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BOUNDARY LAYER ANALYSIS FOR THE FAST HORIZONTAL ROTATING FLUIDS*

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Abstract. It is well known that, for fast rotating fluids with the axis of rotation being perpendicular to the boundary, the boundary layer is of Ekman-type, described by a linear ODE system. In this paper we consider fast rotating fluids, with the axis of rotation being parallel to the boundary. We show that, for certain initial data with special asymptotic expansion, the corresponding boundary layer is described by a nonlinear, degenerated PDE system which is similar to the $2D$ Prandtl system. Finally, we prove the well-posedness of the governing system of the boundary layer in the space of analytic functions with respect to tangential variable.

Keywords. Incompressible Navier Stokes equation; boundary layer; rotating fluids.

AMS subject classifications. 35M13; 35Q30; 35Q35; 76U05.

1. Introduction

The incompressible Navier-Stokes equation coupled with a large Coriolis term reads

$$\begin{cases}\partial_t u^\varepsilon - \nu\Delta u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon + \dfrac{\omega\times u^\varepsilon}{\varepsilon} + \nabla p^\varepsilon = 0,\\ \mathrm{div}\ u^\varepsilon = 0,\\ u^\varepsilon|_{t=0} = u_0^\varepsilon,\end{cases}$$

with Dirichlet boundary condition, where $\frac{\omega\times u_\varepsilon}{\varepsilon}$ stands for the Coriolis force and ω is the rotation vector, ε^{-1} the rescaled speed of rotation, ν the viscosity coefficient. The above system is sufficient to describe the rotating fluids which is a significant part of geophysics. Due to the earth's self-rotation, we cannot neglect the Coriolis force in order to model the oceanography and meteorology dealing with large-scale magnitude. When the fluid is between a strip and the direction of rotation is not parallel to the boundary, we have the well-developed Ekman layers to match the interior flow with Dirichlet boundary condition, cf. [6, 7, 17, 26] and the references therein. The situation will be more complicated when the direction of rotation is parallel to the boundary, considering cylinder for instance and letting the fluid rotate around the vertical axis. Then we will have two types of boundaries, the perpendicular (with respect to the rotation axis, also called horizontal as in [7]) boundary layer which is Ekman layer and the parallel (also called vertical as in [7]) boundary layers for which much less is known, despite various studies [7, 35, 38]. We refer to [7] for detailed discussions on the problem of parallel (or vertical) boundary layers.

In this paper, we study the parallel boundary layers for the fast rotating viscous fluids with a certain class of well chosen initial data. We want to show the similarity of the governing equations for the parallel boundary layers, comparing to the classical

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two-dimensional Prandtl boundary layer system, and we prove the well-posedness of these parallel boundary layers in the space of analytic functions. As mentioned in [7,34] and [2], in the case where both parallel and perpendicular boundary layers exist, we have to take into account the interactions of these layers and also the effect of the domain's corners. Here, as a preliminary step we first consider the half-space case $\mathbb{R}^3_+ = \mathbb{R}^2_h \times \mathbb{R}_+$. The reason is to isolate the effect of the parallel boundary layers, since in this case, the Ekman perpendicular boundary layers do not exist. In the more complicate case, the interactions between parallel layers and Ekman layers will be studied in a forthcoming work. More precisely, we consider the following system

$$\begin{cases} \partial_t u^\varepsilon - \nu \Delta u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \dfrac{e_2 \times u^\varepsilon}{\varepsilon} + \nabla p^\varepsilon = 0 & \text{in } \mathbb{R}^2_h \times \mathbb{R}_+, \forall t \geq 0 \\ \text{div } u^\varepsilon = 0 & \text{in } \mathbb{R}^2_h \times \mathbb{R}_+, \forall t \geq 0 \\ u^\varepsilon|_{x_3=0} = 0 & \text{on } \mathbb{R}^2_h \\ u^\varepsilon|_{t=0} = u^\varepsilon_0, & \text{in } \mathbb{R}^2_h \times \mathbb{R}_+, \end{cases} \tag{N-S$_\varepsilon$}$$

where $e_2 = (0,1,0)$ is the unit horizontal vector, $\nu > 0$ the coefficient of viscosity of fluids and ε the Rossby number. We suppose the initial data admit the asymptotic expansion.

$$\begin{cases} u^\varepsilon_0(x_1,x_2,x_3) = \displaystyle\sum_{j=0}^{1} \varepsilon^{\frac{j}{2}} \left[u_0^{I,j}(x_1,x_2,x_3) + u_0^{B,j}\left(x_1,x_2,\frac{x_3}{\sqrt{\varepsilon}}\right)\right] + R_1(\varepsilon), \\ p^\varepsilon_0(x_1,x_2,x_3) = \displaystyle\sum_{j=-2}^{1} \varepsilon^{\frac{j}{2}} \left[p_0^{I,j}(x_1,x_2,x_3) + p_0^{B,j}\left(x_1,x_2,\frac{x_3}{\sqrt{\varepsilon}}\right)\right] + R_2(\varepsilon), \end{cases} \tag{1.1}$$

where $u_0^{I,j}$, $u_0^{B,j}$, $p_0^{I,j}$, $p_0^{B,j}$ will be determined later. In this paper we only consider the case when

$$R_j(\varepsilon) = \mathcal{O}\left(\varepsilon^2\right), \quad j = 1,2. \tag{1.2}$$

So we impose a jump from the order $\varepsilon^{\frac{1}{2}}$ to ε^2 in the above asymptotic expansion. Note that these initial data are quite special and they do not contain the orders ε^1 and $\varepsilon^{\frac{3}{2}}$. For example, we can consider such initial data that the remainder terms $R_1(\varepsilon)$ and $R_2(\varepsilon)$ in (1.1) vanish. For this class of initial data, we will explain in Section 2 how to close the limiting system as ε goes to zero, which enables the study of the parallel boundary layers.

These equations describe the evolution of an incompressible three-dimensional viscous fluid in a fast rotating frame of angular velocity ε^{-1}. According to the Taylor-Proudman theorem [37], the fast rotation penalizes the movement of the fluid in the direction of the rotation axis. As a consequence, the fluid has tendency to move in columns, parallel to the rotation axis, which are widely known as the Taylor columns. This phenomenon is well-known in oceanography and meteorology, which is observed in many large-scale atmospheric and oceanic flows. From a mathematical point of view, when ε goes to zero, the rotation term $\frac{e_2 \times u^\varepsilon}{\varepsilon}$ becomes large and can only be balanced by the pressure. This means that, if u is the (formal) limit of u^ε, as $\varepsilon \to 0$, then $e_2 \times u$ needs to be a gradient term, which implies that u is independent of x_2 (more explanations will be found in Section 2). In this paper, we will only consider the case where the initial data are well prepared, *i.e.* u_0^ε do not depend on x_2.

When there is no Coriolis force, the zero-viscosity limit for the Navier-Stokes equations for incompressible fluids in a domain with boundary, with non-slip boundary conditions, is a challenging problem due to the formation of a boundary layer which is governed by the Prandtl equations ([31]). The mathematical analysis theory of Prandtl equation is also a challenging problem, see [1, 10, 11, 14, 28] and references therein. Far from the boundary, the inviscid limit problem was treated by several authors; we can refer, for instance, to Swann [36] and Kato [22]. In another work, Kato [21] gives some equivalent formulations of this problem in the case of bounded domains, showing that the convergence to the Euler system is equivalent to the fact that the L^2 strength of the boundary layer goes to 0. Caflisch & Sammartino [33] solved the problem for analytic solutions on a half-space by solving the Prandtl equations via abstract Cauchy-Kowaleskaya theorem. We also refer to [15, 18, 25] and the references therein for the recent progress on the inviscid limit of the Navier-Stokes equations when the initial vorticity is located away from the boundary. On the other hand, another commonly used boundary conditions are Navier-type slip boundary conditions, in which case the vanishing viscosity limit is rigorously justified; cf. [24, 39–41] and references therein.

We want to say a few words to compare the system (N-S$_\varepsilon$) with the case where the rotation axis is perpendicular to the boundary (the rotation axis is in the direction of $e_3=(0,0,1)$ instead of e_2). If the domain considered is between two parallel plates ($\mathbb{T}^2\times[0,1]$ or $\mathbb{R}^2\times[0,1]$), it was proved in Grenier & Masmoudi [17], Masmoudi [26, 27] and Chemin *et al.* [6] that for the rotating fluids with anisotropic viscosity $-\nu\Delta_h-\varepsilon\partial_{x_3}^2$, all the weak solutions of Navier-Stokes equation converge to the solution of the 2D Euler or 2D Navier-Stokes system (with damping term - effect of the Ekman pumping). The vertical rotation and the specific form of the domain (between two parallel plates) permit to explicitly construct the boundary layer velocity term from the interior velocity term (which satisfies a 2D damped Euler system), without using the Prandtl equations. The case of fast rotating fluids around e_3 in the cylinder $\Omega\times[0,1]$ was studied by Bresch *et al.* in [2] where Ω is a bounded domain of $\mathbb{R}^2$. To avoid parallel boundary layers near $\partial\Omega\times[0,1]$, the authors considered anisotropic viscosity, where the horizontal viscosity is supposed to be fixed. Thus, the main difficulty in [2] is to construct "corrector" layers near the domain's corners $\partial\Omega\times\{0\}$ and $\partial\Omega\times\{1\}$. We also want to mention the work of Dalibard and Gérard-Varet [9] in the case of fast rotating fluids on a rough domain with non-slip boundary conditions. The boundary layer is also proved to be of size ε (contrary to the case of Prandtl equations where the boundary layer is of size $\sqrt{\varepsilon}$). We also refer to a series of work for the rotating fluids with anisotopic viscosity (see for example [4, 5, 12, 13, 16, 20, 29, 30]).

We want to emphasize that the formation of the boundary layers in the case where the rotation axis is perpendicular to the boundary is due to the incompatibility of the Dirichlet boundary conditions with the columnar movement of the limit fluid (as $\varepsilon\to 0$). Indeed, as the rotation axis is e_3, the limiting velocity of the fluid is independent of x_3, and so, the Dirichlet boundary conditions imply that the limit velocity should be zero. This incompatibility leads to the fact that a thin layer (Ekman's layer) is formed near the boundary, and the fluid's evolution is violent in this small scale zone, in a way that stops the fluid on the boundary.

In the case of horizontal rotation axis (in the direction of e_2), the incompatibility of boundary conditions will be more complicated, because of the fact that the limit velocity is independent of x_2 instead of x_3. In Section 2, we prove that the limit system is a 2D Euler-like system. This means that we are no longer in the case considered by Ekman. The techniques of [17] and [6] do not work and we can not explicitly calculate the boundary layer. The fast rotation only penalizes the fluid motion in the x_2 direction, and leads to a problem very close to the inviscid limit of two-dimensional Navier-Stokes system. It is then relevant to look for a boundary layer of size $\sqrt{\varepsilon}$ and we will show in Section 2 that in this boundary layer of size $\sqrt{\varepsilon}$, the fluid velocity actually satisfies a two-dimensional Prandtl-like system. Finally, we remark that in this paper, we only consider the case where $\nu=\varepsilon$. Indeed, as explained in [17] and also in [7], if the ratio ν/ε goes to infinity, the fluid rapidly stops after a few evolutions. It is then more interesting to consider the case where $\nu\lesssim\varepsilon$, which moreover better fits physical observations.

In this work, we study the formation of the boundary layer when $\nu=\varepsilon\to 0$. We suppose the existence of a boundary layer of size $\sqrt{\varepsilon}$ near the boundary $\{x_3=0\}$ of $\mathbb{R}^3_+$. Our goal is to derive the limit equation and the boundary layer equation by using a formal asymptotic expansion in the Section 2. We refer to the book of Pedlovsky [32] for more detail about this formal expansion. Let us recall that in this paper, we only consider a class of initial data in the form (1.1) with the assumption (1.2). Then, by continuity in short time, we suppose that the solution of (N-S$_\varepsilon$) also accepts the same asymptotic expansion

$$u^\varepsilon(t,x_1,x_2,x_3)=\sum_{j=0}^{1}\varepsilon^{\frac{j}{2}}\left[u^{I,j}(t,x_1,x_2,x_3)+u^{B,j}\left(t,x_1,x_2,\frac{x_3}{\sqrt{\varepsilon}}\right)\right]+\mathcal{O}\left(\varepsilon^2\right), \tag{1.3}$$

$$p^\varepsilon(t,x_1,x_2,x_3)=\sum_{j=-2}^{1}\varepsilon^{\frac{j}{2}}\left[p^{I,j}(t,x_1,x_2,x_3)+p^{B,j}\left(t,x_1,x_2,\frac{x_3}{\sqrt{\varepsilon}}\right)\right]+\mathcal{O}\left(\varepsilon^2\right), \tag{1.4}$$

where $u^{B,j}(t,x_1,x_2,y)$ and $p^{B,j}(t,x_1,x_2,y)$ exponentially go to zero as $y\stackrel{\text{def}}{=}\frac{x_3}{\sqrt{\varepsilon}}\to+\infty$. Remark that, similar to the initial data, we have a jump from the order $\varepsilon^{\frac{1}{2}}$ to ε^2. This assumption is very important for our study in order to close the limiting system. We emphasize that we only study the short-time existence of the boundary layer systems, so this hypothesis about the asymptotic expansion is reasonable with respect to the initial data of the form (1.1) with the condition (1.2). If we do not have this hypothesis and the expansion is continuous with all the orders from $\varepsilon^{\frac{1}{2}}$ to ε^2, the problem will be much more involved and we cannot obtain a closed limiting system. For more details, we refer to Remarks 2.2, 2.3 and 2.5.

Throughout this paper, we will always use ∂_t, ∂_i (or ∂_{x_i}), $i=1,2,3$, and ∂_y to respectively denote the derivatives with respect to the time variable t, the space variables x_i, $i=1,2,3$, and the boundary layer variable $y=\frac{x_3}{\sqrt{\varepsilon}}$. Using the same approach as in the case of 2D Prandtl equations, we present the new unknown functions

$$\begin{cases}\mathcal{U}_j^{p,0}=u_j^{B,0}+\overline{u_j^{I,0}}, & j=1,2\\ \mathcal{U}_3^{p,1}=u_3^{B,1}+\overline{u_3^{I,1}}+y\overline{\partial_3 u_3^{I,0}}\end{cases} \tag{1.5}$$

where $\overline{u^{I,k}},\overline{p^{I,k}},k=0,1$ are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli-type law

$$\begin{cases}\partial_t\overline{u_1^{I,0}}+\overline{u_1^{I,0}}\partial_1\overline{u_1^{I,0}}+\overline{\partial_1 p^{I,0}}=0\\ \partial_t\overline{u_2^{I,0}}+\overline{u_1^{I,0}}\partial_1\overline{u_2^{I,0}}+\overline{\partial_2 p^{I,0}}=0\\ \partial_t\overline{u_3^{I,1}}+\overline{u_1^{I,0}}\partial_1\overline{u_3^{I,1}}+\overline{u_3^{I,1}}\partial_3\overline{u_3^{I,0}}+\overline{\partial_3 p^{I,1}}=0\end{cases}$$

which is the restriction of the Euler system and linearized Euler system on the boundary $x_3=0$, so that they depend only on the variables (t,x_1). In Section 2, using the asymptotic expansions (1.3) and (1.4), we can deduce that the behavior of the fluid near the boundary is governed by the following 2D Prandtl-like system

$$
\begin{cases}
\partial_t \mathcal{U}_1^{p,0} - \partial_y^2 \mathcal{U}_1^{p,0} + \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_1^{p,0} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_1^{p,0} + \partial_1 p^{B,0} + \overline{\partial_1 p^{I,0}} = 0, \\
\partial_t \mathcal{U}_3^{p,1} - \partial_y^2 \mathcal{U}_3^{p,1} + \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_3^{p,1} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_3^{p,1} + \overline{\partial_3 p^{I,1}} + y\overline{\partial_3^2 p^{I,0}} = 0, \\
\partial_1 \mathcal{U}_1^{p,0} + \partial_y \mathcal{U}_3^{p,1} = 0, \\
\mathcal{U}_1^{p,0}|_{y=0} = 0, \quad \lim_{y\to+\infty} \mathcal{U}_1^{p,0}(t,x_1,y) = \overline{u_1^{I,0}}, \\
\mathcal{U}_3^{p,1}|_{y=0} = 0, \quad \lim_{y\to+\infty} \mathcal{U}_3^{p,1}(t,x_1,y) = \overline{u_3^{I,1}}, \\
(\mathcal{U}_1^{p,0}, \mathcal{U}_3^{p,1})|_{t=0} = (\mathcal{U}_{1,0}^{p,0}, \mathcal{U}_{3,0}^{p,1}),
\end{cases}
\tag{P1}
$$

with the unknown functions $\left(\mathcal{U}_1^{p,0}, \mathcal{U}_3^{p,1}, p^{B,0}\right)$, and the horizontal second component satisfies a parabolic-type equation

$$
\begin{cases}
\partial_t \mathcal{U}_2^{p,0} - \partial_y^2 \mathcal{U}_2^{p,0} + \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_2^{p,0} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_2^{p,0} = 0 \\
\mathcal{U}_2^{p,0}|_{y=0} = 0, \quad \lim_{y\to+\infty} \mathcal{U}_2^{p,0}(t,x_1,y) = \overline{u_2^{I,0}} \\
\mathcal{U}_2^{p,0}|_{t=0} = \mathcal{U}_{2,0}^{p,0}.
\end{cases}
\tag{P2}
$$

Here the Taylor columns are represented by the condition

$$\partial_2 \mathcal{U}_1^{p,0} = \partial_2 \mathcal{U}_2^{p,0} = \partial_2 \mathcal{U}_3^{p,1} = 0.$$

Let us make a few remarks concerning our model. At the first sight, one can say that (P1) is very similar to the 2D Prandtl system, *i.e.* as the Prandtl equation, the first equation in (P1) admits the same degeneracy in x_1 coupled with the nonlocal property arising from the term $\mathcal{U}_3^{p,1}\partial_y \mathcal{U}_1^{p,0}$. One can also say that the second equation of (P1) is redundant since we can obtain $\mathcal{U}_3^{p,1}$ from $\mathcal{U}_1^{p,0}$ using the incompressibility condition given in the third equation of (P1). That is not really true because the system (P1) is really a system of 3 equations with both the velocity $(\mathcal{U}_1^{p,0}, \mathcal{U}_3^{p,1})$ and the boundary pressure $p^{B,0}$ to be determined. This unknown pressure $p^{B,0}$ is the crucial difference between Prandtl equation and the first equation in (P1). We recall that the pressure term in Prandtl equation comes from outflow and can be defined by the Bernoulli law, so that the pressure therein is a given function and Prandtl equation is a kind of degenerate parabolic equation. But here the situation is quite complicated since the unknown pressure $p^{B,0}$ in (P1) arises because of the fast rotation parallel to the boundary, and cannot be defined by the Bernoulli law. Thus, the classical theory for Prandtl equation is not directly applicable to our case and we cannot follow the same strategy as in Prandtl equation to treat the the first equation in (P1). In order to overcome this difficulty, the idea is to invert the method used to solve the Prandtl system: we first solve the second equation to find $\mathcal{U}_3^{p,1}$ and then, we can obtain $\mathcal{U}_1^{p,0}$ using the divergence-free condition (see Section 3 for details) and directly calculate the unknown pressure term $\partial_1 p^{B,0}$ using the first equation of (P1). Finally we mention that the mathematical justification of the inviscid limit for solutions to (N-S$_\varepsilon$), is also complicated as classical Prandtl boundary layer theory. We only concentrate in this work on the well-posedness of boundary layer and will investigate this inviscid limit problem in the future work.

We remark that there is no coupling between $(\mathcal{U}_1^{p,0},\mathcal{U}_3^{p,1})$ and $\mathcal{U}_2^{p,0}$ and so, we can separately solve the systems (P1) and (P2). Using the definition (1.5), our strategy can be expressed in the following steps:

(1) Find $u^{I,0}$ by solving the limiting system (1.6) (of order ε^0), which is a 2D Euler system, with three components.
(2) Find $u_3^{B,1}$ by solving the second equation of (P1) (see Sections 3 and 4).
(3) The value of $u_3^{B,1}$ on the boundary allows to determine the boundary condition of the limiting system 1.7 (of order $\varepsilon^{\frac{1}{2}}$), which is a 2D linearized Euler system, with three components.
(4) The first three steps give $\mathcal{U}_3^{p,1}$ and so, using the incompressibility condition, we can determine $\mathcal{U}_1^{p,0}$ and then calculate $p^{B,0}$.
(5) Solve the system (P2) to find $\mathcal{U}_2^{p,0}$.

In Section 2, we will prove that the limiting velocity of the outer flow satisfies the following 2D Euler-type system with three components, which is,

$$\begin{cases} \partial_t u_1^{I,0} + u_1^{I,0}\partial_1 u_1^{I,0} + u_3^{I,0}\partial_3 u_1^{I,0} + \partial_1 p^{I,0} = 0 \\ \partial_t u_2^{I,0} + u_1^{I,0}\partial_1 u_2^{I,0} + u_3^{I,0}\partial_3 u_2^{I,0} = 0 \\ \partial_t u_3^{I,0} + u_1^{I,0}\partial_1 u_3^{I,0} + u_3^{I,0}\partial_3 u_3^{I,0} + \partial_3 p^{I,0} = 0 \\ \partial_2 u_1^{I,0} = \partial_2 u_2^{I,0} = \partial_2 u_3^{I,0} = \partial_2 p^{I,0} = 0 \\ \partial_1 u_1^{I,0} + \partial_3 u_3^{I,0} = 0 \\ u_3^{I,0}|_{x_3=0} = 0 \\ u^{I,0}|_{t=0} = u_0^{I,0}(x_1,x_3). \end{cases} \tag{1.6}$$

In the system (1.6), the components $(u_1^{I,0},u_3^{I,0},p^{I,0})$ satisfy exactly a 2D incompressible Euler system on the half-plane, so that the existence and regularity in Gevery class of local-in-time solution is well known, (see Vicol [23] and references therein), but in the study of boundary layer equation, we need some weighted analytic function spaces on the tangential variables, as in Definition 1.1, and we cite in particular the results of [8].

The second step is much more difficult, which consists in the construction of boundary velocity $u_3^{B,1}$ - the main result of this paper. By simplification, using the equations satisfied by $u^{I,0}$ and $u^{I,1}$, we can deduce from (P1) that $u_3^{B,1}$ is solution of the system (see Section 3 for more details)

$$\begin{cases} \left(\partial_t - \partial_y^2 + \overline{\partial_3 u_3^{I,0}} y \partial_y\right) u_3^{B,1} + \left(u_1^{B,0} + \overline{u_1^{I,0}}\right) \partial_1 u_3^{B,1} \\ \qquad + \left(u_3^{B,1} + \overline{u_3^{I,1}}\right) \partial_y u_3^{B,1} + \overline{\partial_3 u_3^{I,0}} u_3^{B,1} + \left(-\partial_1 \overline{u_3^{I,1}} + y \overline{\partial_1 \partial_3 u_3^{I,0}}\right) u_1^{B,0} = 0, \\ \partial_1 u_1^{B,0} + \partial_y u_3^{B,1} = 0, \\ u_3^{B,1}|_{y=0} = \overline{u_3^{I,1}}, \ \lim_{y\to+\infty} u_3^{B,1} = 0, \\ u_3^{B,1}|_{t=0} = u_{3,0}^{B,1}. \end{cases}$$

Note that at this step, we do not know anything about $u^{I,1}$. This means that the boundary condition $u_3^{B,1}|_{y=0} = \overline{u_3^{I,1}}$ has no sense and we need to replace this by another boundary condition to complete the above system. Remark that the incompressibility condition gives $\partial_y u_3^{B,1} = -\partial_1 u_1^{B,0}$, then on the boundary, we have

$$\partial_y u_3^{B,1}|_{y=0} = -\partial_1 u_1^{B,0}|_{x_3=0} = \overline{u_1^{I,0}}.$$

The system for $u_3^{B,1}$ becomes

$$\begin{cases} \left(\partial_t - \partial_y^2 + \overline{\partial_3 u_3^{I,0}} y \partial_y\right) u_3^{B,1} + \left(u_1^{B,0} + \overline{u_1^{I,0}}\right) \partial_1 u_3^{B,1} + \left(u_3^{B,1} - u_3^{B,1}(t,x_1,0)\right) \partial_y u_3^{B,1} \\ \qquad\qquad + \overline{\partial_3 u_3^{I,0}} u_3^{B,1} + \left(\partial_1 u_3^{B,1}(t,x_1,0) + y \overline{\partial_1 \partial_3 u_3^{I,0}}\right) u_1^{B,0} = 0, \\ \partial_1 u_1^{B,0} + \partial_y u_3^{B,1} = 0, \\ \partial_y u_3^{B,1}|_{y=0} = \overline{u_1^{I,0}}, \quad \lim_{y\to+\infty} u_3^{B,1} = 0, \\ u_3^{B,1}|_{t=0} = u_{3,0}^{B,1}. \end{cases}$$

This system will be studied in Sections 3 and 4.

The third step consists in the study of the following linearized Euler system, which describes the evolution of the fluids in the interior part of the domain, far from the boundary, at the order $\sqrt{\varepsilon}$.

$$\begin{cases} \partial_t u_1^{I,1} + u_1^{I,0} \partial_1 u_1^{I,1} + u_3^{I,0} \partial_3 u_1^{I,1} + u_1^{I,1} \partial_1 u_1^{I,0} + u_3^{I,1} \partial_3 u_1^{I,0} + \partial_1 p^{I,1} = 0 \\ \partial_t u_2^{I,1} + u_1^{I,0} \partial_1 u_2^{I,1} + u_3^{I,0} \partial_3 u_2^{I,1} + u_1^{I,1} \partial_1 u_2^{I,0} + u_3^{I,1} \partial_3 u_2^{I,0} = 0 \\ \partial_t u_3^{I,1} + u_1^{I,0} \partial_1 u_3^{I,1} + u_3^{I,0} \partial_3 u_3^{I,1} + u_1^{I,1} \partial_1 u_3^{I,0} + u_3^{I,1} \partial_3 u_3^{I,0} + \partial_3 p^{I,1} = 0 \\ \partial_2 u_1^{I,1} = \partial_2 u_2^{I,1} = \partial_2 u_3^{I,1} = \partial_2 p^{I,1} = 0 \\ \partial_1 u_1^{I,1} + \partial_3 u_3^{I,1} = 0 \\ u_3^{I,1}|_{x_3=0} = -u_3^{B,1}(t,x_1,0) \\ u^{I,1}|_{t=0} = u_0^{I,1}(x_1,x_3). \end{cases} \tag{1.7}$$

We remark that the compatibility conditions ask

$$u_{3,0}^{I,1}(x_1,0) = -u_{3,0}^{B,1}(x_1,0).$$

It is exactly the non-slip condition of (N-S$_\varepsilon$) at order 1. Because of its linearity, treating the system (1.7) is still much easier than treating the system (1.6), even with the presence of the given boundary function $u_3^{B,1}(t,x_1,0)$. So, to prove Theorem 2.2, we can simply follow the lines of the proof of Theorem 2.1 as in [8].

Finally, the fourth and fifth steps of our study will be expressed at the end of Section 2 and at Section 5.

To resume, we prove the well-posedness results on (P1) and (P2) in the following weighted analytic function spaces in tangential variables.

Definition 1.1. *Let $\frac{1}{2}<\ell\leq 1$ be given throughout the paper. We denote by $\mathcal{A}_\tau$ the space of analytic functions with analytic radius $\tau>0$, which consists of all functions $f\in L^2(\mathbb{R}^2_+)$ such that*

$$\|f\|_{\mathcal{A}_\tau}\stackrel{\text{def}}{=}\sup_{|\alpha|\geq 0}\frac{\tau^{|\alpha|}}{|\alpha|!}\|\langle z\rangle^{\ell}\partial_z^{\alpha}f\|_{L^2(\mathbb{R}^2_+)}<+\infty.$$

Definition 1.2. *Let $1/2<\ell\leq 1$ be given. With each pair (ρ,a) with $\rho>0$ and $a>0$ we associate a space $X_{\rho,a}$ of all functions $u(x_1,y)\in H^\infty(\mathbb{R}_{x_1};\ H^2(\mathbb{R}_+))$ such that*

$$\sum_{\substack{m\leq 2\\ 0\leq j\leq 1}}\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^j u\|^2_{L^2(\mathbb{R}^2_+)}+\sum_{\substack{m\geq 3\\ 0\leq j\leq 1}}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^j u\|^2_{L^2(\mathbb{R}^2_+)}<+\infty,$$

where we use the convention $0!=1$. We endow $X_{\rho,a}$ with the norm

$$|u|^2_{X_{\rho,a}}=\sum_{\substack{m\leq 2\\ 0\leq j\leq 1}}\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^j u\|^2_{L^2(\mathbb{R}^2_+)}+\sum_{\substack{m\geq 3\\ 0\leq j\leq 1}}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^j u\|^2_{L^2(\mathbb{R}^2_+)}.$$

Here, we want to make a few comments about the $X_{\rho,a}$, which will be used to overcome the technical difficulties in order to obtain the uniform energy estimates for our model. Unlike the 2D Prandtl equations, in our model, we have to deal with terms that are linear in the normal variable y. To be able to balance this linear growth in y, we consider data that decay in y with a speed of e^{-y^2}. We remark that this kind of data is relevant for general boundary theories as mentioned in the work of Oleinik and Samokhin [31]. Another difficulty comes from the loss of one derivative in x_1 and the nonlocal character of the term $\mathcal{U}_1^{B,0}\partial_1\mathcal{U}_3^{B,1}$, which is very similar the 2D Prandtl equations. This difficulty logically leads to the consideration of an analytic norm with a certain weight in x_1, since we do not require the monotonicity of the data.

Our novelty in using this technique relies in the fact that the analytic bandwidth ρ and the auxiliary parameter a are time-depending functions that will be precisely controlled. The idea comes from the fact that if we differentiate, with respect to the time variable, a function of the type $\rho(t)e^{a(t)y^2}\Phi(t)$, we will obtain two additional "good terms", provided that $\rho(t)$ and $a(t)$ are well chosen. More precisely, we have

$$\frac{d}{dt}\Big(\rho(t)e^{a(t)y^2}\Phi(t)\Big)=\rho'(t)e^{a(t)y^2}\Phi(t)+a'(t)y^2\rho(t)e^{a(t)y^2}\Phi(t)+\rho(t)e^{a(t)y^2}\Phi'(t).$$

We remark that this technique was already used in the article of Chemin [3], where the author provides a new type of global existence result for Navier-Stokes equations in some new classes of data. Then, if we choose $a'(t)<0$ and $\rho'(t)<0$, in our energy-type estimates, these two additional terms provide some "smoothing effects", allowing one to absorb the linearly growing term (in y) and on another hand, adapt the abstract Cauchy-Kowalewski theorem to our system. For more details, we send the reader to Sections 3 and Section 4.

The well-posedness of the system (P1) can be stated as follows.

Theorem 1.1. *Suppose that the initial data*

$$\mathcal{U}_{3,0}^{p,1}=u_{3,0}^{B,1}+\overline{u_{3,0}^{I,1}}+y\overline{\partial_3 u_{3,0}^{I,0}}$$

in (P1) satisfies that

$$u_{3,0}^{B,1} \in X_{\rho_0,a_0}, \quad u_{3,0}^{I,1},\, u_{3,0}^{I,0} \in \mathcal{A}_{\tau_0}$$

for some $a_0 > 0$, $\rho_0 > 0$ *and* $\tau_0 > 0$ *and*

$$\mathcal{U}_{1,0}^{p,0}(x_1,y) = -\int_{-\infty}^{x_1} \partial_y u_{3,0}^{B,1}(z,y)dz + \overline{u_{1,0}^{I,0}}(x_1).$$

Then there exist $T>0$, $\tau>0$ *and a pair* (ρ,a) *with* $\rho,a>0$, *such that the system* (P1) *admits a unique solution* $(\mathcal{U}_1^{p,0},\mathcal{U}_3^{p,1},\partial_1 p^{B,0})$, *and moreover*

$$\begin{aligned}
&\mathcal{U}_3^{p,1} = u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}\\
&\mathcal{U}_1^{p,0}(t,x_1,y) = -\int_{-\infty}^{x_1} \partial_y u_3^{B,1}(t,z,y)dz + \overline{u_1^{I,0}}(t,x_1),
\end{aligned}$$

with $u_3^{B,1} \in L^\infty([0,T];\ X_{\rho,a})$ *and* $u_1^{I,0}, u_3^{I,0}, u_3^{I,1} \in L^\infty([0,T];\ \mathcal{A}_\tau)$.

REMARK 1.1. Here we consider the well prepared initial data, that is the initial data are independent of x_2.

Let $\mathcal{U}_1^{p,0},\mathcal{U}_3^{p,1}$ be the solutions to the system (P1) given by the theorem above. Then, (P2) is just a linear parabolic equation, and we have the following theorem concerned with its well-posedness.

THEOREM 1.2. *Let* $\rho_0>0$, $a_0>0$, $\tau_0>0$ *be given. For any initial data*

$$\mathcal{U}_{2,0}^{p,0} = u_{2,0}^{B,0} + \overline{u_{2,0}^{I,0}}$$

where $u_{2,0}^{B,0} \in X_{\rho_0,a_0}$ *and* $u_{2,0}^{I,0} \in \mathcal{A}_{\tau_0}$, *there exist* $T>0$, $0<\tau<\tau_0$ *and* $0<a<a_0$, *such that the Equation* (P2) *admits a unique solution* $\mathcal{U}_2^{p,0}$ *satisfying* $\mathcal{U}_2^{p,0} = u_2^{B,0} + \overline{u_2^{I,0}}$ *with*

$$u_2^{B,0} \in L^\infty([0,T],X_{\rho_0,a}), \quad u_2^{I,0} \in L^\infty([0,T],\mathcal{A}_\tau).$$

By the two above theorems, we obtain the well-posedness for the boundary layer equation of the system (N-S$_\varepsilon$) in the frame of analytic space in tangential variables.

The paper is organized as follows. In Section 2, we formally derive the governing equations of the outer flow inside the domain and the systems (P1) and (P2) which describe the fluid motion inside the boundary layer. The Sections 3-4 are devoted to proving the well-posedness of the system (P1). Finally, we give some brief ideas of the proof of Theorem 1.2 for the well-posedness of Equation (P2) in the Section 5.

2. Formal asymptotic expansion

First of all, we want to give a few words to explain our special choice of the order of the expansions of the velocity and the pressure. Indeed, we remark that as for the formulation of Prandtl boundary layer equations, we are only interested in the leading orders which are necessary to allow us to formally obtain the governing equations of the evolution of the boundary layer. By using the asymptotic expansions (1.3) and (1.4), we have the following asymptotic identities for the leading terms up to order $\varepsilon^{1/2}$ and all the remaining terms are of higher order in ε.

$$\begin{cases} \partial_t u^\varepsilon = \sum_{j=0}^{1} \varepsilon^{\frac{j}{2}} \left(\partial_t u^{I,j} + \partial_t u^{B,j} \right) + \mathcal{O}(\varepsilon^2) \\ -\varepsilon \Delta u^\varepsilon = -\partial_y^2 u^{B,0} - \varepsilon^{\frac{1}{2}} \partial_y^2 u^{B,1} - \sum_{j=0}^{1} \varepsilon^{1+\frac{j}{2}} \left(\Delta u^{I,j} + \Delta_h u^{B,j} \right) + \mathcal{O}(\varepsilon^2) \\ u^\varepsilon \cdot \nabla u^\varepsilon = \sum_{j=0}^{1} \varepsilon^{\frac{j-1}{2}} \left[\sum_{k=0}^{j} \left(u_3^{B,k} + u_3^{I,k} \right) \partial_y u^{B,j-k} \right] + \sum_{j=0}^{1} \varepsilon^{\frac{j}{2}} \left[\sum_{k=0}^{j} \left(u_h^{B,k} + u_h^{I,k} \right) \cdot \nabla_h u^{B,j-k} \right] \\ \qquad + \sum_{j=0}^{1} \varepsilon^{\frac{j}{2}} \left[\sum_{k=0}^{j} \left(u^{B,k} + u^{I,k} \right) \cdot \nabla u^{I,j-k} \right] + \mathcal{O}(\varepsilon) \\ \dfrac{e_2 \times u^\varepsilon}{\varepsilon} = \sum_{j=0}^{1} \varepsilon^{\frac{j}{2}-1} \left[\begin{pmatrix} u_3^{B,j} \\ 0 \\ -u_1^{B,j} \end{pmatrix} + \begin{pmatrix} u_3^{I,j} \\ 0 \\ -u_1^{I,j} \end{pmatrix} \right] + \mathcal{O}(\varepsilon) \\ \nabla p^\varepsilon = \varepsilon^{-\frac{3}{2}} \begin{pmatrix} 0 \\ 0 \\ \partial_y p^{B,-1} \end{pmatrix} + \sum_{j=-2}^{0} \varepsilon^{j} \begin{pmatrix} \partial_1 p^{B,j} \\ \partial_2 p^{B,j} \\ \partial_y p^{B,j+1} \end{pmatrix} + \varepsilon^{\frac{1}{2}} \begin{pmatrix} \partial_1 p^{B,1} \\ \partial_2 p^{B,1} \\ 0 \end{pmatrix} + \sum_{j=-2}^{1} \varepsilon^{\frac{j}{2}} \nabla p^{I,j} + \mathcal{O}(\varepsilon). \end{cases} \tag{2.1}$$

2.1. Formal derivation of the fluid behavior far from the boundary. We put all the asymptotic identities (2.1) into the system (N-S$_\varepsilon$) and we deduce that

$$\sum_{j=0}^{1} \varepsilon^{\frac{j}{2}} \partial_t u^{I,j} - \sum_{j=0}^{1} \varepsilon^{1+\frac{j}{2}} \Delta u^{I,j} + \sum_{j=0}^{1} \varepsilon^{\frac{j}{2}} \sum_{k=0}^{j} u^{I,k} \cdot \nabla u^{I,j-k} \\ + \sum_{j=0}^{1} \varepsilon^{\frac{j}{2}-1} \begin{pmatrix} u_3^{I,j} \\ 0 \\ -u_1^{I,j} \end{pmatrix} + \sum_{j=-2}^{1} \varepsilon^{\frac{j}{2}} \nabla p^{I,j} = \mathcal{O}(\varepsilon). \tag{2.2}$$

Taking the limit $y = \frac{x_3}{\sqrt{\varepsilon}} \to +\infty$ $(\varepsilon \to 0)$, the divergence-free property writes

$$\operatorname{div} u^{I,j} = 0, \qquad \forall j \geq 0. \tag{2.3}$$

At the leading term of ε^{-1} in (2.2), we simply have

$$\begin{pmatrix} u_3^{I,0} \\ 0 \\ -u_1^{I,0} \end{pmatrix} + \begin{pmatrix} \partial_1 p^{I,-2} \\ \partial_2 p^{I,-2} \\ \partial_3 p^{I,-2} \end{pmatrix} = 0. \tag{2.4}$$

Then, classical calculations (see Grenier-Masmoudi [17] or Chemin *et al.* [7]) give

$$\partial_2 p^{I,-2} = \partial_2 u_1^{I,0} = \partial_2 u_2^{I,0} = \partial_2 u_3^{I,0} = 0. \tag{2.5}$$

At the order $\varepsilon^{-1/2}$ in (2.2), we have

$$\begin{pmatrix} u_3^{I,1} \\ 0 \\ -u_1^{I,1} \end{pmatrix} + \begin{pmatrix} \partial_1 p^{I,-1} \\ \partial_2 p^{I,-1} \\ \partial_3 p^{I,-1} \end{pmatrix} = 0, \tag{2.6}$$

which imply

$$\partial_2 p^{I,-1} = \partial_2 u_1^{I,1} = \partial_2 u_2^{I,1} = \partial_2 u_3^{I,1} = 0. \tag{2.7}$$

REMARK 2.1. Identities (2.5) and (2.7) mean that the limit behaviour of the outer flow is two-dimensional, as predicts the Taylor-Proudman theorem.

At the order ε^0 in (2.2), taking into account (2.5) and the divergence-free condition (2.3), we obtain

$$\begin{cases} \partial_t u_1^{I,0} + u_1^{I,0}\partial_1 u_1^{I,0} + u_3^{I,0}\partial_3 u_1^{I,0} + \partial_1 p^{I,0} = 0 \\ \partial_t u_2^{I,0} + u_1^{I,0}\partial_1 u_2^{I,0} + u_3^{I,0}\partial_3 u_2^{I,0} + \partial_2 p^{I,0} = 0 \\ \partial_t u_3^{I,0} + u_1^{I,0}\partial_1 u_3^{I,0} + u_3^{I,0}\partial_3 u_3^{I,0} + \partial_3 p^{I,0} = 0 \\ \partial_2 u_1^{I,0} = \partial_2 u_2^{I,0} = \partial_2 u_3^{I,0} = 0 \\ \partial_1 u_1^{I,0} + \partial_3 u_3^{I,0} = 0 \end{cases} \tag{2.8}$$

Now, by applying ∂_2 to the second equation of the system (2.8), we obtain

$$\partial_2^2 p^{I,0} = 0,$$

which means that there exist $g_1(x_1,x_3)$ and $g_2(x_1,x_3)$ such that

$$p^{I,0} = x_2 g_1 + g_2.$$

Now, differentiating the first and third equations of (2.8) with respect to x_2, we obtain

$$\partial_1 g_1 = \partial_3 g_1 = 0.$$

By taking $|x| \to +\infty$ in the second equation of (2.8), we conclude that $g_1 \equiv 0$. Thus, the system (2.8) becomes the system (1.6), which is the following 2D Euler-type system with three components in the half-plane and which is the formal limiting system of (N-S$_\varepsilon$) far from the boundary as $\varepsilon \to 0$

$$\begin{cases} \partial_t u_1^{I,0} + u_1^{I,0}\partial_1 u_1^{I,0} + u_3^{I,0}\partial_3 u_1^{I,0} + \partial_1 p^{I,0} = 0 \\ \partial_t u_2^{I,0} + u_1^{I,0}\partial_1 u_2^{I,0} + u_3^{I,0}\partial_3 u_2^{I,0} = 0 \\ \partial_t u_3^{I,0} + u_1^{I,0}\partial_1 u_3^{I,0} + u_3^{I,0}\partial_3 u_3^{I,0} + \partial_3 p^{I,0} = 0 \\ \partial_2 u_1^{I,0} = \partial_2 u_2^{I,0} = \partial_2 u_3^{I,0} = \partial_2 p^{I,0} = 0 \\ \partial_1 u_1^{I,0} + \partial_3 u_3^{I,0} = 0 \\ u_3^{I,0}|_{x_3=0} = 0. \end{cases}$$

Since this system is independent of x_2, for the compatibility, we need to impose the well prepared initial data, which means that

$$u^{I,0}(0,x_1,x_3) = u_0^{I,0}(x_1,x_3).$$

The boundary condition will be discussed in (2.25).
The system (1.6) will be completed with a boundary condition for the second component $u_2^{I,0}$. In fact, the trace function $\overline{u_2^{I,0}}(t,x_1)$ on the boundary $\{x_3=0\}$ satisfies the following system

$$\begin{cases} \partial_t \overline{u_2^{I,0}} + \overline{u_1^{I,0}} \partial_1 \overline{u_2^{I,0}} = 0 \\ \overline{u_2^{I,0}}(0,x_1) = u_{0,2}^{I,0}(x_1,0). \end{cases}$$

REMARK 2.2. We want to recall that the hypothesis of a jump from the order $\varepsilon^{\frac{1}{2}}$ to ε^2 in the asymptotic expansion (1.1) is very important at this step. Indeed, if this hypothesis fails, then in the system (2.8) above, we have an additional term of the form $\begin{pmatrix} u_3^{I,2} \\ 0 \\ -u_1^{I,2} \end{pmatrix}$. In the case of Ekman boundary layers, this term turns out to be a dissipative term, known as the so-called Ekman pumping. However, in our model, this term cannot be determined and the system (2.8) cannot be closed.

Using Definition 1.1, we can obtain the following estimates, which are immediate consequences of the definition of $\|\cdot\|_{\mathcal{A}_\tau}$ and Sobolev inequalities. For $u_1^{I,0} \in L^\infty([0,T];\ \mathcal{A}_\tau)$, we have, for all $p,q \geq 0$,

$$\left\| \langle x_1 \rangle^\ell \partial_1^p \partial_3^q u_3^{I,0}(x_1,x_3) \right\|_{L^\infty\left(\mathbb{R}_+;\ L^2(\mathbb{R}_{x_1})\right)} \leq C \left\| u_3^{I,0} \right\|_{\mathcal{A}_\tau} \frac{(p+q+3)!}{\tau^{p+q+3}}. \tag{2.9}$$

Then, the equation

$$\partial_t u_1^{I,0} + u_1^{I,0} \partial_1 u_1^{I,0} + u_3^{I,0} \partial_3 u_1^{I,0} + \partial_1 p^{I,0} = 0,$$

and Leibniz formula give

$$\begin{aligned} &\left\| \langle x_1 \rangle^\ell \partial_t \partial_1^p \partial_3^q u_3^{I,0} \right\|_{L^\infty\left(\mathbb{R}_+;\ L^2(\mathbb{R}_{x_1})\right)} \\ &\quad \leq C_\tau \left(\left\| u_1^{I,0} \right\|_{\mathcal{A}_\tau}^2 + \left\| u_1^{I,0} \right\|_{\mathcal{A}_\tau} \left\| u_3^{I,0} \right\|_{\mathcal{A}_\tau} + \left\| p^{I,0} \right\|_{\mathcal{A}_\tau} \right) \frac{2^{p+q}(p+q)!}{\tau^{p+q}}. \end{aligned} \tag{2.10}$$

The construction of the components $(u_1^{I,0}, u_3^{I,0}, p^{I,0})$ is given in [8]. The construction of $u_2^{I,0}$ is standard, using the classical theory of transport equation. The wellposedness of the system (1.6), needed in our study, is given in the following theorem

THEOREM 2.1 ([8]). *Suppose that the initial data $u_0^{I,0} = (u_{1,0}^{I,0}, u_{2,0}^{I,0}, u_{3,0}^{I,0})$ in (1.6) satisfies*

$$u_{1,0}^{I,0},\ u_{2,0}^{I,0},\ u_{3,0}^{I,0} \in \mathcal{A}_{\tau_0}$$

for some $\tau_0 > 0$, the divergence-free condition and the compatibility condition. Then Euler-type system (1.6) admits a unique solution

$$(u_1^{I,0}, u_2^{I,0}, u_3^{I,0}) \in L^\infty([0,T];\ \mathcal{A}_\tau)$$

for some $T > 0$ and $\tau > 0$.

At the order $\varepsilon^{1/2}$ in (2.2), using (2.7) and the divergence-free condition (2.3), we obtain the system

$$\begin{cases}\partial_t u_1^{I,1} + u_1^{I,0}\partial_1 u_1^{I,1} + u_3^{I,0}\partial_3 u_1^{I,1} + u_1^{I,1}\partial_1 u_1^{I,0} + u_3^{I,1}\partial_3 u_1^{I,0} + \partial_1 p^{I,1} = 0\\ \partial_t u_2^{I,1} + u_1^{I,0}\partial_1 u_2^{I,1} + u_3^{I,0}\partial_3 u_2^{I,1} + u_1^{I,1}\partial_1 u_2^{I,0} + u_3^{I,1}\partial_3 u_2^{I,0} + \partial_2 p^{I,1} = 0\\ \partial_t u_3^{I,1} + u_1^{I,0}\partial_1 u_3^{I,1} + u_3^{I,0}\partial_3 u_3^{I,1} + u_1^{I,1}\partial_1 u_3^{I,0} + u_3^{I,1}\partial_3 u_3^{I,0} + \partial_3 p^{I,1} = 0\\ \partial_2 u_1^{I,1} = \partial_2 u_2^{I,1} = \partial_2 u_3^{I,1} = \partial_2 p^{I,1} = 0\\ \partial_1 u_1^{I,1} + \partial_3 u_3^{I,1} = 0\end{cases}$$

We also remark that we can not obtain any determined boundary condition for $u^{I,1}$, but only a condition depending on the boundary condition of $u^{B,1}$. Indeed, on the boundary, we recall the value of $u_i^{I,j}$ is related to the value of $u_i^{B,j}$ by the equation

$$u_i^{I,j}(t,x_1,0) + u_i^{B,j}(t,x_1,0) = 0 \qquad j = 0,1; \quad i = 1,2,3.$$

Using the same argument as for the order ε^0, we can prove that $\partial_2 p^{I,1} = 0$, and we obtain the system (1.7), which is the following 2D linearized Euler-type system with three components in the half-plane

$$\begin{cases}\partial_t u_1^{I,1} + u_1^{I,0}\partial_1 u_1^{I,1} + u_3^{I,0}\partial_3 u_1^{I,1} + u_1^{I,1}\partial_1 u_1^{I,0} + u_3^{I,1}\partial_3 u_1^{I,0} + \partial_1 p^{I,1} = 0\\ \partial_t u_2^{I,1} + u_1^{I,0}\partial_1 u_2^{I,1} + u_3^{I,0}\partial_3 u_2^{I,1} + u_1^{I,1}\partial_1 u_2^{I,0} + u_3^{I,1}\partial_3 u_2^{I,0} = 0\\ \partial_t u_3^{I,1} + u_1^{I,0}\partial_1 u_3^{I,1} + u_3^{I,0}\partial_3 u_3^{I,1} + u_1^{I,1}\partial_1 u_3^{I,0} + u_3^{I,1}\partial_3 u_3^{I,0} + \partial_3 p^{I,1} = 0\\ \partial_2 u_1^{I,1} = \partial_2 u_2^{I,1} = \partial_2 u_3^{I,1} = \partial_2 p^{I,1} = 0\\ \partial_1 u_1^{I,1} + \partial_3 u_3^{I,1} = 0\\ u_3^{I,1}(t,x_1,0) = -u_3^{B,1}(t,x_1,0)\\ u^{I,1}(0,x_1,x_3) = u_0^{I,1}(x_1,x_3).\end{cases}$$

Here, we also suppose that the initial data are well prepared, *i.e.* independent of x_2.

Remark 2.3. Here, as for the system (2.8), the hypothesis of a jump from the order $\varepsilon^{\frac{1}{2}}$ to ε^2 in the asymptotic expansion (1.1) allows to close the above system, if not, we will have to deal with the additional term of the form $\begin{pmatrix} u_3^{I,3}\\ 0\\ -u_1^{I,3}\end{pmatrix}$, which is undetermined.

For this linearized Euler system (1.7), we have

Theorem 2.2. *Let* $\ell > 1/2$, $\tau_0 > 0$ *and* $u_3^{B,1}(t,x_1,0)$ *a given function such that*

$$\sum_{m\le 2} \left\| \langle x_1\rangle^{\ell}\, \partial_1^m u_3^{B,1}(t,x_1,0)\right\|^2_{L^2(\mathbb{R}_{x_1})} + \sum_{m\ge 3}\left[\frac{\tau_0^{m-1}}{(m-3)!}\right]^2 \left\| \langle x_1\rangle^{\ell}\, \partial_1^m u_3^{B,1}(t,x_1,0)\right\|^2_{L^2(\mathbb{R}_{x_1})} < +\infty.$$

Suppose that the initial data $u_0^{I,1}=(u_{1,0}^{I,1},u_{2,0}^{I,1},u_{3,0}^{I,1})$ in (1.7) satisfies the divergence-free condition, the compatibility condition and

$$u_{1,0}^{I,1},\, u_{2,0}^{I,1},\, u_{3,0}^{I,1}\in\mathcal{A}_{\tau_0}.$$

Then the linearized Euler system (1.7) *admits a unique solution*

$$(u_1^{I,1},u_2^{I,1},u_3^{I,1})\in L^\infty([0,T];\ \mathcal{A}_\tau)$$

for some $T>0$ and $\tau>0$.

2.2. Formal asymptotic expansions inside the boundary layer. Inside the boundary layer (in the domain $0<x_3\le\sqrt{\varepsilon}$), we consider the Taylor expansions

$$u_i^{I,j}(t,x_h,x_3)=u_i^{I,j}(t,x_h,0)+x_3\partial_3u_i^{I,j}(t,x_h,0)+\frac{x_3^2}{2}\partial_3^2u_i^{I,j}(t,x_h,0)+\dots$$
$$p^{I,j}(t,x_h,x_3)=p^{I,j}(t,x_h,0)+x_3\partial_3p^{I,j}(t,x_h,0)+\frac{x_3^2}{2}\partial_3^2p^{I,j}(t,x_h,0)+\dots$$

Performing the change of variable $y=\frac{x_3}{\sqrt{\varepsilon}}$, we have

$$\begin{cases} u_i^{I,j}(t,x_h,x_3)=\overline{u_i^{I,j}}+\varepsilon^{\frac12}y\overline{\partial_3u_i^{I,j}}+\dfrac{\varepsilon y^2}{2!}\overline{\partial_3^2u_i^{I,j}}+\mathcal{O}\left(\varepsilon^{\frac32}\right)\\ p^{I,j}(t,x_h,x_3)=\overline{p^{I,j}}+\varepsilon^{\frac12}y\overline{\partial_3p^{I,j}}+\dfrac{\varepsilon y^2}{2!}\overline{\partial_3^2p^{I,j}}+\mathcal{O}\left(\varepsilon^{\frac32}\right)\end{cases}\tag{2.11}$$

where $\overline{f}=f(t,x_1,x_2,0)$ is the trace of f on $\{x_3=0\}$. Now, we will rewrite the identities (2.1), taking into account the expansion (2.11). First, we have

$$\begin{aligned} u^\varepsilon=&\left(u^{B,0}+\overline{u^{I,0}}\right)+\varepsilon^{\frac12}\left(u^{B,1}+\overline{u^{I,1}}+y\overline{\partial_3u^{I,0}}\right)\\ &+\sum_{k=2}^{3}\varepsilon^{\frac{k}{2}}\left(\frac{y^{k-1}}{(k-1)!}\overline{\partial_3^{k-1}u^{I,1}}+\frac{y^k}{k!}\overline{\partial_3^ku^{I,0}}\right)+\mathcal{O}\left(\varepsilon^2\right)\\ =&\,\mathcal{U}^{p,0}+\varepsilon^{\frac12}\mathcal{U}^{p,1}+\sum_{k=2}^{3}\varepsilon^{\frac{k}{2}}\left(\frac{y^{k-1}}{(k-1)!}\overline{\partial_3^{k-1}u^{I,1}}+\frac{y^k}{k!}\overline{\partial_3^ku^{I,0}}\right)+\mathcal{O}\left(\varepsilon^2\right)\end{aligned}\tag{2.12}$$

where we note

$$\mathcal{U}^{p,0}=u^{B,0}+\overline{u^{I,0}},\qquad \mathcal{U}^{p,1}=u^{B,1}+\overline{u^{I,1}}+y\overline{\partial_3u^{I,0}}.\tag{2.13}$$

The derivatives of u^ε with respect to tangential variables write

$$\begin{aligned}\partial_{t,1,2}^mu^\varepsilon=&\,\partial_{t,1,2}^m\mathcal{U}^{p,0}+\varepsilon^{\frac12}\partial_{t,1,2}^m\mathcal{U}^{p,1}\\ &+\sum_{k=2}^{3}\varepsilon^{\frac{k}{2}}\partial_{t,1,2}^m\left(\frac{y^{k-1}}{(k-1)!}\overline{\partial_3^{k-1}u^{I,1}}+\frac{y^k}{k!}\overline{\partial_3^ku^{I,0}}\right)+\mathcal{O}\left(\varepsilon^2\right).\end{aligned}\tag{2.14}$$

where $m=1,2$. For the normal variable, we have

$$\partial_3u^\varepsilon=\varepsilon^{-\frac12}\partial_yu^{B,0}+\left(\partial_yu^{B,1}+\overline{\partial_3u^{I,0}}\right)+\sum_{k=1}^{3}\varepsilon^{\frac{k}{2}}\left(\frac{y^{k-1}}{(k-1)!}\overline{\partial_3^ku^{I,1}}+\frac{y^k}{k!}\overline{\partial_3^{k+1}u^{I,0}}\right)+\mathcal{O}\left(\varepsilon^2\right)$$

and

$$\partial_3^2 u^\varepsilon = \varepsilon^{-1}\partial_y^2 u^{B,0} + \varepsilon^{-\frac{1}{2}}\partial_y^2 u^{B,1} + \overline{\partial_3^2 u^{I,0}} + \sum_{k=1}^{3}\varepsilon^{\frac{k}{2}}\left(\frac{y^{k-1}}{(k-1)!}\overline{\partial_3^{k+1}u^{I,1}} + \frac{y^k}{k!}\overline{\partial_3^{k+2}u^{I,0}}\right) + \mathcal{O}\left(\varepsilon^2\right).$$

Thus,

$$-\varepsilon\Delta u^\varepsilon = -\varepsilon\Delta_h \mathcal{U}^{p,0} - \varepsilon^{\frac{3}{2}}\Delta_h\mathcal{U}^{p,1} - \partial_y^2\mathcal{U}^{p,0} - \varepsilon^{\frac{1}{2}}\partial_y^2\mathcal{U}^{p,1} - \varepsilon\overline{\partial_3^2 u^{I,0}} - \varepsilon^{\frac{3}{2}}\left(\frac{y^{k-1}}{(k-1)!}\overline{\partial_3^{k+1}u^{I,1}} + \frac{y^k}{k!}\overline{\partial_3^{k+2}u^{I,0}}\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{2.15}$$

For the non-linear term, we only give the explicit calculations for the first orders of its expansion. We write

$$u^\varepsilon\cdot\nabla u^\varepsilon = u_h^\varepsilon\cdot\nabla_h u^\varepsilon + u_3^\varepsilon\partial_3 u^\varepsilon.$$

Then, we have

$$u_h^\varepsilon\cdot\nabla_h u^\varepsilon = \mathcal{U}_h^{p,0}\cdot\nabla_h\mathcal{U}_h^{p,0} + \varepsilon^{\frac{1}{2}}\mathcal{U}_h^{p,0}\cdot\nabla_h\mathcal{U}^{p,1} + \varepsilon^{\frac{1}{2}}\mathcal{U}_h^{p,1}\cdot\nabla_h\mathcal{U}_h^{p,0} + \mathcal{O}(\varepsilon) \tag{2.16}$$

and

$$u_3^\varepsilon\partial_3 u^\varepsilon = \mathcal{U}_3^{p,1}\partial_y\mathcal{U}_h^{p,0} + \varepsilon^{\frac{1}{2}}\mathcal{U}_3^{p,1}\partial_y\mathcal{U}^{p,1} + \varepsilon^{\frac{1}{2}}\left(y\overline{\partial_3 u_3^{I,1}} + \frac{y^2}{2}\overline{\partial_3^2 u_3^{I,0}}\right)\partial_y\mathcal{U}_h^{p,0} + \mathcal{O}(\varepsilon). \tag{2.17}$$

For the Coriolis forcing term (the rotation term), we have

$$\frac{e_2\times u^\varepsilon}{\varepsilon} = \varepsilon^{-1}\begin{pmatrix}0\\0\\-\mathcal{U}_1^{p,0}\end{pmatrix} + \varepsilon^{-\frac{1}{2}}\begin{pmatrix}\mathcal{U}_3^{p,1}\\0\\-\mathcal{U}_1^{p,1}\end{pmatrix} + \sum_{k=2}^{3}\varepsilon^{\frac{k}{2}-1}\left[\frac{y^{k-1}}{(k-1)!}\begin{pmatrix}\overline{\partial_3^{k-1}u_3^{I,1}}\\0\\-\overline{\partial_3^{k-1}u_1^{I,1}}\end{pmatrix} + \frac{y^k}{k!}\begin{pmatrix}\overline{\partial_3^k u_3^{I,0}}\\0\\-\overline{\partial_3^k u_1^{I,0}}\end{pmatrix}\right] + \mathcal{O}(\varepsilon). \tag{2.18}$$

Finally, the pressure term is

$$\nabla p^\varepsilon = \varepsilon^{-\frac{3}{2}}\begin{pmatrix}0\\0\\\partial_y\mathcal{P}^{p,-2}\end{pmatrix} + \sum_{j=-2}^{0}\varepsilon^{\frac{j}{2}}\begin{pmatrix}\partial_1\mathcal{P}^{p,j}\\\partial_2\mathcal{P}^{p,j}\\\partial_y\mathcal{P}^{p,j+1}\end{pmatrix} + \varepsilon^{\frac{1}{2}}\left[\begin{pmatrix}\partial_1\mathcal{P}^{p,1}\\\partial_2\mathcal{P}^{p,1}\\0\end{pmatrix} + \sum_{k=1}^{4}\frac{y^k}{k!}\begin{pmatrix}0\\0\\\overline{\partial_3^k p^{I,2-k}}\end{pmatrix}\right] + \mathcal{O}(\varepsilon). \tag{2.19}$$

where

$$\mathcal{P}^{p,-2} = p^{B,-2} + \overline{p^{I,-2}}, \qquad \mathcal{P}^{p,-1} = p^{B,-1} + \overline{p^{I,-1}} + y\overline{\partial_3 p^{I,-2}} \tag{2.20}$$

$$\mathcal{P}^{p,0} = p^{B,0} + \overline{p^{I,0}} + y\overline{\partial_3 p^{I,-1}} + \frac{y^2}{2}\overline{\partial_3 p^{I,-2}} \tag{2.21}$$

$$\mathcal{P}^{p,1} = p^{B,1} + \overline{p^{I,1}} + y\overline{\partial_3 p^{I,0}} + \frac{y^2}{2}\overline{\partial_3 p^{I,-1}} + \frac{y^3}{6}\overline{\partial_3^2 p^{I,-2}} \tag{2.22}$$

2.3. Incompressibility and boundary conditions. The divergence-free property of the velocity field is rewritten as follows

$$0=\text{div } u^\varepsilon=\varepsilon^{-\frac{1}{2}}\partial_y u_3^{B,0}\left(t,x_h,\frac{x_3}{\sqrt{\varepsilon}}\right)+\left[\text{div } u^{I,0}+\partial_1 u_1^{B,0}+\partial_2 u_2^{B,0}+\partial_y u_3^{B,1}\left(t,x_h,\frac{x_3}{\sqrt{\varepsilon}}\right)\right]$$
$$+\varepsilon^{\frac{1}{2}}\left[\text{div } u^{I,1}+\partial_1 u_1^{B,1}+\partial_2 u_2^{B,1}\right]+\cdots.$$

Inside the boundary layer, using the expansion (2.3) and (2.11), we deduce the following divergence-free condition

$$\varepsilon^{-\frac{1}{2}}\partial_y u_3^{B,0}+\left(\partial_1 u_1^{B,0}+\partial_2 u_2^{B,0}+\partial_y u_3^{B,1}\right)+\varepsilon^{\frac{1}{2}}\left(\partial_1 u_1^{B,1}+\partial_2 u_2^{B,1}\right)=0.$$

Thus, we obtain the incompressibility of the boundary layer

$$\begin{cases}\partial_1 u_1^{B,0}+\partial_2 u_2^{B,0}+\partial_y u_3^{B,1}=0,\\ \partial_1 u_1^{B,1}+\partial_2 u_2^{B,1}=0.\end{cases}\tag{2.23}$$

Moreover, we have

$$\partial_y u_3^{B,0}=0,$$

which, by taking $y\to+\infty$, gives

$$u_3^{B,0}=0.$$

For the boundary condition in (N-S$_\varepsilon$) on $\{x_3=0\}$, we have

$$\sum_{j=0}^{1}\varepsilon^{\frac{j}{2}}\left[u^{I,j}(t,x_h,0)+u^{B,j}(t,x_h,0)\right]=0,$$

which implies that

$$\begin{cases}\overline{u^{I,0}(t)}+u^{B,0}(t,x_h,0)=0,\\ \overline{u^{I,1}(t)}+u^{B,1}(t,x_h,0)=0.\end{cases}\tag{2.24}$$

In particular, $u_3^{B,0}=0$ imply

$$u_3^{I,0}|_{x_3=0}=\overline{u_3^{I,0}}=0,\tag{2.25}$$

which is the boundary condition for Euler equation in (1.6), and the third component in (2.24) gives the boundary condition of linearized Euler equation in (1.7).

2.4. Formal derivation of the governing equations of the fluid in the boundary layer. Now, we consider the system (N-S$_\varepsilon$) near $\{x_3=0\}$, using the asymptotic formal (2.12) - (2.19).

At the order $\varepsilon^{-\frac{3}{2}}$, we have

$$\partial_y p^{B,-2}=0,$$

which implies that $p^{B,-2}=0$ because $p^{B,-2}$ goes to zero as $y\to+\infty$. Using the new notation of the pressure defined in (2.20), we get

$$\partial_y \mathcal{P}^{p,-2}=0.$$

At the order ε^{-1}, using the fact that $\overline{u_3^{I,0}}=0$, $u_3^{B,0}=0$ and $p^{B,-2}=0$, we get

$$\begin{pmatrix} 0 \\ 0 \\ -u_1^{B,0} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\overline{u_1^{I,0}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \partial_y p^{B,-1} \end{pmatrix} + \overline{\nabla p^{I,-2}} = 0,$$

which implies that $\overline{\partial_1 p^{I,-1}} = \overline{\partial_2 p^{I,-1}} = 0$ and

$$-u_1^{B,0} - \overline{u_1^{I,0}} + \partial_y p^{B,-1} + \overline{\partial_3 p^{I,-2}} = 0. \tag{2.26}$$

Using the new velocity and pressure defined in (2.20) and taking into account the fact that $\mathcal{U}_3^{p,0}=0$, we can also write

$$\begin{pmatrix} 0 \\ 0 \\ -\mathcal{U}_1^{p,0} \end{pmatrix} + \begin{pmatrix} \partial_1 \mathcal{P}^{p,-2} \\ \partial_2 \mathcal{P}^{p,-2} \\ \partial_y \mathcal{P}^{p,-1} \end{pmatrix} = 0. \tag{2.27}$$

At the order $\varepsilon^{-1/2}$, we have

$$\begin{pmatrix} u_3^{B,1} \\ 0 \\ -u_1^{B,1} \end{pmatrix} + \begin{pmatrix} \overline{u_3^{I,1}} \\ 0 \\ -\overline{u_1^{I,1}} \end{pmatrix} + y \begin{pmatrix} \overline{\partial_3 u_3^{I,0}} \\ 0 \\ -\overline{\partial_3 u_1^{I,0}} \end{pmatrix} + \begin{pmatrix} \partial_1 p^{B,-1} \\ \partial_2 p^{B,-1} \\ \partial_y p^{B,0} \end{pmatrix} + \begin{pmatrix} \overline{\partial_1 p^{I,-1}} \\ \overline{\partial_2 p^{I,-1}} \\ \overline{\partial_3 p^{I,-1}} \end{pmatrix} + y \begin{pmatrix} \overline{\partial_1 \partial_3 p^{I,-2}} \\ \overline{\partial_2 \partial_3 p^{I,-2}} \\ \overline{\partial_3^2 p^{I,-2}} \end{pmatrix} = 0,$$

or in a equivalent way, using the new velocity and pressure defined in (2.21),

$$\begin{pmatrix} \mathcal{U}_3^{p,1} \\ 0 \\ -\mathcal{U}_1^{p,1} \end{pmatrix} + \begin{pmatrix} \partial_1 \mathcal{P}^{p,-1} \\ \partial_2 \mathcal{P}^{p,-1} \\ \partial_y \mathcal{P}^{p,0} \end{pmatrix} = 0. \tag{2.28}$$

then

$$\partial_2 \mathcal{P}^{p,-1} = 0.$$

and

$$\begin{aligned} \partial_2 \mathcal{U}_1^{p,0} &= \partial_2 \partial_y \mathcal{P}^{p,-1} = \partial_y \partial_2 \mathcal{P}^{p,-1} = 0 \\ \partial_2 \mathcal{U}_3^{p,1} &= -\partial_2 \partial_1 \mathcal{P}^{p,-1} = -\partial_1 \partial_2 \mathcal{P}^{p,-1} = 0. \end{aligned}$$

Using the divergence-free properties (2.3) and (2.23), we also have

$$\partial_2 \mathcal{U}_2^{p,0} = -\partial_1 \mathcal{U}_1^{p,0} - \partial_y \mathcal{U}_3^{p,1} = -\partial_1 \partial_y \mathcal{P}^{p,-1} - \left(-\partial_y \partial_1 \mathcal{P}^{p,-1}\right) = 0.$$

We deduce that $(\mathcal{U}_1^{p,0}, \mathcal{U}_2^{p,0}, \mathcal{U}_3^{p,1})$ is a divergence-free vector field which is independent on x_2. The fact that $\partial_2 u^{I,0} = \partial_2 u^{I,1} = 0$ implies that

$$\partial_2 u_1^{B,0} = \partial_2 u_2^{B,0} = \partial_2 u_3^{B,1} = 0. \tag{2.29}$$

Remark 2.4. The leading order of the velocity of the fluid inside the boundary layer also obeys the Taylor-Proudman theorem.

At the order ε^0, recalling that $u_3^{B,0} = \overline{u_3^{I,0}} = 0$, we get the following equation

$$\partial_t\left(u_h^{B,0}+\overline{u_h^{I,0}}\right)-\partial_y^2 u_h^{B,0}+\left(u_h^{B,0}+\overline{u_h^{I,0}}\right)\cdot\nabla_h\left(u_h^{B,0}+\overline{u_h^{I,0}}\right)$$
$$+\left(u_3^{B,1}+\overline{u_3^{I,1}}+y\overline{\partial_3 u_3^{I,0}}\right)\partial_y u_h^{B,0}+y\begin{pmatrix}\overline{\partial_3 u_3^{I,1}}\\ 0\\ \overline{-\partial_3 u_1^{I,1}}\end{pmatrix}+\frac{y^2}{2}\begin{pmatrix}\overline{\partial_3^2 u_3^{I,0}}\\ 0\\ \overline{-\partial_3^2 u_1^{I,0}}\end{pmatrix}$$
$$+\begin{pmatrix}\partial_1 p^{B,0}\\ \partial_2 p^{B,0}\\ \partial_y p^{B,1}\end{pmatrix}+\begin{pmatrix}\overline{\partial_1 p^{I,0}}\\ \overline{\partial_2 p^{I,0}}\\ \overline{\partial_3 p^{I,0}}\end{pmatrix}+y\begin{pmatrix}\overline{\partial_1\partial_3 p^{I,-1}}\\ \overline{\partial_2\partial_3 p^{I,-1}}\\ \overline{\partial_3^2 p^{I,-1}}\end{pmatrix}+\frac{y^2}{2}\begin{pmatrix}\overline{\partial_1\partial_3^2 p^{I,-2}}\\ \overline{\partial_2\partial_3^2 p^{I,-2}}\\ \overline{\partial_3^3 p^{I,-2}}\end{pmatrix}=0.$$

From (2.4) and (2.6), we deduce that

$$-y\overline{\partial_3 u_1^{I,1}}-\frac{y^2}{2}\overline{\partial_3^2 u_1^{I,0}}+y\overline{\partial_3^2 p^{I,-1}}+\frac{y^2}{2}\overline{\partial_3^3 p^{I,-2}}=0.$$

We also remark that the boundary condition applying to the third equation of the Euler system implies that

$$\overline{\partial_3 p^{I,0}}=0,$$

and so, $\partial_y p^{B,1}=0$, which means that

$$p^{B,1}=0,$$

since $\lim_{y\to+\infty} p^{B,1}=0$. Then, using the new velocity and pressure defined in (2.13) and (2.20), we get

$$\partial_t\mathcal{U}_h^{p,0}-\partial_y^2\mathcal{U}_h^{p,0}+\mathcal{U}_h^{p,0}\cdot\nabla_h\mathcal{U}_h^{p,0}+\mathcal{U}_3^{p,1}\partial_y\mathcal{U}_h^{p,0}+\begin{pmatrix}\partial_1 p^{B,0}+\overline{\partial_1 p^{I,0}}\\ \partial_2\mathcal{P}^{p,0}\end{pmatrix}=0.$$

Taking into account the divergence-free condition (2.23), the identities (2.26) and (2.27), and $(\mathcal{U}_1^{p,0},\mathcal{U}_2^{p,0},\mathcal{U}_3^{p,1})$ is independent on x_2, we deduce that $(\mathcal{U}_1^{p,0},\mathcal{U}_2^{p,0},\mathcal{U}_3^{p,1})$ satisfies the following system

$$\begin{cases}\partial_t\mathcal{U}_1^{p,0}-\partial_y^2\mathcal{U}_1^{p,0}+\mathcal{U}_1^{p,0}\partial_1\mathcal{U}_1^{p,0}+\mathcal{U}_3^{p,1}\partial_y\mathcal{U}_1^{p,0}+\partial_1 p^{B,0}+\overline{\partial_1 p^{I,0}}=0\\ \partial_t\mathcal{U}_2^{p,0}-\partial_y^2\mathcal{U}_2^{p,0}+\mathcal{U}_1^{p,0}\partial_1\mathcal{U}_2^{p,0}+\mathcal{U}_3^{p,1}\partial_y\mathcal{U}_2^{p,0}+\partial_2\mathcal{P}^{p,0}=0\\ \partial_1\mathcal{U}_1^{p,0}+\partial_y\mathcal{U}_3^{p,1}=0,\\ \partial_2\mathcal{U}_1^{p,0}=\partial_2\mathcal{U}_2^{p,0}=\partial_2\mathcal{U}_3^{p,1}=0.\end{cases}$$

We remark that the above system is not complete, since we need another equation for the component $\mathcal{U}_3^{p,1}$.

At the order $\varepsilon^{1/2}$, using the fact that $p^{B,1}=0$, we have

$$\partial_t\mathcal{U}^{p,1}-\partial_y^2\mathcal{U}^{p,1}+\mathcal{U}_h^{p,0}\cdot\nabla_h\mathcal{U}^{p,1}+\mathcal{U}_h^{p,1}\cdot\nabla_h\mathcal{U}_h^{p,0}+\mathcal{U}_3^{p,1}\partial_y\mathcal{U}^{p,1}+\left(y\overline{\partial_3 u_3^{I,1}}+\frac{y^2}{2}\overline{\partial_3^2 u_3^{I,0}}\right)\partial_y\mathcal{U}_h^{p,0}$$
$$+\left[\frac{y^2}{2}\begin{pmatrix}\overline{\partial_3^2 u_3^{I,1}}\\ 0\\ \overline{-\partial_3^2 u_1^{I,1}}\end{pmatrix}+\frac{y^3}{6}\begin{pmatrix}\overline{\partial_3^3 u_3^{I,0}}\\ 0\\ \overline{-\partial_3^3 u_1^{I,0}}\end{pmatrix}\right]+\sum_{k=0}^{3}\frac{y^k}{k!}\begin{pmatrix}\overline{\partial_1\partial_3^k p^{I,1-k}}\\ \overline{\partial_2\partial_3^k p^{I,1-k}}\\ \overline{\partial_3^{k+1} p^{I,1-k}}\end{pmatrix}=0.$$

Here, we are only interested in the component $\mathcal{U}_3^{p,1}$. Using the fact that $\partial_2 \mathcal{U}_3^{p,1} = 0$, we obtain

$$\partial_t \mathcal{U}_3^{p,1} - \partial_y^2 \mathcal{U}_3^{p,1} + \mathcal{U}_1^{p,0} \partial_1 \mathcal{U}_3^{p,1} + \mathcal{U}_3^{p,1} \partial_y \mathcal{U}_3^{p,1} + \overline{\partial_3 p^{I,1}} + y \overline{\partial_3^2 p^{I,0}} = 0.$$

REMARK 2.5. The hypothesis of a jump from the order $\varepsilon^{\frac{1}{2}}$ to ε^2 in the asymptotic expansion (1.1) also allows to close the equations of $\mathcal{U}^{p,0}$ and $\mathcal{U}^{p,1}$. Without it, there will be undetermined terms of the form $\begin{pmatrix} \mathcal{U}_3^{p,2} \\ 0 \\ -\mathcal{U}_1^{p,2} \end{pmatrix}$ and $\begin{pmatrix} \mathcal{U}_3^{p,3} \\ 0 \\ -\mathcal{U}_1^{p,3} \end{pmatrix}$.

Collecting all the above formal calculations, we deduce the following governing equations of the boundary layer

$$\begin{cases} \partial_t \mathcal{U}_1^{p,0} - \partial_y^2 \mathcal{U}_1^{p,0} + \mathcal{U}_1^{p,0} \partial_1 \mathcal{U}_1^{p,0} + \mathcal{U}_3^{p,1} \partial_y \mathcal{U}_1^{p,0} + \partial_1 p^{B,0} + \overline{\partial_1 p^{I,0}} = 0 \\ \partial_t \mathcal{U}_3^{p,1} - \partial_y^2 \mathcal{U}_3^{p,1} + \mathcal{U}_1^{p,0} \partial_1 \mathcal{U}_3^{p,1} + \mathcal{U}_3^{p,1} \partial_y \mathcal{U}_3^{p,1} + \overline{\partial_3 p^{I,1}} + y \overline{\partial_3^2 p^{I,0}} = 0 \\ \partial_1 \mathcal{U}_1^{p,0} + \partial_y \mathcal{U}_3^{p,1} = 0 \\ \partial_2 \mathcal{U}_1^{p,0} = \partial_2 \mathcal{U}_3^{p,1} = 0 \\ \mathcal{U}_1^{p,0}(t,x_1,0) = 0, \quad \lim_{y \to +\infty} \mathcal{U}_1^{p,0}(t,x_1,y) = \overline{u_1^{I,0}}(x_1) \\ \mathcal{U}_3^{p,1}(t,x_1,0) = 0, \quad \partial_y \mathcal{U}_3^{p,1}(t,x_1,0) = 0 \\ \mathcal{U}_1^{p,0}(0,x_1,y) = u_{0,1}^{B,0}(x_1,y) + \overline{u_{0,1}^{I,0}}(x_1) \\ \mathcal{U}_3^{p,1}(0,x_1,y) = u_{0,3}^{B,1}(x_1,y) + \overline{u_{0,3}^{I,1}}(x_1) + y \overline{\partial_3 u_{0,3}^{I,0}}(x_1). \end{cases} \tag{P1}$$

and

$$\begin{cases} \partial_t \mathcal{U}_2^{p,0} - \partial_y^2 \mathcal{U}_2^{p,0} + \mathcal{U}_1^{p,0} \partial_1 \mathcal{U}_2^{p,0} + \mathcal{U}_3^{p,1} \partial_y \mathcal{U}_2^{p,0} + \partial_2 \mathcal{P}^{p,0} = 0 \\ \mathcal{U}_2^{p,0}(0,x_1,y) = u_{0,2}^{B,0}(x_1,y) + \overline{u_{0,2}^{I,0}}(x_1) \\ \mathcal{U}_2^{p,0}(t,x_1,0) = 0, \quad \lim_{y \to +\infty} \mathcal{U}_2^{p,0}(t,x_1,y) = \overline{u_2^{I,0}}(x_1) \\ \mathcal{U}_2^{p,0}(0,x_1,y) = u_{0,2}^{B,0}(x_1,y) + \overline{u_{0,2}^{I,0}}(x_1). \end{cases} \tag{P2}$$

Claim: *The pressure term of the* (P2) *satisfies* $\partial_2 \mathcal{P}^{p,0} = 0$.

Indeed, applying ∂_2 to the first equation of the systems (P1) and (P2), and using the fact that

$$\partial_2 \mathcal{U}_1^{p,0} = \partial_2 \mathcal{U}_2^{p,0} = \partial_2 \mathcal{U}_3^{p,1} = 0,$$

we deduce that

$$\partial_1 \partial_2 \mathcal{P}^{p,0} = \partial_2^2 \mathcal{P}^{p,0} = 0.$$

This means that, modulo a constant, we have

$$\mathcal{P}^{p,0} = x_2 G_1(t,y) + \int_{-\infty}^{x_1} \tilde{f}(t,x,y)dx,$$

where

$$G_1 = -\left(\partial_t \mathcal{U}_2^{p,0} - \partial_y^2 \mathcal{U}_2^{p,0} + \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_2^{p,0} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_2^{p,0}\right)$$

is to be determined and

$$\tilde{f} = \partial_1 \mathcal{P}^{p,0} = -\partial_t \mathcal{U}_1^{p,0} + \partial_y^2 \mathcal{U}_1^{p,0} - \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_1^{p,0} - \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_1^{p,0} - \left(\frac{y^2}{2}\overline{\partial_3^2 u_3^{I,0}} + y\overline{\partial_3 u_3^{I,1}}\right).$$

We recall that, from (2.28), we have

$$\partial_y \mathcal{P}^{p,0} = \mathcal{U}_1^{p,1},$$

where $\mathcal{U}_1^{p,1}$ is the solution of the system

$$\begin{cases} \partial_t \mathcal{U}_1^{p,1} - \partial_y^2 \mathcal{U}_1^{p,1} + \mathcal{U}_1^{p,0}\cdot\partial_1 \mathcal{U}_1^{p,1} + \mathcal{U}_2^{p,0}\cdot\partial_2 \mathcal{U}_1^{p,1} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_1^{p,1} + \mathcal{U}_1^{p,1}\partial_1 \mathcal{U}_1^{p,0} \\ \qquad + \left(y\overline{\partial_3 u_3^{I,1}} + \frac{y^2}{2}\overline{\partial_3^2 u_3^{I,0}}\right)\partial_y \mathcal{U}_1^{p,0} + \left[\frac{y^2}{2}\overline{\partial_3^2 u_3^{I,1}} + \frac{y^3}{6}\overline{\partial_3^3 u_3^{I,0}}\right] + \sum_{k=0}^{3}\frac{y^k}{k!}\overline{\partial_1\partial_3^k p^{I,1-k}} = 0 \\ \mathcal{U}_1^{p,1}(0,x_1,x_2,y) = u_{0,1}^{B,1}(x_1,y) + \overline{u_{0,1}^{I,1}}(x_1) + y\overline{\partial_3 u_{0,1}^{I,0}}(x_1) + \alpha_1(y)x_2 \\ \mathcal{U}_1^{p,1}(t,x_1,0) = 0. \end{cases}$$

We remark that $\partial_y G_1(t,y) = \partial_2 \mathcal{U}_1^{p,1}$ and we recall that $\partial_1\partial_2 \mathcal{U}_1^{p,1} = \partial_2^2 \mathcal{U}_1^{p,1}$. So, in fact, we will find $\partial_y G_1$ by solving the following system

$$\begin{cases} \partial_t(\partial_y G_1) - \partial_y^2(\partial_y G_1) + (\partial_1 \mathcal{U}_1^{p,0})(\partial_y G_1) + \mathcal{U}_3^{p,1}\partial_y(\partial_y G_1) = 0 \\ \partial_y G_1(0,y) = \alpha_1(y) \\ \partial_y G_1(t,0) = 0 \end{cases} \tag{2.30}$$

where α_1 is a given function, with $\alpha_1(0) = 0$. For the case of well prepared data, we consider the initial data to be independent of x_2, so $\alpha_1 \equiv 0$ and it is easy to see that the system (2.30) admits 0 as a trivial solution. Then, the uniqueness of this solution implies $\partial_y G_1(t,.) \equiv 0$. Replacing $y = 0$ in (2.30), we obtain $G_1(t,0) = 0$, and so $G_1(t,.) \equiv 0$, for any $t \in \mathbb{R}_+$.

3. Well-posedness of the boundary layer system

In this section we will prove the well-posedness for system (P1). Since the pressure term in the first equation of (P1) is unknown, we begin with handling the second one to prove the existence of $\mathcal{U}_3^{p,1}$ and then use the divergence-free property to find $\mathcal{U}_1^{p,0}$. To do so we insert the representations

$$\mathcal{U}_1^{p,0} = u_1^{B,0} + \overline{u_1^{I,0}}, \quad \mathcal{U}_3^{p,1} = u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}$$

into the second equation of (P1), and then make use of the Equations (1.6) and (1.7) of $u_3^{I,0}$ and $u_3^{I,1}$. It then follows that the unknowns $u_3^{B,1}, u_1^{B,0}$ and $u_3^{I,1}$ satisfy the equation

$$\left(\partial_t-\partial_y^2+\overline{\partial_3 u_3^{I,0}}y\partial_y\right)u_3^{B,1}+\left(u_1^{B,0}+\overline{u_1^{I,0}}\right)\partial_1 u_3^{B,1}$$
$$+\left(u_3^{B,1}+\overline{u_3^{I,1}}\right)\partial_y u_3^{B,1}+\overline{\partial_3 u_3^{I,0}}u_3^{B,1}+\left(-\partial_1\overline{u_3^{I,1}}+y\overline{\partial_1\partial_3 u_3^{I,0}}\right)u_1^{B,0}=0,$$

and the divergence-free properties (2.23) and (2.29) yield

$$u_1^{B,0}=-\int_{-\infty}^{x_1}\partial_y u_3^{B,1}(t,z,y)dz.$$

Thus the above is just an equation for $u_3^{B,1}$. To solve the system (P1), we consider the following nonlinear initial-boundary problem,

$$\begin{cases}\left(\partial_t-\partial_y^2+\overline{\partial_3 u_3^{I,0}}y\partial_y\right)u+\left(v+\overline{u_1^{I,0}}\right)\partial_1 u\\ \qquad +(u-u(t,x_1,0))\partial_y u+\overline{\partial_3 u_3^{I,0}}u+\left(\partial_1 u(t,x_1,0)+y\overline{\partial_1\partial_3 u_3^{I,0}}\right)v=0,\\ \partial_y u|_{y=0}=-\overline{\partial_3 u_3^{I,0}}(t,x_1),\qquad \lim_{y\to+\infty}u(t,x_1,y)=0,\\ u|_{t=0}=u_0(x_1,y),\end{cases}\tag{3.1}$$

where the unknown functions u and v are linked by the relation

$$v(t,x_1,y)=-\int_{-\infty}^{x_1}\partial_y u(t,z,y)dz.\tag{3.2}$$

Recall the functions $u_1^{I,0},u_3^{I,0}$ are the solutions to the Euler-type system (1.6). By Theorem 2.1, we see $u_1^{I,0},u_3^{I,0}\in\mathcal{A}_\tau$ for some $\tau>0$.

The main result of this section can be stated as follows.

THEOREM 3.1. *Suppose the initial data $u_0\in X_{\rho_0,a_0}$ for some $\rho_0>0$ and $a_0>0$ and satisfies the compatibility conditions. Then the system (3.1) admits a unique solution*

$$u\in L^\infty([0,T_*];X_{\rho_*,a})$$

for some $\rho_*>0$, $a>0$ and $T_*>0$.

We now proceed to the proof of the Theorem 3.1 through the following parabolic approximations.

The approximate solutions. Consider the following regularized system, for $\varepsilon>0$,

$$\begin{cases}\left(\partial_t-\varepsilon\partial_1^2-\partial_y^2+\overline{\partial_3 u_3^{I,0}}y\partial_y\right)u^\varepsilon+\left(v^\varepsilon+\overline{u_1^{I,0}}\right)\partial_1 u^\varepsilon\\ \qquad +(u^\varepsilon-u^\varepsilon(t,x_1,0))\partial_y u^\varepsilon+\overline{\partial_3 u_3^{I,0}}u^\varepsilon+\left(\partial_1 u(t,x_1,0)+y\overline{\partial_1\partial_3 u_3^{I,0}}\right)v=0,\\ \partial_y u^\varepsilon(t,x_1,0)=\overline{\partial_1 u_1^{I,0}}(t,x_1),\qquad \lim_{y\to+\infty}u^\varepsilon(t,x_1,y)=0,\\ u^\varepsilon|_{t=0}=u_0(x_1,y).\end{cases}\tag{3.3}$$

The above is a nonlinear parabolic equation, and from classical theory we can deduce the following local well-posedness result.

THEOREM 3.2. *Suppose the initial data $u_0\in X_{2\rho_0,a_0}$ for some $\rho_0>0$, $a_0>0$ and satisfies the compatibility conditions. Then the system (3.3) admits a unique solution*

$$u^\varepsilon\in L^\infty([0,T_\varepsilon];X_{\rho_0,a})$$

for some $0<a<a_0$ *independent of* ε *and* $T_\varepsilon>0$ *depends on* ε.

Uniform estimates for the approximate solutions. We will perform the uniform estimates with respect to ε for the approximate solutions u^ε given in the previous Theorem. The main result here can be stated as follows.

PROPOSITION 3.1. *Suppose* $u^\varepsilon\in L^\infty([0,T_\varepsilon];X_{\rho_0,a})$ *is a solution to the initial-boundary problem* (3.3). *Then there exists* $0<\rho_*\le\rho_0$, *depending only on* $|u_0|_{X_{\rho_0,a_0}}$, *such that* $u^\varepsilon\in L^\infty([0,T_\varepsilon];X_{\rho_*,a})$ *for all* $\varepsilon>0$. *Moreover*

$$\left\|u^\varepsilon\right\|_{L^\infty([0,T_\varepsilon];X_{\rho_*,a})}\le C|u_0|_{X_{\rho_0,a_0}}, \tag{3.4}$$

where C *is a constant depending only on* $a_0,\rho_0,\tau,\left\|u_3^{I,0}\right\|_{\mathcal{A}_\tau}$ *and* $\left\|u_1^{I,0}\right\|_{\mathcal{A}_\tau}$, *but are independent of* ε.

To prove the above proposition, we need another two *auxiliary norms* $|\cdot|_{Y_{\rho,a}}$ and $|\cdot|_{Z_{\rho,a}}$ which are defined by

$$\begin{aligned}|u|_{Y_{\rho,a}}^2=&\sum_{m\le 2}\left(\sum_{0\le j\le 1}\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y^j u\right\|_{L^2(\mathbb{R}_+^2)}\right)^2\\&+\sum_{m\ge 3}\left(\sum_{0\le j\le 1}(m-1)^{1/2}\rho^{-1/2}\frac{\rho^{m-1}}{(m-3)!}\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y^j u\right\|_{L^2(\mathbb{R}_+^2)}\right)^2,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}|u|_{Z_{\rho,a}}^2=&\sum_{m\le 2}\left(\sum_{1\le j\le 2}\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y^j u\right\|_{L^2(\mathbb{R}_+^2)}\right)^2\\&+\sum_{m\ge 3}\left(\sum_{1\le j\le 2}\frac{\rho^{m-1}}{(m-3)!}\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y^j u\right\|_{L^2(\mathbb{R}_+^2)}\right)^2.\end{aligned}$$

The following energy estimate is a key part to prove Proposition 3.1.

PROPOSITION 3.2. *Let* $u^\varepsilon\in L^\infty([0,T_\varepsilon];X_{\rho_0,a})$ *be a solution to the initial-boundary problem* (3.3) *and let* $0<\rho(t)\le\min\{\rho_0/2,\tau/3\}$ *be a smooth function. Then for any* $t\in[0,T_\varepsilon]$,

$$\begin{aligned}&|u^\varepsilon(t)|_{X_{\rho(t),a}}^2+\int_0^{T_\varepsilon}|u^\varepsilon(t)|_{Z_{\rho(t),a}}^2\,dt-\int_0^T\rho'(t)|u^\varepsilon(t)|_{Y_{\rho(t),a}}^2\,dt\\\le&|u_0|_{X_{\rho_0,a_0}}^2+C\int_0^{T_\varepsilon}\left(|\rho'(t)|\rho(t)^{-2}|u^\varepsilon(t)|_{X_{\rho(t),a}}+|u^\varepsilon(t)|_{X_{\rho(t),a}}^2+|u^\varepsilon(t)|_{X_{\rho(t),a}}^4\right)dt\\&+C\int_0^{T_\varepsilon}|u^\varepsilon(t)|_{Z_{\rho(t),a}}|u^\varepsilon(t)|_{Y_{\rho(t),a}}^2\,dt.\end{aligned}\tag{3.6}$$

The proof of the proposition above is postponed to the next section, and we now use it to prove Proposition 3.1.

Proof. **(Proof of Proposition 3.1.)** To simplify the notations we will use C in the following discussion to denote different suitable constants, which depend only on $a_0,\rho_0,\tau,\left\|u_3^{I,0}\right\|_{\mathcal{A}_\tau}$ and $\left\|u_1^{I,0}\right\|_{\mathcal{A}_\tau}$, but are independent of ε.

Let ρ_ε be the solution to the differential equation:

$$\begin{cases} \rho_\varepsilon'(t) = -|u^\varepsilon(t)|_{Z_{\rho_\varepsilon(t),a}}, \\ \rho|_{t=0} = \min\{\rho_0/2,\tau/3\}, \end{cases} \tag{3.7}$$

or equivalently

$$\rho_\varepsilon(t) = \min\{\rho_0/2,\tau/3\} - \int_0^t |u^\varepsilon(s)|_{Z_{\rho_\varepsilon(s),a}}\,ds. \tag{3.8}$$

Observe, for any $0<\rho,\tilde\rho\le\rho_0/2$, we have

$$\Big||u^\varepsilon|_{Z_{\rho,a}} - |u^\varepsilon|_{Z_{\tilde\rho,a}}\Big| \le C|u^\varepsilon|_{Z_{\rho_0,a}}|\rho-\tilde\rho|,$$

which along with Cauchy-Lipschitz Theorem gives the existence of ρ_ε to Equation (3.7). Now choosing $\rho(t)=\rho_\varepsilon(t)$ in (3.6) and observing (3.7), we can rewrite (3.6) as

$$\begin{aligned}&|u^\varepsilon(t)|^2_{X_{\rho_\varepsilon,a}} + \int_0^{T_\varepsilon} |u^\varepsilon(t)|^2_{Z_{\rho_\varepsilon,a}}\,dt \\ &\le |u_0|^2_{X_{\rho_0,a_0}} + C\int_0^{T_\varepsilon}\Big(|\rho_\varepsilon'(t)|\rho_\varepsilon^{-2}|u^\varepsilon|_{X_{\rho,a}} + |u^\varepsilon|^2_{X_{\rho_\varepsilon,a}} + |u^\varepsilon|^4_{X_{\rho_\varepsilon,a}}\Big)\,dt.\end{aligned}$$

Thus, using (3.7),

$$\begin{aligned}&|u^\varepsilon(t)|^2_{X_{\rho,a}} + \frac12\int_0^{T_\varepsilon} |u^\varepsilon(t)|^2_{Z_{\rho_\varepsilon,a}}\,dt \\ &\le |u_0|^2_{X_{\rho_0,a_0}} + C\int_0^{T_\varepsilon}\Big(\rho_\varepsilon^{-4}|u^\varepsilon|^2_{X_{\rho_\varepsilon,a}} + |u^\varepsilon|^2_{X_{\rho_\varepsilon,a}} + |u^\varepsilon|^4_{X_{\rho_\varepsilon,a}}\Big)\,dt.\end{aligned} \tag{3.9}$$

In view of (3.8) for T_ε be small sufficiently, we have

$$\forall\ t\in[0,T_\varepsilon],\quad \rho_\varepsilon(t)\ge\frac18\min\{\rho_0,\tau/3\},$$

and thus it follows from (3.9) that, for any $t\in[0,T_\varepsilon]$,

$$|u^\varepsilon(t)|^2_{X_{\rho,a}} + \frac12\int_0^{T_\varepsilon}|u^\varepsilon(t)|^2_{Z_{\rho_\varepsilon,a}}\,dt \le |u_0|^2_{X_{\rho_0,a_0}} + C\int_0^{T_\varepsilon}\Big(|u^\varepsilon|^2_{X_{\rho_\varepsilon,a}} + |u^\varepsilon|^4_{X_{\rho_\varepsilon,a}}\Big)\,dt,$$

with C depending only on $a_0,\rho_0,\tau,\left\|u_3^{I,0}\right\|_{\mathcal{A}_\tau}$ and $\left\|u_1^{I,0}\right\|_{\mathcal{A}_\tau}$, but independent of ε. Thus by general Grönwall's inequality, we conclude

$$|u^\varepsilon(t)|^2_{X_{\rho_\varepsilon,a}} \le C|u_0|^2_{X_{\rho_0,a_0}}, \tag{3.10}$$

and

$$\int_0^{T_\varepsilon}|u^\varepsilon(t)|^2_{Z_{\rho_\varepsilon,a}}\,dt \le 3|u_0|^2_{X_{\rho_0,a_0}} + |u_0|^4_{X_{\rho_0,a_0}}.$$

As a result, in view of (3.8) we see

$$\begin{aligned}\rho_\varepsilon(t) &= \min\{\rho_0/2,\tau/3\} - \int_0^t |u^\varepsilon(s)|_{Z_{\rho_\varepsilon(s),a}}\, ds \\ &\geq \min\{\rho_0/2,\tau/3\} - t^{1/2}\left(\int_0^{T_\varepsilon} |u^\varepsilon(t)|^2_{Z_{\rho_\varepsilon,a}}\, dt\right)^{1/2} \\ &\geq \min\{\rho_0/2,\tau/3\} - t^{1/2}\left(2|u_0|^2_{X_{\rho_0,a_0}} + |u_0|^4_{X_{\rho_0,a_0}}\right)^{1/2}.\end{aligned}$$

So if we choose T_* such that

$$T_* = 4^{-1}\left(3|u_0|^2_{X_{\rho_0,a_0}} + |u_0|^4_{X_{\rho_0,a_0}}\right)^{-1}\left(\min\{\rho_0/2,\tau/3\}\right)^2. \tag{3.11}$$

Then

$$\forall\, t \in [0,T_\varepsilon] \subset [0,T_*], \quad \rho_\varepsilon(t) \geq \rho_* \overset{\text{def}}{=} \frac{1}{4}\min\{\rho_0,\tau/3\}.$$

By (3.10), it follows that

$$\forall\, t \in [0,T_\varepsilon] \subset [0,T_*], \quad |u^\varepsilon(t)|^2_{X_{\rho_*,a}} \leq C|u_0|^2_{X_{\rho_0,a_0}}.$$

This completes the proof of Proposition 3.1. □

Remark 3.1. The main difficulty in the proof of Proposition 3.1 is that, when applying the standard energy method, to obtain a control of u in $X_{\rho_0,a}$-norm, we need a control of u in $Y_{\rho_0,a}$-norm, which in turn needs a control of u in $Z_{\rho_0,a}$-norm, ... So the main idea to close the process is to take advantage of the additional "good term"

$$-\int_0^T \rho'(t)|u^\varepsilon(t)|^2_{Y_{\rho(t),a}}\, dt,$$

and to precisely choose the auxiliary function $\rho(t)$ as in (3.7) in order to cancel the uncontrollable term which contains $|u^\varepsilon(t)|_{Z_{\rho(t),a}}$ on the right-hand side of (3.6). We remark that since $\rho(t)$ is bounded from below in $[0,T_*]$, we have a good control of the width of the analyticity band, and assure that does not shrink to zero.

Proof. **(Completion of the proof of Theorem 3.1.)** Due to the uniform estimate (3.4), we can extend the lifespan T_ε to T_* with T_* defined in (3.11), following the standard bootstrap arguments. Thus we see for any $\varepsilon > 0$ the system (3.3) admits a unique solution $u^\varepsilon \in L^\infty([0,T_*];X_{\rho_*,a})$ such that

$$\|u^\varepsilon\|_{L^\infty([0,T_\varepsilon];X_{\rho_*,a})} \leq C|u_0|_{X_{\rho_0,a_0}},$$

with T_*,ρ_*,a,C independent of ε. Thus letting $\varepsilon \to 0$, the compactness arguments show that the limit $u \in L^\infty([0,T_*];X_{\rho_*,a})$ solves the system (3.1), proving Theorem 3.1. □

Proof. **(Proof of Theorem 1.1.)** Taking

$$u = u_3^{B,1}, \quad v = -\int_{-\infty}^{x_1} \partial_y u_3^{B,1}(t,z,y)dz,$$

the system (3.1) implies that the function

$$\mathcal{U}_3^{p,1} = u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}$$

satisfies

$$\begin{cases} \partial_t \mathcal{U}_3^{p,1} - \partial_y^2 \mathcal{U}_3^{p,1} + \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_3^{p,1} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_3^{p,1} + \overline{\partial_3 p^{I,1}} + y\overline{\partial_3^2 p^{I,0}} = 0, \\ \partial_y \mathcal{U}_3^{p,1}(t,x_1,0) = 0, \\ \mathcal{U}_3^{p,1}(0,x_1,y) = u_{0,3}^{B,1}(x_1,y) + \overline{u_{0,3}^{I,1}}(x_1) + y\overline{\partial_3 u_{0,3}^{I,0}}(x_1), \end{cases}$$

with

$$\mathcal{U}_1^{p,0} = -\int_{-\infty}^{x_1} \partial_y u_3^{B,1}(t,z,y)dz + \overline{u_1^{I,0}}.$$

So we need to check the boundary condition

$$\mathcal{U}_3^{p,1}|_{y=0} = u_3^{B,1}(t,x_1,0) + \overline{u_3^{I,1}}(t,x_1) = 0. \tag{3.12}$$

For this purpose, we first use Theorem 3.1 to determine $u_3^{B,1}$, then use Theorem 2.2 to solve the linearized Euler system (1.7) with the boundary condition

$$u_3^{I,1}|_{x_3=0} = -u_3^{B,1}(t,x_1,0).$$

For the component $\mathcal{U}_1^{p,0}$, using the divergence-free properties of $u^{I,0}$, we have firstly

$$\begin{aligned} \mathcal{U}_1^{p,0}|_{y=0} &= -\int_{-\infty}^{x_1} \partial_y u_3^{B,1}(t,z,0)dz + \overline{u_1^{I,0}}(t,x_1) \\ &= \int_{-\infty}^{x_1} \overline{\partial_3 u_3^{I,0}}(t,z)dz + \overline{u_1^{I,0}}(t,x_1) \\ &= -\int_{-\infty}^{x_1} \overline{\partial_1 u_1^{I,0}}(t,z)dz + \overline{u_1^{I,0}}(t,x_1) \\ &= 0. \end{aligned}$$

On the other hand, since $u_3^{B,1} \in L^\infty([0,T_*];X_{\rho_*,a})$, we have the limit

$$\begin{aligned} \lim_{y\to+\infty} \mathcal{U}_1^{p,0}(t,x_1,y) &= -\lim_{y\to+\infty} \int_{-\infty}^{x_1} \partial_y u_3^{B,1}(t,z,y)dz + \overline{u_1^{I,0}}(t,x_1) \\ &= \overline{u_1^{I,0}}(t,x_1). \end{aligned}$$

So the boundary conditions for $\mathcal{U}_1^{p,0}$ are satisfied. Finally, for the pressure term of the first equation in (P1), once we obtain $\mathcal{U}_3^{p,1}$, $\mathcal{U}_1^{p,0}$ and $\overline{\partial_1 p^{I,0}}$, it is enough to put

$$\partial_1 p^{B,0} = -\partial_t \mathcal{U}_1^{p,0} + \partial_y^2 \mathcal{U}_1^{p,0} - \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_1^{p,0} - \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_1^{p,0} - \overline{\partial_1 p^{I,0}}.$$

We then complete the proof of Theorem 1.1. □

4. Uniform energy estimates

In this section we proceed through the following lemmas to prove Proposition 3.2. To simplify the notations in the following proof we will write u instead of u^ε, omitting the superscript ε, and use C in the following discussion to denote different suitable constants, which depend only on $a_0,\rho_0,\tau,\|u_3^{I,0}\|_{\mathcal{A}_\tau}$ and $\|u_1^{I,0}\|_{\mathcal{A}_\tau}$.

In view of the definition of $|\cdot|_{X_{\rho,a}}$ it suffices to estimate terms

$$\sum_{m\le 2}\left(\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m u\right\|_{L^2(\mathbb{R}^2_+)}\right)+\sum_{m\ge 3}\left(\frac{\rho^{m-1}}{(m-3)!}\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m u\right\|_{L^2(\mathbb{R}^2_+)}\right) \tag{4.1}$$

and

$$\sum_{m\le 2}\left(\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y u\right\|_{L^2(\mathbb{R}^2_+)}\right)+\sum_{m\ge 3}\left(\frac{\rho^{m-1}}{(m-3)!}\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y u\right\|_{L^2(\mathbb{R}^2_+)}\right) \tag{4.2}$$

Here we first treat the terms in (4.2), and the ones in (4.1) can be deduced similarly with simpler arguments. To do so, we use the notation $\omega=\partial_y u$. Then it follows from (3.1) that

$$\begin{cases}\left(\partial_t-\partial_y^2+\overline{\partial_3 u_3^{I,0}}y\partial_y\right)\omega+\left(v+\overline{u_1^{I,0}}\right)\partial_1\omega\\ \qquad +(u-u(t,x_1,0))\partial_y\omega+2\overline{\partial_3 u_3^{I,0}}\omega+\left(\partial_1 u(t,x_1,0)+y\overline{\partial_1\partial_3 u_3^{I,0}}\right)\partial_y v\\ \qquad +(\partial_y v)\partial_1 u+\omega^2+\overline{\partial_1\partial_3 u_3^{I,0}}v=0,\\ \omega|_{y=0}=\overline{\partial_1 u_1^{I,0}}(t,x_1),\\ \omega|_{t=0}=\partial_y u_{3,0}^{B,1}.\end{cases} \tag{4.3}$$

Thus the function, defined by

$$\varphi_m=\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\omega(t)=\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y u(t), \tag{4.4}$$

solves the equation

$$\begin{cases}\left(\partial_t-\partial_y^2+\overline{\partial_3 u_3^{I,0}}y\partial_y\right)\varphi_m-a'(t)y^2\varphi_m+\left(v+\overline{u_1^{I,0}}\right)\partial_1\varphi_m\\ \qquad\qquad +(u-u(t,x_1,0))\partial_y\varphi_m=\mathcal{R}^m(t),\\ \varphi_m\big|_{y=0}=\langle x_1\rangle^\ell\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1),\\ \varphi_m\big|_{t=0}=\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y u_{3,0}^{B,1},\end{cases}$$

where

$$\mathcal{R}^m(t)=\sum_{j=1}^{11}\mathcal{R}_j^m(t)$$

with

$$\begin{aligned}\mathcal{R}_1^m&=-4ay\partial_y\varphi_m+4a^2y^2\varphi_m-2a\varphi_m,\\ \mathcal{R}_2^m&=2ay^2\overline{\partial_3 u_3^{I,0}}\varphi_m+2ay(u-u(t,x_1,0))\varphi_m,\end{aligned}$$

$$\mathcal{R}_3^m = \left(\partial_1 \langle x_1\rangle^\ell\right) e^{ay^2} \left(v + \overline{u_1^{I,0}}\right) \partial_1^m \omega,$$
$$\mathcal{R}_4^m = -\sum_{k=1}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\partial_1^k \overline{\partial_3 u_3^{I,0}}\right) y\partial_y \partial_1^{m-k}\omega,$$
$$\mathcal{R}_5^m = -\sum_{k=1}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\partial_1^k v + \partial_1^k \overline{u_1^{I,0}}\right) \partial_1^{m-k+1}\omega,$$
$$\mathcal{R}_6^m = -\sum_{k=1}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\partial_1^k u - \partial_1^k u(t,x_1,0)\right) \partial_1^{m-k}\partial_y\omega,$$
$$\mathcal{R}_7^m = -\sum_{k=0}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\partial_1^k \overline{\partial_3 u_3^{I,0}}\right) \partial_1^{m-k}\omega,$$
$$\mathcal{R}_8^m = -\sum_{k=0}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\partial_1^{k+1} u(t,x_1,0) + y\overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\right) \partial_1^{m-k}\partial_y v,$$
$$\mathcal{R}_9^m = -\sum_{k=0}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\partial_1^{k+1} u\right) \partial_1^{m-k}\partial_y v,$$
$$\mathcal{R}_{10}^m = -\sum_{k=0}^m \binom{m}{k} \langle x_1\rangle^\ell e^{ay^2} \left(\left(\partial_1^k\omega\right) \partial_1^{m-k}\omega + \overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\partial_1^{m-k} v\right).$$

From the first equation in (4.3), it follows that

$$\begin{aligned}
&\left(\left(\partial_t - \partial_y^2 + \overline{\partial_3 u_3^{I,0}} y\partial_y\right)\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)} - \left(a'(t)y^2\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}\\
&\quad + \left(\left(v + \overline{u_1^{I,0}}\right)\partial_1\varphi_m + (u - u(t,x_1,0))\partial_y\varphi_m(t),\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}\\
&= \left(\mathcal{R}^m(t),\ \varphi_m(t)\right)_{L^2(\mathbb{R}_+^2)},
\end{aligned} \tag{4.5}$$

with $\mathcal{R}^m$ given above. Here, the main difficulty comes from the terms which contain $m+1$ derivatives in x_1, say $\mathcal{R}_8^m$ and $\mathcal{R}_9^m$. Then, the main part of this section consists in the proof of Lemma 4.2, which gives an estimate of $\mathcal{R}_8^m$ term. The estimate of $\mathcal{R}_9^m$ term will be given in Lemma 4.3. The estimates of other terms are almost straightforward.

In the following lemmas, let $0 < a(t) < a_0$ to be determined later, and let

$$0 < \rho = \rho(t) \le \min\{\rho_0/2, \tau/3\}$$

be an arbitrary smooth function of t.

Lemma 4.1. *A constants C exists such that for any $N \ge 3$,*

$$\sum_{m=3}^N \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \left(\mathcal{R}_1^m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)} \le C\sum_{m=3}^N \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \left\|y\varphi_m\right\|_{L^2(\mathbb{R}_+^2)}^2,$$

and

$$\sum_{m=3}^N \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \left(\mathcal{R}_2^m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)} \le C\sum_{m=3}^N \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \left\|y\varphi_m\right\|_{L^2(\mathbb{R}_+^2)}^2 + C|u|_{X_{\rho,a}}^4.$$

Proof. We have, integrating by parts,

$$(\mathcal{R}_1^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)} = \big(4a^2y^2\varphi_m,\ \varphi_m\big)_{L^2(\mathbb{R}^2_+)} = 4a^2\big\|y\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)}$$

Direct verification shows

$$(\mathcal{R}_2^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)} \le \Big(2a\big\|\overline{\partial_3 u_3^{I,0}}\big\|_{L^\infty} + a^2\Big)\,\big\|y\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)} + 4\big\|u\big\|^2_{L^\infty(\mathbb{R}^2_+)}\big\|\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)}.$$

Observe

$$\big\|\overline{\partial_3 u_3^{I,0}}\big\|_{L^\infty} \le C\big\|u_3^{I,0}\big\|_{G_\tau}$$

and

$$\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\big\|\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)} \le \sum_{m=3}^{\infty}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\big\|\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)} \le |u|_{X_{\rho,a}},$$

and thus the desired results follow, completing the proof. □

LEMMA 4.2. *There exists a constant C such that for any ρ with $0<\rho\le\tau/3$, we have*

$$\begin{aligned}&\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_8^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)}\\ &\le \frac{1}{8}|u|^2_{Z_{\rho,a}} + C\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\big\|y\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)} + C\,|u|_{Z_{\rho,a}}\,|u|^2_{Y_{\rho,a}}.\end{aligned}$$

Proof. Recall $\mathcal{R}_8^{m,j}$ can be written as, for any $\tilde\varepsilon>0$,

$$\begin{aligned}(\mathcal{R}_8^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)} &= \left(-\sum_{k=0}^{m}\binom{m}{k}\langle x_1\rangle^{\ell}e^{ay^2}\left(\partial_1^{k+1}u(x_1,0)\right)\partial_1^{m-k}\partial_y v,\ \varphi_m\right)_{L^2(\mathbb{R}^2_+)}\\ &\quad+\left(-\sum_{k=0}^{m}\binom{m}{k}\langle x_1\rangle^{\ell}e^{ay^2}\left(y\overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\right)\partial_1^{m-k}\partial_y v,\ \varphi_m\right)_{L^2(\mathbb{R}^2_+)}\\ &\le\sum_{k=0}^{m}\binom{m}{k}\big\|\langle x_1\rangle^{\ell}\partial_1^{k+1}u\big\|_{L^\infty_y L^2_{x_1}}\big\|e^{ay^2}\partial_1^{m-k}\partial_y v\big\|_{L^2_y L^\infty_{x_1}}\big\|\varphi_m\big\|_{L^2}\\ &\quad+\tilde\varepsilon\left[\sum_{k=0}^{m}\binom{m}{k}\big\|\langle x_1\rangle^{\ell}\overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\big\|_{L^2_{x_1}(\mathbb{R})}\big\|e^{ay^2}\partial_1^{m-k}\partial_y v\big\|_{L^2_y L^\infty_{x_1}(\mathbb{R}^2_+)}\right]^2\\ &\quad+C_{\tilde\varepsilon}\big\|y\varphi_m\big\|^2_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

Then it suffices to show that

$$\begin{aligned}&\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\sum_{k=0}^{m}\binom{m}{k}\big\|\langle x_1\rangle^{\ell}\partial_1^{k+1}u\big\|_{L^\infty_y L^2_{x_1}}\big\|e^{ay^2}\partial_1^{m-k}\partial_y v\big\|_{L^2_y L^\infty_{x_1}}\big\|\varphi_m\big\|_{L^2}\\ &\le C\,|u|_{Z_{\rho,a}}\,|u|^2_{Y_{\rho,a}},\end{aligned}\tag{4.6}$$

and

$$\sum_{m=3}^{N}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\left[\sum_{k=0}^{m}\binom{m}{k}\left\|\langle x_1\rangle^{\ell}\overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\right\|_{L^2_{x_1}}\left\|e^{ay^2}\partial_1^{m-k}\partial_y v\right\|_{L^2_y L^\infty_{x_1}}\right]^2\leq C|u|^2_{Z_{\rho,a}}. \quad (4.7)$$

We will proceed to prove the above estimate through the following steps.

Step (a) We begin with several estimates to be used later in the proof. Firstly in view of the definition of $|\cdot|_{Y_{\rho,a}}$ given in (3.5), we may write

$$|u|^2_{Y_{\rho,a}}=\sum_{m=0}^{+\infty}|u|^2_{Y_{\rho,a,m}}$$

where $|u|_{Y_{\rho,a,m}}$ is defined by

$$|u|_{Y_{\rho,a,m}}=\begin{cases}\sum_{0\leq j\leq 1}\left\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^j u\right\|_{L^2(\mathbb{R}^2_+)}, & 0\leq m\leq 2\\ \sum_{0\leq j\leq 1}(m-1)^{1/2}\rho^{-/2}\frac{\rho^{m-1}}{(m-3)!}\left\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^j u\right\|_{L^2(\mathbb{R}^2_+)}, & m\geq 3.\end{cases}$$

Thus

$$\|\varphi_m\|_{L^2(\mathbb{R}^2_+)}\leq\begin{cases}|u|_{Y_{\rho,a}}, & 0\leq m\leq 2,\\ |u|_{Y_{\rho,a,m}}m^{-1/2}\rho^{1/2}\frac{(m-3)!}{\rho^{m-1}}, & m\geq 3,\end{cases}\quad (4.8)$$

Next, from the relations (3.2), it follows that

$$\left\|e^{ay^2}\partial_y v\right\|_{L^2_y(\mathbb{R}_+;\ L^\infty_{x_1}(\mathbb{R}))}\leq C\left\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_y^2 u\right\|_{L^2(\mathbb{R}^2_+)}\leq C|u|_{Z_{\rho,a}},$$

and that for $j\geq 1$,

$$\begin{aligned}\left\|e^{ay^2}\partial_1^j\partial_y v\right\|_{L^2_y(\mathbb{R}_+;\ L^\infty_{x_1}(\mathbb{R}))}&=\left\|e^{ay^2}\partial_1^{j-1}\partial_y^2 u\right\|_{L^2_y(\mathbb{R}_+;\ L^\infty_{x_1}(\mathbb{R}))}\\&\leq C\left\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^j\partial_y^2 u\right\|_{L^2(\mathbb{R}^2_+)}\\&\leq\begin{cases}C|u|_{Z_{\rho,a}}, & 1\leq j\leq 2,\\ C|u|_{Z_{\rho,a,j}}\frac{(j-3)!}{\rho^{j-1}}, & j\geq 3,\end{cases}\end{aligned}$$

where $|u|_{Z_{\rho,a,k}}$ is defined by the relation $|u|_{Z_{\rho,a}}=\sum_{k\geq 0}|u|^2_{Z_{\rho,a,k}}$, so that

$$|u|_{Z_{\rho,a,k}}=\begin{cases}\sum_{1\leq j\leq 2}\left\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^k\partial_y^j u\right\|_{L^2(\mathbb{R}^2_+)}, & 0\leq k\leq 2\\ \sum_{1\leq j\leq 2}\frac{\rho^{k-1}}{(k-3)!}\left\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^k\partial_y^j u\right\|_{L^2(\mathbb{R}^2_+)}, & k\geq 3.\end{cases}$$

Thus we conclude

$$\left\|e^{ay^2}\partial_1^j\partial_y v\right\|_{L^2_y(\mathbb{R}_+;\ L^\infty_{x_1}(\mathbb{R}))}\leq\begin{cases}C|u|_{Z_{\rho,a}}, & 0\leq j\leq 2,\\ C|u|_{Z_{\rho,a,j}}\frac{(j-3)!}{\rho^{j-1}}, & j\geq 3.\end{cases}\quad (4.9)$$

Using the Sobolev inequality

$$\left\|\langle x_1\rangle^{\ell}\partial_1^j u\right\|_{L^\infty_y(\mathbb{R}_+;\ L^2_{x_1}(\mathbb{R}))}\leq C\left\|\langle x_1\rangle^{\ell}\partial_1^j u\right\|_{L^2(\mathbb{R}_+)}+C\left\|\langle x_1\rangle^{\ell}\partial_1^j\partial_y u\right\|_{L^2(\mathbb{R}_+)},$$

gives

$$\big\| \langle x_1\rangle^{\ell}\, \partial_1^j u\big\|_{L^\infty_y(\mathbb{R}_+;\ L^2_{x_1}(\mathbb{R}))} \leq \begin{cases} C\,|u|_{Y_{\rho,a}}, & \text{if } 0\leq j\leq 2,\\ C\,|u|_{Y_{\rho,a,j}}\, j^{-1/2}\rho^{1/2}\frac{(j-3)!}{\rho^{j-1}}, & \text{if } j\geq 3. \end{cases} \tag{4.10}$$

Finally,

$$\forall\ k\geq 0,\quad \big\| \langle x_1\rangle^{\ell}\, \overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\big\|_{L^2_{x_1}(\mathbb{R})} \leq C\big\|u_3^{I,0}\big\|_{\mathcal{A}_\tau}\frac{(k+3)!}{\tau^{k+3}} \tag{4.11}$$

due to (2.9).

Step (b). We now prove (4.7). For this purpose we use (4.11) and (4.9) to calculate

$$\begin{aligned}
&\sum_{k=0}^{m}\binom{m}{k}\big\| \langle x_1\rangle^{\ell}\, \overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\big\|_{L^2_{x_1}(\mathbb{R})}\big\|e^{ay^2}\partial_1^{m-k}\partial_y v\big\|_{L^2_y(\mathbb{R}_+;\ L^\infty_{x_1}(\mathbb{R}))}\\
&\leq C\big\|u_3^{I,0}\big\|_{\mathcal{A}_\tau}\sum_{k=0}^{m-3}\frac{m!}{k!(m-k)!}\frac{(k+3)!}{\tau^{k+3}}\frac{(m-k-3)!}{\rho^{m-k-1}}|u|_{Z_{\rho,a,m-k}}\\
&\quad +C\big\|u_3^{I,0}\big\|_{\mathcal{A}_\tau}\sum_{k=m-2}^{m}\frac{m!}{k!(m-k)!}\frac{(k+3)!}{\tau^{k+3}}|u|_{Z_{\rho,a,m-k}}\\
&\leq C\frac{(m-3)!}{\rho^{m-1}}\sum_{k=0}^{m-3}\frac{m^3}{k^3(m-k-2)^3}\frac{2^k\rho^k}{\tau^{k+3}}|u|_{Z_{\rho,a,m-k}}+C\frac{(m-3)!}{\rho^{m-1}}\sum_{k=m-2}^{m}\frac{2^k\rho^{m-1}}{\tau^{k+3}}|u|_{Z_{\rho,a,m-k}}\\
&\leq C\tau^{-3}\frac{(m-3)!}{\rho^{m-1}}\sum_{k=0}^{m-3}\frac{2^k\rho^k}{\tau^k}|u|_{Z_{\rho,a,m-k}}+C\tau^{-3}\frac{(m-3)!}{\rho^{m-1}}\sum_{k=m-2}^{m}\frac{2^k\rho^{m-1}}{\tau^k}|u|_{Z_{\rho,a,m-k}},
\end{aligned}$$

which yields

$$\begin{aligned}
&\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left[\sum_{k=0}^{m}\binom{m}{k}\big\| \langle x_1\rangle^{\ell}\, \overline{\partial_1^{k+1}\partial_3 u_3^{I,0}}\big\|_{L^2_{x_1}(\mathbb{R})}\big\|e^{ay^2}\partial_1^{m-k}\partial_y v\big\|_{L^2_y(\mathbb{R}_+;\ L^\infty_{x_1}(\mathbb{R}))}\right]^2\\
&\leq C\sum_{m=3}^{N}\left(\sum_{k=0}^{m-3}\frac{2^k\rho^k}{\tau^k}|u|_{Z_{\rho,a,m-k}}\right)^2+C\sum_{m=3}^{N}\left(\sum_{k=m-2}^{m}\frac{2^k\rho^{m-1}}{\tau^k}|u|_{Z_{\rho,a,m-k}}\right)^2.
\end{aligned}$$

On the other hand, by virtue of Young's inequality for discrete convolution (cf. [19, Theorem 20.18]) we have

$$\sum_{m=3}^{N}\left(\sum_{k=0}^{m-3}\frac{2^k\rho^k}{\tau^k}|u|_{Z_{\rho,a,m-k}}\right)^2\leq C\left(\sum_{k=0}^{N}\frac{2^k\rho^k}{\tau^k}\right)^2\sum_{k=0}^{N}|u|^2_{Z_{\rho,a,k}}\leq C|u|^2_{Z_{\rho,a}},$$

since $\rho\leq\tau/3$. And direct computation yields

$$\sum_{m=3}^{N}\left(\sum_{k=m-2}^{m}\frac{2^k\rho^{m-1}}{\tau^k}|u|_{Z_{\rho,a,m-k}}\right)\leq C|u|^2_{Z_{\rho,a}}.$$

Then we obtain (4.7), combining the above inequalities.

Step (c). Now we check (4.6) and write

$$\sum_{k=0}^{m}\binom{m}{k}\left\|\langle x_1\rangle^\ell\partial_1^{k+1}u\right\|_{L_y^\infty(\mathbb{R}_+;\ L_{x_1}^2(\mathbb{R}))}\left\|e^{ay^2}\partial_1^{m-k}\partial_y v\right\|_{L_y^2(\mathbb{R}_+;\ L_{x_1}^\infty(\mathbb{R}))}\left\|\varphi_m\right\|_{L^2(\mathbb{R}_+^2)}$$
$$\le S_1+S_2+S_3$$

with

$$S_1=\sum_{k=0}^{2}\binom{m}{k}\left\|\langle x_1\rangle^\ell\partial_1^{k+1}u\right\|_{L_y^\infty(\mathbb{R}_+;\ L_{x_1}^2(\mathbb{R}))}\left\|e^{ay^2}\partial_1^{m-k}\partial_y v\right\|_{L_y^2(\mathbb{R}_+;\ L_{x_1}^\infty(\mathbb{R}))}\left\|\varphi_m\right\|_{L^2(\mathbb{R}_+^2)},$$
$$S_2=\sum_{k=3}^{m-3}\binom{m}{k}\left\|\langle x_1\rangle^\ell\partial_1^{k+1}u\right\|_{L_y^\infty(\mathbb{R}_+;\ L_{x_1}^2(\mathbb{R}))}\left\|e^{ay^2}\partial_1^{m-k}\partial_y v\right\|_{L_y^2(\mathbb{R}_+;\ L_{x_1}^\infty(\mathbb{R}))}\left\|\varphi_m\right\|_{L^2(\mathbb{R}_+^2)}$$

and

$$S_3=\sum_{k=m-2}^{m}\binom{m}{k}\left\|\langle x_1\rangle^\ell\partial_1^{k+1}u\right\|_{L_y^\infty(\mathbb{R}_+;\ L_{x_1}^2(\mathbb{R}))}\left\|e^{ay^2}\partial_1^{m-k}\partial_y v\right\|_{L_y^2(\mathbb{R}_+;\ L_{x_1}^\infty(\mathbb{R}))}\left\|\varphi_m\right\|_{L^2(\mathbb{R}_+^2)}.$$

For the term $S_{2,m}$, we use (4.8), (4.9) and (4.10) to compute

$$\begin{aligned}
S_{2,m}&=\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\sum_{k=3}^{m-3}\binom{m}{k}\left\|\langle x_1\rangle^\ell\partial_1^{k+1}u\right\|_{L_y^\infty(\mathbb{R}_+;\ L_{x_1}^2(\mathbb{R}))}\\
&\qquad\qquad\times\left\|e^{ay^2}\partial_1^{m-k}\partial_y v\right\|_{L_y^2(\mathbb{R}_+;\ L_{x_1}^\infty(\mathbb{R}))}\left\|\varphi_m\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\sum_{k=3}^{m-3}\frac{m!}{k!(m-k)!}\left[k^{-\frac12}\rho^{\frac12}\frac{(k-2)!}{\rho^k}|u|_{Y_{\rho,a,k+1}}\right]\\
&\qquad\qquad\times\frac{(m-k-3)!}{\rho^{m-k-1}}|u|_{Z_{\rho,a,m-k}}m^{-\frac12}\rho^{\frac12}\frac{(m-3)!}{\rho^{m-1}}|u|_{Y_{\rho,a,m}}\\
&\le C\rho|u|_{Y_{\rho,a,m}}\sum_{k=3}^{m-3}\frac{m^3}{k^2(m-k-2)^3}k^{-\frac12}m^{-\frac12}|u|_{Y_{\rho,a,k+1}}|u|_{Z_{\rho,a,m-k}}\\
&\le C\rho|u|_{Y_{\rho,a,m}}\left(\sum_{k=3}^{m-3}\frac{m^3}{k^2(m-k-2)^3}k^{-\frac12}m^{-\frac12}|u|^2_{Y_{\rho,a,k+1}}\right)^{\frac12}\\
&\qquad\qquad\times\left(\sum_{k=3}^{m-3}\frac{m^3}{k^2(m-k-2)^3}k^{-\frac12}m^{-\frac12}|u|^2_{Z_{\rho,a,m-k}}\right)^{1/2}\\
&\le C\rho|u|_{Z_{\rho,a,m}}|u|^2_{Y_{\rho,a,m}},
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{m=3}^{N}S_{2,m}&\le C\rho\left[\sum_{m=3}^{N}|u|^2_{Y_{\rho,a,m}}\right]^{\frac12}\left(\sum_{m=3}^{N}\left[\sum_{k=3}^{m-3}\frac{m^3}{k^2(m-k-2)^3}k^{-\frac12}m^{-\frac12}|u|_{Y_{\rho,a,k+1}}|u|_{Z_{\rho,a,m-k}}\right]^2\right)^{\frac12}\\
&\le C\rho|u|_{Y_{\rho,a}}\left(\sum_{m=3}^{N}\left[\sum_{k=3}^{m-3}\frac{1}{k^2}|u|_{Y_{\rho,a,k+1}}|u|_{Z_{\rho,a,m-k}}\right]^2\right)^{\frac12}
\end{aligned}$$

$$+C\rho|u|_{Y_{\rho,a}}\left(\sum_{m=3}^{N}\left[\sum_{k=3}^{m-3}\frac{1}{(m-k-2)^3}|u|_{Y_{\rho,a,k+1}}|u|_{Z_{\rho,a,m-k}}\right]^2\right)^{\frac{1}{2}}$$

the last inequality following from the fact that

$$\forall\ 3\le k\le m-3,\quad \frac{m^3}{k^2(m-k-2)^3}k^{-\frac{1}{2}}m^{-\frac{1}{2}}\le C\left(\frac{1}{k^2}+\frac{1}{(m-k-2)^3}\right).$$

Moreover, by virtue of Young's inequality for discrete convolution (cf. [19, Theorem 20.18]) we obtain

$$\begin{aligned}\left(\sum_{m=3}^{N}\left[\sum_{k=3}^{m-3}\frac{1}{k^2}|u|_{Y_{\rho,a,k+1}}|u|_{Z_{\rho,a,m-k}}\right]^2\right)^{1/2} &\le C\left(\sum_{m=3}^{+\infty}|u|^2_{Z_{\rho,a,m}}\right)^{1/2}\sum_{k=3}^{+\infty}\frac{1}{k^2}|u|_{Y_{\rho,a,k}}\\ &\le C|u|_{Z_{\rho,a}}\left(\sum_{k=1}^{+\infty}|u|^2_{Y_{\rho,a,k}}\right)^{1/2}\left(\sum_{k=1}^{+\infty}\frac{1}{k^4}\right)^{1/2}\\ &\le C|u|_{Z_{\rho,a}}|u|_{Y_{\rho,a}}.\end{aligned}$$

Similarly

$$\left(\sum_{m=3}^{N}\left[\sum_{k=3}^{m-3}\frac{1}{(m-k-2)^3}|u|_{Y_{\rho,a,k+1}}|u|_{Z_{\rho,a,m-k}}\right]^2\right)^{1/2}\le C|u|_{Z_{\rho,a}}|u|_{Y_{\rho,a}}.$$

Combining these inequalities we conclude

$$\sum_{m=3}^{N}S_{2,m}\le C\rho|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}}\le C|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}}.$$

The estimates on the rest two terms S_1 and S_3 can be deduced similarly and directly, and we have

$$\sum_{m=3}^{N}(S_{1,m}+S_{3,m})\le C|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}},$$

proving (4.6). The proof of Lemma 4.2 is complete. □

LEMMA 4.3. *A constant C exists such that*

$$\begin{aligned}&\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_3^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)}\le C\left(|u|^3_{X_{\rho,a,m}}+|u|^2_{X_{\rho,a,m}}\right),\\ &\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_4^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)}\le \frac{1}{8}|u|^2_{Z_{\rho,a}}+C\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left\|y\varphi_m\right\|^2_{L^2(\mathbb{R}^2_+)},\\ &\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_5^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)}\le C\left(|u|^3_{X_{\rho,a,m}}+|u|^2_{X_{\rho,a,m}}\right),\\ &\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_6^m,\ \varphi_m)_{L^2(\mathbb{R}^2_+)}\le C|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}},\end{aligned}$$

$$\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_7^m,\ \varphi_m)_{L^2(\mathbb{R}_+^2)}\leq C|u|_{X_{\rho,a}}^2,$$
$$\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_9^m,\ \varphi_m)_{L^2(\mathbb{R}_+^2)}\leq C|u|_{Z_{\rho,a}}|u|_{Y_{\rho,a}}^2,$$
$$\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2(\mathcal{R}_{10}^m,\ \varphi_m)_{L^2(\mathbb{R}_+^2)}\leq C\left(|u|_{X_{\rho,a}}^3+|u|_{X_{\rho,a}}^2\right).$$

Proof. The treatment of $\mathcal{R}_6,\mathcal{R}_9$ is exactly the same as in the proof of (4.6). The other terms can be deduced similarly by following the proof in Lemma 4.2 with slight changes, and the arguments here will be simpler since there is no term with the highest derivative ∂_1^{m+1} involved. This means we can perform the estimates with the norm $Y_{\rho,a}$ in Lemma 4.2 replaced by $X_{\rho,a}$ here. So we omit the proof for brevity. □

Combining the estimates in Lemma 4.1-Lemma 4.3, we have

COROLLARY 4.1. *There are two constants C,C_0 such that*

$$\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\sum_{k=1}^{10}(\mathcal{R}_k^m,\ \varphi_m)_{L^2(\mathbb{R}_+^2)}$$
$$\leq\frac{1}{4}|u|_{Z_{\rho,a}}^2+C_0\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|y\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2+C|u|_{Z_{\rho,a}}|u|_{Y_{\rho,a}}^2+C\left(|u|_{X_{\rho,a}}^2+|u|_{X_{\rho,a}}^4\right).$$

LEMMA 4.4. *We have*

$$\left(\left(\partial_t-\partial_y^2+\overline{\partial_3u_3^{I,0}}y\partial_y\right)\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}-\left(a'(t)y^2\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}$$
$$+\left(\left(v+\overline{u_1^{I,0}}\right)\partial_1\varphi_m+(u-u(t,x_1,0))\partial_y\varphi_m(t),\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}$$
$$\geq\frac{1}{2}\frac{d}{dt}\|\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2+\|\partial_y\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2-a'(t)\|y\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2$$
$$+\frac{d}{dt}\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\,dx_1-C\left(\frac{(m-3)!}{\rho^{m-1}}\right)^2|u|_{X_{\rho,a,m}}^2.$$

Proof. Firstly we calculate, integrating by parts and using the relation (3.2),

$$\left|\left(\overline{\partial_3u_3^{I,0}}y\partial_y\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}\right|+\left|\left(\left(v+\overline{u_1^{I,0}}\right)\partial_1\varphi_m+(u-u(t,x_1,0))\partial_y\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}\right|$$
$$\leq\frac{1}{2}\left(\|\overline{\partial_3u_3^{I,0}}\|_{L^\infty}+\|\overline{\partial_3u_1^{I,0}}\|_{L^\infty}\right)\|\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2.\tag{4.12}$$

Integrating by parts and using the boundary condition in (4.3), we have

$$\left(\left(\partial_t-\partial_y^2\right)\varphi_m,\ \varphi_m\right)_{L^2(\mathbb{R}_+^2)}=\frac{1}{2}\frac{d}{dt}\|\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2+\|\partial_y\varphi_m\|_{L^2(\mathbb{R}_+^2)}^2$$
$$+\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)(\partial_y\varphi_m)(t,x_1,0)\,dx_1.\tag{4.13}$$

Now we check the boundary value of $\partial_y\varphi$. In view of (4.4) we see

$$\partial_y\varphi_m\big|_{y=0}=\langle x_1\rangle^{\ell}\partial_1^m\partial_y^2 u\big|_{y=0}.$$

And moreover, using the relation

$$\langle x_1\rangle^{\ell}\partial_y^2 u\big|_{y=0}=\partial_t u(x_1,0)-\overline{\partial_3 u_3^{I,0}}u(x_1,0)+\overline{u_1^{I,0}}(\partial_1 u)(x_1,0)$$

which follows from (3.1), we conclude

$$\begin{aligned}\partial_y\varphi_m\big|_{y=0}=&\,\partial_t\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\\ &-\langle x_1\rangle^{\ell}\partial_1^m\left(\overline{\partial_3 u_3^{I,0}}u(t,x_1,0)\right)+\langle x_1\rangle^{\ell}\partial_1^m\left(\overline{u_1^{I,0}}(\partial_1 u)(t,x_1,0)\right).\end{aligned}$$

As a result,

$$\begin{aligned}&\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}(\partial_y\varphi_m)(t,x_1,0)\,dx_1\\ =&\frac{d}{dt}\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\,dx_1\\ &-\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_t\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\,dx_1\\ &-\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\partial_1^m\left(\overline{\partial_3 u_3^{I,0}}u(x_1,0)\right)dx_1\\ &+\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\partial_1^m\left(\overline{u_1^{I,0}}(\partial_1 u)(t,x_1,0)\right)dx_1.\end{aligned}$$

Moreover, In view of (2.10), we can repeat the arguments in Lemma 4.2 and Lemma 4.3, to obtain, observing $\rho<\tau/4$,

$$\left|\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_t\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\,dx_1\right|\leq C\left(\frac{(m-3)!}{\rho^{m-1}}\right)^2\|u_1^{I,0}\|_{\mathcal{A}_\tau}|u|_{X_{\rho,a,m}},$$

$$\begin{aligned}&\left|\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\partial_1^m\left(\overline{\partial_3 u_3^{I,0}}u(x_1,0)\right)dx_1\right|\\ \leq&\,C\left(\frac{(m-3)!}{\rho^{m-1}}\right)^2\|u_1^{I,0}\|_{\mathcal{A}_\tau}\|u_3^{I,0}\|_{\mathcal{A}_\tau}|u|_{X_{\rho,a,m}}\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\partial_1^m\left(\overline{u_1^{I,0}}(\partial_1 u)(t,x_1,0)\right)dx_1\right|\\ \leq&\,C\left(\frac{(m-3)!}{\rho^{m-1}}\right)^2\|u_1^{I,0}\|_{\mathcal{A}_\tau}^2|u|_{X_{\rho,a,m}}.\end{aligned}$$

Combing these inequalities above, we conclude

$$\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,(\partial_y\varphi_m)(t,x_1,0)\,dx_1$$
$$\geq\frac{d}{dt}\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1-C\left(\frac{(m-3)!}{\rho^{m-1}}\right)^2|u|_{X_{\rho,a,m}},$$

which, along with (4.12) and (4.13), yields the conclusion, completing the proof. □

LEMMA 4.5. *Let $a(t)=a_0-\left(2a_0^2+C_0\right)t$ with C_0 the constants given in Corollary 4.1. Then for any N,*

$$\frac{1}{2}\frac{d}{dt}\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+\frac{1}{2}\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left\|\langle x_1\rangle^\ell e^{ay^2}\partial_1^m\partial_y^2u(t)\right\|^2_{L^2(\mathbb{R}^2_+)}$$
$$-\rho'(t)\sum_{m=3}^{N}\left((m-1)^{1/2}\rho^{-1/2}\frac{\rho^{m-1}}{(m-3)!}\|\varphi_m\|_{L^2(\mathbb{R}^2_+)}\right)^2$$
$$\leq\frac{1}{4}|u|^2_{Z_{\rho,a}}-\frac{d}{dt}\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1$$
$$+C\left(|\rho'|\rho^{-2}|u|_{X_{\rho,a}}+|u|^2_{X_{\rho,a}}+|u|^4_{X_{\rho,a}}\right).$$

Proof. Using the equality (4.5) and Lemma 4.4, we obtain

$$\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left[\frac{1}{2}\frac{d}{dt}\|\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+\|\partial_y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}-a'(t)\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}\right]$$
$$\leq-\frac{d}{dt}\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1$$
$$+\sum_{m=3}^{N}(2m-2)\frac{\rho'(t)\rho^{2m-3}}{[(m-3)!]^2}\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1+|u|_{\rho,a}$$
$$+\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left(\sum_{k=1}^{10}(\mathcal{R}_k^m(t),\ \varphi_m(t))_{L^2(\mathbb{R}^2_+)}\right)$$
$$\leq-\frac{d}{dt}\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1$$
$$+\sum_{m=3}^{N}(2m-2)\frac{\rho'(t)\rho^{2m-3}}{[(m-3)!]^2}\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1+|u|_{\rho,a}$$
$$+\frac{1}{4}|u|^2_{Z_{\rho,a}}+C_0\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+C|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}}+C\left(|u|^2_{X_{\rho,a}}+|u|^4_{X_{\rho,a}}\right),$$

the last inequality following from Corollary 4.1. On the other hand,

$$\sum_{m=3}^{N}(2m-2)\frac{\rho'(t)\rho^{2m-3}}{[(m-3)!]^2}\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\,\langle x_1\rangle^\ell\,\partial_1^m u(t,x_1,0)\,dx_1$$
$$\leq C\sum_{m=3}^{N}\frac{2^m\rho'(t)\rho^{2m-3}}{[(m-3)!]^2}\left\|\langle x_1\rangle^\ell\,\overline{\partial_1^{m+1}u_1^{I,0}}\right\|_{L^2(\mathbb{R}_{x_1})}\left(\left\|\langle x_1\rangle^\ell\partial_1^m u\right\|_{L^2}+\left\|\langle x_1\rangle^\ell\partial_1^m\partial_y u\right\|_{L^2}\right)$$
$$\leq C\sum_{m=3}^{N}\frac{2^m\rho'(t)\rho^{m-2}}{\tau^{m+3}}\left\|u_1^{I,0}\right\|_{G_\tau}\left[\frac{\rho^{m-1}}{(m-3)!}\left(\left\|\langle x_1\rangle^\ell\partial_1^m u\right\|_{L^2}+\left\|\langle x_1\rangle^\ell\partial_1^m\partial_y u\right\|_{L^2}\right)\right]$$

$$\leq C|\rho'|\rho^{-2}|u|_{X_{\rho,a}}$$

the last inequality used the fact that $\rho<\tau/3$. As a result, combining the inequalities above yields

$$\begin{aligned}
&\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left[\frac{1}{2}\frac{d}{dt}\|\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+\|\partial_y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}-a'(t)\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}\right]\\
&\leq \frac{1}{4}|u|^2_{Z_{\rho,a}}-\frac{d}{dt}\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\,dx_1\\
&+C_0\sum_{m=3}^{N}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+C|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}}+C\left(|\rho'|\rho^{-2}|u|_{X_{\rho,a}}+|u|^2_{X_{\rho,a}}+|u|^4_{X_{\rho,a}}\right).
\end{aligned}$$

Moreover from the relations

$$\|\partial_y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}\geq\frac{1}{2}\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^2u(t)\|^2_{L^2(\mathbb{R}^2_+)}-2a_0^2\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|\partial_y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}\\
&\quad-a'(t)\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}-\rho'(t)\left((m-1)^{1/2}\rho^{-1/2}\frac{\rho^{m-1}}{(m-3)!}\|\varphi_m\|_{L^2(\mathbb{R}^2_+)}\right)^2\\
&=\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\left[\frac{1}{2}\frac{d}{dt}\|\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+\|\partial_y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}-a'(t)\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}\right],
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=3}^{N}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\|\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+\frac{1}{2}\sum_{m=3}^{N}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^2u(t)\|^2_{L^2(\mathbb{R}^2_+)}\\
&-\left(a'(t)+2a_0^2\right)\sum_{m=3}^{N}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}-\rho'(t)\sum_{m=3}^{N}\left((m-1)^{\frac{1}{2}}\rho^{-\frac{1}{2}}\frac{\rho^{m-1}}{(m-3)!}\|\varphi_m\|_{L^2(\mathbb{R}^2_+)}\right)^2\\
&\leq\frac{1}{4}|u|^2_{Z_{\rho,a}}-\frac{d}{dt}\sum_{m=3}^{N}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\int_{\mathbb{R}_{x_1}}\langle x_1\rangle^{\ell}\overline{\partial_1^{m+1}u_1^{I,0}}(t,x_1)\langle x_1\rangle^{\ell}\partial_1^m u(t,x_1,0)\,dx_1\\
&+C_0\sum_{m=3}^{N}\left[\frac{\rho^{m-1}}{(m-3)!}\right]^2\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}+C|u|_{Z_{\rho,a}}|u|^2_{Y_{\rho,a}}+C\left(|\rho'|\rho^{-2}|u|_{X_{\rho,a}}+|u|^2_{X_{\rho,a}}+|u|^4_{X_{\rho,a}}\right).
\end{aligned}$$

Now observing $a(t)=a_0-\left(2a_0^2+C_0\right)t$, we complete the proof. □

REMARK 4.1. Here, we are using the fact that the time derivative $a'(t)=-(2a_0^2+C_0)<0$ to be able to cancel all the terms which contain the linearly growing term in y, say $\|y\varphi_m\|^2_{L^2(\mathbb{R}^2_+)}$.

Proof. **(Completion of the proof of Proposition 3.2.)** By Lemma 4.5, we integrate both sides over $[0,t]\subset[0,T]$ and then let $N\to+\infty$, to obtain that for any $t\in[0,T]$,

$$\sum_{m=3}^{+\infty}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|\varphi_m(t)\|^2_{L^2(\mathbb{R}^2_+)}+\int_0^T\sum_{m=3}^{+\infty}\left(\frac{\rho^{m-1}}{(m-3)!}\right)^2\|\langle x_1\rangle^{\ell}e^{ay^2}\partial_1^m\partial_y^2u(t)\|^2_{L^2(\mathbb{R}^2_+)}dt$$

$$
\begin{aligned}
&-\int_0^T \rho'(t) \sum_{m=3}^{+\infty} \frac{m-1}{\rho} \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \|\varphi_m(t)\|_{L^2(\mathbb{R}^2_+)}^2 dt \\
\leq & |u_0|_{X_{\rho_0,a_0}}^2 + \frac{1}{2}\int_0^T |u(t)|_{Z_{\rho,a}}^2 dt + C\int_0^T \left(|\rho'(t)|\rho^{-2}|u(t)|_{X_{\rho,a}} + |u(t)|_{X_{\rho,a}}^2 + |u(t)|_{X_{\rho,a}}^4\right) dt \\
&+ C\int_0^T |u(t)|_{Z_{\rho,a}} |u(t)|_{Y_{\rho,a}}^2 dt.
\end{aligned}
$$

Direct computation also gives

$$
\begin{aligned}
&\sum_{m\leq 2} \|\varphi_m(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \int_0^T \sum_{m\leq 2} \|\langle x_1\rangle^\ell e^{ay^2} \partial_1^m \partial_y^2 u(t)\|_{L^2(\mathbb{R}^2_+)}^2 dt - \int_0^T \rho'(t) \sum_{m\leq 2}^{+\infty} \|\varphi_m(t)\|_{L^2(\mathbb{R}^2_+)}^2 dt \\
\leq & |u_0|_{X_{\rho_0,a_0}}^2 + \frac{1}{2}\int_0^T |u(t)|_{Z_{\rho,a}}^2 dt + C\int_0^T \left(|\rho'(t)|\rho^{-2}|u(t)|_{X_{\rho,a}} + |u(t)|_{X_{\rho,a}}^2 + |u(t)|_{X_{\rho,a}}^4\right) dt \\
&+ C\int_0^T |u(t)|_{Z_{\rho,a}} |u(t)|_{Y_{\rho,a}}^2 dt.
\end{aligned}
$$

Similarly, using the notation

$$
\psi_m = \langle x_1\rangle^\ell e^{ay^2} \partial_1^m u(t),
$$

we can deduce, following the proof of the above two inequalities with slight modification and simpler arguments,

$$
\begin{aligned}
&\sum_{m\leq 2} \|\psi_m(t)\|_{L^2}^2 + \sum_{m=3}^{+\infty} \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \|\psi_m(t)\|_{L^2(\mathbb{R}^2_+)}^2 \\
&+ \int_0^T \left(\sum_{m\leq 2} \|\langle x_1\rangle^\ell e^{ay^2} \partial_1^m \partial_y u(t)\|_{L^2(\mathbb{R}^2_+)}^2 + \sum_{m=3}^{+\infty} \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \|\langle x_1\rangle^\ell e^{ay^2} \partial_1^m \partial_y u(t)\|_{L^2(\mathbb{R}^2_+)}^2 \right) dt \\
&- \int_0^T \rho'(t) \left(\sum_{m\leq 2} \|\psi_m\|_{L^2}^2 + \sum_{m=3}^{+\infty} \frac{m-1}{\rho} \left(\frac{\rho^{m-1}}{(m-3)!}\right)^2 \|\psi_m\|_{L^2(\mathbb{R}^2_+)}^2 \right) dt \\
\leq & |u_0|_{X_{\rho_0,a_0}}^2 + \frac{1}{2}\int_0^T |u(t)|_{Z_{\rho,a}}^2 dt + C\int_0^T \left(|\rho'(t)|\rho^{-2}|u(t)|_{X_{\rho,a}} + |u(t)|_{X_{\rho,a}}^2 + |u(t)|_{X_{\rho,a}}^4\right) dt \\
&+ C\int_0^T |u(t)|_{Z_{\rho,a}} |u(t)|_{Y_{\rho,a}}^2 dt
\end{aligned}
$$

Combining these inequalities we conclude, observing the definition of $|\cdot|_{X_{\rho,a}}, |\cdot|_{Y_{\rho,a}}$ and $|\cdot|_{Z_{\rho,a}}$ and any $\rho \leq \min\{\rho_0, \tau/3\}$,

$$
\begin{aligned}
&|u(t)|_{X_{\rho,a}}^2 + \int_0^T |u(t)|_{Z_{\rho,a}}^2 dt - \int_0^T \rho'(t) |u(t)|_{Y_{\rho,a}}^2 dt \\
\leq & |u_0|_{X_{\rho_0,a_0}}^2 + \frac{1}{2}\int_0^T |u(t)|_{Z_{\rho,a}}^2 dt + C\int_0^T \left(|\rho'(t)|\rho^{-2}|u(t)|_{X_{\rho,a}} + |u(t)|_{X_{\rho,a}}^2 + |u(t)|_{X_{\rho,a}}^4\right) dt \\
&+ C\int_0^T |u(t)|_{Z_{\rho,a}} |u(t)|_{Y_{\rho,a}}^2 dt
\end{aligned}
$$

Thus Claim (3.6) follows and the proof is complete. ∎

5. Existence of solution for second component

In this section, we determine the second component $\mathcal{U}_2^{p,0}$ by solving the parabolic-type equation

$$\begin{cases} \partial_t \mathcal{U}_2^{p,0} - \partial_y^2 \mathcal{U}_2^{p,0} + \mathcal{U}_1^{p,0}\partial_1 \mathcal{U}_2^{p,0} + \mathcal{U}_3^{p,1}\partial_y \mathcal{U}_2^{p,0} = 0, \\ \mathcal{U}_2^{p,0}(t,x_1,0) = 0, \quad \lim_{y\to+\infty} \mathcal{U}_2^{p,0}(t,x_1,y) = \overline{u_2^{I,0}}(x_1), \\ \mathcal{U}_2^{p,0}(0,x_1,y) = u_{0,2}^{B,0}(x_1,y) + \overline{u_{0,2}^{I,0}}(x_1). \end{cases}$$

We recall that

$$\partial_t \overline{u_2^{I,0}} + \overline{u_1^{I,0}}\partial_1 \overline{u_2^{I,0}} = 0. \tag{5.1}$$

Then, the system (P2) becomes

$$\begin{cases} \partial_t u_2^{B,0} - \partial_y^2 u_2^{B,0} + \left(u_1^{B,0} + \overline{u_1^{I,0}}\right)\partial_1 u_2^{B,0} \\ \qquad\qquad + \left(u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}\right)\partial_y u_2^{B,0} + \overline{\partial_1 u_2^{I,0}} u_1^{B,0} = 0, \\ \partial_2 u_2^{B,0} = 0, \\ u_2^{B,0}(t,x_1,0) = -\overline{u_2^{I,0}}, \quad \lim_{y\to+\infty} u_2^{B,0}(t,x_1,y) = 0, \\ u_2^{B,0}(0,x_1,y) = u_{0,2}^{B,0}(x_1,y). \end{cases} \tag{P2bis}$$

We have the following results

THEOREM 5.1. *Let $\rho_0 > 0$, $a_0 > 0$. For any initial data $u_{2,0}^{B,0} \in X_{\rho_0,a_0}$, there exist $T > 0$, $\tau > 0$ and $0 < a < a_0$, such that the system (P2bis) admits a unique solution*

$$u_2^{B,0} \in L^\infty([0,T], X_{\rho_0,a}).$$

Proof. In order to prove Theorem 5.1, the idea is to define an auxiliary function

$$v = u_2^{B,0} + e^{-2a_0 y^2}\overline{u_2^{I,0}},$$

which satisfies the following boundary conditions

$$v(t,x_1,0) = \lim_{y\to+\infty} v(t,x_1,y) = 0.$$

Then, the first equation of the system (P2bis) becomes

$$\partial_t\left(v - e^{-2a_0y^2}\overline{u_2^{I,0}}\right) - \partial_y^2\left(v - e^{-2a_0y^2}\overline{u_2^{I,0}}\right) + \left(u_1^{B,0} + \overline{u_1^{I,0}}\right)\partial_1\left(v - e^{-2a_0y^2}\overline{u_2^{I,0}}\right)$$
$$+ \left(u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}\right)\partial_y\left(v - e^{-2a_0y^2}\overline{u_2^{I,0}}\right) + \overline{\partial_1 u_2^{I,0}} u_1^{B,0} = 0.$$

Using (5.1), we can rewrite the system (P2bis) as

$$\begin{cases} \partial_t v - \partial_y^2 v + \left(u_1^{B,0} + \overline{u_1^{I,0}}\right)\partial_1 v + \left(u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}\right)\partial_y v + R = 0, \\ \partial_2 v = 0, \\ v(t,x_1,0) = 0, \quad \lim_{y\to+\infty} u_2^{B,0}(t,x_1,y) = 0, \\ v(0,x_1,y) = u_{0,2}^{B,0}(x_1,y) + e^{-2a_0y^2}\overline{u_{0,2}^{I,0}}(x_1), \end{cases} \tag{$P2_v$}$$

where

$$R = \left(16a_0^2y^2 - 4a_0\right)e^{-2a_0y^2}\overline{u_2^{I,0}} + 4a_0\left(u_3^{B,1} + \overline{u_3^{I,1}} + y\overline{\partial_3 u_3^{I,0}}\right)ye^{-2a_0y^2}\overline{u_2^{I,0}} + \left(1 - e^{-2a_0y^2}\right)\overline{\partial_1 u_2^{I,0}}u_1^{B,0}.$$

We remark that the system ($P2_v$) is in the same form as the system (3.1) with Dirichlet boundary conditions. Thus, we can prove Theorem 5.1 in the same way (with a lot of simplifications) as we did to prove Theorem 3.1. □

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