothesis C, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|v(t)-v(t_k^+)|^2\Big)\\
\leq&\mathbb{E}\Big(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}\int_{t_k}^t\Big[|\langle v(s)-v(t_k^+),Av(s)\rangle|+|\langle v(s)-v(t_k^+),B(v(s),v(s))\rangle|\Big]ds\Big)\\
&+\mathbb{E}\Big(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}\int_{t_k}^t\Big[(R_0+R_1|v(s)|)|v(s)-v(t_k^+)|\\
&+K_0+K_1|v(s)|^2+\varepsilon K_2\|v(s)\|^2\Big]ds\Big)\\
&+\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}|v(t)-v(t_k^+)|^2(K_0+K_1|v(s)|^2+\varepsilon K_2\|v(s)\|^2)ds\Big)^{\frac{1}{2}}\\
\leq&\mathbb{E}\Big[\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}\int_{t_k}^t\Big(-2\|v(s)\|^2+2\|v(s)\|\|v(t_k^+)\|+|v(s)|_4^2\|v(s)-v(t_k^+)\|\\
&+|\partial_z(v(s)-v(t_k^+))|\|v(s)\|^{\frac{3}{2}}|v(s)|^{\frac{1}{2}}\Big)ds\Big]\\
&+\mathbb{E}\Big(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}\int_{t_k}^t\Big[(R_0+R_1|v(s)|)|v(s)-v(t_k^+)|\\
&+K_0+K_1|v(s)|^2+\varepsilon K_2\|v(s)\|^2\Big]ds\Big)\\
&+\mathbb{E}\Big(\int_{t_k\wedge\tau}^{t_{k+1}\wedge\tau}|v(s)-v(t_k^+)|^2(K_0+K_1|v(s)|^2+\varepsilon K_2\|v(s)\|^2)ds\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{1}{2}\mathbb{E}(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|v(t)-v(t_k^+)|^2)$$
$$+\frac{C}{n}\Big(1+\sup_{t\in[0,T\wedge\tau)}\mathbb{E}\|v(t)\|^2+\sup_{t\in[0,T\wedge\tau)}\mathbb{E}(|\partial_z v(t)|^8+|v(t)|^4)\Big).$$

Hence, by Lemmas 4.2, 4.4, 4.5, we get

$$\mathbb{E}\left(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|v(t)-v(t_k^+)|^2\right)\leq \frac{C(T)}{n}.$$

Using Lemma 4.6, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[t_k\wedge\tau,t_{k+1}\wedge\tau)}|v^n(t)-v^n(t_k^+)|^2\right)\\ &\leq \frac{C}{n}\Big(1+\sup_{t\in[0,T\wedge\tau)}\mathbb{E}(\|v^n\|^4+\|\eta^n\|^4)\Big)\\ &\leq \frac{C(T)\tilde{K}(N)}{n}.\end{aligned}$$

We complete the proof. ∎

For any $n\geq 1$, define the error term

$$e_n(T)\stackrel{\text{def}}{=}\sup_{k=0,\cdots,n}\Big(|v^n(t_k^+)-v(t_k)|+|\eta^n(t_k^-)-v(t_k)|\Big)$$
$$+\left(\int_0^T\|v^n(s)-v(s)\|^2ds\right)^{\frac{1}{2}}+\left(\int_0^T\|\eta^n(s)-v(s)\|^2ds\right)^{\frac{1}{2}}.$$

Now, we can prove the strong speed of the convergence in probability.

Proof. **(Proof of Theorem 1.1)**. Fix a sequence $l(n)\to\infty$, as $n\to\infty$. Let $M(n)=\ln(\ln(\ln(l(n))))$, $N(n)=\ln(\ln(l(n)))$, then $M(n)\to\infty$ and $N(n)\to\infty$. Note that

$$\begin{aligned}&\mathbb{P}\Big((\Omega_n^{M(n),N(n)})^c(T)\Big)\\ &\leq \mathbb{P}\Big(\sup_{i=0,\cdots,n-1}\int_{t_i\wedge T}^{t_{i+1}\wedge T}(|v^n(s)|^2\|v^n(s)\|^2+|r^n(s)|\|r^n(s)\|)ds>\frac{N(n)}{n}\Big)\\ &\quad+\mathbb{P}\Big(\int_0^T(\|v(s)\|+\|v^n(s)\|^2+|r^n|^4)ds>M(n)\Big).\end{aligned}$$

Clearly, by Lemmas 4.2, 4.4, we have

$$\begin{aligned}&\mathbb{P}\Big(\sup_{i=0,\cdots,n-1}\int_{t_i\wedge T}^{t_{i+1}\wedge T}(|v^n(s)|^2\|v^n(s)\|^2+|r^n(s)|\|r^n(s)\|)ds>\frac{N(n)}{n}\Big)^c\\ &=\mathbb{P}\Big(\sup_{i=0,\cdots,n-1}\int_{t_i\wedge T}^{t_{i+1}\wedge T}(|v^n(s)|^2\|v^n(s)\|^2+|r^n(s)|\|r^n(s)\|)ds\leq\frac{N(n)}{n}\Big)\\ &\leq \mathbb{P}\Big(\int_0^T(|v^n(s)|^2\|v^n(s)\|^2+|r^n(s)|\|r^n(s)\|)ds\leq N(n)\Big)\\ &\to 1 \quad as \quad n\to\infty.\end{aligned}$$

Using Theorem 3.1 and Lemmas 4.1, 4.4, we obtain

$$\mathbb{P}\Big[\int_0^T (\|v(s)\|+\|v^n(s)\|^2+|r^n|^4)ds>M(n)\Big]\to 0 \qquad as \quad n\to\infty.$$

Hence, when $n\to\infty$,

$$\mathbb{P}\Big((\Omega_n^{M(n),N(n)})^c(T)\Big)\to 0. \tag{5.5}$$

Now, we deduce from Chebyshev's inequality and Theorem 5.1 that

$$\begin{aligned}
&\mathbb{P}\Big(e_n(T)\geq \frac{l(n)}{\sqrt{n}}\Big)\\
&\leq \mathbb{P}\Big((\Omega_n^{M(n),N(n)})^c(T)\Big)+\frac{n}{l^2(n)}\mathbb{E}\Big(I_{\Omega_n^{M(n),N(n)}(T)}e_n^2(T)\Big)\\
&\leq \mathbb{P}\Big((\Omega_n^{M(n),N(n)})^c(T)\Big)+C(T)\frac{1}{N(n)}e^{C(T)N(n)}\frac{n}{l^2(n)}\frac{1}{n}\exp\Big\{C(T)(\ln(\ln(l(n))))^{C(b_0)}\Big\}\\
&\leq \mathbb{P}\Big((\Omega_n^{M(n),N(n)})^c(T)\Big)+C(T)\frac{1}{N(n)}\frac{n}{l^2(n)}\frac{1}{n}\exp\Big\{C(T)(\ln(\ln(l(n))))^{C(b_0)\vee 1}\Big\}.
\end{aligned}$$

Since $C(T)(\ln(\ln(l(n))))^{C(b_0)\vee 1}-2\ln(l(n))\to-\infty$, we have

$$C(T)\frac{1}{N(n)}\frac{n}{l^2(n)}\frac{1}{n}\exp\Big\{C(T)(\ln(\ln(l(n))))^{C(b_0)\vee 1}\Big\}\to 0. \tag{5.6}$$

Combining (5.5) and (5.6), we get

$$\mathbb{P}\Big(e_n(T)\geq \frac{l(n)}{\sqrt{n}}\Big)\to 0, \quad as\ n\to\infty.$$

We complete the proof. □

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