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DIFFUSION PROBLEMS IN MULTI-LAYER MEDIA WITH NONLINEAR INTERFACE CONTACT RESISTANCE*

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Abstract. The purpose is a finite element approximation of the heat diffusion problem in composite media, with non-linear contact resistance at the interfaces. As already explained in [Journal of Scientific Computing, **63**, 478-501 (2015)], hybrid dual formulations are well fitted to complicated composite geometries and provide tractable approaches to variationally express the jumps of the temperature. The finite elements spaces are standard. Interface contributions are added to the variational problem to account for the contact resistance. This is an important advantage for computing code developers. We undertake the analysis of the non-linear heat problem for a large range of contact resistances and we investigate its discretization by hybrid dual finite element methods. Numerical experiments are presented at the end to support the theoretical results.

Keywords. Thermal contact resistance; semi-linear problem; Dual Hybrid formulation; finite elements.

AMS subject classifications. 35J61; 80M10.

1. Introduction

Let Ω be a connected bounded domain in $\mathbb{R}^d$, $d=2$ or 3, with a Lipschitz-continuous boundary $\Gamma=\partial\Omega$. Assume that a finite number i^* of connected sub-domains ω_i, called capsules (or cells), are entirely contained in Ω. The capsules do not touch each other, meaning that the intersection of $\overline{\omega}_i$ and $\overline{\omega}_j$ is empty if $i\neq j$. Next, we set:

$$\Omega_I=\cup_{1\leq i\leq i^*}\omega_i, \qquad \Omega_E=\Omega\setminus\overline{\Omega}_I, \qquad \gamma=\cup_{1\leq i\leq i^*}\gamma_i=\cup_{1\leq i\leq i^*}(\partial\omega_i).$$

The indices I and E are respectively for Internal sub-domains, the set of capsules, and for the External sub-domains, the environmental media of the capsules. The generic point in Ω is $\boldsymbol{x}$ and the symbol $\boldsymbol{\tau}$ is used for the generic point in γ.

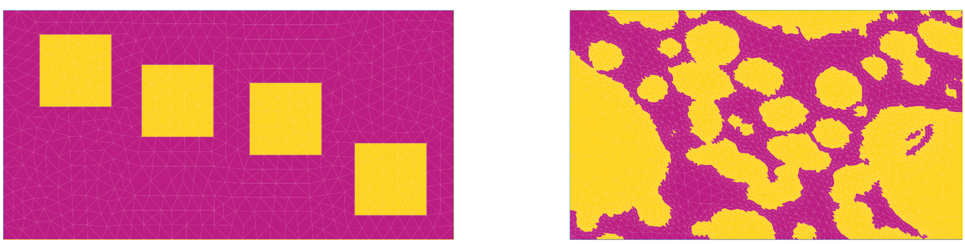

Fig. 1.1. *Examples of composite media.*

The geometry features fixed, we turn to the boundary value problem to deal with here. It is defined by the Laplace partial differential operator with some transmission

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conditions at the interface γ. Hence, the scalar unknown u is the solution to

$$\begin{aligned} & -\operatorname{div}(\alpha\nabla u) = F && \text{in } \Omega_E \cup \Omega_I, \\ [\alpha\partial_{\boldsymbol{n}} u] = 0 \qquad & (\alpha\partial_{\boldsymbol{n}} u) = \beta(\boldsymbol{\tau}, [u])[u] && \text{on } \gamma, \\ & u = f && \text{on } \Gamma. \end{aligned} \tag{1.1}$$

The vector $\boldsymbol{n}$ is the unit normal to external boundary Γ or to the interface γ, oriented outwardly, from Ω_I toward Ω_E. The symbol $[\cdot]$ is for the jump through γ, equal to $(u_E - u_I)$, with obvious notations. The parameter $\beta(\boldsymbol{\tau}, [u])$ represents the conductance and α is the conductivity. The data are the source F and the Dirichlet boundary f. The solution u is then discontinuous as jumps are allowed across the interface γ.

The linear problem, when the conductance $\beta = \beta(\boldsymbol{\tau})$ does not vary with the solution u, has been thoroughly investigated in [3], in two different variational formulations. Each of them is approximated by a specific finite element method. As observed by the authors in there, the choice recommended to practitioners is tied to the geometry of the composite media. The Lagrangian finite element method, with a special handling of the jumps, can be successfully used for macro-encapsulated geometries, that is, when the number of capsules is low (see [15]). They can hardly be extended to micro-encapsulated domains, that is, when the capsules are large in number. Indeed, implementing local continuous/global discontinuous solutions for a dense composite media generates programing difficulties. Hybrid dual finite element method is a tractable alternative that should be preferred (see [22]); discontinuous Galerkin methods or unfitted finite elements bring about similar advantages and are also suitable (see [16]). The appealing feature of it is that the interfacial thermal (or electrical) resistance is taken into account in the hybrid dual variational formulation. The subject of this contribution is twofold: the analysis of the well posedness of the model (1.1) and the study of its discretization by hybrid dual finite element.

The steady problem under scrutiny arises in many physical or biological models and is even at the heart of some of them. Designing performing numerical methods to solve it may be highly useful for some applications. Two examples may be briefly described.

One is picked up in cells electrophysiolgy. The unsteadiness in (1.1) is located on the interface γ, to provide a model for electroporation phenomenon within a cell

$$\begin{aligned} & -\operatorname{div}(\alpha\nabla u) = 0 \qquad \text{in } \Omega_E \cup \Omega_I, \\ [\alpha\partial_n u] = 0 \qquad & (\alpha\partial_n u) = \partial_t[u] + \beta(\boldsymbol{\tau}, [u])[u] + j_p \qquad \text{on } \gamma, \\ & u = f \qquad \text{on } \Gamma. \end{aligned} \tag{1.2}$$

The function u stands for the electrical potential in the cell. In this model Ω_I represents the cells and Ω_E is the external medium. This model considers the cells membranes γ have no thickness; $\beta(\boldsymbol{\tau}, [u])$ is the surface conductance of the membrane γ. The datum f is the effect of the external electric field and j_p describes the current through electropores. We refer to [17, 24] for more details on the modeling and to [2, 12, 18–20] for the mathematical justification and numerical simulations. Further illustrations can be found in cardio-physiology and the bi-domains problem where Ω_I is for the heart surrounded by the torso Ω_E. The transmission condition at pericardial interface γ is modeling an electric resistor-capacitor effect (see [4]).

The other problem comes from the heat conduction in composite media with a thermal contact resistant interface γ. The dynamical term has a different nature as it is present on the whole multilayered domain. The thermal model is therefore given by

$$\begin{aligned} & \partial_t u - \operatorname{div}(\alpha \nabla u) = F \qquad \text{in } \Omega_E \cup \Omega_I, \\ [\alpha \partial_n u] = 0 \qquad & \rho(\boldsymbol{\tau}, [u])(\alpha \partial_n u) = [u] \qquad \text{on } \gamma, \\ & u = f \qquad \text{on } \Gamma. \end{aligned} \tag{1.3}$$

The solution u is the temperature field in the composite media. The thermal contact resistance $\rho(\boldsymbol{\tau}, [u])$ is generated by the imperfection of the contact between the medium filling the capsules Ω_I and the one of the matrix Ω_E. This imperfection causes a drop in the temperature u across the interface γ. The conductance $\beta(\boldsymbol{\tau}, [u]) = 1/\rho(\boldsymbol{\tau}, [u])$ takes most often a constant value at the interface solid/fluid and a small perturbation of a constant in solid/solid contact (see [10, 27]).

The contents of the paper are as follows. Section 2 is dedicated to the formulation of the model (1.1) as a semi linear condensed problem, set at the interface γ. This approach is based on the Dirichlet-to-Neumann operator. The analysis of the well posedness of the reduced problem is the purpose of Section 3. Minty-Browder's theory enables to derive existence and uniqueness results. In Section 4, hybrid finite elements are used for the approximation of the value problem (1.1). The study of the discrete problem, through Minty-Browder theorem together with a regularizing procedure, is undertaken in Section 5. We provide some relevant illustrations, obtained in `FreeFem++` program in the last Section 6.

Functional spaces and Notations — We use the standard notations for the Sobolev spaces $L^2(\Omega), L^2(\gamma)$ and $H^1(\Omega)$. Bold symbols $\boldsymbol{L}^2(\Omega), \boldsymbol{H}^1(\Omega)$ are for vector Sobolev spaces $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$. $H_0^1(\Omega)$ is defined as the closure in $H^1(\Omega)$ of the space $\mathscr{D}(\Omega)$ of infinitely differentiable functions with a compact support in Ω. The traces over γ of the functions of $H^1(\Omega)$ defines the fractionary Sobolev space $H^{1/2}(\gamma)$ and $H^{-1/2}(\gamma)$ is its dual (see [1, 21]). We will need some specific spaces. The broken H^1 Hilbert spaces are defined to be

$$\mathbb{V} = \Big\{ v \in L^2(\Omega);\ \ v_I = v_{|\Omega_I} \in H^1(\Omega_I),\ \ v_E = v_{|\Omega_E} \in H^1(\Omega_E) \Big\}.$$

$$\mathbb{V}_0 = \Big\{ v \in \mathbb{V}, \qquad v_{|\Gamma} = 0 \Big\}.$$

They are naturally endowed with the broken norm

$$\|v\|_{\mathbb{V}} = \Big(\|v_E\|^2_{H^1(\Omega_E)} + \|v_I\|^2_{H^1(\Omega_I)} \Big)^{1/2}.$$

We will also use the non-standard space

$$\mathbb{X} = \Big\{ \boldsymbol{q} \in H(\operatorname{div}, \Omega); \qquad \boldsymbol{q} \cdot \boldsymbol{n} \in L^2(\gamma) \Big\},$$

equipped with the image norm

$$\|\boldsymbol{q}\|_{\mathbb{X}} = \Big(\|\boldsymbol{q}\|^2_{H(\operatorname{div},\Omega)} + \|\boldsymbol{q} \cdot \boldsymbol{n}\|^2_{L^2(\gamma)} \Big)^{1/2}.$$

Finally, we use the particular and possibly non-standard notation $H^{\tau-}(\gamma)$, for the intersection of all Sobolev spaces $H^{\sigma}(\gamma)$, with $\sigma < \tau$. Thus, the significance of $v \in H^{\tau-}(\gamma)$ is that $v \in H^{\sigma}(\gamma)$ for any $\sigma < \tau$.

2. Interface problem

To focus on the difficulties inherent to our problem; we choose to put aside non-essential technicalities. Accordingly, before laying down the functional and variational framework that we consider, we observe that the boundary value problem to be investigated can be rewritten as

$$\begin{aligned} -\operatorname{div}(\alpha \nabla u) = 0 \qquad & \text{in } \Omega_E \cup \Omega_I, \\ [\alpha \partial_{\boldsymbol{n}} u] = 0 \qquad (\alpha \partial_{\boldsymbol{n}} u) = \beta(\boldsymbol{\tau},[u])[u] + g \qquad & \text{on } \gamma, \\ u = 0 \qquad & \text{on } \Gamma. \end{aligned} \tag{2.1}$$

It is straightforwardly derived by shifting the solution as follows $u := u + u_{F,f}$, where $u_{F,f}$ does not suffer from any discontinuity across γ and is the unique solution to the Poisson equation (first equation in (1.1)) set on the whole domain Ω with the Dirichlet condition on the external boundary Γ (third equation in (1.1)). The data g along γ coincides with the normal derivative $(\partial_{\boldsymbol{n}} u_{F,f})$ and belongs at least to $H^{-1/2}(\gamma)$. Different liftings may of course be adopted. The advantage of the one retained here is that the expression of the conductance $\beta(\boldsymbol{\tau},[u])$ is not changed.

We need to make assumptions on the diffusivity $\alpha(\cdot)$ and the interface contact conductance parameters $\beta(\cdot,\cdot)$. As usual, the conductivity α is taken in $L^\infty(\Omega)$, it is positive and bounded away from zero ; there exists a constant $a > 0$ such that $\alpha(\cdot) \geq a$ in Ω. The contact conductance $\beta = \beta(\boldsymbol{\tau}, v)$ is a positive Caratheodory function defined in $\gamma \times \mathbb{R}$, measurable with respect to $\boldsymbol{\tau}$ and continuous and even in v. Furthermore, the odd function $\varphi(\boldsymbol{\tau},\cdot) : v \mapsto \beta(\boldsymbol{\tau}, v)v$ is assumed uniformly positive in the sense that for almost every $\boldsymbol{\tau} \in \gamma$, there holds that

$$(\varphi(\boldsymbol{\tau}, v) - \varphi(\boldsymbol{\tau}, w))(v - w) \geq b(v - w)^2, \qquad \forall v, w \in \mathbb{R}. \tag{2.2}$$

$b > 0$ is a positive constant (does not depend on x). We need also to fix the behavior of the conductance function $\beta(\boldsymbol{\tau}, v)$ for large values of v. We assume the following growth condition

$$\beta(\boldsymbol{\tau}, v) \leq b' + b_r |v|^r, \qquad \forall v \in \mathbb{R}. \tag{2.3}$$

b' and b_r are positive constants independent of $\boldsymbol{\tau}$. The exponent r may be an arbitrary positive real number $r \geq 0$ for $d = 2$ and equals two, $r = 2$ for $d = 3$. Notice that the Sobolev embedding $H^{1/2}(\gamma) \subset L^{r+2}(\gamma)$ holds.

Remark 2.1. The class of conductances we consider is currently used, especially in the modeling of the electro-physiology interaction of cells with their environment *(see [17, 18])*, or in bi-domain model for the heart electrical activities *(see [4])*. The behavior at infinity of $\beta(\boldsymbol{\tau}, v)$ includes a wide variety of conductances. That $\beta(\boldsymbol{\tau}, v)$ takes large values means that the membrane is no longer resistant if the gap $[u]$ grows high. This does not often occur. Nevertheless, we choose assumption (2.3) to embrace a large class of conductances.

The core of the study we have in mind consists in realizing a condensation on problem (2.1). We derive hence a differential equation on the interface γ. Let us choose $\lambda = [u] \in H^{1/2}(\gamma)$ as the main unknown to work with. The milestone of the formulation is the operator

$$\mathcal{N}\lambda = -\alpha(\partial_{\boldsymbol{n}}(u_\lambda)_I),$$

where $u_\lambda \in \mathbb{V}_0$ is the unique solution of

$$\begin{aligned} -\operatorname{div}(\alpha\nabla u_\lambda) &= 0 && \text{in } \Omega_E\cup\Omega_I,\\ [\alpha\partial_{\boldsymbol{n}} u_\lambda]=0 \qquad [u_\lambda] &= \lambda && \text{on } \gamma,\\ u_\lambda &= 0 && \text{on } \Gamma. \end{aligned} \tag{2.4}$$

The existence and uniqueness of $u_\lambda = ((u_\lambda)_I, (u_\lambda)_E)$ does not cause any special difficulty. The operator $\mathcal{N}$ is a Dirichlet-Neumann operator of a particular type. Next, the condensed problem to solve reads as: *find* $\lambda \in H^{1/2}(\gamma)$ *such that*

$$\mathcal{T}\lambda := \mathcal{N}\lambda + \varphi(\boldsymbol{\tau},\lambda) = \mathcal{N}\lambda + \beta(\boldsymbol{\tau},\lambda)\lambda = g, \qquad \text{in } H^{-1/2}(\gamma) \tag{2.5}$$

The purpose of the subsequent is to establish the existence and uniqueness result for (2.5) by Minty-Browder's theorem (see [23]). The properties of the linear operator $\mathcal{N}$ influences the analysis to a large extent. Let us first investigate them.

Lemma 2.1. *The operator $\mathcal{N}$ mapping $H^{1/2}(\gamma)$ in $H^{-1/2}(\gamma)$ is self-adjoint and non-negative. The kernel of $\mathcal{N}$ is the space of piecewise constant functions on the capsule boundaries $(\gamma_i = \partial\omega_i)_{1\le i\le i^*}$ and its dimension is therefore equal to i^*, the number of capsules.*

Proof. Let λ and μ be given in $H^{1/2}(\gamma)$. We have that

$$0 = \int_{\Omega_I\cup\Omega_E} (-\operatorname{div}(\alpha\nabla u_\lambda))u_\mu\, d\boldsymbol{x} = \int_{\Omega_I\cup\Omega_E} \alpha\nabla u_\lambda \nabla u_\mu\, d\boldsymbol{x} + \langle \alpha\partial_{\boldsymbol{n}}(u_\lambda)_I, [u_\mu]\rangle_{1/2}.$$

In view of (2.4), there comes out that

$$\langle \mathcal{N}\lambda,\mu\rangle_{1/2} = -\langle \alpha\partial_{\boldsymbol{n}}(u_\lambda)_I, [u_\mu]\rangle_{1/2} = \int_\Omega \alpha\nabla u_\lambda\nabla u_\mu\, d\boldsymbol{x}.$$

$\mathcal{N}$ is therefore self-adjoint and non-negative.

Now, assume that $\mathcal{N}\lambda = 0$. Then, $\nabla u_\lambda = 0$ in any ω_i and in Ω_E. This yields that u_λ is piecewise constant in $(\omega_i)_i$ and Ω_E. The boundary condition along Γ says that $u_\lambda = 0$ in Ω_E. This makes λ be constant along each $\gamma_i = \partial\omega_i$. The proof is complete. ◻

As a result of the lemma, $\mathcal{N}$ determines a semi-norm in $H^{1/2}(\gamma)$. It can be completed so as to obtain a norm. Let thus $\varrho > 0$ be given and define the bilinear form $\mathcal{N}_\varrho$ as follows

$$\langle \mathcal{N}_\varrho\lambda,\mu\rangle_{1/2} = \langle \mathcal{N}\lambda,\mu\rangle_{1/2} + \varrho(\lambda,\mu)_{L^2(\gamma)} \qquad \forall\lambda,\mu\in H^{1/2}(\gamma).$$

Lemma 2.2. *The operator $\mathcal{N}_\varrho$ is continuous and elliptic on $H^{1/2}(\gamma)$. There exists a couple of constants $\tau'_\varrho \ge \tau_\varrho > 0$ such that*

$$\tau_\varrho \|\mu\|^2_{H^{1/2}(\gamma)} \le \langle \mathcal{N}_\varrho\mu,\mu\rangle_{1/2} \le \tau'_\varrho \|\mu\|^2_{H^{1/2}(\gamma)}.$$

Proof. Given $\mu \in H^{1/2}(\gamma)$, we have that

$$\langle \mathcal{N}_\varrho\mu,\mu\rangle_{1/2} = \int_{\Omega_E\cup\Omega_I} \alpha(\nabla u_\mu)^2\, d\boldsymbol{x} + \int_\gamma \varrho\mu^2\, d\boldsymbol{\tau} = \int_{\Omega_E\cup\Omega_I} \alpha(\nabla u_\mu)^2\, d\boldsymbol{x} + \int_\gamma \varrho[u_\mu]^2\, d\boldsymbol{\tau}.$$

After recalling that $u_\mu \in \mathbb{V}_0$, using [3, Lemma 2.2] we derive that

$$\langle \mathcal{N}_\varrho\mu,\mu\rangle_{1/2} \ge \tau_\varrho(\|u_\mu\|^2_{H^1(\Omega_E)} + \|u_\mu\|^2_{H^1(\Omega_I)}).$$

Applying the trace theorem on γ produces the left inequality of the lemma.

We turn now to the right inequality. Owing to the expansion theorem, we can find $U_I \in H^1(\Omega_I)$ such that $(U_I)_{|\gamma} = \mu$ and (see [1]).

$$\|U_I\|_{H^1(\Omega_I)} \leq \tau \|\mu\|_{H^{1/2}(\gamma)}.$$

Setting $u'_\mu = ((u_\mu)_I + U_I, (u_\mu)_E)$, it is direct that $u'_\mu \in H^1_0(\Omega)$ and

$$\int_{\Omega_I \cup \Omega_E} \alpha \nabla u'_\mu \nabla v \, d\boldsymbol{x} = \int_{\Omega_I} \alpha \nabla U_I \nabla v \, d\boldsymbol{x}, \qquad \forall v \in H^1_0(\Omega).$$

This problem is well posed. u'_μ exists and is unique (and so does u_μ) with

$$\|u'_\mu\|_{H^1(\Omega)} \leq \tau' \|U_I\|_{H^1(\Omega_I)} \leq \tau'' \|\mu\|_{H^{1/2}(\gamma)}.$$

Both bounds on U_I and u'_μ yield directly the right inequality. The proof is complete. □

3. Well posedness

Minty-Browder's theorem turns out to be the suitable tool for the study of problem (2.5). Applying it requires some technical tools. Prior to all, we need to see whether $\mathcal{T}$ determines a non-linear operator that maps $H^{1/2}(\gamma)$ *to* its dual $H^{-1/2}(\gamma)$. This may be achieved owing to assumption (2.3) together with the Sobolev embeddings $H^{1/2}(\gamma) \subset L^q(\gamma)$, for any $q \geq 1$, in two dimensions and for any $q \leq 4$ in three dimensions. This proof is not addressed directly and will appear implicitly during the forthcoming proof of the continuity.

To prepare the sufficient assumptions for Minty-Browder's theorem, let us start by the continuity before stating the monotonicity of $\mathcal{T}$.

LEMMA 3.1. *The operator $\mathcal{T}$ mapping $H^{1/2}(\gamma)$ in $H^{-1/2}(\gamma)$ is continuous.*

Proof. The operator $\mathcal{T}$ may be decomposed as follows

$$\mathcal{T}\mu = (\mathcal{N}_b + b')\mu + (\beta(\cdot,\mu) - b')\mu =: \mathcal{N}_{b'}\mu + \beta'(\cdot,\mu)\mu.$$

According to Lemma 2.2, the linear part $\mathcal{N}_{b'}$ of $\mathcal{T}$ is continuous, it only remains to check the continuity for the non-linear part $\beta'(\cdot,\mu)\mu$. The approach we follow is classical in a way. Let λ be given and $(\mu_k)_{k\in\mathbb{N}}$ a sequence in $H^{1/2}(\gamma)$ that converges toward zero. The aim is to prove that the sequence $(\beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k))_{k\in\mathbb{N}}$ is precompact.

Consider $p \geq 1$ and $q \geq 1$ that are conjugate that is $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\chi \in H^{1/2}(\gamma)$, applying Hölder's inequality gives

$$\begin{aligned}\langle \beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k)) - \beta'(\cdot,\lambda)\lambda, \chi\rangle_{1/2} &= (\beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k)) - \beta'(\cdot,\lambda)\lambda, \chi)_{L^2(\gamma)} \\ &\leq \|\beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k)) - \beta'(\cdot,\lambda)\lambda\|_{L^p(\gamma)} \|\chi\|_{L^q(\gamma)}.\end{aligned}$$

The triangular inequality yields

$$\begin{aligned}\|\beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k)) - \beta'(\cdot,\lambda)\lambda\|_{L^p(\gamma)} \leq \|[\beta'(\cdot,\lambda+\mu_k) - \beta'(\cdot,\lambda)]\lambda\|_{L^p(\gamma)} \\ + \|\beta'(\cdot,\lambda+\mu_k)\mu_k\|_{L^p(\gamma)}.\end{aligned} \tag{3.1}$$

The aim is to prove that, for a subsequence of $(\mu_n)_{n\in\mathbb{N}}$, each term in the bound (3.1) decays toward zero. The continuous Sobolev embedding $H^{1/2}(\gamma) \subset L^{(r+1)p}(\gamma)$ yields that $(\mu_k)_{n\in\mathbb{N}}$ tends towards zero in $L^{(r+1)p}(\gamma)$. r is the exponent in formula (2.3).

The partial converse of the Lebesgue dominated theorem implies the existence of a subsequence $(\mu_{n(k)})_{k\in\mathbb{N}}$, abusively denoted $(\mu_k)_{k\in\mathbb{N}}$, that converges pointwise towards 0, for almost every $\boldsymbol{x}\in\Omega$, and which is uniformly dominated by a non negative-function $\overline{\mu}\in L^{(r+1)p}(\Omega)$. From the fact that β' is a Caratheodory function, we ensue that for almost any $\boldsymbol{\tau}\in\gamma$,

$$\lim_{k\to\infty}\beta'(\boldsymbol{\tau},\lambda(\boldsymbol{\tau})+\mu_k(\boldsymbol{\tau}))=\beta'(\boldsymbol{\tau},\lambda(\boldsymbol{\tau})).$$

Then, we need to show that the first term in (3.1) is uniformly bounded with respect to k. Inequality (2.3) gives

$$\begin{aligned}\int_\gamma|(\beta'(\boldsymbol{\tau},\lambda+\mu_k)-\beta'(\boldsymbol{\tau},\lambda))\lambda|^p\,d\gamma &\le c\int_\gamma[(\lambda+\mu_k)^r+\lambda^r]^p|\lambda|^p\,d\gamma\\ &\le c\int_\gamma[|\lambda|^{rp}+|\mu_k|^{rp}]|\lambda|^p\,d\gamma\\ &\le c\int_\gamma[|\lambda|^{(r+1)p}+|\mu_k|^{rp}|\lambda|^p]\,d\gamma.\end{aligned}$$

Calling once again for Hölder's inequality, we get

$$\int_\gamma|(\beta'(\boldsymbol{\tau},\lambda+\mu_k)-\beta'(\boldsymbol{\tau},\lambda))\lambda|^p\,d\gamma\le c(\|\lambda\|^{(r+1)p}_{L^{(r+1)p}(\gamma)}+\|\mu_k\|^{rp}_{L^{(r+1)p}(\gamma)}\|\lambda\|^p_{L^{(r+1)p}(\gamma)}).$$

Let us focus first on the two dimensional case. By the Sobolev embedding $H^{1/2}(\gamma)\subset L^{(r+1)p}(\gamma)$ we obtain

$$\|(\beta'(\boldsymbol{\tau},\lambda+\mu_k)-\beta'(\boldsymbol{\tau},\lambda))\lambda\|_{L^p(\gamma)}\le c(\|\lambda\|^{(r+1)}_{H^{1/2}(\gamma)}+\|\overline{\mu}\|^r_{L^{(r+1)p}}\|\lambda\|_{H^{1/2}(\gamma)}).$$

Arguing in the same lines we show that $\|\beta'(\cdot,\lambda+\mu_k)\mu_k\|_{L^p(\gamma)}$ is uniformly bounded with respect to k. Now, putting every thing together and using once again the Sobolev embedding $H^{1/2}(\gamma)\subset L^q(\gamma)$, we derive that

$$\begin{aligned}\sup_{\chi\in H^{1/2}(\gamma)}\frac{\langle\beta'(\boldsymbol{\tau},\lambda+\mu_k)(\lambda+\mu_k)-\beta'(\boldsymbol{\tau},\lambda)\lambda,\chi\rangle_{1/2}}{\|\chi\|_{H^{1/2}(\gamma)}}&\le\|[\beta'(\cdot,\lambda+\mu_k)-\beta'(\cdot,\lambda)]\lambda\|_{L^p(\gamma)}\\ &\quad+\|\beta'(\cdot,\lambda+\mu_k)\mu_k\|_{L^p(\gamma)}.\end{aligned}$$

Going to the limit $k\to\infty$ gives the desired result and $(\beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k))_{n\in\mathbb{N}}$ is pre-compact. The fact that $\beta'(\cdot,\lambda)\lambda$ is the unique limit point is straightforward. The whole sequence $(\beta'(\cdot,\lambda+\mu_k)(\lambda+\mu_k))_{k\in\mathbb{N}}$ converges therefore towards $\beta'(\cdot,\lambda)\lambda$ in $H^{-1/2}(\gamma)$.

In three dimensions, we have that $r=2$. Then p and q are chosen so that $p=4/3$ and $q=4$. This yields the formula

$$\|(\beta'(\boldsymbol{\tau},\lambda+\mu_k)-\beta'(\boldsymbol{\tau},\lambda))\lambda\|_{L^{4/3}(\gamma)}\le c(\|\lambda\|_{L^4(\gamma)}+\|\mu_k\|^{\frac{2}{3}}_{L^4(\gamma)}\|\lambda\|^{\frac{1}{3}}_{L^4(\gamma)}).$$

We achieve as above after using the Sobolev embedding $H^{1/2}(\gamma)\subset L^4(\gamma)$. The proof is complete. ∎

Lemma 3.2. *The operator $\mathcal{T}$ is strongly monotone, that is for all $\lambda,\mu\in H^{1/2}(\gamma)$*

$$\langle\mathcal{T}\lambda-\mathcal{T}\mu,\lambda-\mu\rangle_{1/2}\ge\tau_b\|\lambda-\mu\|^2_{H^{1/2}(\gamma)}.$$

τ_b *is the constant given in* Lemma 2.2.

Proof. Let λ and μ be given in $H^{1/2}(\gamma)$. We have that

$$\langle \mathcal{T}\lambda - \mathcal{T}\mu, \lambda-\mu\rangle_{1/2} = \langle \mathcal{N}\lambda - \mathcal{N}\mu, \lambda-\mu\rangle_{1/2} + (\beta(\cdot,\lambda)\lambda - \beta(\cdot,\mu)\mu, \lambda-\mu)_{L^2(\gamma)}.$$

On account of (2.2), we derive

$$\begin{aligned}\langle \mathcal{T}\lambda - \mathcal{T}\mu, \lambda-\mu\rangle_{1/2} &\geq \langle \mathcal{N}\lambda - \mathcal{N}\mu, \lambda-\mu\rangle_{1/2} + b\|\lambda-\mu\|^2_{L^2(\gamma)}\\ &= \langle \mathcal{N}_b\lambda - \mathcal{N}_b\mu, \lambda-\mu\rangle_{1/2}.\end{aligned}$$

Using Lemma 2.2 completes the proof. ∎

PROPOSITION 3.1. *The operator $\mathcal{T}$ is invertible and its inverse $\mathcal{T}^{-1}$ is a Lipschitz operator*

$$\|\mathcal{T}^{-1}g_1 - \mathcal{T}^{-1}g_2\|_{H^{1/2}(\gamma)} \leq \frac{1}{\tau_b}\|g_1-g_2\|_{H^{-1/2}(\gamma)}.$$

Proof. Since $\mathcal{T}$ is a continuous operator from $H^{1/2}(\gamma)$ into $H^{-1/2}(\gamma)$ and is strongly monotonous, it is therefore bijective according to Minty-Browder theorem (see [7,23]). The stability directly ensues from Lemma 3.2. The proof is complete. ∎

A straighforward consequence is the well-posedness of our problem with a Lipschitz-type stability.

THEOREM 3.1. *Problem* (1.1) *has only one solution $u\in\mathbb{V}$ and we have the stability*

$$\|u_1-u_2\|_{\mathbb{V}} \leq C(\|F_1-F_2\|_{L^2(\Omega)} + \|f_1-f_2\|_{H^{1/2}(\Gamma)}).$$

4. Hybrid dual finite elements

Discontinuity across the boundaries of the capsules implies a special treatment of Lagrange finite elements, if they are selected, at those interfaces. When the capsules grow high in number, implementing the jump of the solution u is a pain in the neck. Even harder is the monitoring of the discontinuity for the temperature, solution of the heat equation, computed in composite media when the geometry is particularly complicated. Studies and comments in [3] recommend that Lagrange finite elements be used only for macro-encapsulated domain and be preferably avoided for micro-encapsulated media. On the contrary, hybrid dual finite elements are well fitted, almost naturally, to the problem. They have been experimented and assessed for thermal modeling in the linear context (see [22]). Indeed, the flux conservation across γ makes the standard space of the Raviart-Thomas finite elements relevant for the gradient tensor $(\alpha\nabla u)$ and the discontinuous finite elements space for the solution u. The resistance condition at the interface is accounted for in the resulting mixed variational form. The appealing feature is that the core libraries of finite element computing codes are spared. Users only intervene at a higher level as they are left with the construction of the matrices linked to the variational problem they have to solve. Work to do avoids tinkering the low level of the computing programs and is therefore easier to achieve.

We limit our exposition to two dimensions for clarity. Extension to 3 dimensions can be achieved with no other specific difficulty than adding the third dimension.

Assume hence that Ω is a polygon. We suppose that each $\gamma_i = \partial\omega_i$ is a closed polygonal line. Let $(\mathcal{T}_h)_h$ be a regular family of triangulations of Ω by triangles (see [5,8]), in the sense that

- The intersection of any two triangles of $\mathcal{T}_h$ is either empty, a vertex or a common edge to both triangles.
- The ratio of the diameter h_K of any element K of $\mathcal{T}_h$ to the diameter of its inscribed circle is smaller than a constant independent of h.
- Each γ_i is the union of edges of elements of $\mathcal{T}_h$.

The symbol h stands for the maximum of the diameters h_K, $K\in\mathcal{T}_h$. An illustration of the mesh is provided in Figure 1.1

The finite element framework needed for the hybrid dual formulation is standard, not specifically affected by the discontinuity of the solution. The space where u is sought for is defined to be

$$\mathbb{M}_h=\big\{v_h\in L^2(\Omega);\quad \forall K\in\mathcal{T}_h,\quad v_h|_K\in\mathcal{P}_0(K)\big\},$$

where $\mathcal{P}_0(K)$ stands for the space of constant functions on K. The other space where $\boldsymbol{p}=\alpha\nabla u$ is approximated, is built-up on basic Raviart–Thomas elements (see [25])

$$\mathbb{X}_h=\big\{\boldsymbol{q}_h\in H(\text{div};\Omega);\qquad \forall K\in\mathcal{T}_h,\qquad \boldsymbol{q}_h|_K\in\mathcal{RT}_0(K)\big\}.$$

where $RT_0(K)$ is the space of restrictions to K of polynomials of the form $\boldsymbol{a}+a'\boldsymbol{x}$, $\boldsymbol{a}\in\mathbb{R}^d$, $a'\in\mathbb{R}$. It is well-known to be $H(\text{div};\Omega)$-conforming. The lowest degree is retained for simplicity.

The discrete problem is now constructed from problem (1.2) following the Ritz-Galerkin procedure, and reads as: *Find* $(u_h,\boldsymbol{p}_h)$ *in* $\mathbb{M}_h\times\mathbb{X}_h$ *such that*

$$\begin{aligned}\forall\boldsymbol{q}_h\in\mathbb{X}_h,\quad &\int_\Omega\frac{1}{\alpha}\boldsymbol{p}_h\cdot\boldsymbol{q}_h\,d\boldsymbol{x}+\int_\gamma\frac{1}{\beta(\boldsymbol{\tau},[u_h])}(\boldsymbol{p}_h\cdot\boldsymbol{n})(\boldsymbol{q}_h\cdot\boldsymbol{n})\,d\boldsymbol{\tau}\\&\qquad+\int_\Omega u_h(\text{div}\,\boldsymbol{q}_h)\,d\boldsymbol{x}=\langle f,\boldsymbol{q}_h\cdot\boldsymbol{n}\rangle_{1/2,\Gamma},\\\forall v_h\in\mathbb{M}_h,\quad &\int_\Omega v_h(\text{div}\,\boldsymbol{p}_h)\,d\boldsymbol{x}=\int_\Omega Fv_h\,d\boldsymbol{x}.\end{aligned}\tag{4.1}$$

This formulation is chosen for a while so to see how the contact resistance at the interface $(1/\beta)$ influences the variational problem — to be compared with the formulation studied in [3], in the linear context. Note that, due to the choice of $\mathbb{M}_h$ and $\mathbb{X}_h$, the discretization is fully conforming. The main point we are dedicated to is whether this non-linear mixed problem has a solution $(u_h,\boldsymbol{p}_h)$ in $\mathbb{M}_h\times\mathbb{X}_h$ and if it is unique.

Remark 4.1. The existence for the dual hybrid problem (4.1) is questionable in the continuous setting, at first consideration. Does it have a solution in the natural frame work that is $(u,\boldsymbol{p})\in L^2(\Omega)\times\mathbb{X}$? The answer is positive and is a direct consequence of Theorem 3.1 where $\boldsymbol{p}$ is put equal to $(\alpha\nabla u)$. It may be checked out in particular that $\boldsymbol{p}\cdot\boldsymbol{n}=\alpha\partial_{\boldsymbol{n}}u$ belongs to $L^2(\gamma)$ and that $(u,\boldsymbol{p})$ fulfills the variational model (4.1). We shall explain later on why the study of the existence conducted in the dual hybrid framework may fail.

In the discrete frame, the existence for the hybrid dual formulation has to be proved directly. The analysis needs some preliminaries. The model (4.1) will be transformed, so that Minty-Browder theory is enabled.

The continuity and monotonicity of $\varphi(\boldsymbol{\tau},v)$, for a.e $\boldsymbol{\tau}\in\gamma$, implies its invertibility. Then, one can find a monotonous function ψ such that

$$\varphi(\boldsymbol{\tau},\psi(\boldsymbol{\tau},v))=\psi(\boldsymbol{\tau},\varphi(\boldsymbol{\tau},v))=v,\qquad\forall v\in\mathbb{R},\text{a.e. }\boldsymbol{\tau}\in\gamma.$$

It may be verified that $\psi(\cdot,\cdot)$ is a Caratheodory function and that $v\psi(\boldsymbol{\tau},v)>0$ for all $v\in\mathbb{R}\setminus\{0\}$. Furthermore, formulae (2.2) and (2.3) imply some useful properties on ψ. The most interesting is the global Lipschitz continuity

$$|\psi(\boldsymbol{\tau},v)-\psi(\boldsymbol{\tau},w)|\leq\frac{1}{b}|v-w|,\qquad\forall v,w\in\mathbb{R},\text{a.e. }\boldsymbol{\tau}\in\gamma.$$

It will be useful in some place to note that $\psi(\boldsymbol{\tau},v)=v\eta(\boldsymbol{\tau},v)$, where η is of Caratheodory type with $0<\eta(\boldsymbol{\tau},v)\leq 1/b$.

The resistant interface condition can be expressed using the function ψ. Indeed, we have

$$[u_h(\boldsymbol{\tau})]=\psi(\boldsymbol{\tau},(\boldsymbol{p}_h\cdot\boldsymbol{n})(\boldsymbol{\tau})).$$

Plugging it in the first equation of the mixed problem (4.1) yields the new formulation

$$\begin{aligned}&\forall\boldsymbol{q}_h\in\mathbb{X}_h,\quad\int_\Omega\alpha^{-1}\boldsymbol{p}_h\cdot\boldsymbol{q}_h\,d\boldsymbol{x}+\int_\gamma\psi(\boldsymbol{\tau},\boldsymbol{p}_h\cdot\boldsymbol{n})(\boldsymbol{q}_h\cdot\boldsymbol{n})\,d\boldsymbol{\tau}\\&\qquad\qquad\qquad+\int_\Omega u_h(\operatorname{div}\boldsymbol{q}_h)\,d\boldsymbol{x}=\langle f,\boldsymbol{q}_h\cdot\boldsymbol{n}\rangle_{1/2,\Gamma},\\&\forall v_h\in\mathbb{M}_h,\quad\int_\Omega v_h(\operatorname{div}\boldsymbol{p}_h)\,d\boldsymbol{x}=\int_\Omega Fv_h\,d\boldsymbol{x}.\end{aligned}\tag{4.2}$$

At this stage, we recall a highly useful result that has to do with the differential divergence operator (see [25, 26]).

LEMMA 4.1. *The operator* div *is surjective from* $\mathbb{X}_h$ *into* $\mathbb{M}_h$. *It has a right inverse mapping* $\mathbb{M}_h$ *in* $\mathbb{X}_h$ *that is uniformly bounded, with respect to* h. *This means that for all* $v_h\in\mathbb{M}_h$ *there exists* $\boldsymbol{q}_h\in\mathbb{X}_h$ *such that*

$$\|\boldsymbol{q}_h\|_{\mathbb{X}}\leq c\|v_h\|_{L^2(\Omega)}.$$

The constant does not depend on h.

Proof. The proof that div is surjective from $\mathbb{X}_h$ into $\mathbb{M}_h$ can be found in [25, 26]. The uniform stability with respect to the norm of $\mathbb{X}$ has been established in [3]. ∎

5. Uniqueness and existence

This section opens with the uniqueness; it is proceeded directly on problem (4.2). As a matter of convenience, notice that the index $_h$ is dropped off during the proofs.

PROPOSITION 5.1. *The discrete hybrid dual problem* (4.2) *has at most one solution in* $\mathbb{M}_h\times\mathbb{X}_h$.

Proof. Let $(u,\boldsymbol{p})$ and $(u',\boldsymbol{p}')$ be two solutions to problem (4.2) in $\mathbb{M}_h\times\mathbb{X}_h$. Necessarily, we have that for all $\boldsymbol{q}\in\mathbb{X}_h$ and $v\in\mathbb{M}_h$,

$$\begin{aligned}&\forall\boldsymbol{q}_h\in\mathbb{X}_h,\quad\int_\Omega\alpha^{-1}(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{q}\,d\boldsymbol{x}+\int_\gamma[\psi(\boldsymbol{\tau},\boldsymbol{p}\cdot\boldsymbol{n}))-\psi(\boldsymbol{\tau},\boldsymbol{p}'\cdot\boldsymbol{n})](\boldsymbol{q}\cdot\boldsymbol{n})\,d\boldsymbol{\tau}\\&\qquad\qquad\qquad+\int_\Omega(u-u')(\operatorname{div}\boldsymbol{q})\,d\boldsymbol{x}=0,\\&\forall v_h\in\mathbb{M}_h,\quad\int_\Omega v(\operatorname{div}(\boldsymbol{p}-\boldsymbol{p}'))\,d\boldsymbol{x}=0.\end{aligned}$$

Concluding to the uniqueness is done in two steps.

We select $\boldsymbol{q}=(\boldsymbol{p}-\boldsymbol{p}')$ in the first equation, fix $v=(u-u')$ in the second and subtract. Thus, the following formula holds true

$$\int_\Omega \alpha^{-1}(\boldsymbol{p}-\boldsymbol{p}')^2\,d\boldsymbol{x}+\int_\gamma [\psi(\boldsymbol{\tau},\boldsymbol{p}\cdot\boldsymbol{n})-\psi(\boldsymbol{\tau},\boldsymbol{p}'\cdot\boldsymbol{n})](\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}'\cdot\boldsymbol{n})\,d\boldsymbol{\tau}=0.$$

The monotonicity of ψ yields that $(\boldsymbol{p}-\boldsymbol{p}')=0$ and thus $\boldsymbol{p}=\boldsymbol{p}'$. A straightforward consequence is

$$\forall \boldsymbol{q}\in\mathbb{X}_h,\quad \int_\Omega (u-u')(\operatorname{div}\boldsymbol{q})\,d\boldsymbol{x}=0.$$

Owing to the surjectivity of the operator div mapping $\mathbb{X}_h$ onto $\mathbb{M}_h$, one can choose $\boldsymbol{q}$ such that $\operatorname{div}\boldsymbol{q}=(u-u')$. This implies that $u=u'$. The proof is complete. □

When it comes to the existence, a transformation of the mixed variational formulation is needed so as to prepare ultimately the application of the Minty-Browder theory. We introduce the space $\boldsymbol{X}_h=\mathbb{M}_h\times\mathbb{X}_h$, and denote $\underline{\boldsymbol{q}}_h=(v_h,\boldsymbol{q}_h)\in\boldsymbol{X}_h$. Then, define the operators $\mathcal{A}_h:\boldsymbol{X}_h\to\boldsymbol{X}_h'$ and $\mathcal{F}_h\in\boldsymbol{X}_h'$ in such a way that

$$\begin{aligned}\langle\mathcal{A}_h(\underline{\boldsymbol{p}}_h),\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h}&=\int_\Omega \alpha^{-1}\boldsymbol{p}_h\cdot\boldsymbol{q}_h\,d\boldsymbol{x}+\int_\gamma \psi(\boldsymbol{\tau},\boldsymbol{p}_h\cdot\boldsymbol{n})(\boldsymbol{q}_h\cdot\boldsymbol{n})\,d\boldsymbol{\tau}\\&\quad+\int_\Omega u_h(\operatorname{div}\boldsymbol{q}_h)\,d\boldsymbol{x}-\int_\Omega v_h(\operatorname{div}\boldsymbol{p}_h)\,d\boldsymbol{x},\\ \langle\mathcal{F}_h,\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h}&=\langle f,\boldsymbol{q}_h\cdot\boldsymbol{n}\rangle_{1/2,\Gamma}-\int_\Omega F v_h\,d\boldsymbol{x}.\end{aligned}$$

The variational problem (4.2) may equivalently be reworded in the following terms: *Find* $\underline{\boldsymbol{p}}_h$ *in* $\boldsymbol{X}_h$ *such that*

$$\langle\mathcal{A}_h(\underline{\boldsymbol{p}}_h),\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h}=\langle\mathcal{F}_h,\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h},\qquad \forall\underline{\boldsymbol{q}}_h\in\boldsymbol{X}_h. \tag{5.1}$$

Although the non-linear operator $\mathcal{A}_h$ is monotonous, it fails to fulfill the coerciveness criterion to ensure the existence of a solution. A remedy to this failure is to add a regularizing term depending on a parameter $\epsilon>0$, expected to decay to zero. We show that the set of new problems fit with the Minty-Browder framework. Then, we will carry out the analysis to find out what happens when ϵ goes to the limit zero. The new operator to focus on is determined by

$$\langle\mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{p}}_h),\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h}=\langle\mathcal{A}_h(\underline{\boldsymbol{p}}_h),\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h}+\epsilon(u_h,v_h)_{L^2(\Omega)}.$$

The regularized problem reads therefore as follows: *Find* $\underline{\boldsymbol{p}}_{h,\epsilon}$ *in* $\boldsymbol{X}_h$ *such that*

$$\langle\mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{p}}_{h,\epsilon}),\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h}=\langle\mathcal{F}_h,\underline{\boldsymbol{q}}_h\rangle_{\boldsymbol{X}_h',\boldsymbol{X}_h},\qquad \forall\underline{\boldsymbol{q}}_h\in\boldsymbol{X}_h. \tag{5.2}$$

PROPOSITION 5.2. *The regularized problem* (5.2) *has only one solution* $\underline{\boldsymbol{p}}_{h,\epsilon}\in\boldsymbol{X}_h$.

Proof. The continuity of the operator $\mathcal{A}_{h,\epsilon}$ can be proven following the same lines as in the proof of Lemma 3.1 and is skipped over. It is actually based on the fact that ψ is Lipschitz. Another important point is the strict monotonicity. We have, after activating the monotonicity of $\psi()$ ([1])

$$\langle \mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{p}}) - \mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{q}}), \underline{\boldsymbol{p}} - \underline{\boldsymbol{q}} \rangle_{\boldsymbol{X}'_h, \boldsymbol{X}_h} \geq \int_\Omega \alpha^{-1} (\boldsymbol{p} - \boldsymbol{q})^2 \, d\boldsymbol{x} + \epsilon \int_\Omega (u - v)^2 \, d\boldsymbol{x}.$$

Due to the equivalence of all the norms on the finite dimensional space $\boldsymbol{X}_h$, we derive that

$$\langle \mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{p}}) - \mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{q}}), \underline{\boldsymbol{p}} - \underline{\boldsymbol{q}} \rangle_{\boldsymbol{X}'_h, \boldsymbol{X}_h} \geq b'' \|\underline{\boldsymbol{p}} - \underline{\boldsymbol{q}}\|^2_{\boldsymbol{X}}.$$

The constant $b'' > 0$ is of course dependent on (ϵ, h). This technical work done, we are now allowed to apply Minty-Browder's theorem The regularized problem has therefore a unique solution $\underline{\boldsymbol{p}}_{h,\epsilon} \in \boldsymbol{X}_h$. The proof is complete. ∎

A decisive result, in view of passing to the limit in the regularized problem, are some a priori estimates on $\boldsymbol{p}_{h,\epsilon}$ that are uniform with respect to ϵ.

PROPOSITION 5.3. *The following stability holds*

$$\|\underline{\boldsymbol{p}}_{h,\epsilon}\|_{\boldsymbol{X}} \leq C(\|f\|_{H^{1/2}(\gamma)} + \|F\|_{L^2(\Omega)}).$$

The constant does not depend on ϵ.

Proof. We still maintain the convention that $_h$ is canceled for a clear exposition. The proof is obtained after three steps.

(*i.*) We fix in (5.2) the test function $\underline{\boldsymbol{q}} = (\operatorname{div} \boldsymbol{p}_\epsilon, 0) \in \boldsymbol{X}_h$. We straightforwardly ensue the bound

$$\|\operatorname{div} \boldsymbol{p}_\epsilon\|_{L^2(\Omega)} \leq \epsilon \|u_\epsilon\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}. \tag{5.3}$$

(*ii.*) Now, we select $\underline{\boldsymbol{q}} = \underline{\boldsymbol{p}}_\epsilon = (u_\epsilon, p_\epsilon) \in \boldsymbol{X}_h$. As a result, we have in particular that

$$\left\|(\sqrt{\alpha})^{-1} \boldsymbol{p}_\epsilon\right\|^2_{L^2(\Omega)^2} + \left\|\sqrt{\eta(\cdot, \boldsymbol{p}_\epsilon \cdot \boldsymbol{n})} (\boldsymbol{p}_\epsilon \cdot \boldsymbol{n})\right\|^2_{L^2(\gamma)} \leq \langle f, \boldsymbol{p}_\epsilon \cdot \boldsymbol{n} \rangle_{1/2,\Gamma}, -(F, u_\epsilon)_{L^2(\Omega)}.$$

Combining with (5.3) and using Young's inequality, we deduce that for any small $\kappa > 0$,

$$\begin{aligned} &\|\boldsymbol{p}_\epsilon\|_{H(\operatorname{div},\Omega)} + \left\|\sqrt{\eta(\cdot, \boldsymbol{p}_\epsilon \cdot \boldsymbol{n})} (\boldsymbol{p}_\epsilon \cdot \boldsymbol{n})\right\|_{L^2(\gamma)} \\ &\leq c \left[(\epsilon + \kappa) \|u_\epsilon\|_{L^2(\Omega)} + \|f\|_{H^{1/2}(\Gamma)} + (1 + \frac{1}{4\kappa}) \|F\|_{L^2(\Omega)} \right]. \end{aligned} \tag{5.4}$$

To control $(\boldsymbol{p}_\epsilon \cdot \boldsymbol{n})_{|\Gamma}$, the trace theorem in $H(\operatorname{div}\Omega)$ is applied here. Moreover, we use that $\alpha(\cdot)$ is bounded away from zero.

(*iii.*) At last, putting $\underline{\boldsymbol{q}} = (0, \boldsymbol{q}) \in \boldsymbol{X}_h$ produces the identity

$$\begin{aligned} (\operatorname{div} \boldsymbol{q}, u_\epsilon)_{L^2(\Omega)} = \langle f, \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{1/2,\Gamma} &- ((\sqrt{\alpha})^{-1} \boldsymbol{p}_\epsilon, \boldsymbol{q})_{L^2(\Omega)^2} \\ &- (\eta(\cdot, \boldsymbol{p}_\epsilon \cdot \boldsymbol{n})(\boldsymbol{p}_\epsilon \cdot \boldsymbol{n}), \boldsymbol{q} \cdot \boldsymbol{n}_{L^2(\gamma)})_{L^2(\gamma)}. \end{aligned}$$

[1]The index h is dropped.

Choose $\boldsymbol{q}$ as in Lemma 4.1, we derive that

$$\begin{aligned}\|u_\epsilon\|_{L^2(\Omega)} &\le c\Big[\|f\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{p}_\epsilon\|_{L^2(\Omega)} + \|\eta(\cdot,\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})(\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})\|_{L^2(\gamma)}\Big]\\ &\le c'\left[\|f\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{p}_\epsilon\|_{L^2(\Omega)} + \left\|\sqrt{\eta(\cdot,\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})}(\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})\right\|_{L^2(\gamma)}\right].\end{aligned}$$

The boundedness of $\eta(\cdot,\cdot)$ is used. Next, estimate (5.4) implies that

$$\|u_\epsilon\|_{L^2(\Omega)} \le c''\left[(\epsilon+\kappa)\|u_\epsilon\|_{L^2(\Omega)} + \|f\|_{H^{1/2}(\Gamma)} + (1+\frac{1}{4\kappa})\|F\|_{L^2(\Omega)}\right].$$

Given that we are interested in ϵ only when it decays toward zero, after choosing κ small enough, we get the final bound

$$\|u_\epsilon\|_{L^2(\Omega)} \le c'''\Big[\|f\|_{H^{1/2}(\Gamma)} + \|F\|_{L^2(\Omega)}\Big].$$

Accounting for this stability on (5.4) yields finally

$$\|\underline{\boldsymbol{p}}_{h,\epsilon}\|_{H(\mathrm{div},\Omega)} + \|u_{h,\epsilon}\|_{L^2(\Omega)} \le C(\|f\|_{H^{1/2}(\gamma)} + \|F\|_{L^2(\gamma)}).$$

After observing that all the norms are equivalent on the finite dimensional space $\boldsymbol{X}_h$, the proof is complete. ∎

Remark 5.1. The constant in *Proposition 5.3* may depend on h. It actually does unless the conductance function $\beta(\cdot,[u])$ is bounded. This means that, the constant b_r in (2.3) is zero which is pretty restrictive. If $\beta(\cdot,[u])$ is not bounded, then $\eta(\cdot,\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})$ in (5.4) decays close to zero. The failure to control $\boldsymbol{p}_\epsilon\cdot\boldsymbol{n}$ along the interface γ is hence the consequence and we cannot then go effectively to the limit on $(\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})_{\epsilon>0}$. That is the reason why the analysis of the continuous model has been conducted in a different and successful formulation.

Technical tools necessary to the existence result are at our disposal; we can annouce the final result for the approximated problem (4.1).

Theorem 5.1. *The discrete mixed problem* (4.1) *has only one solution* $(u_h,\boldsymbol{p}_h)\in \mathbb{M}_h\times\mathbb{X}_h$.

Proof. The aim is to show that the sequence $(\underline{\boldsymbol{p}}_\epsilon)_{\epsilon>0}$ fulfills Cauchy's criterion of convergence in $\boldsymbol{X}_h$. We proceed as in Proposition 5.1. The key in our argumentation is the identity

$$\forall \underline{\boldsymbol{q}}\in\boldsymbol{X}_h,\qquad \langle\mathcal{A}_{h,\epsilon}(\underline{\boldsymbol{p}}_\epsilon)-\mathcal{A}_{h,\zeta}(\underline{\boldsymbol{p}}_\zeta),\underline{\boldsymbol{q}}\rangle_{\boldsymbol{X}'_h,\boldsymbol{X}_h}=0. \tag{5.5}$$

Choosing $\underline{\boldsymbol{q}}=(\mathrm{div}(\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta),0)$ produces

$$\|\mathrm{div}(\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta)\|_{L^2(\Omega)} \le (\epsilon+\zeta)(\|u_\epsilon\|_{L^2(\Omega)}+\|u_\zeta\|_{L^2(\Omega)}).$$

On the other hand, taking $\underline{\boldsymbol{q}}=(\underline{\boldsymbol{p}}_\epsilon-\underline{\boldsymbol{p}}_\zeta)$ in (5.5), we have after simplification

$$\begin{aligned}&\left\|(\sqrt{\alpha})^{-1}(\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta)\right\|^2_{L^2(\Omega)^2}+(\psi(\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})-\psi(\boldsymbol{p}_\zeta\cdot\boldsymbol{n}),\boldsymbol{p}_\epsilon\cdot\boldsymbol{n}-\boldsymbol{p}_\zeta\cdot\boldsymbol{n})_{L^2(\gamma)}\\ &=(\epsilon u_\epsilon-\zeta u_\zeta,\mathrm{div}(\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta))\\ &\le\|\epsilon u_\epsilon-\zeta u_\zeta\|_{L^2(\Omega)}\|\mathrm{div}(\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta)\|_{L^2(\Omega)}.\end{aligned}$$

The monotonicity of ψ, together with the stability in Proposition 5.3 shows that $(\underline{\boldsymbol{p}}_\epsilon)_{\epsilon>0}$ satisfies Cauchy's criterion in $H(\mathrm{div},\Omega)$. This sequence has then a limit $\boldsymbol{p}\in\mathbb{X}_h$, with respect to the norm of $H(\mathrm{div},\Omega)$, when ϵ tends toward zero. As a result, $(p_\epsilon\cdot\boldsymbol{n})_{\epsilon>0}$ converges towards $p\cdot\boldsymbol{n}$ with respect to $L^2(\gamma)$-norm. Moreover, $(\psi(\cdot,p_\epsilon\cdot\boldsymbol{n}))_{\epsilon>0}$ converges towards $\psi(\cdot,p_\epsilon\cdot\boldsymbol{n})$ because ψ is Lipschitz-continuous. We stress the fact that the last two convergences may not be uniform with respect to the mesh size h. Appositely, the convergence of $(\underline{\boldsymbol{p}}_\epsilon)_{\epsilon>0}$ towards $\boldsymbol{p}$ in $H(\mathrm{div},\Omega)$ is uniform with respect to h.

The only proof that remains is the convergence for $(u_\epsilon)_{\epsilon>0}$ which requires to get back (5.5) and select $\underline{\boldsymbol{q}}=(0,\boldsymbol{q})\in\boldsymbol{X}_h$. We deduce that

$$(\mathrm{div}\,\boldsymbol{q},(u_\epsilon-u_\zeta))_{L^2(\Omega)}=\int_\Omega\frac{1}{\alpha}(\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta)\cdot\boldsymbol{q}\,d\boldsymbol{x}+\int_\gamma[\psi(\boldsymbol{\tau},\boldsymbol{p}_\epsilon\cdot\boldsymbol{n})-\psi(\boldsymbol{\tau},\boldsymbol{p}_\zeta\cdot\boldsymbol{n})](\boldsymbol{q}\cdot\boldsymbol{n})\,d\boldsymbol{\tau}$$
$$\le C\left[\|\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta\|_{L^2(\Omega)^2}+\|\psi(\cdot,p_\epsilon\cdot\boldsymbol{n})-\psi(\cdot,p_\zeta\cdot\boldsymbol{n})\|_{L^2(\Omega)^2}\right]\|\boldsymbol{q}\|_{\mathbb{X}}.$$

Choosing $\boldsymbol{q}$ such that $\mathrm{div}\,\boldsymbol{q}=(u_\epsilon-u_\zeta)$ as in Lemma 4.1 we ensue that

$$\|u_\epsilon-u_\zeta\|_{L^2(\Omega)}\le C\left[\|\boldsymbol{p}_\epsilon-\boldsymbol{p}_\zeta\|_{L^2(\Omega)^2}+\|\psi(\cdot,p_\epsilon\cdot\boldsymbol{n})-\psi(\cdot,p_\zeta\cdot\boldsymbol{n})\|_{L^2(\Omega)^2}\right].$$

The sequence $(u_\epsilon)_{\epsilon>0}$ is then a Cauchy sequence and converges therefore towards some $u\in\mathbb{M}_h$ with respect to $L^2(\Omega)$–norm. Hence, the last result that $(\underline{\boldsymbol{p}}_\epsilon)_{\epsilon>0}$ converges toward $\underline{\boldsymbol{p}}_\epsilon$ in $\boldsymbol{X}$. We need to pass to the limit in (5.2) to be complete. It is quite easy to achieve it. This proves that $\underline{\boldsymbol{p}}$ is the solution to problem (5.1). The uniqueness in Proposition 5.2, completes the proof. ∎

6. Computational examples

The computations are carried out within the code `FreeFem++` (see [13, 14]) where various finite elements have been developed such as $\mathcal{P}_0,\mathcal{P}_1,\mathcal{RT}_0$, and $\mathcal{RT}_1$, to cite only those we need (see [6, 26]). Scripts to take into account the contact resistance have been developed by us. The non-linearity of the contact resistance is handled by Newton's algorithm.

One capsule model — In the first example, the square $\Omega=[-0.5,0.5]\times[-0.5,0.5]$ is the domain, with a single circular capsule Ω_I, centered at the origin with radius 0.3. The sub-domain surrounding Ω_I is the matrix Ω_E. The interface γ is hence the circular boundary of the capsule. The horizontal edges are considered adiabatic and Dirichlet conditions are enforced along both vertical edges. The left one is subjected to a homogeneous data, $f(-0.5,x_2)=0$, while on the right edge a non-homogeneous condition $f(0.5,x_2)=10(1+x_2(1-x_2))$ is prescribed. The load source F is constant in Ω_E. In Ω_I, it is supported in the left half of the concentric disc with half radius 0.15. The isolines of F are plotted in Figure 6.1. The important fact to be noticed is that while F is regular in Ω_E, it has a limited smoothness within Ω_I since $F\in H^{(1/2)-}(\Omega_I)$, because of the drop at $x_2=0$. The conductivities are chosen so that $(\alpha_E,\alpha_I)=(689,1)$ and the conductance is given by

$$\beta_1(v)=4+3(1+\tanh[10(\sqrt{v^2+0.01}-0.25)].$$

The shape of β_1 is similar to that of the conductances used in Electroporation (see [18]).

Given the unavailability of an explicit expression for u, we compute a reference solution $(u_r,\boldsymbol{p}_r)$ by the mixed finite elements $\mathcal{RT}_1\times\mathcal{P}_1$ with a high resolution mesh, that is $h_r=0.0066$. The reference field $(u_r,\boldsymbol{p}_r)$ is hence a sharp approximation of the exact solution and is assimilated to (u,p) for our purposes. Then, the discrete solutions

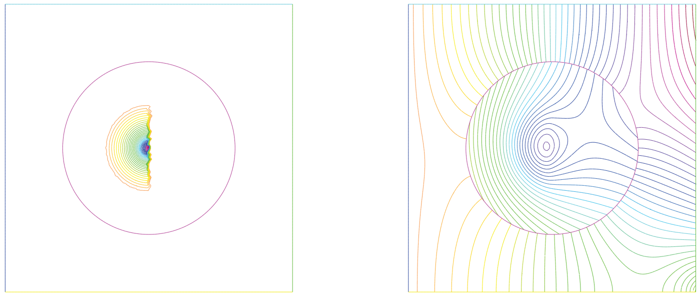

FIG. 6.1. *The data F (left) and the solution u (right).*

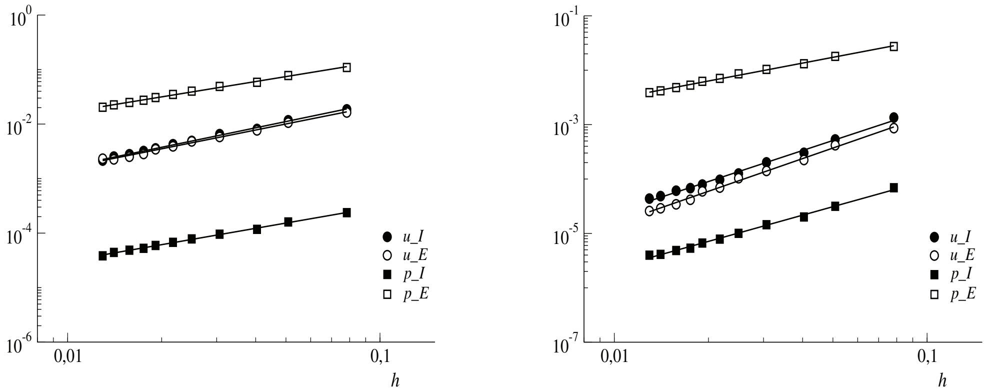

FIG. 6.2. *Convergence curves for $\mathcal{RT}_0 \times \mathcal{P}_0$ (left) and $\mathcal{RT}_1 \times \mathcal{P}_1$ (right).*

$(u_h, \boldsymbol{p}_h)$ are computed by the $\mathcal{RT}_0 \times \mathcal{P}_0$ finite element method, for meshes with size $h \in [0.06, 0.4]$, and they are compared to $(u_r, \boldsymbol{p}_r)$. Figure 6.2 depicts, in logarithmic scales, different errors evaluated in the L^2-norm. These errors are computed separately within each sub-domain Ω_I and Ω_E.

The convergence rates are obtained by calculating the coefficient ν in the regressions $e = \eta h^\nu$ of the error curves (linear regression in log. scales). The evaluations yield to the rates $(1.18, 0.99)$ for $(u, \boldsymbol{p}_h)$ in Ω_I and $(1.04, 0.93)$ in Ω_E. The errors decay approximately like h in both sub-domains Ω_I and Ω_E and for both unknowns. This predicts that $(u, \boldsymbol{p})$, restricted to each sub-domain belong to $H^1 \times \boldsymbol{H}^1$, for the least. Before attempting an explanation to these convergence rates, we run additional simulations for the same meshes where $(u_h, \boldsymbol{p}_h)$ is approximated by $\mathcal{RT}_1 \times \mathcal{P}_1$ finite elements. Error curves are displayed in the right panel in Figure 6.2. The convergence rates are augmented to $(1.90, 1.60)$ in Ω_I and $(1.96, 0.93)$ in Ω_E.

As is well known, for a given finite element method, the error estimates, and then the convergence rates, are tightly related to the regularity of the solution u (and then $\boldsymbol{p} = \alpha(\nabla u)$). One can see two possible restrictions to the smoothness of u. The first cause is the low regularity of the $F_{|\Omega_I}$ which prevent the solution from being in $H^{5/2}(\Omega_I)$. The second limitation has to do with the corners of Ω_E.

The regularity study starts from the fact that $u \in \mathbb{V}$, then follows a bootstrapping argument. We notice that

$$\alpha_I(\partial_{\boldsymbol{n}} u_I) = \alpha_E(\partial_{\boldsymbol{n}} u_E) = \beta_1([u])[u] \in H^{1/2}(\gamma). \tag{6.1}$$

This tells that $u_I \in H^2(\Omega_I)$ and $u_E \in H^{2-}(\Omega_E)$ according to [9, 11]. We cannot hope for more regularity on u_E due to the singularity born at the corner $(0.5, -0.5)$. The smoothness is limited because the boundary conditions change when passing the vertex, aggravated by the non-homogeneous Dirichlet data. In fact, Neumann and Dirichlet data do not match at that corner since $\partial_{x_2} f(0.5, -0.5) \neq 0$. Thus, the solution cannot even be in the Sobolev space H^2 in the vicinity of $(0.5, -0.5)$ (see [11]). At the remaining corners, boundary conditions match together. This does not create stiff singularities on the solution u. Therefore, we know so far that definitely $u_E \in H^{2-}(\Omega_E)$ and $\boldsymbol{p}_E \in \boldsymbol{H}^{1-}(\Omega_E)$. Now, back to (6.1), we have a further regularity on the flux, indeed $\alpha_I(\partial_{\boldsymbol{n}} u_I) \in H^{3/2}(\gamma)$ and $F \in H^{(1/2)-}(\Omega_I)$. According to the elliptic regularity, we obtain that $u_I \in H^{(5/2)-}(\Omega_I)$ and $p_I \in \boldsymbol{H}^{(3/2)-}(\Omega_I)$. The $\mathcal{RT}_1 \times \mathcal{P}_1$ method is expected to comply with the error estimates (see [25])

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{2-}, \qquad \|\boldsymbol{p} - \boldsymbol{p}_h\|_{\boldsymbol{L}^2(\Omega_I)} \leq Ch^{(3/2)-}, \qquad \|\boldsymbol{p} - \boldsymbol{p}_h\|_{\boldsymbol{L}^2(\Omega_E)} \leq Ch^{1-}.$$

The effective rates calculated above are not far away from the theoretical ones, if one admits the following approximations $(1.90 \approx 1.99 \approx 2)$, $(1.60 \approx 3/2)$ and $(0.93 \approx 1)$.

Multi-capsules model— A more elaborate example is involved with the domain represented in Figure 1.1, the rectangle $\Omega = [-1.2, -1.2 + 7.5] \times [-2.8, -2.8 + 3.8]$. Four square capsules $(\omega_i)_{1 \leq i \leq 4}$ are contained in Ω; their union is Ω_I. The edges of the capsules are all equal to 1.2. Ω_E is the sub-domain external to the capsules. The interface γ is the union of the four square boundaries of $(\omega_i)_{1 \leq i \leq 4}$. The conductivities are still $(\alpha_E, \alpha_I) = (689, 1)$. We realize simulations with the conductance

$$\beta_2(v) = \varrho(1 + \frac{|v|}{2}).$$

The boundary conditions are analogous to the first example on Γ. The horizontal edges are again adiabatic and the vertical ones are submitted to Dirichlet conditions. The load data F is represented in the left plot of Figure 6.3. Counting from the left, the second capsule is heated while the third is cooled. This data has a reduced smoothness in each subdomain Ω_I and in Ω_E. It is however enough so that it will not be a limiting factor of smoothness to u.

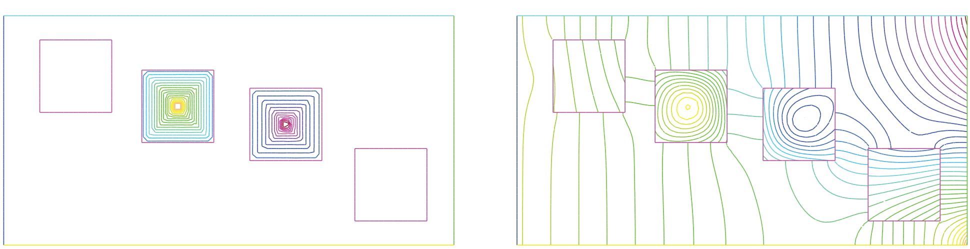

FIG. 6.3. *The data F (left) and the solution u (right).*

The reference solution $(u_r, \boldsymbol{p}_r)$ is obtained, for $\varrho = 4$, by the method $\mathcal{RT}_1 \times \mathcal{P}_1$. The size of the reference mesh $h_r = 0.027$. The discrete solutions $(u_h, \boldsymbol{p}_h)$ are obtained

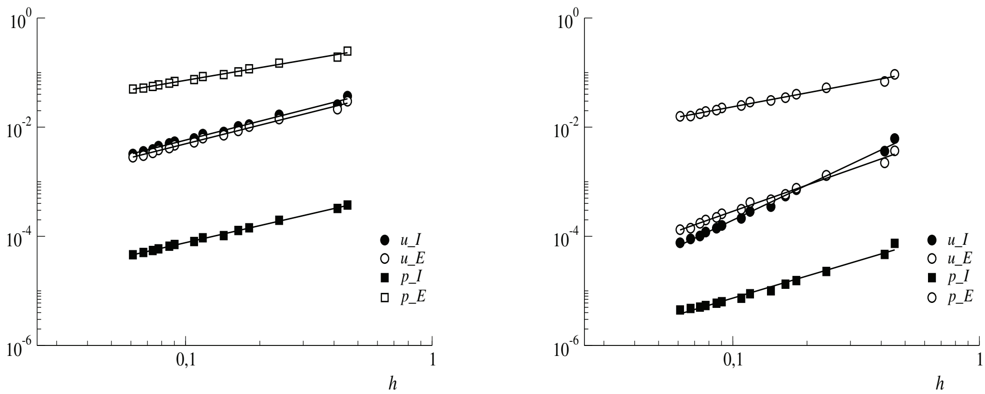

FIG. 6.4. *Convergence curves for* $\mathcal{RT}_0\times\mathcal{P}_0$ *(left) and* $\mathcal{RT}_1\times\mathcal{P}_1$ *(right).*

by $\mathcal{RT}_0\times\mathcal{P}_0$ and $\mathcal{RT}_1\times\mathcal{P}_1$ methods, for meshes with sizes in $h\in[0.06,0.45]$ and are compared to $(u_r,\boldsymbol{p}_r)$. In logarithmic scales, Figure 6.4 provides different errors evaluated in the L^2-norm. The calculations of the regressions result in the following convergence rates (1.15,1.03) in Ω_I and (1.13,0.73) in Ω_E for the $\mathcal{RT}_0\times\mathcal{P}_0$ method while for the $\mathcal{RT}_1\times\mathcal{P}_1$ method, they are (2.13,1.34) in Ω_I and (1.59,0.84) in Ω_E.

To find out the regularity of $(u,\boldsymbol{p})$, we scan the flux across the boundary of each capsule,

$$\alpha_I(\partial_{\boldsymbol{n}}u_I)=\beta_2([u])[u]\in H^{1/2}(\gamma_i),\qquad 1\le i\le 4. \tag{6.2}$$

As a result, we have $u_I\in H^2(\Omega_I)$. At this step, the right angles (when seen from inside Ω_I) induce no harm on the smoothness of u_I. Things are different, when inspecting the external side of these corners, we are therefore in Ω_E. These external angles are reflex, their common measure is $\frac{3\pi}{2}$. Thus, u_E cannot be more regular than being in $H^{(1+2/3)-}(\Omega_E)$. Back to u_I, we dispose of further regularity on the flux $\alpha_I(\partial_{\boldsymbol{n}}u_I)$, owing to (6.2). It belongs in fact to $H^{(1+1/6)-}(\gamma_i)$. This implies that $u_I\in H^{(1+1/6+3/2)-}(\Omega_I)$. The limitation one would expect due to the internal corner $\frac{\pi}{2}$ is not reached yet. That reduction is active starting from $H^{3-}(\Omega_I)$. To summarize, the following regularities are available

$$(u_I,p_I)\in H^{8/3}(\Omega_I)\times H^{5/3}(\Omega_E),\qquad (u_E,p_E)\in H^{5/3}(\Omega_I)\times H^{2/3}(\Omega_E).$$

The table below provides, for both Raviart-Thomas finite elements, the convergence rates we observe and the rates predicted by theory. We point out that the limitation of the convergence order to 2 for u_I, with the $\mathcal{RT}_1\times\mathcal{P}_1$-approximation, is inherent to the degree of the finite elements and not to any lack of smoothness of u_I. A careful scrutiny of Table 6.1 show a satisfactory agreement between computations and the theory.
Micro-encapsulated model— We close with the rectangular sample with a complex granularity, depicted in Figure 1.1 (right panel). The capsules Ω_I, constituted of salt, a low (thermal) conductive material, are in grey while a high conducting graphite foam is the surrounding dark (or black) media Ω_E. The hybrid dual method is still tractable for this kind of geometry. The integral term in (4.1), responsible for the contact resistance, does not create any notable difficulty in $\texttt{FreeFem}_{++}$. Using the indicator function of

	u_I	$\boldsymbol{p}_I$	u_E	$\boldsymbol{p}_E$
$\mathcal{RT}_0 \times \mathcal{P}_0$	1.15 (1.00)	1.03 (1.00)	1.13 (1.00)	0.73 (0.66)
$\mathcal{RT}_1 \times \mathcal{P}_1$	2.13 (2.00)	1.34 (1.66)	1.59 (1.66)	0.84 (0.66)

TABLE 6.1. *Computational convergence rates versus (theoretical rates).*

the geometry, equal to one in Ω_I and to zero in Ω_E, allows for an easy evaluation of the interface integral contribution to the variational problem. `FreeFem++` offers a variety of procedures and functions making it easy to compute those type of integrals (see [14]).

The composite domain is rectangular. Its width is 1.692×10^{-3} and its length equals 2.772×10^{-3}. The sample is differentially heated at the vertical walls while the horizontal walls remain adiabatic. The Dirichlet data is fixed to $f = 585.15$ on the left wall and $f = 575.15$ on the right wall. The conductivities are still $(\alpha_I, \alpha_E) = (1, 689)$. The conductance is given by β_2 given above where $\varrho = 100$. The mesh we use is depicted in the left panel in Figure 6.5 and the computed temperature is plotted on the right panel. In the last simulations, the interface is getting strongly resistant. The conductance is then gradually weaker and corresponds to $\varrho = 1$ and then to $\varrho = 0.01$ in the expression of β_2. The diffusion of the heat is slowed at the interface which explains why the most visible isolines start to embrace the shape of the interfaces especially in the hot part of the sample.

The aim of these illustrations is to display the complex structure (consider the isolines) of the temperature successfully captured here by the hybrid dual finite elements.

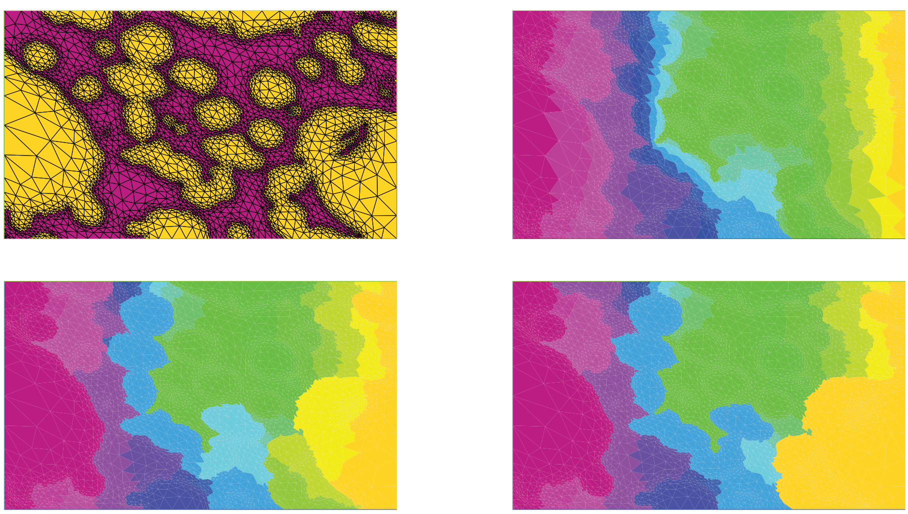

FIG. 6.5. *The mesh in the composite sample. The computed temperature with decreasing conductances,* $\varrho = 100, 1, 0.01$.

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REFERENCES

[1] Robert A. Adams and J.J.F. Fournier, *Sobolev Spaces*, Academic Press, 85, 2003.

[2] H. Ammari, T. Widlak, and W. Zhang, *Towards monitoring critical microscopic parameters for electropermeabilization*, Quart. Appl. Math., 75:1–17, 2017.

[3] F. Ben Belgacem, C. Bernardi, F. Jelassi, and M. Mint Brahim, *Finite element methods for the temperature in composite media with contact resistance*, J. Sci. Comput., 63:478–501, 2015.

[4] M. Boulakia, S. Cazeau, M.A. Fernández, J.-F. Gerbeau, and N. Zemzemi, *Mathematical modeling of electrocardiograms: a numerical study*, Annals of Biomedical Engineering, 38:1071–1097, 2010.

[5] S.C. Brenner and L.R. Scott, *Mathematical Theory of Finite Element Methods*, Springer, 2008.

[6] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.

[7] F.E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc., 69:862–874, 1963.

[8] Ph. Ciarlet, *The finite element method for elliptic problems*, J. Appl. Mech., 45(4):968–969, 1978.

[9] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Mathematics, Springer Berlin, 1341, 1988.

[10] L.S. Fletcher, *Conduction in solids, Imperfect metal-to-metal contacts: Thermal contact resistance*, Section 502.5, Heat Transfer and Fluid Mechanics Data Books, Genium Publishing Company, Schenectady, New York, 1991.

[11] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics in Applied Mathematics, Soc. Indust. Appl. Math., 2011.

[12] A. Guittet, A.C. Poignard, and F. Gibouab, *A Voronoi Interface approach to cell aggregate electropermeabilization*, J. Comput. Phys., 332:143–159, 2017.

[13] F. Hecht, *FreeFem++, User's Guide*, Université Pierre et Marie Curie, Paris, 2011.

[14] F. Hecht, *New development in FreeFem++*, J. Numer. Math., 20:251–265, 2012.

[15] F. Jelassi, M. Azaïez, and E. Palomo Del Barrio, *A substructuring method for phase change modelling in hybrid media*, Computers & Fluids, 88:81–92, 2013.

[16] H. Ji, F. Wang, and J. Chen, *Unfitted finite element methods for the heat conduction in composite media with contact resistance*, Numer. Meth. Part. Diff. Eqs., 33:354–380, 2017.

[17] W. Krassowska and P.D. Filev, *Modeling electroporation in a single cell*, Biophys. J., 92:404–417, 2007.

[18] O. Kavian, M. Leguèbe, C. Poignard, and L. Weynans, *"Classical" electropermeabilization modeling at the cell scale*, J. Math. Bio., 68:235–265, 2014.

[19] M. Léguèbe, *Modélisation de l'électroperméabilisation à l'échelle cellulaire*, PhD Thesis, Institut de Mathmatiques de Bordeaux, Université de Bordeaux, 2014.

[20] M. Leguèbe, L.M. Silve, L.M. Mir, and C. Poignard, *Conducting and permeable states of cell membrane submitted to high voltage pulses: Mathematical and numerical studies validated by the experiments*, J. Theor. Biol., 360:83–94, 2014.

[21] J.L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications*, Dunod, 1968.

[22] M. Mint Brahim, *Méthodes d'éléments finis pour le problème de changement de phase en milieux composites*, PhD Thesis, Institut de Mécanique et d'Ingénierie de Bordeaux, Université de Bordeaux, 2016.

[23] G.J. Minty, *On a monotonicity method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A., 50:1038–1041, 1963.

[24] J.C. Neu and W. Krassowska, *Asymptotic model of electroporation*, Phys. Rev. E, 59:3471–3482, 1999.

[25] P.-A. Raviart and J.-M. Thomas, *A mixed finite element method for 2-nd order elliptic problems*, in Lecture Notes in Mathematics, Springer, 606:292–315, 1977.

[26] J.E. Roberts and J.-M. Thomas, *Mixed and hybrid methods*, in Handbook of Numerical Analysis, North-Holland, Amsterdam, 2:523–639, 1991.

[27] E.T. Swartz and R.O. Pohl, *Thermal boundary resistance*, Rev. Mod. Phys., 61:605–668, 1989.