

THE GLOBAL ATTRACTOR FOR THE 3-D VISCOUS PRIMITIVE EQUATIONS OF LARGE-SCALE MOIST ATMOSPHERE*

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Abstract. Absorbing ball in $H^1(\mathbb{U})$ is obtained for the strong solution to the three dimensional viscous moist primitive equations under the natural assumption $Q_1, Q_2 \in L^2(\mathbb{U})$ which is weaker than the assumption $Q_1, Q_2 \in H^1(\mathbb{U})$ in the previous works. In view of the structure of the manifold and the special geometry involved with vertical velocity, the continuity of the strong solution in $H^1(\mathbb{U})$ is established with respect to time and initial data. To obtain the existence of the global attractor for the moist primitive equations, the common method is to obtain the absorbing ball in $H^2(\mathbb{U})$ for the strong solution to the equations. But it is difficult due to the complex structure of the moist primitive equations. To overcome the difficulty, we try to use Aubin-Lions lemma and the continuous property of the strong solutions to the moist primitive equations to prove the existence of the global attractor which improves the result obtained before, namely, the existence of weak attractor.

Keywords. Moist primitive equations; uniform estimates; global attractor.

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1. Introduction

The paper is concerned with the 3-dimensional viscous primitive equations in the pressure coordinate system (see e.g. [19, 34, 36, 37] and the references therein).

$$\partial_t v + \nabla_v v + w \partial_\xi v + \frac{f}{R_0} v^\perp + \text{grad} \Phi + L_1 v = 0, \quad (1.1)$$

$$\partial_\xi \Phi + \frac{bP}{p} (1 + aq) T = 0, \quad (1.2)$$

$$\text{div} v + \partial_\xi w = 0, \quad (1.3)$$

$$\partial_t T + \nabla_v T + w \partial_\xi T - \frac{bP}{p} (1 + aq) w + L_2 T = Q_1, \quad (1.4)$$

$$\partial_t q + \nabla_v q + w \partial_\xi q + L_3 q = Q_2. \quad (1.5)$$

The unknowns for the primitive equations are the fluid velocity field $(v, w) = (v_\theta, v_\varphi, w) \in \mathbb{R}^3$ with $v = (v_\theta, v_\varphi)$ and $v^\perp = (-v_\varphi, v_\theta)$ being horizontal, the temperature T , q the mixing ratio of water vapor in the air and the geopotential Φ . $f = 2\cos\theta$ is the given Coriolis parameter, Q_1 corresponds to the sum of the heating of the sun and the heat added or removed by condensation or evaporation, Q_2 represents the amount of water added or removed by condensation or evaporation, a and b are positive constants with $a \approx 0.618$, R_0 is the Rossby number, P stands for an approximate value of pressure at the surface of the earth, p_0 is the pressure of the upper atmosphere with $p_0 > 0$ and the variable ξ satisfies $p = (P - p_0)\xi + p_0$ where $0 < p_0 \leq p \leq P$. The viscosity, the heat and the water vapor diffusion operators L_1 , L_2 and L_3 are given respectively as the following:

$$L_i = -\nu_i \Delta - \mu_i \partial_{zz}, i = 1, 2, 3.$$

Here the positive constants ν_1, μ_1 are the horizontal and vertical viscosity coefficients; the positive constants ν_2, μ_2 are the horizontal and vertical heat diffusivity coefficients;

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while the positive constants ν_3, μ_3 are the horizontal and vertical water vapor diffusivity coefficients. The definitions of $\nabla_v v, \Delta v, \Delta T, \Delta q, \nabla_v q, \nabla_v T, \text{div} v, \text{grad} \Phi$ will be given in Section 2.

The space domain of equations: (1.1)-(1.5) is

$$\mathcal{U} = S^2 \times (0, 1),$$

where S^2 is two-dimensional unit sphere. The boundary value conditions are given by

$$\xi = 1 (p = P) : \partial_\xi v = 0, \quad w = 0, \quad \partial_\xi T = \alpha_s (T_s - T), \quad \partial_\xi q = \beta_s (q_s - q), \quad (1.6)$$

$$\xi = 0 (p = p_0) : \partial_\xi v = 0, \quad w = 0, \quad \partial_\xi T = 0, \quad \partial_\xi q = 0, \quad (1.7)$$

where α_s, β_s are positive constants, T_s is the given temperature on the surface of the earth, q_s is the given mixing ratio of water vapor on the surface of the earth. To simplify the notations, we set $T_s = 0$ and $q_s = 0$ without losing any generality. For the case $T_s \neq 0$ and $q_s \neq 0$, we can homogenize the boundary value conditions for T, q ; see [19] for detailed discussion on this issue. Moreover, using (1.2), (1.3) and the boundary conditions (1.6)-(1.7), we have

$$w(t; \theta, \varphi, \xi) = \int_\xi^1 \text{div} v(t; \theta, \varphi, \xi') d\xi', \quad (1.8)$$

$$\int_0^1 \text{div} v d\xi = 0, \quad (1.9)$$

$$\Phi(t; \theta, \varphi, \xi) = \Phi_s(t; \theta, \varphi) + \int_\xi^1 \frac{bP}{p} (1 + aq) T d\xi', \quad (1.10)$$

where $\Phi_s(t; \theta, \varphi)$ is a certain unknown function at the isobaric surface $\xi = 1$. In this article, we assume that the constants $v_i = \mu_i = 1, i = 1, 2, 3$. For the general case, the results will still be valid. Then using (1.8)-(1.10), we obtain the following equivalent formulation for system (1.1)-(1.7) with initial conditions

$$\begin{aligned} \partial_t v + \nabla_v v + \left(\int_\xi^1 \text{div} v(t; \theta, \varphi, \xi') d\xi' \right) \partial_\xi v + \frac{f}{R_0} v^\perp + \text{grad} \Phi_s \\ + \int_\xi^1 \frac{bP}{p} \text{grad} [(1 + aq) T] d\xi' - \Delta v - \partial_{\xi\xi} v = 0, \end{aligned} \quad (1.11)$$

$$\begin{aligned} \partial_t T + \nabla_v T + \left(\int_\xi^1 \text{div} v(t; \theta, \varphi, \xi') d\xi' \right) \partial_\xi T \\ - \frac{bP}{p} (1 + aq) \left(\int_\xi^1 \text{div} v(t; \theta, \varphi, \xi') d\xi' \right) - \Delta T - \partial_{\xi\xi} T = Q_1, \end{aligned} \quad (1.12)$$

$$\partial_t q + \nabla_v q + \left(\int_\xi^1 \text{div} v(t; \theta, \varphi, \xi') d\xi' \right) \partial_\xi q - \Delta q - \partial_{\xi\xi} q = Q_2, \quad (1.13)$$

$$\int_0^1 \operatorname{div} v d\xi = 0, \tag{1.14}$$

$$\xi = 1 : \partial_\xi v = 0, \quad w = 0, \quad \partial_\xi T = -\alpha_s T, \quad \partial_\xi q = -\beta_s q, \tag{1.15}$$

$$\xi = 0 : \partial_\xi v = 0, \quad w = 0, \quad \partial_\xi T = 0, \quad \partial_\xi q = 0, \tag{1.16}$$

$$v(0; \theta, \phi, \xi) = v_0(\theta, \phi, \xi), T(0; \theta, \phi, \xi) = T_0(\theta, \phi, \xi), q(0; \theta, \phi, \xi) = q_0(\theta, \phi, \xi). \tag{1.17}$$

In order to understand the mechanism of long-term weather prediction, one can take advantage of the historical records and numerical computations to detect the future weather. Alternatively, one should also study the long-time behavior mathematically using the equations and models governing the motion. The primitive equations represent the classical model for the study of climate and weather prediction, describing the motion of the atmosphere when the hydrostatic assumption is enforced [18, 23, 24, 40, 43]. But the resulting flow or the atmosphere is rich in its organization and complexity (see [18, 23, 24]), the full governing equations are too complicated to be treatable both from the theoretical and the computational side. To overcome this difficulty, some simple numerical models were introduced. The 2-D and 3-D quasi-geostrophic models have been the subject of analytical mathematical study (see e.g., [3, 5, 10, 11, 15, 16, 39, 48–50] and references therein). To the best of our knowledge, the mathematical framework of primitive equations was formulated in [36–38], where the definitions of weak and strong solutions were given and the existence of weak solution was proven, leaving the uniqueness of weak solution as an open problem for now. Local well-posedness of strong solutions was obtained in [21, 47]. If the domain was thin, the global well-posedness of 3D primitive equations was shown in [26]. Taking advantage of the fact that the pressure is essentially two-dimensional in the primitive equations, global well-posedness of the full three-dimensional case was established in [12] and independently in [29, 30]. In the subsequent work [31] a different proof was developed which allows one to treat non-rectangular domains. Recently, the results were improved in [7–9, 13] by considering the system with partial dissipation, i.e., with only partial viscosities or only partial diffusion. For the inviscid primitive equations, finite-time blowup was established in [6]. To study the long-term behavior of primitive equations, the existence of global attractor was established in [27] and dimensions were proven to be finite in [28]. When moisture is included, an equation for the conservation of water must be added, which is the case in e.g. [19, 20, 36, 42]. In [51], global well-posedness of quasi-strong and strong solutions was obtained for the primitive equations of atmosphere in presence of vapour saturation.

The understanding of asymptotic behavior of a dynamical system is one of the most important topics of modern mathematical physics. One way to solve the problem for a dissipative deterministic dynamical system is to consider its global attractor (see its definition in Section 2). Thus, in order to capture the dynamical features of moist primitive equations, Guo and Huang in [20] proved the existence of universal attractor which is weakly closed in V (see the notations in Section 2) and attracts any orbit in V -weak topology, when time goes to ∞ . Then, one natural question arising from it is the existence of global attractor which is compact in V and attracts any orbit in V -strong topology, as time goes to ∞ .

As it is stated in [20] that lacking information for the time derivative of the vertical velocity in the moist primitive equations leads to the failure of establishing the existence

of the global attractor. How to overcome the difficulty? Inspired by [27], we try to use Aubin-Lions lemma combined with continuity of the strong solution in V with respect to time and initial data to prove the solution operator is compact in V . Then the compact property of the solution operator implies the existence of the global attractor, i.e., the universal attractor is indeed compact in V and attracts all bounded sets in V with respect to V -strong topology, when $t \rightarrow \infty$.

In this article, we first try to obtain time-uniform *a priori* estimates in various function spaces under the natural assumption $Q_1, Q_2 \in L^2(\mathcal{U})$, reducing the stronger assumption $Q_1, Q_2 \in H^1(\mathcal{U})$ in [20]. Then, by making delicate and careful estimates of L^4 norm for the strong solution, estimates about L^3 norm, which are required in [20], are omitted. To obtain uniform boundedness with respect to time t in V , estimates about $\partial_\xi T, \partial_\xi q$ are carefully considered, which is more complex than oceanic primitive equations. We recall that the continuity of the strong solution with respect to initial data was shown to be true in weak solution space H in previous work due to which no boundedness is available for the derivatives of the vertical velocity. This is not sufficient for our purpose. To overcome the difficulty, the structure of the manifold and the special geometry involved with the vertical velocity are used to obtain the continuity of the strong solution with respect to initial data in V , improving the results obtained before. Finally, in order to prove that the absorbing ball is compact in V , the common method is to show that the ball is uniformly bounded with respect to time t in $H^2(\mathcal{U})$. But it is difficult to achieve because of the high nonlinearity of the moist primitive equations. We try to use an Aubin-Lions compactness lemma combined with a continuity argument to show that the solution operator is compact in V for every time $t > 0$, which further implies the existence of the global attractor for the dynamical system generated by the primitive equations of large-scale moist atmosphere.

The remaining part of the paper is organized as follows. In Section 2, we present the notations and recall some important facts which are crucial to later analysis. Absorbing ball is obtained in Section 3. Section 4 and Section 5 are for continuity of the strong solution with respect to time t and initial data in V respectively. Finally, in Section 6, using Aubin-Lions lemma and the continuity properties of the strong solution, we prove the existence of the global attractor. As usual, the positive constants c may change from one line to the next unless we give a special declaration.

2. Preliminaries

In this section we collect some preliminary results that will be used in the rest of this paper, and we start with the following notations which will be used throughout this work. Denote

$$\bar{v} = \int_0^1 v d\xi, \quad \tilde{v} = v - \bar{v}.$$

Then we have

$$\nabla \cdot \bar{v} = 0, \quad \bar{\tilde{v}} = 0. \tag{2.1}$$

Now we give the definitions of some differential operators. Firstly, the natural generalization of the directional derivative on the Euclidean space to the covariant derivative on S^2 is given as follows. Let $T, q, \in C^\infty(\mathcal{U}), \Phi_s \in C^\infty(S^2)$ and

$$v = v_\theta e_\theta + v_\varphi e_\varphi, \quad u = u_\theta e_\theta + u_\varphi e_\varphi \in C^\infty(T\mathcal{U}|TS^2),$$

where $C^\infty(T\mathcal{U}|TS^2)$ represents the first two components of smooth vector fields on \mathcal{U} . We define the covariant derivative of u, T and q with respect to v as follows

$$\nabla_v u = (v_\theta \partial_\theta u_\theta + \frac{v_\varphi}{\sin\theta} \partial_\varphi u_\theta - v_\varphi u_\varphi \cot\theta) e_\theta + (v_\theta \partial_\theta u_\varphi + \frac{v_\varphi}{\sin\theta} \partial_\varphi u_\varphi + v_\varphi u_\theta \cot\theta) e_\varphi,$$

$$\nabla_v T = v_\theta \partial_\theta T + \frac{v_\varphi}{\sin\theta} \partial_\varphi T,$$

$$\nabla_v q = v_\theta \partial_\theta q + \frac{v_\varphi}{\sin\theta} \partial_\varphi q.$$

We give the definition of the horizontal gradient $\nabla = \text{grad}$ for T and Φ_s on S^2 by

$$\nabla T = \text{grad}T = (\partial_\theta T) e_\theta + \frac{1}{\sin\theta} (\partial_\varphi T) e_\varphi,$$

$$\nabla \Phi_s = \text{grad}\Phi_s = (\partial_\theta \Phi_s) e_\theta + \frac{1}{\sin\theta} (\partial_\varphi \Phi_s) e_\varphi.$$

We define the divergence of v by

$$\text{div}v = \text{div}(v_\theta e_\theta + v_\varphi e_\varphi) = \frac{1}{\sin\theta} (\partial_\theta (v_\theta \sin\theta) + \partial_\varphi v_\varphi).$$

The horizontal Laplace-Beltrami operator of scalar functions T and q are

$$\Delta T = \text{div}(\text{grad}T) = \frac{1}{\sin\theta} [\partial_\theta (\sin\theta \partial_\theta T) + \frac{1}{\sin\theta} \partial_\varphi \varphi T],$$

$$\Delta q = \text{div}(\text{grad}q) = \frac{1}{\sin\theta} [\partial_\theta (\sin\theta \partial_\theta q) + \frac{1}{\sin\theta} \partial_\varphi \varphi q].$$

We define the horizontal Laplace-Beltrami operator Δ for vector functions on S^2 as

$$\Delta v = (\Delta v_\theta - \frac{2\cos\theta}{\sin^2\theta} \partial_\varphi v_\varphi - \frac{v_\theta}{\sin^2\theta}) e_\theta + (\Delta v_\varphi + \frac{2\cos\theta}{\sin^2\theta} \partial_\varphi v_\theta - \frac{v_\varphi}{\sin^2\theta}) e_\varphi.$$

Consequently, by integration by parts, we have

$$\int_0^1 w \partial_\xi v d\xi = \int_0^1 v \text{div}v d\xi = \int_0^1 \tilde{v} \text{div}\tilde{v} d\xi, \tag{2.2}$$

$$\int_0^1 \nabla_v v d\xi = \int_0^1 \nabla_{\tilde{v}} \tilde{v} d\xi + \nabla_{\bar{v}} \bar{v}. \tag{2.3}$$

Taking the average of Equations (1.11) in the z direction, over the interval $(0,1)$ and using (2.1)-(2.3) and the boundary conditions (1.15)-(1.16), we arrive at

$$\begin{aligned} \partial_t \bar{v} + \nabla_{\bar{v}} \bar{v} + \overline{\tilde{v} \text{div}\tilde{v} + \nabla_{\tilde{v}} \tilde{v}} + \frac{f}{R_0} \bar{v}^\perp + \text{grad}\Phi_s + \int_0^1 \int_\xi^1 \frac{bP}{p} \text{grad}[(1+aq)T] d\xi' d\xi \\ - \Delta \bar{v} = 0 \quad \text{in } S^2. \end{aligned} \tag{2.4}$$

By subtracting (2.4) from (1.11), we obtain the following equation

$$\begin{aligned} & \partial_t \tilde{v} + \nabla_{\tilde{v}} \tilde{v} + \left(\int_{\xi}^1 \operatorname{div} \tilde{v} d\xi' \right) \partial_{\xi} \tilde{v} + \nabla_{\tilde{v}} \tilde{v} + \nabla_{\tilde{v}} \tilde{v} - \overline{(\tilde{v} \operatorname{div} \tilde{v} + \nabla_{\tilde{v}} \tilde{v})} + \frac{f}{R_0} \tilde{v}^{\perp} \\ & + \int_{\xi}^1 \frac{bP}{p} \operatorname{grad}[(1+aq)T] d\xi' - \int_0^1 \int_{\xi}^1 \frac{bP}{p} \operatorname{grad}[(1+aq)T] d\xi' d\xi \\ & - \Delta \tilde{v} - \partial_{\xi\xi} \tilde{v} = 0 \quad \text{in } \mathcal{U}, \end{aligned} \tag{2.5}$$

with the following boundary value conditions

$$\partial_{\xi} \tilde{v} = 0 \quad \text{on } \xi = 1 \text{ and } \xi = 0. \tag{2.6}$$

Let e_{θ}, e_{φ} and e_{ξ} be the unit vectors in θ, φ and ξ directions of the space domain \mathcal{U} respectively,

$$e_{\theta} = \partial_{\theta}, \quad e_{\varphi} = \frac{1}{\sin \theta} \partial_{\varphi}, \quad e_{\xi} = \partial_{\xi}.$$

The inner product and norm on $T_{(\theta, \varphi, \xi)} \mathcal{U}$ (the tangent space of \mathcal{U} at the point (θ, φ, ξ)) are defined by

$$(u, v) = u \cdot v = \sum_{i=1}^3 u_i v_i, \quad |u| = (u, u)^{\frac{1}{2}},$$

where $u = u_1 e_{\theta} + u_2 e_{\varphi} + u_3 e_{\xi} \in T_{(\theta, \varphi, \xi)} \mathcal{U}$ and $v = v_1 e_{\theta} + v_2 e_{\varphi} + v_3 e_{\xi} \in T_{(\theta, \varphi, \xi)} \mathcal{U}$. For $1 \leq p \leq \infty$, let $L^p(\mathcal{U})$ and $L^p(S^2)$ be the usual Lebesgue spaces with the norm $|\cdot|_p$ and $|\cdot|_{L^p(S^2)}$ respectively. If there is no confusion, we will write $|\cdot|_p$ instead of $|\cdot|_{L^p(S^2)}$. $L^2(T\mathcal{U}|TS^2)$ represents the first two components of L^2 vector fields on \mathcal{U} with the norm $|v|_2 = (\int_{\mathcal{U}} (|v_{\theta}|^2 + |v_{\varphi}|^2) d\mathcal{U})^{\frac{1}{2}}$, where $v = (v_{\theta}, v_{\varphi}) : \mathcal{U} \rightarrow TS^2$. Denote by $C^{\infty}(S^2)$ the functions of all smooth functions from S^2 to \mathbb{R} . Similarly, we can define $C^{\infty}(\mathcal{U})$. $H^m(\mathcal{U})$ is the Sobolev space of functions which are in L^2 , together with all their covariant derivatives with respect to $e_{\theta}, e_{\varphi}, e_{\xi}$ of order $\leq m$, with the norm

$$\|h\|_m = \left[\int_{\mathcal{U}} \left(\sum_{1 \leq k \leq m} \sum_{i_j=1,2,3; j=1, \dots, k} |\nabla_{i_1} \cdots \nabla_{i_k} h|^2 + |h|^2 \right) \right]^{\frac{1}{2}},$$

where $\nabla_1 = \nabla_{e_{\theta}}, \nabla_2 = \nabla_{e_{\varphi}}$ and $\nabla_3 = \partial_{\xi}$ which are defined above. Denote $H^m(T\mathcal{U}|TS^2) = \{v; v = (v_{\theta}, v_{\varphi}) : \mathcal{U} \rightarrow TS^2, \|v\|_m < \infty\}$, where the norm is similar to that of $H^m(\mathcal{U})$ (i.e., let $h = (v_{\theta}, v_{\varphi}) = v_{\theta} e_{\theta} + v_{\varphi} e_{\varphi}$.) We will conduct our work in the following functional spaces. Let

$$\mathcal{V}_1 := \{v; v \in C^{\infty}(T\mathcal{U}|TS^2), \partial_{\xi} v|_{\xi=0} = 0, \partial_{\xi} v|_{\xi=1} = 0, \int_0^1 \operatorname{div} v d\xi = 0\},$$

$$\mathcal{V}_2 := \{T; T \in C^{\infty}(\mathcal{U}), \partial_{\xi} T|_{\xi=0} = 0, \partial_{\xi} T|_{\xi=1} = -\alpha_s T\},$$

$$\mathcal{V}_3 := \{q; q \in C^{\infty}(\mathcal{U}), \partial_{\xi} q|_{\xi=0} = 0, \partial_{\xi} q|_{\xi=1} = -\beta_s q\}.$$

We denote by V_1, V_2 and V_3 the closure spaces of $\mathcal{V}_1, \mathcal{V}_2$ and \mathcal{V}_3 in $H^1(\mathcal{U})$ under H^1 -topology, respectively. In addition, we denote by H_1, H_2 and H_3 the closure of $\mathcal{V}_1, \mathcal{V}_2$

and \mathcal{V}_3 in $L^2(\mathcal{U})$ under L^2 - topology. Let $H := H_1 \times H_2 \times H_3$ and $V = V_1 \times V_2 \times V_3$ with V' being the dual space of V . By definition, the inner products and norms on V_1, V_2 and V_3 are given by

$$\langle v, v_1 \rangle_{V_1} = \int_{\mathcal{U}} (\nabla_{e_\theta} v \cdot \nabla_{e_\theta} v_1 + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} v_1 + \partial_\xi v \partial_\xi v_1 + v \cdot v_1) d\mathcal{U},$$

$$\|v\|_1 = \langle v, v \rangle_{V_1}^{\frac{1}{2}}, \quad \forall v, v_1 \in V_1,$$

$$\langle T, T_1 \rangle_{V_2} = \int_{\mathcal{U}} (\text{grad}T \cdot \text{grad}T_1 + \partial_\xi T \partial_\xi T_1) d\mathcal{U} + \alpha_s \int_{S^2} TT_1 dS^2,$$

$$\|T\|_1 = \langle T, T \rangle_{V_2}^{\frac{1}{2}}, \quad \forall T, T_1 \in V_2,$$

$$\langle q, q_1 \rangle_{V_3} = \int_{\mathcal{U}} (\text{grad}q \cdot \text{grad}q_1 + \partial_\xi q \partial_\xi q_1) d\mathcal{U} + \beta_s \int_{S^2} qq_1 dS^2,$$

$$\|q\|_1 = \langle q, q \rangle_{V_3}^{\frac{1}{2}}, \quad \forall q, q_1 \in V_3.$$

Let $V'_i (i=1,2,3)$ be the dual space of V_i with $\langle \cdot, \cdot \rangle$ being the inner products between V'_i and V_i . Without confusion, we also denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathcal{U})$ and $L^2(S^2)$. Define the linear operator $A_i : V_i \mapsto V'_i, i=1,2,3$:

$$\langle A_1 u_1, u_2 \rangle = \langle u_1, u_2 \rangle_{V_1}, \quad \forall u_1, u_2 \in V_1;$$

$$\langle A_2 \theta_1, \theta_2 \rangle = \langle \theta_1, \theta_2 \rangle_{V_2}, \quad \forall \theta_1, \theta_2 \in V_2;$$

$$\langle A_3 q_1, q_2 \rangle = \langle q_1, q_2 \rangle_{V_3}, \quad \forall q_1, q_2 \in V_3.$$

Denote $D(A_i) = \{\eta \in V_i, A_i \eta \in H_i\}$. Since A_i is positive self-adjoint with compact resolvent, according to the classical spectral theory we can define the power A_i^s for any $s \in \mathbb{R}$. Then we have $D(A_i^{\frac{1}{2}}) = V_i$ and $D(A_i^{-\frac{1}{2}}) = V'_i$. Moreover,

$$D(A_i) \subset V_i \subset H_i \subset V'_i \subset D(A_i)',$$

where $D(A_i)'$ is the dual space of $D(A_i)$ and the embeddings above are all compact. In the following, we state some lemmas including integrations by parts and the uniform Gronwall lemma, which are frequently used in our paper. For the proof of Lemma 2.1-Lemma 2.3, we can see [19]. The proof of the uniform Gronwall lemma was given in [17, 46].

LEMMA 2.1. *Let $u = (u_\theta, u_\varphi), v = (v_\theta, v_\varphi) \in C^\infty(T\mathcal{U}|TS^2)$ and $p \in C^\infty(S^2)$. Then*

$$\int_{S^2} p \operatorname{div} u dS^2 = - \int_{S^2} \nabla p \cdot u dS^2,$$

$$\int_{\mathcal{U}} \nabla p \cdot v d\mathcal{U} = 0 \quad \text{for any } v \in V_1,$$

and

$$\int_{\mathcal{U}} (-\Delta u) \cdot v d\mathcal{U} = \int_{\mathcal{U}} (\nabla_{e_\theta} u \cdot \nabla_{e_\theta} v + \nabla_{e_\varphi} u \cdot \nabla_{e_\varphi} v + u \cdot v) d\mathcal{U}.$$

LEMMA 2.2. For any $h \in C^\infty(S^2), v \in C^\infty(T\mathcal{U}|TS^2)$, we have

$$\int_{S^2} \nabla_v h dS^2 + \int_{S^2} h \operatorname{div} v dS^2 = \int_{S^2} \operatorname{div}(hv) dS^2 = 0.$$

LEMMA 2.3. Let $u, v \in V_1, T \in V_2, q \in V_3$. Then we have

$$\int_{\mathcal{U}} [\nabla_u v + (\int_{\xi}^1 \operatorname{div} u d\xi') \partial_{\xi} v] v d\mathcal{U} = 0,$$

$$\int_{\mathcal{U}} [\nabla_u g + (\int_{\xi}^1 \operatorname{div} u d\xi') \partial_{\xi} g] g d\mathcal{U} = 0, \quad \text{for } g = T \text{ or } g = q,$$

$$\int_{\mathcal{U}} \left(\int_{\xi}^1 \frac{bP}{p} \operatorname{grad}[(1+aq)T] d\xi' \cdot u - \frac{bP}{p} (1+aq)T (\int_{\xi}^1 \operatorname{div} u d\xi') \right) = 0.$$

LEMMA 2.4. Let f, g and h be three non-negative locally integrable functions on (t_0, ∞) such that

$$\frac{df}{dt} \leq gf + h, \quad \forall t \geq t_0,$$

and

$$\int_t^{t+r} f(s) ds \leq a_1, \quad \int_t^{t+r} g(s) ds \leq a_2, \quad \int_t^{t+r} h(s) ds \leq a_3, \quad \forall t \geq t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$f(t+r) \leq \left(\frac{a_1}{r} + a_3\right) e^{a_2}, \quad \forall t \geq t_0.$$

Before considering the long-time behavior of the dynamics, we recall the definitions of strong solution to (1.11)-(1.17).

DEFINITION 2.1. Suppose $Q_1, Q_2 \in L^2(\mathcal{U}), (v_0, T_0, q_0) \in V$ and $\tau > 0$. (v, T, q) is called a strong solution of (1.11)-(1.17) on the time interval $[0, \tau]$ if it satisfies (1.11)-(1.13) in a weak sense, and also

$$v \in C([0, \tau]; V_1) \cap L^2([0, \tau]; H^2(\mathcal{U})),$$

$$T \in C([0, \tau]; V_2) \cap L^2([0, \tau]; H^2(\mathcal{U})),$$

$$q \in C([0, \tau]; V_3) \cap L^2([0, \tau]; H^2(\mathcal{U})),$$

$$\partial_t v, \partial_t T, \partial_t q \in L^1([0, \tau]; L^2(\mathcal{U})).$$

Now we state the global well-posedness theorem for the strong solution as follows. For the proof of the theorem, one can refer to [20].

PROPOSITION 2.1. *Let $Q_1, Q_2 \in H^1(\mathcal{U}), U_0 = (v_0, T_0, q_0) \in V$. Then for any $\tau > 0$ given, the global strong solution U of the system (1.11)-(1.17) is unique on the interval $[0, \tau]$. Moreover, the strong solution U is continuous with respect to initial data in H .*

REMARK 2.1. Notice that there are some gaps between the Proposition 2.1 and Definition 2.1. In fact, in Section 3 of this work we will find that if the condition $Q_1, Q_2 \in H^1(\mathcal{U})$ is relaxed as $Q_1, Q_2 \in L^2(\mathcal{U})$, the result of Proposition 2.1 still holds.

REMARK 2.2. In the Proposition 2.1, the result that the strong solution is continuous with respect to initial data in H is not sufficient for our purpose. We will improve the result in Section 5 by establishing that the strong solution is continuous with respect to initial data in V .

To see the difference between global attractor and universal attractor, we introduce the two definitions in the following. For more details, we refer to [20, 22, 46] and other references. Let (X, d) be a separable metric space and $S(t): X \rightarrow X, 0 \leq t < \infty$, be a semigroup satisfying:

- (i) $S(t)S(s)x = S(t+s)x$, for all $t, s \in \mathbb{R}_+$ and $x \in X$;
- (ii) $S(0) = I$ (Identity in X);
- (iii) $S(t)$ is continuous in X for all $t \geq 0$.

Typically, $S(t)$ is associated with an autonomous differential equation; $S(t)x$ is the state at time t of the solution whose initial data is x .

DEFINITION 2.2. *A subset \mathcal{A} in X is said to be a global attractor if it satisfies the following properties:*

- (i) \mathcal{A} is compact in X ;
- (ii) for every $t \geq 0, S(t)\mathcal{A} = \mathcal{A}$;
- (iii) for every bounded set B in X , the set $S(t)B$ converges to \mathcal{A} in X , when $t \rightarrow \infty$, i.e.,

$$\lim_{t \rightarrow \infty} d(S(t)B, \mathcal{A}) = 0.$$

Here, and in the following, for A and B subsets of X , $d(A, B)$ is the semi-distance given by

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

DEFINITION 2.3. *A subset \mathcal{A} in X is said to be a universal attractor or weak attractor if it satisfies the following properties:*

- (i) \mathcal{A} is bounded and weakly closed in X ;
- (ii) for every $t \geq 0, S(t)\mathcal{A} = \mathcal{A}$;
- (iii) for every bounded set B in X , the set $S(t)B$ converges to \mathcal{A} with respect to X -weak topology, when $t \rightarrow \infty$, i.e.,

$$\lim_{t \rightarrow \infty} d_X^w(S(t)B, \mathcal{A}) = 0$$

where the distance d_X^w is induced by the X -weak topology.

3. Uniform estimates and absorbing balls

In this section, we will obtain some useful uniform *a priori* estimates about the solution to (1.11)-(1.17) under the natural assumption $Q_1, Q_2 \in L^2(\mathcal{U})$ and give a proof of the existence of the absorbing ball in V for the solution to the moist primitive equation. The estimates of this section are rigorous without justification using Galerkin approximation due to Proposition 2.1.

3.1. L^2 estimates of v, T, q . Taking inner product of (1.13) with q in $L^2(\mathcal{U})$, by Lemma 2.3 we have

$$\frac{1}{2} \frac{d|q|_2^2}{dt} + |\nabla q|_2^2 + |\partial_\xi q|_2^2 + \beta_s |q|_{\xi=1}|_2^2 = \int_{\mathcal{U}} q Q_2 d\mathcal{U}. \quad (3.1)$$

Since $q(\theta, \varphi, \xi) = -\int_\xi^1 \partial_\xi q d\xi' + q|_{\xi=1}$, by Hölder's inequality and the Cauchy-Schwarz inequality we have

$$|q|_2^2 \leq 2|\partial_\xi q|_2^2 + 2|q_{\xi=1}|_2^2, \quad (3.2)$$

which together with (3.1) implies that there exists a positive constant c such that

$$\frac{d}{dt} |q|_2^2 + c|q|_2^2 \leq |Q_2|_2^2.$$

Therefore, we have

$$|q(t)|_2^2 \leq e^{-ct} |q_0|_2^2 + c|Q_2|_2^2. \quad (3.3)$$

By (3.1) and (3.3), for arbitrary $t_0 \geq 0$ we have

$$\begin{aligned} & \int_{t_0}^{t_0+1} (|\nabla q(t)|_2^2 + |\partial_\xi q(t)|_2^2 + \beta_s |q|_{\xi=1}(t)|_2^2) dt \\ & \leq |q(t_0)|_2^2 + |Q|_2^2 \leq e^{-ct_0} |q_0|_2^2 + c|Q_2|_2^2. \end{aligned} \quad (3.4)$$

Taking an analogous argument as (3.1), we have

$$\frac{1}{2} \frac{d|T|_2^2}{dt} + |\nabla T|_2^2 + |\partial_\xi T|_2^2 + \alpha_s |T|_{\xi=1}|_2^2 = \int_{\mathcal{U}} \frac{bP}{p} (1+aq) T w d\mathcal{U} + \int_{\mathcal{U}} Q_1 T d\mathcal{U}. \quad (3.5)$$

Multiplying v with respect to (1.11) and integrating on \mathcal{U} ; by Lemma 2.1, Lemma 2.3 and $(\frac{f}{R_0} \times v) \cdot v = 0$ we have

$$\frac{1}{2} \frac{d|v|_2^2}{dt} + |\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2 + |v|_2^2 + |\partial_\xi v|_2^2 = - \int_{\mathcal{U}} \left(\int_\xi^1 \frac{bP}{p} \text{grad}[(1+aq)T] d\xi' \right) \cdot v. \quad (3.6)$$

Combining (3.5)-(3.6) and Lemma 2.3 yields

$$\begin{aligned} & \frac{1}{2} \frac{d(|v|_2^2 + |T|_2^2)}{dt} + |\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2 + |\partial_\xi v|_2^2 + |v|_2^2 \\ & + |\nabla T|_2^2 + |\partial_\xi T|_2^2 + \alpha_s |T|_{\xi=1}|_2^2 = \int_{\mathcal{U}} Q_1 T d\mathcal{U}. \end{aligned} \quad (3.7)$$

Similar to the deduction of (3.2), we have

$$|T|_2^2 \leq 2|\partial_\xi T|_2^2 + 2|T|_{\xi=1}|_2^2,$$

which, together with (3.6) and Hölder's inequality, implies

$$\begin{aligned} \frac{d(|v|_2^2 + |T|_2^2)}{dt} &+ |\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2 + |\partial_\xi v|_2^2 + |v|_2^2 \\ &+ |\nabla T|_2^2 + |\partial_\xi T|_2^2 + \alpha_s |T|_{\xi=1}|_2^2 \leq c|Q_1|_2^2. \end{aligned} \tag{3.8}$$

Therefore, we conclude that

$$|v(t)|_2^2 + |T(t)|_2^2 \leq e^{-ct}(|v_0|_2^2 + |T_0|_2^2) + c|Q_1|_2^2. \tag{3.9}$$

Combining (3.7) and (3.8) we arrive at

$$\begin{aligned} &\int_{t_0}^{t_0+1} (|\nabla_{e_\theta} v(t)|_2^2 + |\nabla_{e_\varphi} v(t)|_2^2 + |\partial_\xi v(t)|_2^2 + |v(t)|_2^2) dt \\ &+ \int_{t_0}^{t_0+1} (|\nabla T(t)|_2^2 + |\partial_\xi T(t)|_2^2 + \alpha_s |T|_{\xi=1}(t)|_2^2) dt \leq e^{-ct}(|v_0|_2^2 + |T_0|_2^2) + c|Q_1|_2^2. \end{aligned} \tag{3.10}$$

3.2. L^4 estimates of q . Multiplying q^3 on both sides of (1.13) and integrating on \mathcal{U} yields

$$\begin{aligned} &\frac{1}{4} \frac{d|q|_4^4}{dt} + 3\|\nabla q|q|_2^2 + 3\|\partial_\xi q|q|_2^2 + \beta_s |q|_{\xi=1}|_4^4 \\ &= \int_{\mathcal{U}} Q_2 q^3 d\mathcal{U} - \int_{\mathcal{U}} [\nabla_v q + (\int_{\xi}^1 \operatorname{div} v d\xi')] \partial_\xi q|q|^3. \end{aligned} \tag{3.11}$$

Since by Lemma 2.2 or Lemma 2.3 we have

$$\int_{\mathcal{U}} [\nabla_v q + (\int_{\xi}^1 \operatorname{div} v d\xi')] \partial_\xi q|q|^3 = 0. \tag{3.12}$$

Then (3.11) and (3.12) imply

$$\begin{aligned} &\frac{1}{4} \frac{d|q|_4^4}{dt} + 3\|\nabla q|q|_2^2 + 3\|\partial_\xi q|q|_2^2 + \beta_s |q|_{\xi=1}|_4^4 \\ &= \int_{\mathcal{U}} Q_2 q^3 d\mathcal{U} \leq |Q_2|_2 |q^2|_3^{\frac{3}{2}} \leq c|Q_2|_2 |q^2|_2^{\frac{3}{4}} \|q^2\|_1^{\frac{3}{4}} \\ &= c|Q_2|_2 |q|_4^{\frac{3}{2}} (|q|_4^{\frac{3}{2}} + \|\nabla q|q|_2^{\frac{3}{2}} + |q\partial_\xi q|_2^{\frac{3}{2}}) \\ &\leq \varepsilon (\|\nabla q|q|_2^2 + |q\partial_\xi q|_2^2) + c|Q_2|_2^{\frac{8}{5}} |q|_4^{\frac{12}{5}} + c|Q_2|_2 |q|_4^3. \end{aligned} \tag{3.13}$$

Since $q^4(\theta, \varphi, \xi) = -\int_{\xi}^1 \partial_{\xi'} q^4 d\xi' + q^4|_{\xi=1}$, by Hölder's inequality we get

$$|q|_4^4 \leq c|q\partial_\xi q|_2^2 + \frac{1}{2}|q|_4^4 + |q|_{\xi=1}|_4^4,$$

which implies that there exists a positive constant c_1 such that

$$\frac{d}{dt}|q|_4^4 + c_1|q|_4^4 \leq c|Q_2|_2^{\frac{8}{5}} |q|_4^{\frac{12}{5}} + c|Q_2|_2 |q|_4^3.$$

That is

$$\frac{d}{dt}|q|_4^2 + c|q|_4^2 \leq c|Q_2|_2^2.$$

Therefore, we have

$$|q(t)|_4^2 \leq e^{-ct} |q_0|_4^2 + c|Q_2|_2^2. \quad (3.14)$$

Combining (3.13) and (3.14), we arrive at

$$\int_{t_0}^{t_0+1} (|\nabla q(t)|_2^2 + |\partial_\xi q(t)|_2^2 + \beta_s |q|_{\xi=1}^4) dt \leq ce^{-c_1 t_0} |q_0|_4^2 + c|Q_2|_2^2. \quad (3.15)$$

3.3. L^4 estimate for T . Taking inner product of (1.12) with T^3 in $L^2(\mathcal{U})$ yields,

$$\begin{aligned} & \frac{1}{4} \frac{|T|_4^4}{dt} + 3|\nabla T|_2^2 + 3|\partial_\xi T|_2^2 + \alpha_s |T|_{\xi=1}^4 \\ &= \int_{\mathcal{U}} \frac{bP}{p} \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) |T|^2 T d\mathcal{U} + \int_{\mathcal{U}} \frac{abP}{p} \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) q |T|^2 T d\mathcal{U} \\ & \quad - \int_{\mathcal{U}} \left[\nabla_v T + \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) \partial_\xi T \right] |T|^2 T d\mathcal{U} + \int_{\mathcal{U}} Q_1 |T|^2 T d\mathcal{U}. \end{aligned} \quad (3.16)$$

Using the estimate in [20], we know that

$$\begin{aligned} & \left| \int_{\mathcal{U}} \frac{bP}{p} \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) |T|^2 T \right| \\ & \leq c(|\nabla_{e_\theta} v|_2 + |\nabla_{e_\varphi} v|_2) |T|_4^2 (|\nabla T|_2 + |T|_2). \end{aligned} \quad (3.17)$$

Taking a similar argument as in [20], we get

$$\begin{aligned} & \left| \int_{\mathcal{U}} \frac{abP}{p} q \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) T^3 \right| \\ & \leq c \int_0^1 \left[\left(\int_{S^2} q^4 dS^2 \right)^{\frac{1}{4}} \left(\int_{S^2} T^{12} dS^2 \right)^{\frac{1}{4}} \right] d\xi \sup_{\xi \in [0,1]} \left| \frac{bP}{p} \int_{\xi}^1 \operatorname{div} v d\xi' \right|_{L^2(S^2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left| \int_{\mathcal{U}} \frac{abP}{p} q \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) T^3 \right| \\ & \leq c \int_0^1 |q|_{L^4(S^2)} |T^2|_{L^6(S^2)}^{\frac{3}{2}} d\xi |\operatorname{div} v|_2 \\ & \leq c \int_0^1 |q|_{L^4(S^2)} |T^2|_{L^2(S^2)}^{\frac{1}{2}} |\nabla T^2|_{L^2(S^2)} d\xi |\operatorname{div} v|_2 \\ & \leq \varepsilon |T|\nabla T|_2^2 + c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2) |T|_4^2 |q|_4^2. \end{aligned} \quad (3.18)$$

To estimate the last term on the right of (3.16), we use the interpolation inequality and Hölder's inequality to have

$$\begin{aligned} \int_{\mathcal{U}} Q_1 T^3 d\mathcal{U} & \leq |Q_1|_2 |T|_6^3 = |Q_1|_2 |T^2|_3^{\frac{3}{2}} \\ & \leq c |Q_1|_2 |T^2|_2^{\frac{3}{4}} (|T|\nabla T|_2^{\frac{3}{4}} + |T|\partial_\xi T|_2^{\frac{3}{4}} + |T^2|_2^{\frac{3}{4}}) \\ & \leq \varepsilon (|T|\nabla T|_2^2 + |T|\partial_\xi T|_2^2) + c |Q_1|_2 |T|_4^3 + c |Q_1|_2^{\frac{8}{5}} |T|_4^{\frac{12}{5}}. \end{aligned} \quad (3.19)$$

By virtue of (3.16)-(3.19), we have

$$\begin{aligned} \frac{|T|_4^2}{2} \frac{d|T|_4^2}{dt} &\leq c(|\nabla_{e_\theta} v|_2 + |\nabla_{e_\varphi} v|_2) |T|_4^2 (|\nabla T|_2 + |T|_2) \\ &\quad + c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2) |T|_4^2 |q|_4^2 \\ &\quad + c|Q_1|_2 |T|_4^3 + c|Q_1|_2^{\frac{8}{5}} |T|_4^{\frac{12}{5}}, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d|T|_4^2}{dt} &\leq c(|\nabla_{e_\theta} v|_2 + |\nabla_{e_\varphi} v|_2) (|\nabla T|_2 + |T|_2) \\ &\quad + c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2) |q|_4^2 \\ &\quad + c|Q_1|_2 |T|_4 + c|Q_1|_2^{\frac{8}{5}} |T|_4^{\frac{6}{5}}. \end{aligned} \tag{3.20}$$

Since $|T|_4 \leq c\|T\|_1$, by (3.9), (3.10), (3.14) and the uniform Gronwall lemma, we obtain the desired uniform boundedness of $|T|_4$ and it gives the absorbing ball of T in $L^4(\mathcal{U})$. That is to say there exists a constant c independent of t such that

$$|T|_4(t) \leq c \tag{3.21}$$

for all $t \geq 0$. Moreover, by (3.16) we have the following uniform bound on the time average

$$\begin{aligned} &\int_{t_0}^{t_0+1} (|\nabla T(t)|_2 |T(t)|_2^2 + |\partial_\xi T(t)|_2 |T(t)|_2^2 + |T(t)|_{\xi=1}^4) dt \\ &\leq c|T(t_0)|_4^4 + c \int_{t_0}^{t_0+1} (|\nabla_{e_\theta} v(t)|_2 + |\nabla_{e_\varphi} v(t)|_2) |T(t)|_4^2 (|\nabla T(t)|_2 + |T(t)|_2) dt \\ &\quad + c \int_{t_0}^{t_0+1} (|\nabla_{e_\theta} v(t)|_2^2 + |\nabla_{e_\varphi} v(t)|_2^2) |T(t)|_4^2 |q(t)|_4^2 dt \\ &\quad + c \int_{t_0}^{t_0+1} |Q_1|_2 |T(t)|_4^3 dt + c \int_{t_0}^{t_0+1} |Q_1|_2^{\frac{8}{5}} |T(t)|_4^{\frac{12}{5}} dt \leq c, \end{aligned} \tag{3.22}$$

where the constant c is independent of t_0 .

REMARK 3.1. In this part, we try to use the uniform Gronwall lemma to get the uniform boundedness of $|T|_4^2$ instead of $|T|_4^4$. Therefore, from (3.20)-(3.22) we can see that the estimate of $|T|_3^3$ is not necessary, which simplifies the proof of the existence of the absorbing ball in space $L^4(\mathcal{U})$ in [20]. This technique will be used again in the following to get the uniform estimates for v in $L^4(\mathcal{U})$ with respect to time.

3.4. L^4 estimate for v . Taking inner product of Equation (2.5) with $|\tilde{v}|^2 \tilde{v}$ in $L^2(\mathcal{U})$, we get

$$\begin{aligned} &\frac{1}{4} \frac{d|\tilde{v}|_4^4}{dt} + \int_{\mathcal{U}} (|\nabla_{e_\theta} \tilde{v}|^2 |\tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2 |\tilde{v}|^2 + \frac{1}{2} |\nabla_{e_\theta} \tilde{v}|^2 + \frac{1}{2} |\nabla_{e_\varphi} \tilde{v}|^2 + |\tilde{v}|^4) d\mathcal{U} \\ &\quad + \int_{\mathcal{U}} (|\tilde{v}_\xi|^2 |\tilde{v}|^2 + \frac{1}{2} |\partial_\xi |\tilde{v}|^2|^2) d\mathcal{U} = - \int_{\mathcal{U}} [\nabla_{\tilde{v}} \tilde{v} + \left(\int_{\xi}^1 \operatorname{div} \tilde{v} d\xi' \right) \partial_\xi \tilde{v}] \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} \\ &\quad - \int_{\mathcal{U}} (\nabla_{\tilde{v}} \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} - \int_{\mathcal{U}} (\nabla_{\tilde{v}} \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} \end{aligned}$$

$$\begin{aligned}
& - \int_{\mathcal{U}} \left(\int_{\xi}^1 \frac{bP}{p} \operatorname{grad}[(1+aq)T] d\xi' \cdot |\tilde{v}|^2 \tilde{v} \right) d\mathcal{U} \\
& + \int_{\mathcal{U}} \left(\int_0^1 \int_{\xi} \frac{bP}{p} \operatorname{grad}[(1+aq)T] d\xi' d\xi \right) \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} \\
& + \int_{\mathcal{U}} (\tilde{v} \operatorname{div} \tilde{v} + \nabla_{\tilde{v}} \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} - \int_{\mathcal{U}} \left(\frac{f}{R_0} k \times \tilde{v} \right) \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U}, \tag{3.23}
\end{aligned}$$

where $\tilde{v}_{\xi} = \partial_{\xi} \tilde{v}$. By Lemma 2.3, we have

$$\int_{\mathcal{U}} \left[\nabla_{\tilde{v}} \tilde{v} + \left(\int_{\xi}^1 \operatorname{div} \tilde{v} d\xi' \right) \partial_{\xi} \tilde{v} \right] |\tilde{v}|^2 \tilde{v} d\mathcal{U} = 0. \tag{3.24}$$

Using Lemma 2.3 again and (2.1), we obtain

$$\int_{\mathcal{U}} (\nabla_{\tilde{v}} \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} = \frac{1}{4} \int_{\mathcal{U}} \nabla_{\tilde{v}} |\tilde{v}|^4 d\mathcal{U} = -\frac{1}{4} \int_{\mathcal{U}} |\tilde{v}|^4 \operatorname{div} \tilde{v} d\mathcal{U} = 0. \tag{3.25}$$

By virtue of Lemma 2.2, we have

$$\begin{aligned}
0 & = \int_{\mathcal{U}} \operatorname{div} [(|\tilde{v}|^2 \tilde{v} \cdot \tilde{v}) \tilde{v}] d\mathcal{U} = \int_{\mathcal{U}} \nabla_{\tilde{v}} (|\tilde{v}|^2 \tilde{v} \cdot \tilde{v}) d\mathcal{U} + \int_{\mathcal{U}} |\tilde{v}|^2 \tilde{v} \cdot \tilde{v} \operatorname{div} \tilde{v} d\mathcal{U} \\
& = \int_{\mathcal{U}} [|\tilde{v}|^2 \tilde{v} \cdot \nabla_{\tilde{v}} \tilde{v} + \tilde{v} \cdot \nabla_{\tilde{v}} (|\tilde{v}|^2 \tilde{v})] d\mathcal{U} + \int_{\mathcal{U}} |\tilde{v}|^2 \tilde{v} \cdot \tilde{v} \operatorname{div} \tilde{v} d\mathcal{U}.
\end{aligned}$$

Therefore,

$$\int_{\mathcal{U}} [|\tilde{v}|^2 \tilde{v} \cdot \tilde{v} \operatorname{div} \tilde{v} + \tilde{v} \cdot \nabla_{\tilde{v}} (|\tilde{v}|^2 \tilde{v})] d\mathcal{U} = - \int_{\mathcal{U}} |\tilde{v}|^2 \tilde{v} \cdot \nabla_{\tilde{v}} \tilde{v} d\mathcal{U}. \tag{3.26}$$

Using integration by parts, we obtain

$$\begin{aligned}
\int_{\mathcal{U}} \left[\int_0^1 (\tilde{v} \operatorname{div} \tilde{v} + \nabla_{\tilde{v}} \tilde{v}) d\xi \right] \cdot |\tilde{v}|^2 \tilde{v} d\mathcal{U} & = - \int_{\mathcal{U}} \left(\int_0^1 \tilde{v}_{\theta} \tilde{v} d\xi \right) \cdot \nabla_{e_{\theta}} (|\tilde{v}|^2 \tilde{v}) d\mathcal{U} \\
& - \int_{\mathcal{U}} \left(\int_0^1 \tilde{v}_{\varphi} \tilde{v} d\xi \right) \cdot \nabla_{e_{\varphi}} (|\tilde{v}|^2 \tilde{v}) d\mathcal{U}. \tag{3.27}
\end{aligned}$$

Note that the minus sign on the right-hand side of (3.27) was missed in [20]. In view of (3.23)-(3.27) combined with $(\frac{f}{R_0} \times \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} = 0$, we have

$$\begin{aligned}
& \frac{1}{4} \frac{d|\tilde{v}|_4^4}{dt} + \int_{\mathcal{U}} \left(|\nabla_{e_{\theta}} \tilde{v}|^2 |\tilde{v}|^2 + |\nabla_{e_{\varphi}} \tilde{v}|^2 |\tilde{v}|^2 + \frac{1}{2} |\nabla_{e_{\theta}} \tilde{v}|^2 + \frac{1}{2} |\nabla_{e_{\varphi}} \tilde{v}|^2 + |\tilde{v}|^4 \right) d\mathcal{U} \\
& + \int_{\mathcal{U}} \left(|\tilde{v}_{\xi}|^2 |\tilde{v}|^2 + \frac{1}{2} |\partial_{\xi} |\tilde{v}|^2|^2 \right) d\mathcal{U} \\
& = \int_{\mathcal{U}} [|\tilde{v}|^2 \tilde{v} \cdot \tilde{v} \operatorname{div} \tilde{v} + \tilde{v} \cdot \nabla_{\tilde{v}} (|\tilde{v}|^2 \tilde{v})] d\mathcal{U} \\
& + \int_{\mathcal{U}} \left(\int_0^1 \tilde{v}_{\theta} \tilde{v} d\xi \right) \cdot \nabla_{e_{\theta}} (|\tilde{v}|^2 \tilde{v}) d\mathcal{U} + \int_{\mathcal{U}} \left(\int_0^1 \tilde{v}_{\varphi} \tilde{v} d\xi \right) \cdot \nabla_{e_{\varphi}} (|\tilde{v}|^2 \tilde{v}) d\mathcal{U} \\
& + \int_{\mathcal{U}} \left(\int_{\xi}^1 \frac{bP}{p} [(1+aq)T] d\xi' \cdot \operatorname{div} (|\tilde{v}|^2 \tilde{v}) \right) d\mathcal{U}
\end{aligned}$$

$$- \int_{\mathcal{U}} \left(\int_0^1 \int_{\xi}^1 \frac{bP}{p} [(1+aq)T] d\xi' d\xi \cdot \operatorname{div}(|\tilde{v}|^2 \tilde{v}) \right) d\mathcal{U}.$$

Then by Hölder's inequality and the argument above, we obtain

$$\begin{aligned} & \frac{1}{4} \frac{d|\tilde{v}|_4^4}{dt} + \int_{\mathcal{U}} \left(|\nabla_{e_\theta} \tilde{v}|^2 |\tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2 |\tilde{v}|^2 + \frac{1}{2} |\nabla_{e_\theta} |\tilde{v}|^2|^2 + \frac{1}{2} |\nabla_{e_\varphi} |\tilde{v}|^2|^2 + |\tilde{v}|^4 \right) d\mathcal{U} \\ & + \int_{\mathcal{U}} \left(|\tilde{v}_\xi|^2 |\tilde{v}|^2 + \frac{1}{2} |\partial_\xi |\tilde{v}|^2|^2 \right) d\mathcal{U} \\ \leq & c \int_{S^2} |\bar{v}| \left(\int_0^1 |\tilde{v}|^3 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right) dS^2 \\ & + c \int_{S^2} \left(\int_0^1 |\tilde{v}|^2 d\xi \right) \left(\int_0^1 |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right) dS^2 \\ & + c \int_{S^2} \left((|\bar{T}| + |\overline{qT}|) \int_0^1 |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right) dS^2 \\ =: & I_1 + I_2 + I_3. \end{aligned} \tag{3.28}$$

In the following, we estimate $I_i, i=1,2,3$, separately. By the interpolation inequality, Hölder's inequality and Minkowski's inequality, we have

$$\begin{aligned} I_1 & \leq \int_{S^2} \left[|\bar{v}| \left(\int_0^1 |\tilde{v}|^4 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\xi \right)^{\frac{1}{2}} \right] dS^2 \\ & \leq |\bar{v}|_4 \left(\int_{S^2} \left(\int_0^1 |\tilde{v}|^4 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \left(\int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} \right)^{\frac{1}{2}} \\ & \leq |\bar{v}|_4 \left(\int_0^1 \left(\int_{S^2} |\tilde{v}|^{2 \times 4} dS^2 \right)^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} \right)^{\frac{1}{2}} \\ & \leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c |\bar{v}|_4^2 \int_0^1 \|\tilde{v}\|_2 (|\tilde{v}\|_2 + |\nabla |\tilde{v}||_2) d\xi \\ & \leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c |\bar{v}|_4^2 \|\tilde{v}\|_2 (|\tilde{v}\|_2 + |\nabla |\tilde{v}||_2) \\ & \leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + \varepsilon |\nabla |\tilde{v}||_2^2 + c (\|\bar{v}\|_1^2 + |\bar{v}|_2^2 \|\bar{v}\|_1^2) |\bar{v}|_4^4. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 & \leq c \int_{S^2} \left(\int_0^1 |\tilde{v}|^2 d\xi \right) \left(\int_0^1 |\tilde{v}|^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\xi \right)^{\frac{1}{2}} dS^2 \\ & \leq c \left(\int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} \right)^{\frac{1}{2}} \left(\int_{S^2} \left(\int_0^1 |\tilde{v}|^2 d\xi \right)^3 dS^2 \right)^{\frac{1}{2}} \\ & \leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c \left(\int_0^1 \left(\int_{S^2} |\tilde{v}|^6 dS^2 \right)^{\frac{1}{3}} d\xi \right)^3 \\ & \leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c \left(\int_0^1 |\tilde{v}|^{\frac{4}{3}} (|\tilde{v}|^{\frac{2}{3}} + |\nabla_{e_\theta} \tilde{v}|^{\frac{2}{3}} + |\nabla_{e_\varphi} \tilde{v}|^{\frac{2}{3}}) d\xi \right)^3 \\ & \leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c |\tilde{v}|_4^4 |\tilde{v}|_2^2 + c |\tilde{v}|_4^4 (|\nabla_{e_\theta} \tilde{v}|_2^2 + |\nabla_{e_\varphi} \tilde{v}|_2^2). \end{aligned}$$

Analogously, we have

$$\begin{aligned}
I_3 &\leq \int_{S^2} (|\bar{T}| + |\overline{qT}|) \left(\int_0^1 |\tilde{v}|^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\xi \right)^{\frac{1}{2}} dS^2 \\
&\leq \left(\int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} \right)^{\frac{1}{2}} \left(\int_{S^2} \left(\int_0^1 |\tilde{v}|^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} (|\bar{T}|_4 + |\overline{qT}|_4) \\
&\leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c \int_0^1 \left(\int_{S^2} |\tilde{v}|^4 dS^2 \right)^{\frac{1}{2}} d\xi (|\bar{T}|_4^2 + |\overline{qT}|_4^2) \\
&\leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + c|\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + |\tilde{v}|_4^2 \left[|\bar{T}|_4^2 + \left(\int_{S^2} \left(\int_0^1 qT d\xi \right)^4 dS^2 \right)^{\frac{1}{2}} \right] \\
&\leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + c|\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c|\tilde{v}|_4^2 |T|_4^2 + c|\tilde{v}|_4^2 \left(\int_0^1 |q|_8 |T|_8 d\xi \right)^2 \\
&\leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + c|\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c|\tilde{v}|_4^2 |T|_4^2 \\
&\quad + c|\tilde{v}|_4^2 \left(\int_0^1 |q|_4^{\frac{1}{2}} (|q|_2^{\frac{1}{2}} + |\nabla q|_2^{\frac{1}{2}}) |T|_4^{\frac{1}{2}} (|T|_2^{\frac{1}{2}} + |\nabla T|_2^{\frac{1}{2}}) d\xi \right)^2 \\
&\leq \varepsilon \int_{\mathcal{U}} |\tilde{v}|^2 (|\nabla_{e_\theta} \tilde{v}|^2 + c|\nabla_{e_\varphi} \tilde{v}|^2) d\mathcal{U} + c|\tilde{v}|_4^2 |T|_4^2 \\
&\quad + c|\tilde{v}|_4^2 (|q|_4^2 + |q|_4 |\nabla q|_2) (|T|_4^2 + |T|_4 |\nabla T|_2).
\end{aligned}$$

By the estimates of $I_1 - I_3$, we have

$$\frac{d|\tilde{v}|_4^2}{dt} \leq c(\|v\|_1^2 + |v|_2^2 \|v\|_1^2) |\tilde{v}|_4^2 + c|T|_4^2 + c(|q|_4^2 + |q|_4 |\nabla q|_2) (|T|_4^2 + |T|_4 |\nabla T|_2).$$

Therefore, by (3.4), (3.9), (3.10), (3.14), (3.21) and the uniform Gronwall lemma, we have

$$|\tilde{v}(t)|_4 \leq c, \tag{3.29}$$

where $t \geq 0$ and c is independent of t . Furthermore, by (3.28) and the estimates of $I_1 - I_3$, we get

$$\begin{aligned}
&\int_{t_0}^{t_0+1} \left(\|\nabla_{e_\theta} \tilde{v}(t)\|_2^2 + \|\nabla_{e_\varphi} \tilde{v}(t)\|_2^2 \right) dt \\
&\leq |\tilde{v}(t_0)|_4^4 + c \int_{t_0}^{t_0+1} \left((\|v(t)\|_1^2 + |v(t)|_2^2 \|v(t)\|_1^2) |\tilde{v}(t)|_4^2 + |T(t)|_4^2 \right) dt \\
&\quad + c \int_{t_0}^{t_0+1} \left(|q(t)|_4^2 + |q(t)|_4 |\nabla q(t)|_2 \right) (|T(t)|_4^2 + |T(t)|_4 |\nabla T(t)|_2) dt \leq c,
\end{aligned} \tag{3.30}$$

where c is independent of t_0 .

3.5. H^1 estimates of v, T, q . From [20], we have

$$\frac{d\|\tilde{v}\|_1}{dt} + |\Delta \tilde{v}|_2^2 \leq c(\|\tilde{v}\|_1^2 + |\tilde{v}|_2^2 \|\tilde{v}\|_1^2) \|\tilde{v}\|_1^2 + c\|\tilde{v}\|_1 \|\nabla_{e_\theta} \tilde{v}\|_2^2 + c\|\tilde{v}\|_1 \|\nabla_{e_\varphi} \tilde{v}\|_2^2. \tag{3.31}$$

By the uniform Gronwall lemma and (3.9)-(3.10), we obtain

$$\|\tilde{v}(t)\|_1 \leq c, \tag{3.32}$$

where $t \geq 0$ and c is independent of t . By Sobolev's inequality, we have that for all $t \geq 0$

$$|\bar{v}(t)|_{L^4(S^2)} \leq c|\bar{v}(t)|_{L^2(S^2)} + c|\nabla \bar{v}(t)|_{L^2(S^2)} \leq c,$$

which implies

$$|v(t)|_4 \leq |\tilde{v}(t)|_4 + |\bar{v}(t)|_4 \leq c. \tag{3.33}$$

From [20], we have

$$\begin{aligned} & \frac{d|v_\xi|_2^2}{dt} + |\nabla_{e_\theta} v_\xi|_2^2 + |\nabla_{e_\varphi} v_\xi|_2^2 + |v_\xi|_2^2 + |v_{\xi\xi}|_2^2 \\ & \leq c(\|\bar{v}\|_1^8 + |\tilde{v}|_4^8)|v_\xi|_2^2 + c|T|_2^2 + c|q|_4^4 + c|T|_4^4, \end{aligned} \tag{3.34}$$

where $\partial_{\xi\xi} v = v_{\xi\xi}$. By the uniform Gronwall lemma, we obtain

$$|v_\xi(t)|_2 \leq c, \tag{3.35}$$

where $t \geq 0$ and c is independent of t . Therefore, by (3.33)-(3.35) and the uniform Gronwall lemma we have

$$\int_{t_0}^{t_0+1} (|\nabla_{e_\theta} v_\xi|_2^2 + |\nabla_{e_\varphi} v_\xi|_2^2 + |v_{\xi\xi}|_2^2) dt \leq c, \tag{3.36}$$

where $t_0 \geq 0$ and c is independent of t_0 . Taking inner product with $-\Delta v$ in $L^2(\mathcal{U})$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2 + |v|_2^2)}{dt} + |\Delta v|_2^2 + |\nabla_{e_\theta} v_\xi|_2^2 + |\nabla_{e_\varphi} v_\xi|_2^2 + |v_\xi|_2^2 \\ & = \int_{\mathcal{U}} \nabla_v v \cdot \Delta v d\mathcal{U} + \int_{\mathcal{U}} \left(\int_{\xi}^1 \operatorname{div} v d\xi' \right) \partial_\xi v \cdot \Delta v d\mathcal{U} \\ & \quad + \int_{\mathcal{U}} \left(\int_{\xi}^1 \frac{bP}{p} \operatorname{grad}(1+aq)T d\xi' \right) \Delta v d\mathcal{U} \\ & \quad + \int_{\mathcal{U}} \frac{f}{R_0} v^\perp \cdot \Delta v d\mathcal{U} + \int_{\mathcal{U}} \operatorname{grad} \Phi_s \cdot \Delta v d\mathcal{U} \\ & = J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \tag{3.37}$$

By Hölder's inequality, the interpolation inequality and Young's inequality, we have

$$\begin{aligned} J_1 & \leq |\Delta v|_2 |v|_4 (|\nabla_{e_\theta} v|_4 + |\nabla_{e_\varphi} v|_4) \\ & \leq c|\Delta v|_2 |v|_4 (|\nabla_{e_\theta} v|_2^{\frac{1}{2}} + |\nabla_{e_\varphi} v|_2^{\frac{1}{2}}) (|\Delta v|_2^{\frac{3}{4}} + |\nabla_{e_\theta} v_\xi|_2^{\frac{3}{4}} + |\nabla_{e_\varphi} v_\xi|_2^{\frac{3}{4}} + |\nabla_{e_\varphi} v|_2^{\frac{3}{4}} + |\nabla_{e_\theta} v|_2^{\frac{3}{4}}) \\ & \leq \varepsilon |\Delta v|_2^2 + \varepsilon |\nabla_{e_\theta} v_\xi|_2^2 + \varepsilon |\nabla_{e_\varphi} v_\xi|_2^2 + c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2) (|v|_4^8 + |v|_4^2) \\ & \leq \varepsilon |\Delta v|_2^2 + \varepsilon |\nabla_{e_\theta} v_\xi|_2^2 + \varepsilon |\nabla_{e_\varphi} v_\xi|_2^2 + c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2). \end{aligned}$$

To estimate J_2 , we have

$$\begin{aligned} J_2 & \leq \int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \right) \left(\int_0^1 |\partial_\xi v| |\Delta v| d\xi \right) dS^2 \\ & \leq \int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \right) \left(\int_0^1 |\partial_\xi v|^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |\Delta v|^2 d\xi \right)^{\frac{1}{2}} dS^2 \end{aligned}$$

$$\begin{aligned}
&\leq c|\Delta v|_2 \left(\int_{S^2} \left(\int_0^1 |\partial_\xi v|^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \left(\int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \right)^4 dS^2 \right)^{\frac{1}{4}} \\
&\leq c|\Delta v|_2 \left(\int_0^1 \left(\int_{S^2} |\partial_\xi v|^4 dS^2 \right)^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \left(\int_0^1 \left(\int_{S^2} |\operatorname{div} v|^4 dS^2 \right)^{\frac{1}{4}} d\xi \right) \\
&\leq c|\Delta v|_2 \left(\int_0^1 |v_\xi|_2 (|v_\xi|_2 + |\nabla_{e_\theta} v_\xi|_2 + |\nabla_{e_\varphi} v_\xi|_2) d\xi \right)^{\frac{1}{2}} \\
&\quad \left(\int_0^1 |\operatorname{div} v|_2^{\frac{1}{2}} (|\operatorname{div} v|_2^{\frac{1}{2}} + |\Delta v|_2^{\frac{1}{2}}) d\xi \right) \\
&\leq c|\Delta v|_2 |v_\xi|_2^{\frac{1}{2}} (|v_\xi|_2^{\frac{1}{2}} + |\nabla_{e_\theta} v_\xi|_2^{\frac{1}{2}} + |\nabla_{e_\varphi} v_\xi|_2^{\frac{1}{2}}) \\
&\quad \cdot |\operatorname{div} v|_2^{\frac{1}{2}} (|\operatorname{div} v|_2^{\frac{1}{2}} + |\Delta v|_2^{\frac{1}{2}}) \\
&\leq \varepsilon |\Delta v|_2^2 + c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2) \\
&\quad \cdot (1 + |v_\xi|_2^4 + |v_\xi|_2^2 |\nabla_{e_\theta} v_\xi|_2^2 + |v_\xi|_2^2 |\nabla_{e_\varphi} v_\xi|_2^2). \tag{3.38}
\end{aligned}$$

Concerning estimate of J_3 , there are some typos in (5.6) of [20]. In the first inequality of (5.6), the integral region should be S^2 instead of Ω . For the reader's convenience, we estimate J_3 again. Using Hölder's inequality, Minkowski's inequality and the interpolation inequality, we have

$$\begin{aligned}
J_3 &\leq c|\nabla T|_2 |\Delta v|_2 + c \int_{S^2} \left(\int_0^1 |\operatorname{grad} q|^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 T^2 d\xi \right)^{\frac{1}{2}} \int_0^1 |\Delta v| d\xi dS^2 \\
&\quad + c \int_{S^2} \left(\int_0^1 |\operatorname{grad} T|^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 q^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |\Delta v| d\xi \right) dS^2 \\
&\leq c|\nabla T|_2 |\Delta v|_2 + c|\Delta v|_2 |T|_4 \left(\int_0^1 \left(\int_{S^2} |\operatorname{grad} q|^4 dS^2 \right)^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \\
&\quad + c|\Delta v|_2 |q|_4 \left(\int_0^1 \left(\int_{S^2} |\operatorname{grad} T|^4 dS^2 \right)^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \\
&\leq c|\nabla T|_2 |\Delta v|_2 + c|\Delta v|_2 |T|_4 |\nabla q|_2^{\frac{1}{2}} (|\nabla q|_2^{\frac{1}{2}} + |\Delta q|_2^{\frac{1}{2}}) \\
&\quad + c|\Delta v|_2 |q|_4 |\nabla T|_2^{\frac{1}{2}} (|\nabla T|_2^{\frac{1}{2}} + |\Delta T|_2^{\frac{1}{2}}) \\
&\leq \varepsilon |\Delta v|_2^2 + \varepsilon |\Delta T|_2^2 + \varepsilon |\Delta q|_2^2 + c|\nabla T|_2^2 + c|q|_4^2 |\nabla T|_2^2 \\
&\quad + c|T|_4^2 |\nabla q|_2^2 + c|q|_4^4 |\nabla T|_2^2 + c|T|_4^4 |\nabla q|_2^2. \tag{3.39}
\end{aligned}$$

By Hölder's inequality, we have

$$J_4 \leq \varepsilon |\Delta v|_2^2 + c|v|_2^2.$$

In [20], the authors thought $J_4 = 0$. But we do not think it is rigorous. By Lemma 2.1, we infer that $J_5 = 0$. From (3.37) and estimates of $J_1 - J_5$, we obtain

$$\begin{aligned}
&\frac{d(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2 + |v|_2^2)}{dt} \\
&\leq c(|\nabla_{e_\theta} v|_2^2 + |\nabla_{e_\varphi} v|_2^2) (1 + |v_\xi|_2^4 + |v_\xi|_2^2 |\nabla_{e_\theta} v_\xi|_2^2 + |v_\xi|_2^2 |\nabla_{e_\varphi} v_\xi|_2^2) \\
&\quad + c + c|\nabla T|_2^2 + c|\nabla q|_2^2. \tag{3.40}
\end{aligned}$$

In view of (3.4), (3.10), (3.35), (3.36) and the uniform Gronwall lemma, we get that for all $t \geq 0$

$$|\nabla_{e_\theta} v(t)|_2^2 + |\nabla_{e_\varphi} v(t)|_2^2 \leq c, \tag{3.41}$$

where c is a positive constant which is independent of t . Furthermore, by virtue of (3.37) we infer that for all $t_0 \geq 0$

$$\int_{t_0}^{t_0+1} |\Delta v(t)|_2^2 + |\nabla_{e_\theta} v_\xi(t)|_2^2 + |\nabla_{e_\varphi} v_\xi(t)|_2^2 dt \leq c, \tag{3.42}$$

where c is independent of t_0 .

REMARK 3.2. To get the estimates of T in V_2 space, we should first take the derivative with respect to ξ in the temperature equation and then estimate T_ξ in $L^2(\mathcal{U})$. If we take inner product of Equation (1.12) with $-\partial_{\xi\xi}T$ in $L^2(\mathcal{U})$, it is difficult to obtain the energy estimates of T_ξ because of its higher nonlinear structure than the oceanic primitive equation. Therefore, one can not even prove the global existence of the strong solution in this way.

By integration by parts,

$$\begin{aligned} \int_{\mathcal{U}} Q_{1\xi} T_\xi d\mathcal{U} &= \int_{\mathcal{U}} [\partial_\xi(Q_1 T_\xi) - Q_1 T_{\xi\xi}] d\mathcal{U} \\ &= -\alpha_s \int_{S^2} Q_1|_{\xi=1} T|_{\xi=1} dS^2 - \int_{\mathcal{U}} Q_1 T_{\xi\xi} d\mathcal{U} \\ &\leq \varepsilon |T_{\xi\xi}|_2^2 + c|Q_1|_{\xi=1}|_2^2 + c|T|_{\xi=1}|_2^2 + |Q_1|_2^2. \end{aligned}$$

In view of estimates in [20] and the above argument, we have

$$\begin{aligned} &\frac{1}{2} \frac{d(|T_\xi|_2^2 + \alpha_s |T|_{\xi=1}|_2^2)}{dt} + |\nabla T_\xi|_2^2 + |T_{\xi\xi}|_2^2 + \alpha_s |\nabla T|_{\xi=1}|_2^2 \\ &\leq \varepsilon (|T_{\xi\xi}|_2^2 + |\nabla T_\xi|_2^2 + |q_{\xi\xi}|_2^2 + |\nabla q_\xi|_2^2) + c|T_\xi|_2^2 + c|q_\xi|_2^2 + c\|v_\xi\|_1^2 + c\|v\|_1^2 \\ &\quad + c|\nabla q|_2^2 + |T|_{\xi=1}|_4^4 + c|T|_{\xi=1}|_2^2 + c|Q_1|_{\xi=1}|_2^2 + c|Q_1|_2^2 + c. \end{aligned} \tag{3.43}$$

Similarly, by the estimates in [20], we also have

$$\begin{aligned} &\frac{1}{2} \frac{d(|q_\xi|_2^2 + \beta_s |q|_{\xi=1}|_2^2)}{dt} + |\nabla q_\xi|_2^2 + |q_{\xi\xi}|_2^2 + \beta_s |\nabla q|_{\xi=1}|_2^2 \\ &\leq \varepsilon (|q_{\xi\xi}|_2^2 + |\nabla q_\xi|_2^2) + c|q_\xi|_2^2 + c\|v\|_1^2 + c\|v_\xi\|_1^2 \\ &\quad + c|q|_{\xi=1}|_4^4 + c|q|_{\xi=1}|_2^2 + c|Q_2|_{\xi=1}|_2^2 + c|Q_2|_2^2. \end{aligned} \tag{3.44}$$

Combining (3.43) and (3.44) yields,

$$\begin{aligned} &\frac{d(|T_\xi|_2^2 + |q_\xi|_2^2 + \alpha_s |T|_{\xi=1}|_2^2 + \beta_s |q|_{\xi=1}|_2^2)}{dt} + |\nabla T_\xi|_2^2 + |T_{\xi\xi}|_2^2 \\ &\quad + |\nabla q_\xi|_2^2 + |q_{\xi\xi}|_2^2 + \alpha_s |\nabla T|_{\xi=1}|_2^2 + \beta_s |\nabla q|_{\xi=1}|_2^2 \\ &\leq c + c(|T_\xi|_2^2 + |q_\xi|_2^2) + c\|v_\xi\|_1^2 + c\|v\|_1^2 + c\|q\|_1^2 + c|T|_{\xi=1}|_4^4 + c|T|_{\xi=1}|_2^2 \\ &\quad + c|q|_{\xi=1}|_4^4 + c|q|_{\xi=1}|_2^2 + c(|Q_1|_{\xi=1}|_2^2 + |Q_2|_{\xi=1}|_2^2) + c(|Q_1|_2^2 + |Q_2|_2^2). \end{aligned} \tag{3.45}$$

Then in view of the uniform Gronwall lemma, (3.4), (3.10), (3.15), (3.22) and (3.36), we obtain for arbitrary $t_0 \geq 0$

$$|T_\xi(t_0)|_2^2 + |q_\xi(t_0)|_2^2 + \int_{t_0}^{t_0+1} (\|T_\xi(t)\|_1^2 + \|q_\xi(t)\|_1^2 + |\nabla T|_{\xi=1}(t)|_2^2 + |\nabla q|_{\xi=1}(t)|_2^2) dt \leq c, \tag{3.46}$$

where c is independent of t_0 . By taking inner product of Equation (1.12) with $-\Delta T$, in $L^2(\mathcal{U})$, we reach

$$\begin{aligned} & \frac{1}{2} \frac{d|\nabla T|_2^2}{dt} + |\Delta T|_2^2 + |\nabla T_\xi|_2^2 + \alpha_s |\nabla T|_{\xi=1}|_2^2 \\ &= \int_{\mathcal{U}} \nabla_v T \Delta T d\mathcal{U} + \int_{\mathcal{U}} \int_{\xi} \operatorname{div} v d\xi' T_\xi \Delta T d\mathcal{U} - \int_{\mathcal{U}} \frac{bP}{p} (1+aq) w \Delta T d\mathcal{U} - \int_{\mathcal{U}} Q_1 \Delta T d\mathcal{U} \\ &= l_1 + l_2 + l_3 + l_4. \end{aligned} \tag{3.47}$$

To estimate l_1 , using Hölder’s inequality, the interpolation inequality and Young’s inequality we have

$$\begin{aligned} l_1 &\leq |\Delta T|_2 |\nabla T|_4 |v|_4 \\ &\leq c |\Delta T|_2 [|\nabla T|_2^{\frac{1}{4}} (|\nabla T|_2^{\frac{3}{4}} + |\Delta T|_2^{\frac{3}{4}} + |\nabla T_\xi|_2^{\frac{3}{4}})] |v|_4 \\ &\leq \varepsilon |\Delta T|_2^2 + \varepsilon |\nabla T_\xi|_2^2 + c |\nabla T|_2^2 |v|_4^2 + c |\nabla T|_2^2 |v|_4^8 \\ &\leq \varepsilon |\Delta T|_2^2 + \varepsilon |\nabla T_\xi|_2^2 + c |\nabla T|_2^2. \end{aligned}$$

To estimate l_2 , using Hölder’s inequality, Minkowski’s inequality, the interpolation inequality and Young’s inequality we obtain

$$\begin{aligned} l_2 &\leq \int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \int_0^1 |T_\xi| |\Delta T| d\xi \right) dS^2 \\ &\leq \int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \left(\int_0^1 T_\xi^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |\Delta T|^2 d\xi \right)^{\frac{1}{2}} \right) dS^2 \\ &\leq |\Delta T|_2 \left(\int_{S^2} \left(\int_0^1 T_\xi^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \left(\int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \right)^4 dS^2 \right)^{\frac{1}{4}} \\ &\leq |\Delta T|_2 \left(\int_0^1 \left(\int_{S^2} T_\xi^4 dS^2 \right)^{\frac{1}{2}} d\xi \right)^{\frac{1}{2}} \left(\int_0^1 \left(\int_{S^2} |\operatorname{div} v|^4 dS^2 \right)^{\frac{1}{4}} d\xi \right) \\ &\leq c |\Delta T|_2 \left(\int_0^1 |T_\xi|_2 (|T_\xi|_2 + |\nabla T_\xi|_2) d\xi \right)^{\frac{1}{2}} \cdot \left(\int_0^1 |\operatorname{div} v|_2^{\frac{1}{2}} (|\operatorname{div} v|_2^{\frac{1}{2}} + |\Delta v|_2^{\frac{1}{2}}) d\xi \right) \\ &\leq c |\Delta T|_2 |T_\xi|_2^{\frac{1}{2}} (|T_\xi|_2^{\frac{1}{2}} + |\nabla T_\xi|_2^{\frac{1}{2}}) \cdot |\operatorname{div} v|_2^{\frac{1}{2}} (|\operatorname{div} v|_2^{\frac{1}{2}} + |\Delta v|_2^{\frac{1}{2}}) \\ &\leq \varepsilon |\Delta T|_2^2 + c (|T_\xi|_2^2 + |T_\xi|_2 |\nabla T_\xi|_2) (|\operatorname{div} v|_2^2 + |\operatorname{div} v|_2 |\Delta v|_2) \\ &\leq \varepsilon |\Delta T|_2^2 + c (1 + |\nabla T_\xi|_2) (1 + |\Delta v|_2). \end{aligned}$$

Taking an analogous argument as above we get

$$l_3 \leq \varepsilon |\Delta T|_2^2 + c (1 + |\nabla q|_2) (1 + |\Delta v|_2).$$

Similarly, we have

$$l_4 \leq \varepsilon |\Delta T|_2^2 + c |Q_1|_2^2.$$

Combining (3.47) and estimates of $l_1 - l_4$ we get

$$\begin{aligned} & \frac{1}{2} \frac{d|\nabla T|_2^2}{dt} + |\Delta T|_2^2 + |\nabla T_\xi|_2^2 + \alpha_s |\nabla T|_{\xi=1}|_2^2 \\ &\leq c |\nabla T|_2^2 + c |Q_1|_2^2 + c (1 + |\nabla T_\xi|_2) (1 + |\Delta v|_2) + c (1 + |\nabla q|_2) (1 + |\Delta v|_2). \end{aligned} \tag{3.48}$$

According to (3.4), (3.10), (3.42), (3.46) and the uniform Gronwall lemma, we have for all $t \geq 0$

$$|\nabla T(t)|_2^2 + \int_t^{t+1} (|\Delta T(t)|_2^2 + |\nabla T_\xi(t)|_2^2) dt \leq c, \tag{3.49}$$

where c is a positive constant independent of t . Analogous to the deduction of (3.48), we have

$$\begin{aligned} & \frac{1}{2} \frac{d|\nabla q|_2^2}{dt} + |\Delta q|_2^2 + |\nabla q_\xi|_2^2 + \beta_s |\nabla q|_{\xi=1}|_2^2 \\ & \leq \varepsilon |\Delta q|_2^2 + c|Q_2|_2^2 + c|\nabla q|_2^2 + c(1 + |\nabla T_\xi|_2)(1 + |\Delta v|_2). \end{aligned} \tag{3.50}$$

According to (3.4), (3.42), (3.49) and the uniform Gronwall lemma, we have for all $t \geq 0$

$$|\nabla q(t)|_2^2 \leq c, \tag{3.51}$$

where c is a positive constant independent of t .

Combining the above results, we have proved the existence of an absorbing ball for the strong solution (v, T, q) in the solution space V for the 3D viscous primitive equations of large-scale moist atmosphere.

4. Continuity of strong solution with respect to t

In this section, we will consider the continuity of the strong solution with respect to time t in V , which will be helpful for proving the existence and connectedness of the global attractor for moist primitive equations. To establish our result of this section, we need the following lemma which is a consequence of a general result of [35]. For the proof, one can refer to [46].

LEMMA 4.1. *Let V, H, V' be three Hilbert spaces such that $V \subset H = H' \subset V'$, where H' and V' are the dual spaces of H and V respectively. Suppose $u \in L^2([0, T]; V)$ and $u' \in L^2([0, T]; V')$. Then u is almost everywhere equal to a function continuous from $[0, T]$ into H .*

From (1.11) and Lemma 2.1, we obtain for $\eta \in V_1 = D(A_1^{\frac{1}{2}})$,

$$\begin{aligned} \langle \partial_t A_1^{\frac{1}{2}} v, \eta \rangle &= \langle \partial_t v, A_1^{\frac{1}{2}} \eta \rangle \\ &= -\langle A_1 v, A_1^{\frac{1}{2}} \eta \rangle - \langle \nabla_v v, A_1^{\frac{1}{2}} \eta \rangle \\ &\quad - \left\langle \left(\int_\xi^1 \operatorname{div} v(x, y, \xi, t) d\xi \right) v_\xi, A_1^{\frac{1}{2}} \eta \right\rangle \\ &\quad - \left\langle \int_\xi^1 \frac{bP}{p} \operatorname{grad}[(1 + aq)T] d\xi', A_1^{\frac{1}{2}} \eta \right\rangle \\ &\quad - \left\langle \frac{f}{R_0} v^\perp, A_1^{\frac{1}{2}} \eta \right\rangle. \end{aligned}$$

Using the Hölder inequality, Agmon's inequality and the interpolation inequality yields

$$\begin{aligned} |\langle A_1 v, A_1^{\frac{1}{2}} \eta \rangle| &\leq |A_1 v|_2 \|\eta\|_1, \\ |\langle (v \cdot \nabla) v, A_1^{\frac{1}{2}} \eta \rangle| &\leq c \|v\|_\infty \|v\|_1 |A_1^{\frac{1}{2}} \eta|_2 \leq c \|v\|_1^{\frac{3}{2}} \|v\|_2^{\frac{1}{2}} |A_1^{\frac{1}{2}} \eta|_2. \end{aligned}$$

By Hölder's inequality, Minkowski's inequality and the interpolation inequality, we have

$$\begin{aligned}
& - \left\langle \int_{\xi}^1 \frac{bP}{p} \operatorname{grad}[(1+aq)T] d\xi', A_1^{\frac{1}{2}} \eta \right\rangle \\
& \leq c \int_{S^2} \left(\int_0^1 (|\operatorname{grad}T| + |\operatorname{grad}(Tq)|) d\xi \int_0^1 A_1^{\frac{1}{2}} \eta d\xi \right) dS^2 \\
& \leq c \|T\|_1 |A_1^{\frac{1}{2}} \eta|_2 + c |A_1^{\frac{1}{2}} \eta|_2 \int_0^1 |\operatorname{grad}T|_4 |q|_4 d\xi + c |A_1^{\frac{1}{2}} \eta|_2 \int_0^1 |\operatorname{grad}q|_4 |T|_4 d\xi \\
& \leq c \|T\|_1 |A_1^{\frac{1}{2}} \eta|_2 + c |A_1^{\frac{1}{2}} \eta|_2 \int_0^1 |\operatorname{grad}T|_2^{\frac{1}{2}} \cdot (|\Delta T|_2^{\frac{1}{2}} + |\operatorname{grad}T|_2^{\frac{1}{2}}) |q|_4 d\xi \\
& \quad + c |A_1^{\frac{1}{2}} \eta|_2 \int_0^1 |\operatorname{grad}q|_2^{\frac{1}{2}} \cdot (|\Delta q|_2^{\frac{1}{2}} + |\operatorname{grad}q|_2^{\frac{1}{2}}) |T|_4 d\xi \\
& \leq c \|T\|_1 |A_1^{\frac{1}{2}} \eta|_2 + c |A_1^{\frac{1}{2}} \eta|_2 \|T\|_1^{\frac{1}{2}} \|T\|_2^{\frac{1}{2}} |q|_4 + c |A_1^{\frac{1}{2}} \eta|_2 \|q\|_1^{\frac{1}{2}} \|q\|_2^{\frac{1}{2}} |T|_4.
\end{aligned}$$

Similarly, we have

$$\left| \left\langle \left(\int_{\xi}^1 \operatorname{div}v(x, y, \xi', t) d\xi' \right) v_{\xi}, A_1^{\frac{1}{2}} \eta \right\rangle \right| \leq c \|v\|_1 \|v\|_2 \|\eta\|_1.$$

In view of the above arguments, we conclude that

$$\begin{aligned}
\|\partial_t(A_1^{\frac{1}{2}}v)\|_{V_1'} & \leq c \|v\|_2 + c \|v\|_1 \|v\|_2 + c \|T\|_1 \\
& \quad + c (\|T\|_1^{\frac{1}{2}} \|T\|_2^{\frac{1}{2}} |q|_4 + \|q\|_1^{\frac{1}{2}} \|q\|_2^{\frac{1}{2}} |T|_4).
\end{aligned}$$

Since

$$v \in L^{\infty}([0, \tau]; V_1) \cap L^2([0, \tau]; H^2(\mathcal{U})), \quad \forall \tau > 0,$$

we have

$$A_1^{\frac{1}{2}}v \in L^2([0, \tau]; V_1), \quad \partial_t A_1^{\frac{1}{2}}v \in L^2([0, \tau]; V_1'), \quad \forall \tau > 0.$$

Therefore, by Lemma 4.1, we infer that $v \in C([0, \tau]; V_1)$. For any $\phi \in V_2 = D(A_2^{\frac{1}{2}})$, we have

$$\begin{aligned}
\langle \partial_t A_2^{\frac{1}{2}}T, \phi \rangle & = \langle \partial_t T, A_2^{\frac{1}{2}}\phi \rangle \\
& = - \langle A_2 T, A_2^{\frac{1}{2}}\phi \rangle - \langle \nabla_v T, A_2^{\frac{1}{2}}\phi \rangle - \left\langle \left(\int_{\xi}^1 \operatorname{div}v(x, y, \xi', t) d\xi' \right) \partial_{\xi} T, A_2^{\frac{1}{2}}\phi \right\rangle \\
& \quad + \left\langle \frac{bP}{p} (1+aq) \left(\int_{\xi}^1 \operatorname{div}v(x, y, \xi', t) d\xi' \right), A_2^{\frac{1}{2}}\phi \right\rangle + \langle Q_1, A_2^{\frac{1}{2}}\phi \rangle.
\end{aligned}$$

By Hölder's inequality, Minkowski's inequality and the interpolation inequality, we have

$$\begin{aligned}
& \left\langle \frac{bP}{p} (1+aq) \left(\int_{\xi}^1 \operatorname{div}v(x, y, \xi', t) d\xi' \right), A_2^{\frac{1}{2}}\phi \right\rangle \\
& \leq c \|v\|_1 \|\phi\|_1 + c \int_{S^2} \left(\int_0^1 |\operatorname{div}v| d\xi \right) \left(\int_0^1 |q| |A_1^{\frac{1}{2}}\phi| d\xi \right) dS^2
\end{aligned}$$

$$\begin{aligned} &\leq c\|v\|_1\|\phi\|_1 + c \int_{S^2} \left(\int_0^1 |\operatorname{div} v| d\xi \right) \left(\int_0^1 q^2 d\xi \right)^{\frac{1}{2}} \left(\int_0^1 |A_2^{\frac{1}{2}} \phi|^2 d\xi \right)^{\frac{1}{2}} dS^2 \\ &\leq c\|v\|_1\|\phi\|_1 + c\|\phi\|_1 \left(\int_0^1 |\operatorname{div} v|_4 d\xi \right) \left(\int_0^1 |q|_4^2 d\xi \right)^{\frac{1}{2}} \\ &\leq c\|v\|_1\|\phi\|_1 + c\|\phi\|_1 |q|_4 \|v\|_1^{\frac{1}{2}} \|v\|_2^{\frac{1}{2}}. \end{aligned}$$

Consequently, taking an analogous argument about $\|\partial_t A_1^{\frac{1}{2}} v\|_{V'_1}$, we have

$$\begin{aligned} \|\partial_t A_2^{\frac{1}{2}} T\|_{V'_2} &\leq c(\|T\|_2 + \|v\|_1 + \|v\|_1^{\frac{1}{2}} \|v\|_2^{\frac{1}{2}} \|T\|_1 + |q|_4 \|v\|_1^{\frac{1}{2}} \|v\|_2^{\frac{1}{2}} \\ &\quad + \|T\|_1^{\frac{1}{2}} \|T\|_2^{\frac{1}{2}} \|v\|_1^{\frac{1}{2}} \|v\|_2^{\frac{1}{2}} + |Q_1|_2). \end{aligned}$$

Similarly, we can infer that

$$\|\partial_t A_3^{\frac{1}{2}} q\|_{V'_2} \leq c(\|q\|_2 + \|v\|_1^{\frac{1}{2}} \|v\|_2^{\frac{1}{2}} \|q\|_1 + \|q\|_1^{\frac{1}{2}} \|q\|_2^{\frac{1}{2}} \|v\|_1^{\frac{1}{2}} \|v\|_2^{\frac{1}{2}} + |Q_2|_2).$$

Since

$$(v, T, q) \in L^\infty([0, \tau]; V) \cap L^2([0, \tau]; H^2(\mathcal{U})), \quad \forall \tau > 0,$$

we have

$$A_2^{\frac{1}{2}} T \in L^2([0, \tau]; V_2), \quad \partial_t A_2^{\frac{1}{2}} T \in L^2([0, \tau]; V'_2)$$

and

$$A_3^{\frac{1}{2}} q \in L^2([0, \tau]; V_3), \quad \partial_t A_3^{\frac{1}{2}} q \in L^2([0, \tau]; V'_3).$$

By Lemma 4.1, we infer that $T \in C([0, +\infty); V_2)$ and $q \in C([0, +\infty); V_3)$. So far, we obtain that

$$(v, T, q) \in C([0, +\infty); V).$$

5. Continuity in V with respect to initial data

It is shown in [20] that the strong solution to the 3D viscous primitive equations of large-scale moist atmosphere is unique and Lipschitz continuous with respect to the initial condition in H . But what we need to do here is to show the continuity property in V .

In the following, will prove that for any fixed $t > 0$, the mapping $(v_0, T_0, q_0) \mapsto (v(t), T(t), q(t))$ is Lipschitz continuous from V into itself for all the strong solutions.

Assume $(v_i, T_i, q_i), i = 1, 2$, are two strong solutions to the Equations (1.11)-(1.17) with initial data $(v_{0,i}, T_{0,i}, q_{0,i}) \in V$. Let

$$u = v_1 - v_2, \quad \tau = T_1 - T_2, \quad q = q_1 - q_2, \quad \Phi_s(t; \theta, \varphi) = \Phi_{1,s}(t; \theta, \varphi) - \Phi_{2,s}(t; \theta, \varphi).$$

Then we derive from (1.11)-(1.17) that

$$\begin{aligned} \partial_t u + L_1 u + \nabla_{v_1} u + \nabla_u v_2 + \left(\int_\xi^1 \operatorname{div} v_1(x, y, \xi', t) d\xi' \right) \partial_\xi v \\ + \left(\int_\xi^1 \operatorname{div} v(x, y, \xi', t) d\xi' \right) \partial_\xi v_2 + \frac{f}{R_0} v^\perp + \operatorname{grad} \Phi_s \end{aligned}$$

$$+ \int_{\xi}^1 \frac{bP}{p} \text{grad} \tau d\xi' + \int_{\xi}^1 \frac{abP}{p} \text{grad}(q_1 \tau) d\xi' + \int_{\xi}^1 \frac{abP}{p} \text{grad}(qT_2) d\xi' = 0, \tag{5.1}$$

$$\begin{aligned} & \partial_t \tau + L_2 \tau + \nabla_{v_1} \tau + \nabla_u T_2 + \left(\int_{\xi}^1 \text{div} v_1(x, y, \xi', t) d\xi' \right) \partial_{\xi} \tau \\ & + \left(\int_{\xi}^1 \text{div} u(x, y, \xi', t) d\xi' \right) \partial_{\xi} T_2 - \frac{bP}{p} \left(\int_{\xi}^1 \text{div} u(x, y, \xi', t) d\xi' \right) \\ & - \frac{abP}{p} q_1 \left(\int_{\xi}^1 \text{div} u(x, y, \xi', t) d\xi' \right) - \frac{abP}{p} q \left(\int_{\xi}^1 \text{div} v_2(x, y, \xi', t) d\xi' \right) = 0, \end{aligned} \tag{5.2}$$

$$\begin{aligned} & \partial_t q + L_3 q + \nabla_{v_1} q + \nabla_u q_2 + \left(\int_{\xi}^1 \text{div} v_1(x, y, \xi', t) d\xi' \right) \partial_{\xi} q \\ & + \left(\int_{\xi}^1 \text{div} u(x, y, \xi', t) d\xi' \right) \partial_{\xi} q_2 = 0, \end{aligned} \tag{5.3}$$

$$u|_{t=0} = v_0^1 - v_0^2, \quad \tau|_{t=0} = T_0^1 - T_0^2, \quad q|_{t=0} = q_0^1 - q_0^2, \tag{5.4}$$

$$\xi = 1: \quad \partial_{\xi} u = 0, \quad \partial_{\xi} \tau = -\alpha_s \tau, \quad \partial_{\xi} q = -\beta_s q, \tag{5.5}$$

$$\xi = 0: \quad \partial_{\xi} u = 0, \quad \partial_{\xi} \tau = 0, \quad \partial_{\xi} q = 0. \tag{5.6}$$

Taking inner product of (5.1) with $A_1 u$ in $L^2(\mathcal{U})$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_1^2 + |A_1 u|_2^2 &= - \langle \nabla_{v_1} u, A_1 u \rangle - \langle \nabla_u v_2, A_1 u \rangle \\ &- \left\langle \left(\int_{\xi}^1 \text{div} u(x, y, \xi', t) d\xi' \right) \partial_{\xi} v_2, A_1 u \right\rangle \\ &- \left\langle \left(\int_{\xi}^1 \text{div} v_1(x, y, \xi', t) d\xi' \right) \partial_{\xi} u, A_1 u \right\rangle \\ &- \left\langle \int_{\xi}^1 \frac{bP}{p} \text{grad} \tau d\xi', A_1 u \right\rangle - \left\langle \left(\frac{f}{R_0} u^{\perp} + \text{grad} \Phi_s \right), A_1 u \right\rangle \\ &- \left\langle \int_{\xi}^1 \frac{abP}{p} \text{grad}(q_1 \tau) d\xi', A_1 u \right\rangle - \left\langle \int_{\xi}^1 \frac{abP}{p} \text{grad}(qT_2) d\xi', A_1 u \right\rangle \\ &= \sum_{i=1}^8 k_i. \end{aligned} \tag{5.7}$$

By Hölder’s inequality, Agmon’s inequality and Young’s inequality , we have

$$\begin{aligned} k_1 &\leq |v_1|_{\infty} (|\nabla_{e_{\theta}} u|_2 + |\nabla_{e_{\varphi}} u|_2) |A_1 u|_2 \\ &\leq c \|v_1\|_1^{\frac{1}{2}} |A_1 v_1|_2^{\frac{1}{2}} \|u\|_1 |A_1 u|_2 \\ &\leq \varepsilon |A_1 u|_2^2 + c \|v_1\|_1 |A_1 v_1|_2 \|u\|_1^2. \end{aligned}$$

Similarly, we obtain

$$k_2 \leq \|u\|_{\infty} \|v_2\|_1 |A_1 u|_2 \leq \varepsilon |A_1 u|_2^2 + c \|u\|_1^2 \|v_2\|_1^4.$$

Taking an analogous argument as (3.38), we have

$$k_3 + k_4 \leq \varepsilon |A_1 u|_2^2 + c \|u\|_1^2 \|v_2\|_1^2 \|v_2\|_2^2 + c \|u\|_1^2 \|v_1\|_1^2 \|v_1\|_2^2.$$

In view of Lemma 2.1 and Hölder's inequality, we obtain

$$k_5 + k_6 \leq \varepsilon |A_1 u|_2^2 + c \|\tau\|_1^2 + c |u|_2^2.$$

To estimate k_7 , by Hölder's inequality, Minkowski's inequality and the interpolation inequality we have

$$\begin{aligned} k_7 &\leq c \int_{S^2} \left(\int_0^1 |\text{grad} q_1| |\tau| d\xi \int_0^1 |A_1 u| d\xi \right) dS^2 \\ &\quad + c \int_{S^2} \left(\int_0^1 |q_1| |\text{grad} \tau| d\xi \int_0^1 |A_1 u| d\xi \right) dS^2 \\ &\leq c |A_1 u|_2 \left(\int_{S^2} \left(\int_0^1 |\text{grad} q_1|^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \left(\int_{S^2} \left(\int_0^1 |\tau|^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \\ &\quad + c |A_1 u|_2 \left(\int_{S^2} \left(\int_0^1 |q_1|^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \left(\int_{S^2} \left(\int_0^1 |\text{grad} \tau|^2 d\xi \right)^2 dS^2 \right)^{\frac{1}{4}} \\ &\leq \varepsilon |A_1 u|_2^2 + c |\text{grad} q_1|_2 (|\text{grad} q_1|_2 + |\Delta q_1|_2) (\|\tau\|_2^2 + \|\tau\|_2 \|\nabla \tau\|_2) \\ &\quad + c |\text{grad} \tau|_2 (|\text{grad} \tau|_2 + |\Delta \tau|_2) \|q_1\|_4^2 \\ &\leq \varepsilon |A_1 u|_2^2 + \varepsilon |\Delta \tau|_2^2 + c \|\tau\|_1^2 (\|q_1\|_1^4 + \|q_1\|_2^2). \end{aligned}$$

Similarly, we have

$$\begin{aligned} k_8 &\leq \varepsilon |A_1 u|_2^2 + c |\text{grad} q|_2 (|\text{grad} q|_2 + |\Delta q|_2) \|T_2\|_4^2 \\ &\quad + c |\text{grad} T_2|_2 (|\text{grad} T_2|_2 + |\Delta T_2|_2) (\|q\|_2^2 + \|q\|_2 \|\nabla q\|_2) \\ &\leq \varepsilon |A_1 u|_2^2 + \varepsilon |\Delta q|_2^2 + c \|q\|_1^2 (\|T_2\|_1^4 + \|T_2\|_2^2). \end{aligned}$$

By (5.7) and estimates of $k_1 - k_8$, we get

$$\begin{aligned} \frac{1}{2} \frac{d\|u\|_1^2}{dt} + |A_1 u|_2^2 &\leq \varepsilon |A_1 u|_2^2 + \varepsilon |A_2 \tau|_2^2 + \varepsilon |A_3 q|_2^2 \\ &\quad + c \|u\|_1^2 (\|v_1\|_1^2 \|v_1\|_2^2 + \|v_2\|_1^2 \|v_2\|_2^2 + \|v_2\|_1^4 + 1) \\ &\quad + c \|\tau\|_1^2 (\|q_1\|_1^4 + \|q_1\|_2^2) \\ &\quad + c \|q\|_1^2 (\|T_2\|_1^4 + \|T_2\|_2^2). \end{aligned} \tag{5.8}$$

Taking an analogous argument as above, from (5.2) and (5.3) we have

$$\begin{aligned} \frac{1}{2} \frac{d\|\tau\|_1^2}{dt} + |A_2 \tau|_2^2 &\leq \varepsilon |A_2 \tau|_2^2 + \varepsilon |A_1 u|_2^2 + c \|q\|_1^2 \|v_2\|_1 \|v_2\|_2 \\ &\quad + c \|\tau\|_1^2 (1 + \|v_1\|_1^2 \|v_1\|_2^2) + c \|u\|_1^2 (1 + \|q_1\|_1^4 + \|T_2\|_2^2 + \|T_2\|_1^2 \|T_2\|_2^2) \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} \frac{1}{2} \frac{d\|q\|_1^2}{dt} + |A_3 q|_2^2 &\leq \varepsilon |A_3 q|_2^2 + c \|q\|_1^2 (1 + \|v_1\|_1^2 \|v_1\|_2^2) \\ &\quad + c \|u\|_1^2 (\|q_2\|_2^2 + \|q_2\|_1^2 \|q_2\|_2^2). \end{aligned} \tag{5.10}$$

Let

$$g_1 := 1 + \|v_1\|_1^2 \|v_1\|_2^2 + \|v_2\|_1^4 + \|v_2\|_1^2 \|v_2\|_2^2 + \|T_2\|_2^2 + \|T_2\|_1^2 \|T_2\|_2^2 + \|q_1\|_1^4 + \|q_2\|_2^2 + \|q_2\|_1^2 \|q_2\|_2^2,$$

$$g_2 := 1 + \|q_1\|_2^2 + \|q_1\|_1^4 + \|v_1\|_1^2 \|v_1\|_2^2$$

and

$$g_3 := 1 + \|T_2\|_2^2 + \|T_2\|_1^4 + \|v_2\|_1 \|v_2\|_2 + \|v_1\|_1^2 \|v_1\|_2^2.$$

Obviously, for arbitrary $0 \leq a < b < \infty$, we have

$$\int_a^b (g_1(t) + g_2(t) + g_3(t)) dt < \infty.$$

Thereby, in view of (5.8)-(5.10) we get

$$\frac{d(\|u\|_1^2 + \|\tau\|_1^2 + \|q\|_1^2)}{dt} \leq c(g_1(t) + g_2(t) + g_3(t))(\|u\|_1^2 + \|\tau\|_1^2 + \|q\|_1^2),$$

which combined with the Gronwall lemma implies

$$\begin{aligned} & \|u(t)\|_1^2 + \|\tau(t)\|_1^2 + \|q(t)\|_1^2 \\ & \leq c(\|v_{0,1} - v_{0,2}\|_1^2 + \|T_{0,1} - T_{0,2}\|_1^2 + \|q_{0,1} - q_{0,2}\|_1^2) e^{\int_0^t (g_1(s) + g_2(s) + g_3(s)) ds}. \end{aligned}$$

So far, we have shown that for $t > 0$, $(v(t), T(t), q(t))$ is Lipschitz continuous in V with respect to the initial data $(v(0), T(0), q(0))$.

6. The global attractor

In this section, we present our main result, the existence of the global attractor, of this paper. To show our main result, we make use of the following theorem from Teman [46]. For more details about the theorem, see, e.g., [4, 14, 22, 32, 44–46] and the references therein.

THEOREM 6.1. *Suppose that X is a metric space and semigroup $\{S(t)\}_{t \geq 0}$ is a family of operators from X into itself such that*

- (i) *for any fixed $t > 0$, $S(t)$ is continuous from X into itself;*
- (ii) *for some t_0 , $S(t_0)$ is compact from X into itself;*
- (iii) *there exists a subset B_0 of X which is bounded, a subset U of X is open, such that*

$B_0 \subseteq U \subseteq X$, and B_0 is the absorbing set of U , i.e. for any bounded subset $B \subset U$, there

is a $t_0 = t_0(B)$, such that

$$S(t)B \subset B_0, \quad \forall t > t_0(B).$$

Then $\mathcal{A} := \omega(B_0)$, the ω -limit set of B_0 , is a compact attractor which attracts all the bounded sets of U , i.e. for any $x \in U$,

$$\lim_{t \rightarrow \infty} \text{dist}(S(t)x, \mathcal{A}) = 0.$$

The set \mathcal{A} is the maximal bounded attractor in U for the inclusion relation.

Suppose in addition that X is a Banach space, U is convex and

(iv) $\forall x \in X, S(t)x: \mathbb{R}_+ \mapsto X$ is continuous.

Then $\mathcal{A} := \omega(B_0)$ is also connected.

If $U = X, \mathcal{A}$ is the global attractor of the semigroup $\{S(t)\}_{t \geq 0}$ in X .

Next, we give our main result of the paper and complete the proof of the theorem.

THEOREM 6.2. Assume $Q_1, Q_2 \in L^2(\mathbb{U})$ and $Q_1|_{\xi=1}, Q_2|_{\xi=1} \in L^2(S^2)$. Then, for $t \geq 0$, the solution operator $\{S(t)\}_{t \geq 0}$ of the 3D viscous primitive equations of large-scale moist atmosphere (1.11)-(1.17): $S(t)(v_0, T_0, q_0) = (v(t), T(t), q(t))$ defines a semigroup in the space V . Furthermore, the results below hold:

- (1) For any $(v_0, T_0, q_0) \in V, t \rightarrow S(t)(v_0, T_0, q_0)$ is a continuous map from \mathbb{R}_+ into V .
- (2) For any $t > 0, S(t)$ is a continuous map in V .
- (3) For any $t > 0, S(t)$ is a compact map in V .
- (4) $\{S(t)\}_{t \geq 0}$ possesses a global attractor \mathcal{A} in V . The global attractor in \mathcal{A} is compact and connected in V and is the maximal bounded attractor in V in the sense of set inclusion relation; \mathcal{A} attracts all bounded subsets in V in the norm of V .

To prove Theorem 6.2, we need to check the conditions (i)-(iv) in Theorem 6.1. First, condition (i), the continuous dependence on initial data of the solution, is verified in Section 5. Second, condition (iii), the existence of an absorbing ball in V , is proved in Section 3. Third, condition (iv), the regularity of the solution, is shown in Section 4. Finally, only the condition (ii), compactness of the solution operator $\{S(t)\}_{t \geq 0}$, is left to be checked. We will use Aubin-Lions lemma stated below and continuity argument to verify condition (ii). For more details of the lemma, see [1], [33] and references therein.

LEMMA 6.1. Let $\mathcal{H}_0, \mathcal{H}, \mathcal{H}_1$ be Banach spaces such that $\mathcal{H}_0, \mathcal{H}_1$ are reflexive and $\mathcal{H}_0 \overset{c}{\subset} \mathcal{H} \subset \mathcal{H}_1$. Define, for $0 < \tau < \infty$,

$$X := \left\{ u \mid u \in L^2([0, \tau]; \mathcal{H}_0), \frac{du}{dt} \in L^2([0, \tau]; \mathcal{H}_1) \right\}.$$

Then X is a Banach space equipped with the norm $\|u\|_{L^2([0, \tau]; \mathcal{H}_0)} + \|u'\|_{L^2([0, \tau]; \mathcal{H}_1)}$. Moreover, $X \overset{c}{\subset} L^2([0, \tau]; \mathcal{H})$.

Proof. (**Proof of Theorem 6.1.**) By the argument above, we only have to check condition (ii), the compactness of the solution operator. For any fixed $\tau > 0$, let \mathcal{B} be a bounded subset of V and A_τ denote the subset of the space $L^2([0, T]; H)$:

$$A_\tau := \left\{ \left(A_1^{\frac{1}{2}} v, A_2^{\frac{1}{2}} T, A_3^{\frac{1}{2}} q \right) \mid (v_0, T_0, q_0) \in \mathcal{B}, (v(t), T(t), q(t)) = S(t)(v_0, T_0, q(0)), t \in [0, \tau] \right\}.$$

For $(v_0, T_0, q(0)) \in \mathcal{B}$, it has been shown previously that the strong solution (v, T, q) satisfies

$$(A_1^{\frac{1}{2}} v, A_2^{\frac{1}{2}} T, A_3^{\frac{1}{2}} q) \in L^2([0, \tau]; V), \quad (\partial_t A_1^{\frac{1}{2}} v, \partial_t A_2^{\frac{1}{2}} T, \partial_t A_3^{\frac{1}{2}} q) \in L^2([0, \tau]; V').$$

If we denote

$$\mathcal{H}_0 = V, \mathcal{H} = H, \mathcal{H}_1 = V',$$

from Lemma 6.1 we infer that A_τ is compact in $L^2([0, \tau]; H)$.

To prove that the solution operator $\{S(t)\}_{t \geq 0}$ is compact in V , for any bounded sequence $\{(v_{0,n}, T_{0,n}, q_{0,n})\}_0^n \subset \mathcal{B}$, we should show that there exists a convergent subsequence of $\{S(t)(v_{0,n}, T_{0,n}, q_{0,n})\}_0^\infty$ in V .

Since A_τ is compact in $L^2([0, \tau]; H)$, there exists a function $(v_*, T_*, q_*) \in L^2([0, \tau]; V)$ such that there is a subsequence of $\{S(\cdot)(v_{0,n}, T_{0,n}, q_{0,n})\}_0^\infty$, still denoted as $\{S(\cdot)(v_{0,n}, T_{0,n}, q_{0,n})\}_0^\infty$, satisfying

$$\lim_{n \rightarrow \infty} \int_0^\tau \|S(t)(v_{0,n}, T_{0,n}, q_{0,n}) - (v_*(t), T_*(t), q_*(t))\|_1^2 dt = 0,$$

which implies that there is a subsequence of $\{S(\cdot)(v_{0,n}, T_{0,n}, q_{0,n})\}_0^\infty$, still denoted as $\{S(\cdot)(v_{0,n}, T_{0,n}, q_{0,n})\}_0^\infty$ for simplicity of notation, converging to (v_*, T_*, q_*) in V a.e. in $(0, \tau)$:

$$\lim_{n \rightarrow \infty} \|S(t)(v_{0,n}, T_{0,n}, q_{0,n}) - (v_*(t), T_*(t), q_*(t))\|_1 = 0, \quad \text{a.e. } t \text{ in } (0, \tau).$$

For any $t \in (0, \tau)$, we can choose a $t_0 \in (0, t)$ such that

$$\lim_{n \rightarrow \infty} \|S(t_0)(v_{0,n}, T_{0,n}, q_{0,n}) - (v_*(t_0), T_*(t_0), q_*(t_0))\|_1 = 0.$$

Therefore, for any $t > 0$, by the continuity of $S(t)$ in V , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|S(t)(v_{0,n}, T_{0,n}, q_{0,n}) - S(t-t_0)(v_*(t_0), T_*(t_0), q_*(t_0))\|_1 \\ &= \lim_{n \rightarrow \infty} \|S(t-t_0)S(t_0)(v_{0,n}, T_{0,n}, q_{0,n}) - S(t-t_0)(v_*(t_0), T_*(t_0), q_*(t_0))\|_1 = 0, \end{aligned}$$

which implies that $S(t)$ is a compact map in V for any $t > 0$. Then by the above argument of this section and Theorem 6.1, we obtain the existence of the global attractor for the moist primitive Equations (1.11)-(1.17). \square

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REFERENCES

- [1] J. Aubin, *Un théorème de compacité*, C.R. Acad. Sci. Paris, 256:5042–5044, 1963.
- [2] V. Bjerknes, *Das Problem der Wettervorhersage betrachtet vom Standpunkte der Mechanik und der Physik*, Meteorol. Z., 21:1–7, 1904.
- [3] A. J. Bourgeois and J. T. Beale, *Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean*, SIAM J. Math. Anal., 25:1023–1068, 1994.
- [4] A. Babin and M. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.
- [5] D. Cordoba, *Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation*, Ann. of Math., 148:1135–1152, 1998.
- [6] C. Cao, S. Ibrahim, K. Nakanishi, and E. S. Titi, *Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics*, Comm. Math. Phys., 337:473–482, 2015.
- [7] C. Cao, J. Li, and E. S. Titi, *Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity*, Arch. Anal. Ration. Mech., 214:35–76, 2014.
- [8] C. Cao, J. Li, and E. S. Titi, *Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity*, J. Diff. Eqs., 257:4108–4132, 2014.
- [9] C. Cao, J. Li, and E. S. Titi, *Global well-posedness for the 3D primitive equations with only horizontal viscosity and diffusion*, Commun. Pure Appl. Math., 69(8):1492–1531, 2016.
- [10] P. Constantin, A. Majda, and E. Tabak, *Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar*, Nonlinearity, 7:1495–1533, 1994.
- [11] P. Constantin, A. Majda, and E. Tabak, *Singular front formation in a model for quasigeostrophic flow*, Phys. Fluids, 6:9–11, 1994.

- [12] C. Cao and E. S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. Math., 166:245–267, 2007.
- [13] C. Cao and E. S. Titi, *Global well-posedness of the three-dimensional primitive equations with partial vertical turbulence mixing heat diffusion*, Commun. Math. Phys., 310:537–568, 2012.
- [14] V. Chepyzhov and M. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society Colloquium Publications, Amer. Math. Soc., Providence, RI, 49, 2002.
- [15] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal., 30:937–948, 1999.
- [16] P. F. Embid and A. J. Majda, *Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity*, Comm. Part. Diff. Eqs., 21:619–658, 1996.
- [17] C. Foias and G. Prodi, *Sur le comportement global des solutions non-stationnaires deséquations de Navier-Stokes en dimension 2*, Rend. Sem. Mat. Univ. Padova, 39:1–34, 1967.
- [18] A. E. Gill, *Atmosphere–Ocean Dynamics*, (International Geophysics Series, San Diego, CA: Academic, 30, 1982.
- [19] B. Guo and D. Huang, *Existence of weak solutions and trajectory attractors for the moist atmospheric equations in geophysics*, J. Math. Phys., 47:083508, 2006.
- [20] B. Guo and D. Huang, *Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere*, J. Diff. Eqs., 251:457–491, 2011.
- [21] F. Guillén–González, N. Masmoudi, and M. A. Rodríguez–Bellido, *Anisotropic estimates and strong solutions of the primitive equations*, Diff. Integral Eq., 14:1381–1408, 2001.
- [22] J. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, 1988.
- [23] G. J. Haltiner, *Numerical Weather Prediction*, New York: John Wiley, 1971.
- [24] G. J. Haltiner and R. T. Williams, *Numerical Prediction and Dynamic Meteorology*, New York: John Wiley, 1980.
- [25] J. R. Holton, *An Introduction to Dynamic Meteorology*, Third Edition, Academic Press, 1992.
- [26] C. Hu, R. Temam, and M. Ziane, *The primitive equations on the large scale ocean under the small depth hypothesis*, Discrete Contin. Dyn. Syst., 9:97–131, 2003.
- [27] N. Ju, *The global attractor for the solutions to the 3D viscous primitive equations*, Discrete Contin. Dyn. Syst., 17:159–179, 2007.
- [28] N. Ju and R. Temam, *Finite dimensions of the global attractor for 3D viscous primitive equations with viscosity*, J. Nonlinear Sci., 25:131–155, 2015.
- [29] G. M. Kobelkov, *Existence of a solution ‘in the large’ for the 3D large-scale ocean dynamics equations*, C.R. Math. Acad. Sci. Paris, 343:283–286, 2006.
- [30] G. M. Kobelkov, *Existence of a solution “in the large” for ocean dynamics equations*, J. Math. Fluid Mech., 9:588–610, 2007.
- [31] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity, 20:2739–2753, 2007.
- [32] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, 1991.
- [33] J. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.
- [34] J. Li and J. Chou, *Asymptotic behavior of solutions of the moist atmospheric equations*, Acta Meteor. Sinica, 56:61–72, 1998.
- [35] J. Lions and B. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer–Verlag, New York, 1972.
- [36] J. L. Lions, R. Temam, and S. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity, 5:237–288, 1992.
- [37] J. L. Lions, R. Temam, and S. Wang, *On the equations of the large-scale ocean*, Nonlinearity, 5:1007–1053, 1992.
- [38] J. L. Lions, R. Temam, and S. Wang, *Models of the coupled atmosphere and ocean*, Comput. Mech. Adv., 1:1–54, 1993.
- [39] A. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lect. Notes Math., 9, 2003.
- [40] A. Miranville and R. Temam, *Mathematical Modeling in Continuum Mechanics*, Cambridge University Press, 2005.
- [41] J. Pedlosky, *Geophysical Fluid Dynamics*, Second Edition, Springer-Verlag, Berlin/New York, 1987.
- [42] M. Petcu, R. M. Temam, and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handb. Numer. Anal., 14:577–750, 2009.
- [43] L. F. Richardson, *Weather Prediction by Numerical Process*, Cambridge University, 2007.
- [44] J. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, Cambridge, 2001.

- [45] G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer-Verlag, New York, 2002.
- [46] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Second Edition, Springer Verlag, 1988, 1997.
- [47] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of Mathematical Fluid Dynamics, 3:535–658, 2005.
- [48] S. Wang, *Attractors for the 3-D baroclinic quasi-geostrophic equations of large-scale atmosphere*, J. Math. Anal. Appl., 165:266–283, 1992.
- [49] J. Wang, *Global solutions of the 2D dissipative quasi-geostrophic equations in Besov spaces*, SIAM J. Math. Anal., 36:1014–1030, 2004.
- [50] J. Wang, *The two-dimensional quasi-geostrophic equation with critical or supercritical dissipation*, Nonlinearity, 18:139–154, 2005.
- [51] M. C. Zelati, A. Huang, L. Kukavica, R. Teman, and M. Ziane, *The primitive equations of the atmosphere in presence of vapour saturation*, Nonlinearity, 28:625–668, 2015.