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A FIRST-ORDER REDUCTION OF THE CUCKER-SMALE MODEL ON THE REAL LINE AND ITS CLUSTERING DYNAMICS*

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Abstract. We present a first-order reduction for the Cucker-Smale (C-S) model on the real line, and discuss its clustering dynamics in terms of spatial configurations and system parameters. In previous literature, flocking estimates for the C-S model were mainly focused on the relaxation dynamics of the particle's velocities toward the common velocity. In contrast, the relaxation dynamics of spatial configurations was treated as a secondary issue except for the uniform boundedness of the spatial diameter. In this paper, we first derive a first-order system for the spatial coordinate that can be rewritten as a gradient flow, and then use this first-order formulation to derive several sufficient conditions on the clustering dynamics based on the spatial positions depending on the natural velocities characterized by initial position-velocity configurations.

Keywords. Clustering; collective dynamics; Cucker-Smale model; flocking; synchronization.

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1. Introduction

Collective behaviors of complex systems are ubiquitous in our nature, to name a few, the aggregation of bacteria, flocking of birds, and swarming of fish in biological systems, herding of volatilities in financial markets and formation of dominant opinions in social systems [16–18, 38, 45, 48–51] etc. Despite its universal appearance in our daily lives, systematic studies began only about half century ago. In [37, 51], Winfree and Kuramoto introduced continuous-time dynamical systems for synchronization. The research on the collective dynamics became popularized in the physics community due to Vicsek's work [50]. Recently, several mechanical models were proposed for the collective motions of multi-agent systems in the engineering community due to the applications of unmanned aerial vehicles and sensor networks [38,44,45] etc. Among them, our main interest lies in the Newton-like system on the real line introduced by Cucker and Smale in [15]. Next, we briefly describe the second-order C-S model. Let $(x_i,v_i)\in\mathbb{R}^2$ be the spatial position-velocity coordinate of the i-th particle. Then, the dynamics of (x_i,v_i) is governed by the C-S model:

$$\begin{aligned} \dot{x}_i &= v_i, \quad t>0, \quad i=1,2,\cdots,N, \\ \dot{v}_i &= \frac{\kappa}{N}\sum_{k=1}^{N}\psi(x_k-x_i)\big(v_k-v_i\big), \end{aligned} \tag{1.1}$$

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where κ is a positive coupling strength, and ψ is a communication weight satisfying parity, positivity, continuity and monotonicity conditions: there exists a positive constant $\psi_0 > 0$ such that

$$\begin{aligned} &\psi(-r) = \psi(r), \quad 0 \le \psi(r) \le \psi_0, \quad \forall\, r \in \mathbb{R}, \quad \psi(\cdot) \in \mathrm{Lip}(\mathbb{R};\mathbb{R}_+), \\ &\psi \text{ is monotonically decreasing (increasing) for } r \ge 0 \ (r \le 0). \end{aligned} \tag{1.2}$$

The C-S model (1.1)-(1.2) has been extensively studied in flocking literature [1,2,5,6,8–15,21–23,26–29,39,43,46,47] from diverse perspectives. In aforementioned literature, sufficient frameworks for mono-cluster and multi-cluster flockings were proposed in terms of spatial-velocity configurations (see Theorem 2.1 for the emergence of mono-cluster flocking). Since the C-S model (1.1)-(1.2) is translation invariant, any translational motions are solutions (relative equilibria) of (1.1)-(1.2). As aforementioned, all literature were mainly focused on the velocity configurations. In this paper, we shift our attention from the velocity dynamics to the spatial dynamics, and then study the spatial configurations in the asymptotic flocking state resulting from initial configurations. This is the contrasted difference between this work and all aforementioned literature on the C-S model. In previous literature, even if the mono-cluster and multi-cluster flocking occur asymptotically, we do not have much information on the spatial patterns in relative equilibria (flocking state) emerging from the given initial data. Thus, in this paper, we are interested in the relative equilibria resulting from given initial data.

The main results of this paper are two-fold. First, we present a first-order formulation for the spatial configuration from the second-order model (1.1). In this first-order formulation, we can study more detailed dynamics for the spatial position of the C-S particles. Let $\{(x_i, v_i)\}$ be the ensemble of the C-S spatial-velocity configuration whose dynamics is governed by system (1.1). Then, we show that the dynamics of the spatial position x_i on the real line is governed by the gradient flow (see Section 2.2). More precisely, we first show that the spatial configuration x_i satisfies

$$\dot{x}_i = \Omega_i + \frac{\kappa}{N} \sum_{k=1}^{N} \Psi(x_k - x_i), \quad t > 0, \quad i = 1, \cdots, N, \tag{1.3}$$

where $\Omega_i = \Omega_i(X^0, V^0)$ and Ψ are the natural velocity of the i-th particle and anti-derivative of ψ, respectively:

$$\Omega_i(X^0, V^0) := v_i^0 - \frac{\kappa}{N} \sum_{k=1}^{N} \Psi(x_k^0 - x_i^0), \qquad \Psi(r) := \int_0^r \psi(y)\, dy. \tag{1.4}$$

Then, it is easy to see that the first-order system (1.3)-(1.4) can be rewritten as a gradient flow on $\mathbb{R}^N$ with a scalar potential ϕ:

$$\begin{cases} \dot{X} = -\nabla_X \phi(X), \quad t > 0, \\ \phi(X) := -\displaystyle\sum_{k=1}^{N} \Omega_k x_k + \frac{\kappa}{2N} \sum_{i,k=1}^{N} \int_0^{x_i - x_k} \Psi(y)\, dy, \end{cases} \tag{1.5}$$

where X and V are position and velocity vectors in $\mathbb{R}^N$:

$$X := (x_1, \cdots, x_N) \quad \text{and} \quad V := (v_1, \cdots, v_N).$$

Second, we present clustering dynamics of the first-order model (1.3) depending on the effective range of the communication function ψ in the large and intermediate coupling strength regimes (see Theorem 4.1, Proposition 5.1, Theorem 5.1). For the long-range case with $\|\psi\|_{L^1(\mathbb{R}_+)} = \infty$, the second-order model (1.1) exhibits mono-cluster(global) flocking for any initial data. In Theorem 4.1, we will show that asymptotic spatial configuration of (1.1) with zero spatial and velocity sum conditions will be a unique solution of the equilibrium system:

$$\Omega_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(x_k^\infty - x_i^\infty) = 0, \qquad \sum_{k=1}^{N} x_k^\infty = 0.$$

During the spatial relaxation process, particles with different natural velocities will undergo at most one collision, and the distances between particles having the same natural velocities will decrease to zero asymptotically. Thus, the resulting asymptotic spatial configuration will be arranged according to the order of corresponding natural velocities. In other words, initial configuration will determine natural velocities via the relation (1.4). Then, the asymptotic spatial configuration will be arranged by the order of natural velocities. On the other hand, for a short-range case with $\|\psi\|_{L^1(\mathbb{R}_+)} = \infty$, depending on the initial data, multi-cluster flocking can emerge asymptotically (see [11,12]). Thus, the asymptotic spatial configuration would be more delicate and difficult to imagine only with initial data. In Proposition 3.1, we will show that how the natural velocities affect the collisions in the spatial relaxation process. In particular, the particles with the same natural velocities will approach so that the distance between particles decreases to zero exponentially fast. In Theorem 5.1, we will provide a sufficient and necessary condition for the complete segregation. See Section 3.2 for related results on other first-order models.

The rest of this paper is organized as follows. In Section 2, we study the relative equilibria of the C-S model. In particular, we present an explicit dynamics for the C-S model with all-to-all coupling $\psi \equiv 1$. In Section 3, we present a gradient flow formulation for the C-S model which is the first-order system for the position, and briefly discuss the relations with other first-order consensus models. In Section 4, we study the structure of the relative equilibria for long-range communications. In Section 5, we present the structure of the relative equilibria for short-range communications. Finally, Section 6 is devoted to a brief summary of our paper and discussion of future directions.

2. Preliminaries

In this section, we study the basic estimates for the C-S model (1.1) such as associated conservation laws, relative equilibria and clustering dynamics of the first-order model (1.3) for the all-to-all coupling $\psi \equiv 1$. First, we briefly discuss the relative equilibria for the C-S model.

Lemma 2.1. *Let* $\{(x_i, v_i)\}$ *be a global solution to* (1.1)-(1.2). *Then, we have the following assertions.*

(1) The C-S model (1.1) *is Galilean invariant in the sense that it is invariant under the Galilean transformation: for any* $c \in \mathbb{R}$*:*

$$(x_i,\ v_i) \to (x_i + tc,\ v_i + c).$$

(2) The total momentum is conserved and the average position moves with constant velocity:

$$\sum_{j=1}^N v_i(t) = \sum_{j=1}^N v_i^0, \qquad \sum_{j=1}^N x_i(t) = \sum_{j=1}^N x_i^0 + t\sum_{j=1}^N v_i^0, \quad t\geq 0.$$

Proof. Since the right-hand side of (1.1) is given by the relative distances and velocities, i.e., $x_i - x_j$ and $v_i - v_j$, system (1.1) is clearly invariant under the Galilean transformation. For the second assertion, we add $(1.1)_2$ with respect to i and use the evenness of the communication function ψ to derive the desired result. ∎

REMARK 2.1. Conservation of total momentum and relation (1.4) yield

$$\sum_{i=1}^N \Omega_i = \sum_{i=1}^N v_i^0.$$

Note that the equilibrium (X^∞, V^∞) for (1.1) corresponds to the state with zero velocity vector:

$$v_i^\infty = 0, \quad i = 1, \cdots, N.$$

Thus, the traveling state with nonzero common velocity $v^\infty = c \neq 0$ is not an equilibrium for (1.1). Hence, we need to relax the concept of equilibrium as in the N-body system in celestial mechanics.

DEFINITION 2.1. *Let* (X^∞, V^∞) *be a relative equilibrium for* (1.1)-(1.2) *if it can be represented by the following relations: there exist constant vectors* $\tilde{X} := (\tilde{x}_1, \cdots, \tilde{x}_N)$ *and* $\tilde{V} = \tilde{v}(1, \ldots, 1)$ *in* $\mathbb{R}^N$ *such that*

$$X^\infty = \tilde{X} + t\tilde{V}, \qquad V^\infty = \tilde{V}, \quad t\in\mathbb{R}.$$

Next, we study the structure of the relative equilibria for system (1.1) with $\psi \equiv 1$. Consider the C-S model with $\psi\equiv 1$. In this case, the C-S model becomes

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{\kappa}{N}\sum_{k=1}^N (v_k - v_i), \quad t>0. \tag{2.1}$$

Then, its corresponding first-order model is given by

$$\dot{x}_i = \Omega_i + \frac{\kappa}{N}\sum_{k=1}^N (x_k - x_i), \quad \text{where} \quad \Omega_i := v_i^0 - \frac{\kappa}{N}\sum_{k=1}^N (x_k^0 - x_i^0). \tag{2.2}$$

We set the spatial and velocity differences:

$$x_{ij} := x_i - x_j, \quad \Omega_{ij} := \Omega_i - \Omega_j.$$

Then, it is easy to see that x_{ij} satisfies

$$\dot{x}_{ij} = \Omega_i - \Omega_j + \frac{\kappa}{N}\sum_{k=1}^N (x_k - x_i) - \frac{\kappa}{N}\sum_{k=1}^N (x_k - x_j) = \Omega_{ij} - \kappa x_{ij}. \tag{2.3}$$

This yields

$$x_{ij}(t) = \frac{\Omega_{ij}}{\kappa} + \Big(x_{ij}^0 - \frac{\Omega_{ij}}{\kappa}\Big)e^{-\kappa t}, \quad t>0.$$

Thus, we have the following equivalence:

$$\begin{aligned}&\text{the existence of a finite-time collision between } x_i \text{ and } x_j \text{ at } t=t_*\\ \Longleftrightarrow\quad & t_* = -\frac{1}{\kappa}\ln\Big(\frac{\Omega_{ij}}{\Omega_{ij}-x_{ij}^0\kappa}\Big)\ge 0.\end{aligned} \tag{2.4}$$

Below, we are interested in the formation and the asymptotic distribution of the equilibrium state.

PROPOSITION 2.1. *Let X be a solution to system (2.2) with initial data X^0. For $i,j\in\{1,\cdots,N\}$ with $x_j^0<x_i^0$, the following assertions hold.*

(1) If $\Omega_i<\Omega_j$, the i-th particle and j-th particle will collide exactly once.

(2) If $\Omega_j=\Omega_i$, the asymptotic positions of i and j-particles coincide.

$$\lim_{t\to+\infty}(x_i(t)-x_j(t))=0.$$

Proof.
(i) Suppose that $\Omega_j>\Omega_i$, i.e., $\Omega_{ij}<0$. Note that as long as

$$x_i(t_*)\ge x_j(t_*),\quad \text{i.e.,}\quad x_{ij}(t_*)\ge 0.$$

we have

$$\dot{x}_{ij}(t_*)=\Omega_{ij}-\kappa x_{ij}(t_*)<0.$$

Thus, x_i-x_j will keep decreasing and collide in finite time, i.e., $x_i(t_*)=x_j(t_*)$ for some instant t_*. After the instant t_*, the right side is still negative and thus the distance between the two particles will decrease until $x_j-x_i=\frac{\Omega_j-\Omega_i}{\kappa}$, which is the equilibrium state. Then, it is obvious that they will collide exactly once before they reach the equilibrium state.

(ii) It follows from (2.3) that we have

$$\dot{x}_{ij}=-\kappa x_{ij},\quad t\ge 0.$$

This yields

$$x_{ij}(t)=x_{ij}^0e^{-\kappa t},\quad t\ge 0.$$

Thus, we have

$$\lim_{t\to\infty}x_{ij}(t)=0.$$

□

PROPOSITION 2.2. *Let (X,V) be a solution to system (2.1) with initial data (X^0,V^0) satisfying*

$$\sum_{k=1}^N x_k^0=0,\quad \sum_{k=1}^N v_k^0=0.$$

If there exists a relative equilibrium X^∞ and V^∞ such that

$$\lim_{t\to\infty}(X(t),V(t))=(X^\infty,V^\infty),$$

then, the limit position and velocity vectors are given by the following explicit relations:

$$x_i^\infty=\frac{\Omega_i}{\kappa}\quad \textit{and}\quad v_i^\infty=0,\quad 1\le i\le N.$$

Proof.
(i) We set

$$v^\infty = v_i^\infty, \quad 1 \le i \le N.$$

Then, it follows from Lemma 2.1 that we have

$$0 = \sum_{k=1}^N v_k^0 = \lim_{t\to\infty} \sum_{k=1}^N v_k(t) = N v^\infty.$$

This yields

$$v^\infty = 0.$$

(ii) Again, it follows from Lemma 2.1 that we have

$$\sum_{k=1}^N x_k^0 = \sum_{k=1}^N x_k^\infty = 0.$$

On the other hand, it follows from Definition 2.1 that

$$0 = \Omega_i + \frac{\kappa}{N} \sum_{k=1}^N (x_k^\infty - x_i^\infty) = \Omega_i - \kappa x_i^\infty.$$

This yields the desired estimate. ∎

Next, we summarize asymptotic flocking estimate for system (1.1) as follows. For $X = (x_1, \cdots, x_N)$, $V = (v_1, \cdots, v_N) \in \mathbb{R}^N$, we set

$$\mathcal{D}_X(t) := \max_{1\le i,j\le N} |x_i - x_j|, \quad \mathcal{D}_V(t) := \max_{1\le i,j\le N} |x_i - x_j|.$$

By slight modification of the arguments in [27, 43], asymptotic flocking estimate can be provided in terms of $\mathcal{D}_X$ and $\mathcal{D}_V$ as follows.

THEOREM 2.1. [24, 27, 43] *Suppose that the coupling strength and initial data satisfy*

$$\mathcal{D}_V(0) < \kappa \int_{\mathcal{D}_X(0)}^\infty \psi(s) ds. \tag{2.5}$$

Then, for any solution $\{(x_i, v_i)\}$ to (1.1), there exists a positive constant $\mathcal{D}_x^\infty$ such that

$$\sup_{0\le t<\infty} \mathcal{D}_X(t) \le \mathcal{D}_x^\infty \quad \text{and} \quad \mathcal{D}_V(t) \le \mathcal{D}_V(0) e^{-\kappa\psi(\mathcal{D}_x^\infty)t}, \quad t > 0,$$

where $\mathcal{D}_x^\infty$ is uniquely determined by the relation:

$$\int_{\mathcal{D}_X(0)}^{\mathcal{D}_x^\infty} \psi(s) ds = \frac{\mathcal{D}_V(0)}{\kappa}.$$

Proof. For the proof, we refer to Theorem 2.1 in [24]. ∎

REMARK 2.2. Note that the proof of Theorem 2.1 is based on the differential inequalities:

$$\left|\frac{d}{dt}\mathcal{D}_X(t)\right| \le \mathcal{D}_V(t), \qquad \frac{d}{dt}\mathcal{D}_V(t) \le -\kappa\psi\Big(\mathcal{D}_X(t)\Big)\mathcal{D}_V(t), \quad \text{a.e.} \quad t>0.$$

Several sufficient conditions have also been proposed in earlier works [1,15,27,29] in ℓ_2-framework. The condition (2.5) is sufficient for the mono-cluster flocking. Thus, if initial data (X^0,V^0) do not satisfy this sufficient condition, there might be the emergence of local flockings as noticed in [11,12].

3. The first-order reduction of the C-S model

In this section, we discuss an equivalence between the second-order model (1.1) and the first-order system (1.5).

3.1. A gradient-flow formulation. Let $x_i = x_i(t)$ be a quantifiable state of the i-th agent exhibiting a flocking dynamics with a natural velocity Ω_i:

$$\begin{cases} \dot{x}_i = \Omega_i + \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^N \Psi(x_k - x_i), \quad \Psi(r) = \displaystyle\int_0^r \psi(y)\,dy \\ x_i(0) = x_i^0, \end{cases} \tag{3.1}$$

where $(\Omega_1,\cdots\Omega_N)$ is the natural velocity vector. In order to associate (3.1) and a second-order system, we set

$$v_i := \Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i), \quad i=1,\cdots,N. \tag{3.2}$$

Consider a second-order system:

$$\begin{cases} \dot{x}_i = v_i, \quad t>0, \quad i=1,\cdots,N, \\ \dot{v}_i = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^N \psi(x_k - x_i)(v_k - v_i), \\ (x_i(0),v_i(0)) = (x_i^0, v_i^0), \end{cases} \tag{3.3}$$

In the sequel, we discuss the relation between the first-order model (3.1) and the second-order model (3.3).

THEOREM 3.1. *The following assertions hold.*

(1) The first-order model (3.1) *and the second-order model* (3.3) *are equivalent.*

(2) The system (3.1) *can be rewritten as a gradient flow:*

$$\dot{X} = -\nabla_X \phi(X), \qquad \text{where} \quad \phi(X) := -\sum_{i=1}^N \Omega_i x_i + \frac{\kappa}{2N}\sum_{i,j}\int_0^{x_i - x_j} \Psi(y)dy.$$

Proof.

(i) • (From (3.1) to (3.3)): Suppose that x_i is a solution to (3.1). Then, we use the relation (3.2) by letting $t \to 0+$:

$$v_i^0 := \Omega_i + \frac{\kappa}{N}\sum_{j=1}^N \Psi(x_j^0 - x_i^0).$$

We differentiate the equation (3.1) with respect to t to obtain (3.3):

$$\dot{v}_i = \frac{d}{dt}\Big(\Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i)\Big) = \frac{\kappa}{N}\sum_{k=1}^N \psi(x_k - x_i)(v_k - v_i).$$

• (From (3.3) to (3.1)): Note that the equation $(2.4)_2$ can be rewritten in terms of x_i:

$$\frac{d^2 x_i}{dt^2} = \frac{\kappa}{N}\sum_{k=1}^N \psi(x_k - x_i)(v_k - v_i) = \frac{d}{dt}\Big(\frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i)\Big).$$

Next, we integrate the above relation with respect to t to obtain

$$\frac{dx_i}{dt} = v_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k^0 - x_i^0) + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i).$$

Thus, we set

$$\Omega_i := v_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k^0 - x_i^0)$$

to derive the system (3.1).
(ii) Define a potential function ϕ as follows.

$$\phi(X) := -\sum_{i=1}^N \Omega_i x_i + \frac{\kappa}{2N}\sum_{i,k}\int_0^{x_i - x_k} \Psi(y)\,dy.$$

For given $j \in \{1,\dots,N\}$, we can separate this relation into two parts:

$$\begin{aligned}\phi(X) = &\Big[-\Omega_j x_j + \frac{\kappa}{2N}\sum_{k=1}^N \int_0^{x_j - x_k}\Psi(y)\,dy + \frac{\kappa}{2N}\sum_{i=1}^N \int_0^{x_i - x_j}\Psi(y)\,dy\Big] \\ &+ \Big[-\sum_{i\neq j}\Omega_i x_i + \frac{\kappa}{2N}\sum_{i,k\neq j}\int_0^{x_i - x_k}\Psi(y)\,dy\Big].\end{aligned}$$

We take a partial derivative with respect to x_j, and then the second term vanishes and we obtain

$$\begin{aligned}\frac{\partial \phi}{\partial x_j} &= -\Omega_j + \frac{\kappa}{2N}\sum_{k=1}^N \Psi(x_j - x_k) - \frac{\kappa}{2N}\sum_{i=1}^N \Psi(x_i - x_j) \\ &= -\Omega_j - \frac{\kappa}{N}\sum_{i=1}^N \Psi(x_i - x_j) = -\dot{x}_j.\end{aligned}$$

Thus, the system (3.1) has a gradient flow structure:

$$\dot{X} = -\nabla_X \phi(X).$$

□

We next present a property of collisions between particles.

PROPOSITION 3.1. *Let (X,V) be a solution to (1.1) with initial data (X^0,V^0). Suppose that the initial positions of the i-th and j-th particles satisfies*

$$x_i^0 < x_j^0.$$

Then the following trichotomy holds.

(1) If $\Omega_i < \Omega_j$, then, x_i and x_j will never collide in finite time:

$$|\{t_* \in (0,\infty) \ : \ x_i(t_*) = x_j(t_*)\}| = 0.$$

(2) If $\Omega_i > \Omega_j$, then, x_i and x_j will collide at most once:

$$|\{t_* \in (0,\infty) \ : \ x_i(t_*) = x_j(t_*)\}| \le 1.$$

(3) If $\Omega_i = \Omega_j$, then, the relative distance $|x_i - x_j|$ satisfies

$$|x_i^0 - x_j^0| e^{-\kappa\psi_0 t} \le |x_i(t) - x_j(t)| \le |x_i^0 - x_j^0| \exp\Big[-\frac{\kappa}{N}\psi(|x_i^0 - x_j^0|)t\Big], \quad t \ge 0.$$

Proof.
(i) Suppose that $\Omega_i < \Omega_j$. Then, we claim that there are no collisions between x_j and x_i:

$$x_i(t) < x_j(t), \quad t \in [0,\infty).$$

Suppose not, i.e., there does exist $t_* \in (0,\infty)$ such that

$$x_i(t) < x_j(t), \quad t \in (0,t_*) \quad \text{and} \quad x_i(t_*) = x_j(t_*). \tag{3.4}$$

Note that $x_j - x_i$ at $t = t_*$ satisfies

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_*}(x_j - x_i) &= \Omega_j - \Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Big[\Psi(x_k(t_*) - x_j(t_*)) - \Psi(x_k(t_*) - x_i(t_*))\Big] \\ &= \Omega_j - \Omega_i > 0. \end{aligned} \tag{3.5}$$

Thus, $x_j - x_i$ is in a strict increasing mode at $t = t_*$, i.e., due to the continuity of $\dot{x}_j - \dot{x}_i$, there exists $\delta > 0$ such that

$$x_j(t) - x_i(t) < 0, \quad \text{i.e.,} \quad x_j(t) < x_i(t), \qquad t \in (t_* - \delta, t_*).$$

This contradicts to (3.4). Thus, j-th particle will be always behind i-th particle.
(ii) Suppose that $\Omega_i > \Omega_j$. Then, we claim that there will be at most one collision if they exist. If there are no collisions, we are done. Thus, it remains that there can be at most one collision. Suppose that there exists a positive constant $t_* \in (0,\infty)$ such that

$$x_i(t) < x_j(t), \quad t \in (0,t_*) \qquad \text{and} \qquad x_i(t_*) = x_j(t_*).$$

Then, by the same argument as in (3.5), we have

$$\frac{d}{dt}\Big|_{t=t_*}(x_j - x_i) = \Omega_j - \Omega_i < 0.$$

Thus, we have $\delta > 0$ such that

$$x_j(t) < x_i(t) \quad \text{for} \quad t \in (t_*, t_* + \delta). \tag{3.6}$$

Then, the relation (3.6) and $\Omega_j < \Omega_i$ implies that the situation is returned to the first case, and we do not have collision afterward.

(iii) By applying the results of (i) and (ii), without loss of generality, we may assume

$$\{x_k \mid x_i < x_k < x_j\} = \emptyset,$$

i.e., we can assume that x_i and x_j are two particles which are immediately next to each other. Then, we use the mean value theorem for Ψ to obtain

$$\begin{aligned}\frac{d}{dt}|x_i - x_j| &= \frac{\kappa}{N}\sum_{k=1}^{N} \mathrm{sgn}(x_i - x_j)\Big(\Psi(x_k - x_i) - \Psi(x_k - x_j)\Big)\\ &= -\frac{\kappa}{N}\sum_{k=1}^{N}\psi(x^*_{k,ij})|x_i - x_j|, \end{aligned} \tag{3.7}$$

where $x^*_{k,ij}$ is the value between $x_k - x_i$ and $x_k - x_j$. Thus, the relation (3.7) implies that the relative distance $|x_i - x_j|$ is non-increasing in time:

$$|x_i(t) - x_j(t)| \le |x_i^0 - x_j^0|, \quad t \ge 0. \tag{3.8}$$

• Case A (upper bound estimate): We use (3.8) to obtain

$$\sum_{k=1}^{N}\psi(x^*_{k,i,j}) \ge \psi(x^*_{i,i,j}) \ge \psi(|x_i - x_j|) \ge \psi(|x_i^0 - x_j^0|) > 0.$$

Now, we use the above relation and (3.7) to see

$$\frac{d}{dt}|x_i - x_j| \le -\frac{\kappa}{N}\psi(|x_i^0 - x_j^0|)|x_i - x_j|.$$

This leads to

$$|x_i(t) - x_j(t)| \le |x_i^0 - x_j^0| \exp\Big(-\frac{\kappa}{N}\psi(|x_i^0 - x_j^0|)t\Big), \quad t \ge 0.$$

• Case B (lower bound estimate): We use the upper bound ψ_0 for ψ to find

$$\frac{d}{dt}|x_i - x_j| \ge -\kappa\psi_0|x_i - x_j|, \quad t > 0.$$

This implies

$$|x_i(t) - x_j(t)| \ge |x_i^0 - x_j^0| e^{-\kappa\psi_0 t}, \quad t \ge 0.$$

□

As a direct application of Proposition 3.1, we have the following results.

COROLLARY 3.1. *Let (X,V) be a solution to (1.1) with initial data (X^0,V^0).*

(1) If $x_i^0 = x_j^0$ and $\Omega_i = \Omega_j$, x_i and x_j will stay together:

$$x_i(t) = x_j(t), \quad t \in [0,\infty).$$

(2) If $x_i^0 \ne x_j^0$ and $\Omega_i = \Omega_j$, the relative distance $|x_i - x_j|$ will converge to zero exponentially fast, and they will not collide in finite time.

3.2. A unified platform for collective dynamics. In this subsection, we present a unified platform for the first-order modeling on the collective dynamics such as aggregation, synchronization and opinion consensus depending on the choices of dynamic quantity and defining relation of interaction forces. Consider a finite ensemble of particles on a given manifold $\mathcal{M}$, and let $q_i = q_i(t)$ be the generalized coordinate of the i-th particle exhibiting collective dynamics. Then, we propose a governing system for q_i by the first-order system:

$$\dot{q}_i = \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} F(q_k, q_i), \quad q_i \in \mathcal{M}, \quad t > 0, \tag{3.9}$$

where $F(q_k, q_i)$ represents an *attractive* interaction force between the k-th and i-th particles, which is responsible for a collective group dynamics. Below, we show that some models arising from the aggregation, flocking and synchronization can be put into a special form of (3.9).

Example 1 (Inviscid Keller-Segel model): Consider a group of bacteria moving on the plane $\mathbb{R}^2$ and assume that they are attracted by the chemotactic substances, and let $\rho = \rho(x,t)$ and $S = S(x,t)$ be the local mass densities of bacteria and chemotactic substances, respectively. Then, one of the Keller-Segel-type models [35, 36] is given by the coupled parabolic-elliptic system:

$$\partial_t \rho + \nabla \cdot (\rho \nabla S) = \sigma \Delta \rho, \quad -\Delta S = \rho, \tag{3.10}$$

and its corresponding stochastic particle system can be given as follows.

$$dx_i(t) = \frac{\kappa}{N}\sum_{k \neq i} \frac{x_k(t) - x_i(t)}{|x_k(t) - x_i(t)|^2} dt + \sqrt{2\sigma} dB_i(t), \tag{3.11}$$

where $B_i(t)$ is the standard Brownian motion on $\mathbb{R}^2$. Thus, the coupled system (3.10) can be obtained as a mean-field limit of the stochastic particle system (3.11). In the absence of noise $\sigma = 0$, system (3.11) becomes

$$\dot{x}_i = \frac{\kappa}{N}\sum_{k \neq i} \frac{x_k - x_i}{|x_k - x_i|^2}.$$

This corresponds to (3.9) with the following special choices:

$$\mathcal{M} = \mathbb{R}^2, \quad \nu_i = 0, \quad F(q_k, q_i) = \frac{q_k - q_i}{|q_k - q_i|^2}.$$

Example 2 (The Cucker-Smale model on the line): As we have seen in Section 3.1, the Cucker-Smale model on the line can be written as the first-order model:

$$\dot{x}_i = \Omega_i + \frac{\kappa}{N}\sum_{k=1}^{N} \Psi(x_k - x_i), \quad \Psi(r) = \int_0^r \psi(y)\, dy.$$

Thus, the C-S model on the line can be put into the form of (3.9) with the following special choices:

$$\mathcal{M} = \mathbb{R}, \quad \nu_i = \Omega_i, \quad F(q_k, q_i) = \int_0^{q_k - q_i} \psi(y) dy.$$

Example 3 (The Kuramoto model): As one of the well-known synchronization models, the Kuramoto model exhibits the synchronous dynamics of the phase-coupled oscillator. Let θ_i be the phase of the i-th oscilltor. Then, the Kuramoto model reads as follows.

$$\dot{\theta}_i = \Omega_i + \frac{\kappa}{N}\sum_{k=1} \sin(\theta_k - \theta_i), \tag{3.12}$$

where Ω_i is the natural frequency of the i-th oscillator. Then, this can be put into the form (3.9) with the following special choices:

$$\mathcal{M} = \mathbb{S}^1, \quad \nu_i = \Omega_i, \quad F(q_k, q_i) = \sin(q_k - q_i).$$

For the emergent dynamics of (3.12), we refer to recent survey articles [20, 25] and references therein. Note that other than the above three examples, some consensus models [32, 34, 42], particle models for clustering dynamics [3, 19, 30, 41, 42] can also be put into the form (3.9) with suitable $q, \mathcal{M}$ and F as well. Recently, Jabin and Motsch [32] investigated the formation of mono-cluster and multi-clusters using the consensus model with non-symmetric and regular communication weight.

In the following two sections, we study the structure of relative equilibria which emerges from initial data depending on the integrability of ψ:

$$\int_0^\infty \psi(y)\,dy < \infty \quad \text{and} \quad \int_0^\infty \psi(y)\,dy = \infty.$$

We call the former and latter cases as short-range and long-range communications, respectively.

4. Relative equilibria I: Long-range communication

In this section, we study the relative equilibria for the Cucker-Smale model (1.1) with $\int_0^\infty \psi(r)\,dr = \infty$. Let (x_i, v_i) be a solution to the system (1.1). Then, it follows from Theorem 3.1 that the spatial position x_i satisfies

$$\dot{x}_i = \Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_i), \quad \text{for} \quad i = 1, \cdots, N.$$

Note that the constant Ω_i plays a role as a natural frequency in the Kuramoto model, and it is explicitly determined by initial data (X^0, V^0) via the relation (3.2).

LEMMA 4.1. *Suppose that ψ satisfies the relations* (1.2) *and is long-ranged. Then,* $\Psi(r) = \int_0^r \psi(y)\,dy$ *satisfies the following assertions.*

(*i*) $\Psi(-r) = -\Psi(r)$.
(*ii*) $\Psi = \Psi(r)$ *is increasing for* $r \geq 0$ *and* $\lim_{r\to\infty} \Psi(r) = \infty$.
(*iii*) $\Psi(r)$ *is concave for* $r \geq 0$ *and convex for* $r \leq 0$.

Proof.
(i) We use the evenness of ψ to see that

$$\Psi(-r) = \int_0^{-r} \psi(y)\,dy = -\int_0^r \psi(-z)\,dz = -\int_0^r \psi(z)\,dz = -\Psi(r).$$

(ii) Since ψ is nonnegative and long-ranged, its anti-derivative Ψ is clearly increasing and tends to ∞.

(iii) For $r \geq 0$,

$$\Psi''(r) = \psi'(r) \leq 0, \quad \text{for } r \geq 0.$$

Thus, Ψ is concave for $r \geq 0$. Similarly, we can show that Ψ is convex for the region $r \leq 0$. ∎

Next, we present an estimate on the structure of the relative equilibria of (1.1) for the long-ranged case.

THEOREM 4.1. *Suppose that κ and Ψ satisfy positivity and asymptotic limit:*

$$\kappa > 0 \quad \textit{and} \quad \lim_{r\to\infty} \Psi(r) = \infty,$$

and let (X, V) be a solution to system (1.1) *with initial data (X^0, V^0) such that*

$$\sum_{k=1}^N x_k^0 = 0, \quad \sum_{k=1}^N v_k^0 = 0. \tag{4.1}$$

Then, the asymptotic state $(X^\infty, V^\infty) = \lim_{t\to\infty}(X(t), V(t))$ is a unique solution to the equilibrium system:

$$\Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k^\infty - x_i^\infty) = 0, \qquad \sum_{k=1}^N x_k^\infty = 0. \tag{4.2}$$

Proof.

• (Existence): It follows from [29] that we have

$$\begin{aligned} &\dot{x}_i = v_i \to v_i^\infty = 0, \quad \text{as } t \to \infty, \\ &x_i^\infty = x_i^0 + \int_0^\infty v_i(s)\,ds, \quad i = 1, \cdots, N. \end{aligned} \tag{4.3}$$

On the other hand, it follows from (1.3) that we have

$$v_i(t) = \Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k(t) - x_i(t)).$$

Now, we use (4.2) and (4.3) to see that the asymptotic state is an equilibrium solution to the equilibrium system:

$$0 = \Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k^\infty - x_i^\infty), \qquad \sum_{k=1}^N x_k^\infty = 0.$$

We next show that the equilibrium system (4.2) is a *unique* equilibrium.

• (Uniqueness): Let $\bar{X}^\infty = (\bar{x}_1^\infty, \cdots, \bar{x}_N^\infty)$ be a solution satisfying the relation:

$$\Omega_i + \frac{\kappa}{N}\sum_{k=1}^N \Psi(\bar{x}_k^\infty - \bar{x}_i^\infty) = 0 \quad \text{and} \quad \sum_{k=1}^N \bar{x}_k^\infty = 0.$$

We next claim:

$$x_i^\infty = \bar{x}_i^\infty, \qquad v_i^\infty = \bar{v}_i^\infty, \quad 1 \leq i \leq N. \tag{4.4}$$

Proof of claim: It follows from Proposition 3.1 that we may assume

$$x_1^\infty \le x_2^\infty \le \cdots \le x_N^\infty \quad \text{and} \quad \bar{x}_1^\infty \le \bar{x}_2^\infty \le \cdots \le \bar{x}_N^\infty.$$

Suppose that the two equilibria X^∞ and $\bar{X}^\infty$ are different. Then, we consider the three index sets:

$$\mathcal{A} := \{i \mid \bar{x}_i^\infty > x_i^\infty\}, \qquad \mathcal{B} := \{i \mid \bar{x}_i^\infty < x_i^\infty\}, \qquad \mathcal{C} := \{i \mid \bar{x}_i^\infty = x_i^\infty\}.$$

Since $\bar{X}^\infty$ is different from X^∞, at least one of the sets $\mathcal{A}$ and $\mathcal{B}$ should be non-empty. Without loss of generality, we assume that $\mathcal{A} \neq \emptyset$. Then, we will show $\mathcal{B} = \emptyset$.
Suppose that $\mathcal{B} \neq \emptyset$. Note that for any $i \in \mathcal{B}$ we have

$$\Omega_i = \frac{\kappa}{N} \sum_{k=1}^N \Psi(\bar{x}_i^\infty - \bar{x}_k^\infty). \tag{4.5}$$

Then, we add equation (4.5) for all $i \in \mathcal{B}$ to obtain

$$\sum_{i\in\mathcal{B}} \Omega_i = \frac{\kappa}{N} \sum_{i\in\mathcal{B}} \sum_{k=1}^N \Psi(\bar{x}_i^\infty - \bar{x}_k^\infty) = \frac{\kappa}{N} \sum_{i\in\mathcal{B}} \sum_{k\notin\mathcal{B}} \Psi(\bar{x}_i^\infty - \bar{x}_k^\infty), \tag{4.6}$$

where we use the oddness property of Ψ to see the cancellation in the double sum:

$$\sum_{i\in\mathcal{B}} \sum_{k\in\mathcal{B}} \Psi(\bar{x}_i^\infty - \bar{x}_k^\infty) = 0.$$

Note that for $i \in \mathcal{B}$ and $k \notin \mathcal{B}$, we have

$$\bar{x}_i^\infty < x_i^\infty \quad \text{and} \quad \bar{x}_k^\infty \ge x_k^\infty.$$

This yields

$$\bar{x}_i^\infty - \bar{x}_k^\infty < x_i^\infty - x_k^\infty. \tag{4.7}$$

If $i < k$, then the relation (4.7) implies

$$\bar{x}_i^\infty - \bar{x}_k^\infty < x_i^\infty - x_k^\infty < 0.$$

On the other hand, if $i > k$, then the relation (4.7) implies

$$0 < \bar{x}_i^\infty - \bar{x}_k^\infty < x_i^\infty - x_k^\infty.$$

Thus, in both cases, we have

$$\Psi(\bar{x}_i^\infty - \bar{x}_k^\infty) < \Psi(x_i^\infty - x_k^\infty).$$

It follows from (4.6) and the monotonicity of Ψ that we have

$$\sum_{i\in\mathcal{B}} \Omega_i = \frac{\kappa}{N} \sum_{i\in\mathcal{B}} \sum_{k\notin\mathcal{B}} \Psi(\bar{x}_i^\infty - \bar{x}_k^\infty) < \frac{\kappa}{N} \sum_{i\in\mathcal{B}} \sum_{k\notin\mathcal{B}} \Psi(x_i^\infty - x_k^\infty) = \sum_{i\in\mathcal{B}} \Omega_i.$$

This is clearly contradictory. Thus, the set $\mathcal{B} = \emptyset$.

Now, we have $\mathcal{A}\cup\mathcal{C}=\{1,\ldots,N\}$. If the set $\mathcal{A}$ is not empty, then we have

$$0=\sum_{k=1}^{N}\bar{x}_k^{\infty}>\sum_{k=1}^{N}x_k^{\infty}=0$$

which is again contradictory. Hence, we can conclude $\mathcal{A}=\emptyset$ and we have $\mathcal{C}=\{1,\cdots,N\}$. Finally, we have (4.4). □

REMARK 4.1. Theorem 4.1 shows how to construct the unique asymptotic state for a long-range communication, and Proposition 3.1 shows the rearrangement of the particles according to the order of natural velocities:

$$\begin{aligned}\Omega_i=\Omega_j &\implies x_i^{\infty}=x_j^{\infty},\\ \Omega_i>\Omega_j &\implies x_i^{\infty}>x_j^{\infty}.\end{aligned}$$

Note that all these constructions only depend on the natural velocities Ω_i and communication weight function ψ. Thus, for second-order C-S model (1.1) with a long-ranged communication on a line, the initial data and the communication function will eventually determine the final structure of resulting asymptotic states.

5. Relative equilibria II: Short-range communication

In this section, we study the structure of relative equilibria for the C-S model (1.1) with a short-ranged communication, $\int_0^{\infty}\psi(r)\,dr<\infty$. For a short-range communication weight, we will generically have multi-clusters asymptotically. It follows from Proposition 3.1 that there are at most M clusters with $M=|\{\Omega_i\}_{i=1}^{N}|$.

Note that we can see that the asymptotic positions of particles are positively correlated to the order of natural velocities, which is the same as the long-range case. However, the asymptotic relocation for the short-range case is much more difficult than that of the long-range case due to the emergence of multi-clusters. We have already shown the positive correlation between the asymptotic position x_i^{∞} and the natural velocities Ω_i in Proposition 3.1 . Thus, we have the following ramifications:

(1) If $\Omega_i<\Omega_j$ and x_i and x_j aggregate with $\limsup_{t\to+\infty}|x_j(t)-x_i(t)|<\infty$, then all particles with natural velocity between Ω_i and Ω_j will aggregate: for k with $\Omega_i\le\Omega_k\le\Omega_j$,

$$\limsup_{t\to+\infty}|x_k(t)-x_i(t)|<\infty.$$

(2) If $\Omega_i<\Omega_j$ and x_i and x_j segregate with $\liminf_{t\to+\infty}|x_i(t)-x_j(t)|=\infty$, then we have

$$\begin{aligned}&\liminf_{t\to+\infty}|x_k(t)-x_j(t)|=\infty, \quad \text{for} \quad \Omega_k\le\Omega_i,\\ &\liminf_{t\to+\infty}|x_k(t)-x_i(t)|=\infty, \quad \text{for} \quad \Omega_k\ge\Omega_j.\end{aligned}$$

Thus, it is very interesting to find sufficient conditions for the aggregation or segregation of two particles.

Next, we present a simple calculus-type lemma to be used later.

LEMMA 5.1. *Let y be a positive solution to the following nonlinear ODE:*

$$y'=\alpha-\kappa F(y), \quad t>0,$$

where $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a bounded and strictly increasing function such that*

$$F(0)=0<F(y)<F^\infty, \quad y\in(0,\infty).$$

If $\kappa > \dfrac{\alpha}{F^\infty}$*, then* y *is uniformly bounded and has a limit as* $t\to\infty$*:*

$$\sup_{t\geq 0} y(t) \leq \max\{y_0, y_*\} \quad \textit{and} \quad \lim_{t\to\infty} y(t) = y_* := F^{-1}\Big(\frac{\alpha}{\kappa}\Big).$$

Proof. Suppose that κ is sufficiently large so that $\kappa > \dfrac{\alpha}{F^\infty}$. Then, there exists a positive solution $y_* \in (0,\infty)$ such that

$$\alpha - \kappa F(y_*) = 0, \quad \text{i.e.,} \quad y_* = F^{-1}\Big(\frac{\alpha}{\kappa}\Big).$$

If $y_0 < y_*$, then the solution $y(t)$ is strictly increasing to y_* as $t\to\infty$. If $y_0 > y_*$, then $y(t)$ is strictly decreasing to y_* as $t\to\infty$. □

PROPOSITION 5.1. *Let* (X,V) *be a solution to* (1.1) *with initial data* (X^0,V^0)*:*

$$x_1^0 < x_2^0 < \cdots < x_N^0, \qquad \Omega_1 < \Omega_2 < \cdots < \Omega_N.$$

(1) If the coupling strength κ *is sufficiently large such that*

$$\kappa > \frac{N|\Omega_i - \Omega_j|}{2\Psi^\infty}, \quad \Psi^\infty := \lim_{t\to\infty}\Psi(t),$$

then, x_i *and* x_j *will aggregate asymptotically:*

$$\limsup_{t\to+\infty} |x_i(t) - x_j(t)| < \infty.$$

(2) If the coupling strength κ *is sufficiently small such that*

$$\kappa < \frac{|\Omega_j - \Omega_i|}{\Big(1+\frac{1}{N}|j-i-1|\Big)\Psi^\infty},$$

then, x_i *and* x_j *will segregate asymptotically:*

$$\liminf_{t\to+\infty} |x_i(t) - x_j(t)| = \infty.$$

Proof. We consider two cases for large and small coupling strengths:

$$\kappa \gg 1, \quad \kappa \ll 1.$$

• Case A (large coupling strength): Suppose that κ is sufficiently large such that

$$\kappa > \frac{N|\Omega_i - \Omega_j|}{2\Psi^\infty}.$$

It follows from Proposition 3.1 that we have

$$x_i(t) < x_j(t), \quad t\geq 0.$$

Next, we use the relations

$$\Psi(x_k - x_j) - \Psi(x_k - x_i) = -\psi(x^*_{k,i,j})|x_i - x_j| \le 0, \quad \forall\ k \in \mathcal{N},$$
$$\Psi(0) = 0, \quad \sup_{r\ge 0} \Psi(r) \le \Psi(\infty) < \infty.$$

to obtain that $x_j - x_i = |x_j - x_i|$ satisfies

$$\begin{aligned}\frac{d}{dt}|x_j - x_i| &= \Omega_j - \Omega_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\Psi(x_k - x_j) - \Psi(x_k - x_i)\Big)\\ &= |\Omega_j - \Omega_i| + \frac{2\kappa}{N}\Psi(x_i - x_j) + \underbrace{\frac{\kappa}{N}\sum_{k\ne i,j}\Big(\Psi(x_k - x_j) - \Psi(x_k - x_i)\Big)}_{\text{non-positvie}}\\ &\le |\Omega_i - \Omega_j| - \frac{2\kappa}{N}\Psi(|x_j - x_i|). \end{aligned} \tag{5.1}$$

We set a positive solution $r = D^\infty_{ij}$ satisfying the equation:

$$|\Omega_i - \Omega_j| - \frac{2\kappa}{N}\Psi(r) = 0, \quad \text{i.e.,} \quad \Psi(D^\infty_{ij}) = \frac{N|\Omega_i - \Omega_j|}{2\kappa}.$$

Then, it follows from Lemma 5.1 and the comparison principle for (5.1) that

$$|x_i(t) - x_j(t)| \le \max\{|x^0_i - x^0_j|,\ D^\infty_{ij}\} \quad \text{for} \quad t \ge 0.$$

Thus, we have aggregation:

$$\limsup_{t\to+\infty}|x_i(t) - x_j(t)| \le \max\{|x^0_i - x^0_j|,\ D^\infty_{ij}\}.$$

• Case B (small coupling strength): Suppose that κ is sufficiently small to satisfy

$$\kappa < \frac{|\Omega_j - \Omega_i|}{\Big(1 + \frac{1}{N}|j - i - 1|\Big)\Psi^\infty}.$$

Let $\mathcal{M}_{ij}$ be a set of indices satisfying

$$\mathcal{M}_{ij} := \{k\ :\ x_i < x_k < x_j\}.$$

Then, it follows from (1.3) that we have

$$\begin{aligned}\frac{d}{dt}|x_j - x_i| &= \Omega_j - \Omega_i + \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\Psi(x_k - x_j) - \Psi(x_k - x_i)\Big)\\ &\ge \Omega_j - \Omega_i - \frac{\kappa}{N}\sum_{k\in\mathcal{M}_{ij}}|\Psi(x_k - x_j) - \Psi(x_k - x_i)|\\ &\quad - \frac{\kappa}{N}\sum_{k\notin\mathcal{M}_{ij}}|\Psi(x_k - x_j) - \Psi(x_k - x_i)|\\ &=: |\Omega_j - \Omega_i| + \mathcal{I}_{11} + \mathcal{I}_{12}. \end{aligned} \tag{5.2}$$

◇ Case B.1 (estimate for $\mathcal{I}_{11}$): For $k \in \mathcal{M}_{ij}$, we take a rough estimate by

$$|\Psi(x_k - x_j) - \Psi(x_k - x_i)| \le 2\Psi^\infty.$$

Then, we have

$$\mathcal{I}_{11} \geq -\frac{2\kappa|\mathcal{M}_{ij}|}{N}\Psi^\infty. \tag{5.3}$$

$\diamond$ Case B.2 (estimate for $\mathcal{I}_{ij}$): For $k \in \mathcal{M}_{ij}$, we have

$$\text{either} \quad x_k < x_i < x_j \quad \text{or} \quad x_i < x_j < x_k.$$

In either case, $\Psi(x_k - k_j)$ and $\Psi(x_k - x_i)$ have the same sign. Thus, we have

$$|\Psi(x_k - x_j) - \Psi(x_k - x_i)| \leq \max\{|\Psi(x_k - k_j)|, |\Psi(x_k - x_i)|\} \leq \Psi^\infty.$$

This implies

$$\mathcal{I}_{12} \geq -\frac{\kappa(N - |\mathcal{M}_{ij}|)}{N}\Psi^\infty. \tag{5.4}$$

Finally, in (5.2), we combine (5.3) and (5.4) to obtain

$$\begin{aligned}\frac{d}{dt}|x_j - x_i| &\geq |\Omega_j - \Omega_i| - \frac{2\kappa|\mathcal{M}_{ij}|}{N}\Psi^\infty - \frac{\kappa(N - |\mathcal{M}_{ij}|)}{N}\Psi^\infty \\ &= |\Omega_j - \Omega_i| - \frac{\kappa(N + |\mathcal{M}_{ij}|)}{N}\Psi^\infty \\ &=: \Lambda(\kappa, N, |\mathcal{M}_{ij}|). \end{aligned} \tag{5.5}$$

If the coupling strength κ is sufficiently small such that

$$\kappa < \frac{|\Omega_j - \Omega_i|N}{(N + |\mathcal{M}_{ij}|)\Psi^\infty}, \quad \text{i.e.,} \quad \Lambda(\kappa, N, |\mathcal{M}_{ij}|) > 0,$$

then, it follows from (5.5) that we have

$$\frac{d}{dt}|x_j - x_i| \geq \Lambda(\kappa, N, |\mathcal{M}_{ij}|) > 0.$$

This implies

$$\liminf_{t \to +\infty} |x_j(t) - x_i(t)| = \infty.$$

$\square$

REMARK 5.1. Combining the sufficient condition for aggregation or segregation of two particles in Proposition 5.1, we can construct a sufficient condition for m-cluster emergence provided $1 \leq m \leq M$, where M is the number of distinct natural frequencies, $M = |\{\Omega_j\}_{i=1}^N|$. However, these conditions are far from optimal. Among multi-cluster emergence, flocking and complete segregation are two extreme cases. We will show a sufficient condition for flocking and a sufficient and necessary condition for the complete segregation in the following two lemmas.

For next lemma, we set

$$\Omega_M := \max_{1 \leq i \leq N} \Omega_i, \quad \Omega_m := \min_{1 \leq i \leq N} \Omega_i, \quad D(\Omega) := \Omega_M - \Omega_m.$$

LEMMA 5.2. *Suppose that the coupling strength κ is sufficiently large such that*

$$\kappa > \frac{\mathcal{D}(\Omega)}{\Psi^\infty}, \quad \text{or equivalently} \quad \kappa\Psi^\infty > \mathcal{D}(\Omega). \tag{5.6}$$

Then, for any solution X to (1.3) *with initial data* (X^0, V^0) *such that*

$$\sum_{k=1}^N x_k^0 = 0, \quad \sum_{k=1}^N v_k^0 = 0,$$

we have a mono-cluster flocking:

$$\sup_{t\geq 0} |x_i(t) - x_j(t)| < \Psi^{-1}\Big(\frac{D(\Omega)}{\kappa}\Big), \quad 1 \leq i,j \leq N.$$

Proof. For $t \geq 0$, we set

$$x_m(t) := \min_{1\leq i\leq N} x_i(t) \quad \text{and} \quad x_M(t) := \max_{1\leq i\leq N} x_i(t).$$

Then, it follows from (1.3) that we have

$$\begin{aligned}\frac{d}{dt}(x_M - x_m) &= \Omega_M - \Omega_m - \frac{\kappa}{N}\sum_{k=1}^N |\Psi(x_k - x_M) - \Psi(x_k - x_m)| \\ &\leq D(\Omega) - \kappa\Psi(x_M - x_m).\end{aligned} \tag{5.7}$$

It follows from Lemma 5.1 and the comparison principle for (5.7) that we have

$$|x_M(t) - x_m(t)| \leq \Psi^{-1}\Big(\frac{D(\Omega)}{\kappa}\Big), \quad t \geq 0.$$

□

REMARK 5.2. We briefly comment on the conditions (5.6) and (2.5) employed in Theorem 2.1. Note that the sufficient condition (2.5) can be rewritten in terms of Ψ:

$$\kappa\Psi^\infty > \mathcal{D}_V(0) + \kappa\Psi(\mathcal{D}_X(0)). \tag{5.8}$$

On the other hand, recall a defining relation for Ω_i in (1.4):

$$\Omega_i = v_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k^0 - x_i^0). \tag{5.9}$$

We choose indices M and m such that

$$\Omega_M := \max_i \Omega_i, \quad \Omega_m := \min_i \Omega_i.$$

Then, it follows from $(5.9)_1$ that we have

$$\begin{aligned}\mathcal{D}(\Omega) &= v_M^0 - v_m^0 - \frac{\kappa}{N}\sum_{k=1}^N \Big(\Psi(x_k^0 - x_M^0) - \Psi(x_k^0 - x_m^0)\Big) \\ &\leq \mathcal{D}_V(0) + 2\kappa\Psi(\mathcal{D}_X(0)) \leq 2(\mathcal{D}_V(0) + \kappa\Psi(\mathcal{D}_X(0))).\end{aligned} \tag{5.10}$$

Suppose that the relation (5.8) holds. Then, it follows from the relation (5.10) that we have

$$\mathcal{D}(\Omega) \le 2\kappa\Psi^\infty.$$

Thus, it seems to us that the relation (5.10) does not have a direct relation with (5.6).

Next, we consider the emergence of the complete segregation i.e., all particles depart from each other. More precisely, we want to investigate the case

$$\liminf_{t\to+\infty} |x_i(t) - x_j(t)| = +\infty, \quad 1 \le i < j \le N.$$

In the following theorem, we have the critical configuration for the complete segregation.

THEOREM 5.1. *Suppose that the natural velocity Ω_i is well-ordered and has mean zero:*

$$\Omega_1 < \Omega_2 \cdots < \Omega_N \quad \textit{and} \quad \sum_{i=1}^N \Omega_i = 0.$$

Then, N particles are completely segregated in the sense that

$$\limsup_{t\to+\infty} x_1(t) = -\infty, \quad \liminf_{t\to+\infty} x_N(t) = \infty,$$
$$\liminf_{t\to+\infty} |x_{i+1}(t) - x_i(t)| = \infty, \quad 1 \le i \le N-1$$

if and only if the coupling strength κ is sufficiently small in the sense that

$$\kappa < \min\Big\{ -\frac{N\Omega_1}{(N-1)\Psi^\infty}, \frac{N(\Omega_2-\Omega_1)}{2\Psi^\infty}, \dots, \frac{N(\Omega_N-\Omega_{N-1})}{2\Psi^\infty}, \frac{N\Omega_N}{(N-1)\Psi^\infty} \Big\}. \tag{5.11}$$

Proof. Because of the mean zero assumption on natural frequencies, it is obvious that

$$\Omega_1 < 0 \quad \text{and} \quad \Omega_N > 0.$$

We will prove the equivalance in two directions.

(i) ($\Longleftarrow$): Suppose that the coupling strength κ satisfies (5.11). Then, the spatial configuration $\{x_i\}$ will be ordered in finite time. More precisely, we may assume that

$$x_1(t) < x_2(t) < \cdots < x_N(t), \quad t \ge 0.$$

Then, we consider the following three cases.

- Case A (dynamics of x_1): It follows from (5.11) that we have

$$\kappa < -\frac{N\Omega_1}{(N-1)\Psi^\infty}.$$

Note that x_1 satisfies

$$\dot{x}_1 = \Omega_1 + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_1) \le \Omega_1 + \frac{\kappa(N-1)}{N}\Psi^\infty < 0.$$

Thus, we have

$$\limsup_{t\to+\infty} x_1(t) = -\infty.$$

• Case B (dynamics of x_N): It follows from (5.11) that we have

$$\kappa < \frac{N\Omega_N}{(N-1)\Psi^\infty}.$$

Note that x_N satisfies

$$\dot{x}_N = \Omega_N + \frac{\kappa}{N}\sum_{k=1}^N \Psi(x_k - x_N) \geq \Omega_N - \frac{\kappa(N-1)}{N}\Psi^\infty > 0.$$

This implies

$$\liminf_{t\to+\infty} x_N(t) = +\infty.$$

• Case C (dynamics of $x_{i+1}-x_i$): Next, we prove that the complete segregation emerges by the proof of contradiction. Suppose that the complete segregation does not emerge. Then, it follows from Proposition 3.1 that we can find an index set $\mathcal{A}$ with $|\mathcal{A}| \geq 2$ such that

$$\liminf_{t\to+\infty} |x_L(t) - x_S(t)| = C < \infty, \quad S = \min\mathcal{A}, \quad L = \max\mathcal{A}, \quad \text{and}$$
$$\liminf_{t\to+\infty} |x_i(t) - x_j(t)| = +\infty, \quad \text{for} \quad i \notin \mathcal{A}, \quad \text{and} \quad j \in \mathcal{A}.$$

We consider the equation for $x_L - x_S$:

$$\begin{aligned}\frac{d}{dt}(x_L - x_S) =& \Omega_L - \Omega_S + \frac{\kappa}{N}\sum_{i=1}^N \Big(\Psi(x_i - x_L) - \Psi(x_i - x_S)\Big)\\ =& \Omega_L - \Omega_S + \frac{\kappa}{N}\sum_{i\in\mathcal{A}} \Big(\Psi(x_i - x_L) - \Psi(x_i - x_S)\Big)\\ &+ \frac{\kappa}{N}\sum_{i\notin\mathcal{A}} \Big(\Psi(x_i - x_L) - \Psi(x_i - x_S)\Big).\end{aligned} \tag{5.12}$$

Since $\liminf_{t\to+\infty} |x_i(t) - x_j(t)| = +\infty$ for $i \notin \mathcal{A}$, we have

$$\lim_{t\to+\infty} \Psi(x_i - x_L) = \lim_{t\to+\infty} \Psi(x_i - x_S) = \Psi^\infty \text{ or } -\Psi^\infty, \quad i \notin \mathcal{A}.$$

Then, for any $\varepsilon > 0$, we can find a large T and a sequence $\{t_k\}$ such that

$$|\Psi(x_i(t) - x_L(t)) - \Psi(x_i(t) - x_S(t))| \leq \varepsilon, \quad i \notin \mathcal{A}, \quad t > T.$$

Then it follows from (5.12) that for $t > T$

$$\begin{aligned}\frac{d}{dt}(x_L(t) - x_S(t)) \geq& \Omega_L - \Omega_S + \frac{\kappa}{N}\sum_{i\in\mathcal{A}} \Big(\Psi(x_i(t) - x_L(t)) - \Psi(x_i(t) - x_S(t))\Big) - \frac{2\kappa}{N}\sum_{i\notin\mathcal{A}}\varepsilon\\ \geq& \Omega_L - \Omega_S - \frac{2\kappa(|\mathcal{A}|-1)}{N}\Psi(x_L(t) - x_S(t)) - \frac{2\kappa}{N}\sum_{i\notin\mathcal{A}}\varepsilon.\end{aligned}$$

$$\geq \Omega_L - \Omega_S - \frac{2\kappa(|\mathcal{A}|-1)\Psi\infty}{N} - \frac{2\kappa}{N}\sum_{i\notin\mathcal{A}}\varepsilon.$$

Since κ satisfies (5.11), we have

$$\Omega_L - \Omega_S = \sum_{j=S}^{L-1}\Omega_{j+1} - \Omega_j > \sum_{j=S}^{L-1}\frac{2\kappa\Psi^\infty}{N} = \frac{2\kappa(|\mathcal{A}|-1)\Psi^\infty}{N}.$$

Thus, we can choose small ε so that

$$\gamma := \Omega_L - \Omega_S - \frac{2\kappa(|\mathcal{A}|-1)\Psi^\infty}{N} - \frac{2\kappa}{N}\sum_{i\notin\mathcal{A}}\varepsilon > 0.$$

Thus, for $t > T$, we have

$$\frac{d}{dt}(x_L(t) - x_S(t)) \geq \gamma > 0 \tag{5.13}$$

and this yields

$$x_L(t) - x_S(t) \geq (x_L(T) - x_S(T)) + \gamma(t-T) \to \infty \quad \text{as} \quad t \to \infty.$$

This contradicts with

$$\liminf_{t\to+\infty}|x_L(t) - x_S(t)| = C < \infty.$$

Hence, we can conclude

$$\liminf_{t\to+\infty}|x_i - x_j| = +\infty, \quad 1 \leq i < j \leq N.$$

(ii) ($\Longrightarrow$): Suppose that the complete segregation does happen. Then, it follows from Proposition 5.1 that

$$\kappa \leq \min\left\{\frac{N(\Omega_2 - \Omega_1)}{2\Psi^\infty}, \cdots, \frac{N(\Omega_N - \Omega_{N-1})}{2\Psi^\infty}\right\}.$$

Otherwise, at least two adjacent particles will aggregate. Thus, we have

$$\liminf_{t\to+\infty}|x_i(t) - x_j(t)| = +\infty, \quad 1 \leq i < j \leq N.$$

Therefore, we have

$$\lim_{t\to+\infty}\Psi(x_i - x_j) = \operatorname{sgn}(\Omega_i - \Omega_j)\Psi^\infty, \quad 1 \leq i < j \leq N.$$

Now, consider the equation for x_N:

$$\frac{dx_N}{dt} = \Omega_N + \frac{\kappa}{N}\sum_{k=1}^{N}\Psi(x_k - x_N) \to \Omega_N - \frac{\kappa(N-1)}{N}\Psi^\infty \quad \text{as} \quad t \to \infty.$$

If $\Omega_N < \frac{\kappa(N-1)\Psi^\infty}{N}$, there exists a finite time $\tilde{t}$ such that

$$\frac{d}{dt}x_N(t) < 0 \quad \text{for} \quad t > \tilde{t}.$$

Thus, x_N cannot tend to infinity which contradicts the emergence of segregation. Hence we have

$$\Omega_N \geq \frac{\kappa(N-1)\Psi^\infty}{N}.$$

By analogous argument for x_1, we also get

$$\Omega_1 \leq -\frac{\kappa(N-1)\Psi^\infty}{N}.$$

□

REMARK 5.3.

(1) For the complete segregation, we can calculate the asymptotic velocity. In fact, we have

$$v_i^\infty = \Omega_i - \frac{\kappa(2i-N-1)\Psi^\infty}{N}.$$

(2) Thanks to Theorem 5.1, we can construct the transition of coupling strength.

- $\frac{(\Omega_N-\Omega_1)}{\Psi^\infty} < \kappa$: flocking phase.
- $\frac{N(\Omega_N-\Omega_1)}{2(N-1)\Psi^\infty} < \kappa \leq \frac{(\Omega_N-\Omega_1)}{\Psi^\infty}$: transition phase i.e., flocking/multi-cluster conditionally.
- $\min\{\frac{N\Omega_N}{(N-1)\Psi^\infty}, \frac{N(\Omega_{i+1}-\Omega_i)}{2\Psi^\infty}, \frac{-N\Omega_1}{(N-1)\Psi^\infty}\} < \kappa \leq \frac{N(\Omega_N-\Omega_1)}{2(N-1)\Psi^\infty}$: multi-cluster phase with at most $N-1$ clusters emergence.
- $0 \leq \kappa \leq \min\{\frac{N\Omega_N}{(N-1)\Psi^\infty}, \frac{N(\Omega_{i+1}-\Omega_i)}{2\Psi^\infty}, \frac{-N\Omega_1}{(N-1)\Psi^\infty}\}$: N-cluster phase.

6. Conclusion

In this paper, we presented a correspondence relation between the second-order C-S model on the real line and the first-order C-S model by utilizing the special structure in the velocity equation of the C-S model. In previous studies on the C-S model, the velocity relaxation has been the only focal point in flocking analysis, in contrast, the spatial relaxation was treated marginally. In this work, we switch our attention from the velocity relaxation to the spatial asymptotic configuration and how information encoded in the initial data determines the resulting asymptotic configuration. By the first-order formulation of the C-S model, we provide some criterion to detemine whether two initial configurations lead to the same asymptotic spatial configuration or not. Of course, there are several remaining interesting issues in relation to this work. Apparently, our analysis in this work is valid only for one-dimension setting. However, if we look at the C-S model, the component dynamics is weakly one-dimensional in the sense that component dynamics is weakly coupled through the communication function. At present, we do not have any idea how to overcome this weak coupling between component dynamics. Although we did not treat the corresponding mean-field equation, it will be interesting to investigate the emergence of multi-clustering using the kinetic C-S model itself. These issues will be addressed in future works.

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