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A QUADRATIC SPLINE LEAST SQUARES METHOD FOR COMPUTING ABSOLUTELY CONTINUOUS INVARIANT MEASURES*

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Abstract. We develop a quadratic spline approximation method for the computation of absolutely continuous invariant measures of one dimensional mappings, based on the orthogonal projection of L^2 spaces. We prove the norm convergence of the numerical scheme and present the numerical experiments.

Keywords. Frobenius-Perron Operators; Invariant Measures; Least Squares Approximations; Spline Functions.

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1. Introduction

In the past several decades, the statistical or stochastic study of chaotic deterministic systems has attracted much attention of the scientific community. In such approaches, the existence and computation of absolutely continuous invariant measures associated with the underlying transformations are two important aspects of the applications of ergodic theory to applied areas. The densities of such invariant measures are fixed points of the Frobenius-Perron operator with respect to the given measurable transformation.

Since Ulam [16] proposed the first numerical scheme for the computation of absolutely continuous invariant measures using piecewise-constant functions, and starting from Li's paper [13] on the convergence of Ulam's method for the Lasota-Yorke class of interval mappings [12], developing efficient higher order computational methods for the statistical study of dynamical systems has been a main topic in computational ergodic theory. Different approaches to the computation of fixed points of Frobenius-Perron operators have appeared in the literature, which have resulted in several classes of numerical methods: Markov finite approximations that include Ulam's original piecewise-constant method [13,16] and higher order ones [4,6], and Galerkin projections [3,5]. The paper [14] contains a unified approach to the rigorous numerical analysis of Frobenius-Perron operators.

While the original Ulam's method has the L^1-norm convergence rate of only $O(\ln n/n)$ [2], the piecewise-linear Markov method has achieved the first-order convergence rate under the BV-norm. Among various efficient algorithms for the fixed point computation of Frobenius-Perron operators, a piecewise-linear least squares method proposed in [7,8] has turned out to be a fast one since the numerical results have shown that it has a higher convergence rate than the piecewise-linear Markov approximations method from [6]. This method employs continuous piecewise-linear functions and uses

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the same projection principle as the previous projection methods [3, 5] in which discontinuous piecewise polynomials were employed. The minimal distance property under the L^2-norm may explain why it outperforms the Markov approximations method, which is integral-preserving and positivity-preserving but does not possess the best approximation property.

Inspired by the use of continuous piecewise-linear functions, a piecewise-quadratic projection method was proposed in [15] that employed continuous piecewise-quadratic functions. The numerical results of [15] show that it outperforms both the continuous piecewise-quadratic Markov approximation method [4] and the discontinuous piecewise-quadratic projection method [3]. On the other hand, because of the use of higher degree polynomials, the piecewise-quadratic scheme from [15] also outperforms the piecewise-linear projection method [7, 8], although the increase of the convergence rate is not significant. One possible reason is that the approximating functions are only continuous in both methods, so the lack of smoothness of the method proposed in [15] may explain the minor improvement in the convergence rate over the piecewise-linear method.

In this paper we continue to study the least squares method, but we now use the quadratic spline functions. The continuous piecewise-linear functions used before are just the linear splines, which are not differentiable. The quadratic spline functions are continuously differentiable. The higher order smoothness of the quadratic spline functions compared with continuous piecewise-quadratic functions and linear spline functions is expected to increase the order of convergence rate.

A general convergence analysis for projection methods and its application to Frobenius-Perron operators was given in [3], but it is not easily used for a concrete projection method such as the one that we consider here. On the other hand, we have found that by using techniques from matrix analysis, it is more straightforward to obtain the norm convergence of the method for a class of Frobenius-Perron operators. For this purpose, we can establish the uniform boundedness of a sequence of the inverses of the matrices resulting from the numerical scheme, from which some other inequalities of the uniform boundedness can be deduced. In other words, we prove in Section 4 below that

$$\sup_n \|T_n^{-1}\| < \infty, \ \sup_n \|Q_n\|_1 < \infty, \text{ and} \sup_n \|Q_n\|_{BV} < \infty \tag{1.1}$$

in succession, where $\|f\|_{BV} = \bigvee_0^1 f + \|f\|_1$ with $\|f\|_1 = \int_0^1 |f(x)|dx$. Then the norm convergence of our method is a natural consequence for the Lasota-Yorke class of interval maps.

As an outline of the paper, in the next section we shall describe the basic splines of order two. The projection method using quadratic spline functions will be developed in Section 3. We analyze the consistency, stability, and convergence properties of the method in Section 4. Numerical experiments will be given in Section 5, and we conclude in Section 6.

2. The quadratic spline functions

Generally speaking, spline functions associated with a partition of the domain are piecewise polynomials locally and smooth to a certain order globally. A simple example of spline functions is the class of continuous piecewise-linear functions. The degree-k spline functions can be defined via an inductive process. In other words, if the class of degree $k-1$ is available, one can define degree-k spline functions via a kind of "linear combination" of the lower degree ones. The splines that will be used here are the so-called B-splines.

Suppose the interval $[0,1]$ is divided by a finite sequence of nodes, $0=x_0<x_1<\cdots<x_{n-1}<x_n=1$. For the convenience of presentation and analysis, we assume that outside the interval there are nodes $x_{-1}>x_{-2}>\cdots$ to the left of x_0 and $x_{n+1}<x_{n+2}<\cdots$ to the right of x_n. Thus, we can think that there is given an infinite sequence

$$\cdots<x_{-2}<x_{-1}<x_0<x_1<x_2<\cdots$$

on the real line.

The B-splines of degree 0, denoted as B_i^0, are special piecewise-constant functions and are defined on each subinterval as

$$B_i^0(x)=\begin{cases}1, & x_i\le x<x_{i+1},\\ 0, & \text{otherwise.}\end{cases}$$

They are just the characteristic functions of $[x_i,x_{i+1})$ and act as the starting point for the recursive definition of higher degree B-splines by recursive formulas. For $k=1,2,\ldots,$ the degree-k B-splines are defined as

$$B_i^k(x)=\frac{x-x_i}{x_{i+k}-x_i}\;B_i^{k-1}(x)+\frac{x_{i+k+1}-x}{x_{i+k+1}-x_{i+1}}\;B_{i+1}^{k-1}(x)$$

for each i. Clearly B_i^k is a piecewise polynomial of degree k, which is called a B-spline of degree k. Some basic properties of B-splines are summarized in the following proposition; more properties can be referred to in [1].

Proposition 2.1. *(i) If $x\notin[x_i,x_{i+k+1})$, then $B_i^k(x)=0$.*

(ii) If $x\in(x_i,x_{i+k+1})$, then $B_i^k(x)>0$.

(iii) $\sum_i B_i^k(x)\equiv 1$ for all x.

(iv) The B-splines of degree-k

$$B_{-k}^k,B_{-k+1}^k,\ldots,B_{n-2}^k,B_{n-1}^k$$

constitute a basis for the space S_n^k of all functions in $C^{k-1}[0,1]$ which are piecewise polynomials of degree $\le k$ on the n subintervals

$$[x_0,x_1],\ [x_1,x_2],\ \ldots,\ [x_{n-1},x_n].$$

In particular, each B-spline of degree k is $k-1$ times continuously differentiable on the whole domain $[0,1]$, and the dimension of S_n^k is $n+k$.

For the convenience of practical computation, we let the nodes x_i be evenly distributed, so that the resulting subintervals all have the same length $h=1/n$. The corresponding B-splines can be expressed more concisely using scaling and translation techniques applied to a single basic function. For example, the continuous piecewise-linear functions, which are the B-splines of degree one and used in [7,8], can be expressed as

$$B_i^1(x)=l\left(\frac{x-x_i}{h}\right),$$

where

$$l(x)=\begin{cases}x, & 0\le x\le 1,\\ 2-x, & 1<x\le 2,\\ 0, & x\notin[0,2]\end{cases}$$

is the standard tent function. In the case of degree two splines,

$$B_i^2(x) = q\left(\frac{x - x_i}{h}\right),$$

where

$$q(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x \le 1, \\ \frac{3}{4} - \left(x - \frac{3}{2}\right)^2, & 1 < x \le 2, \\ \frac{1}{2}(x-3)^2, & 2 < x \le 3, \\ 0, & x \notin [0,3]. \end{cases}$$

It is the above B-splines of degree two that we shall use to develop the higher order method below for the computation of fixed points of Frobenius-Perron operators.

The Frobenius-Perron operator associated with a nonsingular transformation S: $[0,1] \to [0,1]$ is defined by

$$Pf(x) = \frac{d}{dx} \int_{S^{-1}([0,x])} f(t)\ dt \tag{2.1}$$

for all $f \in L^1(0,1)$. The nonsingularity of S means that S is Lebesgue-measurable and $m(S^{-1}(A)) = 0$ whenever $m(A) = 0$, where m denotes the Lesbegue measure. The Frobenius-Perron operator P, which maps $L^1(0,1)$ into itself, is linear and positive. In fact it is a Markov operator, that is, it preserves the positivity and integral of nonnegative functions [10, 11].

One property of Frobenius-Perron operators, which will be used in the proof of the convergence theorem in the following, is that, if f is a fixed point of P, then its positive part f^+ and negative part f^- are also fixed points of P, where

$$f^+(x) \equiv \max\{f(x), 0\}, \ \ f^-(x) = \max\{-f(x), 0\}.$$

The dual of the Frobenius-Perron operator is the Koopman operator associated with S, defined by

$$Ug(x) = g(S(x)), \forall\, g \in L^\infty(0,1).$$

The duality implies the equality

$$\int_0^1 Pf(x) \cdot g(x) dx = \int_0^1 f(x) g(S(x)) dx$$

for all $f \in L^1(0,1)$ of L^1-norm $\|f\|_1 = \int_0^1 |f(x)| dx$ and $g \in L^\infty(0,1)$ of L^∞-norm $\|g\|_\infty =$ $\operatorname{ess\,sup} |g(x)|$. In particular, the above equality implies, by letting $g = 1$, that

$$\int_0^1 Pf(x) dx = \int_0^1 f(x) dx.$$

That is, the Frobenius-Perron operator preserves the integral of functions. A comprehensive study of the Frobenius-Perron operator and the Koopman operator can be found in [11], and [10] contains more materials on the numerical analysis of the Frobenius-Perron operator.

We are looking for a density function such that $Pf^* = f^*$. This f^* is called a stationary density for P. Assume that P has a unique stationary density f^*. We want to approximate f^* from the spline space S_n^2, using the least squares technique.

3. A quadratic spline least squares method

To develop our quadratic spline least squares method, we divide $[0,1]$ into n equal subintervals $I_i=[x_{i-1},x_i]$ for $i=1,2,\ldots,n$ with the length $h=1/n$. The basis of S_n^2 is

$$B_{-2}^2, B_{-1}^2, \ldots, B_{n-2}^2, B_{n-1}^2.$$

For the simplicity of notation, we write $\Delta_n=S_n^2$ and $\phi_i=B_i^2$ for all i. It is easy to see that $\|\phi_{-2}\|_1=\|\phi_{n-1}\|_1=h/6$, $\|\phi_{-1}\|_1=\|\phi_{n-2}\|_1=5h/6$, and

$$\|\phi_i\|_1=h,\ i=0,1,\ldots,n-3.$$

Given $f\in L^1(0,1)$, we define a function $Q_nf\in\Delta_n$ such that

$$(Q_nf,\phi_i)=(f,\phi_i),\ \forall\, i=-2,-1,\ldots,n-1,$$

where $(f,g)=\int_0^1 f(x)g(x)dx$ for all functions f and g such that the integral is well defined. If we write $Q_nf=\sum_{j=-2}^{n-1}c_j\phi_j$, then the coefficients $c_{-2},c_{-1},\ldots,c_{n-1}$ are determined uniquely from the nonsingular system of linear equations

$$\sum_{j=-2}^{n-1} c_j(\phi_j,\phi_i)=(f,\phi_i),\ \forall\, i=-2,-1,\ldots,n-1. \tag{3.1}$$

In the following we always use B_n to denote the $(n+2)\times(n+2)$ coefficient matrix of the linear system (3.1). The matrix B_n depends only on the basis functions $\phi_{-2},\phi_{-1},\ldots,\phi_{n-1}$, and a direct computation gives

$$B_n=\frac{h}{120}\begin{bmatrix} 6 & 13 & 1 & & & & & & \\ 13 & 60 & 26 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 26 & 60 & 13 \\ & & & & & & 1 & 13 & 6 \end{bmatrix}\in\mathbf{R}^{(n+2)\times(n+2)}.$$

When restricted to $L^2(0,1)$, it is easy to see that $Q_n:L^2(0,1)\to L^2(0,1)$ is actually the orthogonal projection from $L^2(0,1)$ onto Δ_n. Thus, it defines the corresponding least squares approximation Q_nf of f by

$$\|f-Q_nf\|_2=\min\left\{\left\|f-\sum_{i=-2}^{n-1}c_i\phi_i\right\|_2 : \forall\, c_i\in\mathbf{R}\right\}.$$

Now we can use the finite dimensional projection operator Q_n to develop the least squares method for the computation of the stationary density f^* of the Frobenius-Perron operator P. In order to solve the infinite dimensional fixed point equation

$$Pf=f,\ \ f\in L^1(0,1)$$

numerically, we apply Q_n to both the sides of the above equation and solve the resulting discretized operator equation

$$Q_nPf=Q_nf,\ \ f\in\Delta_n, \tag{3.2}$$

which is, since $(Q_n f, \phi_i) = (f, \phi_i)$, equivalent to

$$(Q_n P f, \phi_i) = (f, \phi_i),\ i = -2, -1, \ldots, n-1.$$

It follows from $(Q_n P f, \phi_i) = (P f, \phi_i)$ that

$$(P f, \phi_i) = (f, \phi_i),\ i = -2, -1, \ldots, n-1.$$

By writing $f = \sum_{j=-2}^{n-1} d_j \phi_j$, we obtain the following equations

$$\sum_{j=-2}^{n-1} d_j (P\phi_j, \phi_i) = \sum_{j=-2}^{n-1} d_j (\phi_j, \phi_i),\ i = -2, -1, \ldots, n-1. \tag{3.3}$$

If we define the $(n+2) \times (n+2)$ matrix $A_n = (a_{ij})$ with

$$a_{ij} = (P\phi_j, \phi_i) = (\phi_j, U\phi_i),$$

then the system (3.3) can be written as

$$(A_n - B_n) d = 0, \tag{3.4}$$

where $d = (d_{-2}, d_{-1}, \ldots, d_{n-1})^T \in \mathbf{R}^{n+2}$.

Because of the partition of unity property of the basis functions ϕ_i, we can prove the following existence result of the homogeneous Equation (3.4).

PROPOSITION 3.1. *There is a nonzero function $f_n \in \Delta_n$ that solves the Equation (3.2).*

Proof. Using Proposition 2.1 (iii), for each j we have

$$\sum_{i=-2}^{n-1} a_{ij} = \sum_{i=-2}^{n-1} (P\phi_j, \phi_i) = (P\phi_j, 1)$$
$$= (\phi_j, 1) = \sum_{i=-2}^{n-1} (\phi_j, \phi_i) = \sum_{i=-2}^{n-1} b_{ij}.$$

This shows that the row vector $(1, 1, \ldots, 1)$ is a left eigenvector of $A_n - B_n$ corresponding to the eigenvalue 0, thus $A_n - B_n$ is singular. Hence, there is a nonzero vector d such that $(A_n - B_n) d = 0$, so the nonzero function

$$f_n = \sum_{j=-2}^{n-1} d_j \phi_j$$

solves (3.2). ∎

If we normalize $f_n \in \Delta_n$ so that $\|f_n\|_1 = 1$, then f_n is a degree-2 spline least squares approximation of the stationary density f^* of the original Frobenius-Perron operator P.

When implementing the least squares method, the main numerical work is the formation of the matrix A. There are two ways to evaluate the entries of matrix A: either from P or from U, so

$$a_{ij} = \int_{\mathrm{supp}(\phi_i)} \phi_i(x) P\phi_j(x)\, dx = \int_{\mathrm{supp}(\phi_j)} \phi_i(S(x))\, \phi_j(x)\, dx.$$

4. Convergence analysis

In order to give the convergence analysis of the least squares method, we need to investigate the projection operators Q_n. For this purpose, the sequence of the matrices B_n in the previous section will be studied now since it will play a key role in the following. We first show that the matrix $T_n \equiv (120/h)B_n$ is nonsingular and the norm of its inverse is uniformly bounded.

LEMMA 4.1. *Let $A=(a_{ij}) \in \mathbf{C}^{r\times r}$ be strictly diagonally dominant. Then*

$$\|A^{-1}\|_\infty \le \max_{1\le i\le r} \frac{1}{|a_{ii}| - \sum_{j\neq i}|a_{ij}|}.$$

Proof. There exists $y=(y_1,y_2,\dots,y_r)^T \in \mathbf{C}^r$, $\|y\|_\infty = 1$, such that

$$\|A^{-1}\|_\infty = \|A^{-1}y\|_\infty.$$

Let $x=(x_1,x_2,\dots,x_r)^T = A^{-1}y$. Assume that $|x_{i_0}| = \max_{1\le i\le r}|x_i| = \|x\|_\infty$. Since $Ax=y$, we get

$$\sum_{j=1}^{r} a_{i_0 j}x_j = y_{i_0}.$$

Then, we have

$$\begin{aligned} |x_{i_0}|\left(|a_{i_0 i_0}| - \sum_{j\neq i_0}|a_{i_0 j}|\right) &\le |a_{i_0 i_0}x_{i_0}| - \sum_{j\neq i_0}|a_{i_0 j}x_j| \\ &\le \left|\sum_{j=1}^{r} a_{i_0 j}x_j\right| = |y_{i_0}| \le 1. \end{aligned}$$

That is,

$$\|A^{-1}\|_\infty = \|x\|_\infty = |x_{i_0}| \le \max_{1\le i\le r} \frac{1}{|a_{ii}| - \sum_{j\neq i}|a_{ij}|}.$$

∎

LEMMA 4.2. *The matrix T_n is nonsingular and $\|T_n^{-1}\|_1 \le 95/54$ for all n.*

Proof. Let

$$J_n = \begin{bmatrix} 1 & & & & & & & \\ -\frac{13}{6} & 1 & & & & & & \\ -\frac{1}{6} & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \in \mathbf{R}^{(n+2)\times(n+2)},$$

$$K_n = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & -\frac{1}{6} \\ & & & & & & 1 & -\frac{13}{6} \\ & & & & & & & 1 \end{bmatrix} \in \mathbf{R}^{(n+2)\times(n+2)}.$$

Then

$$M_n \equiv K_n J_n T_n J_n^T K_n^T = \begin{bmatrix} 6 & 0 & 0 & & & & & & & \\ 0 & \frac{191}{6} & \frac{143}{6} & 1 & & & & & & \\ 0 & \frac{143}{6} & \frac{395}{6} & 26 & 1 & & & & & \\ 0 & 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 26 & 66 & 26 & 1 & 0 \\ & & & & & 1 & 26 & \frac{395}{6} & \frac{143}{6} & 0 \\ & & & & & & 1 & \frac{143}{6} & \frac{191}{6} & 0 \\ & & & & & & & 0 & 0 & 6 \end{bmatrix}.$$

It is easy to check that the symmetric matrix M_n is strictly diagonally dominant, thus it is positive definite, and so is T_n. We also have $\|M_n^{-1}\|_1 = \|M_n^{-1}\|_\infty$. According to Lemma 4.1, we see that $\|M_n^{-1}\|_\infty \le 1/6$. So

$$\begin{aligned} \|T_n^{-1}\|_1 &= \|J_n^T K_n^T M_n^{-1} K_n J_n\|_1 \\ &\le \|J_n^T K_n^T\|_1 \|M_n^{-1}\|_1 \|K_n J_n\|_1 \\ &= \|J_n^T K_n^T\|_1 \|M_n^{-1}\|_\infty \|K_n J_n\|_1 \\ &\le \frac{19}{6} \times \frac{1}{6} \times \frac{10}{3} = \frac{95}{54}. \end{aligned}$$

∎

LEMMA 4.3. *The operator sequence Q_n satisfies the following properties:*

(i) $\|Q_n\|_1 \le 1900/9$ *uniformly.*

(ii) $\lim_{n\to\infty} \|Q_n f - f\|_1 = 0$ *for any* $f \in L^1(0,1)$.

Proof.

(i) Since

$$b = ((f,\phi_{-2}),(f,\phi_{-1}),\ldots,(f,\phi_{n-1}))^T$$

for any $f \in L^1(0,1)$, we have

$$\|b\|_1 = \sum_{i=-2}^{n-1} |(f,\phi_i)| \le \sum_{i=-2}^{n-1} (|f|,\phi_i) = \left(|f|, \sum_{i=-2}^{n-1} \phi_i\right) = (|f|,1) = \|f\|_1. \tag{4.1}$$

Since $B_n = (h/120)T_n$, by Lemma 4.2, $\|B_n^{-1}\|_1 \le 1900/(9h)$. It follows from (4.1) and $\|\phi_i\|_1 \le h$ for all i that

$$\|Q_n f\|_1 = \left\|\sum_{i=-2}^{n-1} c_i\phi_i\right\|_1 \le \sum_{i=-2}^{n-1} |c_i|\|\phi_i\|_1 \le h\sum_{i=-2}^{n-1}|c_i|$$
$$= h\|c\|_1 \le h\|B_n^{-1}\|_1\|b\|_1 \le \frac{1900}{9}\|f\|_1.$$

(ii) If $f \in C^3[0,1]$, then $\lim_{n\to\infty}\|Q_n f - f\|_1 = 0$ from the approximation theory of spline functions [1]. Since $C^3[0,1]$ is dense in $L^1(0,1)$, (ii) follows from (i). □

Now we establish the stability result for the sequence Q_n. Before we prove the next lemma, note that $\phi_i'(x) = h^{-1}q'((x-x_i)/h)$, where

$$q'(x) = \begin{cases} x, & 0 \le x \le 1, \\ -2x+3, & 1 < x \le 2, \\ x-3, & 2 < x \le 3, \\ 0, & x \notin [0,3]. \end{cases} \tag{4.2}$$

LEMMA 4.4. *For any $f \in L^1(0,1)$,*

$$\bigvee_0^1 Q_n f \le \sum_{i=-1}^{n-1} |c_i - c_{i-1}|.$$

Proof. Since $Q_n f$ is continuously differentiable,

$$\bigvee_0^1 Q_n f = \int_0^1 |(Q_n f)'(x)|dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |(Q_n f)'(x)|dx.$$

On the subinterval $[x_i, x_{i+1}]$, there are only three terms in the expression of $Q_n f$ which are nonzero; these terms are $c_{i-2}\phi_{i-2}$, $c_{i-1}\phi_{i-1}$, and $c_i\phi_i$. So using (4.2), we have

$$\bigvee_0^1 Q_n f = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |c_{i-2}\phi_{i-2}'(x) + c_{i-1}\phi_{i-1}'(x) + c_i\phi_i'(x)|dx$$
$$= \frac{1}{h}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \left|c_{i-2}\left(\frac{x}{h} - i - 1\right) + c_{i-1}\left(2i+1-2\frac{x}{h}\right) + c_i\left(\frac{x}{h} - i\right)\right|dx$$
$$= \frac{1}{h}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \left|(c_i - 2c_{i-1} + c_{i-2})\frac{x}{h} + (c_{i-1} - c_{i-2})(i+1) + (c_{i-1} - c_i)i\right|dx$$
$$\le \frac{1}{h}\sum_{i=0}^{n-1}\frac{1}{2}h(|c_i - c_{i-1}| + |c_{i-1} - c_{i-2}|)$$
$$= \frac{1}{2}\sum_{i=0}^{n-1}(|c_i - c_{i-1}| + |c_{i-1} - c_{i-2}|) \le \sum_{i=-1}^{n-1}|c_i - c_{i-1}|.$$

The first inequality in the above is from the simple fact that

$$\int_r^s |\xi t + \eta|dt \le \frac{1}{2}(s-r)(|\xi r + \eta| + |\xi s + \eta|).$$ □

In the equalities in $B_n c = b$, subtracting 5 times the first one from the second, subtracting the sum of the first two from the third, subtracting the i-th equality from the $(i+1)$-th with $i = 3, 4, \ldots, n-1$, adding up the last two equalities and then subtracting the third from the last, and 5 times the last equality and then subtracting the second from the last in succession, we obtain the following equalities for $c_{-1} - c_{-2}, c_0 - c_{-1}, c_1 - c_0, \ldots, c_{n-1} - c_{n-2}$:

$$\begin{bmatrix} 17 & 22 & 1 & & & & & & \\ 18 & 65 & 26 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 26 & 65 & 18 \\ & & & & & & 1 & 22 & 17 \end{bmatrix} \begin{bmatrix} c_{-1} - c_{-2} \\ c_0 - c_{-1} \\ c_1 - c_0 \\ \vdots \\ c_{n-3} - c_{n-4} \\ c_{n-2} - c_{n-3} \\ c_{n-1} - c_{n-2} \end{bmatrix} = \frac{120}{h} \begin{bmatrix} b_{-1} - 5b_{-2} \\ b_0 - b_{-1} - b_{-2} \\ b_1 - b_0 \\ \vdots \\ b_{n-3} - b_{n-4} \\ b_{n-1} + b_{n-2} - b_{n-3} \\ 5b_{n-1} - b_{n-2} \end{bmatrix}. \tag{4.3}$$

The following lemma gives a uniform upper bound for the inverse of the $(n+1) \times (n+1)$ coefficient matrix E_n of the above system.

LEMMA 4.5. *E_n^{-1} exists and $\|E_n^{-1}\|_1 \le 117/289$ for all n.*

Proof. Let

$$U_n = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & 1 & 0 & -\frac{1}{17} \\ & & & & & & 1 & -\frac{18}{17} \\ & & & & & & & 1 \end{bmatrix} \in \mathbf{R}^{(n+1)\times(n+1)},$$

$$V_n = \begin{bmatrix} 1 & & & & & & & \\ -\frac{18}{17} & 1 & & & & & & \\ -\frac{1}{17} & 0 & 1 & & & & & \\ & & & 1 & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \in \mathbf{R}^{(n+1)\times(n+1)},$$

$$W_n = \begin{bmatrix} 1 & -\frac{22}{17} & -\frac{1}{17} & & & & & \\ & 1 & 0 & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \in \mathbf{R}^{(n+1)\times(n+1)},$$

$$
Z_n = \begin{bmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& \ddots &&&& \\
&&&& 1 &&& \\
&&&&& 1 && \\
&&&&& 0 & 1 & \\
&&&&& -\frac{1}{17} & -\frac{22}{17} & 1
\end{bmatrix} \in \mathbf{R}^{(n+1)\times(n+1)}.
$$

Then

$$
N_n \equiv U_n V_n E_n W_n Z_n = \begin{bmatrix}
17 & 0 & 0 &&&&&&&&& \\
0 & \frac{709}{17} & \frac{424}{17} & 1 &&&&&&&& \\
0 & \frac{420}{17} & \frac{1121}{17} & 26 & 1 &&&&&&& \\
0 & 1 & 26 & 66 & 26 & 1 &&&&&& \\
&&& \ddots & \ddots & \ddots & \ddots & \ddots &&&& \\
&&&&&& 1 & 26 & 66 & 26 & 1 & 0 \\
&&&&&&& 1 & 26 & \frac{1121}{17} & \frac{420}{17} & 0 \\
&&&&&&&& 1 & \frac{424}{17} & \frac{709}{17} & 0 \\
&&&&&&&&& 0 & 0 & 17
\end{bmatrix}.
$$

Since N_n is strictly diagonally dominant, it is nonsingular. So E_n is nonsingular. According to Lemma 4.1, we have $\|N_n^{-1}\|_1 = \|N_n^{-T}\|_\infty \le 1/12$. Thus

$$
\begin{aligned}
\|E_n^{-1}\|_1 &= \|W_n Z_n N_n^{-1} U_n V_n\|_1 \\
&\le \|W_n Z_n\|_1 \|N_n^{-1}\|_1 \|U_n V_n\|_1 \\
&= \|W_n Z_n\|_1 \|N_n^{-1}\|_\infty \|U_n V_n\|_1 \\
&\le \frac{39}{17} \times \frac{1}{12} \times \frac{36}{17} = \frac{117}{289}.
\end{aligned}
$$

□

For the simplicity of notation, let $u \in \mathbf{R}^{n+1}$ be such that $u_i = c_i - c_{i-1}$ for $i = -1, 0, \ldots, n-1$ and denote the right-hand side vector of (4.3) by v. Then from $E_n u = v$, and Lemmas 4.4 and 4.5, we have

$$
\bigvee_0^1 Q_n f \le \sum_{i=-1}^{n-1} |c_i - c_{i-1}| = \|u\|_1 \le \|E_n^{-1}\|_1 \|v\|_1 \le \frac{117}{289} \|v\|_1. \tag{4.4}
$$

Now we let $f \in C^1[0,1]$ and estimate $\|v\|_1$ in terms of $\bigvee_0^1 f$. In the following we often use the equality $\bigvee_a^b f = \int_a^b |f'(x)| dx$ for $f \in C^1[a,b]$. For $i = 1, 2, \ldots, n-3$, since

$$
b_i = (f, \phi_i) = \int_{x_i}^{x_{i+3}} f(x) q\left(\frac{x - x_i}{h}\right) dx = h \int_0^3 f(x_i + ht) q(t) dt,
$$

by changing the order of integration and noting that the integral of q over any subinterval of $[0,3]$ with length 1 is at most $\frac{2}{3}$, we obtain that

$$
h^{-1} |b_i - b_{i-1}| = \left| \int_0^3 [f(x_i + ht) - f(x_{i-1} + ht)] q(t) dt \right|
$$

$$
\begin{aligned}
&= \left| \int_0^3 \int_{x_{i-1}+ht}^{x_i+ht} f'(x)dxq(t)dt \right| \le \int_0^3 \int_{x_{i-1}+ht}^{x_i+ht} |f'(x)|dxq(t)dt \\
&= \int_{x_{i-1}}^{x_i} \int_0^{\frac{x-x_{i-1}}{h}} q(t)dt|f'(x)|dx + \int_{x_i}^{x_{i+2}} \int_{\frac{x-x_i}{h}}^{\frac{x-x_{i-1}}{h}} q(t)dt|f'(x)|dx \\
&\quad + \int_{x_{i+2}}^{x_{i+3}} \int_{\frac{x-x_i}{h}}^{3} q(t)dt|f'(x)|dx \\
&\le \frac{1}{6}\int_{x_{i-1}}^{x_i} |f'(x)|dx + \frac{2}{3}\int_{x_i}^{x_{i+2}} |f'(x)|dx + \frac{1}{6}\int_{x_{i+2}}^{x_{i+3}} |f'(x)|dx \\
&= \frac{1}{6}\bigvee_{x_{i-1}}^{x_i} f + \frac{2}{3}\bigvee_{x_i}^{x_{i+2}} f + \frac{1}{6}\bigvee_{x_{i+2}}^{x_{i+3}} f.
\end{aligned}
$$

On the other hand, from the fact that $\|\phi_{-1}\|_1 = 5h/6$ and $\|\phi_{-2}\|_1 = h/6$, we have $\int_0^1 [\phi_{-1}(x) - 5\phi_{-2}(x)]dx = 0$, so

$$
\begin{aligned}
|b_{-1} - 5b_{-2}| &= \left| \int_0^{2h} f(x)[\phi_{-1}(x) - 5\phi_{-2}(x)]dx - \int_0^{2h} f(h)[\phi_{-1}(x) - 5\phi_{-2}(x)]dx \right| \\
&\le \int_0^{2h} |f(x) - f(h)| \cdot |\phi_{-1}(x) - 5\phi_{-2}(x)|dx \\
&\le \bigvee_0^{2h} f \int_0^{2h} |\phi_{-1}(x) - 5\phi_{-2}(x)|dx \\
&= \frac{4}{5}h \bigvee_0^{2h} f.
\end{aligned}
$$

Similarly, we get

$$
|5b_{n-1} - b_{n-2}| < \frac{4}{5}h \bigvee_{(n-2)h}^{1} f.
$$

By the same token, it follows from $\int_0^1 [\phi_0(x) - \phi_{-1}(x) - \phi_{-2}(x)]dx = 0$ that

$$
\begin{aligned}
|b_0 - b_{-1} - b_{-2}| &= \Bigg| \int_0^{3h} f(x)\,[\phi_0(x) - \phi_{-1}(x) - \phi_{-2}(x)]\,dx \\
&\qquad - \int_0^{3h} f(h)\,[\phi_0(x) - \phi_{-1}(x) - \phi_{-2}(x)]\,dx \Bigg| \\
&\le \int_0^{3h} |f(x) - f(h)| \cdot |\phi_0(x) - \phi_{-1}(x) - \phi_{-2}(x)|dx \\
&\le \bigvee_0^{3h} f \int_0^{3h} |\phi_0(x) - \phi_{-1}(x) - \phi_{-2}(x)|dx \\
&= \frac{4}{3}h \bigvee_0^{3h} f.
\end{aligned}
$$

Similarly,

$$|b_{n-1}+b_{n-2}-b_{n-3}| \le \frac{4}{3}h \bigvee_{(n-3)h}^{1} f.$$

In summary, we have

$$\begin{aligned}\|v\|_1 &= \frac{120}{h}\Big(|b_{-1}-5b_{-2}|+|b_0-b_{-1}-b_{-2}|+\sum_{i=1}^{n-3}|b_i-b_{i-1}| \\ &\quad +|b_{n-1}+b_{n-2}-b_{n-3}|+|5b_{n-1}-b_{n-2}|\Big) \\ &\le 96\bigvee_{x_0}^{x_2} f+160\bigvee_{x_0}^{x_3} f+\sum_{i=1}^{n-3}\left(20\bigvee_{x_{i-1}}^{x_i} f+80\bigvee_{x_i}^{x_{i+2}} f+20\bigvee_{x_{i+2}}^{x_{i+3}} f\right)+160\bigvee_{x_{n-3}}^{x_n} f+96\bigvee_{x_{n-2}}^{x_n} f \\ &\le 356\bigvee_0^1 f.\end{aligned}$$

The above inequality and (4.4) establish the following stability result.

LEMMA 4.6. *Let $f \in C^1[0,1]$. Then for all n,*

$$\bigvee_0^1 Q_n f \le 145 \bigvee_0^1 f.$$

We are ready to prove a convergence theorem for a class of piecewise onto, C^2, and stretching mappings $S:[0,1]\to[0,1]$. The Lasota-Yorke inequality [10, 11] for this class of mappings says that, there are two constants α and $\beta>0$ such that for all functions f of bounded variation,

$$\bigvee_0^1 Pf \le \alpha \bigvee_0^1 f+\beta\|f\|_1. \tag{4.5}$$

As usual, we assume that the corresponding Frobenius-Perron operator P has a unique stationary density f^* for the purpose of convergence of our numerical method.

THEOREM 4.1. *Let $S:[0,1]\to[0,1]$ be such that its corresponding Frobenius-Perron operator P maps C^1-functions to C^1-functions and satisfies (4.5) with $\alpha<1/145$. Then for any sequence $\{f_n\}$ of the piecewise-quadratic least squares approximations of f^* with $\|f_n\|_1 \equiv 1$,*

$$\lim_{n\to\infty}\|f_n-f^*\|_1=0.$$

Proof. Since $f_n=Q_nPf_n$ and note the fact that f_n and Pf_n are both in $C^1[0,1]$, Lemma 4.6 and (4.5) imply that

$$\begin{aligned}\bigvee_0^1 f_n &= \bigvee_0^1 Q_nPf_n \le 145\bigvee_0^1 Pf_n \\ &\le 145\left(\alpha\bigvee_0^1 f_n+\beta\|f_n\|_1\right)=145\alpha\bigvee_0^1 f_n+145\beta.\end{aligned}$$

Since $145\alpha < 1$,

$$\bigvee_0^1 f_n \le \frac{145\beta}{1-145\alpha}, \ \forall n,$$

so by Helly's lemma there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges to some $g \in L^1(0,1)$. Clearly $\|g\|_1 = 1$. Since $\|P\|_1 = 1$, from Lemma 4.3 (i), $\|Q_n P\|_1 \le \|Q_n\|_1 \le 1900/9$ for all n. Consequently,

$$\begin{aligned}\|g - Pg\|_1 \le &\|g - f_{n_k}\|_1 + \|f_{n_k} - Q_{n_k} P f_{n_k}\|_1 \\ &+ \|Q_{n_k} P f_{n_k} - Q_{n_k} P g\|_1 + \|Q_{n_k} P g - P g\|_1 \\ \le &\|g - f_{n_k}\|_1 + \frac{1900}{9}\|f_{n_k} - g\|_1 + \|Q_{n_k} P g - P g\|_1.\end{aligned}$$

By letting $k \to \infty$ in the above inequality, we get $Pg = g$ by Lemma 4.3 (ii), so $Pg^+ = g^+$ and $Pg^- = g^-$. Since f^* is the unique stationary density of P, there must be $g = f^*$ or $g = -f^*$. Without loss of generality we may assume that $g = f^*$. This proves the theorem since every convergent subsequence of $\{f_n\}$ converges to f^*. ∎

Before ending the section, we present a weak convergence result for another class of Frobenius-Perron operators. In the next theorem, we assume that the Frobenius-Perron operator P maps L^2 functions to L^2 functions and satisfies

$$\|Pf\|_2 \le \alpha \|f\|_2 + C\|f\|_1, \forall f \in L^2(0,1), \tag{4.6}$$

where $\alpha < 1$ and C are two constants. There exists a stationary density $f^* \in L^2(0,1)$ of P by Theorem 2.1 of [9]. By the same technique of [7], we can prove the following result.

THEOREM 4.2. *Suppose the stationary density f^* of P is unique. Let $f_n \in \Delta_n$ be such that $Q_n P f_n = f_n$ and $\|f_n\|_1 = 1$ for all n. Then $\{f_n\}$ converges weakly to f^*.*

Proof. Since Q_n is an orthogonal projection on $L^2(0,1)$ by definition, $\|Q_n\|_2 \equiv 1$. From $f_n = Q_n P f_n$ and (4.6),

$$\begin{aligned}\|f_n\|_2 = &\|Q_n P f_n\|_2 \le \|P f_n\|_2 \\ \le &\alpha \|f_n\|_2 + C\|f_n\|_1 = \alpha\|f_n\|_2 + C,\end{aligned}$$

which implies that

$$\|f_n\|_2 \le \frac{C}{1-\alpha}$$

uniformly. Thus the sequence $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ which converges weakly to some $g \in L^1(0,1)$ [11]. Since P is weakly continuous [10], it follows that $Pg = g$. The assumption that f^* is the only stationary density of P ensures that $g = f^*$. So f_n converges to f^* weakly. ∎

5. Numerical results

In this section we present some numerical experiment results on the performance of the new quadratic spline least squares method (QS-LSM) and compare them with those of the continuous piecewise-quadratic least squares method (CPQ-LSM) obtained from [15] and the linear spline least squares method (LS-LSM), studied in [7, 8]. We also present the piecewise-constant Markov method (Ulam), commonly known as Ulam's method [13, 16], and the linear spline Markov method (LS-MM), studied in [6], for the purpose of comparison. The results are reported in Tables 5.1 and 5.2 below. The test mappings are

$$S_1(x) = \begin{cases} \frac{2x}{1-x^2}, & 0 \le x \le \sqrt{2}-1, \\ \frac{1-x^2}{2x}, & \sqrt{2}-1 \le x \le 1, \end{cases}$$
$$S_2(x) = \begin{cases} \frac{2x}{1-x}, & 0 \le x \le \frac{1}{3}, \\ \frac{1-x}{2x}, & \frac{1}{3} \le x \le 1, \end{cases}$$
$$S_3(x) = 4x(1-x),$$
$$S_4(x) = \left(\frac{1}{8} - 2\left|x - \frac{1}{2}\right|^3\right)^{1/3} + \frac{1}{2}.$$

Since they are piecewise onto and C^2, from their respective explicit expressions and by the definition (2.1) of the corresponding Frobenius-Perron operator P, it is easy to check that P maps C^1-functions to C^1-functions. The unique fixed densities of S_i are given by

$$f_1^*(x) = \frac{4}{\pi(1+x^2)},$$
$$f_2^*(x) = \frac{2}{(1+x)^2},$$
$$f_3^*(x) = \frac{1}{\pi\sqrt{x(1-x)}},$$
$$f_4^*(x) = 12\left(x - \frac{1}{2}\right)^2.$$

In the computation with MATLAB, we divided the interval $[0,1]$ into 2^k subintervals for $k = 2, 3, \ldots, 8$. For the comparison of the errors we used

$$\epsilon_n \equiv \|f_n - f^*\|_1 = \int_0^1 |f_n(x) - f^*(x)|dx.$$

n	Ulam	LS-MM	LS-LSM	CPQ-LSM	QS-LSM
4	5.3×10^{-2}	1.1×10^{-2}	2.7×10^{-3}	4.4×10^{-4}	5.1×10^{-4}
8	2.4×10^{-2}	4.3×10^{-3}	6.5×10^{-4}	4.6×10^{-5}	4.9×10^{-5}
16	1.2×10^{-2}	1.6×10^{-3}	1.7×10^{-4}	6.1×10^{-6}	6.2×10^{-6}
32	5.5×10^{-3}	5.3×10^{-4}	4.3×10^{-5}	8.2×10^{-7}	7.3×10^{-7}
64	2.7×10^{-3}	1.7×10^{-4}	1.1×10^{-5}	3.4×10^{-7}	8.4×10^{-8}
128	1.3×10^{-3}	5.0×10^{-5}	2.7×10^{-6}	2.3×10^{-7}	1.0×10^{-8}
256	6.6×10^{-4}	1.5×10^{-5}	6.4×10^{-7}	1.8×10^{-7}	1.3×10^{-9}

TABLE 5.1. *L^1−norm errors comparison for S_1.*

It is clear from Tables 5.1 and 5.2 that the errors in QS-LSM are considerably smaller than those of LS-LSM in the case of f_1^* and f_2^*. Also those tables demonstrate

n	Ulam	LS-MM	LS-LSM	CPQ-LSM	QS-LSM
4	1.0×10^{-1}	4.2×10^{-2}	7.7×10^{-3}	8.0×10^{-4}	9.6×10^{-4}
8	5.1×10^{-2}	1.8×10^{-2}	1.9×10^{-3}	1.3×10^{-4}	1.3×10^{-4}
16	2.6×10^{-2}	6.8×10^{-3}	5.4×10^{-4}	1.9×10^{-5}	1.9×10^{-5}
32	1.3×10^{-2}	2.3×10^{-3}	1.4×10^{-4}	2.3×10^{-6}	2.2×10^{-6}
64	6.6×10^{-3}	7.5×10^{-4}	3.6×10^{-5}	5.1×10^{-7}	3.0×10^{-7}
128	3.3×10^{-3}	2.3×10^{-4}	8.5×10^{-6}	2.5×10^{-7}	3.8×10^{-8}
256	1.6×10^{-3}	6.9×10^{-5}	2.2×10^{-6}	2.2×10^{-7}	4.6×10^{-9}

TABLE 5.2. *L^1-norm errors comparison for S_2.*

n	Ulam	LS-MM	LS-LSM	QS-LSM
4	4.3×10^{-1}	3.7×10^{-1}	4.2×10^{-1}	3.2×10^{-1}
8	3.7×10^{-1}	3.0×10^{-1}	2.7×10^{-1}	2.5×10^{-1}
16	2.8×10^{-1}	2.2×10^{-1}	2.4×10^{-1}	1.8×10^{-1}
32	2.2×10^{-1}	1.6×10^{-1}	1.7×10^{-1}	1.4×10^{-1}
64	1.6×10^{-1}	1.2×10^{-1}	1.3×10^{-1}	1.1×10^{-1}
128	1.2×10^{-1}	8.5×10^{-2}	9.7×10^{-2}	8.1×10^{-2}
256	9.2×10^{-2}	6.1×10^{-2}	6.9×10^{-2}	6.2×10^{-2}

TABLE 5.3. *L^1-norm errors comparison for S_3.*

that the order of convergence of QS-LSM is $O(h^3)$. The function f_3^* is unbounded on $[0,1]$ so the order of error doesn't follow any definite rule and all the methods have large errors (see Table 5.3).

Under the same partition of $[0,1]$ into n subintervals, the dimension of the space of the corresponding spline functions is $n+2$, roughly half of that of the space of continuous piecewise-quadratic ones. Even with much less computational cost and with the same partition, the quadratic spline least squares method has smaller errors, especially when n becomes larger and larger, which can be seen clearly from Tables 5.1 and 5.2 .

Finally, we discuss about f_4^*. Since $P_S f_4^* = f_4^*$ and f_4^* is quadratic, we have $Q_n P_S f_4^* = f_4^*$ for any n because Q_n is a projection onto the space of degree-2 splines.

6. Conclusions

We have developed a fast convergent numerical method based on the least squares approximation via the projection of integrable functions to the subspace of quadratic spline functions, which has a much better performance in the convergence rate than all the current numerical schemes that have appeared in the literature of computational ergodic theory. In particular, the choice of the quadratic splines in the least squares approach is much less costly than the method employing continuous piecewise-quadratic functions with the same subdivisions of the domain interval. Furthermore, the new computational scheme converges faster than the others even under the same partition of the interval. The numerical experiments indicate that the quadratic spline method has a convergence rate of about order three under the L^1-norm when the stationary density is smooth. If the stationary density is already a quadratic spline, the method can compute it exactly when the partition of the interval matches that of the spline function. A general convergence analysis based on the same matrix analysis technique with other projection methods that satisfy the uniform boundedness condition (1.1) will be studied in the future.

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