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DYNAMICAL BEHAVIORS OF A SYSTEM MODELING WAVE BIFURCATIONS*

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Abstract. We rigorously show that a class of systems of partial differential equations (PDEs) modeling wave bifurcations supports stationary equivariant bifurcation dynamics through deriving its full dynamics on the center manifold(s). This class of systems is related to the theory of hyberbolic conservation laws and supplies a new class of PDE examples for stationary $O(2)$-bifurcation. A direct consequence of our result is that the oscillations of the dynamics are *not* due to rotation waves though the system exhibits Euclidean symmetries. The main difficulties of carrying out the program are: 1) the system under study contains multi bifurcation parameters and we do not know *a priori* how they come into play in the bifurcation dynamics. 2) the representation of the linear operator on the center space is a 2×2 zero matrix, which makes the characteristic condition in the well-known normal form theorem trivial. We overcome the first difficulty by using projection method. We managed to overcome the second subtle difficulty by using a conjugate pair coordinate for the center space and applying duality and projection arguments. Due to the specific complex pair parametrization, we could naturally obtain a form of the center manifold reduction function, which makes the study of the current dynamics on the center manifold possible. The symmetry of the system plays an essential role in excluding the possibility of bifurcating rotation waves.

Keywords. Spectrum; resolvent; equivariant bifurcation; center manifold; symmetry; Implicit function theorem.

AMS subject classifications. 37G; 35P; 34G; 34B.

1. Introduction

The current work is a continuation of our works [32] and [22] on dynamical behaviors of solutions to partial differential equations with symmetries. Here our main goals or motivations include the study of a new bifurcation mechanism based on a different spectral scenario, the demonstration of another viewpoint of computing the dynamics on center manifold, the illustration of the use of symmetry, and a comparison between the current work and our former works on equivariant Hopf bifurcations. More importantly, we assert that though there are still oscillations in our current dynamics and though the system exhibits Euclidean symmetry, more precisely, the system is $O(2)$-equivariant, the oscillations are not due to bifurcated rotation waves. We actually rigorously show that the oscillations are associated with a stationary equivariant bifurcation and the resulting S^1-family of waves can be associated with arbitrary nonzero wave numbers. The current study and our former studies in [22,32] cover the generic bifurcation dynamics for the system (1.1) below, demonstrate that this model for bifurcating waves has abundant interesting dynamics. We also hope that our current study could shed some light on our further investigations of equivariant bifurcation dynamics of partial differential equations (see also the discussion and future work). One common feature in our work here and [22,32] is that we use Fourier analysis to decompose the problem to get admissible critical configuration points for our bifurcation analysis. This feature has some similarity with E. Hopf's study [15] on fluid turbulence and with L. Rayleigh's

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study [29] on fluid stability. With the above-mentioned goals in mind, we will adopt a similar organizational structure as in our former works for the ease of comparisons.

1.1. The system. Let us first recall the system we are going to study and some physical background (see also [22, 32]). In this work, we continue to study the following class of systems in continuum mechanics

$$\begin{cases} \partial_t \tau - \partial_x u = -a\partial_x^4 \tau, \\ \partial_t u - \partial_x \sigma(\tau) = -\delta\partial_x^2 u - \varepsilon\partial_x^4 u \end{cases} \tag{1.1}$$

on the spatial periodic domain $\mathbb{T}^1 = \mathbb{R}^1/[-M, M]$ where M is any positive constant. In system (1.1), $\tau = \tau(x,t)$ and $u = u(x,t)$ are the scalar unknown functions. The scalar function $\sigma(\tau)$ is usually called flux function in mathematics and pressure law in physics. The three parameters a, δ, ε are called diffusion coefficients. We emphasize that the domain is $\mathbb{R}^1/[-M, M]$, which means that we have periodic boundary conditions.

Without loss of generality, we can always consider the case $M = \pi$ and $\varepsilon = 1$ or else we can do the following two successive scalings and renamings given by

$$t \mapsto \bar{t} = \frac{\pi}{M}t, x \mapsto \bar{x} = \frac{\pi}{M}x, a \mapsto \bar{a} = \frac{\pi^3}{M^3}a, \delta \mapsto \bar{\delta} = \frac{\pi}{M}\delta, \varepsilon \mapsto \bar{\varepsilon} = \frac{\pi^3}{M^3}\varepsilon.$$

and

$$t \mapsto \tilde{t} = \varepsilon t, x \mapsto \tilde{x} = x, u \mapsto \tilde{u} = u, \tau \mapsto \tilde{\tau} = \varepsilon\tau,$$
$$a \mapsto \tilde{a} = \varepsilon^{-1}a, \sigma(\tau) \mapsto \tilde{\sigma}(\tilde{\tau}) = \varepsilon^{-1}\sigma(\varepsilon^{-1}\tilde{\tau}) = \varepsilon^{-1}\sigma(\tau), \delta \mapsto \tilde{\delta} = \varepsilon^{-1}\delta.$$

Meanwhile, we observe that any constant state (τ_0, u_0) is a solution to system (1.1). We consider the constant state $(0,0)$ without loss of generality. This is obvious by first choosing (τ_0, u_0) in the physical range and then using the invariance of system (1.1) in the translation group actions $u \mapsto u + h$ for any $h \in \mathbb{R}^1$ and redefining $\sigma(\tau)$ by $\sigma(\tau_0 + \tau)$. The above constant states are usually referred to as uniform states or homogeneous states in physics literature.

Systems of form (1.1) are generic in classical continuum mechanics (see [1, 7, 10, 11] and the references therein) as they describe Newton's second law of motion, in gas dynamics, for example the p-system (see in particular Chapter 2 of Dafermos [7] and Nishida [27]). When $M = \infty$, systems of form (1.1) are also connected with the Kuramoto-Sivashinsky and related systems when we seek traveling wave solutions (see [10, 12, 19, 20, 30, 31]).

If we regard the term $\begin{pmatrix} -a\partial_x^4 \tau \\ -\varepsilon\partial_x^4 u \end{pmatrix}$ as vanishing viscosity term in the system (1.1) and study the process $a \to 0+, \varepsilon \to 0+$, which is the vanishing viscosity method in partial differential equations for the well-known p-system in gas dynamics. Naturally, this requires $a > 0$ and $\varepsilon > 0$. However, in our work here, we are not restricted to the case with $a > 0$ and $\varepsilon > 0$ which gives dissipation. We will see later that as long as (a_c, δ_c) is an admissible critical configuration point, we can obtain bifurcation dynamics under necessary conditions to be specified later. If there are admissible critical configuration points (a_c, δ_c) such that $a_c > 0$ after scalings and renaming, we could scale back and analyze the vanishing viscosity limit process.

1.2. Main results. In order to state our main results clearly, we first introduce the admissible critical configuration set $\mathcal{A}(k_0)$ for an arbitrary nonzero integer k_0. Actually, we will see that $\mathcal{A}(k_0) = \mathcal{A}(-k_0)$. Hence we may regard k_0 as a positive integer in the remaining parts of the paper.

DEFINITION 1.1. *For a given positive integer k_0, we say that the pair $(a_c,\delta_c)\in\mathbb{R}^2$ is an admissible critical configuration point if the following three conditions are fulfilled:*

(a) $a_ck_0^4(k_0^2-\delta_c)+\sigma'(0)=0.$

(b) for any nonzero integer k such that $|k|\neq|k_0|$ and nonzero real number ω,

$$[a_ck^4(k^4-\delta_ck^2)+\sigma'(0)-\omega^2]+i[(a_c+1)k^4-\delta_ck^2]\neq 0.$$

(c) $(a_c+1)k_0^2-\delta_c\neq 0.$

Consequently, $\mathcal{A}(k_0)$ is the collection of all admissible critical configuration points (a_c,δ_c) associated with k_0.

The conditions (a)-(c) characterize a spectral scenario of the linear operator $\mathcal{L}$ (see Section 3) and exactly make all the denominators appearing later non-vanishing. Now we are in a position to state our main results.

THEOREM 1.1. *Let (a_c,δ_c) be any admissible critical configuration point, $\mu=(\delta_c-k_0^2)(a-a_c)+a_c(\delta-\delta_c)$. System* (1.1) *or equivalently* (2.1) *admits a center manifold reduction and supports a stationary equivariant $O(2)$-bifurcation from any uniform solution (take $(\tau,u)=(0,0)$ without loss of generality) near $\mu=0$ in the Hilbert space Y consisting of functions in $L^2_{per}(-\pi,\pi)$ with zero mean over one period when the parameters a and δ vary around (a_c,δ_c). The full dynamics on the center manifold(s) has the following form AAA*

$$\frac{dA}{dt}=\frac{k_0^4\mu A}{(a_c+1)k_0^2-\delta_c}+\frac{1}{(a_c+1)k_0^2-\delta_c}\Big(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}\Big)A^2A^*+O(|A|^5), \tag{1.2}$$

where $A=A(t)\in\mathbb{C}^1$ and the complex conjugate pair $(A(t),A(t)^*)$ gives the coordinate of the center space and represents the dynamics for (τ,u) in the center manifold and the $O(|A|^5)$ term is given by $\sum_{j=2}^m \mathbb{c}_j|A|^{2j+1}A+O(|A|^{2m+3})$ with all the $\mathbb{c}_j$ real if $\sigma(\cdot)\in C^{2m+1}$ around 0.

In view of the coefficients $\frac{k_0^4}{(a_c+1)k_0^2-\delta_c}$ and $\frac{1}{(a_c+1)k_0^2-\delta_c}\Big(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}\Big)$ of μA and A^2A^* term above and that the product of them is proportional to $\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}$ with positive proportional constant, we have the following bifurcation diagram for the above bifurcation dynamics:

THEOREM 1.2. *Let (a_c,δ_c) be any admissible critical configuration point and μ be as above and small. Parameterizing the solution by μ, the system* (1.1) *undergoes a supercritical (resp. subcritical) stationary equivariant $O(2)$-bifurcation around $\mu=0$ when $\frac{1}{(a_c+1)k_0^2-\delta_c}\Big(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}\Big)<0$ (resp., $\frac{1}{(a_c+1)k_0^2-\delta_c}\Big(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}\Big)>0$), more precisely, the following properties hold in a neighborhood of 0 of $\mathbb{R}^1$ for sufficiently small μ when (a,δ) varies around (a_c,δ_c):*

(i) If $\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}<0$ (resp., $\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}>0$), (1.1) *has precisely one trivial equilibrium $U=0$ for $\mu<0$ (resp., $\mu>0$). This equilibrium is stable when $\frac{1}{(a_c+1)k_0^2-\delta_c}\Big(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}\Big)<0$ and unstable when $\frac{1}{(a_c+1)k_0^2-\delta_c}\Big(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}\Big)>0$,*

(ii) If $\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}<0$ (resp., $\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)}-\frac{\sigma'''(0)}{2}>0$), (1.1) *possesses, for $\mu>0$ (resp., $\mu<0$), the trivial solution $U=0$ and a family of nontrivial equilibrium*

$U = U_\epsilon^\theta$ parametrized by $\epsilon = \mu^{1/2}$ and phase constant $\theta \in \mathbb{R}^1/2\pi\mathbb{Z}$ and the dependence of U on ϵ is smooth in a neighborhood of $(0,0)$, and $U_\epsilon = O(\epsilon)$, i.e., of magnitude $O(|\mu|^{1/2})$. The bifurcated family of equilibria is stable if $\frac{1}{(a_c+1)k_0^2-\delta_c}\left(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)} - \frac{\sigma'''(0)}{2}\right) < 0$ and unstable if $\frac{1}{(a_c+1)k_0^2-\delta_c}\left(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)} - \frac{\sigma'''(0)}{2}\right) > 0$.

The quotient of the coefficient of μA term and that of A^2A^* term is given by $k_0^4\left(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)} - \frac{\sigma'''(0)}{2}\right)^{-1}$. Therefore, each bifurcated equilibrium U_ϵ^θ above corresponds to a bifurcated oscillation wave of (1.1) given as

$$k_0^2\epsilon\sqrt{\left(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)} - \frac{\sigma'''(0)}{2}\right)^{-1}}(e^{i\theta}\xi + e^{-i\theta}\xi^*) + O(|\epsilon|^3), \tag{1.3}$$

or more precisely,

$$k_0^2\epsilon\sqrt{\left(\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2-5\delta_c)} - \frac{\sigma'''(0)}{2}\right)^{-1}}\left(e^{i(\theta+k_0)x}\begin{pmatrix}1\\ -ia_ck_0^3\end{pmatrix} + e^{-i(\theta+k_0)x}\begin{pmatrix}1\\ ia_ck_0^3\end{pmatrix}\right) + O(|\epsilon|^3). \tag{1.4}$$

The family of bifurcated oscillation waves is homeomorphic to S^1 due to the range of the phase constant θ and forms a closed orbit around the trivial equilibrium. This closed orbit has diameter $O(|\mu|^{1/2}) = O(|(\delta_c - k_0^2)(a-a_c) + a_c(\delta-\delta_c)|^{1/2})$ in the bifurcation range for (1.1) when the parameters a and δ vary around any admissible critical configuration point. Please see Section 6.6 and Section 7 for more details. Here we remark that the oscillations in the current dynamics driven by the system of partial differential equation (1.1) is not due to the bifurcation of rotation waves though the system exhibits Euclidean symmetry – the $O(2)$ symmetry. We also point out that the wave number k_0 enters the dynamics of (1.1) on the center manifold(s) through k_0^2. This is natural due to symmetry of the system and in particular implies that the dynamics of the system (1.1) are exactly the same for $\pm k_0$. Furthermore, as the sign of k_0^2 is always the same for any nonzero wave number k_0, we could conclude by Theorem 1.2 that the bifurcation dynamics of (1.1) under the spectral scenario in the current paper are topologically equivalent in the sense of continuous dynamical systems for any nonzero wave number k_0.

We shall also emphasize that we get the dynamics on the center manifold to arbitrary orders allowed by the smoothness of the flux functions. Hence we get, by using symmetry, the full dynamics on the center manifold in the sense of the unique finite-order approximation property of the center manifold reduction function(s). From the dynamical behaviors of the system on center manifold, we could observe that the third-order term in the flux also plays an important role for the current study. This is in sharp contrast to our former study on the equivariant Hopf bifurcation case in which the third-order term only enters the angular equations and hence does not influence the stability. Besides, the computations in the current study are different from our former methods in which normal form theory is involved. As σ'' is not involved in the definition of an admissible critical configuration point, a direct consequence of the above observation is that genuine nonlinearity of the corresponding first-order system in (1.1) is not needed to support the current dynamics. More precisely, system (1.1) still supports the current stationary equivariant bifurcation dynamics even if $\sigma''(0) = 0$. Of course, we need $\sigma'''(0) \neq 0$ to avoid degeneracy. As we can also allow $\sigma'(0) = 0$ (see Defintion 1.1 or Section 3 below), it is fair to see that the current bifurcation mechanism is not solely driven by the first-order hyperbolic system but also by the interplay between the diffusions and nonlinearity in the flux function. This is another difference of the current result with our former results on equivariant Hopf bifurcation for the same system (1.1). Surprisingly, $\sigma''(0)$ also enters the bifurcation dynamics on the center manifold(s) through squares. In the setting of hyperbolic conservation laws, we know that the convex flux and concave flux induce different behaviors of wave phenomena and their stability property. However, here the sign of $\sigma'(0)$ if it is non-vanishing (or equivalently the local convexity or concavity of the flux function around the reference uniform solution), is not essential.

1.3. Discussion and future work. The series of research works (the current work, the previous work of [32] and [22]) aim to study the equivariant dynamics driven by partial differential equations which exhibit various symmetries from dynamical system point of view and using techniques in partial differential equations. These symmetries are typically represented by certain isometry group actions. See, for example, [28] for some physical systems such as compressible Navier-Stokes equations, magnetohydrodynamics (MHD) and viscoelasticity models. As most physical systems exhibit symmetry, we shall further explore the role of the symmetry in the study of dynamics of partial differential equations. We shall also study similar models in higher spatial dimensions, especially the most interesting two and three dimensional ones. A direct obstruction for carrying out similar programs is that the dimensions of the center spaces are much larger due to higher spatial dimensions and symmetry.

After the completion of the current work, we found another elegant way to get the current equivariant bifurcation dynamics through Lyapunov-Schmidt reduction and normal form arguments. And actually, the "spectum+normal form+dynamical decomposition" argument can apply to various dynamical systems driven by partial differential equations with symmetries. Related work will be reported elsewhere.

We refer the reader to the works of Bressan [2–4], Carr [6], Chicone [9], Golubitsky-Stewart-Schaeffer [13], Haragus and Iooss [8], Henry [14], Iooss and Adelmeyer [16], Jia and Sverak [17, 18], Moehlis-Knobloch [23], Ma and Wang [24, 25], Nakanishi and Schlag [26] and the references therein for similar and related studies in a variety of settings.

Convention. We will use "$*$" to denote "complex conjugate", i.e., for $z\in\mathbb{C}$, z^* means the complex conjugate of z; "$\int f\,dx$" means "$\int_{-\pi}^{\pi} f\,dx$" for the integrand f; for two nonnegative quantities, "$A\lesssim B$" means "$A\le CB$" for some constant $C>0$. For $U=\begin{pmatrix}u_1\\u_2\end{pmatrix}, V=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in\mathbb{C}^2$, "$\langle U,V\rangle''$ means "$\int_{-\pi}^{\pi} u_1v_1^*+u_2v_2^*\,dx$". We also adopt the standard big O "$\mathcal{O}$" and small o "o" notations for limiting processes. For a vector V, we use v_j or $v^{(j)}$ to represent its components. We use "$[\cdot,\cdot]$" to denote commutator: $[F,G]=FG-GF$ for F, G being functions, symbols or operators. For a linear operator $\mathcal{L}$: $\mathcal{X}\mapsto\mathcal{X}$ on some Banach space $\mathcal{X}$, we use $\sigma(\mathcal{L})$ and $\rho(\mathcal{L})$ to denote its spectral set and resolvent set. Further, $\sigma(\mathcal{L})=\sigma_c(\mathcal{L})\cup\sigma_s(\mathcal{L})\cup\sigma_u(\mathcal{L})$, i.e., $\sigma(\mathcal{L})$ is the union of the center spectral set $\sigma_c(\mathcal{L})$, the stable spectral set $\sigma_s(\mathcal{L})$ and the unstable spectral set $\sigma_u(\mathcal{L})$. The associated space decomposition is $\mathcal{X}=\mathcal{X}_c\cup\mathcal{X}_s\cup\mathcal{X}_u$ where $\mathcal{X}_c:=\left(\frac{1}{2\pi i}\oint_{\gamma_c}(\lambda-\mathcal{L})^{-1}\,d\lambda\right)\mathcal{X}$ is the center space, $\mathcal{X}_s:=\left(\frac{1}{2\pi i}\oint_{\gamma_s}(\lambda-\mathcal{L})^{-1}\,d\lambda\right)\mathcal{X}$ is the stable space, $\mathcal{X}_u:=\left(\frac{1}{2\pi i}\oint_{\gamma_u}(\lambda-\mathcal{L})^{-1}\,d\lambda\right)\mathcal{X}$ is the unstable space; the hyperbolic space is given by $\mathcal{X}_h:=\mathcal{X}_s\cup\mathcal{X}_u$. The γ_j above is any closed simple curve in the complex plane containing only spectrum of type $j=c,s,u$.

2. Functional analytic setting

Now we start our analysis by settling down the functional analytic framework.

First, we write the system (1.1) in the form of nonlinear perturbation system. For this purpose, we need to regard (τ,u) as the perturbation variable around the state (0,0) and write (1.1) as the following system

$$\begin{cases} \partial_t\tau - \partial_x u = -a\partial_x^4\tau \\ \partial_t u - \sigma'(0)\partial_x\tau = -\delta\partial_x^2 u - \partial_x^4 u + \partial_x\left(\frac{\sigma''(0)}{2}\tau^2 + \frac{\sigma'''(0)}{6}\tau^3 + \Gamma(\tau)\right) \end{cases} \tag{2.1}$$

where $\Gamma(\tau) := \sigma(\tau) - \sigma(0) - \frac{\sigma''(0)}{2}\tau^2 - \frac{\sigma'''(0)}{6}\tau^3$ and $\Gamma(\tau) = \mathcal{O}(|\tau|^4)$ when $|\tau|$ is small. We shall keep in mind that the system (2.1) is equivalent to the system (1.1).

Second, we write the nonlinear perturbation system (2.1) into operator equation form. For this purpose, we denote $U = \begin{pmatrix}\tau \\ u\end{pmatrix}$ and $\mathcal{L},\mathcal{N}$ as follows:

$$\mathcal{L} := \begin{pmatrix} -a\partial_x^4 & \partial_x \\ \sigma'(0)\partial_x & -\delta\partial_x^2 - \partial_x^4 \end{pmatrix} \tag{2.2}$$

and

$$\mathcal{N}\begin{pmatrix}\tau \\ u\end{pmatrix} := \begin{pmatrix} 0 \\ \partial_x\left(\frac{\sigma''(0)}{2}\tau^2 + \frac{\sigma'''(0)}{6}\tau^3 + \Gamma(\tau)\right) \end{pmatrix}. \tag{2.3}$$

With the above notations, the nonlinear perturbation system can be written symbolically as

$$\partial_t U = \mathcal{L}U + \mathcal{N}(U). \tag{2.4}$$

In order to emphasize the linear operator and nonlinear term, we may also write the system (1.1) as

$$\partial_t\begin{pmatrix}\tau \\ u\end{pmatrix} = \begin{pmatrix} -a\partial_x^4 & \partial_x \\ \sigma'(0)\partial_x & -\delta\partial_x^2 - \partial_x^4 \end{pmatrix}\begin{pmatrix}\tau \\ u\end{pmatrix} + \mathcal{N}\begin{pmatrix}\tau \\ u\end{pmatrix}. \tag{2.5}$$

Third, we need to decide the space triplet under which we work and achieve our specific goals. The choice of working space triplet is nontrivial (See also Section 7). Here $\mathcal{L}$ is a fourth-order linear differential operator on the periodic domain $\mathbb{R}^1/[-\pi,\pi]$. We may consider it as a linear operator on the space $L^2_{per}(-\pi,\pi)$ with domain $H^4_{per}(-\pi,\pi)$. Meanwhile, if we seek solutions (τ,u) on the periodic Sobolev spaces, the quantities $\int\tau\,dx$ and $\int u\,dx$ are conserved due to the conservative form of the original system (1.1) as follows:

$$\partial_t\begin{pmatrix}\tau \\ u\end{pmatrix} = \partial_x\begin{pmatrix} u - a\partial_x^3 \\ \sigma(\tau) - \delta\partial_x u - \partial_x^3 u \end{pmatrix}. \tag{2.6}$$

Taking into account of the above considerations and the requirement of center manifold theory, we will first (but also see Section 7) work on the space triplet $Z \subset Y \subset X$ given below:

$$X := \{U \in L^2_{per}(-\pi,\pi); \int_{-\pi}^{\pi} U\,dx = 0\},$$

$$Y = X = \{U \in L^2_{per}(-\pi,\pi); \int_{-\pi}^{\pi} U\,dx = 0\},$$

$$Z := \{U \in H^4_{per}(-\pi,\pi); \int_{-\pi}^{\pi} U\,dx = 0\}.$$

It is important to notice that the mean zero restriction which comes naturally from the conservative structure of system (1.1) also has influence on the spectra of the linear operator $\mathcal{L}(a_c,\delta_c)$.

3. Spectral analysis I

With the functional analytic preparations above, we are now ready to study the spectra of the linear operator $\mathcal{L}$ and give explanations of the admissible configuration set associated with an arbitrarily fixed positive interger k_0. Based on our choice of space triplet, we shall regard

$$\mathcal{L} = \mathcal{L}(a,\delta) := \begin{pmatrix} -a\partial_x^4 & \partial_x \\ \sigma'(0)\partial_x & -\delta\partial_x^2 - \partial_x^4 \end{pmatrix} \tag{3.1}$$

as a linear operator on the space X with domain Z to study its spectra. To this end, we can proceed by Fourier analysis as we are working on periodic domains. After Fourier transformation, the differential operator is represented by

$$M_k = \begin{pmatrix} -ak^4 & ik \\ \sigma'(0)ki & \delta k^2 - k^4 \end{pmatrix}, \, k \in \mathbb{Z}, k \neq 0.$$

Therefore, we have

$$\sigma(\mathcal{L}) = \cup_{k\in\mathbb{Z},\, k\neq 0}\sigma(M_k).$$

The mode $k=0$ is not included in the above union because we have mean 0 restriction in the definition of X. The eigenvalues λ of M_k for $k\neq 0$ are given by

$$\begin{aligned} \det(\lambda - M_k) &= \det\begin{pmatrix} \lambda + ak^4 & -ik \\ -\sigma'(0)ki & \lambda - \delta k^2 + k^4 \end{pmatrix} \\ &= (\lambda + ak^4)(\lambda - \delta k^2 + k^4) + \sigma'(0)k^2 \\ &= \lambda^2 + \Big((a+1)k^4 - \delta k^2\Big)\lambda + ak^4(k^4 - \delta k^2) + \sigma'(0)k^2 \\ &= 0. \end{aligned}$$

Before further analysis, we first notice that in the formula for the eigenvalues of M_k, i.e.,

$$\lambda^2 + \Big((a+1)k^4 - \delta k^2\Big)\lambda + ak^4(k^4 - \delta k^2) + \sigma'(0)k^2 = 0, \tag{3.2}$$

k enters the equation through k^2, which in particular implies $\sigma(M_k) = \sigma(M_{-k})$. There is no surprise here as this is a direct consequence of the $O(2)$-symmetry exhibited by the system (1.1) (see Section 7).

Now we explain the admissible critical configuration sets $\mathcal{A}(k_0)$ associated with a nonzero (positive) integer k_0 in this paper. As we have studied the equivariant Hopf bifurcation spectral scenario in [22, 32], we study here the spectral scenario that there is one and only one spectral curve crossing the imaginary axis in $\mathbb{C}^1$ through the origin with the purpose of covering the generic cases.

Now let us fix the nonzero (positive) integer k_0. To achieve the above spectral crossing scenario based on the wave number k_0, necessarily we need that M_{k_0} contributes zero spectrum for the linear operator $\mathcal{L}$ while all the other M_k for $|k| \neq k_0$ do not. In other words, we require the following necessary conditions (a) and (b) in (3.2):

(a) *for the nonzero integer* k_0,

$$a_c k_0^4 (k_0^2 - \delta_c) + \sigma'(0) = 0.$$

(b) *for any nonzero integer* k *such that* $|k| \neq |k_0|$, *and* $\omega \in \mathbb{R}^1 - \{0\}$,

$$[a_c k^4 (k^4 - \delta_c k^2) + \sigma'(0) - \omega^2] + i[(a_c + 1)k^4 - \delta_c k^2] \neq 0.$$

The above conditions (a) and (b) give the necessary conditions for the spectral crossing scenario. We write condition (b) in the above form for the ease of verifying consistency. As we may expect or guess, it is not sufficient for our bifurcation analysis as we need the crossing to be transverse in order to complete the bifurcation analysis. It turns out that conditions (a), (b) together with the following nondegeneracy condition (c) are sufficient for our purpose:

(c) *for the above* k_0, $(a_c + 1)k_0^2 - \delta_c \neq 0$.

The reasons that we call condition (c) a nondegeneracy condition lie in the following two observations: (1) the condition $(a_c + 1)k_0^2 - \delta_c \neq 0$ means that M_{k_0} does not contribute repeated zero spectrum for $\mathcal{L}$ as $(a + 1)k^4 - \delta k^2 = ((a + 1)k^2 - \delta)k^2$ is the coefficient of the first-order term in the spectral variable λ in (3.2); (2) when we compute the dual kernel of the linear operator $\mathcal{L}(a_c, \delta_c)$ during the reduction procedure, $(a_c + 1)k_0^2 - \delta_c$ appears in the denominator, see Remark 6.1.

We may notice that the expression $a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0)$ also appears as a denominator during the reduction procedure, see in particular (6.7). However, we do not need to worry about if $a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0)$ is nonzero or not. Actually, under conditions (a), (b) and (c), the expression $a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0)$ is automatically nonzero, which is a direct consequence of the paragraph above (6.7) (see also Remark 6.3). Therefore, in the definition of $\mathcal{A}(k_0)$, the following condition (d) is implicitly included:

(d) $a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0) \neq 0$.

Before we do further spectral analysis, we need to first verify that there exists flux function $\sigma(\tau)$ such that $\mathcal{A}(k_0)$ is nonempty for some nonzero (positive) integer k_0, i.e., (a), (b) and (c) are consistent for some k_0 as above, which is enough for us to carry out the remaining parts of the program. Actually, we can easily show that there exist flux functions $\sigma(\tau)$ such that $\mathcal{A}(k_0)$ is nonempty for any nonzero (positive) integer k_0. For this purpose, we claim:

(*Consistency*) *conditions* (a)-(d) *are consistent* in the sense above.

This is easily seen by considering the special case $\sigma'(0) = 0$. Then (a) and (d) require that $a_c \neq 0$ and $\delta_c = k_0^2$. As long as $a_c \neq 0$ and $\delta_c = k_0^2$, (a), (c) and (d) are satisfied. We can then adjust $a_c > 0$ to satisfy (b), which is easy. In particular if $k_0 = \pm 1$, (b) is satisfied since $(a_c + 1)k_0^4 - \delta_c k_0^2 = a_c \neq 0$. Notice that we have no requirement on the sign of a_c here. Alternatively, we can let $a_c < -1$ to satisfy (b). In particular, the operator $\mathcal{L}(a_c,\delta_c)$ is not a sectoral operator if $a_c < -1$.

Another easy way to see the consistency is to choose $\delta_c = 0$, $a_c \neq 0, -1$ and $a_c k_0^6 + \sigma'(0) = 0$.

Next we give several remarks based on the above analysis and the whole paper.

REMARK 3.1. From the analysis in the above two paragraphs, we see that $\mathcal{A}(k_0)$ is generically nonempty for any nonzero integer k_0 and the conditions (a), (b) and (c) are actually rather mild though we may regard on first sight that they impose strong constraints on the flux functions or the parameters.

REMARK 3.2. Without condition (c), we can still show the existence of center manifold reduction but we have some problems in computation as this appears as a denominator.

REMARK 3.3. As long as $a_c \neq 0$, the resolvent estimate in Lemma 5.1 is valid, which will be sufficient to guarantee the existence of center manifold. $a_c \neq 0$ as long as $(a_c,\delta_c) \in \mathcal{A}(k_0)$, which is easily seen from conditions (a) and (d).

4. Spectral analysis II

From now on, we let (a_c,δ_c) be an admissible critical configuration point. Due to the spectral preparations in the last section, we are now on a sound foundation to do the bifurcation analysis. In this section, we will introduce the bifurcation system and make some spectral preparations.

First, we introduce several notations:

$$\nu = (\nu_1,\nu_2) := (a - a_c, \delta - \delta_c), \quad \mu = \mu(\nu) := (\delta_c - k_0^2)\nu_1 + a_c \nu_2. \tag{4.1}$$

Technically, we can do the bifurcation analysis for the following three cases: (1) δ varies around δ_c with $a = a_c$ being fixed; (2) a varies around a_c with $\delta = \delta_c$ being fixed; (3) (a,δ) vary around (a_c,δ_c). Formally, case (3) covers cases (1) and (2) and we will first do the analysis based on case (3). However, there are points to be remarked about cases (1) and (2) from both the mathematical and applicational points of view, see Section 7.

Second, we isolate the bifurcation parameter μ to get the bifurcation system. For this purpose, we write system (2.1) in the following form

$$\partial_t U = \mathcal{L}(a_c,\delta_c)U + (\mathcal{L}(a,\delta) - \mathcal{L}(a_c,\delta_c))U + \mathcal{N}(U), \tag{4.2}$$

where we have the obvious identifications:

$$\mathcal{L}(a_c,\delta_c) = \begin{pmatrix} -a_c\partial_x^4 & \partial_x \\ \sigma'(0)\partial_x & -\delta_c\partial_x^2 - \partial_x^4 \end{pmatrix}, \tag{4.3}$$

$$\mathcal{L}(a,\delta) - \mathcal{L}(a_c,\delta_c) = \begin{pmatrix} -\nu_1\partial_x^4 & 0 \\ 0 & -\nu_2\partial_x^2 \end{pmatrix} \tag{4.4}$$

and

$$\mathcal{N}(U) = \begin{pmatrix} 0 \\ \partial_x\left(\frac{\sigma''(0)}{2}\tau^2 + \frac{\sigma'''(0)}{6}\tau^3 + \Gamma(\tau)\right) \end{pmatrix} \tag{4.5}$$

Now we introduce $R(U,\nu) := R_{11}(U,\nu) + R_{20}(U,U) + R_{30}(U,U,U) + \tilde{R}(U)$ where

$$R_{11}(U,\nu) := \begin{pmatrix} -\nu_1\partial_x^4 U^{(1)} \\ -\nu_2\partial_x^2 U^{(2)} \end{pmatrix}, R_{20}(U,V) := \begin{pmatrix} 0 \\ \frac{\sigma''(0)}{2}\partial_x(U^{(1)}V^{(1)}) \end{pmatrix},$$

$$R_{30}(U,V,W) := \begin{pmatrix} 0 \\ \frac{\sigma'''(0)}{6}\partial_x(U^{(1)}V^{(1)}W^{(1)}) \end{pmatrix}, \; \tilde{R}(U) := \begin{pmatrix} 0 \\ \partial_x\Gamma(U^{(1)}) \end{pmatrix},$$

and write the nonlinear perturbation system (4.2) around $\mu = 0$ as

$$\partial_t U = \mathcal{L}(a_c,\delta_c)U + R(U,\nu). \tag{4.6}$$

The above equation (4.6) is the *bifurcation equation or system.*

Finally, we make some spectral preparations for the use of center manifold theory.

LEMMA 4.1. $\sigma_c(\mathcal{L}(a_c,\delta_c)) = \{0\}$.

Proof. This is a direct consequence of the definition of the admissible critical configuration set $\mathcal{A}(k_0)$. ∎

Concerning the spectrum of $\sigma(\mathcal{L}(a_c,\delta_c))$, we also have the following lemma:

LEMMA 4.2. *There exists a positive constant $\gamma > 0$, such that* $\sup\{Re\lambda; \lambda \in \sigma_s(\mathcal{L}(a_c,\delta_c))\} < -\gamma$ *and* $\inf\{Re\lambda; \lambda \in \sigma_u(\mathcal{L}(a_c,\delta_c))\} > \gamma$.

Proof. We need to consider the distribution of the roots of equation (3.2) for $k \neq k_0$. By symmetry, we just need to consider the case $|k_0| \neq k \in \mathbb{N}$. By Lemma 4.1, we know that roots of equation (3.2) for $k \neq k_0$ do not lie on the imaginary axis. Hence we just need to show that there is no accumulation of spectra to the imaginary axis when $k \to +\infty$. This is obvious by writing down the solutions explicitly through quadratic formula: the real parts of the roots of (3.2) can only tend to $\pm\infty$. ∎

REMARK 4.1. $\cup_{0 \neq k_0 \in \mathbb{Z}} \mathcal{A}(k_0)$ contains all the admissible critical configuration points. Our bifurcation analysis will be done in particular around $\nu = (\nu_1,\nu_2) = (0,0)$. The role of the parameter μ will be self-evident after we get the reduced dynamics (see (6.12)).

5. Existence of parameter-dependent center manifold

In this section, we show the existence of parameter-dependent center manifold for the bifurcation system (4.6) or equivalently the system (4.2) or (1.1). The main ingredient that remains to be shown is a resolvent estimate. A similar estimate was first done in [32]. However, we will reproduce it here to see the roles of the defining conditions in $\mathcal{A}(k_0)$ and correct some misprints in [32] and also for completeness. Meanwhile, we will add symmetries into consideration in order to make comparisons with our former works (see Section 7).

Let us first state a version of the parameter-dependent center manifold theorem with group actions.

THEOREM 5.1 (Parameter-dependent center manifold theorem with symmetries, see [8, 13, 32]). *Let the inclusions in the Banach space triplet $\mathcal{Z} \subset \mathcal{Y} \subset \mathcal{X}$ be continuous. Consider a differential equation in a Banach space $\mathcal{X}$ of the form*

$$\frac{du}{dt} = \mathcal{L}u + \mathcal{R}(u,\nu)$$

and assume that

(1) (Assumption on linear operator and nonlinearity) $\mathcal{L} : \mathcal{Z} \mapsto \mathcal{X}$ is a bounded linear map and for some $k \geq 2$, there exist neighborhoods $\mathscr{V} \subset \mathcal{Z}$ and $\mathscr{V}_\nu \subset \mathbb{R}^m$ of $(0,0)$ such that $\mathcal{R} \in C^k(\mathscr{V}_u \times \mathscr{V}_\nu, \mathcal{Y})$ and

$$\mathcal{R}(0,0) = 0, \; D_u\mathcal{R}(0,0) = 0.$$

(2) (Spectral decomposition) there exists some constant $\gamma > 0$ *such that*

$$\inf\{Re\lambda; \lambda \in \sigma_u(\mathcal{L})\} > \gamma, \sup\{Re\lambda; \lambda \in \sigma_s(\mathcal{L})\} < -\gamma,$$

and the set $\sigma_c(\mathcal{L})$ *consists of a finite number of eigenvalues with finite algebraic multiplicities.*

(3) (Resolvent estimates) For Hilbert space triplet $\mathcal{Z} \subset \mathcal{Y} \subset \mathcal{X}$*, assume there exists a positive constant* $\omega_0 > 0$ *such that* $i\omega \in \rho(\mathcal{L})$ *for all* $|\omega| > \omega_0$ *and* $\|(i\omega - \mathcal{L})^{-1}\|_{\mathcal{X}\mapsto\mathcal{X}} \lesssim \frac{1}{|\omega|}$*. For Banach space triplet, we need further* $\|(i\omega - \mathcal{L})^{-1}\|_{\mathcal{Y}\mapsto\mathcal{X}} \lesssim \frac{1}{|\omega|^{\alpha}}$ *for some* $\alpha \in [0,1)$*.*

Then there exists a map $\Psi \in \mathcal{C}^k(Z_c, Z_h)$ *and a neighborhood* $\mathscr{O}_u \times \mathscr{O}_\nu$ *of* $(0,0)$ *in* $\mathcal{Z} \times \mathbb{R}^m$ *such that*

(a) (Tangency) $\Psi(0,0) = 0$ *and* $D_u\Psi(0,0) = 0$*.*

(b) (Local flow invariance) the manifold $\mathcal{M}_0(\nu) = \{u_0 + \Psi(u_0,\nu); u_0 \in Z_c\}$ *has the properties. (i)* $\mathcal{M}_0(\nu)$ *is locally invariant, i.e., if* u *is a solution satisfying* $u(0) \in \mathcal{M}_0(\nu) \cap \mathscr{O}_\nu$ *and* $u(t) \in \mathcal{O}_u$ *for all* $t \in [0,T]$*, then* $u(t) \in \mathcal{M}_0(\nu)$ *for all* $t \in [0,T]$*; (ii)* $\mathcal{M}_0(\nu)$ *contains the set of bounded solutions staying in* $\mathcal{O}_u$ *for all* $t \in \mathbb{R}^1$*, i.e., if* u *is a solution satisfying* $u(t) \in \mathcal{O}_u$ *for all* $t \in \mathbb{R}^1$*, then* $u(0) \in \mathcal{M}_0(\nu)$*.*

(c) (Symmetry) Moreover, if the vector field is equivariant in the sense that there exists an isometry $\mathscr{T} \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Z})$ *which commutes with the vector field in the original system,*

$$[\mathscr{T}, \mathcal{L}] = 0, \, [\mathscr{T}, \mathcal{R}] = 0,$$

then Ψ *commutes with* $\mathscr{T}$ *on* Z_c*:* $[\Psi, T] = 0$*.*

It is easy to see that in the above Theorem 5.1, the linear operator is not required to be sectorial. Another subtle point is that the center manifolds are generally not unique but the center manifold reduction function can be approximated uniquely up to any finite order (hence we use "manifold" and "manifolds" interchangeably) as long as the nonlinearity has enough smoothness (see for example, Theorem 2.5 on page 35 and Theorem 10 on page 120 of [6]). More insight can be drawn here. In particular, this uniqueness of approximation to any finite order enables us to get full dynamics on center manifold through center manifold reduction. Of course, the distinct properties of the systems under consideration should be taken into account. We have the following lemma regarding system (4.6).

Lemma 5.1 (Existence of parameter-dependent center manifolds). *For system* (4.6)*, there exists a map* $\Psi \in \mathcal{C}^k(Z_c \times \mathbb{R}^1, Z_h)$*, with*

$$\Psi(0,0) = 0, \, D_U\psi(0,0) = 0,$$

and a neighborhood of $\mathcal{O}_U \times \mathcal{O}_\nu$ *of* $(0,0)$ *such that for* $\nu \in \mathcal{O}\mu$*, the manifold*

$$\mathcal{M}_0(\nu) := \{U_0 + \Psi(U_0, \nu); \, U_0 \in Z_c\}$$

is locally invariant and contains the set of bounded solutions of the nonlinear perturbation system in $\mathcal{O}_U$ *for all* $t \in \mathbb{R}$*.*

Proof. From the definition of the operator $\mathcal{L}(a_c, \delta_c)$ and $R(U,\nu)$ in (4.6), we know that the assumptions (1) and (2) in Theorem 5.1 on the linear operator and nonlinearity hold. *A subtle point here is that the highest order of derivatives in* $R(U,\nu)$ *is four,* which is allowed by our space triplet choice $Z \subset Y \subset X$ in Section 2 and Theorem

5.1. The spectral decomposition assumption is a direction consequence of the analysis in Section 4, see Lemma 4.1 and Lemma 4.2. It is obvious that $i\omega \in \rho(\mathcal{L}(a_c,\delta_c))$ for $|\omega|>0$. To show the resolvent estimate in the current Hilbert space triplet setting, we write $(i\omega-\mathcal{L}(a_c,\delta_c))U=\tilde{U}$ for $\tilde{U}=\begin{pmatrix}\tilde{\tau}\\ \tilde{u}\end{pmatrix}\in X$ and $U=\begin{pmatrix}\tau\\ u\end{pmatrix}\in Z$ and show that $\|U\|_X \lesssim \frac{1}{|\omega|}\|\tilde{U}\|_X$ for $|\omega|>\omega_0>0$ where $\omega_0 \gg 1$ is a large constant. Without loss of generality, we just need to prove for $\omega \geq \omega_0$. Multiplying the equation $(i\omega-\mathcal{L}(a_c,\delta_c))U=\tilde{U}$ by U^*, integrating over $[-\pi,+\pi]$ and integrating by parts, we arrive at

$$i\omega|\tau|_{L^2}^2+a_c|\partial_x^2\tau|_{L^2}^2-\int \partial_x u\tau^* = \int \tilde{\tau}\tau^*;$$

$$\int -\sigma'(0)\partial_x\tau u^* + i\omega|u|_{L^2}^2 - \delta_c\int|\partial_x u|^2+\int|\partial_x^2 u|^2=\int \tilde{u}u^*.$$

Taking the imaginary and real parts respectively, we see

$$\omega|\tau|_{L^2}^2 = Im\int \partial_x u\tau^* + Im\int \tilde{\tau}\tau^*,$$

$$\omega|u|_{L^2}^2=\sigma'(0)Im\int \partial_x\tau u^* + Im\int \tilde{u}u^*$$

and

$$a_c|\partial_x^2\tau|_{L^2}^2 = Re\int \partial_x u\tau^* + Re\int \tilde{\tau}\tau^*,$$

$$-\delta_c|\partial_x u|_{L^2}^2+\int|\partial_x^2 u|^2 = Re\int \tilde{u}u^* + Re\int \sigma'(0)\partial_x\tau u^*.$$

From the imaginary part equations, we get by using elementary inequalities, Fourier analysis and mean zero property for elements in the space X, that

$$\omega|\tau|_{L^2}^2 \lesssim \frac{1}{\omega}\Big(|\partial_x u|_{L^2}^2+|\tilde{\tau}|_{L^2}^2\Big) \leq \frac{1}{\omega}\Big(|\partial_x^2 u|_{L^2}^2+|\tilde{\tau}|_{L^2}^2\Big), \tag{5.1}$$

$$\omega|u|_{L^2}^2 \lesssim \frac{1}{\omega}\Big(|\partial_x \tau|_{L^2}^2+|\tilde{u}|_{L^2}^2\Big) \leq \frac{1}{\omega}\Big(|\partial_x^2 \tau|_{L^2}^2+|\tilde{u}|_{L^2}^2\Big). \tag{5.2}$$

In view of $a_c \neq 0$ (see Remark (3.3)), we obtain from the real part equations that

$$|\partial_x^2\tau|_{L^2}^2 \lesssim |\tau|_{L^2}\Big(|\tilde{\tau}|_{L^2}+|\partial_x u|_{L^2}\Big), \tag{5.3}$$

$$|\partial_x^2 u|_{L^2}^2 \lesssim |u|_{L^2}\Big(|\tilde{u}|_{L^2}+|\partial_x\tau|_{L^2}\Big)+|\partial_x u|_{L^2}^2. \tag{5.4}$$

Notice that (5.4) is valid for any real number δ_c. In particular, δ_c can be 0. By interpolation, we know that for any $\epsilon>0$, the following holds

$$|\partial_x u|_{L^2}^2 \leq \epsilon|\partial_x^2 u|_{L^2}^2 + C(\epsilon)|u|_{L^2}^2.$$

We may pick ϵ so small that we can conclude from (5.4) that

$$|\partial_x^2 u|_{L^2}^2 \lesssim |u|_{L^2}\Big(|\tilde{u}|_{L^2} + |\partial_x \tau|_{L^2} + |u|_{L^2}\Big).$$

By interpolation in $|\partial_x \tau|_{L^2}$,

$$|\partial_x \tau|_{L^2}^2 \lesssim \epsilon |\partial_x^2 \tau|_{L^2}^2 + C(\epsilon)|\tau|_{L^2}^2,$$

we can get from (5.3) and (5.4) that

$$\begin{cases} |\partial_x^2 \tau|_{L^2}^2 \lesssim |\tau|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + \epsilon|\partial_x^2 u|_{L^2}^2 + |u|_{L^2}^2, \\ |\partial_x^2 u|_{L^2}^2 \lesssim |u|_{L^2}^2 + |\tilde{u}|_{L^2}^2 + \epsilon|\partial_x^2 \tau|_{L^2}^2 + \epsilon|\partial_x^2 u|_{L^2}^2. \end{cases} \tag{5.5}$$

Adding the two equations in (5.5) together and choosing ϵ smaller if necessary, we get

$$|\partial_x^2 \tau|_{L^2}^2 + |\partial_x^2 u|_{L^2}^2 \lesssim |\tau|_{L^2}^2 + |u|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + |\tilde{u}|_{L^2}^2. \tag{5.6}$$

Now we have in view of (5.1), (5.2) and (5.6) that

$$\begin{aligned} \omega|\tau|_{L^2}^2 &\lesssim \frac{1}{\omega}|\tilde{\tau}|_{L^2}^2 + \frac{1}{\omega}|\partial_x^2 u|_{L^2}^2 \\ &\lesssim \frac{1}{\omega}|\tilde{\tau}|_{L^2}^2 + \frac{1}{\omega}\Big(|\tau|_{L^2}^2 + |u|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + |\tilde{u}|_{L^2}^2\Big). \end{aligned}$$

$$\begin{aligned} \omega|u|_{L^2}^2 &\lesssim \frac{1}{\omega}|\tilde{u}|_{L^2}^2 + \frac{1}{\omega}|\partial_x^2 \tau|_{L^2}^2 \\ &\lesssim \frac{1}{\omega}|\tilde{u}|_{L^2}^2 + \frac{1}{\omega}\Big(|\tau|_{L^2}^2 + |u|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + |\tilde{u}|_{L^2}^2\Big). \end{aligned}$$

Adding the above two inequalities together, we get

$$(-\frac{1}{\omega} + \omega)|U|_{L^2}^2 \lesssim \frac{1}{\omega}|\tilde{U}|_{L^2}^2,$$

which implies

$$|U|_{L^2}^2 \lesssim \frac{1}{\omega(\omega - \frac{1}{\omega})}|\tilde{U}|_{L^2}^2 = \frac{1}{\omega^2 - 1}|\tilde{U}|_{L^2}^2 \lesssim \frac{1}{\omega^2}|\tilde{U}|_{L^2}^2.$$

for $\omega \geq \omega_0$ large. Hence we arrive at the estimate of desired form $|U|_{L^2} \lesssim \frac{1}{\omega}|\tilde{U}|_{L^2}$. ∎

REMARK 5.1. The second inequalities in (5.1) and (5.2) hold with our choice of space triplet as there are derivatives acting on the function τ and u.

Now we define the following operations

$$R_\phi \begin{pmatrix} \tau(x) \\ u(x) \end{pmatrix} = \begin{pmatrix} \tau(x+\phi) \\ u(x+\phi) \end{pmatrix}, \quad S\begin{pmatrix} \tau(x) \\ u(x) \end{pmatrix} = \begin{pmatrix} \tau(-x) \\ -u(-x) \end{pmatrix}, \tag{5.7}$$

for any $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. We can easily verify that system (1.1) (equivalently (4.2)) is equivariant:

$$[R_\phi, \mathcal{L}(a_c,\delta_c)] = 0, [R_\phi, R(\cdot,\nu)] = 0, [S, R(\cdot,\nu)] = 0, [S, \mathcal{L}(a_c,\delta_c)] = 0, R_\phi S = SR_{-\phi}. \tag{5.8}$$

Therefore, the center manifold function in Lemma 5.1 also inherits the above symmetries.

6. Dynamics on center manifold

In this section, we compute and analyze the dynamics on the center manifold for our bifurcation system (4.6). As we want to demonstrate some subtle points during the process and also to make the exposition clear, we divide this section into subsections.

6.1. Center space and parametrization. First, we compute the center space of the operator $\mathcal{L}(a_c,\delta_c)$. Recall that

$$\mathcal{L}(a_c,\delta_c):=\begin{pmatrix} -a_c\partial_x^4 & \partial_x \\ \sigma'(0)\partial_x & -\delta_c\partial_x^2-\partial_x^4 \end{pmatrix}.$$

For the wave number k_0, letting $\xi=e^{ik_0x}V=e^{ik_0x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, we get in the Fourier side the system satisfied by V:

$$\begin{pmatrix} -a_ck_0^4 & ik_0 \\ \sigma'(0)ik_0 & \delta_ck_0^2-k_0^4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6.1}$$

which is equivalent to

$$\begin{cases} -a_ck_0^4v_1+ik_0v_2=0 \\ \sigma'(0)ik_0v_1+(\delta_ck_0^2-k_0^4)v_2=0. \end{cases} \tag{6.2}$$

From condition (a) in the definition of $\mathcal{A}(k_0)$, we know that

$$\det\begin{pmatrix} -a_ck_0^4 & ik_0 \\ \sigma'(0)ik_0 & \delta_ck_0^2-k_0^4 \end{pmatrix}=k_0^2[a_ck_0^4(k_0^2-\delta_c)+\sigma'(0)]=0$$

and get

$$a_ck_0^4v_1=ik_0v_2, \quad i.e., \quad v_2=\frac{a_ck_0^4}{ik_0}v_1=-ia_ck_0^3v_1.$$

Consequently, we can choose

$$\xi=e^{ik_0x}\begin{pmatrix} 1 \\ -ia_ck_0^3 \end{pmatrix}.$$

By conjugacy, for the wave number $-k_0$, we seek solutions of the form $e^{-ik_0x}V=e^{-ik_0x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and get that

$$\xi^*=e^{-ik_0x}\begin{pmatrix} 1 \\ ia_ck_0^3 \end{pmatrix}.$$

Then the center space Z_h which is nothing but $\ker\mathcal{L}(a_c,\delta_c)$ can be parametrized by

$$Z_h=\{A\xi+A^*\xi^*; A\in\mathbb{C}^1\}.$$

This parametrization will be important for the later computations.

6.2. The dual kernel $\mathcal{L}^*(a_c,\delta_c)$. Next, we compute the kernel of the conjugate operator $\mathcal{L}^*(a_c,\delta_c)$. By simple checking, based on the definition of conjugate operator, we get

$$\mathcal{L}^*(a_c,\delta_c) := \begin{pmatrix} -a_c\partial_x^4 & -\sigma'(0)\partial_x \\ -\partial_x & -\delta_c\partial_x^2 - \partial_x^4 \end{pmatrix}.$$

Seeking elements of the form $\eta = \kappa e^{ik_0x}V = \kappa e^{ik_0x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in the dual kernel where κ is a renomalization constant to be picked, we get

$$\begin{pmatrix} -a_ck_0^4 & \sigma'(0)ik_0 \\ -ik_0 & \delta_ck_0^2 - k_0^4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6.3}$$

which is equivalent to

$$\begin{cases} -a_ck_0^4v_1 - \sigma'(0)ik_0v_2 = 0 \\ -ik_0v_1 + (\delta_ck_0^2 - k_0^4)v_2 = 0. \end{cases} \tag{6.4}$$

Since $\det\begin{pmatrix} -a_ck_0^4 & ik_0 \\ \sigma'(0)ik_0 & \delta_ck_0^2 - k_0^4 \end{pmatrix} = 0$, we get the relation between v_1 and v_2 as

$$-a_ck_0^4v_1 - \sigma'(0)ik_0v_2 = 0.$$

Since $a_c \neq 0$, we know $v_1 = \frac{\sigma'(0)ik_0}{-a_ck_0^4}v_2 = -i\frac{\sigma'(0)}{a_ck_0^3}v_2 = -ik_0(\delta_c - k_0^2)v_2$. Consequently, we get

$$\eta = \kappa e^{ik_0x}\begin{pmatrix} -ik_0(\delta_c - k_0^2) \\ 1 \end{pmatrix}.$$

For computational convenience, we choose κ such that $\langle\eta,\xi\rangle = 1$. By direct computation, we get

$$\begin{aligned} \langle\eta,\xi\rangle &= \langle \kappa e^{ik_0x}\begin{pmatrix} -ik_0(\delta_c - k_0^2) \\ 1 \end{pmatrix}, e^{ik_0x}\begin{pmatrix} 1 \\ -ia_ck_0^3 \end{pmatrix}\rangle \\ &= \int \kappa e^{ik_0x}\begin{pmatrix} -ik_0(\delta_c - k_0^2) \\ 1 \end{pmatrix} e^{-ik_0x}\begin{pmatrix} 1 \\ ia_ck_0^3 \end{pmatrix} \\ &= 2\pi[-ik_0(\delta_c - k_0^2) + ia_ck_0^3]\kappa \\ &= 2\pi ik_0[(a_c+1)k_0^2 - \delta_c]\kappa \\ &= 1, \end{aligned}$$

which suggests that

$$\kappa = \Big(2\pi ik_0[(a_c+1)k_0^2 - \delta_c]\Big)^{-1}$$

and

$$\eta = \Big(2\pi ik_0[(a_c+1)k_0^2 - \delta_c]\Big)^{-1} e^{ik_0x}\begin{pmatrix} -ik_0(\delta_c - k_0^2) \\ 1 \end{pmatrix}.$$

By our choice of κ and conjugacy of inner product, we have the following duality numerical relations:

$$\langle \eta,\xi\rangle = 1, \langle \eta^*,\xi^*\rangle = 1, \langle \eta^*,\xi\rangle = 0, \langle \eta,\xi^*\rangle = 0.$$

Also, we have obtained by now

$$\ker \mathcal{L}^*(a_c,\delta_c) = \{B\eta + B^*\eta^*; B \in \mathbb{C}^1\}.$$

We end this subsection by the following remark:

REMARK 6.1. The denominator in κ which is the coefficient of the first-order term in the characteristic equation (3.2) up to a nonzero constant involving k_0^2 is not zero for $(a_c,\delta_c) \in \mathcal{A}(k_0)$.

6.3. The projection operator $\mathbb{P}$. In this subsection, we will introduce the projection operator $\mathbb{P}$ to make the remaining computations elegant and efficient. To do so, we decompose the space triplet $Z \subset Y \subset X$ as follows:

$$Z = Z_c \oplus Z_h, Y = Y_c \oplus Y_h, X = X_c \oplus X_h,$$

where the subindices "c" and "h" stand for center and hyperbolic spaces respectively and

$$Z_c = Y_c = X_c = \{A\xi + A^*\xi^*; A \in \mathbb{C}^1\}.$$

For an element $U \in Z$ or in Y, X, we decompose it according to the above space decompositions as

$$\begin{aligned} U =& A\xi + A^*\xi^* + (U - A\xi - A^*\xi^*) \\ =& \langle \eta^*,U\rangle\xi + \langle \eta,U\rangle\xi^* + (U - \langle \eta^*,U\rangle\xi - \langle \eta,U\rangle\xi^*) \\ =& \mathbb{P}U + (1-\mathbb{P})U \end{aligned}$$

where the projection operator $\mathbb{P}$ is defined as

$$\mathbb{P}\cdot = \langle \eta^*,\cdot\rangle\xi + \langle \eta,\cdot\rangle\xi^*.$$

6.4. The center manifold reduction function. In this subsection, we will compute the second-order approximation of the center manifold reduction function Ψ to make preparations for the study of full dynamics of (1.1) on center manifolds. By now, we have shown the existence of center manifold and parametrized the center space of $\mathcal{L}(a_c,\delta_c)$ in terms of the complex conjugate pair (A,A^*). Hence we can correspondingly decompose an element $U = U(x,t)$ by $U = A(t)\xi + A(t)\xi^* + \Psi(A,A^*)$ where $A = A(t) \in \mathbb{C}^1$ is a complex function in time variable t and $\Psi(A,A^*) \in Y_h$ is the center manifold function.

For brevity, we define

$$R_{20}(A,A^*) := R_{20}(A(t)\xi + A(t)\xi^*, A(t)\xi + A(t)\xi^*)$$

and

$$R_{30}(A,A^*) := R_{20}(A(t)\xi + A(t)\xi^*, A(t)\xi + A(t)\xi^*, A(t)\xi + A(t)\xi^*).$$

The second-order approximation of the center manifold function $\Psi(A,A^*)$ is given by $\Psi(A,A^*) = (-\mathcal{L}(a_c,\delta_c))^{-1}(1-\mathbb{P})R_{20}(A,A^*) + O(|A|^3)$. Simple computation yields

$$R_{20}(A,A^*) = \begin{pmatrix} 0 \\ \frac{\sigma''(0)}{2}\partial_x[(Ae^{ik_0x} + A^*e^{-ik_0x})^2] \end{pmatrix}$$

$$= ik_0\sigma''(0)\begin{pmatrix} 0 \\ e^{2ik_0x}A^2 - e^{-2ik_0x}A^{*2} \end{pmatrix}$$
$$= ik_0\sigma''(0)[e^{2ik_0x}\begin{pmatrix} 0 \\ A^2 \end{pmatrix} - e^{-2ik_0x}\begin{pmatrix} 0 \\ A^{*2} \end{pmatrix}]$$

We can easily check

$$\begin{aligned}\langle\eta^*, R_{20}(A,A^*)\rangle &= \int \kappa e^{-ik_0x}\begin{pmatrix} ik_0(\delta_c - k_0^2) \\ 1 \end{pmatrix}\cdot[(-ik_0\sigma''(0))[e^{2ik_0x}\begin{pmatrix} 0 \\ A^2 \end{pmatrix} - e^{-2ik_0x}\begin{pmatrix} 0 \\ A^{*2} \end{pmatrix}] \\ &= \int \kappa\begin{pmatrix} ik_0(\delta_c - k_0^2) \\ 1 \end{pmatrix}\cdot[(-ik_0\sigma''(0))[e^{ik_0x}\begin{pmatrix} 0 \\ A^2 \end{pmatrix} - e^{-3ik_0x}\begin{pmatrix} 0 \\ A^{*2} \end{pmatrix}] \\ &= 0.\end{aligned}$$

Similarly, we can also check that $\langle\eta, R_{20}(A,A^*)\rangle = 0$. Hence we conclude

$$\mathbb{P}R_{20}(A,A^*) = \langle\eta^*, R_{20}(A,A^*)\rangle\xi + \langle\eta, R_{20}(A,A^*)\rangle\xi^* = 0.$$

Consequently,

$$\Psi(A,A^*) = (-\mathcal{L}(a_c,\delta_c))^{-1}(1-\mathbb{P})R_{20}(A,A^*) = (-\mathcal{L}(a_c,\delta_c))^{-1}R_{20}(A,A^*) + O(|A|^3).$$

To proceed further, we compute $(-\mathcal{L}(a_c,\delta_c))^{-1}R_{20}(A,A^*)$ which we define by $\Phi(A,A^*)$. As $R_{20}(A,A^*)$ is in the hyperbolic space, the equality

$$(-\mathcal{L}(a_c,\delta_c))^{-1}R_{20}(A,A^*) = \Phi(A,A^*)$$

is equivalent to

$$-\mathcal{L}(a_c,\delta_c)\Phi(A,A^*) = R_{20}(A,A^*),$$

and hence can be written specifically as

$$-\mathcal{L}(a_c,\delta_c)\Phi(A,A^*) = ik_0\sigma''(0)[e^{2ik_0x}\begin{pmatrix} 0 \\ A^2 \end{pmatrix} - e^{-2ik_0x}\begin{pmatrix} 0 \\ A^{*2} \end{pmatrix}]. \tag{6.5}$$

Due to the form in the right-hand side of the above equation and the fact that $\Phi(A,A^*)$ is real-valued, we should assume that

$$\Phi(A,A^*) = i(e^{2ik_0x}V - e^{-2ik_0x}V^*),$$

where $V \in \mathbb{C}^2$ is to be determined. The above observation enables us to carry out the current program. Now we observe that (6.5) is equivalent to

$$-\mathcal{L}(a_c,\delta_c)e^{2ik_0x}V = k_0\sigma''(0)e^{2ik_0x}\begin{pmatrix} 0 \\ A^2 \end{pmatrix}.$$

To solve for the complex vector $V \in \mathbb{C}^2$, we write the above equation specifically as

$$\begin{pmatrix} 16a_ck_0^4 & -2ik_0 \\ -2ik_0\sigma'(0) & -4\delta_ck_0^2 + 16k_0^4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ k_0\sigma''(0)A^2 \end{pmatrix}. \tag{6.6}$$

Before we solve this elementary system of algebraic equations, let us first claim that

$$\det\begin{pmatrix} 16a_ck_0^4 & -2ik_0 \\ -2ik_0\sigma'(0) & -4\delta_ck_0^2+16k_0^4 \end{pmatrix} \neq 0.$$

The above claim is a simple consequence of the facts that $k_0\sigma''(0)e^{2ik_0x}\begin{pmatrix} 0 \\ A^2 \end{pmatrix} \in X_h$, $\Phi(A,A^*)\in X_h$, and $-\mathcal{L}(a_c,\delta_c)|_{X_h}$ is invertible. Actually, we can also directly compute it out as

$$\det\begin{pmatrix} 16a_ck_0^4 & -2ik_0 \\ -2ik_0\sigma'(0) & -4\delta_ck_0^2+16k_0^4 \end{pmatrix} = (2k_0)^2[\sigma'(0)+a_c(2k_0)^4((2k_0)^2-\delta_c)] \neq 0$$

at any admissible critical configuration pairs. Now simple computations yield

$$\begin{cases} v_1 = \frac{i\sigma''(0)A^2}{2[\sigma'(0)+a_c(2k_0)^4((2k_0)^2-\delta_c)]} \\ v_2 = \frac{8a_ck_0^3\sigma''(0)A^2}{2[\sigma'(0)+a_c(2k_0)^4((2k_0)^2-\delta_c)]}. \end{cases} \tag{6.7}$$

Noticing the relation $a_ck_0^4(k_0^2-\delta_c)+\sigma'(0)=0$, we get

$$\sigma'(0)+a_c(2k_0)^4((2k_0)^2-\delta_c) = 3a_ck_0^4(21k_0^2-5\delta_c) \neq 0.$$

Then we can simplify the expressions of v_1 and v_2 further as

$$\begin{cases} v_1 = \frac{i\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \\ v_2 = \frac{8a_ck_0^3\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)}. \end{cases} \tag{6.8}$$

To avoid ambiguity and for later reference, we will denote from now on that

$$\phi_1 = \frac{i\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)}, \ \phi_2 = \frac{8a_ck_0^3\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)}.$$

To sum up, we have computed that

$$\Phi(A,A^*) = i(e^{2ik_0x}\begin{pmatrix} \frac{i\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \\ \frac{8a_ck_0^3\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \end{pmatrix} - e^{-2ik_0x}\begin{pmatrix} \frac{-i\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \\ \frac{8a_ck_0^3\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \end{pmatrix}),$$

and

$$\Psi(A,A^*) = i(e^{2ik_0x}\begin{pmatrix} \frac{i\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \\ \frac{8a_ck_0^3\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \end{pmatrix} - e^{-2ik_0x}\begin{pmatrix} \frac{-i\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \\ \frac{8a_ck_0^3\sigma''(0)A^2}{6a_ck_0^4(21k_0^2-5\delta_c)} \end{pmatrix}) + O(|A|^3).$$

We will again end this subsection with remarks:

REMARK 6.2. The form of $\Phi(A,A^*)$ is inherited from the symmetry of the system (4.6).

REMARK 6.3. The claim right before the equation (6.7) in particular shows that condition (d) is a consequence of conditions (a), (b) and (c) in the definition of $\mathcal{A}(k_0)$.

6.5. Dynamics on center manifold. In this subsection, we compute the reduced dynamics for the system (1.1) or equivalently the system (4.2) or (4.6).

By our specific parametrization of the center space through complex-conjugate coordinates, the full dynamics of the system (1.1) or equivalently the system (4.2) or (4.6) on center manifold is given by

$$\begin{aligned}
&\frac{d}{dt}(A\xi+A^*\xi+\Psi(A,A^*))\\
=&\mathcal{L}(a,\delta)(A\xi+A^*\xi+\Psi(A,A^*))+\mathcal{N}(A\xi+A^*\xi+\Psi(A,A^*))\\
=&\mathcal{L}(a,\delta)(A\xi+A^*\xi^*)+R_{20}(A\xi+A^*\xi^*+\Psi(A,A^*),A\xi+A^*\xi^*+\Psi(A,A^*))\\
&+R_{30}(A\xi+A^*\xi^*+\Psi(A,A^*),A\xi+A^*\xi^*+\Psi(A,A^*),A\xi+A^*\xi^*\\
&+\Psi(A,A^*))+\tilde{R}(A\xi+A^*\xi^*+\Psi(A,A^*))
\end{aligned}$$

In view of flow invariance and by direct computations through expanding the left-hand side or directly referring to page 245 of [25] (or pages 56-62 of [24]), we know that the second-order approximation of the dynamics on the center manifold is given by

$$\begin{aligned}
&\frac{d}{dt}(A(t)\xi+A(t)\xi^*)\\
=&\mathbb{P}\mathcal{L}(a,\delta)(A\xi+A\xi^*+\Phi(A,A^*))+\mathbb{P}R_{20}(A\xi+A^*\xi^*+\Phi(A,A^*))+\sum_{j=2}^{3}\mathbb{P}R_{j0}(A,A^*).
\end{aligned}\tag{6.9}$$

Next, we proceed to compute the dynamics on the center manifold. During the procedure, we make use of the trivial fact that $\int_{-\pi}^{\pi}e^{ikx}\,dx=0$ for any nonzero integer k. Later, we will also see that the second approximation together with the latter fact will give enough insight for the full dynamics on the center manifold(s).

First, we compute $\mathbb{P}R_{20}(A,A^*)$. From former analysis, we know that $R_{20}\in Z_h$, which yields $\mathbb{P}R_{20}(A,A^*)=0$.

Second, we compute $\mathbb{P}R_{20}(A\xi+A^*\xi^*+\Phi(A,A^*),A\xi+A^*\xi^*+\Phi(A,A^*))$. To proceed, we first observe that

$$(A\xi+A^*\xi^*+\Phi(A,A^*))^{(1)}=Ae^{ik_0x}+A^*e^{-ik_0x}+i(e^{2ik_0x}\phi_1-e^{-2ik_0x}\phi_1^*);$$

$$\partial_x(A\xi+A^*\xi^*+\Phi(A,A^*))^{(1)}=ik_0(Ae^{ik_0x}-A^*e^{-ik_0x})-2k_0(e^{2ik_0x}\phi_1+e^{-2ik_0x}\phi_1^*).$$

Consequently, we have

$$(A\xi+A^*\xi^*+\Phi(A,A^*))^{(1)}\partial_x(A\xi+A^*\xi^*+\Phi(A,A^*))^{(1)}=\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3+\mathcal{Q}_4$$

where

$$\begin{aligned}
\mathcal{Q}_1&=ik_0(Ae^{ik_0x}+A^*e^{-ik_0x})(Ae^{ik_0x}-A^*e^{-ik_0x})\\
&=ik_0(A^2e^{2ik_0x}-A^{*2}e^{-2ik_0x});
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_2&=-2k_0(Ae^{ik_0x}+A^*e^{-ik_0x})(e^{2ik_0x}\phi_1+e^{-2ik_0x}\phi_1^*)\\
&=-2k_0(A\phi_1e^{3ik_0x}+A\phi_1^*e^{-ik_0x}+A^*\phi_1e^{ik_0x}+A^*\phi_1^*e^{-3ik_xx});
\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_3 &= -k_0(e^{2ik_0x}\phi_1 - e^{-2ik_0x}\phi_1^*)(Ae^{ik_0x} - A^*e^{-ik_0x})\\ &= -k_0(A\phi_1 e^{3ik_0x} - A^*\phi_1 e^{ik_0x} - A\phi_1^* e^{-ik_0x} + A^*\phi_1^* e^{-3ik_x x});\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_4 &= -2ik_0(e^{2ik_0x}\phi_1 - e^{-2ik_0x}\phi_1^*)(e^{2ik_0x}\phi_1 + e^{-2ik_0x}\phi_1^*)\\ &= -2ik_0(e^{4ik_0x}\phi_1^2 - e^{-4ik_0x}\phi_1^{*2}).\end{aligned}$$

Now we compute a specific form of $R_{20}(A\xi + A^*\xi^* + \Phi(A,A^*), A\xi + A^*\xi^* + \Phi(A,A^*))$ as follows

$$\begin{aligned}&R_{20}(A\xi + A^*\xi^* + \Phi(A,A^*), A\xi + A^*\xi^* + \Phi(A,A^*))\\ =&\frac{\sigma''(0)}{2}\partial_x\begin{pmatrix}0\\ A\xi + A^*\xi^* + \Phi(A,A^*)^2\end{pmatrix}\\ =&\sigma''(0)\begin{pmatrix}0\\ (A\xi + A^*\xi^* + \Phi(A,A^*))^{(1)}\partial_x(A\xi + A^*\xi^* + \Phi(A,A^*))^{(1)}\end{pmatrix}\\ =&\sigma''(0)\begin{pmatrix}0\\ \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4\end{pmatrix}.\end{aligned}$$

Now, it is easy to observe that

$$\langle \eta^*, \begin{pmatrix}0\\ \mathcal{Q}_1\end{pmatrix}\rangle = 0,\ \langle \eta^*, \begin{pmatrix}0\\ \mathcal{Q}_4\end{pmatrix}\rangle = 0;$$

$$\langle \eta^*, \begin{pmatrix}0\\ \mathcal{Q}_2\end{pmatrix}\rangle = \int_{-\pi}^{\pi}(-\kappa e^{-ik_0x})(-2k_0A^*\phi_1 e^{ik_0x})\,dx = 4\pi\kappa k_0 A^*\phi_1;$$

$$\langle \eta^*, \begin{pmatrix}0\\ \mathcal{Q}_3\end{pmatrix}\rangle = \int_{-\pi}^{\pi}(-\kappa e^{-ik_0x})(k_0A^*\phi_1 e^{ik_0x})\,dx = -2\pi\kappa k_0 A^*\phi_1.$$

From the expression of $\mathcal{Q}_j$ for $1 \le j \le 4$ and the above expression on $R_{20}(A\xi + A^*\xi^* + \Phi(A,A^*), A\xi + A^*\xi^* + \Phi(A,A^*))$, we obtain

$$\begin{aligned}&\mathbb{P}R_{20}(A\xi + A^*\xi^* + \Phi(A,A^*), A\xi + A^*\xi^* + \Phi(A,A^*))\\ =&\langle \eta^*, R_{20}(A\xi + A^*\xi^* + \Phi(A,A^*))\rangle\xi + \langle \eta, R_{20}(A\xi + A^*\xi^* + \Phi(A,A^*))\rangle\xi^*\\ =&\sigma''(0)\sum_{j=1}^{4}\langle \eta^*, \begin{pmatrix}0\\ \mathcal{Q}_j\end{pmatrix}\rangle\xi + \sigma''(0)\sum_{j=1}^{4}\langle \eta, \begin{pmatrix}0\\ \mathcal{Q}_j\end{pmatrix}\rangle\xi^*\\ =&2\pi k_0\sigma''(0)(\kappa A^*\phi_1\xi + \kappa^* A\phi_1^*\xi^*)\end{aligned}$$

Third, we compute $\mathbb{P}R_{30}(A,A^*)$. First, we note that

$$\begin{aligned}R_{30}(A,A^*) &= \begin{pmatrix}0\\ \frac{\sigma'''(0)}{6}\partial_x((A\xi + A^*\xi^*)^{(1)3})\end{pmatrix}\\ &= \begin{pmatrix}0\\ \frac{\sigma'''(0)}{2}(A\xi + A^*\xi^*)^{(1)2}\partial_x((A\xi + A^*\xi^*)^{(1)})\end{pmatrix}\\ &= \begin{pmatrix}0\\ \frac{ik_0\sigma'''(0)}{2}(Ae^{ik_0x} + A^*e^{-ik_0x})(A^2e^{2ik_0x} - A^{*2}e^{-2ik_0x})\end{pmatrix}\\ &= \begin{pmatrix}0\\ \frac{ik_0\sigma'''(0)}{2}(A^3e^{3ik_0x} - AA^{*2}e^{-ik_0x} + A^2A^*e^{ik_0x} - A^{*3}e^{-3ik_0x})\end{pmatrix}.\end{aligned}$$

Now, we have

$$\begin{aligned}\langle \eta^*, R_{30}(A,A^*)\rangle &= \langle \eta^*, \begin{pmatrix} 0 \\ \frac{ik_0\sigma'''(0)}{2}(-AA^{*2}e^{-ik_0x})\end{pmatrix}\rangle \\ &= \int_{\pi}^{\pi}(-\kappa e^{ik_0x})(\frac{ik_0\sigma'''(0)}{2}A^2A^*e^{ik_0x})\,dx \\ &= -i\pi\kappa\sigma'''(0)k_0A^2A^*.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\mathbb{P}R_{30}(A,A^*) &= \langle \eta^*, R_{30}(A,A^*)\rangle\xi + \langle \eta, R_{30}(A,A^*)\rangle\xi^* \\ &= -i\pi\kappa\sigma'''(0)k_0A^2A^*\xi - i\pi\kappa\sigma'''(0)k_0AA^{*2}\xi^*\end{aligned}$$

where we have used the fact that $i\kappa$ is real.

Fourth, we compute the remaining term

$$\mathbb{P}(\mathcal{L}(a,\delta)-\mathcal{L}(a_c,\delta_c))(A\xi + A\xi^* + \Phi(A,A^*)).$$

For this purpose, we recall that

$$\mathcal{L}(a,\delta)-\mathcal{L}(a_c,\delta_c) = \begin{pmatrix} -\nu_1\partial_x^4 & 0 \\ 0 & -\nu_2\partial_x^2\end{pmatrix}.$$

We note that

$$\begin{aligned}&\partial_x^4(A\xi + A^*\xi^* + \Phi(A,A^*))^{(1)} \\ =&\partial_x^4(Ae^{ik_0x} + A^*e^{-ik_0x} + i(e^{2ik_0x}\phi_1 - e^{-2ik_0x}\phi_1^*)) \\ =&k_0^4Ae^{ik_0x} + k_0^4A^*e^{-ik_0x} + i((2k_0)^4e^{2ik_0x}\phi_1 - (2k_0)^4e^{-2ik_0x}\phi_1^*)\end{aligned}$$

and

$$\begin{aligned}&\partial_x^2(A\xi + A^*\xi^* + \Phi(A,A^*))^{(2)} \\ =&\partial_x^2[Ae^{ik_0x}(-ia_ck_0^3) + A^*e^{-ik_0x}(ia_ck_0^3) + i(e^{2ik_0x}\phi_2 - e^{-2ik_0x}\phi_2^*)] \\ =&ia_ck_0^5Ae^{ik_0x} - ia_ck_0^5A^*e^{-ik_0x} + i[-(2k_0)^2e^{2ik_0x}\phi_2 + (2k_0)^2e^{-2ik_0x}\phi_2^*].\end{aligned}$$

Hence, we can obtain

$$\begin{aligned}&\langle \eta^*, (\mathcal{L}(a,\delta)-\mathcal{L}(a_c,\delta_c))(A\xi + A\xi^* + \Phi(A,A^*))\rangle \\ =&\langle \eta^*, \begin{pmatrix} -\nu_1\partial_x^4(A\xi + A^*\xi^* + \Phi(A,A^*))^{(1)} \\ -\nu_2\partial_x^2(A\xi + A^*\xi^* + \Phi(A,A^*))^{(2)}\end{pmatrix}\rangle \\ =&\langle \eta^*, e^{-ik_0x}\begin{pmatrix} (-\nu_1)k_0^4A^* \\ (-\nu_2)(-ia_ck_0^5A^*)\end{pmatrix}\rangle \\ =&2\pi(i\kappa)k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]A,\end{aligned}$$

which yields

$$\mathbb{P}(\mathcal{L}(a,\delta)-\mathcal{L}(a_c,\delta_c))(A\xi + A\xi^* + \Phi(A,A^*)) = 2\pi(i\kappa)k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2](A\xi + A^*\xi^*). \tag{6.10}$$

To sum up, we get the reduced dynamics on the center manifold given by the following equation on $A = A(t) \in \mathbb{C}^1$

$$\begin{aligned}\frac{d}{dt}A(t) =& 2\pi(i\kappa)k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]A(t) \\ &+ 2\pi\sigma''(0)\kappa A^*(t)\phi_1 - i\pi\kappa\sigma'''(0)k_0A^2(t)A^*(t) + O(|A|^4) \\ =& \frac{1}{(a_c+1)k_0^2 - \delta_c}\Big(k_0^4[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]A(t) \\ &+ [\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2 - 5\delta_c)} - \frac{\sigma'''(0)}{2}]A(t)^2A(t)^*\Big) + O(|A|^4),\end{aligned}$$

and its complex conjugate equation on A^*. In next subsection, we will strengthen the dynamics to be (1.2) with $g(r) \equiv 0$ where $g(r)$ is the derivative of the angular variable θ with respect to the time variable t (see also (6.15)).

6.6. Analysis of the dynamics on center manifold. In this subsection, we shall analyze the full dynamics of (1.1) by using the dynamics on center manifolds and therefore prove our main results. To analyze the dynamics of the system (4.6), it is sufficient to analyze the $A(t)$ equation or equivalently $A^*(t)$ equation. If we introduce polar coordinates $A(t) = r(t)e^{i\theta(t)}$, we get after simple computations that

$$\begin{cases}\frac{d}{dt}r(t) = \frac{1}{(a_c+1)k_0^2 - \delta_c}\Big(k_0^4[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]r(t) + [\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2 - 5\delta_c)} - \frac{\sigma'''(0)}{2}]r(t)^3\Big) + O(|r|^4) \\ \frac{d}{dt}\theta(t) = O(r^4).\end{cases} \tag{6.11}$$

When we approximate the center manifold to an arbitrary order for sufficient smooth flux function $\sigma(\tau)$, using the cancellation given by $\int_{-\pi}^{\pi} e^{ikx}\,dx = 0$ for any nonzero integer k, we obtain that the higher-order terms (≥ 4) in the center manifold dynamics $\frac{d}{dt}A(t)$ appear as $AP(|A|^2)$ where $P(\cdot)$ is a one variable polynomial with in general complex coefficients and without constant term and first-order term. As a demonstration to make this point clear, we can consider typical terms to appear in the $A(t)$ equation, say $\langle \eta, \mathbb{P}R_{n0}(A, A^*)\rangle$ where $R_{n0}(A, A^*) = R_{n0}(A\xi + A^*\xi^*, ..., A\xi + A^*\xi^*)$. In view of the definitions of η, ξ and the inner product, we easily see that $\langle \eta, \mathbb{P}R_{n0}(A, A^*)\rangle = 0$ for n even and $\langle \eta, \mathbb{P}R_{n0}(A, A^*)\rangle = \Omega A^{*k}A^{k+1} = \Omega|A|^{2k}A$ for $n = 2k+1$ for $k \geq 2$ for some number Ω.

Denote $\mathbb{a} := \mathbb{a}(k_0, a_c, \delta_c) = \frac{k_0^4}{(a_c+1)k_0^2 - \delta_c}$ which is nonzero and real, and

$$\mathbb{b} := \mathbb{b}(k_0, a_c, \delta_c, \sigma''(0), \sigma'''(0)) = \frac{1}{(a_c+1)k_0^2 - \delta_c}[\frac{\sigma''(0)^2}{6a_ck_0^4(21k_0^2 - 5\delta_c)} - \frac{\sigma'''(0)}{2}].$$

Now it is natural to choose the bifurcation parameter μ as

$$\mu = \mu(\nu) = \mu(\nu_1, \nu_2) = (\delta_c - k_0^2)\nu_1 + a_c\nu_2. \tag{6.12}$$

Due to the above analysis, we conclude that the full $A(t)$ dynamics on center manifold are actually given by

$$\frac{d}{dt}A(t) = \mathbb{a}\mu A + \mathbb{b}|A|^2A + O(|A|^5) \tag{6.13}$$

in which the $O(|A|^5)$ term is given by $AP(|A|^2)$. If further $\sigma(\tau)$ is assumed to be smooth (which is a technical assumption), then $P(z) = \sum_{j\geq 2}\mathbb{c}_j z^j$ with $\mathbb{c}_j$ being complex in general.

Using subindices r and i to represent real and imaginary parts respectively, the dynamics of equation (6.13) can always be written in polar coordinates as

$$\begin{cases} \frac{d}{dt}r(t) = \mathbb{a}_r\mu r + \mathbb{b}_r r^3 + O(r^5) \\ \frac{d}{dt}\theta(t) = \mathbb{a}_i\mu r + \mathbb{b}_i r^3 + O(r^5) \end{cases} \tag{6.14}$$

where the $O(r^5)$ term in the $\frac{d}{dt}r(t)$-equation is given by $\sum_{j\geq 2}\mathbb{c}_{jr}r^{2j+1}$ while the $O(r^5)$ term in the $\frac{d}{dt}\theta(t)$-equation is given by $\sum_{j\geq 2}\mathbb{c}_{ji}r^{2j+1}$ if $\sigma(\tau)$ is smooth.

Noticing that $\mathbb{a}$ and $\mathbb{b}$ are real, i.e., $\mathbb{a}_r = \mathbb{a}, \mathbb{a}_i = 0$ and $\mathbb{b}_r = \mathbb{b}, \mathbb{b}_i = 0$, we know that the above system takes the following form

$$\begin{cases} \frac{d}{dt}r(t) = f(r,\mu) \\ \frac{d}{dt}\theta(t) = g(r) \end{cases} \tag{6.15}$$

where $f(r,\mu) := \mathbb{a}\mu r + \mathbb{b}r^3 + O(|r|^5)$ and $g(r) = \sum_{j\geq 2}\mathbb{c}_{ji}r^{2j+1} = O(r^5)$. Notice that $f(r,\mu)$ and $g(r)$ are both real polynomials.

Let us first analyze the radial equation, which is independent of the angular equation. Consider the radial equation $f(r,\mu) = \mathbb{a}\mu r + \mathbb{b}r^3 + O(|r|^5) = 0$. From our former analysis, we can write $f(r,\mu)$ as $f(r,\mu) = rh(r^2,\mu)$ where $h(r^2,\mu) = \mathbb{a}\mu + \mathbb{b}r^2 + o(|\mu| + r^2)$ with h being at least a C^1-function. First, we observe that 0 is always an equilibrium. Noticing that $h(0,0) = 0$ and $\frac{\partial}{\partial(r^2)}h(0,0) = \mathbb{a} \neq 0$, we conclude by the implicit function theorem that μ is a function of r^2 in a neighborhood of 0, i.e., $\mu = \tilde{g}(r^2)$ with $g(0) = 0$ for some function $\tilde{g}$ in a neighborhood of (0,0). In view that $h(0,0) = 0$, we know that the Taylor expansion of $\tilde{g}$ is $\mu = \tilde{g}(r^2) = -\frac{\mathbb{a}}{\mathbb{b}}r^2 + o(r^4)$. Then we obtain that there is a curve of nontrivial equilibria in the (μ,r)-plane that has a second order tangency at (0,0) to the graph of $\mu = -\frac{\mathbb{b}}{\mathbb{a}}r^2 + o(r^4)$.

Moreover, for the truncated equation $f_0(r,\nu) := \mathbb{a}\mu r + \mathbb{b}r^3 = 0$, we observe that $r = 0$ is always a solution. Meanwhile, $r = \sqrt{\frac{-\mathbb{a}\mu}{\mathbb{b}}} > 0$, i,e., $\mu = -\frac{\mathbb{b}}{\mathbb{a}}r^2$ is another solution in the parameter range such that $\frac{-\mathbb{a}\mu}{\mathbb{b}} > 0$. In the parameter range $\frac{-\mathbb{a}\mu}{\mathbb{b}} < 0$, 0 is the only solution. The truncated differential equation $\frac{dr}{dt} = \mathbb{a}r + \mathbb{b}r^3$ can be solved explicitly: $r^2(t) = \frac{\mathbb{a}\mu r_0^2}{a\mu e^{-2\mathbb{a}\mu t} + \mathbb{b}r_0^2(e^{-2\mathbb{a}\mu t} - 1)}$.

The above two paragraphs show that the truncated equation and the full equation have the same number of equilibria in a neighborhood of origin $\mu = 0$, which are $o(|\mu|^{1/2})$-close to each other. Another fact is that the dynamics of the r-equation is exactly that of a standard pitchfork bifurcation if we allow $r < 0$. As in our case here $r(t) = |A(t)|$, there is only one bifurcated nonzero equilibrium, which is different from the standard pitchfork bifurcation with two bifurcated equilibria. Hence, we may regard the dynamic here as *half pitchfork bifurcation.*

Now, we analyze the angular equation. We need to distinguish two different dynamics: (i) $g(r) = 0$, in other words, all the $\mathbb{c}_j$ are real and the angular equation is trivial, (ii) there is some $\mathbb{c}_{j_0 i} \neq 0$. Next, we let the nontrivial equilibrium of radial equation be r_μ. If the angular equation were nontrivial, then $\theta(t) - \theta_0 = g(r_\mu)t \bmod 2\pi = O(|\mu|^{5/2})$ and due to $SO(2)$-symmery (obviously, $SO(2)$ is a subgroup of $O(2)$), i.e., rotational symmetry, any nontrivial equilibrium r_μ would correspond to a *rotation wave* with radius r_μ and angular speed $O(|\mu|^{5/2})$, which is slower in order as compared with amplitude. However, next we will use symmetry to exclude the possibility (ii), i.e., we will show $g(r) \equiv 0$. The main idea is that the reduced equation on the center manifold(s) inherit the symmetry of the original system by applying (c) of Theorem 5.1.

In view of our choice of parametrization of the center space and that $R_\phi \xi = e^{i\phi}\xi, S\xi = \xi^*$, we know on the center space coordinated by (A, A^*), the operators R_ϕ and S act as the following 2×2 matrices respectively:

$$\begin{pmatrix} e^{k_0\phi} & 0 \\ 0 & e^{-k_0\phi} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, due to the inherited symmetry, (6.13) must have the form $\frac{d}{dt}A(t) = F(A, A^*, \mu)$ with F satisfying for any ϕ,

$$F(e^{ik_0\phi}A, e^{-ik_0\phi}A^*, \mu) = e^{ik_0\phi}F(A, A^*, \mu), \quad F(A^*, A, \mu) = F(A, A^*, \mu)^*.$$

By choosing $\phi = -\frac{\arg A}{k_0}$ and then $\phi = \frac{\pi}{k_0} - \frac{\arg A}{k_0}$, we find $F(|A|, |A|, \mu) = e^{-i\arg A}F(A, A^*, \mu)$ and $F(-|A|, -|A|, \mu) = -e^{-i\arg A}F(A, A^*, \mu)$ respectively. Therefore, $F(|A|, |A|, \mu)$ is odd in $|A|$. This implies that $F(|A|, |A|, \mu) = |A|G(|A|, \mu)$ for some even function $G(|A|, \mu)$ in its first argument. Therefore, we have $F(A, A^*, \mu) = e^{i\arg A}|A|G(|A|, \mu) = AG(|A|, \mu)$. For our analysis of dynamics on center manifolds, it suffices to consider the case when F is a polynomial. In this case, the analysis above implies that $G(|A|, \mu)$ is an even polynomial in $|A|$. Further, the condition $F(A^*, A, \mu) = F(A, A^*, \mu)^*$ forces the coefficient of this even polynomial to be real. Hence, we have shown that $g(r) \equiv 0$.

By now, we have proved our main theorem-Theorem 1.1.

7. Related issues

We discuss here some other related issues from both applicational and mathematical points of view.

7.1. Isolation, bifurcation parameters and measurement. In our bifurcation analysis, we always isolate the bifurcation parameters in our way of computing dynamics on the center manifold, which is natural. Here we introduced the bifurcation parameter μ through the bifurcation vector $\nu \in \mathbb{R}^2$. Mathematically, the results involving this parameter vector take care of variations of the control parameters a and δ in one go. Though wide enough, these results are not easy to measure due to the change of two parameters, and also not sharp mathematically in certain cases. We consider the following two cases:

(i) If we fixed a_c and consider the dynamics of $\partial_t U = \mathcal{L}(a_c, \delta_c)U + (\mathcal{L}(a, \delta) - \mathcal{L}(a_c, \delta_c))U + N(U)$ when δ varies around δ_c. Since $\mathcal{L}(a, \delta_c) - \mathcal{L}(a_c, \delta_c) = \begin{pmatrix} 0 & 0 \\ 0 & -(\delta - \delta_c)\partial_x^2 \end{pmatrix}$, then it is better for the ease of measurement to introduce the bifurcation parameter $\Gamma_2 := a_c k_0^4(k_0^2 - \delta) + \sigma'(0)$. The relation of Γ_2 and ν_2 is given by $\Gamma_2 = -a_c k_0^4 \nu_2$ through the difference $a_c k_0^4(k_0^2 - \delta_c) + \sigma'(0) = 0$ and $a_c k_0^4(k_0^2 - \delta) + \sigma'(0) = \Gamma_2$. In this case, we have $R(U, \Gamma_2) = \begin{pmatrix} 0 & 0 \\ 0 & -(\delta - \delta_c) \end{pmatrix} U + N(U) \in H^2_{per}(-\pi, \pi)$. As a consequence, we can choose space triplet $Z \subset Y \subset X$ as $Z = H^4_{per}(-\pi, \pi)$, $Y = H^2_{per}(-\pi, \pi)$ and $X = L^2_{per}(-\pi, \pi)$. Then the corresponding bifurcation occurs in $Y = H^2_{per}(-\pi, \pi)$. This is sharp.

(ii) Similarly, if we fixed δ_c and consider the dynamics of $\partial_t U = \mathcal{L}(a_c, \delta_c)U + (\mathcal{L}(a, \delta) - \mathcal{L}(a_c, \delta_c))U + N(U)$ when a varies around a_c. Since $\mathcal{L}(a, \delta_c) - \mathcal{L}(a_c, \delta_c) = \begin{pmatrix} -(a - a_c)\partial_x^4 & 0 \\ 0 & 0 \end{pmatrix}$, then it is better for the ease of measurement to introduce the bifurcation parameter $\Gamma_1 := a k_0^4(k_0^2 - \delta_c) + \sigma'(0)$. The relation of Γ_1 and ν_1 is given by $\Gamma_1 =$

$k_0^4(k_0^2-\delta_c)\nu_1$ through the difference $a_c k_0^4(k_0^2-\delta_c)+\sigma'(0)=0$ and $ak_0^4(k_0^2-\delta_c)+\sigma'(0)=\Gamma_1$. In this case, we have $R(U,\Gamma_2)=\begin{pmatrix}0 & 0\\ 0 & -(\delta-\delta_c)\end{pmatrix}U+N(U)\in L^2_{per}(-\pi,\pi)$. As a consequence, we can not upgrade the space triplet choice $Z\subset Y\subset X$ as in (i) above but shall choose $Z=H^4_{per}(-\pi,\pi)$, $Y=X=L^2_{per}(-\pi,\pi)$. Of course, the bifurcation occurs in $Y=L^2_{per}(-\pi,\pi)$ in this case. A drawback is that the observational bifurcation parameter Γ_1 is degenerate if $k_0^2=\delta_c$ which is allowed.

7.2. Symmetry and computations. For our system (4.6), we can verify that it is equivariant under the $O(2)$-group action (hence under $SO(2)$-group action) entirely similarly as in [22] (see Proposition 5.1 and Section 9 of [22]). In the current paper, we see that symmetry does play its role and the effect is embodied by the fact $\mathcal{A}(k_0)=\mathcal{A}(-k_0)$, by the form of the eigenvalue equation(3.2) for M_k where k enters the equation through k^2, by the parametrization of center space, during the computation of the dynamics on center manifold (for example, the form of $\Phi(A,A^*)$), by the paragraph below (6.11). However, the computations of the dynamics on center manifold in the current work did not refer to normal form theory. The reason is that the representation of the linear operator $\mathcal{L}(a_c,\delta_c)$ on the center space coordinated through the conjugate pair (A,A^*) is the two by two zero matrix $0_{2\times 2}$. We refer the reader to Theorem 6.2 in [22] or Theorem 7.11 in [32] for this point. Though we did not use the normal form theory to help us to do computations, it is interesting to compare the $O(2)$-equivariant Hopf bifurcations in [22, 32] and the bifurcations here. In particular, both $\sigma'(0)$ and $\sigma''(0)$ are allowed to be zero in the current paper, which means the important fact that *hyperbolicity* and *genuine nonlinearity* (see page 90 of [4] for the two definitions; see also [5, 21]) are not necessary for the current study.

7.3. Choice of spaces. This can be regarded as a remark for Section 2 and a continuation of Subsection 7.1 and we content ourselves in touching only several points among those we can make. In the following discussion, we talk about infinitely dimensional dynamics. There are two main ways of choosing working spaces in the literature: one is a space pair, say $D\subset X$, and the other is a space triplet $Z\subset Y\subset X$ as we adopted. In the former, one usually chooses D as the domain of the linear operator on a Banach space X which is properly large. While in the latter, one may choose Z to be a subspace of the domain of the linear operator on the large ambient Banach space X while Y to be the interpolation space which takes care of the nonlinearities in a specific problem. Consequently, bifurcations occur in X in the former and in Y in the latter. Obviously, the former way of choosing working spaces can be regarded as a special case of the latter one. It is due to this elementary fact that the space triplet choice way usually locates more precisely the spaces in which bifurcation dynamics occur. Carefully checking the proofs in the center manifold theory, we also see that the theory based on space pairs is more effective if the linear operators involved are sectorial (which is the case for dissipative partial differential equations) while the theory established with space triplets can deal with linear operators that are not sectorial. Hence the latter has more applications besides dissipative partial differential equations.

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