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RESIDUAL DIFFUSIVITY IN ELEPHANT RANDOM WALK MODELS WITH STOPS*

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Abstract. We study the enhanced diffusivity in the so-called elephant random walk model with stops (ERWS) by including symmetric random walk steps at small probability ϵ. At any $\epsilon > 0$, the large-time behavior transitions from sub-diffusive at $\epsilon = 0$ to diffusive in a wedge-shaped parameter regime where the diffusivity is strictly above that in the unperturbed ERWS model in the $\epsilon \downarrow 0$ limit. The perturbed ERWS model is shown to be solvable with the first two moments and their asymptotics calculated exactly in both one and two space dimensions. The model provides a discrete analytical setting of the residual diffusion phenomenon known for the passive scalar transport in chaotic flows (e.g. generated by time periodic cellular flows and statistically sub-diffusive) as molecular diffusivity tends to zero.

Keywords. Elephant random walk with stops; sub-diffusion; moment analysis; residual diffusivity.

AMS subject classifications. 60G50; 60H30; 58J37.

1. Introduction

Residual diffusion is a remarkable phenomenon arising in large scale fluid transport from chaotic flows [2,8,10]. It refers to the positive macroscopic effective diffusivity (D^E) as the microscopic molecular diffusivity (D_0) approaches zero, in the broader context of flow enhanced turbulent diffusion that has been studied for nearly a century [9,14]. An example of a chaotic smooth flow is the particle trajectories of the time periodic cellular flow ($X = (x,y) \in \mathbb{R}^2$):

$$\boldsymbol{v}(X,t) = (\cos(y),\cos(x)) + \theta \cos(t)\,(\sin(y),\sin(x)), \quad \theta \in (0,1]. \tag{1.1}$$

The first term of (1.1) is a steady cellular flow consisting of a periodic array of vortices, and the second term is a time periodic perturbation that introduces an increasing amount of disorder in the flow trajectories as θ becomes larger. At $\theta = 1$, the flow is fully mixing, and empirically sub-diffusive [16]. The flow (1.1) is a simple model of chaotic advection in the Rayleigh-Bénard experiment [3]. The motion of a diffusing particle in the flow (1.1) satisfies the stochastic differential equation (SDE):

$$dX_t = \boldsymbol{v}(X_t,t)\,dt + \sqrt{2\,D_0}\,dW_t,\ X(0) = (x_0,y_0) \in \mathbb{R}^2, \tag{1.2}$$

where $D_0 > 0$ is the molecular diffusivity, W_t is the standard 2-dimensional Wiener process. The mean square displacement in the unit direction e at large times is given by [1]:

$$\lim_{t\uparrow+\infty} E(|(X(t) - X(0))\cdot e|^2)/t = D^E, \tag{1.3}$$

where $D^E = D^E(D_0,e,\theta) > D_0$ is the effective diffusivity. Numerical simulations [2,8,10] based on the associated Fokker-Planck equations suggest that at $e = (1,0)$, $\theta = 1$, $D^E =$

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$O(1)$ as $D_0 \downarrow 0$, the *residual diffusion* occurs. In fact, $D^E = O(1)$ for $e = (0,1)$ and a range of values in $\theta \in (0,1)$ as well [8]. In contrast, at $\theta = 0$, $D^E = O(\sqrt{D_0})$ as $D_0 \downarrow 0$, see [5, 6, 11] for various proofs and generalized steady cellular flows.

Currently, the mathematical theory of residual diffusion for the SDE model (1.2) with the flow (1.1) is lacking. In this paper, we analyze the residual diffusion phenomenon in a random walk model which is solvable in the sense of moments and has certain statistical features of the SDE model (1.2). The baseline random walk model is the so-called elephant random walk model with stops (ERWS) [7] which is non-Markovian and exhibits sub-diffusive, diffusive and super-diffusive regimes. The ERWS plays the role of flow (1.1) in that there is a sub-diffusive statistical regime. For a review on various stochastic models of animal movement (including SDE and random walk models), memory effects and anomalous diffusion, see [13]. The sub-diffusive regime is absent in the earlier version of the ERW model without stops [12]. Stops in random walk models are often interpreted as occasional periods of rest during an animal's movement [15]. Recall that the chaotic system from (1.2) is sub-diffusive [16] at $D_0 = 0$ and transitions to diffusive with residual diffusion at $D_0 > 0$. To mimic this in the ERWS model, we add a small probability of symmetric random walk in the sub-diffusive regime and examine the large-time behavior of the mean square displacement. Interestingly, the sub-diffusive regime also transitions into diffusive regime and a wedge-shaped parameter region appears where the diffusivity is strictly above that of the baseline ERWS model in the zero probability limit of the symmetric random walk (analogue of the zero molecular diffusivity limit). In the context of animal dispersal in ecology, the emergence of residual diffusion indicates that the large-time statistical behavior of the movement can pick up positive normal diffusivity when the animal's rest pattern is slightly disturbed consistently in time. We also extend our analysis to a two-dimensional ERWS model (see [4] for a related solvable model). It is our hope that more diverse mathematical models of residual diffusivity can be developed and analyzed towards gaining understanding of the SDE residual diffusivity problem (1.1)-(1.3) in the future.

The rest of the paper is organized as follows. In Section 2, we present our perturbed model which is ERWS with a small probability (ϵ) of symmetric random walk, analyze the first two moments and derive the large-time asymptotics of the second moment. The residual diffusive behavior follows. In Section 3, we generalize our results to a two-dimensional perturbed ERWS model. Conclusions are in Section 4.

2. Perturbed ERWS and moment analysis

In this section, we show the perturbed ERWS model in one dimension and the analysis of the first two moments leading up to residual diffusivity.

2.1. Perturbed ERWS. Consider a random walker on a one-dimensional lattice with unit distance between adjacent lattice sites. Denote the position of the walker at time t by X_t. Time is discrete ($t = 0, 1, 2, \ldots$) and the walker starts at the origin, $X_0 = 0$. At each time step, $t \to t+1$,

$$X_{t+1} = X_t + \sigma_{t+1},$$

where $\sigma_{t+1} \in \{-1, 0, 1\}$ is a random number depending on $\{\sigma_t\} = (\sigma_1, \ldots, \sigma_t)$ as follows. Let $p, q, r, \epsilon \in (0,1)$ and $p+q+r = 1$. The process is started at time $t = 0$ by allowing the walker to move to the right with probability s and to the left with probability $1-s$, $s \in (0,1)$. For $t \geq 1$, a random previous time $k \in \{1, \ldots, t\}$ is chosen with uniform probability.

(i) If $\sigma_k = \pm 1$,

$$P(\sigma_{t+1} = \sigma_k) = p,\, P(\sigma_{t+1} = -\sigma_k) = q,$$
$$P(\sigma_{t+1} = 0) = r.$$

(ii) If $\sigma_k = 0$,

$$P(\sigma_{t+1} = 1) = P(\sigma_{t+1} = -1) = \epsilon/2,$$
$$P(\sigma_{t+1} = 0) = 1 - \epsilon.$$

When $\epsilon = 0$, the above model reduces to the ERWS model of [7].

2.2. Moment analysis. We calculate the first and second moments of X_t below.

2.2.1. First moment $\langle X_t \rangle$. At $t = 0$, it follows from the initial condition of the model for $\sigma = \pm 1$ that

$$P(\sigma_1 = \sigma) = \frac{1}{2}[1 + (2s-1)\sigma].$$

Let $\gamma = p - q$, for $t \geq 1$, it follows from the probabilistic structure of the model and $\sigma_k \in \{1, -1, 0\}$ that

$$\begin{aligned} P(\sigma_{t+1} = 1 | \{\sigma_t\}) &= \frac{1}{t}\sum_{k=1}^{t}\left[\sigma_k^2 (1+\sigma_k)\frac{p}{2} + \sigma_k^2 (1-\sigma_k)\frac{q}{2} + \left(1-\sigma_k^2\right)\frac{\epsilon}{2}\right] \\ &= \frac{1}{t}\sum_{k=1}^{t}\left[\sigma_k^2 \frac{1-r}{2} + \sigma_k \frac{\gamma}{2} + \left(1-\sigma_k^2\right)\frac{\epsilon}{2}\right] \\ &= \frac{1}{2t}\sum_{k=1}^{t}\left[\sigma_k^2 (1-\epsilon-r) + \sigma_k \gamma\right] + \frac{\epsilon}{2}, \end{aligned}$$

$$\begin{aligned} P(\sigma_{t+1} = -1 | \{\sigma_t\}) &= \frac{1}{t}\sum_{k=1}^{t}\left[\sigma_k^2 (1-\sigma_k)\frac{p}{2} + \sigma_k^2 (1+\sigma_k)\frac{q}{2} + \left(1-\sigma_k^2\right)\frac{\epsilon}{2}\right] \\ &= \frac{1}{t}\sum_{k=1}^{t}\left[\sigma_k^2 \frac{1-r}{2} - \sigma_k \frac{\gamma}{2} + \left(1-\sigma_k^2\right)\frac{\epsilon}{2}\right] \\ &= \frac{1}{2t}\sum_{k=1}^{t}\left[\sigma_k^2 (1-\epsilon-r) - \sigma_k \gamma\right] + \frac{\epsilon}{2}, \end{aligned}$$

$$\begin{aligned} P(\sigma_{t+1} = 0 | \{\sigma_t\}) &= \frac{1}{t}\sum_{k=1}^{t}\left[\sigma_k^2 r + \left(1-\sigma_k^2\right)(1-\epsilon)\right] \\ &= \frac{1}{t}\sum_{k=1}^{t}\left[-\sigma_k^2 (1-\epsilon-r)\right] + 1 - \epsilon \\ &= \frac{1}{2t}\sum_{k=1}^{t}\left[-2\sigma_k^2 (1-\epsilon-r)\right] + 1 - \epsilon. \end{aligned}$$

Therefore, for $\sigma=\pm1,0$,

$$P(\sigma_{t+1}=\sigma|\{\sigma_t\})=\frac{1}{2t}\sum_{k=1}^{t}\left[\sigma_k^2\left(3\sigma^2-2\right)(1-\epsilon-r)+\sigma\sigma_k\gamma\right]+\frac{\sigma^2}{2}\epsilon+\left(1-\sigma^2\right)(1-\epsilon).$$

The conditional mean value of σ_{t+1} for $t\geq 1$ is

$$\begin{aligned}\langle\sigma_{t+1}|\{\sigma_t\}\rangle &= \sum_{\sigma=\pm1,0}\sigma P(\sigma_{t+1}=\sigma|\{\sigma_t\})\\ &= \sum_{\sigma=\pm1}\sigma\left\{\frac{1}{2t}\sum_{k=1}^{t}\left[\sigma_k^2\left(3\sigma^2-2\right)(1-\epsilon-r)+\sigma\sigma_k\gamma\right]+\frac{\sigma^2}{2}\epsilon+\left(1-\sigma^2\right)(1-\epsilon)\right\}\\ &= \sum_{\sigma=\pm1}\sigma\left\{\frac{1}{2t}\sum_{k=1}^{t}\left[\sigma_k^2(1-\epsilon-r)+\sigma\sigma_k\gamma\right]+\frac{\epsilon}{2}\right\}\\ &= \sum_{\sigma=\pm1}\frac{1}{2t}\sum_{k=1}^{t}\sigma^2\sigma_k\gamma,\end{aligned}$$

hence,

$$\langle\sigma_{t+1}|\{\sigma_t\}\rangle=\frac{\gamma}{t}X_t. \tag{2.1}$$

It follows that

$$\langle\sigma_{t+1}\rangle=\frac{\gamma}{t}\langle X_t\rangle,$$

therefore

$$\langle X_{t+1}\rangle=\left(1+\frac{\gamma}{t}\right)\langle X_t\rangle.$$

By the initial condition $\langle X_1\rangle=2s-1$,

$$\langle X_t\rangle=(2s-1)\frac{\Gamma(t+\gamma)}{\Gamma(1+\gamma)\Gamma(t)}.$$

Since $\lim_{t\to\infty}\frac{\Gamma(t+\alpha)}{\Gamma(t)t^\alpha}=1$, $\forall\alpha$,

$$\langle X_t\rangle\sim\frac{2s-1}{\Gamma(1+\gamma)}t^\gamma,\quad t\to\infty.$$

From this point on, *we shall take $s=1/2$, and so $\langle X_t\rangle=0$, the mean square displacement agrees with the second moment.*

2.2.2. Second moment $\langle X_t^2\rangle$. The conditional mean value of σ_{t+1}^2 for $t\geq1$ is

$$\begin{aligned}\langle\sigma_{t+1}^2|\{\sigma_t\}\rangle &= \sum_{\sigma=\pm1,0}\sigma^2 P(\sigma_{t+1}=\sigma|\{\sigma_t\})\\ &= \sum_{\sigma=\pm1}\sigma^2\left\{\frac{1}{2t}\sum_{k=1}^{t}\left[\sigma_k^2\left(3\sigma^2-2\right)(1-\epsilon-r)+\sigma\sigma_k\gamma\right]+\frac{\sigma^2}{2}\epsilon+\left(1-\sigma^2\right)(1-\epsilon)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sigma=\pm 1} \left\{ \frac{1}{2t} \sum_{k=1}^{t} \left[\sigma_k^2 (1-\epsilon-r) + \sigma\sigma_k\gamma \right] + \frac{\epsilon}{2} \right\} \\
&= \sum_{\sigma=\pm 1} \left\{ \frac{1}{2t} \sum_{k=1}^{t} \sigma_k^2 (1-\epsilon-r) + \frac{\epsilon}{2} \right\} \\
&= \frac{1-\epsilon-r}{t} \sum_{k=1}^{t} \sigma_k^2 + \epsilon.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\langle \sigma_{t+1}^2 \big| \{\sigma_t\} \rangle &= \frac{1-\epsilon-r}{t} \sum_{k=1}^{t-1} \sigma_k^2 + \frac{1-\epsilon-r}{t} \sigma_t^2 + \epsilon \\
&= \frac{t-1}{t} \left(\frac{1-\epsilon-r}{t-1} \sum_{k=1}^{t-1} \sigma_k^2 + \epsilon \right) - \frac{t-1}{t}\epsilon + \frac{1-\epsilon-r}{t} \sigma_t^2 + \epsilon \\
&= \frac{t-1}{t} \langle \sigma_t^2 \big| \{\sigma_{t-1}\} \rangle + \frac{1-\epsilon-r}{t} \sigma_t^2 + \frac{\epsilon}{t},
\end{aligned}
$$

so

$$
\begin{aligned}
&\langle \sigma_1^2 \rangle = 1, \\
&\langle \sigma_{t+1}^2 \rangle = \left(1 - \frac{\epsilon+r}{t} \right) \langle \sigma_t^2 \rangle + \frac{\epsilon}{t}.
\end{aligned} \tag{2.2}
$$

Since

$$
\langle X_{t+1}^2 \big| \{\sigma_t\} \rangle = X_t^2 + 2X_t \langle \sigma_{t+1} | \{\sigma_t\} \rangle + \langle \sigma_{t+1}^2 \big| \{\sigma_t\} \rangle,
$$

by (2.1),

$$
\langle X_{t+1}^2 \rangle = \left(1 + \frac{2\gamma}{t} \right) \langle X_t^2 \rangle + \langle \sigma_{t+1}^2 \rangle. \tag{2.3}
$$

To motivate the solution we shall present, let us consider the ODE analogue of the difference Equations (2.2) and (2.3).

$$
\begin{cases}
x' + \dfrac{\epsilon+r}{t} x = \dfrac{\epsilon}{t}, \\
y' - \dfrac{2\gamma}{t} y = x.
\end{cases} \tag{2.4}
$$

The solution to (2.4) is

$$
\begin{cases}
x(t) = \dfrac{C}{t^{\epsilon+r}} + \dfrac{\epsilon}{\epsilon+r}, \\
y(t) = \dfrac{\epsilon}{(1-2\gamma)(\epsilon+r)} t + \dfrac{C}{1-\epsilon-r-2\gamma} t^{1-\epsilon-r} + Dt^{2\gamma},
\end{cases}
$$

if $\gamma \neq \frac{1}{2}$, and

$$
\begin{cases}
x(t) = \dfrac{C}{t^{\epsilon+r}} + \dfrac{\epsilon}{\epsilon+r}, \\
y(t) = \dfrac{\epsilon}{\epsilon+r} t \ln t - \dfrac{C}{\epsilon+r} t^{1-\epsilon-r} + Dt,
\end{cases}
$$

if $\gamma = \frac{1}{2}$, where C and D are constants.

PROPOSITION 2.1. *The solution to* (2.2) *is*

$$\langle \sigma_t^2 \rangle = C \frac{\Gamma(t-\epsilon-r)}{\Gamma(t)} + \frac{\epsilon}{\epsilon+r}, \tag{2.5}$$

where

$$C = \frac{r}{(\epsilon+r)\Gamma(1-\epsilon-r)}.$$

Proof. Clearly,

$$\begin{aligned} \frac{\epsilon}{\epsilon+r} &= \left(1 - \frac{\epsilon+r}{t}\right) \frac{\epsilon}{\epsilon+r} + \frac{\epsilon}{t}, \\ \frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t+1)} &= \left(1 - \frac{\epsilon+r}{t}\right) \frac{\Gamma(t-\epsilon-r)}{\Gamma(t)}, \end{aligned}$$

so a general solution to the recurrence equation in (2.2) is given by (2.5). The initial condition $\langle \sigma_1^2 \rangle = 1$ implies $C = \frac{r}{(\epsilon+r)\Gamma(1-\epsilon-r)}$. □

It follows from Proposition 2.1 and (2.3) that

$$\begin{aligned} \langle X_1^2 \rangle &= 1, \\ \langle X_{t+1}^2 \rangle &= \left(1 + \frac{2\gamma}{t}\right) \langle X_t^2 \rangle + C \frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t+1)} + \frac{\epsilon}{\epsilon+r}. \end{aligned} \tag{2.6}$$

THEOREM 2.1.
(1) If $\gamma \neq \frac{1}{2}$, *the solution to* (2.6) *is*

$$\langle X_t^2 \rangle = \frac{\epsilon}{(1-2\gamma)(\epsilon+r)} t + \frac{C}{1-\epsilon-r-2\gamma} \frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t)} + D \frac{\Gamma(t+2\gamma)}{\Gamma(t)}, \tag{2.7}$$

where

$$D = -\frac{1}{\Gamma(2\gamma)} \left[\frac{\epsilon}{(\epsilon+r)(1-2\gamma)} + \frac{r}{(\epsilon+r)(1-\epsilon-r-2\gamma)} \right].$$

(2) If $\gamma = \frac{1}{2}$, *the solution to* (2.6) *is*

$$\langle X_t^2 \rangle = \frac{\epsilon}{\epsilon+r} t \sum_{k=1}^{t} \frac{1}{k} - \frac{C}{\epsilon+r} \frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t)} + Dt, \tag{2.8}$$

where

$$D = \frac{\epsilon}{(\epsilon+r)^2} - 1.$$

Proof. Motivated by the ODE solution, we check the formula of the solution to (2.6).

If $\gamma \neq \frac{1}{2}$, by the identity $\Gamma(x+1) = x\Gamma(x)$,

$$\frac{\epsilon}{(1-2\gamma)(\epsilon+r)}(t+1) = \left(1+\frac{2\gamma}{t}\right)\frac{\epsilon}{(1-2\gamma)(\epsilon+r)}t + \frac{\epsilon}{\epsilon+r}, \tag{2.9}$$

$$\frac{C}{1-\epsilon-r-2\gamma}\frac{\Gamma(t+2-\epsilon-r)}{\Gamma(t+1)} = \left(1+\frac{2\gamma}{t}\right)\frac{C}{1-\epsilon-r-2\gamma}\frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t)} + C\frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t+1)}, \tag{2.10}$$

$$\frac{\Gamma(t+1+2\gamma)}{\Gamma(t+1)} = \left(1+\frac{2\gamma}{t}\right)\frac{\Gamma(t+2\gamma)}{\Gamma(t)}. \tag{2.11}$$

Hence a general solution to the recurrence equation in (2.6) is given by (2.7) for some constant D. Then $\langle X_1^2\rangle = 1$ and $C = \frac{r}{(\epsilon+r)\Gamma(1-\epsilon-r)}$ imply

$$\frac{\epsilon}{(1-2\gamma)(\epsilon+r)} + \frac{r\Gamma(2-\epsilon-r)}{(\epsilon+r)(1-\epsilon-r-2\gamma)\Gamma(1-\epsilon-r)} + D\Gamma(1+2\gamma) = 1,$$

so

$$D = -\frac{1}{\Gamma(2\gamma)}\left[\frac{\epsilon}{(\epsilon+r)(1-2\gamma)} + \frac{r}{(\epsilon+r)(1-\epsilon-r-2\gamma)}\right].$$

If $\gamma = \frac{1}{2}$, (2.10) and (2.11) still hold,

$$-\frac{C}{\epsilon+r}\frac{\Gamma(t+2-\epsilon-r)}{\Gamma(t+1)} = \left(1+\frac{1}{t}\right)\left(-\frac{C}{\epsilon+r}\frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t)}\right) + C\frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t+1)},$$

$$t+1 = \left(1+\frac{1}{t}\right)t.$$

For the recurrence relation

$$a_{t+1} = \left(1+\frac{1}{t}\right)a_t + \frac{\epsilon}{\epsilon+r},$$

suppose $a_t = tb_t$, then

$$b_{t+1} = b_t + \frac{\epsilon}{\epsilon+r}\frac{1}{t+1},$$

so for $t \geq 1$,

$$b_t = b_0 + \frac{\epsilon}{\epsilon+r}\sum_{k=1}^{t}\frac{1}{k}.$$

Set $b_0 = 0$, then

$$a_t = \frac{\epsilon}{\epsilon+r}t\sum_{k=1}^{t}\frac{1}{k}.$$

Hence a general solution to the recurrence equation in (2.6) in this case is (2.8). Similarly, the initial condition gives

$$D = \frac{\epsilon}{(\epsilon + r)^2} - 1.$$

□

The corollary below follows from (2.7) and (2.8).

Corollary 2.1.
(1) If $\gamma \neq \frac{1}{2}$,

$$\langle X_t^2 \rangle \sim \frac{\epsilon}{(1-2\gamma)(\epsilon+r)} t + \frac{C}{1-\epsilon-r-2\gamma} t^{1-\epsilon-r} + D t^{2\gamma}, \quad t \to \infty.$$

(2) If $\gamma = \frac{1}{2}$,

$$\langle X_t^2 \rangle \sim \frac{\epsilon}{\epsilon+r} t \ln t - \frac{C}{\epsilon+r} t^{1-\epsilon-r} + D t, \quad t \to \infty.$$

2.3. Residual diffusivity. The occurrence of residual diffusivity relies on the choice of γ as a function of ϵ. To this end, we show three cases: 1) case 1 only recovers the unperturbed diffusivity, 2) case 2 reveals the residual diffusivity exceeding the unperturbed diffusivity in the limit of $\epsilon \downarrow 0$, 3) case 3 results in residual superdiffusivity. The cases 2 and 3 are illustrated in Figure 2.1. As $\epsilon \to 0$, the parameter region of the residual diffusion shrinks towards $\gamma = \frac{1}{2}$ while the enhanced diffusivity remains strictly above the unperturbed diffusivity.

2.3.1. Regular diffusivity: $\gamma = \frac{1-\epsilon}{2}$. Let $\gamma = \frac{1-\epsilon}{2}$, then $D = 0$ and

$$\langle X_t^2 \rangle = \frac{1}{(\epsilon+r)} t - \frac{1}{(\epsilon+r)\Gamma(1-\epsilon-r)} \frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t)},$$

so

$$\langle X_t^2 \rangle \sim \frac{1}{(\epsilon+r)} t - \frac{1}{(\epsilon+r)\Gamma(1-\epsilon-r)} t^{1-\epsilon-r}, \quad t \to \infty,$$

and diffusivity equals $\frac{1}{\epsilon+r}$.

For fixed $r \in \left(0, \frac{1}{2}\right)$, let $\epsilon \in (0,1)$, then

$$p = \frac{3-\epsilon-2r}{4}, \quad q = \frac{1+\epsilon-2r}{4}.$$

Recall the second-moment formula of [7] (equation (18)),

$$\begin{aligned} \langle X_t^2 \rangle &= \frac{1}{(2\gamma+r-1)\Gamma(t)} \left(\frac{\Gamma(t+2\gamma)}{\Gamma(2\gamma)} - \frac{\Gamma(1+t-r)}{\Gamma(1-r)} \right) \\ &\sim \frac{1}{(2\gamma+r-1)} \left(\frac{t^{2\gamma}}{\Gamma(2\gamma)} - \frac{t^{1-r}}{\Gamma(1-r)} \right), \end{aligned} \tag{2.12}$$

which is diffusive at $\gamma = 1/2$ with diffusivity $1/r$.

We see that for $\gamma = (1-\epsilon)/2$, $\epsilon \in (0,1)$ and the above (p,q), the diffusivity of the perturbed ERW problem $1/(\epsilon+r)$ approaches $1/r$, the diffusivity of the unperturbed model as $\epsilon \downarrow 0$. Hence no residual diffusivity exists.

2.3.2. Residual diffusivity: $\gamma = \frac{1-\epsilon r}{2}$. Let $\gamma = \frac{1-\epsilon r}{2}$, then

$$\langle X_t^2 \rangle = \frac{1}{r(\epsilon+r)} t - \frac{r\Gamma(t+1-\epsilon-r)}{(\epsilon+r)(\epsilon+r-\epsilon r)\Gamma(1-\epsilon-r)\Gamma(t)} - \frac{1}{\Gamma(1-\epsilon r)} \left[\frac{1}{r(\epsilon+r)} - \frac{r}{(\epsilon+r)(\epsilon+r-\epsilon r)} \right] \frac{\Gamma(t+1-\epsilon r)}{\Gamma(t)},$$

and

$$\langle X_t^2 \rangle \sim \frac{1}{r(\epsilon+r)} t - \frac{r}{(\epsilon+r)(\epsilon+r-\epsilon r)\Gamma(1-\epsilon-r)} t^{1-\epsilon-r} - \frac{1}{\Gamma(1-\epsilon r)} \left[\frac{1}{r(\epsilon+r)} - \frac{r}{(\epsilon+r)(\epsilon+r-\epsilon r)} \right] t^{1-\epsilon r}, \quad t \to \infty.$$

Hence

$$\lim_{t\to\infty} \frac{\langle X_t^2 \rangle}{t} = \frac{1}{r(\epsilon+r)}.$$

The diffusivity $\frac{1}{r(\epsilon+r)}$ can be much larger than $\frac{1}{r}$ in the unperturbed model. In particular, given any $\delta > 0$, let $r_0 = \min\left\{\frac{1}{3}, \frac{1}{\delta}\right\}$, then for $r \in (0, r_0)$, $\epsilon \in \left(0, \frac{1}{6}\right)$,

$$\frac{1}{r(\epsilon+r)} - \frac{1}{r} = \frac{1}{r}\left(\frac{1}{\epsilon+r} - 1\right) > \frac{1}{r_0}\left(\frac{1}{\frac{1}{6}+r_0} - 1\right) \geq \delta\left(\frac{1}{\frac{1}{6}+\frac{1}{3}} - 1\right) = \delta.$$

The **new diffusive region with residual diffusivity** is the **wedge to the left of $\gamma = 1/2$ covered by the dashed lines in Figure 2.1**.

2.3.3. Residual super-diffusivity: $\gamma = \frac{1+\epsilon r}{2}$. If $\gamma = \frac{1+\epsilon r}{2}$, then

$$\langle X_t^2 \rangle = -\frac{1}{r(\epsilon+r)} t - \frac{r\Gamma(t+1-\epsilon-r)}{(\epsilon+r)(\epsilon+r+\epsilon r)\Gamma(1-\epsilon-r)\Gamma(t)} + \frac{1}{\Gamma(1+\epsilon r)} \left[\frac{1}{r(\epsilon+r)} + \frac{r}{(\epsilon+r)(\epsilon+r+\epsilon r)} \right] \frac{\Gamma(t+1+\epsilon r)}{\Gamma(t)},$$

and

$$\langle X_t^2 \rangle \sim -\frac{1}{r(\epsilon+r)} t - \frac{r}{(\epsilon+r)(\epsilon+r+\epsilon r)\Gamma(1-\epsilon-r)} t^{1-\epsilon-r} + \frac{1}{\Gamma(1+\epsilon r)} \left[\frac{1}{r(\epsilon+r)} + \frac{r}{(\epsilon+r)(\epsilon+r+\epsilon r)} \right] t^{1+\epsilon r}, \quad t \to \infty.$$

Thus at any $\epsilon > 0$, super-diffusion arises and

$$\lim_{t\to\infty} \frac{\langle X_t^2 \rangle}{t^{1+\epsilon r}} = \frac{1}{\Gamma(1+\epsilon r)} \left[\frac{1}{r(\epsilon+r)} + \frac{r}{(\epsilon+r)(\epsilon+r+\epsilon r)} \right].$$

As $\epsilon \downarrow 0$, the super-diffusivity tends to $r^{-2} + r^{-1} > r^{-1}$ the limiting super-diffusivity of the unperturbed model as seen from (2.12). The residual super-diffusive region is the wedge covered by lines to the right of $\gamma > 1/2$ in Figure 2.1.

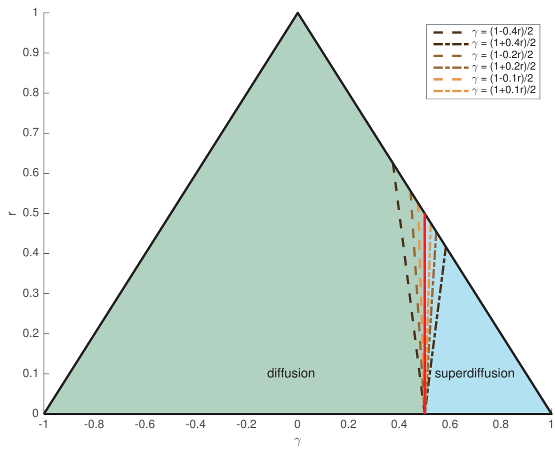

FIG. 2.1. *Regions of residual diffusivity (wedge left of $\gamma = 1/2$) and residual super-diffusivity (wedge right of $\gamma = 1/2$) covered by the dashed lines at $\epsilon = 0.4, 0.2, 0.1$.*

3. 2D perturbed ERWS model

In this section, we generalize our model to the two-dimensional square lattice. Let $\mathbf{i}$,$\mathbf{j}$ be the standard basis in 2D. Denote the position of the walker at time t by $\boldsymbol{X}_t$,

$$\boldsymbol{X}_{t+1} = \boldsymbol{X}_t + \boldsymbol{\sigma}_{t+1},$$

where $\boldsymbol{\sigma}_{t+1} \in \{\mathbf{i}, \mathbf{j}, -\mathbf{i}, -\mathbf{j}\}$. Let $s_i \in (0,1)$, $i = 1, \ldots, 4$ and the process is started by allowing the walker to move to the right, upward, to the left, downward with probability $s_1, \ldots, s_4$. Let $p, q, q', r, \epsilon \in (0,1)$ and $p + q + q' + r = 1$,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For $t \geq 1$, a random $k \in \{1, \ldots, t\}$ is chosen with uniform probability.

(i) If $|\boldsymbol{\sigma}_k| = 1$,

$$\begin{aligned} &P(\boldsymbol{\sigma}_{t+1} = \boldsymbol{\sigma}_k) = p, \\ &P(\boldsymbol{\sigma}_{t+1} = -\boldsymbol{\sigma}_k) = q, \\ &P(\boldsymbol{\sigma}_{t+1} = A\boldsymbol{\sigma}_k) = p', \\ &P\left(\boldsymbol{\sigma}_{t+1} = A^{-1}\boldsymbol{\sigma}_k\right) = q', \\ &P(\boldsymbol{\sigma}_{t+1} = \mathbf{0}) = r. \end{aligned}$$

(ii) If $|\boldsymbol{\sigma}_k| = 0$,

$$\begin{aligned} &P(\boldsymbol{\sigma}_{t+1} = \mathbf{i}) = P(\boldsymbol{\sigma}_{t+1} = \mathbf{j}) = P(\boldsymbol{\sigma}_{t+1} = -\mathbf{i}) = P(\boldsymbol{\sigma}_{t+1} = -\mathbf{j}) = \epsilon/4, \\ &P(\boldsymbol{\sigma}_{t+1} = \mathbf{0}) = 1 - \epsilon. \end{aligned}$$

Let $\gamma = p - q$, $\gamma' = p' - q'$, then for $t \geq 1$,

$$P(\boldsymbol{\sigma}_{t+1} = \boldsymbol{\sigma} | \{\boldsymbol{\sigma}_t\}) = \frac{1}{t} \sum_{k=1}^{t} \Big[\boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma} (\boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma} + 1) \frac{p}{2} + \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma} (\boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma} - 1) \frac{q}{2}$$

$$+\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\left(\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}+1\right)\frac{p'}{2}+\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\left(\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}-1\right)\frac{q'}{2}$$
$$+\left(1-|\boldsymbol{\sigma}_k|^2\right)\frac{\epsilon}{4}\Big]$$
$$=\frac{1}{2t}\sum_{k=1}^{t}\Big[\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\gamma+\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\gamma'+\left(\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\right)^2(p+q)$$
$$+\left(\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\right)^2(p'+q')-\frac{1}{2}|\boldsymbol{\sigma}_k|^2\epsilon\Big]+\frac{\epsilon}{4},$$

for $|\boldsymbol{\sigma}|=1$, and

$$P(\boldsymbol{\sigma}_{t+1}=\mathbf{0}|\{\boldsymbol{\sigma}_t\})=\frac{1}{t}\sum_{k=1}^{t}\left[|\boldsymbol{\sigma}_k|^2 r+\left(1-|\boldsymbol{\sigma}_k|^2\right)(1-\epsilon)\right]$$
$$=\frac{1}{t}\sum_{k=1}^{t}|\boldsymbol{\sigma}_k|^2(r+\epsilon-1)+1-\epsilon.$$

The conditional mean of $\boldsymbol{\sigma}_{t+1}$ for $t\geq 1$ is

$$\langle\boldsymbol{\sigma}_{t+1}|\{\boldsymbol{\sigma}_t\}\rangle=\sum_{|\boldsymbol{\sigma}|=1}P(\boldsymbol{\sigma}_{t+1}=\boldsymbol{\sigma}|\{\boldsymbol{\sigma}_t\})\boldsymbol{\sigma}$$
$$=\frac{1}{2t}\sum_{k=1}^{t}\sum_{|\boldsymbol{\sigma}|=1}\Big[\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\gamma+\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\gamma'+\left(\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\right)^2(p+q)$$
$$+\left(\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\right)^2(p'+q')-\frac{1}{2}|\boldsymbol{\sigma}_k|^2\epsilon\Big]\boldsymbol{\sigma}$$
$$=\frac{1}{2t}\sum_{k=1}^{t}\sum_{|\boldsymbol{\sigma}|=1}\left(\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\gamma+\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\gamma'\right)\boldsymbol{\sigma}$$
$$=\frac{1}{2t}\sum_{k=1}^{t}\sum_{|\boldsymbol{\sigma}|=1}\left(\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\gamma+A\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\gamma'\right)\boldsymbol{\sigma}$$
$$=\frac{1}{2t}\sum_{k=1}^{t}2\left(\gamma\boldsymbol{\sigma}_k+\gamma' A\boldsymbol{\sigma}_k\right)$$
$$=\frac{1}{t}\left(\gamma+\gamma' A\right)\boldsymbol{X}_t.$$

Here the symmetry of $\pm\mathbf{i}$, $\pm\mathbf{j}$ is used. Thus,

$$\langle\boldsymbol{X}_{t+1}\rangle=\left(1+\frac{\gamma}{t}+\frac{\gamma'}{t}A\right)\langle\boldsymbol{X}_t\rangle.$$

The conditional mean of $|\boldsymbol{\sigma}_{t+1}|^2$ for $t\geq 1$ is

$$\langle|\boldsymbol{\sigma}_{t+1}|^2\Big|\{\boldsymbol{\sigma}_t\}\rangle=\sum_{|\boldsymbol{\sigma}|=1}P(\boldsymbol{\sigma}_{t+1}=\boldsymbol{\sigma}|\{\boldsymbol{\sigma}_t\})|\boldsymbol{\sigma}|^2$$
$$=\frac{1}{2t}\sum_{k=1}^{t}\sum_{|\boldsymbol{\sigma}|=1}\Big[\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\gamma+\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma}\gamma'+\left(\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}\right)^2(p+q)$$

$$
\begin{aligned}
&+(\boldsymbol{\sigma}_k\cdot A\boldsymbol{\sigma})^2(p'+q')-\frac{1}{2}|\boldsymbol{\sigma}_k|^2\epsilon\Big]+\epsilon\\
=&\frac{1}{2t}\sum_{k=1}^{t}2(p+q+p'+q'-\epsilon)|\boldsymbol{\sigma}_k|^2+\epsilon\\
=&\frac{1-\epsilon-r}{t}\sum_{k=1}^{t}|\boldsymbol{\sigma}_k|^2+\epsilon.
\end{aligned}
$$

Similar to the 1D case,

$$
\langle\, |\boldsymbol{\sigma}_{t+1}|^2 \Big| \{\boldsymbol{\sigma}_t\}\rangle = \frac{t-1}{t}\langle\, |\boldsymbol{\sigma}_t|^2 \Big| \{\boldsymbol{\sigma}_{t-1}\}\rangle + \frac{1-\epsilon-r}{t}|\boldsymbol{\sigma}_t|^2+\frac{\epsilon}{t},
$$

so

$$
\begin{aligned}
\langle|\boldsymbol{\sigma}_1|^2\rangle &= 1,\\
\langle|\boldsymbol{\sigma}_{t+1}|^2\rangle &= \left(1-\frac{\epsilon+r}{t}\right)\langle|\boldsymbol{\sigma}_t|^2\rangle+\frac{\epsilon}{t}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\langle\, |\boldsymbol{X}_{t+1}|^2 \Big| \{\sigma_t\}\rangle &= |\boldsymbol{X}_t|^2 + 2\boldsymbol{X}_t\cdot\langle\boldsymbol{\sigma}_{t+1}|\{\boldsymbol{\sigma}_t\}\rangle + \langle\boldsymbol{\sigma}_{t+1}^2\big|\{\boldsymbol{\sigma}_t\}\rangle\\
&= |\boldsymbol{X}_t|^2 + 2\boldsymbol{X}_t\cdot\frac{1}{t}(\gamma+\gamma' A)\boldsymbol{X}_t + \langle\boldsymbol{\sigma}_{t+1}^2\big|\{\boldsymbol{\sigma}_t\}\rangle\\
&= \left(1+\frac{2\gamma}{t}\right)|\boldsymbol{X}_t|^2 + \langle\boldsymbol{\sigma}_{t+1}^2\big|\{\boldsymbol{\sigma}_t\}\rangle,
\end{aligned}
$$

hence

$$
\langle|\boldsymbol{X}_{t+1}|^2\rangle = \left(1+\frac{2\gamma}{t}\right)\langle|\boldsymbol{X}_t|^2\rangle + \langle\boldsymbol{\sigma}_{t+1}^2\rangle.
$$

By Proposition 2.1 and Theorem 2.1,

$$
\begin{aligned}
\langle|\boldsymbol{\sigma}_t|^2\rangle &= C\frac{\Gamma(t-\epsilon-r)}{\Gamma(t)}+\frac{\epsilon}{\epsilon+r},\\
\langle|\boldsymbol{X}_t|^2\rangle &= \frac{\epsilon}{(1-2\gamma)(\epsilon+r)}t+\frac{C}{1-\epsilon-r-2\gamma}\frac{\Gamma(t+1-\epsilon-r)}{\Gamma(t)}+D\frac{\Gamma(t+2\gamma)}{\Gamma(t)},
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \frac{r}{(\epsilon+r)\Gamma(1-\epsilon-r)},\\
D &= -\frac{1}{\Gamma(2\gamma)}\left[\frac{\epsilon}{(\epsilon+r)(1-2\gamma)}+\frac{r}{(\epsilon+r)(1-\epsilon-r-2\gamma)}\right].
\end{aligned}
$$

Due to the above moment formulas, the residual diffusivity results in 1D extend verbatim to the 2D model.

4. Conclusions

We found that residual diffusivity occurs in ERWS models in one and two dimensions with an inclusion of small probability of symmetric random walk steps. A wedge-like sub-diffusive parameter region in the (r,γ)-plane transitions into a diffusive region with residual diffusivity in the sense that the enhanced diffusivity strictly exceeds the unperturbed diffusivity in the limit of vanishing symmetric random walks. In future work, we plan to identify other discrete stochastic models for residual diffusivity so that the region where this occurs remains distinct from the unperturbed diffusivity region in the limit of vanishing diffusive perturbations.

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