

CONVERGENCE TO CONSENSUS OF THE GENERAL FINITE-DIMENSIONAL CUCKER-SMALE MODEL WITH TIME-VARYING DELAYS*

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Abstract. We consider the well known finite-dimensional Cucker-Smale system, modelling interacting collective dynamics and their possible convergence to consensus. The objective of this paper is to study the influence of time-delays in the general model on the convergence to consensus. By a Lyapunov functional approach, we establish convergence results to consensus for symmetric and nonsymmetric communication weights under some structural conditions.

Keywords. consensus models; delay; Lyapunov functions.

AMS subject classifications. 34D05; 34D20.

1. Introduction

The study of collective behavior of autonomous agents has recently attracted great interest in various scientific applicative areas, such as biology, sociology, robotics, economics (see [2, 3, 6, 8, 14, 18, 32, 33, 35, 36, 43, 44, 46]). The main motivation is to model and explain the possible emergence of self-organization or global pattern formation in a large group of agents having mutual interactions, where individual agents may interact either globally or even only at the local scale.

The well known Cucker-Smale model has been proposed and studied in [22, 23] as a paradigmatic model for flocking, namely for modelling dynamics where autonomous agents reach a consensus based on limited environmental information. Consider $N \in \mathbb{N}$ agents and let $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$, $i = 1, \dots, N$ be their phase-space coordinates. One can think of $x_i(t) \in \mathbb{R}^d$ as the position of the i^{th} agent and $v_i(t) \in \mathbb{R}^d$ as its velocity, but for instance, in social sciences these variables may stand for other notions such as opinions. The general finite-dimensional Cucker-Smale model is the following:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \frac{\lambda}{N} \sum_{j=1}^N \psi_{ij}(t)(v_j(t) - v_i(t)) \quad \forall i = 1, \dots, N \end{aligned} \tag{1.1}$$

where the real number $\lambda \geq 0$ is a coupling strength and the communication rates $\psi_{ij}(t)$ are of the form

$$\psi_{ij}(t) = \psi(|x_i(t) - x_j(t)|)$$

where the function ψ is called the potential. Here and throughout, the notation $|\cdot|$ stands for the Euclidean norm in \mathbb{R}^d . Along any solution of (1.1), we define the (position

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and velocity) variances

$$X(t) = \frac{1}{2N^2} \sum_{i,j=1}^N |x_i(t) - x_j(t)|^2$$

and

$$V(t) = \frac{1}{2N^2} \sum_{i,j=1}^N |v_i(t) - v_j(t)|^2. \quad (1.2)$$

DEFINITION 1.1. *We say that a solution of (1.1) converges to consensus (or flocking) if*

$$\sup_{t>0} X(t) < +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} V(t) = 0.$$

The potential function ψ initially considered by Cucker and Smale in [22, 23] is the function $\psi(s) = (1 + s^2)^{-b}$ with $b \geq 0$. It is proved in these references that there is unconditional convergence to flocking whenever $b < \frac{1}{2}$. For $b \geq \frac{1}{2}$, there is convergence to flocking under appropriate assumptions on the values of the initial variances on positions and speeds (see [26]). Their analysis relies on a Lyapunov approach with quadratic functionals, which we will refer to in the sequel as an L^2 analysis. This L^2 approach allows to treat symmetric communication rates. An extension of the flocking result to the case of nonsymmetric communication rates has been proposed in [40] with a different approach that we will refer to in the sequel as an L^∞ analysis, which we will describe further.

There have been a number of generalizations and variants of Cucker-Smale models, involving more general potentials (friction, attraction-repulsion), cone-vision constraints, topological interactions (see [5, 31]), leadership (see [16, 20, 29, 39, 41, 49, 51]), clustering emergence (see [34, 40]), social networks (see [4]), pedestrian crowds (see [19, 37]), stochastic or noisy models (see [21, 27]), kinetic models in infinite dimension (see [1, 4, 9, 15, 24, 30, 47]), and the control of such models (see [7, 10–13, 45, 50]).

Cucker-Smale with time-varying delays. In the present paper, we introduce time-delays in the Cucker-Smale model and we perform an asymptotic analysis of the resulting model. Time-delays reflect the fact that, for a given individual agent, information from other agents does not always propagate at infinite speed and is received after a certain time-delay, or reflect the fact that the agent needs time to elaborate a reaction to incoming stimuli.

We assume throughout that the delay $\tau(t) > 0$ is time-varying. This models the fact that the amplitude of the delay may exhibit some seasonal effects or that it depends on the age of the agents for instance. The time-delay function is assumed to be bounded: we assume that there exists $\bar{\tau} > 0$ such that

$$0 \leq \tau(t) \leq \bar{\tau} \quad \forall t > 0 \quad (1.3)$$

and that the function $t \mapsto \tau(t)$ belongs to $W^{1,\infty}([0, T])$, for all $T > 0$, and satisfies

$$\tau'(t) \leq c < 1 \quad \forall t > 0 \quad (1.4)$$

for some $c > 0$. Assumptions (1.3) and (1.4), used to perform our analysis, are standard requirements for problems with time-varying delays in several contexts (see [42, 48]). In

particular, (1.3) says that the time-delay remains in a fixed range, which is a natural requirement when trying to derive a flocking result, while the bound on the derivative (1.4) ensures that $t - \tau(t) \geq -\tau(0)$ and thus that our system below is well-posed with initial conditions given on the interval $[-\tau(0), 0]$. Our model, considered throughout, is the following:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{\lambda}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_i(t)) \end{cases} \quad \forall i = 1, \dots, N \tag{1.5}$$

with initial conditions

$$x_i(t) = f_i(t), \quad v_i(t) = g_i(t), \quad \forall t \in [-\tau(0), 0]$$

where $f_i, g_i: [-\tau(0), 0] \rightarrow \mathbb{R}$ are given functions and $\psi_{ij}(t)$, $i, j = 1, \dots, N$, are suitable communication rates. In the symmetric case, we have

$$\psi_{ij}(t) = \psi(|x_i(t) - x_j(t)|) \quad \forall i, j \in \{1, \dots, N\}. \tag{1.6}$$

The potential function ψ in (1.6) is assumed to be continuous and bounded. Without loss of generality (if necessary, do a time reparametrization), we assume that it takes values in $(0, 1]$, namely $\psi: [0, +\infty) \rightarrow (0, 1]$. This implies

$$\psi_{ij}(t) \leq 1 \quad \forall t \in [-\tau(0), +\infty) \quad \forall i, j \in \{1, \dots, N\}. \tag{1.7}$$

In the model (1.5) above, the delay is time-varying. Note that there is no delay in v_i in the equation for velocity v_i : this reflects the fact that every agent receives information coming from the other agents with a certain delay while its own velocity is known exactly at every time t .

We could also consider time-delays depending on the agents pair, namely

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{\lambda}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \psi_{ij}(t - \tau_{ij}(t))(v_j(t - \tau_{ij}(t)) - v_i(t)) \end{cases} \quad \forall i = 1, \dots, N \tag{1.8}$$

where $\tau_{ij}(t)$ is the (time-varying) time-delay of the agent i in receiving the information from the agent j . Assuming that

$$0 \leq \tau_{ij}(t) \leq \bar{\tau} \quad \forall t > 0 \quad \forall i, j \in \{1, \dots, N\} \tag{1.9}$$

and that the functions $t \mapsto \tau_{ij}(t)$ belong to $W^{1,\infty}([0, T])$ for all $T > 0$ and satisfy

$$\tau'_{ij}(t) \leq c < 1 \quad \forall t > 0 \quad \forall i, j \in \{1, \dots, N\} \tag{1.10}$$

for some $c > 0$, the model (1.8) can be analyzed analogously to (1.5). As it will be clear from our proofs, the arguments in such a case do not require any substantial modifications and thus we keep the model (1.5) for the sake of simplicity.

State of the art. Various delayed Cucker-Smale models have been considered in several contributions, with a constant delay $\tau > 0$.

A time-delayed model has been introduced and studied in [25], where the equation for velocities (which actually also involves noise terms in that paper) is

$$\dot{v}_i(t) = \frac{\lambda}{N} \sum_{j=1}^N \psi_{ij}(t-\tau)(v_j(t-\tau) - v_i(t-\tau))$$

with a constant delay τ . Considering $v_i(t-\tau)$ instead of $v_i(t)$ in the equation for $v_i(\cdot)$ facilitates much the analysis because it allows to keep one of the most important features of the standard Cucker-Smale system (1.1), namely the fact that the mean velocity $\bar{v}(t) = \frac{1}{N} \sum_{i=1}^N v_i(t)$ remains constant in time, i.e., $\dot{\bar{v}}(t) = 0$, as in the undelayed Cucker-Smale model. This fact then simplifies much all arguments in the asymptotic analysis and in the proof of convergence to consensus.

In our model (1.5) above, in contrast, the mean velocity does not remain constant, which complicates the analysis significantly as we shall see.

In [38] the authors consider as equation for the velocities

$$\dot{v}_i(t) = \alpha \sum_{j=1}^N a_{ij}(t-\tau)(v_j(t-\tau) - v_i(t)) \tag{1.11}$$

where $\alpha > 0$ and the coupling coefficients a_{ij} are such that $\sum_{j=1}^N a_{ij} = 1$, $i = 1, \dots, N$. Compared with (1.5), the sum is running over *all* indices j , including $j = i$, and thus (1.11) involves, with respect to (1.5), the additional term $a_{ii}(t-\tau)(v_i(t-\tau) - v_i(t))$ at the right-hand side. Actually, the authors of [38] claim to study (1.5) but their result (unconditional flocking for all delays) only applies to (1.11) (cf. [38, Equation (7)]). Note that (1.11) can indeed be rewritten as

$$\dot{v}_i(t) = \alpha \sum_{j=1}^N a_{ij}(t-\tau)v_j(t-\tau) - \alpha v_i(t) \tag{1.12}$$

with a negative coefficient, independent of the time t , for the undelayed velocity $v_i(t)$ of the i^{th} agent. This leads to a strong stability result: unconditional flocking for all time-delays.

In [17] the authors analyze a Cucker-Smale model with delay and normalized communication weights Φ_{ij} given by

$$\Phi_{ij}(x, \tau) = \begin{cases} \frac{\psi(|x_j(t-\tau) - x_i(t)|)}{\sum_{k \neq i} \psi(|x_k(t-\tau) - x_i(t)|)} & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases} \tag{1.13}$$

where the influence function ψ is assumed to be bounded, nonincreasing, Lipschitz continuous on $[0, +\infty)$, with $\psi(0) = 1$. Since $\sum_{j=1}^N \Phi_{ij} = 1$, their model can be written as

$$\dot{v}_i(t) = \sum_{j=1}^N \Phi_{ij}(x, t-\tau)v_j(t-\tau) - v_i(t)$$

and the same considerations as those for the model (1.12) apply. Moreover, the particular form of the communication weights Φ_{ij} allows to apply some convexity arguments

in order to obtain the flocking result for sufficiently small delays. Then the result relies on the specific form of the interaction between the agents. Note also that the influence function ψ in the Definition (1.13) of Φ_{ij} has as arguments $|x_k(t - \tau) - x_i(t)|$, $k = 1, \dots, N, k \neq i$, with the state of the i^{th} agent at the time t and the states of the other agents at time $t - \tau$. This corresponds to a time-delay in the vision which does not seem appropriate for describing flocks of birds or, in general, groupings of animals, but may be relevant for instance in robotics. Moreover, it allows to easily derive the mean-field limit of the problem at hand by obtaining a nice and tractable kinetic equation. In contrast, putting the time-delay also in the state of i^{th} agent is more suitable to describe the physical model related to groups of animals but it makes unclear (at least to us) the passage to mean-field limit (see Section 5).

Framework and structure of the present paper. In Section 2, we consider the model (1.5) with symmetric interaction weights ψ_{ij} given by (1.6). In this symmetric case, we perform a L^2 analysis, designing appropriate quadratic Lyapunov functionals adapted to the time-delay framework. The main result, Theorem 2.1, establishes convergence to consensus for small enough time-delays.

As in [25], a structural assumption is required on the matrix of communication rates. We define the $N \times N$ Laplacian matrix $L = (L_{ij})$ by

$$L_{ij} = -\frac{\lambda}{N} \psi_{ij} \quad \text{for } i \neq j, \quad L_{ii} = \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}$$

with $\psi_{ij} = \psi(|x_i - x_j|)$. The matrix L is symmetric, diagonally dominant with nonnegative diagonal entries, has nonnegative eigenvalues, and its smallest eigenvalue is zero. Note that for $\mathbf{v} \in (\mathbb{R}^d)^N$, $\mathbf{v} = (v_1, \dots, v_N)$, the matrix notation $L\mathbf{v}$ does not have the usual meaning of a $N \times N$ matrix acting on \mathbb{R}^N . Instead, we have

$$L(v_1, \dots, v_N) = \left(\sum_{j=1}^N l_{1j} v_j, \dots, \sum_{j=1}^N l_{Nj} v_j \right). \tag{1.14}$$

Considering the matrix $L(t)$ along a trajectory solution of (1.5), we denote by $\mu(t)$ its smallest positive eigenvalue, also called the *Fiedler number*. The *structural assumption* along trajectories that we make throughout is the following:

$$\exists \gamma > 0 \mid \mu(t) \geq \gamma \quad \forall t > 0. \tag{1.15}$$

This is a technical but instrumental and standardly used assumption that we use as well to perform our convergence analysis. It is guaranteed for instance if, along trajectories, the communication rates are uniformly bounded away from zero, i.e., if there exists $\psi^* > 0$ such that $\psi_{ij}(t) \geq \psi^*$ for all i, j and for every $t > 0$ (in that case there is already convergence to consensus for the undelayed model). Actually, the precise result established in [22, Proposition 4] states that there exists a constant $C > 0$ such that

$$\mu(t) \geq C\psi(X(t)).$$

This fact, adequately used in the proof of [22], makes it possible to establish that, for the potential function $\psi(s) = (1 + s^2)^{-b}$ with $b \geq 0$, there is unconditional convergence to consensus if $b < \frac{1}{2}$, and (conditional) convergence if $b \geq \frac{1}{2}$ under an additional requirement on initial conditions (thus giving a region of initial conditions for which convergence to consensus is guaranteed, see also [26]).

In Section 3, we consider the model (1.5) with possibly *nonsymmetric* potentials:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_i(t)) \quad \forall i = 1, \dots, N \end{aligned}$$

where the communication rates $a_{ij} > 0$ are arbitrary. They may of course be symmetric as above, for instance,

$$a_{ij}(t) = \psi(|x_i(t) - x_j(t)|) \tag{1.16}$$

or nonsymmetric, for instance

$$a_{ij}(t) = \frac{N\psi(|x_i(t) - x_j(t)|)}{\sum_{k=1}^N \psi(|x_k(t) - x_i(t)|)} \tag{1.17}$$

for a suitable bounded function ψ . To analyze such models, we perform an L^∞ analysis as in [40], by considering, instead of Euclidean norms, the time-evolution of the diameters in position and velocity phase space. The main result, Theorem 3.1, establishes convergence to consensus under the following structural assumption along trajectories:

$$\exists \psi^* > 0 \quad | \quad \frac{1}{N^2} \sum_{i,j=1}^N \min(a_{qi}(t)a_{pj}(t), a_{qj}(t)a_{pi}(t)) \geq \psi^* \quad \forall p, q = 1, \dots, N. \tag{1.18}$$

As above, Assumption (1.18) is instrumental in order to ensure convergence. It corresponds to require that the interactions in the flock are strong enough. In particular, (1.18) is satisfied for the interactions (1.16) and (1.17) if the influence function ψ in the definitions of a_{ij} satisfies the lower bound $\psi(r) \geq \psi_0 > 0$. Indeed, (1.18) is verified in both cases with $\psi^* = \psi_0^2$ and $\psi^* = \left(\frac{\psi_0}{\|\psi\|_\infty}\right)^2$ respectively.

In Section 4, we give some numerical simulations illustrating our results and, finally, in Section 5, we provide a conclusion and further comments.

2. Consensus for symmetric potentials: L^2 analysis

2.1. The main result.

Several notations. Following [22], we set

$$\Delta = \left\{ (v_1, v_2, \dots, v_N) \in (\mathbb{R}^d)^N \mid v_1 = \dots = v_N \right\} = \left\{ (v, v, \dots, v) \mid v \in \mathbb{R}^d \right\}.$$

The set Δ is the eigenspace for the operator L , defined in (1.14), associated with the zero eigenvalue. Its orthogonal in $(\mathbb{R}^d)^N$ is

$$\Delta^\perp = \left\{ (v_1, v_2, \dots, v_N) \in (\mathbb{R}^d)^N \mid \sum_{i=1}^N v_i = 0 \right\}.$$

Given any $\mathbf{v} = (v_1, v_2, \dots, v_N) \in (\mathbb{R}^d)^N$, we denote the mean by $\bar{v} = \frac{1}{N} \sum_{j=1}^N v_j \in \mathbb{R}^d$, and we define $\mathbf{w} = (w_1, \dots, w_N) \in (\mathbb{R}^d)^N$ by $w_i = v_i - \bar{v}$ for $i = 1, \dots, N$, so that

$$\mathbf{v} = (\bar{v}, \dots, \bar{v}) + \mathbf{w} \in \Delta + \Delta^\perp \tag{2.1}$$

and we have $L\mathbf{w} = L\mathbf{v}$. Moreover,

$$\frac{1}{2N^2} \sum_{i,j=1}^N |w_i - w_j|^2 = \frac{1}{N} \|\mathbf{w}\|^2 \tag{2.2}$$

and

$$\langle L\mathbf{v}, \mathbf{v} \rangle = \frac{1}{2} \frac{\lambda}{N} \sum_{i,j=1}^N \psi_{ij} |v_i - v_j|^2. \tag{2.3}$$

THEOREM 2.1. *We set*

$$\chi = \frac{\gamma^2}{2\lambda^2} \frac{1-c}{2\lambda^2 + \gamma^2}. \tag{2.4}$$

Let $\tau_0 > 0$ be the unique¹ positive real number such that

$$\tau_0^2 e^{\tau_0} = \chi. \tag{2.5}$$

Under the structural Assumption (1.15), if $\bar{\tau} < \tau_0$ then every solution of the system (1.5) satisfies

$$V(t) \leq C e^{-rt} \tag{2.6}$$

with

$$r = \min \left\{ \gamma - 4 \frac{\lambda^2}{\gamma} \frac{\lambda^2 \bar{\tau}^2}{(1-c)e^{-\bar{\tau}} - 2\lambda^2 \bar{\tau}^2}, 1 \right\} \tag{2.7}$$

and

$$C = V(0) + \frac{\lambda^2 \bar{\tau}}{\gamma N} \frac{1}{(1-c)e^{-\bar{\tau}} - 2\lambda^2 \bar{\tau}^2} \int_{-\tau(0)}^0 e^s \int_s^0 \sum_{i=1}^N |\dot{v}_i(\sigma)|^2 d\sigma ds.$$

REMARK 2.1. As already said, the structural Assumption (1.15) is either always satisfied for some classes of potentials, or, for other potentials, leads to a region of favorable initial conditions. Under this assumption, for $\bar{\tau} = 0$ (no delay in the model), convergence to consensus is guaranteed. The theorem says that convergence to consensus withstands the introduction of a delay in the model, provided that the maximal delay $\bar{\tau}$ does not exceed the threshold τ_0 defined by (2.4) and (2.5).

Note the interesting fact that the threshold τ_0 depends on the parameter λ and on the lower bound γ in (1.15) for the Fiedler number, but does not depend on the number N of the agents.

2.2. Proof of Theorem 2.1. We start with the following lemma.

LEMMA 2.1. *We consider an arbitrary solution $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$ of (1.5). Setting*

$$R_\tau(t) = \frac{1}{N} \int_{t-\tau(t)}^t \sum_{i=1}^N |\dot{v}_i(s)|^2 ds \tag{2.8}$$

¹The threshold τ_0 is well defined in this way because the function $x \mapsto x^2 e^x$ is continuous and monotone increasing from $(0, +\infty)$ to $(0, +\infty)$.

we have

$$\frac{1}{N} \sum_{i=1}^N |\dot{v}_i(t)|^2 \leq 4 \frac{\lambda^2}{N} \|\mathbf{w}(t)\|^2 + 2\lambda^2 \bar{\tau} R_\tau(t) \tag{2.9}$$

for every $t > 0$.

Proof. First observe that, from (2.1), $v_i - v_j = w_i - w_j$, for all $i, j = 1, \dots, N$. Using (1.5), we then obtain

$$\begin{aligned} \dot{v}_i(t) &= \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(w_j(t) - w_i(t)) + \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \\ &= \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(w_j(t) - w_i(t)) - \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t)) \int_{t-\tau(t)}^t \dot{v}_j(s) ds. \end{aligned}$$

Now, using (1.7), we get that

$$|\dot{v}_i(t)| \leq \frac{\lambda}{N} \sum_{j \neq i} |w_j(t) - w_i(t)| + \frac{\lambda}{N} \sum_{j \neq i} \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds.$$

Then,

$$\begin{aligned} |\dot{v}_i(t)|^2 &\leq 2 \frac{\lambda^2}{N^2} \left(\sum_{j=1}^N |w_i(t) - w_j(t)| \right)^2 + 2 \frac{\lambda^2}{N^2} \left(\sum_{j=1}^N \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \right)^2 \\ &\leq 2 \frac{\lambda^2}{N} \sum_{j=1}^N |w_j(t) - w_i(t)|^2 + 2 \frac{\lambda^2}{N} \sum_{j=1}^N \left(\int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \right)^2. \end{aligned}$$

Using (2.2), the Cauchy-Schwarz inequality and (1.3), we infer that

$$\begin{aligned} \sum_{i=1}^N |\dot{v}_i(t)|^2 &\leq 2 \frac{\lambda^2}{N} \sum_{i,j=1}^N |w_i(t) - w_j(t)|^2 + 2\lambda^2 \tau(t) \int_{t-\tau(t)}^t \sum_{i=1}^N |\dot{v}_i(s)|^2 ds \\ &\leq 4\lambda^2 \|\mathbf{w}(t)\|^2 + 2\lambda^2 \bar{\tau} \int_{t-\tau(t)}^t \sum_{i=1}^N |\dot{v}_i(s)|^2 ds \end{aligned}$$

which gives (2.9). □

REMARK 2.2. The term $R_\tau(t)$ is due to the presence of the time-delay. Indeed, we have two quantities at the right-hand side of the inequality (2.9): the ‘‘classical’’ term $\|w\|^2$ (coming from the undelayed model), and the term $R_\tau(t)$ caused by the delay effect.

LEMMA 2.2. Given any solution $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$ of (1.5), we have

$$\frac{d}{dt} \left(\frac{1}{N} \|\mathbf{w}(t)\|^2 \right) \leq -\frac{\gamma}{N} \|\mathbf{w}(t)\|^2 + \frac{\lambda^2 \bar{\tau}}{\gamma} R_\tau(t) \tag{2.10}$$

for every $t > 0$.

Proof. Using (1.5), we compute

$$\begin{aligned}
 \dot{w}_i(t) &= \dot{v}_i(t) - \frac{1}{N} \sum_{k=1}^N \dot{v}_k(t) \\
 &= \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_i(t)) - \frac{\lambda}{N^2} \sum_{k=1}^N \sum_{j \neq k} \psi_{kj}(t - \tau(t))(v_j(t - \tau(t)) - v_k(t)) \\
 &= \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t) - v_i(t)) + \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \\
 &\quad - \frac{\lambda}{N^2} \sum_{k=1}^N \sum_{j \neq k} \psi_{kj}(t - \tau(t))(v_j(t) - v_k(t)) - \frac{\lambda}{N^2} \sum_{k=1}^N \sum_{j \neq k} \psi_{kj}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \\
 &= \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(w_j(t) - w_i(t)) + \frac{\lambda}{N} \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \\
 &\quad - \frac{\lambda}{N^2} \sum_{k=1}^N \sum_{j \neq k} \psi_{kj}(t - \tau(t))(w_j(t) - w_k(t)) \\
 &\quad - \frac{\lambda}{N^2} \sum_{k=1}^N \sum_{j \neq k} \psi_{kj}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)).
 \end{aligned}$$

Then,

$$\begin{aligned}
 \sum_{i=1}^N w_i(t) \dot{w}_i(t) &= -\frac{1}{2} \frac{\lambda}{N} \sum_{i,j=1}^N \psi_{ij}(t - \tau(t)) |w_i - w_j|^2 \\
 &\quad + \frac{\lambda}{N} \sum_{i=1}^N \sum_{j \neq i} \psi_{ij}(t - \tau(t)) (v_j(t - \tau(t)) - v_j(t)) w_i
 \end{aligned}$$

where we have used that $\sum_i w_i = 0$ and

$$\begin{aligned}
 &\sum_{j \neq i} \psi_{ij}(t - \tau(t))(w_j(t) - w_i(t)) w_i(t) \\
 &= \sum_{j \neq i} \psi_{ij}(t - \tau(t))(w_j(t) - w_i(t))(w_i(t) - w_j(t)) + \sum_{j \neq i} \psi_{ij}(t - \tau(t))(w_j(t) - w_i(t)) w_j(t) \\
 &= -\frac{1}{2} \sum_{j \neq i} \psi_{ij}(t - \tau(t)) |w_i(t) - w_j(t)|^2.
 \end{aligned}$$

Therefore, thanks to (2.3), we infer that

$$\begin{aligned}
 &\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}(t)\|^2 \right) \\
 &= -\langle L(t - \tau(t)) \mathbf{w}(t), \mathbf{w}(t) \rangle + \frac{\lambda}{N} \sum_i \sum_{j \neq i} \psi_{ij}(t - \tau(t)) (v_j(t - \tau(t)) - v_j(t)) w_i(t) \\
 &= -\frac{1}{2} \frac{\lambda}{N} \sum_{i,j=1}^N \psi_{ij}(t - \tau(t)) |w_i(t) - w_j(t)|^2 \\
 &\quad + \frac{\lambda}{N} \sum_i \sum_{j \neq i} \psi_{ij}(t - \tau(t)) (v_j(t - \tau(t)) - v_j(t)) w_i(t).
 \end{aligned}$$

The second term at the right-hand side of the above equality is bounded by

$$\left| \frac{\lambda}{N} \sum_i \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t))w_i(t) \right| \leq \frac{\lambda}{N} \|w(t)\| \|U(t)\|$$

where $U(t) = (U_1(t), \dots, U_N(t))$ is defined by

$$U_i(t) = \sum_{j \neq i} \psi_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)), \quad i = 1, \dots, N$$

and is estimated by

$$\begin{aligned} \|U(t)\| &\leq \sum_{i=1}^N |U_i(t)| \leq \sum_{i=1}^N \sum_{j \neq i} \psi_{ij}(t - \tau(t)) \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \\ &\leq \sum_{ij} \psi_{ij}(t - \tau(t)) \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}(t)\|^2 \right) &\leq -\frac{1}{2} \frac{\lambda}{N} \sum_{i,j=1}^N \psi_{ij}(t - \tau(t)) |w_i(t) - w_j(t)|^2 \\ &\quad + \frac{\lambda}{N} \sum_{ij} \psi_{ij}(t - \tau(t)) \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \|\mathbf{w}(t)\| \\ &\leq -\frac{1}{2} \frac{\lambda}{N} \sum_{i,j=1}^N \psi_{ij}(t - \tau(t)) |w_i(t) - w_j(t)|^2 + \lambda \sum_{j=1}^N \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \|\mathbf{w}(t)\| \\ &\leq -\frac{1}{2} \frac{\lambda}{N} \sum_{i,j=1}^N \psi_{ij}(t - \tau(t)) |w_i(t) - w_j(t)|^2 + \lambda \frac{\delta}{2} \|\mathbf{w}\|^2 \\ &\quad + \frac{\lambda}{2\delta} \left(\sum_{j=1}^N \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \right)^2 \end{aligned}$$

where we have used the Young inequality² for some arbitrary $\delta > 0$. Choosing $\delta = \frac{\gamma}{\lambda}$, where γ is the constant in the structural Assumption (1.15), we infer that

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}(t)\|^2 \right) &\leq -\langle L(t - \tau(t))\mathbf{w}(t), \mathbf{w}(t) \rangle + \frac{\gamma}{2} \|\mathbf{w}(t)\|^2 + \frac{\lambda^2}{2\gamma} \left(\sum_{j=1}^N \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds \right)^2 \\ &\leq -\frac{\gamma}{2} \|\mathbf{w}(t)\|^2 + \frac{\lambda^2 \tau(t)}{2\gamma} \sum_{j=1}^N \int_{t-\tau(t)}^t |\dot{v}_j(s)|^2 ds \end{aligned} \tag{2.11}$$

which, using (1.3) and the Definition (2.8) of $R_\tau(t)$, gives (2.10). □

²This inequality states that, given any positive real numbers a, b and δ , we have $ab \leq \frac{a^2}{2\delta} + \frac{\delta b^2}{2}$.

We are now in a position to prove Theorem 2.1. Let $\beta > 0$ be a positive constant to be chosen later. We consider the Lyapunov functional along solutions of (1.5), defined by

$$\mathcal{L}(t) = \frac{1}{2N} \|\mathbf{w}(t)\|^2 + \frac{\beta}{N} \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \sum_{i=1}^N |\dot{v}_i(\sigma)|^2 d\sigma ds. \tag{2.12}$$

Using (2.11) and Lemma 2.1, we have

$$\begin{aligned} \dot{\mathcal{L}}(t) &\leq -\frac{\gamma}{2N} \|\mathbf{w}(t)\|^2 + \frac{\lambda^2 \bar{\tau}}{2\gamma} R_\tau(t) + \frac{\beta \tau(t)}{N} \sum_{i=1}^N |\dot{v}_i(t)|^2 \\ &\quad - \frac{\beta}{N} (1 - \tau'(t)) e^{-\tau(t)} \int_{t-\tau(t)}^t \sum_{i=1}^N |\dot{v}_j(s)|^2 ds \\ &\quad - \frac{\beta}{N} \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \sum_{i=1}^N |\dot{v}_i(\sigma)|^2 d\sigma ds \\ &\leq -\frac{1}{N} \left(\frac{\gamma}{2} - 4\lambda^2 \beta \bar{\tau} \right) \|\mathbf{w}(t)\|^2 - \left(\beta(1-c)e^{-\bar{\tau}} - \frac{\lambda^2 \bar{\tau}}{2\gamma} - 2\beta \lambda^2 \bar{\tau}^2 \right) R_\tau(t) \\ &\quad - \frac{\beta}{N} \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \sum_{i=1}^N |\dot{v}_i(\sigma)|^2 d\sigma ds, \end{aligned}$$

where we have used (1.3) and (1.4). Convergence to consensus is then ensured if

$$\frac{\gamma}{2} - 4\beta \lambda^2 \bar{\tau} > 0 \quad \text{and} \quad \beta(1-c)e^{-\bar{\tau}} - \frac{\lambda^2 \bar{\tau}}{2\gamma} - 2\beta \lambda^2 \bar{\tau}^2 \geq 0. \tag{2.13}$$

The second inequality of (2.13) gives a first restriction on the size of the delay, namely, that $\bar{\tau}^2 e^{\bar{\tau}} < \frac{1-c}{2\lambda^2}$. Let us now choose the constant $\beta > 0$ in the Definition (2.12) of $\mathcal{L}(\cdot)$ so that both conditions in (2.13) are satisfied:

$$\frac{\lambda^2 \bar{\tau}}{2\gamma} \frac{1}{(1-c)e^{-\bar{\tau}} - 2\lambda^2 \bar{\tau}^2} \leq \beta < \frac{\gamma}{8\lambda^2 \bar{\tau}}.$$

This is possible only if

$$\frac{\lambda^2 \bar{\tau}^2}{(1-c)e^{-\bar{\tau}} - 2\lambda^2 \bar{\tau}^2} < \frac{\gamma^2}{4\lambda^2},$$

which is equivalent to

$$\bar{\tau}^2 e^{\bar{\tau}} < \chi$$

with χ defined by (2.4). We conclude that, if $\bar{\tau}^2 e^{\bar{\tau}} < \chi$, then we can choose β such that

$$\frac{d\mathcal{L}}{dt}(t) \leq -r\mathcal{L}(t) \tag{2.14}$$

for a suitable positive constant r . In particular, in order to obtain the best decay rate with our procedure, we fix

$$\beta = \frac{\lambda^2 \bar{\tau}}{2\gamma} \frac{1}{(1-c)e^{-\bar{\tau}} - 2\lambda^2 \bar{\tau}^2},$$

thus obtaining (2.14) with r as in (2.7).

To conclude, it suffices to write that

$$\frac{1}{N} \|\mathbf{w}(t)\|^2 \leq 2\mathcal{L}(t) \leq 2\mathcal{L}(0)e^{-rt}.$$

Then (2.6) follows from the latter inequality, (1.2) and (2.2) with $C = \mathcal{L}(0)$ as in the statement.

3. Consensus for nonsymmetric potentials: L^∞ analysis

3.1. The main result. In this section, we consider nonsymmetric potentials, and we perform an L^∞ analysis as in [40]. We consider the Cucker-Smale system

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_i(t)) \quad \forall i = 1, \dots, N \end{aligned} \tag{3.1}$$

with initial conditions, for $i = 1, \dots, N$,

$$x_i(t) = f_i(t), \quad v_i(t) = g_i(t) \quad \forall t \in [-\tau(0), 0]$$

where $f_i, g_i : [-\tau(0), 0] \rightarrow \mathbb{R}$ are given functions and $a_{ij} > 0$ quantifies the pairwise influence of j^{th} agent on the alignment of i^{th} agent. By rescaling λ if necessary (or by time reparametrization), we assume that

$$\frac{1}{N} \sum_{j \neq i} a_{ij} < 1. \tag{3.2}$$

This includes, for instance, the case considered in the previous section, that is

$$a_{ij}(t) = \psi(|x_i(t) - x_j(t)|)$$

with $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying $\psi(r) < 1$ for every $r \geq 0$, but we can consider a nonsymmetric interaction, for instance like in (1.17),

$$a_{ij}(t) = \frac{N\psi(|x_i(t) - x_j(t)|)}{\sum_{k=1}^N \psi(|x_k(t) - x_i(t)|)}$$

for a suitable bounded function ψ .

As shortly mentioned before, an analogous delay model has also been investigated in [17] for τ constant and under a restrictive assumption on the potential interaction. Indeed, the authors there consider the problem

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \sum_{j=1}^N \Phi_{ij}(x, \tau)(v_j(t - \tau) - v_i(t)) \quad \forall i = 1, \dots, N \end{aligned} \tag{3.3}$$

where the communication weights are defined in (1.13). With such communication rates, the analysis of convergence to consensus is significantly easier because, using that $\sum_{j=1}^N \Phi_{ij} = 1$ for $i = 1, \dots, N$, one can rewrite the velocity equation as

$$\dot{v}_i(t) = \sum_{j=i}^N \Psi_{ij}(x, \tau)v_j(t - \tau) - v_i(t) \quad \forall i = 1, \dots, N.$$

Namely, the term depending on v_i in the left-hand side of (3.3) is not $\frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau)v_i(t)$ as in the more general model (3.1), but simply $-v_i(t)$.

Following [40], we set $a_{ii} = N - \sum_{j \neq i} a_{ij}$, so that $\sum_{j=1}^N a_{ij} = N, i = 1, \dots, N$. Setting

$$\tilde{v}_i(t) = \frac{1}{N} \sum_{j=1}^N a_{ij}(t - \tau(t))v_j(t) \quad \forall i = 1, \dots, N \tag{3.4}$$

the system (3.1) is written as

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \lambda(\tilde{v}_i(t) - v_i(t)) + \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \quad \forall i = 1, \dots, N. \end{aligned}$$

We denote by $d_X(t)$ and $d_V(t)$ the diameter in position and velocity phase spaces (see [30]), respectively defined by

$$d_X(t) = \max_{i,j} |x_j(t) - x_i(t)|$$

and

$$d_V(t) = \max_{i,j} |v_j(t) - v_i(t)|.$$

A solution of (3.1) converges to consensus if

$$\sup_{t \geq 0} d_X(t) < +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} d_V(t) = 0.$$

Note that the functions d_X and d_V are not of class C^1 in general. We will thus use a suitable notion of generalized gradient, namely the upper Dini derivative, as in [38], in order to perform our computations. We recall that, for a given function F continuous at t , the upper Dini derivative of F at t is defined by

$$D^+ F(t) = \limsup_{h \rightarrow 0^+} \frac{F(t+h) - F(t)}{h}.$$

If F is differentiable at t , then $D^+ F(t) = \frac{dF}{dt}(t)$. However, for all t there exists a sequence $h_n \rightarrow 0^+$ such that

$$D^+ F(t) = \lim_{n \rightarrow +\infty} \frac{F(t+h_n) - F(t)}{h_n}.$$

In particular, from the definition of $d_X(\cdot)$, there exists at most countable increasing sequence $\{t_k\}_k$ such that $d_X(t) = |x_r(t) - x_s(t)|$, for suitable $r, s \in \{1, \dots, N\}$, on $[t_k, t_{k+1})$. Then, for fixed t one can find a sequence $h_n \rightarrow 0^+$ for which $d_X(t+h_n) = |x_r(t+h_n) - x_s(t+h_n)|$, for n large (small h_n), and

$$\begin{aligned} D^+ d_X(t) &= \lim_{n \rightarrow \infty} h_n^{-1} \{|x_r(t+h_n) - x_s(t+h_n)| - |x_r(t) - x_s(t)|\} \\ &\leq \left| \frac{dx_r}{dt}(t) - \frac{dx_s}{dt}(t) \right|. \end{aligned} \tag{3.5}$$

Analogous arguments apply to $D^+d_V(t)$ and $D^+d_V^2(t)$.

THEOREM 3.1. *We set*

$$\chi = \frac{1-c}{\lambda} \frac{\psi^*}{\psi^*+2} \tag{3.6}$$

where ψ^* is the constant in (1.18). Under (1.18), if $\bar{\tau}e^{\bar{\tau}} < \chi$ then every solution of (3.1) satisfies

$$d_V(t) \leq Ce^{-rt} \quad \forall t \geq 0 \tag{3.7}$$

with

$$r = \min \left\{ \lambda \left(\psi^* - \frac{2\lambda\bar{\tau}}{(1-c)e^{-\bar{\tau}} - \lambda\bar{\tau}} \right), 1 \right\} \tag{3.8}$$

and

$$C = d_V(0) + \frac{2\lambda}{(1-c)e^{-\bar{\tau}} - \lambda\bar{\tau}} \int_{-\tau(0)}^0 e^s \int_s^0 \max_{j=1, \dots, N} |\dot{v}_j(\sigma)| d\sigma ds. \tag{3.9}$$

3.2. Proof of Theorem 3.1. We start by establishing several estimates.

LEMMA 3.1 ([40]). *Let $S = (S_{ij})_{1 \leq i, j \leq N}$ be a skew-symmetric matrix such that $|S_{ij}| \leq M$ for all i, j . Let $u, w \in \mathbb{R}^N$ be two given real vectors with nonnegative entries, $u_i, w_i \geq 0$, and let $\bar{U} = \frac{1}{N} \sum_i u_i$ and $\bar{W} = \frac{1}{N} \sum_i w_i$. Then*

$$\frac{1}{N^2} |\langle Su, w \rangle| \leq M \left(\bar{U}\bar{W} - \frac{1}{N^2} \sum_{i,j=1}^N \min(u_i w_j, u_j w_i) \right). \tag{3.10}$$

LEMMA 3.2. *Let $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$ be a solution of (3.1). Setting*

$$\sigma_\tau(t) = \int_{t-\tau(t)}^t \max_{j=1, \dots, N} |\dot{v}_j(s)| ds, \tag{3.11}$$

we have, for every $t \geq 0$,

$$D^+d_X(t) \leq d_V(t), \quad D^+d_V(t) \leq -\lambda\psi^*d_V(t) + 2\lambda\sigma_\tau(t), \tag{3.12}$$

where ψ^* is the constant in (1.18).

Proof. Fix $t \geq 0$ and let p, q, r and s be indices such that $d_X(t) = |x_r(t) - x_s(t)|$ and $d_V(t) = |v_p(t) - v_q(t)|$. Then, from (3.5), we have $D^+d_X(t) \leq |v_r(t) - v_s(t)| \leq d_V(t)$. Also, observing that for $D^+d_V(\cdot)$ and $D^+d_V^2(\cdot)$, analogous considerations to those for $D^+d_X(\cdot)$ (see comments before (3.5)) are valid, we have

$$\begin{aligned} D^+(d_V^2(t)) &= 2\langle v_p(t) - v_q(t), \dot{v}_p(t) - \dot{v}_q(t) \rangle \\ &= 2\lambda\langle v_p(t) - v_q(t), \tilde{v}_p(t) - \tilde{v}_q(t) \rangle - 2\lambda|v_p(t) - v_q(t)|^2 \\ &\quad + 2\frac{\lambda}{N} \left\langle v_p(t) - v_q(t), \sum_{j \neq p} a_{pj}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \right. \\ &\quad \left. - \sum_{j \neq q} a_{qj}(t - \tau(t))(v_j(t - \tau(t)) - v_j(t)) \right\rangle \end{aligned}$$

where $\tilde{v}_i, i = 1, \dots, N$, are defined in (3.4), and then

$$\begin{aligned} D^+(d_V^2(t)) &= 2 \langle v_p(t) - v_q(t), \dot{v}_p(t) - \dot{v}_q(t) \rangle \\ &\leq 2\lambda \langle v_p(t) - v_q(t), \tilde{v}_p(t) - \tilde{v}_q(t) \rangle - 2\lambda |v_p(t) - v_q(t)|^2 \\ &\quad + 4\lambda |v_p(t) - v_q(t)| \int_{t-\tau(t)}^t \max_{j=1, \dots, N} |\dot{v}_j(s)| ds. \end{aligned}$$

But since

$$\begin{aligned} \tilde{v}_p(t) - \tilde{v}_q(t) &= \frac{1}{N} \sum_{j=1}^N a_{pj}(t - \tau(t)) v_j(t) - \frac{1}{N} \sum_{i=1}^N a_{qi}(t - \tau(t)) v_i(t) \\ &= \frac{1}{N^2} \sum_{i=1}^N a_{qi}(t - \tau(t)) \sum_{j=1}^N a_{pj}(t - \tau(t)) v_j(t) \\ &\quad - \frac{1}{N^2} \sum_{j=1}^N a_{pj}(t - \tau(t)) \sum_{i=1}^N a_{qi}(t - \tau(t)) v_i(t) \\ &= \frac{1}{N^2} \sum_{i,j=1}^N a_{qi}(t - \tau(t)) a_{pj}(t - \tau(t)) (v_j(t) - v_i(t)), \end{aligned}$$

we get that

$$\begin{aligned} D^+(d_V^2(t)) &\leq 2 \frac{\lambda}{N^2} \sum_{i,j=1}^N a_{qi}(t - \tau(t)) a_{pj}(t - \tau(t)) \langle v_j(t) - v_i(t), v_p(t) - v_q(t) \rangle \\ &\quad - 2\lambda |v_p(t) - v_q(t)|^2 + 4\lambda |v_p(t) - v_q(t)| \int_{t-\tau(t)}^t \max_{j=1, \dots, N} |\dot{v}_j(s)| ds. \end{aligned} \tag{3.13}$$

We estimate the first term at the right-hand side of (3.13) by applying Lemma 3.1, with $S_{ij} = \langle v_j(t) - v_i(t), v_p(t) - v_q(t) \rangle$ and $u_i = a_{qi}(t - \tau)$ and $w_j = a_{pj}(t - \tau(t))$ for $i, j = 1, \dots, N$. Since $|S_{ij}| \leq d_V^2(t)$ and $\bar{U}, \bar{W} = 1$, using Assumption (1.18), we infer from (3.10) with $\theta = \psi^*$ that

$$\left| \frac{1}{N^2} \sum_{i,j=1}^N a_{qi}(t - \tau(t)) a_{pj}(t - \tau(t)) \langle v_j(t) - v_i(t), v_p(t) - v_q(t) \rangle \right| \leq (1 - \psi^*) d_V^2(t).$$

With the above estimate, we obtain from (3.13) that

$$D^+(d_V^2(t)) \leq 2\lambda(1 - \psi^*) d_V^2(t) - 2\lambda d_V^2(t) + 4\lambda d_V(t) \sigma_\tau(t),$$

from which (3.12) follows. □

LEMMA 3.3. *Let $(\mathbf{x}(\cdot), \mathbf{v}(\cdot))$ be a solution of (3.1). Then*

$$\max_{j=1, \dots, N} |\dot{v}_j(t)| \leq \lambda d_V(t) + \lambda \sigma_\tau(t) \tag{3.14}$$

for every $t \geq 0$.

Proof. Using (3.1), we have

$$\dot{v}_i(t) = \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t)) (v_j(t) - v_i(t)) + \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t)) (v_j(t - \tau(t)) - v_j(t)),$$

from which we infer that

$$|\dot{v}_i(t)| \leq \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t)) d_V(t) + \frac{\lambda}{N} \sum_{j \neq i} a_{ij}(t - \tau(t)) \int_{t-\tau(t)}^t |\dot{v}_j(s)| ds.$$

Then, we have

$$|\dot{v}_i(t)| \leq \lambda d_V(t) + \lambda \int_{t-\tau(t)}^t \max_{j=1, \dots, N} |\dot{v}_j(s)| ds$$

and the lemma follows by taking the maximum in the left-hand side and using the Definition (3.11) of $\sigma_\tau(t)$. \square

We are now in a position to prove the theorem. Let $\beta > 0$ to be chosen later. We consider the Lyapunov functional defined along any solution by

$$\mathcal{F}(t) = d_V(t) + \beta \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \max_{j=1, \dots, N} |\dot{v}_j(\sigma)| d\sigma ds. \tag{3.15}$$

First of all, using (3.12), we have

$$\begin{aligned} D^+ \mathcal{F}(t) &\leq -\lambda \psi^* d_V(t) + 2\lambda \sigma_\tau(t) \\ &\quad - \beta(1 - \tau'(t)) e^{-\tau(t)} \int_{t-\tau(t)}^t \max_{j=1, \dots, N} |\dot{v}_j(s)| ds + \beta \tau(t) \max_{j=1, \dots, N} |\dot{v}_j(t)| \\ &\quad - \beta \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \max_{j=1, \dots, N} |\dot{v}_j(\sigma)| d\sigma ds. \end{aligned}$$

Using (1.3) and (1.4), it follows from Lemma 3.3 that

$$\begin{aligned} D^+ \mathcal{F}(t) &\leq -\lambda \psi^* d_V(t) + (2\lambda - \beta(1 - c)e^{-\bar{\tau}}) \sigma_\tau(t) + \beta \bar{\tau} \lambda d_V(t) \\ &\quad + \beta \bar{\tau} \lambda \int_{t-\tau(t)}^t \max_{j=1, \dots, N} |\dot{v}_j(s)| ds - \beta \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \max_{j=1, \dots, N} |\dot{v}_j(\sigma)| d\sigma ds \\ &\leq -\lambda(\psi^* - \beta \bar{\tau}) d_V(t) - (\beta(1 - c)e^{-\bar{\tau}} - 2\lambda - \beta \bar{\tau} \lambda) \sigma_\tau(t) \\ &\quad - \beta \int_{t-\tau(t)}^t e^{-(t-s)} \int_s^t \max_{j=1, \dots, N} |\dot{v}_j(\sigma)| d\sigma ds. \end{aligned}$$

Convergence to consensus is ensured if

$$\psi^* - \bar{\tau} \beta > 0 \quad \text{and} \quad \beta(1 - c)e^{-\bar{\tau}} - 2\lambda - \beta \bar{\tau} \lambda \geq 0. \tag{3.16}$$

The second inequality of (3.16) gives a first restriction on the size of the delay: $\bar{\tau} e^{\bar{\tau}} < \frac{1-c}{\lambda}$. Let us now choose the constant $\beta > 0$ in the Definition (3.15) of the Lyapunov functional \mathcal{F} so that both conditions in (3.16) are satisfied. We impose that $\beta < \frac{\psi^*}{\bar{\tau}}$ and $\beta \geq \frac{2\lambda}{(1-c)e^{-\bar{\tau}} - \lambda \bar{\tau}}$. This is possible if and only if $\frac{2\lambda}{(1-c)e^{-\bar{\tau}} - \lambda \bar{\tau}} < \frac{\psi^*}{\bar{\tau}}$, that is, equivalently, $\bar{\tau} e^{\bar{\tau}} < \chi$ where χ is defined by (3.6).

We now choose β in the definition of \mathcal{F} such that

$$D^+ \mathcal{F}(t) \leq -r \mathcal{F}(t) \tag{3.17}$$

for a suitable positive constant r . In order to have a better decay estimate, let us fix

$$\beta = \frac{2\lambda}{(1-c)e^{-\bar{\tau}} - \lambda\bar{\tau}}.$$

Then, we obtain (3.17) with r as in (3.8). Therefore,

$$d_V(t) \leq \mathcal{F}(t) \leq \mathcal{F}(0)e^{-rt} \quad \forall t \geq 0.$$

The exponential decay estimate (3.7) is then proved with $C = \mathcal{F}(0)$ as in (3.9).

4. Numerical simulations

We provide here some numerical simulations illustrating our results. Throughout the section, we take $d=2$ and we choose the Cucker and Smale potential

$$\psi(s) = \frac{1}{(1+s^2)^b}$$

with $b=2$. As recalled in the introduction, for this value of b , for the classical model without time-delay, there is a conditional flocking result (see [22, 23, 26]): under a suitable relation on the initial data, itself implying that the structural Assumption (1.15) is satisfied along trajectories, convergence to consensus is ensured (and we find numerically that $\gamma=1$).

Here, we have $\lambda=1$. The value of the threshold τ_0 defined by (2.5) in Theorem 2.1 is

$$\tau_0 \simeq 0.3437.$$

Recall that this threshold does not depend on the number N of agents. According to Theorem 2.1, if the initial conditions are chosen in the favorable region of consensus (determined by the structural Assumption (1.15)) for the undelayed model, then we still do have convergence to consensus for the delayed model provided $\tau < \tau_0$.

Interestingly, we are going to see in our numerical simulations hereafter that the maximal threshold value τ_0 for the delay under which convergence to consensus is kept, seems to be sharp when N is large.

All numerical simulations have been done with `Matlab` on a standard desktop computer. The differential system to be integrated has been discretized with the usual explicit Runge-Kutta method of order 4 (RK4 scheme). The subdivision of the time interval has been chosen regular, and the values of the delays have been chosen such that, at some given time t_i of the subdivision, denoting by τ a delay, the subdivision is such that $t_i - \tau$ belongs to the subdivision: this condition is often referred to as a *commensurability* assumption in the literature treating numerical simulations of time-delayed systems.

4.1. Simulations with 3 agents. We take $N=3$ (3 agents).

For the moment, we do not consider any time-delay in the model, i.e., $\tau(\cdot) \equiv 0$. We take as initial conditions

$$\begin{aligned} x_1^0 &= (0,0), & v_1^0 &= (1,0), \\ x_2^0 &= (0,1), & v_2^0 &= (1,0), \\ x_3^0 &= (1,0), & v_3^0 &= (0.5,0.5). \end{aligned}$$

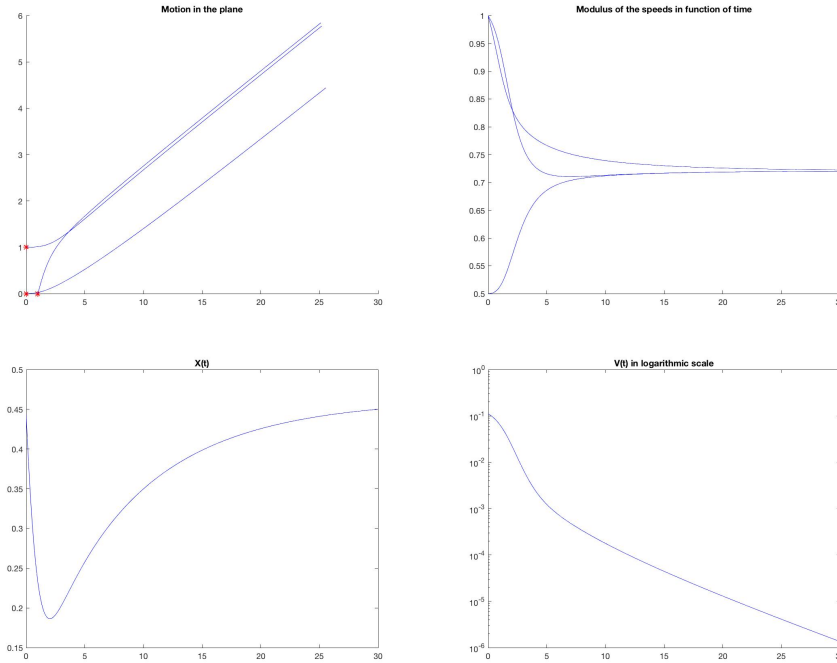


FIG. 4.1. $N=3$, no time-delay ($\tau=0$). Simulation on the time interval $[0,30]$.

These initial conditions are “favorable” in the sense that we have convergence to consensus, as it can be seen on Figure 4.1. At the top left are drawn the curves $t \mapsto x_i(t) \in \mathbb{R}^2$: motion in the plane of the three agents; the initial points are represented with a star. At the top right, one can see the modulus of the speeds $\|v_i(t)\|$, as a function of t . At the bottom left is drawn the time evolution of the position variance $X(t)$, and at the bottom right, the speed variance $V(t)$ in a logarithmic scale.

We now introduce a time-delay which, for simplicity, we take constant: $\tau(t) \equiv \tau$. We take as initial conditions, on $[-\tau, 0]$,

$$x_i(t) = x_i^0 + (t + \tau)v_i^0, \quad v_i(t) = v_i^0, \quad i = 1, \dots, N. \tag{4.1}$$

In other words, along the interval $[-\tau, 0]$ the agents follow the dynamics $\dot{x}_i = v_i$ and $\dot{v}_i = 0$, and thus each agent performs a translation motion, starting at x_i^0 with the speed v_i^0 .

We observe in the numerical simulations that convergence of consensus is obtained for any value of the delay satisfying $\tau < \tau_0$, as expected.

Let us test, however, larger values of τ .

The corresponding solution for $\tau=1$ is drawn on Figure 4.2. For this value of the time-delay, we observe that convergence to consensus is still obtained. This shows that our threshold τ_0 is not sharp for $N=3$. This is not surprising because the region of consensus (set of “favorable” initial conditions, studied in [22, 26]) depends on N .

The corresponding solution for $\tau=5$ is drawn on Figure 4.3. For this value of the time-delay, convergence to consensus is now lost. When time goes to infinity, the agents do not remain grouped, and one can indeed observe that the position variance $X(t)$ tends to $+\infty$.

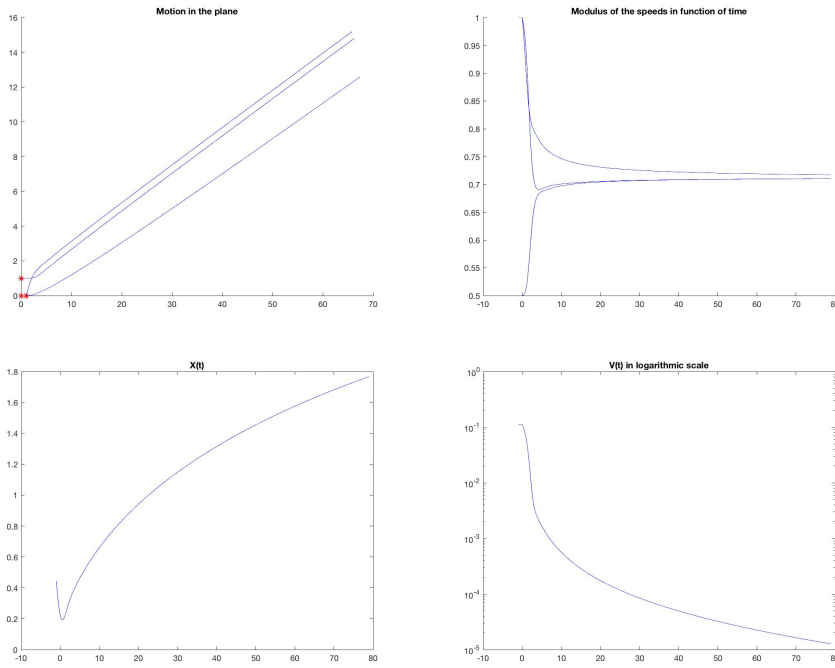


FIG. 4.2. $N=3$, time-delay $\tau=1$. Simulation on the time interval $[-1,80]$.

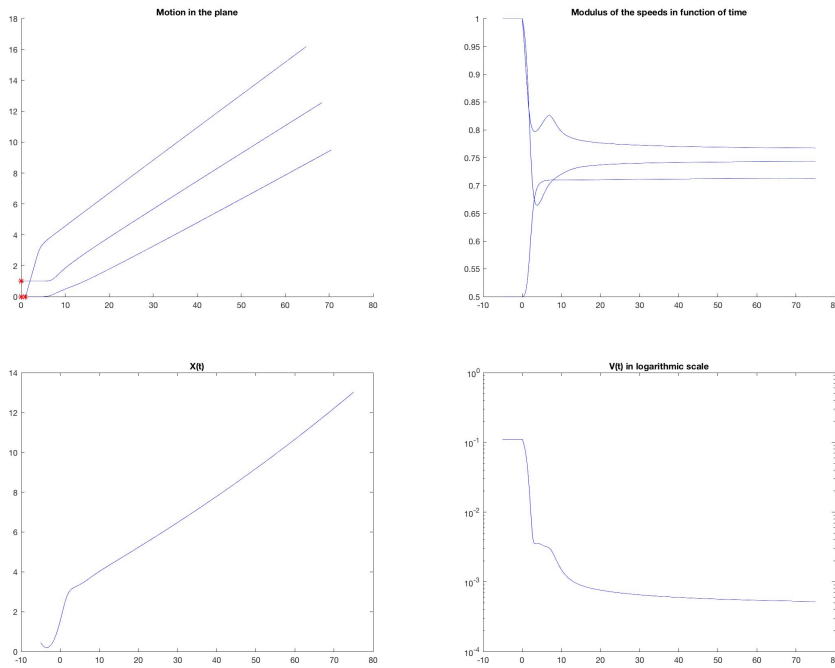


FIG. 4.3. $N=3$, time-delay $\tau=5$. Simulation on the time interval $[-5,80]$.

We do not provide here any numerical simulation with time-varying delays, because they do not provide any new relevant information with respect to that already provided here. Anyway, if one would like to simulate time-varying delays, note that one should in any case take care of the commensurability condition, as already mentioned; therefore the delay function $t \mapsto \tau(t)$ should be discretized as a piecewise constant function in accordance with the subdivision that is used in the RK4 scheme.

4.2. Simulations with 50 agents.

We now take $N = 50$.

As before, we first consider the model without delay, i.e., $\bar{\tau} = 0$. We take as initial conditions

$$\begin{aligned} x_i^0 &= (1.1 \cos(i + \sqrt{2}), 1.1 \cos(i + 2\sqrt{2})) \\ v_i^0 &= (2 + 0.15i \sin(\sqrt{3} - 1), 2 + 0.15i \sin(\sqrt{3} - 2)) \quad \forall i = 1, \dots, N. \end{aligned}$$

These initial conditions are “favorable”: we have convergence to consensus (see Figure 4.4). At the top left of this figure are drawn the curves $t \mapsto x_i(t) \in \mathbb{R}^2$: motion in the plane of the 50 agents (initial points are represented with a star); for better readability, the trajectories are drawn only on the time interval $[0, 3]$. At the top right are drawn the speeds $\|v_i(t)\|$, as a function of t , only on the time interval $[0, 200]$ to have better readability. At the bottom left is drawn the time evolution of the position variance $X(t)$, and at the bottom right, the speed variance $V(t)$ in a logarithmic scale, on the time interval $[0, 2000]$, so that convergence to consensus can clearly be observed.

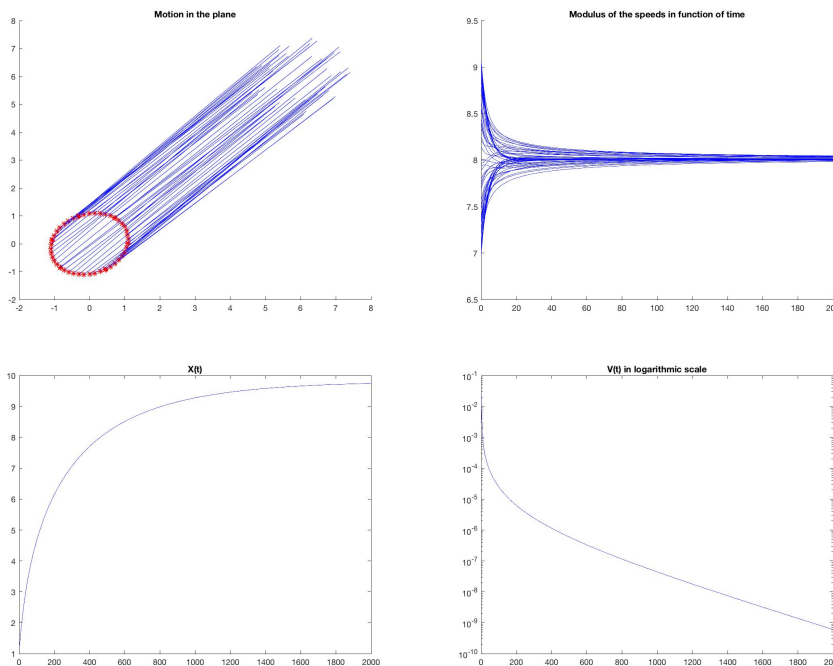


FIG. 4.4. $N = 50$, no time-delay ($\tau = 0$).

We now introduce a constant time-delay ($\tau(t) \equiv \tau$). We take initial conditions (4.1) on $[-\tau, 0]$ as before.

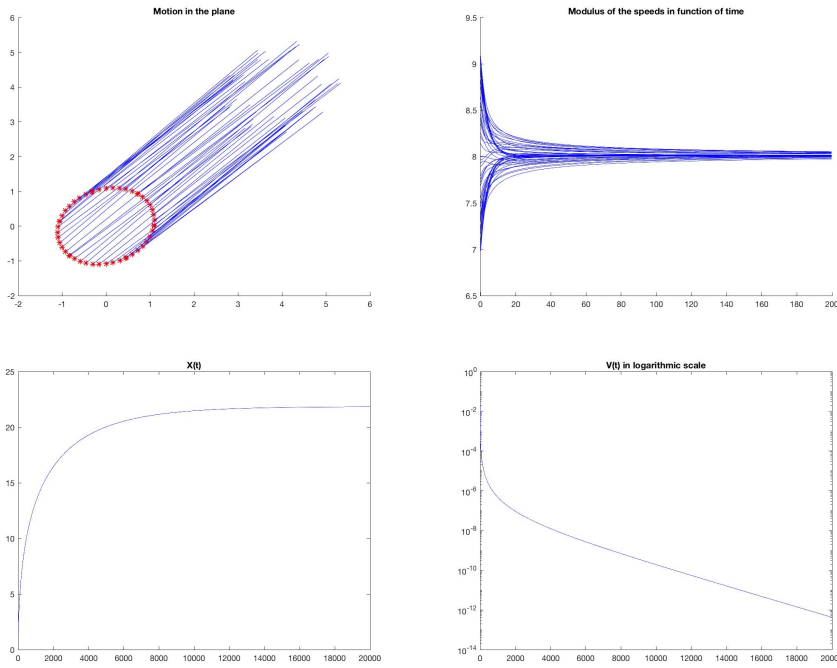


FIG. 4.5. $N = 50$, time-delay $\tau = 0.25$. Simulation on the time interval $[-0.25, 20000]$.

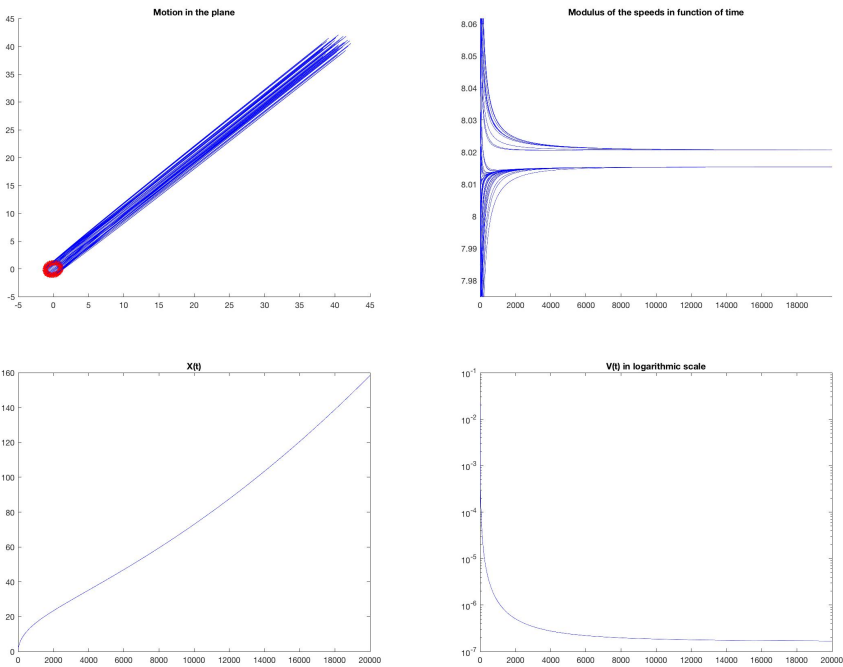


FIG. 4.6. $N = 50$, time-delay $\tau = 0.5$. Simulation on the time interval $[-0.5, 20000]$.

We observe in the numerical simulations that convergence to consensus is obtained for any value of the delay satisfying $\tau < \tau_0$, as expected.

For instance, for $\tau = 0.25$ the corresponding solution is drawn on Figure 4.5. At the bottom left: the position variance $X(t)$; at the bottom right: the speed variance $V(t)$ in a logarithmic scale; both drawn on the time interval $[0, 20000]$, because convergence to consensus is slow. The curves $t \mapsto x_i(t) \in \mathbb{R}^2$ (motion in the plane) and $t \mapsto \|v_i(t)\|$ (modulus of the speeds) are represented within a much smaller time frame to have better readability.

Finally, for $\tau = 0.5$ the corresponding solution is drawn on Figure 4.6. For this value of the time-delay, which is larger than the threshold τ_0 , convergence to consensus does not hold: one can indeed observe that the position variance $X(t)$ tends to $+\infty$, that $V(t)$ converges to a limit that is positive, and that the speeds do not coincide at the limit.

This simulation, as well as others that are not reported here, illustrates that the threshold τ_0 defined by (2.5) in Theorem 2.1 seems to be sharp, at least when N is sufficiently large.

5. Conclusion and further comments

We have analyzed the finite-dimensional general Cucker-Smale model with time-varying time-delays, and we have established rigorous convergence results to consensus under appropriate assumptions on the time-delay function $\tau(\cdot)$. Our results are valid for symmetric as well as for nonsymmetric interaction rates. The symmetric case has been analyzed thanks to a L^2 analysis, in the spirit of the original papers [22, 23], while we were able to deal with the loss of symmetry by carrying out a L^∞ analysis as in [40].

In both cases, we have established convergence to consensus provided the time-delay is below an explicit threshold given by a Lyapunov stability analysis. The bound on the time-delay depends on the coupling strength λ , on the communication weights and on the bound c on the time-derivative of $\tau(\cdot)$, but it does not depend on the number N of the agents. This important fact suggests that it might be possible to extend our analysis performed here on the finite-dimensional Cucker-Smale model to the infinite-dimensional case, as we comment next.

Towards a kinetic extension. The kinetic equation for the undelayed Cucker-Smale model has been derived in [30], using the BBGKY hierarchy, from the Cucker-Smale particle model as a mesoscopic description for flocking (see also [28, 45]). By considering the mean-field limit in the case $\tau = 0$, one obtains the kinetic equation

$$\partial_t \mu + \langle v, \text{grad}_x \mu \rangle + \text{div}_v ((\xi[\mu])\mu) = 0$$

where $\mu(t) = \mu(t, x, v)$ is the density of agents at time t at (x, v) and the interaction field is defined by

$$\xi[\mu](x, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|)(w - v) d\mu(y, w).$$

If we introduce a delay τ in the Cucker-Smale system as in (1.5), even when τ is constant, it is not clear how to deduce the corresponding kinetic model. In contrast, it is easy to pass to the mean-field limit when one considers a Cucker-Smale model with communication weights as in (1.13): indeed, the authors of [17], putting a delay on x_j but not on x_i in the communication weights in the equation for v_i , are able to pass to the mean-field limit and obtain the kinetic equation

$$\partial_t \mu(t) + \langle v, \text{grad}_x \mu(t) \rangle + \text{div}_v ((\xi[\mu(t - \tau)])\mu(t)) = 0.$$

Deriving an appropriate kinetic equation by considering the mean-field limit of (1.5), with communication weights depending on the states at time $t - \tau$ for all the agents; as it is, in our opinion, more adequate from a physical point of view; seems out of reach at this moment. We let it as an open question.

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